WAVEGUIDES

In the electromagnetic spectrum, microwaves are the waves with wavelengths comparable to ordinary laboratory dimensions. Furthermore, smooth surfaces of good conductors form very perfect reflectors for them. As a consequence, an electromagnetic wave in a reflecting pipe is reflected back and forth from wall to wall, so that it can travel to large distances with small attenuation. Hence we have "wave guides" as transmission lines, dealing with electromagnetic fields inside of hollow regions, rather than outside wires for more conventional electric applications, or waves in free space as in optics, and the propagating electromagnetic field is confined to the finite region of the guide by reflecting walls. The present article essentially reviews the basic theory and the different geometries used in the various applications of such hollow waveguides.

FUNDAMENTALS

Historical Evolution

J. C. Maxwell established the fundamentals of electromagnetic theory around 1880. In 1883, F. G. Fitzgerald suggested sources for electromagnetic emission. In 1888, H. Hertz proved that the concept of propagation was included in Maxwell's theory. At the same time, O. Lodge demonstrated the existence of maxima and minima on transmission lines. A group of experimental scientists was formed—the Hertzians—while J. C. Bose was experimenting with millimeter waves in India. In 1894, Lodge demonstrated radiation from circular hollow pipes—waveguides—as well as the effect of irises and resonant cavities, and illustrated the highpass properties of the device. Bose developed a semiconductor detector, rectangular waveguides and horns, at 60 GHz (1–5). In 1893, Heaviside considered various possibilities for waves along wire lines from a theoretical standpoint and concluded that guided waves needed "two leads as a pair of parallel lines; or if but one is be used, there is the earth, or something equivalent, to make another" (6). J. J. Thompson gave a theoretical analysis of electric oscillations within a conducting cylindrical cavity of finite length. He found that there were permissible normal modes that were a function of the radius of the cylinder (7). Shortly thereafter, J. Larmor similarly investigated the theory of resonant structures such as coaxial metallic cylinders and a single dielectric cylinder (8).

In 1897, Rayleigh showed that waves could indeed propagate within a hollow metallic cylinder (9). He found that such waves existed only in a set of well-defined normal modes, with waves of two types: one with a longitudinal component of electric intensity only, while the other had a longitudinal component of magnetic intensity only. Both types had transverse components of both electric and magnetic intensity. He found that a fundamental limitation on the existence of such waves was that the frequency must exceed a lower limit—cut-off frequency—depending on mode number and the cross-sectional dimensions of the cylinder. He published the complete solutions in the case of rectangular and circular cross-section, yielding all possible solutions.

Hertzian links, feeders, detectors, and even radioastronomy with an attempt by Lodge were on the horizon. As a matter of fact, none of this happened, probably because of both the success of the long waves used by Marconi and the difficulty in generating microwaves. Lodge got interested in paranormal phenomena, and Bose in plant growth and biological effects of electromagnetic fields. Microwave electronics fell out of favor.

Almost 40 years later, G. C. Southworth, working with Schelkunoff, of Bell Telephone Laboratories (BTL), and W. L. Barrow, working with Chu, Stratton, and others, of the Massachusetts Institute of Technology (MIT), rediscovered the concept, each working independently for almost five years with no knowledge of the other. The question they investigated was the practical possibility of using waveguides for the transmission of microwave power. This work culminated in almost simultaneous presentations in 1936, when they claimed that they had discovered that electromagnetic waves would propagate in hollow tubes and had experimentally demonstrated the practicality of this phenomenon. They had become aware of each other's work about one month prior to the announcements, after publication of the programs for the meetings at which the presentations were to be made (3).

From the beginning, the most obvious application of waveguide had been as a communications medium. It had been determined by both Schelkunoff and Mead independently in 1933 that an axially symmetric electric wave in circular waveguide would have an attenuation factor that decreased with increased frequency (3). This unique characteristic was believed to offer a great potential for wideband multichannel systems, and for many years to come the development of such a system was a major focus of activity within the waveguide group at BTL. The use of a waveguide as a long transmission line, however, did not prove to be practical and Southworth himself concluded in 1939 that microwave radio with highly directive antennas was preferable, coming to the conclusion that the hollow cylindrical conductor would be valued as a new circuit element, a new type of toll cable.

The years from 1936 to 1940 saw great activity in microwave electronics, but hardly with important practical applications. The great impetus to its further development came with the application to radar in World War II. Generally, the development of experimental radar into an operational system is attributed to R. Watson-Watt, having convinced politicians to install a range of radar protection on the British coast. At the end of the summer of 1938, five stations were protecting London. In July 1940, fifty stations were operational. The efficiency of the system was determinant on the evolution of the war in Europe. During the war, the famous MIT Radiation Laboratory was the place where British and U.S. scientists and technicians worked together, developing microwave technology and electronics as well as the basic theory. The series of twenty-eight books published after the war summarized the tremendous amount of high quality work accomplished at that time (2,10).

Since then, waveguides have been very common in microwaves, especially in the centimeter and millimeter wave ranges, mostly because of their losses, lower than in coaxial cables. Another reason is that the waveguide forms a closed transmission line, which is an advantage in certain environments, in particular in the presence of interferences or adverse tropospheric conditions. The tendency to go to higher frequencies has introduced particular configurations of loaded waveguides, essentially the fin line structure (see FINLINES).

For one or two decades now most commercial applications have been based on planar transmission lines, especially in the lower frequency range, such as the microstrip, the most common planar line, although stripline came first. More recently, other configurations like slotlines and coplanar waveguides have become popular, because of their good properties at higher frequencies, namely in the 30–60 GHz range. Planar technology developed with respect to waveguides for essentially two reasons. One is economical: producing a planar circuit is cheap, which compensates for the cost of research and development. The other is that planar technology easily combines with semiconductors, leading to microwaves monolithic integrated circuits (MMIC), while the combination waveguide-semiconductor has always been rather laborious, because of the significant differences in configuration as well as in impedance level.

Basic Theory

Electromagnetic Field in a Waveguide. Electromagnetic fields within any region of space are determined by solving Maxwell's equations in a coordinate system appropriate to the region. Such regions may be termed as either uniform or nonuniform. Uniform regions are characterized by the fact that cross sections transverse to a given symmetry, or propagation, direction are everywhere identical with one another in both size and shape. Examples of uniform regions are provided by regions cylindrical about the symmetry direction and having planar cross sections with, for example, rectangular or circular peripheries (2,4,5,10–12).

A waveguide is a metallic structure of arbitrary but constant cross section that extends in the direction of propagation of a wave and confines the wave energy within it. It is bounded by a conductor of high conductivity, filled with a dielectric of low loss, with arbitrary cross section. It may have more than one bounding surface, as the coaxial line, which consists of the annular space between two concentric circular cylindrical conductors. In practice the two most common cases are the rectangular and the circular guide. Hollow waveguides cross sections are limited by metallic boundaries. In the basic theory, those metallic boundaries are supposed to be made of perfect electric conductors. At microwaves, the actual losses in actual conductors are very small and are usually evaluated by a perturbation theory.

Within the enclosed region the electromagnetic field may be represented as the superposition of an infinite number of standard functions that form a mathematically complete set. The mathematical representation of the electromagnetic field within a uniform region is in the form of a superposition of an infinite number of modes or wave types. The electric and magnetic field components of each mode are factorable into form functions depending only on the cross-sectional coordinates transverse to the direction of propagation, and into amplitude functions depending only on the coordinate in the propagation direction. The transverse functional form of each mode depends upon the cross-sectional shape of the region and, save for the amplitude factor, is identical at every cross section. As a result the amplitudes of a mode completely characterize the mode at every cross section. The variation of each amplitude along the propagation direction is given implicitly as a solution of a one-dimensional wave or transmission-line equation. According to the mode in question the wave amplitudes may be either propagating or attenuating along the transmission line.

Transverse Electromagnetic, Transverse Magnetic, and Transverse Electric Modes. In physics, waves are usually classified according to two criteria. One is the geometrical form of the wavefront, which describes the surface of constant phase, most generally planar, cylindrical, or spherical, more rarely elliptical or other. The second criterion is that the wave is said to be uniform or nonuniform, depending on whether the field has a constant amplitude in each point of the wavefront or not. As examples, the wave emitted from a point source is a uniform spherical wave while the wave propagating in a coaxial cable is a nonuniform plane wave. The term "pure mode," or simply "mode," refers to a wave whose field structure remains the same along the entire path of propagation. Different modes will, in general, have different field structures and different velocities of propagation. In uniform hollow waveguides, the waves, more often called the *modes,* are usually classified according to their properties with respect to the propagation direction and whether they have longitudinal components or not. A wave with longitudinal components for neither the electric nor the magnetic field is called a transverse electromagnetic (TEM) wave. It has only four field components. A wave with no longitudinal magnetic field component while having a longitudinal electric field component is called a transverse magnetic (TM) wave, and a wave with no longitudinal electric field component while having a longitudinal magnetic field component is called a transverse electric (TE) wave. TM and TE waves, or modes, each have five field components.

For a TEM mode to propagate on a guiding structure, at least two conductors are needed. The equations to be satisfied by a TEM mode *in the transverse plane* of such a structure are indeed identical to those of statics where two conductors at least are necessary, to impose a potential difference as a boundary condition. Hence a TEM mode cannot propagate on or in a hollow one-conductor waveguide, while it can propagate on or in a two-conductor waveguide of any cross section and any geometry, for instance a coaxial cable or waveguide. A TEM mode has no cutoff frequency: because of the two or more conductors it can propagate down to zero frequency.

In a waveguide, an infinite number of TE and of TM modes can propagate. In the usual one-conductor waveguide, those two classes of modes cannot propagate down to zero frequency, since only one conductor is available. All the TE and the TM modes have a cut-off frequency below which they cannot propagate: the waveguide has a high pass characteristic. The cut-off frequencies depend on the dimensions of the waveguide and of the homogeneous medium filling the waveguide. In a waveguide with two or more conductors, like the coaxial cable or waveguide, the TEM, TE, and TM modes can propagate. The number of propagating modes is determined by the frequency: all the modes with a cut-off frequency lower than the generator frequency propagate while the others do not. If a waveguide is excited with a signal whose frequency is lower than the cut-off frequency, then instead of propagating, this signal is attenuated exponentially with distance. This phenomenon of attenuation in a nonabsorbing medium which occurs in a waveguide below cut-off is similar to the phenomenon of total reflection in optics, in which light is reflected going from an optically denser to an optically rarer medium. In the rarer medium there is an attenuated wave of the type discussed here. With a slightly absorbing medium inside the waveguide, the cut-off is no longer as sharp as it is in the nonabsorbing case. The attenuation is small in the range of frequency above cut-off, and rapidly becomes large when going to lower frequencies.

The mode with the lowest cut-off frequency is called the *dominant mode.* In general, most waveguides are operating at a frequency such that only the dominant mode propagates. In that case, the electromagnetic field is characterized by the amplitudes of this dominant mode. The amplitudes that measure the transverse electric and magnetic field intensities of this dominant mode are defined as voltage and current, respectively, as on appropriate transmission line. The knowledge of the wave impedance and propagation constant, that is, propagating, of the transmission line then permits one to describe the propagation of the dominant mode in familiar impedance terms.

When cross-sectional discontinuities are present in the waveguide, they require more boundary conditions to be satisfied than those characterizing the dominant mode, which are imposed by the longitudinal metallic walls (see WAVEGUIDE DISCONTINUITIES). Mode voltages and currents are introduced as measures of the amplitudes of the transverse electric and magnetic field intensities of each of the higher order modes. Each of these is represented by a transmission line, having however a reactive wave impedance and a real propagation constant, that is, attenuating. In this manner the complete description of the electromagnetic field in a waveguide may be represented in terms of the behavior of the voltages and currents on a infinite number of transmission lines.

Orthogonality Properties and Expansion of the Fields in Normal Modes. Solutions of the wave equation always have certain properties of orthogonality, which are particularly important when expanding a general solution as a sum of solutions for the various modes and when considering problems of energy. Several general theorems have been proved independently of the particular geometry of the cross section. The main orthogonality theorems are the following.

1. The integral over the cross section of the scalar product of either the transverse electric fields or the transverse magnetic fields of two different modes is zero.
2. The integral over the cross section of the product of either the longitudinal electric field components or the longitudinal magnetic field components of two different modes is zero.
3. The integral over the cross section of the longitudinal component of the vector product of the transverse electric field of one mode with the transverse magnetic field of another mode is zero.

These theorems hold when the two modes are both TE or both TM, as well as when the two modes are of opposite types. Several other theorems can be derived, relating to the integrals of squares of components of the electric and the magnetic fields, or of products of components with their conjugates.

Maxwell's equations within an one-conductor waveguide with perfectly reflecting walls have an infinite number of possible solutions, representing various normal modes, some TE and some TM. Furthermore, corresponding to each of these modes with propagation along the $+z$ direction, there is a possible wave propagated along $-z$. Each mode has a cut-off frequency, such that a disturbance in that mode at a frequency below cut-off is rapidly attenuated, whereas at a frequency higher than cut-off it is propagated.

The last problem to be considered is that of the most general field which can exist in the waveguide. Maxwell's equations, as well as the boundary conditions are linear. Hence, all the solutions that have been found can be superposed, with appropriate coefficients, to form another solution for the equations submitted to the boundary conditions. It can be proved that the most general solution, subject to the boundary conditions and to the additional condition that the guide contains no volume charge or current except what arises from Ohm's law in the imperfect dielectric filling it, can be represented by such a superposition. As usual when dealing with problems of expansion in orthogonal functions, the magnitude of the functions are normalized by determining their magnitudes so that their squares integrate to some convenient value, and by multiplying these normalized functions by an additional coefficient to secure an arbitrary value for the function. The complete field is obtained by superposing all possible fields, with waves traveling along the $-z$ direction as well as along $+z$ (2).

A quite general approach to waveguide theory is obtained by elevating Maxwell's equations into a dyadic form. This yields the concept of dyadic Green functions in electromagnetic theory. Vector wave functions can then be established for waveguides of specific cross sections (12,13).

THEORY

Electromagnetic Fields in a Waveguide

TEM Mode. TEM waves have longitudinal field components neither for the electric nor for the magnetic field. Hence they are fully determined by four components only, two for the transverse electric field and two for the transverse magnetic field. Along a guiding structure, TEM waves can propagate only if at least two conductors are available, as is the case in a coaxial cable or waveguide. It can be shown indeed that those fields satisfy in the transverse plane the same equations as those of statics. To demonstrate the statement the vectors and operators appearing in Maxwell's equations are decomposed into their tranverse and longitudinal components, which leads to decomposed equations (14). These show that in the absence of the longitudinal components the purely transverse electric and magnetic fields do satisfy the equations of statics in the transverse plane. Furthermore, since neither electric nor magnetic variable field can exist in a perfect electric conductor, boundary conditions to be satisfied at the walls are also identical to those of statics. Hence two conductors at least are necessary for a TEM wave to propagate on a guiding structure: in a coaxial waveguide a TEM wave can propagate, while it cannot in a one-conductor hollow waveguide.

TE and TEM Modes. Only uniform waveguides are considered, where the cross section is identical along the propagation direction and limited by conducting walls surrounding a homogeneous and isotropic medium, usually with small losses. The propagation is interior to the guide. The walls are first supposed to be lossless. They can be either perfect electric or perfect magnetic, which can be used for the ease of calculation in some special cases. In practice, however, there are numerous materials which are very good electric conductors, although not perfect, which is not the case for magnetic conductors. Small losses in the walls will be evaluated later, using a perturbation method. Internal volume charge and current density are assumed to be zero. Steady-state sinusoidal solutions are calculated. Maxwell's equations are solved in the frequency domain and fields are represented by phasors (complex vectors).

Maxwell's equations are written as:

$$
\begin{aligned}
&\nabla \times \overline{E} = -j\omega\mu\overline{H} && \text{or} \quad \nabla \times \overline{E} = -jk\zeta\overline{H}\\
&\nabla \times \overline{H} = (j\omega\epsilon + \sigma)\overline{E} && \text{or} \quad \nabla \times \overline{H} = jk\eta\overline{E}\\
&\nabla \cdot \overline{E} = 0\\
&\nabla \cdot \overline{H} = 0
\end{aligned}
\qquad (1)
$$

by defining:

$$
\begin{aligned}
jk &\triangleq \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} \quad (\mathrm{m}^{-1})\\
\eta &\triangleq 1/\zeta \triangleq \sqrt{(\sigma + j\omega\epsilon)/j\omega\mu} \quad (\mathrm{S})
\end{aligned}
\qquad (2)
$$

Transverse and Longitudinal Components for TE and TM Modes. Rewriting the first two equations of Eq. (1) in detail yields:

$$
\begin{aligned}
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -jk\zeta\, H_x & \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= jk\eta\, E_x\\
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -jk\zeta\, H_y & \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= jk\eta\, E_y\\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -jk\zeta\, H_z & \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= jk\eta\, E_z
\end{aligned}
\qquad (3)
$$

Looking for a solution propagating in the $+z$ direction, the classical transmission line formalism yields a z dependence $e^{-\gamma z}$. Hence derivatives with respect to z are replaced by multiplications by $-\gamma$. TE and TM modes are most easily investigated by obtaining an equation for each of the longitudinal components separately. Extracting for example E_z from the last equation after replacing H_x and H_y by their values from the first two equations yields:

$$E_z = \frac{1}{j^2k^2}\left[\gamma\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) + \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2}\right] \tag{4}$$

Usually the third Maxwell's equation yields:

$$E_z = -\frac{1}{k^2}\left(\gamma^2 E_z + \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2}\right) \tag{5}$$

or

$$[\nabla_t^2 + (k^2 + \gamma^2)]E_z = 0 \tag{6}$$

A similar equation can be obtained for any other component, and in particular for H_z. Defining then:

$$p^2 \triangleq \gamma^2 + k^2 \tag{7}$$

yields:

$$(\nabla_t^2 + p^2)(E_z \quad \text{or} \quad H_z) = 0 \tag{8}$$

scalar eigenvalue Helmholtz equation. The eigenvalues are $p2$, defined by Eq. (7). To each eigenvalue there corresponds at least one eigenfunction, solution of the equation. If there is more than one eigenfunction, the solutions are said to be degenerate. This is the case for instance in a square waveguide for the solutions which have as the only difference the orientation of the electromagnetic field with respect to the two sides of the waveguide.

Equations similar to Eq. (8) can be obtained also for E_x and H_x, and for E_y and H_y, respectively. The fact however that the longitudinal components of electric and magnetic fields are not coupled—which is not the case when the medium filling the guide is inhomogeneous or anisotropic—makes solutions easy to calculate when the boundary conditions do not introduce a coupling between the two longitudinal components. It will be shown that this is the case for uniform waveguides filled with homogeneous and isotropic medium.

Waveguide Modes

The absence of coupling between the two longitudinal components in Eq. (8) yields separate solutions for modes with no H_z called TM modes (also sometimes called E-modes because of nonzero E_z) and modes with no E_z called TE modes (also sometimes called H-modes because of nonzero H_z). The general solution can be any linear combination of any TE and TM modes (2). The separation of solutions for uniform, homogeneous, and isotropic waveguides in two classes, TE and TM modes respectively, is particularly useful because, once solutions have been obtained either for TE or for TM modes, all the transverse components can be calculated from the longitudinal components. The appropriate re-arrangement of Eq. (3) yields as an example:

$$E_x = \frac{k}{p^2}\left(\frac{1}{j\eta}\frac{\partial H_z}{\partial y} - \frac{\gamma}{k}\frac{\partial E_z}{\partial x}\right) \tag{9}$$

and the relation between the transverse and longitudinal components can be rewritten as (14):

$$\begin{aligned}\overline{E}_t &= (1/p^2)(-\gamma\nabla_t E_z + jk\zeta\overline{a}_z \times \nabla_t H_z)\\ \overline{H}_t &= (1/p^2)(-\gamma\nabla_t H_z - jk\eta\overline{a}_z \times \nabla_t E_z)\end{aligned} \tag{10}$$

In summary, one has the general relations:

$$\begin{array}{ll} \text{TE modes } (H) & \text{TM modes } (E) \\ (\nabla_t^2 + p^2)H_z = 0 & (\nabla_t^2 + p^2)E_z = 0 \\ \overline{H}_t = -(\gamma/p^2)\nabla_t H_z & \overline{E}_t = -(\gamma/p^2)\nabla_t E_z \\ \overline{E}_t = (jk\zeta/p^2)\overline{a}_z \times \nabla_t H_z & \overline{H}_t = -(jk\eta/p^2)\overline{a}_z \times \nabla_t E_z \end{array} \tag{11}$$

For both types of modes, boundary conditions are the vanishing of the tangential electric field at perfect electric walls and magnetic field at perfect magnetic walls, respectively.

Equivalent Transmission Lines. The previously obtained equations contain partial derivatives. An equivalence can be found between the expressions relating the transverse components and transmission line equations. Separating indeed the transverse and longitudinal variables by defining:

$$\begin{aligned}\overline{E}_t &\triangleq V(z)\overline{e}_t(\overline{r}_t)\\ \overline{H}_t &\triangleq I(z)\overline{h}_t(\overline{r}_t)\end{aligned} \tag{12}$$

one observes that the vectors $\overline{e}_t$ and $\overline{h}_t$ are transverse but do not vary in z, while $V(z)$ and $I(z)$ are amplitude coefficients as a function of the coordinate z. It should be noted that at the right side of both equations each of the two terms of the products are only defined with respect to a complex arbitrary constant which may multiply one term and divide the other: only their products are defined. Looking after an equivalence with transmission line equations, one has:

$$\begin{aligned}V(z) &= V_+e^{-\gamma z} + V_-e^{+\gamma z}\\ I(z) &= Y_o(V_+e^{-\gamma z} - V_-e^{+\gamma z})\end{aligned} \tag{13}$$

where Y_0 is the characteristic admittance of the transmission line. The parameters Z_0 and γ are now to be calculated, in view of determining the parameters of the line: Z series impedance in Ω/m, and Y shunt admittance in S/m. Those are different for TE and TM modes, respectively.

Investigating first the *TM modes,* the first Maxwell's equation shows that the transverse curl of $\overline{e}_t$ is zero. Hence the transverse field derives in the transverse plane from a complex scalar *electric potential:*

$$\overline{e}_t = -\nabla_t\phi \tag{14}$$

This potential is proportional to E_z. Considering only one traveling wave, and introducing Eq. (12) into Eq. (10), yields:

$$Y_0\overline{h}_t = (jk\eta/\gamma)\overline{a}_z \times \overline{e}_t \tag{15}$$

Defining:

$$Y_0 \triangleq (jk\eta/\gamma)K \qquad \text{(S)} \tag{16}$$

where K is an arbitrary complex constant, depending on the complex constants unwritten in Eq. (12) to be determined, yields:

$$\overline{h}_t = (1/K)\overline{a}_z \times \overline{e}_t \tag{17}$$

from which can be deduced:

$$E_z = (p^2/jk\eta K)I(z)\phi \tag{18}$$

Hence, the potential is proportional to Ez and satisfies the same equation:

$$(\nabla_t^2 + p^2)\phi = 0 \tag{19}$$

When the wall is a perfect electric conductor, the longitudinal electric field vanishes at the wall, and so does the potential. There is a particular case however: the potential may be a nonzero constant when p^2 is zero. This is the case for the TEM mode, which may exist if the guide has more than one conductor: the TEM mode is a special case of a TM mode. The parameters Z and Y of the equivalent transmission line *for the transverse components* can be calculated using:

$$Z = \gamma/Y_0 \quad \text{and} \quad Y = \gamma Y_0 \tag{20}$$

which yields (Fig. 1):

$$Z = \gamma^2/jk\eta K \quad (\Omega/m) \qquad ; \qquad Y = jk\eta K \quad \text{(S/m)} \tag{21}$$

or:

$$Z = \frac{p^2 - k^2}{(\sigma + j\omega\epsilon)K} = \frac{p^2}{P(\sigma + j\omega\epsilon)K} + \frac{j\omega\mu}{K} \qquad Y = (\sigma + j\omega\epsilon)K \tag{22}$$

It is observed that, if there are no losses in the medium filling the guide, the equivalent circuit has a high-pass characteristic with a cut-off frequency given by:

$$\mu\omega_c/K = p^2/\epsilon K\omega_c \text{ from which } \omega_c = pc \tag{23}$$

where c is the phase velocity in the medium supposed to be infinite.

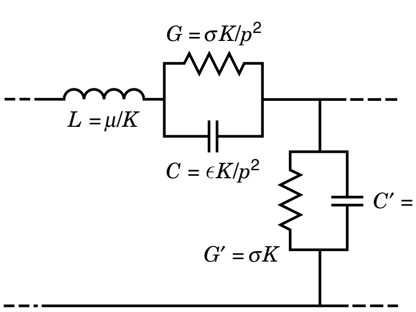

Figure 1. Equivalent circuit of TM modes, illustrating series impedance and parallel admittance.

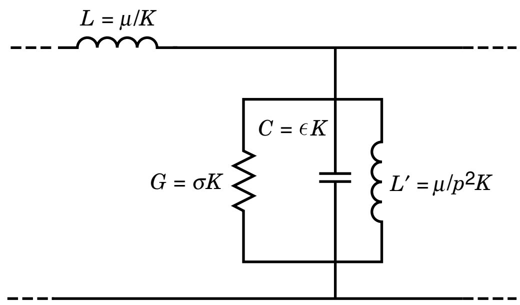

Figure 2. Equivalent circuit of TE modes, illustrating series impedance and parallel admittance.

Proceeding similarly for the *TE modes,* it can be shown that $\overline{h}_t$ now depends upon a complex scalar *magnetic potential,* that the wave admittance also contains an arbitrary complex constant to be determined, that the potential is proportional to H_z, and that it satisfies the same equation as before:

$$(\nabla_t^2 + p^2)\Psi = 0 \tag{24}$$

The boundary conditions to be satisfied on perfect electric walls impose the normal derivative of the magnetic potential to vanish. The parameters of the equivalent transmission line are:

$$\begin{aligned} Z &= \gamma/Y_0 = jk\zeta/K = j\omega\mu/K \\ Y &= \gamma^2 K/jk\zeta = \frac{p^2 - k^2}{ik\zeta}K = \frac{p^2 K}{j\omega\mu} + (\sigma + j\omega\epsilon)K \end{aligned} \tag{25}$$

They are represented in Fig. 2. The equivalent circuit also exhibits a high-pass characteristic, and the cut-off frequency is given by the same expression as for TM modes.

The equivalent circuits at Fig. 1 and Fig. 2 are not dual. This is because magnetic losses have not been introduced for the medium. The circuits would be dual if those losses were introduced. It can be seen that the equivalent circuits for the transverse components of TE and TM modes, respectively, are dual in the absence of any loss in the medium.

Eigenvalues, Power, and Impedance Level. Eqs. (19) and (24) are eigenvalues equations. They only have solutions—eigenfunctions—for an infinite number of discrete values of the eigenvalues p^2. To each eigenvalue correspond one, or more in the case of degeneracy, eigenfunctions which determine the spatial distributions of the transverse fields. Those spatial distributions are called the *modes* of the waveguide. Equation (7) describes the dispersion characteristic of each mode: it expresses the variation of the propagation constant as a function of frequency and of the parameters of the medium. Certain properties of the eigenvalues can be demonstrated using Green's theorems (5). In particular they are equal to the ratio of integrals over the cross section of the square of the absolute values of the gradient of the potential and of the potential, respectively, for TE as well as for TM modes:

$$p^2 = p^{*2} = \int_A |\nabla_t\theta|^2\, da \Big/ \int_A |\theta|^2 da \qquad \theta = \phi \text{ or } \Psi \tag{26}$$

Hence the eigenvalues:

1. Are real and positive, even in the presence of losses in the medium;
2. Do not depend on the properties of the medium filling the guide, as long as the medium is homogeneous and isotropic;
3. And depend entirely upon the geometry of the cross section of the guide.

There is an interest in expressing the complex power traveling along the guide identically to the classical expression of lumped circuit theory $VI^*/2$. Calculating the complex power S in the $+z$ direction by using Poynting's vector integrated over the cross section A yields the classical expression:

$$S = (1/2)VI^* \tag{27}$$

provided one imposes:

$$K^* \triangleq \int_A |\bar{e}_t|^2 da \tag{28}$$

This shows that to obtain Eq. (27) the unknown complex constant K must be real. It also shows that the value of K remains unknown, because there is no more equation which can be used to specify the value of this constant: for TE and TM modes the impedance of the equivalent transmission is not uniquely specified, and its level is arbirtrary. This is why, when calculating the three expressions of power for TE and TM modes:

$$V/I \quad ; \quad 2S/|I|^2 \quad ; \quad |V|^2/2S \tag{29}$$

identical values are not found (15). The TEM mode is the only field configuration for which the concept of characteristic impedance is rigorously valid and the three expressions in Eq. (29) yield the same value. Unfortunately, in most references the existence of the unknown constant for TE and TM modes is not stated explicitly. From now on, it will be put equal to 1. This has consequences:

1. Eq. (28) with $K^* = 1$ can be used as a normalization condition for the fields;
2. And Eq. (26) reduces to

$$p^2 \int_A |\theta|^2 da = 1 \tag{30}$$

which will be used to determine the integration constants when integrating the Helmholtz equation.

TE and TM modes, as well as the TEM mode in a more-than-one conductor waveguide, have been shown to be the solutions of waves propagating in a waveguide. To actually exist however in the guide, an adequate excitation transducer must be available. Its geometry has to be compatible with the geometry of the electromagnetic configuration of the launched modes, as will be demonstrated later.

Dispersion Diagram, Active and Reactive Power. Equation (7), defining the eigenvalues, can be written as:

$$\gamma^2 = p^2 - (\omega/c)^2 \tag{31}$$

in the absence of losses in the medium filling the guide. It relates the propagation constant to the frequency. The propagation constant vanishes at the cut-off frequency, which determines the cut-off wavelength:

$$\omega_c = pc \quad ; \quad \lambda_c = c/f_c = 2\pi/p \tag{32}$$

Variations of γ^2 and $\gamma = \alpha + j\beta$ are represented in Fig. 3. At frequencies below cut-off the propagation constant is real. There is attenuation and the value of the attenuation constant is:

$$\alpha = \sqrt{p^2 - \omega^2\mu\epsilon} \tag{33}$$

At zero frequency the value of the attenuation constant is p and the curve $\alpha - \omega$ is an ellipse. At frequencies above cut-off the propagation constant is imaginary. There is propagation, the value of the propagation constant is:

$$\beta = \sqrt{\omega^2\mu\epsilon - p^2} \tag{34}$$

and the curve $\beta - \omega$ is hyperbolic with the straight line $\beta = \omega/c$ as an asymptote for any value of p. At high frequencies indeed the wavelength decreases, the electric distance be-

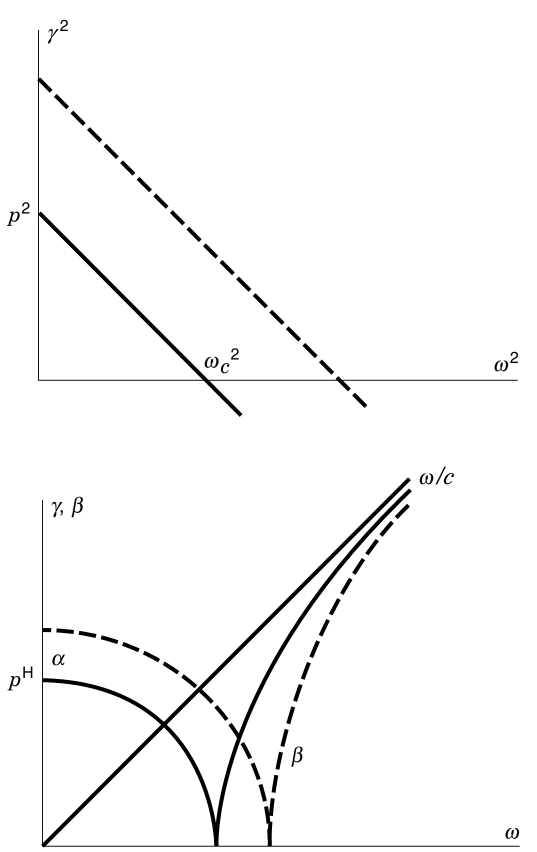

Figure 3. Dispersion diagram of typical modes, illustrating cut-off frequencies.

tween the walls increases, and the propagation approaches free space propagation. At frequencies of operation the waveguide modes are dispersive and the dispersion increases when the frequency decreases down to the cut-off frequency. The mode having the lowest cut-off frequency, hence the smallest eigenvalue p, is called the *dominant mode.* It is most common to operate at a frequency located between the lowest cut-off frequency and the cut-off frequency of the next higher-order mode, so that only one mode propagates, the dominant one. To optimize this frequency range, sometimes called the *bandwidth,* appropriate waveguide dimensions are chosen, in particular for the rectangular waveguide.

From Eq. (34) one determines the guide wavelength, the phase velocity, and the group velocity for each mode:

$$\begin{aligned}\lambda_g &\triangleq \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{(\omega/c)^2 - p^2}} = \frac{1}{\sqrt{(1/\lambda_o)^2 - (p/2\pi)^2}}\\ v_{ph} &= \omega/\beta = c\lambda_g/\lambda_0\\ v_g &= \partial\omega/\partial\beta = c^2\beta/\omega = c\lambda_o/\lambda_g\end{aligned} \tag{35}$$

It is observed that, for each mode, the product of the phase velocity and of the group velocity is equal to the constant $c2$. For only one traveling wave, the complex power of Eq. (27) can be written:

$$S = P + jQ = (1/2)|V + |^2 Y_0^* e^{-2\alpha z} \tag{36}$$

with:

$$P = (1/2)|V_+|^2 e^{-2\alpha z}\mathrm{Re}\,Y_0 \qquad Q = -(1/2)|V_+|^2 e^{-2\alpha z}\mathrm{Im}\,Y_0 \tag{37}$$

where the admittance Y_0 is complex and is equal to:

for TM modes: $Y_{0E} = jk\eta/\gamma = (\sigma + j\omega\epsilon)/(\alpha + j\beta)$
for TE modes: $Y_{0H} = \gamma/jk\zeta = (\sigma + j\beta)/j\omega\mu)$

For no losses in the medium, these expressions reduce to the following.

1. For $\omega > \omega_c$:

Real power is transmitted. There is no energy storage.

2. For $\omega < \omega_c$:

Power is purely imaginary. No real power is transmitted and there is energy storage. The energy is capacitive for TM (E) modes and inductive for TE (H) modes: TM modes below cut-off store electric energy while TE modes below cut-off store magnetic energy. Hence, capacitors and inductors are designed at centimeter and millimeter wavelength by creating a limited region of space along the waveguide which concentrates the appropriate modes below cut-off. A resonant circuit is similarly designed around a frequency where the stored electric and magnetic energies of modes below cut-off compensate. Those regions of stored energy are centered on obstacles of the appropriate geometry, creating supplementary boundary conditions to be satisfied by higher-order modes, which are below cut-off.

Metallic waveguides can transport a significant power. Its value depends upon the medium filling the guide, surface quality, humidity, pressure, possible temperature elevation, and frequency. If the guide is filled with dry air, the electric field may not go beyond 3 MV/m, which corresponds to a power range of 10 MW at 4 GHz and 100 kW at 40 GHz. Discontinuities and irregularities in the waveguide may impose a security factor of 4 or more. Furthermore, losses in copper walls are of the order of 0.03 dB/m at 4 GHz and 0.75 dB/m at 40 GHz (5).

Wall Losses. The losses in a metallic waveguide wall can be calculated considering the metal wall as a perturbation of the perfectly conducting wall and using skin effect formulation. The formulation is valid as long as the skin depth is small compared with a distance around the periphery in which the magnetic field of the lossless guide changes considerably. Hence the formulation is not valid near corners unless that magnetic field is zero at the corner and may not be valid for modes of very high order which, however, are usually not of great interest in practice. A number of results are given in (4) for rectangular and circular metal-walled guides, as well as for coaxial lines and other lines which are not of the guide type.

Specific Geometries

Rectangular Waveguide. The guide with a rectangular cross section has an inside horizontal width a (coordinate x) and vertical height b (coordinate y) with $a > b$. The identical eigenvalue Eq. (19) for TM modes and Eq. (24) for TE modes are solved by separation of variables and the potentials are the Cartesian eigenfunctions sine and cosine. The general solution for the potentials is:

$$\text{potential} = C\,(\cos ux \text{ or } \sin ux)(\cos vy \text{ or } \sin vy)$$

where C is an integration constant to be determined and u and v are constituents of the eigenvalue:

$$p^2 = u^2 + v^2 \neq 0 \tag{38}$$

Investigating first the TM modes, imposing the electric potential to vanish in $x = 0$, a and in $y = 0$, b yields the general solution:

$$\phi_{mn} = C_{Emn}\sin(m\pi x/a)\sin(n\pi y/b) \tag{39}$$

where the product mn has to be different from zero, because of the sine functions. To each mn combination there corresponds a mode called TM_{mn}. The integration constant CE_{mn} is determined by:

$$p_{mn}^2 = u^2 + v^2 = (m\pi/a)^2 + (n\pi/b)^2 \tag{40}$$

and

$$p_{mn}^2\int_A |\phi_{mn}|^2 da = 1 \tag{41}$$

from which:

$$C_{Emn} = 2/(p_{mn}\sqrt{ab}) \tag{42}$$

The transverse components are calculated using Eq. (14):

$$\bar{e}_{tmn} = -C_{mn}\left(\bar{a}_x \frac{m\pi}{a}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y + \bar{a}_y\frac{n\pi}{b}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\right)$$
$$\bar{h}_{tmn} = C_{mn}\left(\bar{a}_x \frac{n\pi}{b}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y - \bar{a}_y\frac{m\pi}{a}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\right) \quad (43)$$

As for TE modes, the boundary conditions impose the normal derivatives of the magnetic potential to vanish at the walls, which yields cosine solutions:

$$\Psi_{mn} = C_{Hmn}\cos(m\pi x/a)\cos(n\pi y/b) \quad (44)$$

where the sum $m + n$ has to be different from zero. To each mn combination corresponds a mode called TE_{mn}. The integration constant CH_{mn} is determined by the same equations as for TM modes, which yields:

$$C_{Hmn} - 2/(p_{mn}\sqrt{ab}) \quad (45)$$

when m and n are different from zero and:

$$C_{Hmn} = \sqrt{2}/(p_{mn}\sqrt{ab}) \quad (46)$$

when m or n is zero. The transverse components are:

$$\bar{e}_{tmn} = C_{mn}\left(\bar{a}_x \frac{n\pi}{b}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y - \bar{a}_y\frac{m\pi}{a}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\right)$$
$$\bar{h}_{tmn} = C_{mn}\left(\bar{a}_x \frac{m\pi}{a}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y + \bar{a}_y\frac{n\pi}{b}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\right) \quad (47)$$

The smallest eigenvalue is obtained when the integer n related to the smallest dimension b is zero. Hence the dominant mode is the TE_{10}. For a lossless medium, its parameters are:

$$\begin{aligned}
\Psi_{10} &= C_{10}\cos(\pi x/a)\\
C_{10} &= \sqrt{2a/b}/\pi\\
p_{10} &= \pi/a\\
\omega_{c10} &= \pi c/a\\
\lambda_{c10} &= 2a\\
\beta_{10} &= [(\omega/c)^2 - (\pi/a)^2]^{1/2} \quad \text{at} \quad \omega > \omega_c\\
\lambda_{g10} &= [(1/\lambda_0)^2 - (1/2a)^2]^{-1/2} = \lambda_0[1-(\lambda_o/2a)^2]^{-1/2}
\end{aligned} \quad (48)$$

The fields of a traveling wave of the dominant mode have only three components. They are:

$$\begin{aligned}
\bar{E}_t &= -\bar{a}_y C_{10}V_+(\pi/a)\sin(\pi x/a)e^{-j\beta_{10}z}\\
\bar{H}_t &= \bar{a}_x C_{10}V_+(\pi/a)(\beta_{10}/\omega\mu)\sin(\pi x/a)e^{-j\beta_{10}z}\\
H_z &= C_{10}V_+(\pi/a)^2(1/j\omega\mu)\cos(\pi x/a)e^{-j\beta_{10}z}
\end{aligned} \quad (49)$$

The transverse fields are in phase in time and in space while H_z is out of phase with them, both in time and in space. Figure 4 represents the fields and surface current of the dominant mode along one-half guide wavelength. The electric field has only a vertical component. Hence the dominant mode is linearly polarized in the electric field. Calculating the cut-off frequencies of the first higher-order modes, it can be seen that to improve the bandwidth the height b must be chosen smaller than half the width a. Calculating the three power expressions Eq. (29) for the dominant mode, normalized with respect to the factor $(b/a)Z_0$, respectively yields the values 1.57, 1.23, and 2.00, which shows that the differences are significant. Hence matching for instance a rectangular to a circular waveguide may offer serious difficulties. It seems that the square root of the product of 1.57 by 2.00 leads to the most convenient result for the power, although the reason is not certain (16). The mode impedances and the parameters of interest for all the modes can be calculated according to the general theory developed earlier. Mode configurations can be found in a number of references, such as (4,10,17,18).

Equation (40) shows that the eigenvalues, hence the cut-off frequencies, depend upon the dimensions of both sides of the cross section of the guide. Furthermore, the height b must be smaller than $a/2$ to optimize the frequency range in which only the dominant mode propagates. For these reasons rectangular waveguides have specified dimensions (17) and corresponding normalized frequency bands have been defined. Most usual bands are the following.

L	from 1.12 to 1.70 GHz
S	from 2.6 to 3.95 GHz
C	from 3.94 to 5.99 GHz
X	from 8.2 to 12.4 GHz

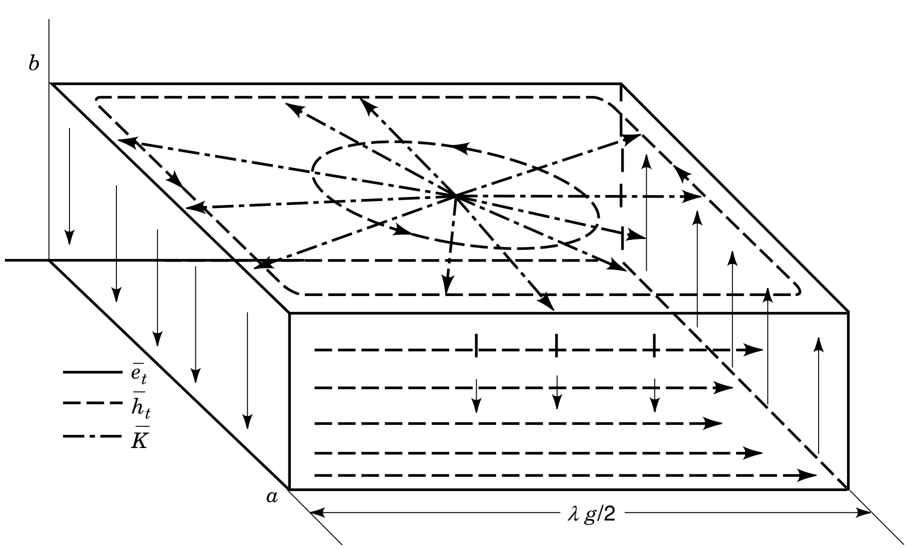

Figure 4. Fields and wall currents of dominant mode in rectangular waveguide, showing current source and sink.

Ku	from 12.4 to 18.8 GHz
K	from 18.0 to 26.5 GHz
Ka (or **Q**)	from 126.5 to 26.5 GHz
O	from 40 to 70 GHz
V	from 50 to 75 GHz
W	from 75 to 110 GHz

Ridge Waveguide. A waveguide that has the cross section as shown in Fig. 5 is called a *ridge waveguide:* It has a central ridge added either to the top or bottom or both of a rectangular section. Its interesting electromagnetic property is that the cut-off frequency is lowered because of the capacitance effect at the center and could in principle be made as low as desired by decreasing the gap width sufficiently. Of course, the impedance of the guide also decreases as the gap is made smaller. Because of the increased effective length of the periphery, the attenuation is larger than in a usual rectangular waveguide. One of the important applications is as a nonuniform transmission system for matching purposes, obtained by varying the depth of ridge as one progresses along the guide (18).

The calculation of cut-off frequencies is rather easy if one remembers that, at cut-off, there is no variation in the z direction, so that waves reflect from sidewall to sidewall in a transverse direction at or below cut-off. Thus, at cut-off, the ridge waveguide can be considered as a short-circuited parallel-plane waveguide with infinite width in the z direction. The case of the dominant mode, like all the modes with no variation in the y direction, is particularly easy to calculate. Starting indeed in the x direction from the vertical sidewall which is a short-circuit at the end of the transverse equivalent transmission line, one first has a parallel-plate waveguide per unit length in the z direction, then a capacitance because of the abrupt change in height (see WAVEGUIDE DISCONTINUITIES), then again a parallel-plate waveguide with reduced height up to the middle of the cross section. There the impedance is either infinity or zero, depending on whether the mode is odd or even as a function of the electric field. The second half of the cross section is identical to the first half just described. The transverse resonance frequency, hence the cut-off frequency of the mode, can easily be calculated in terms of transmission line parameters, as well as wavelength and impedance (19). For the double-ridge waveguide, the impedance is twice that of the simple-ridge waveguide. Fundamental Refs. are 20 and 21.

Circular Waveguide. The circular waveguide has a circular cross section of radius a. The eigenvalue Eq. (19) for TM

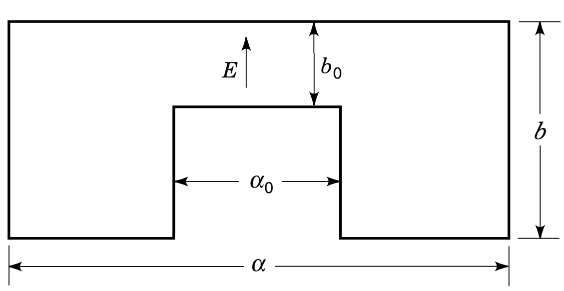

Figure 5. Cross section of a ridge waveguide.

modes and Eq. (24) for TE modes are written in polar form for the transverse coordinates and solved again by separation of variables $r\phi$. The potentials are the polar eigenfunctions sine and cosine. The general solution for the potentials is:

$$\theta = [A_m J_m(pr) + B_m y_m(pr)](\cos m\phi + \alpha \sin m\phi) \qquad (50)$$

where m is an integer, possibly equal to zero, when the guide is of circular symmetry which ensures a periodic solution in the polar angle, and where J_m and Y_m are the Bessel functions of first and second kind, respectively. The functions Y_m are sometimes called Weber or Neumann functions. They are singular at the origin $r = 0$. They have hence to be excluded from the solutions for a hollow guide. They have, however, to be included in the solutions for a coaxial cable or guide, and the general solution in that case will include a linear combination of both functions J and Y. In the case of the one-conductor circular waveguide, the general solution reduces to:

$$\theta(r, \phi) = C_m J_m(pr)(\cos m\phi + \alpha \sin m\phi) \qquad (51)$$

The choice of the cosine or sine function only depends on the polarization of the field, and all the modes are spatially degenerate. An elliptical polarization will be obtained by linearly combining the two orthogonal polarizations. In the following, only the cosine variation will be considered.

For *TM modes* the scalar potential must be zero at the wall:

$$J_m(p_e a) = 0 \qquad (52)$$

which determines the eigenvalues *pemn* from the arguments *emn* = *pemna* for which the J_m functions vanish and yield:

$$\phi_m = C_{em} J_m(p_e r)\cos m\phi \qquad (53)$$

where the first index m is the order of the function J_m and the second n relates to the order of the zeros with increasing argument.

For *TE modes* the normal derivative of the potential must vanish at the wall:

$$J'_m(p_h a) = 0 \qquad (54)$$

which determines the eigenvalues *phmn* from the arguments *hmm* = *phmna* for which the J_m functions are extremum, and yields:

$$\Psi_m = C_{hm} J_m(p_h r)\cos m\phi \qquad (55)$$

where the first index m still is the order of the function J_m and the second n relates to the order of the extrema with increasing argument. There is a degeneracy which is typical of the circular waveguide: the TE_{0n} modes have the same eigenvalue as the TM_{1n} modes since a property of the Bessel functions is:

$$J'_0(pr) = -J_1(pr) \qquad (56)$$

For TM and for TE modes, the field configurations can be calculated from the general theory and the normalization conditions. The following results are obtained.

TE modes

$$\begin{aligned}
\Psi_{mn} &= C_{hmn} J_m(p_{hmn} r)\cos m\phi \\
J'_m(h_{mn}) &= 0 \\
C_{hmn} &= \frac{1}{h_{mn}\sqrt{\pi}\sqrt{1-(m/h_{mn})^2}J_m(h_{mn})} \\
&\quad \text{multiplied by } \sqrt{2} \text{ if } m \neq 0;\ \text{by } 1 \text{ if } m = 0 \\
\overline{h}_t &= C_{hmn}[-\overline{a}_r p_{hmn} J'_m(p_{hmn} r)\cos m\phi \\
&\quad + a_\phi (m/r) J_m(p_{hmn} r)\sin m\phi] \\
\overline{e}_t &= C_{hmn}[\overline{a}_r (m/r) J_m(p_{hmn} r)\sin m\phi \\
&\quad + \overline{a}_\phi p_{hmn} J'_m(p_{hmn} r)\cos m\phi]
\end{aligned}$$

TM modes

$$\begin{aligned}
\phi_{mn} &= C_{emn} J_m(p_{emn} r)\cos m\phi \\
J_{mn}(e_{mn}) &= 0 \\
C_{emn} &= \frac{1}{e_{mn}\sqrt{\pi}\, J'_m(e_{mn})} \\
&\quad \text{multiplied by } \sqrt{2} \text{ if } m \neq 0;\ \text{by } 1 \text{ if } m = 0 \\
\overline{e}_t &= C_{emn}[-\overline{a}_r p_{emn} J'_m(p_{emn} r)\cos m\phi \\
&\quad + a_\phi (m/r) J_m(p_{emn} r)\sin m\phi] \\
\overline{h}_t &= C_{emn}[-\overline{a}_r (m/r) J_m(p_{emn} r)\sin m\phi \\
&\quad - \overline{a}_\phi P_{emn} J^-_m(p_{emn} r)\cos m\phi]
\end{aligned}$$

Comparing the successive order of zeros and extrema of Bessel functions shows that the *dominant mode* is the TE11 mode. Its eigenvalue corresponds to the first maximum of the function *J1* and has the following characteristics:

$$\begin{aligned}
p_{h11} &\cong 1.84/a \\
\Psi_{11} &= C_{h11} J_1(p_{h11} r)\cos\phi \\
\omega_{h11} &= p_{h11} c \cong 1.84 c/a \\
\lambda_c &\cong (2\pi/1.84)a
\end{aligned} \tag{57}$$

It is apparent that the wavelength at the cut-off frequency of the dominant mode is of the order of the guide circumference, while being larger than $2a$. Hence, the cut-off frequency of a circular waveguide of radius a is lower than that of a rectangular waveguide of width a. The transverse fields of the dominant mode are:

$$\begin{aligned}
\overline{h}_t &= -\overline{a}_r C J'_1(pr)\cos\phi + \overline{a}_\phi (C/r) J_1(pr)\sin\phi \\
\overline{e}_t &= \overline{a}_r (C/r) J_1(pr)\sin\phi + \overline{a}_\phi C J'_1(pr)\cos\phi
\end{aligned} \tag{58}$$

The TE_{0n} modes exhibit the interesting property of having losses which decrease when the frequency increases. As an example, the TE_{01} mode has the following characteristics:

$$\begin{aligned}
\Psi_{01} &= C J_0(p_{ho1} r) \text{ with } p_{ho1} \cong 3.83/a \\
\overline{h}_t &= \overline{a}_r C p_{ho1} J_1(pr) \quad ; \quad \overline{e}_t = -\overline{a}_\phi C p_{ho1} J_1(pr)
\end{aligned} \tag{59}$$

Hence the transverse fields vanish at the wall and induce no wall losses. The only wall losses are due to the field longitudinal component. Calculating the proportionality between the H_z component of a TE mode and the scalar potential yields:

$$H_z = (p^2/jk\zeta)kV(z)\Psi \tag{60}$$

which shows that when the medium is lossless H_z is inversely proportional to the frequency. Hence losses decrease with increasing frequency. A number of mode configurations can be found in (4,10,17,18).

Coaxial Waveguide. Coaxial lines are among the most commonly used of all transmission lines, largely because of the convenient construction and the nearly perfect shielding between fields inside and outside of the line. They are commonly used up to 10 to 20 GHz. The range of impedances that may be obtained most conveniently in the TEM mode is about 30 to 100 Ω. Above 15 GHz and up to about 30 GHz the most common coaxial lines are rigid. In addition to the TEM mode, higher-order TE and TM mode solutions can also exist. They are usually cut off and are important essentially as reactive effects near junctions of the line, although they occasionally may enter as additional propagating modes in the transmission system. Higher-order TE and TM modes are calculated as indicated for the circular waveguide and the general solution is Eq. (50), where Bessel functions of both the first and second kind are to be used. The transcendental equations are more complicated than for the one-conductor circular guide. The solutions determine the values of the cut-off frequency, for any mode type and any particular sizes. Solution of the transcendental equations is obtained by graphical methods or by consulting published tables. More information is found in (10,18).

There are a number of normalized connectors for coaxial cables and waveguides. They are characterized by a maximum admissible standing wave ratio, due to very severe tolerances. In rigid coaxial guide with no dielectric the inside conductor is maintained in position by using special techniques (22).

The usual frequency limitation for using coaxial cable at microwaves is its specific attenuation, which is the sum of copper and dielectric losses. Analytical expressions are found for instance in (23). The specific attenuation of a good coaxial cable is of the order of 1 dB/m. When calculating the attenuation, the voltage breakdown, and the maximum power as a function of the ratio of the outside to the inside radii of an air-filled coaxial line, it appears that there is a minimum for the attenuation and a maximum for the voltage breakdown and for the maximum power in the approximate range of 30 to 80 Ω for the characteristic impedance, which explains why most coaxial lines and cables have an impedance of 50 or 75 Ω.

Elliptical Waveguide. An elliptical waveguide is a uniform region in which the transverse cross section is of elliptical form (24). The rectangular coordinates *xy* of the cross section are related to the coordinates of the confocal ellipse and confocal hyperbola. The boundary ellipse is defined by the coordinate as a function of which the major and minor axes as well as the eccentricity and the focal distance can be expressed. Mode functions for TE and TM modes are derived from Mathieu functions, eigenfunctions of Helmholtz equation in elliptical coordinates (25). When the eccentricity decreases, the elliptical boundary reduces to a circular one and the Mathieu functions degenerate into circular functions. Correspondingly, the confocal coordinates become the polar coordinates.

The field components can be calculated for each mode according to the general theory. The cut-off frequencies are expressed in terms of the roots of the functions, determined by the boundary conditions, and the semifocal distance. An alternative expression in terms of the eccentricity of the boundary ellipse is obtained by use of the elliptic integral formula for the length of the boundary ellipse. Computation of power flow and attenuation in elliptical guides involves numerical integration of the Mathieu functions over the guide cross section. Some mode patterns are shown in (10).

Radial Waveguide

Cylindrical Cross Section. The transverse cross section of a radial waveguide with cylindrical cross section is a complete cylindrical surface of a given height. The classical circular-cylindrical coordinate system is appropriate to a region of this type. Radial waveguides are encountered in many of the resonant cavities used in microwave oscillator tubes, filters, and so on. Free space can be regarded as a radial waveguide of infinite height. The transverse electromagnetic field in radial waveguides cannot be represented, in general, as a superposition of transverse vector modes: there exists only a scalar representation that, for no H_z field, is expressible in terms of TM-modes and, for no E_z field, in terms of TE-type modes (10).

In a radial waveguide the concept of guide wavelength loses its customary significance because of the nonperiodic nature of the field variation in the transmission direction. Consequently the usual relation between guide wavelength and cut-off wavelength is no longer valid. The cut-off wavelength however is still useful as an indication of the propagating or nonpropagating character of a mode. The radial waveguide is a two-conductor system and supports a TEM mode. TE and TM modes are calculated according to the general theory (10).

Cylindrical Sector Cross Section. In this case the transverse cross section has a given aperture, limited by vertical metallic planes. Such devices have been used for quite some time as specific radar antennas. They have a wall formed by one conductor only, hence no TEM mode can propagate. Furthermore, there is no periodic symmetry in the horizontal plane, which alters the solutions with respect to the cylindrical situation. TE and TM modes are calculated according to the general theory (10).

Spherical Waveguide. Free space can be considered as a nonuniform transmission region or spherical waveguide. The cross sections transverse to the radial transmission direction are complete spherical surfaces. In practice many spherical cavities may be conveniently regarded as terminated spherical guides. The dominant mode is spherical TEM. TE and TM modes are based on the eigenfunctions in the spherical coordinate system, that is, the associated Legendre functions, or Legendre polynomials (15). The corresponding fields are calculated for both types of modes (10). The r dependence of the mode fields is determined by the spherical transmission-line behavior of the mode voltage and current. As for the modes in radial waveguide, the concepts of cut-off wavelength and guide wavelength lose their customary significance in a spherical waveguide because of the lack of spatial periodicity along the transmission direction. The cut-off wavelength of both TE and TM modes is, however, indicative of the regions wherein these modes are propagating or nonpropagating. For regions such that the wavelength is smaller than a given cut-off wavelength the mode fields decay spatially like $1/r$ and hence may be termed propagating; conversely for wavelength larger than the cut-off wavelength the mode fields decay faster than $1/r$ and may, therefore, be termed nonpropagating.

Conical Waveguide. Conical waveguides are of a two-conductor type. The transmission direction is along the radius r and the cross sections transverse thereto are spherical surfaces bounded by the two cones. The conical waveguide is seen to bear the same relation to a spherical waveguide that a coaxial waveguide bears to a circular guide. Examples of conical guides are provided by tapered sections in coaxial guide, conical antennas (15), and others. Because of the two conductors, the conical waveguide propagates a TEM mode. The r dependence of this dominant mode voltage and current is determined by the spherical transmission-line equations, which reduce in this case to uniform transmission-line equations. The attenuation constant of this mode is given in (10), as well as some expressions related to TE and TM modes.

ADVANCED TOPICS

Periodic Waveguides

Periodic waveguides are periodically loaded waveguides. Hence, strictly speaking, they should not be considered in this article. The importance they have had, the fact that they are "empty" except for a periodic loading by infinitely thin obstacles, in most cases capacitive, and the beauty of their electromagnetic properties certainly deserves some attention (26).

Periodic waveguides have been used in the microwave-tube family as traveling-wave amplifiers and backward-wave oscillators, as microwave and video filters, and as linear particle accelerators. Most of the mathematics and point of view in studying such structures is the same as that used in analyzing the vibrations of a periodically weighted string or in studying the propagation of light or electrons through a crystal lattice (27). The basic theorem is from Floquet (28). The theorem applies to a second-order linear differential equation with periodic coefficients and is quite general (5). It says that a solution of the equation differs from the solution one period away by only a complex factor. Applied to a periodically loaded transmission line it may be translated as follows (26): for a given mode of propagation at a given steady-state frequency the fields at one cross section differ from those one period away only by a complex constant. The theorem is true whether the structure contains loss or not so long as it is periodic.

The study reveals that periodic transmission lines have the following characteristics:

1. There are pass bands and stop bands, which is to say certain frequencies can propagate down the structure with little or no attenuation, whereas other frequencies are attenuated at a rapid rate. These frequencies occur in bands.
2. There is no unique phase velocity. At any frequency for a given mode of propagation there is found to be an in-

finite number of discrete velocities characterizing the mode.

3. The fields may be analyzed in a Fourier series, often called space harmonics or Hartree harmonics, each component of the series having a different phase velocity but all having the same group velocity. Typically, there will be many components with phase velocities less than the velocity of light, yielding slow waves.
4. Structures can be treated from both the field and the circuit standpoint, each approach being useful in certain problems.

To understand what causes pass bands and stop bands, assume that circular infinitely thin irises are placed periodically in a circular waveguide propagating the TM10 mode. This is the structure used in the linear electron accelerator. At each obstacle there will be transmission and reflection. If an observer stations herself in the plane of an iris, she will see that there are certain frequencies for which the reflections from the successive obstacles returning to her add in phase. These frequencies will be nearly equal to the frequencies for which the one-way phase shift between obstacles in the unperturbed guide is any multiple of π. These frequencies are the centers of the stop bands. At other frequencies the accumulated reflections from an infinite length of guide add up to zero, and transmission occurs. Those frequencies make up the pass bands. Figure 6 shows an ω–β diagram for this structure. The dashed line represents the ω–β line for the waveguide without obstacles. Electric field representations illustrate the situation in more detail.

Two important theorems on lossless periodic transmission lines have been derived (26). One is that the time-average electrical stored energy per period is equal to the time-average magnetic stored energy per period in the pass bands. The other is that the time-average power flow in the pass bands is equal to the group velocity times the time-average electrical and magnetic stored energy per period divided by the period.

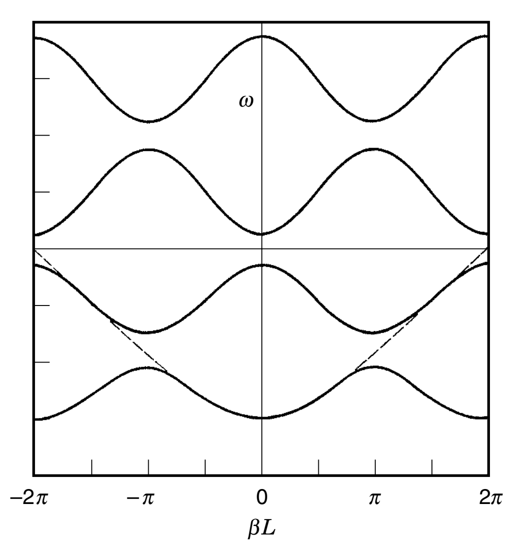

Figure 6. Dispersion diagram (ω–β) for iris-loaded waveguide, with dashed line for unloaded waveguide.

Oversize Waveguides

As has already been mentioned, the TE0n and in particular the TE_{01} modes in circular waveguides have received a good deal of attention for possible long-distance propagation of energy, especially at millimeter waves. The reason is that the amplitude of the fields at the wall decrease when frequency increases, and current and conductor losses approach zero at very high frequency. Attenuations as low as 1.3 dB/km have been attained. Since electric field lines are circular, modes of this class are often described as *circular electric* modes (18). A major problem arises because the mode is not the dominant one. Such guides are known as oversize guides. For the TE01 mode, there are four other propagating modes, with cut-off frequency below the TE_{01} cut-off. Furthermore, since the frequency is well above cut-off, many other modes are in the propagating range. This raises several practical problems. First, the desired mode must be excited with reasonable purity. Secondly, coupling from the desired mode to undesired modes must be avoided. A general solution is to devise mode filters which discriminate against the undesired modes, but cause negligible attenuation to the desired one (18).

Suggestions have been advanced also for using a large-size rectangular waveguide, whose transverse dimensions are just less than $\lambda \times 2\lambda$, to increase the CW power-handling capability. Trapped-mode resonance effects in such a system have been investigated experimentally, and it has been shown how to suppress them (29).

Mode Excitation

The general theory shows that an infinity of TE and TM modes may exist in a hollow waveguide. Furthermore, in a more-than-one conductor waveguide a TEM mode may also exist. This possible TEM mode has a zero cut-off frequency while the TE and TM modes each have a specific cut-off frequency. Each of those modes will attenuate below cut-off, storing energy as has been seen, and propagate above cut-off. The modes however are only the possible solutions of the equations, submitted to the boundary conditions. As mentioned earlier modes will actually exist in the waveguide only if they are properly excited, which requires an adequate transducer at the generator end of the waveguide. To adequately detect the power transmitted by those modes such a transducer is also necessary at the detector end of the waveguide. For the sake of completeness, it should be mentioned that irregularities or nonuniformities in the waveguide may also excite other modes, causing what is called mode conversion. This may be an advantage or a disadvantage, depending upon the application. This subject is however outside of the scope of this article (see WAVEGUIDE DISCONTINUITIES).

Equation (12) and the associated boundary conditions may yield the amplitudes of the fields, under specific excitation conditions. As already stated a linear combination of all the modes, with adequate coefficients, is the general solution of Maxwell's equations for the waveguide (2). If for a given excitation specific mode coefficients are found to be zero because the corresponding modes do not satisfy the boundary conditions of the excitation, those modes will not be excited and will not actually exist in the guide. Using Eq. (12) and adding the expressions of the longitudinal components Eqs. (18) and

(60) for both directions of propagation yields the system:

$$\begin{aligned}\overline{E}_t &= \sum_i (V_{+i}e^{-\gamma_i z} + V_{-i}e^{\gamma_i z})\overline{e}_{ti} \\ \overline{H}_t &= \sum_i Y_{0i}(V_{+i}e^{-\gamma_i z} - v_{-i}e^{\gamma_i z})\overline{h}_{ti} \\ E_z &= \sum_i (p_i^2/jk\eta)Y_{0i}(V_{+i}e^{-\gamma_i z} - V_{-i}e^{\gamma_i z})\phi_i \\ H_z &= \sum_i (p_i^2/jk\zeta)(V_{+i}e^{-\gamma_i z} + V_{-i}e^{\gamma_i z})\Psi_i\end{aligned} \tag{61}$$

The sums are on the indices m and n for TE as well as for TM modes. All the modes are considered, propagating and attenuating. The coefficients V are calculated from the boundary conditions. To obtain a unique solution in a given volume the tangential electric field must be imposed on part of the bounding surface and the tangential magnetic field on the remaining part of the surface. In a waveguide with lossless conducting walls the tangential electric field is known to be zero on the walls. Hence it is sufficient to impose two conditions in reference planes, for instance E_{tan} in $z = 0$ and H_{tan} in $z = L$, or E_{tan} and H_{tan} in $z = 0$, and so on. Use is then made of orthogonality conditions obtained from Green's theorems (see Green's function methods). As an example, if E_{tan} is specified in $z = 0$ and H_{tan} in $z = L$, the following system is obtained:

$$\begin{aligned}V_{+j} =& (e^{\gamma_j L} + e^{-\gamma_j L})^{-1}\left[e^{\gamma_j L}\int_A \overline{E}_{\tan} \right. \\ & \left. \cdot \overline{e}_{tj}^* da + Z_{oj}\int_A \overline{H}_{\tan}\cdot \overline{h}_{tj}^* da\right] \\ V_{-j} =& (e^{\gamma_j L} + e^{-\gamma_j L})^{-1}\left[e^{-\gamma_j L}\int_A \overline{E}_{\tan} \right. \\ & \left. \cdot \overline{e}_{tj}^* da - Z_{oj}\int_A \overline{H}_{\tan}\cdot \overline{h}_{tj}^* da\right]\end{aligned} \tag{62}$$

If E_{tan} and H_{tan} are specified in $z = 0$, for instance by specifying a current sheet K in part of the plane $z = 0$, which could be a transverse electron pencil in that plane, and terming 1 the negative z region and 2 the positive z region yields as source conditions:

$$\overline{a}_z \times (\overline{H}_{t1} - \overline{H}_{t2}) = \overline{K}_t \qquad \text{and} \qquad \overline{E}_{t1} = \overline{E}_{t2} \tag{63}$$

Suppose matching at both ends of the guide finally yields:

$$V_{-j}^{(1)} = V_{+j}^{(2)} = (Z_{oj}/2)\int_A \overline{e}_{tj}^* \cdot \overline{K}_t\, da \tag{64}$$

Hence, specifying K yields the coefficients of the excited modes. Equation (64) clearly shows as an example that the modes for which the transverse fields are perpendicular to the current source are not excited.

BIBLIOGRAPHY

1. K. L. Smith, Victorian microwaves: millimetre transmissions before the Boer war, *IEEE AESS Newsletter,* 1–3 June 1984.
2. J. C. Slater, *Microwave Electronics,* New York: Van Nostrand, 1950.
3. K. S. Packard, The origin of waveguides: a case of multiple rediscovery, *IEEE Trans. Microw. Theory. Tech.,* **MTT-32**: 961–969, Sept. 1984.
4. R. A. Waldron, *Theory of Guided Electromagnetic Waves,* London: Van Nostrand Reinhold, 1969.
5. A. Vander Vorst and D. Vanhoenacker-Janvier, *Bases de l'ingénierie micro-onde,* Brussels: De Boeck, 1996.
6. O. Heaviside, *Electromagnetic Theory,* London: Benn, 1893.
7. J. J. Thompson, *Notes on Recent Researchers in Electricity and Magnetism,* Oxford: Clarendon, 1893.
8. J. Larmor, Electric vibrations in condensing systems, *Proc. London Math. Soc.,* **26**: 119, Dec. 1894.
9. Lord Rayleigh, On the passage of electric waves through tubes, or the vibration of dielectric cylinders, *Phil. Mag.,* **43**: 125–132, 1897.
10. N. Marcuvitz, *Waveguide Handbook,* New York: McGraw-Hill, 1948.
11. J. A. Stratton, *Electromagnetic Theory,* New York: McGraw-Hill, 1941.
12. R. E. Collin, *Field Theory of Guided Waves,* New York: McGraw-Hill, 1960.
13. C. T. Tai, *Dyadic Green Functions in Electromagnetic Theory,* 2nd ed., New York: IEEE Press, 1994.
14. A. Vander Vorst, *Transmission, Propagation et Rayonnement,* Brussels: De Boeck, 1995.
15. S. A. Schelkunoff, *Electromagnetic Waves,* New York: Van Nostrand, 1943.
16. R. M. Walker, Waveguide impedance: too many definitions, *Electronic Commun.,* **1**(3): May 1966.
17. *The Microwave Engineer's Handbook and Buyer's Guide,* Dedham, MA: Horizon House, 1966.
18. S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics,* New York: Wiley, 1965.
19. T. K. Ishii, *Microwave Engineering,* New York: Ronald Press, 1996.
20. S. B. Cohn, Properties of ridge waveguide, *Proc. IRE,* **35**: 783–788, 1947.
21. W. J. Getsinger, Ridge waveguide field description and application to directional couplers, *IRE Trans. Microw. Theory Tech.,* **MTT-10**: 41–50, 1944.
22. G. L. Ragan, *Microwave Transmission Circuits,* New York: Dover, MIT Rad. Lab Series, vol 9, 1965.
23. G. L. Matthaei, L. Young, and E. M. T. Jones, *Microwave Filters, Impedance-Matching Networks, and Coupling Structures,* New York: McGraw-Hill, 1965.
24. L. J. Chu, Electromagnetic waves in elliptic hollow pipes, *J. Applied Phys.,* **9**: Sept. 1938.
25. M. Abramowitz and I. E. Stegun, *Handbook of Mathematical Functions,* New York: Dover, 1965.
26. D. A. Watkins, *Topics in Electromagnetic Theory,* New York: Wiley, 1958.
27. L. Brillouin, *Wave Propagation in Periodic Structures,* New York: Dover, 1953.
28. G. Floquet, Sur les équations différentielles linéaires à coefficients périodiques, *Ann. Ecole Norm. Sup.,* **12**(47): 1883.
29. A. L. Cullen, R. Reitzig, and P. N. Robson, Further considerations of overmoded rectangular waveguide for high-power transmission, *Proc. IEE,* **112** (7): 1301–1310, 1965.

ANDRÉ VANDER VORST
Université catholique de Louvain

WAVEGUIDES, HIGH-FREQUENCY. See HIGH-FREQUENCY TRANSMISSION LINES.

WAVEGUIDES LOADED WITH FERRITES. See FERRITE-LOADED WAVEGUIDES.

WAVEGUIDES, OPTICAL. See INTEGRATED OPTIC WAVEGUIDES.

WAVEGUIDES, PLANAR. See INTEGRATED OPTIC WAVEGUIDES.