FUZZY STATISTICS

The aim of this article is to give a summary view of many concepts, results, and methods to deal with statistical problems in which some elements are either fuzzily perceived, or reported, or valued. Different handy approaches to model and manage univariate problems are examined and a few techniques from them are gathered. Multivariate statistics with fuzzy elements are briefly discussed, and finally two examples illustrating the use of some models and procedures in the article are included.

INTRODUCTION

Applications of statistics occur in many fields, and the general theory of statistics has been developed by considering the common features of these fields. Three major branches of statistics are descriptive statistics, inferential statistics, and statistical decision making.

All of them, and especially the latter two, are closely related to the concept of uncertainty. In making inferential statements or statistical decisions, the statistician or decision maker is usually unsure of the certain characteristics of a random experiment. Uncertainty involved in statistical problems is traditionally assumed to be due to *randomness* (or unpredictability of the outcomes or events occurring in any performance of the experiment). To deal with this type of uncertainty, probability theory has become a well-developed mathematical apparatus.

However, in several fields of applications of statistics, other types of uncertainty often arise. Thus, in social sciences, psychology, engineering, communications, and so on, statistical problems can include observed or reported data like *very long, quite fast, a few people, more or less in agreement,* and *good yield.* The entry of fuzzy set theory has allowed dealing with the type of uncertainty referred to as *fuzziness* or vagueness (or difficulty of defining sharply the elements—outcomes, events or data—in the problem). We are now going to summarize the models and some relevant methods stated in the literature to manage and solve statistical decision problems involving both randomness and fuzziness.

WHY DEVELOP FUZZY STATISTICS?

The basic model in statistics is a mathematical idealization which is used to describe a random experiment. This model is given by a *probability space* $(\Omega, \mathscr{A}, P_\theta)$, $\theta \in \Theta$, where

- Ω is the *sample space,* which is defined so that each element of Ω denotes an experimental outcome, and any experimental performance results in an element of Ω.
- $\mathscr{A}$ is a class of *events of interest* (which are assumed to be identifiable with subsets of Ω), this class being a σ-field of Ω, and
- P_θ is a *probability measure* defined on $\mathscr{A}$ (that is, a real function from $\mathscr{A}$ which is nonnegative, normalized, and σ-additive), which often involves some uncertain elements that will be generically denoted by θ (unknown parameter value, unknown state of nature, or unknown subindex), Θ being the parameter, state, or index space.

The mechanism of this model can be summarized as follows:

$$\Theta \to \{P_\theta, \theta \in \Theta\} \to \Omega \to \mathscr{A}$$

$$\theta \xrightarrow[\text{distribution}]{\text{experimental}} P_\theta \xrightarrow[\text{performance}]{\text{experimental}} \omega \xrightarrow[\text{interest}]{\text{event of}} A \text{ (occurs if } \omega \in A)$$

To develop a more operational model to describe random experiments, the outcomes can be "converted" into numerical values by associating with each outcome $\omega \in \Omega$ a real (or vectorial) value, so that the interest is not focused on the outcomes but on the associated values. The rule formalizing this association is referred to as a *random variable,* and it is assumed to be Borel-measurable to guarantee that many useful probabilities can be computed.

The incorporation of a random variable to the former model induces a probability space, $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, P_\theta^X)$, $\theta \in \Theta$ (or in general $(\mathbb{R}^k, \mathscr{B}_{\mathbb{R}^k}, P_\theta^X)$, $\theta \in \Theta$, with $k \geq 1$), P_θ^X denoting the induced probability.

The mechanism of the induced model can be summarized as follows:

$$\Theta \to \{P_\theta, \theta \in \Theta\} \to \Omega \to \mathbb{R} \to \mathscr{B}_{\mathbb{R}}$$

$$\theta \xrightarrow[\text{distribution}]{\text{experimental}} P_\theta \xrightarrow[\text{performance}]{\text{experimental}} \omega \xrightarrow[\text{variable}]{\text{random}} X(\omega)$$

$$\xrightarrow[\text{interest}]{\text{event of}} [X \in B] \text{ (occurs if } X(\omega) \in B)$$

The preceding two models could be enlarged if a Bayesian context is considered. In this context θ would behave as a random variable, so that the parameter, state, or index would be specified in accordance with a prior distribution. Indeed, the uncertainty involved in the preceding models corresponds to randomness, which arises in the experimental performance (and in the specification of θ if a Bayesian framework is considered).

However, statistical problems can also involve fuzziness. More precisely, either the numerical values associated with experimental outcomes can be fuzzily perceived or reported, or the values associated with experimental outcomes can be fuzzy, or the events of interest can be identifiable with fuzzy subsets of the sample space. We are now going to recall the most well-developed approaches to model and handle problems involving both fuzzy imprecision and probabilistic uncertainty.

FUZZY STATISTICS BASED ON FUZZY PERCEPTIONS OR REPORTS OF EXISTING NUMERICAL DATA

In this section we consider situations in which the experimental outcomes have been converted into numerical values, by means of a classical random variable, but numerical data are fuzzily perceived or reported by the statistician or observer. In this way, when we say that an item is *expensive* or a day is *cool,* there exist some underlying numerical values (the exact price and temperature) so we are considering fuzzy perceptions or reports of some existing real-valued data.

The scheme of such situations is the following:

$$\Theta \to \{P_\theta, \theta \in \Theta\} \to \Omega \to \mathbb{R} \to \mathscr{F}(\mathbb{R})$$

$$\theta \xrightarrow[\text{distribution}]{\text{experimental}} P_\theta \xrightarrow[\text{performance}]{\text{experimental}} \omega \xrightarrow[\text{variable}]{\text{random}} X(\omega)$$

$$\xrightarrow[\text{of report}]{\text{perception}} \tilde{V}$$

where $\mathscr{F}(\mathbb{X})$ means the class of fuzzy subsets $\tilde{V}$ of $\mathbb{X}$.

To deal with these types of situations we can consider two different approaches. The first one is based on the concept of *fuzzy random variable,* as intended by Kwakernaak (1,2), and

Kruse and Meyer (3). The second one is based on the concept of *fuzzy information* [Okuda et al. (4), Tanaka et al. (5)].

The essential differences between these approaches lie in the nature of the parameters and in the probabilistic assessments. Thus, parameters in the first approach are assumed to be either fuzzy (fuzzy perceptions of unknown classical parameters) or crisp, whereas parameters in the second one are always assumed to be crisp (the classical parameters of the original random variable). On the other hand, in the approach based on fuzzy random variables probabilities often refer to fuzzy variable values and inferences are commonly fuzzy, whereas in the approach based on fuzzy information probabilities are initially assessed to the underlying numerical values and inferences are crisp.

Approach Based on Fuzzy Random Variables

Whenever an extension of probability theory to nonstandard data is going to be established, the fundamental aim is to provide an appropriate concept of a generalized random variable that allows one to verify the validity of essential limit theorems such as the strong law of large numbers and the central limit theorem. In the case of set-valued data, the generalized random variable is a *random set* [Matheron (6), Kendall (7), Stoyan et al. (8)], for which a strong law of large numbers was proved in Artstein and Vitale (9). With regard to fuzzy data and the basic notions of a *fuzzy random variable* [Kwakcrnaak (1)], and *random fuzzy set* (which are also referred to in the literature as fuzzy random variables) introduced by Puri and Ralescu (10), an analogous theorem can be formulated [Ralescu (11), Kruse (12), Miyakoshi and Shimbo (13), Klement et al. (14), and Kruse and Meyer (3)]. It supports the development of a fuzzy probability theory and thus, the laying down of the concepts for mathematical statistics on fuzzy sets. The monographs Kruse and Meyer (3), and Bandemer and Näther (15) describe the theoretical and practical methods of fuzzy statistics in much detail. For comparable discussions and alternative approaches we mention, for instance, Gil (16), Czogala and Hirota (17), Hirota (18), Viertl (19,20), Tanaka (21), Kandel (22).

The background of fuzzy statistics can be distinguished from two different viewpoints of modelling imperfect information using fuzzy sets. The first one regards a fuzzy datum as an existing object, for example a physical grey scale picture. Therefore, this view is called the *physical interpretation* of fuzzy data. The second view, the *epistemic interpretation,* applies fuzzy data to imperfectly specify a value that is existing and precise, but not measurable with exactitude under the given observation conditions. Thus, the first view does not examine real-valued data, but objects that are more complex. In the most simple case we need multivalued data, as they turn up in the field of random sets.

In this section we restrict ourselves to considering the second view (since the first view is considered in the next section), namely the extension of traditional probability theory and mathematical statistics from the treatment of real-valued crisp data to handling fuzzy data in their epistemic interpretation as *possibility distributions* [Zadeh (23), Dubois and Prade (24)]. The statistical analysis of this sort of data was first studied by Kwakernaak (1,2). An extensive investigation of relevant aspects of statistical inference in the presence of possibilistic data can be found in Kruse and Meyer (3). The corresponding methods have been incorporated into the software tool SOLD (Statistics on Linguistic Data) that offers several operations for analysing fuzzy random samples [Kruse (25), Kruse and Gebhardt (26)].

In the following sections we introduce the concept of a fuzzy random variable and outline how to use it for the development of a theory of fuzzy probability and fuzzy statistics. Additionally, we show some implementation aspects and features of the mentioned software tool SOLD.

Fuzzy Random Variables. Introducing the concept of a *fuzzy random variable* means that we deal with situations in which two different types of uncertainty appear simultaneously, namely randomness and possibility. *Randomness* refers to the description of a random experiment by a probability space $(\Omega, \mathscr{A}, P\theta)$, and we assume that the whole information that is relevant for further analysis of any outcome of the random experiment can be expressed with the aid of a real number, so that we can specify a mapping $U: \Omega \to \mathbb{R}$, which assigns to each outcome in Ω its random value in $\mathbb{R}$, U being a *random variable.*

Possibility as a second kind of uncertainty in our description of a random experiment has to be involved whenever we are not in the position to fix the random values $U(\omega)$ as crisp numbers in $\mathbb{R}$, but only to imperfectly specify these values by possibility distributions on $\mathbb{R}$. In this case, the random variable $U: \Omega \to \mathbb{R}$ changes into a *fuzzy random variable* $\mathscr{X}: \Omega \to \mathscr{F}(\mathbb{R})$ with $\mathscr{F}(\mathbb{R}) = \{\tilde{V} \mid \tilde{V}: \mathbb{R} \to [0, 1]\}$ denoting the class of all fuzzy subsets (unnormalized possibility distributions) of the real numbers.

A fuzzy random variable $\mathscr{X}: \Omega \to \mathscr{F}(\mathbb{R})$ is interpreted as a (fuzzy) perception of an inaccessible usual random variable $U_0: \Omega \to \mathbb{R}$, which is called the *original* of $\mathscr{X}$. The basic idea is to assume that the considered random experiment is characterized by U_0, but the available description of its attached random values $U_0(\omega)$ is imperfect in the sense that their most specific specification is the possibility distribution $\mathscr{X}_\omega = \mathscr{X}(\omega)$. In this case, for any $r \in \mathbb{R}$, the value $\mathscr{X}_\omega(r)$ quantifies the degree of possibility with which the proposition $U_0(\omega) - r$ is regarded as being true.

More particularly, $\mathscr{X}_\omega(r) - 0$ means that there is no supporting evidence for the possibility of truth of $U_0(\omega) = r$, whereas $\mathscr{X}_\omega(r) - 1$ means that there is no evidence against the possibility of truth of $U_0(\omega) = r$, so that this proposition is fully possible, and $\mathscr{X}_\omega(r) \in (0, 1)$ reflects that there is evidence that supports the truth of the proposition as well as evidence that contradicts it, based on a set of competing contexts for the specification of $U_0(\omega)$.

Recent research activities in possibility theory have delivered a variety of different approaches to the semantic background of a degree of possibility, similar to the several interpretations that have been proposed with respect to the meaning of subjective probabilities [Shafer (27), Nguyen (28), Kampé de Fériet (29), Wang (30), Dubois et al. (31)]. A quite promising way of interpreting a possibility distribution $\mathscr{X}_\omega: \mathbb{R} \to [0, 1]$ is that of viewing $\mathscr{X}_\omega$ in terms of the context approach [Gebhardt (32), Gebhardt and Kruse (33,34)]. It is very important to provide such semantical underpinnings in order to obtain a well-founded concept of a fuzzy random variable. On the other hand, the following results are presented in a way that makes it sufficient for the reader to confine himself to the intuitive view of a possibility distribution $\mathscr{X}_\omega$

as a gradual constraint on the set $\mathbb{R}$ of possible values [Zadeh (23)].

The concept of a fuzzy random variable is a reasonable extension of the concept of a usual random variable in the many practical applications of random experiments where the implicit assumption of data precision seems to be an inappropriate simplification rather than an adequate modelling of the real physical conditions. Considering possibility distributions allows one to involve *uncertainty* (due to the probabilities of occurrence of competing specification contexts) as well as *imprecision* [due to the context-dependent set-valued specifications of $U_0(\omega)$]. For this reason a frequent case in applications, namely using error intervals instead of crisp points for measuring $U_0(\omega)$, is covered by the concept of a fuzzy random variable.

Note that fuzzy random variables describe situations where the uncertainty and imprecision in observing a random value $U_0(\omega)$ is functionally dependent on the respective outcome ω. If observation conditions are not influenced by the random experiment, so that for any $\omega_1, \omega_2, \in \Omega$, the equality of random values $U_0(\omega_1)$ and $U_0(\omega_2)$ does not imply that their imperfect specifications with the aid of possibility functions are the same, then theoretical considerations in fuzzy statistics become much simpler, since in this case we do not need anymore a concept of a fuzzy random variable. It suffices to generalize operations of traditional statistical inference for crisp data to operations on possibility distributions using the well-known extension principle [Zadeh (35)]. We reconsider this topic later.

After the semantical underpinnings and aims of the concept of a fuzzy random variable have been clarified, we will now present its full formal definition and show how to use it for a probability theory based on fuzzy sets. Let $\mathscr{F}_N(\mathbb{R})$ be the class of all normal fuzzy sets of the real line. Moreover, let $\mathscr{F}_c(\mathbb{R})$ denote the class of all upper semicontinuous fuzzy sets $\tilde{V} \in \mathscr{F}_N(\mathbb{R})$, which means that for all $\alpha \in (0, 1]$, the α-cuts $\tilde{V}_\alpha = \{x \in \mathbb{R} \mid \tilde{V}(x) \geq \alpha\}$ are compact real sets.

Definition. Let $(\Omega, \mathscr{A}, P_\theta)$ be a probability space. A function $\mathscr{X}: \Omega \to \mathscr{F}_c(\mathbb{R})$ is called a fuzzy random variable, if and only if

$$\inf \mathscr{X}_\alpha : \Omega \to \mathbb{R},\ \omega \mapsto \inf(\mathscr{X}(\omega))_\alpha$$

and

$$\sup \mathscr{X}_\alpha : \Omega \to \mathbb{R},\ \omega \mapsto \sup(\mathscr{X}(\omega))_\alpha$$

are Borel-measurable for all $\alpha \in (0, 1)$.

The notion of a fuzzy random variable and the related notion of a probabilistic set were introduced by several authors in different ways. From a formal viewpoint, the definition in this section is similar to that of Kwakernaak (1,2) and Miyakoshi and Shimbo (13). Puri and Ralescu (10) and Klement et al. (14) considered fuzzy random variables [which will be hereafter referred to as random fuzzy sets] as measurable mappings whose values are fuzzy subsets of $\mathbb{R}^k$, or, more generally, of a Banach space; this approach involves distances on spaces of fuzzy sets and measurability of random elements valued in a metric space.

Fuzzy Probability Theory and Descriptive Statistics. Our generalization of concepts of traditional probability theory to a fuzzy probability theory is based on the idea that a fuzzy random variable is considered as a (fuzzy) perception of an inaccessible usual random variable $U_0: \Omega \to \mathbb{R}$, which we referred to as the unknown *original* of X.

Let $\mathscr{H} = \{U \mid U: \Omega \to \mathbb{R}$ and U Borel-measurable w.r.t. $(\Omega, \mathscr{A})\}$ be the set of all one-dimensional random variables w.r.t. $(\Omega, \mathscr{A}, P_\theta)$.

If only fuzzy data are available, then it is of course not possible to identify one of the candidates in $\mathscr{U}$ as the true original of $\mathscr{X}$, but we can evaluate the degree of possibility $\mathrm{Orig}_{\mathscr{X}}(U)$ of the truth of the statement "U is the original of $\mathscr{X}$," determined by the following possibility distribution $\mathrm{Orig}_{\mathscr{X}}$ on $\mathscr{U}$:

$$\mathrm{Orig}_{\mathscr{X}}: \mathscr{H} \to [0,1], \quad U \mapsto \inf_{\omega\in\Omega}\{\mathscr{X}_\omega(U(\omega))\}$$

The definition of $\mathrm{Orig}_{\mathscr{X}}$ shows relationships to random set theory (Matheron (6)) in the way that for all $\alpha \in [0, 1]$, $(\mathrm{Orig}_{\mathscr{X}})_\alpha$ coincides with the set of all selectors of the random set $\mathscr{X}_\alpha: \Omega \to \mathscr{B}_{\mathbb{R}}$, $\mathscr{X}_\alpha(\omega) = (\mathscr{X}_\omega)_\alpha$.

Zadeh's extension principle, which can be justified by the context approach mentioned previously [Gebhardt and Kruse (33)], helps us to define fuzzifications of well-known probability theoretical notions. As an example, consider the generalization of characteristic parameters of crisp random variables to fuzzy random variables:

If $\gamma(U)$ is a characteristic of a classical random variable $U: \Omega \to \mathbb{R}$, then

$$\gamma(\mathscr{X}): \mathbb{R} \to [0,1],\ t \mapsto \sup_{U\in\mathscr{U}, \gamma(U)=t} \inf_{\omega\in\Omega}\{\mathscr{X}_\omega(U(\omega))\}$$

turns out to be the corresponding characteristic of a fuzzy random variable. For example, expected value and variance of a fuzzy random variable are defined as follows:

Definition

(a) $\tilde{E}(\mathscr{X}): \mathbb{R} \to [0, 1]$, $t \mapsto \sup\{\mathrm{Orig}_{\mathscr{X}}(U) \mid U \in \mathscr{H}, E(|U|) < \infty, E(U) = t\}$, is called the *expected value of* $\mathscr{X}$.

(b) $\widetilde{\mathrm{Var}}(\mathscr{X}): \mathbb{R} \to [0, 1]$, $t \mapsto \sup\{\mathrm{Orig}_{\mathscr{X}}(U) \mid U \in \mathscr{H}, E(|U - E(U)|^2 < \infty, E[(U - E(U))^2] = t\}$ is the *variance of* $\mathscr{X}$.

There are also definitions for a real-valued variance [see Bandemer and Näther (15), Näther (36), and recently Körner (37)], but they are introduced on the basis of the Fréchet approach and will be concerned in fact with random fuzzy sets.

In a similar way, other notions of probability theory and descriptive statistics can be generalized to fuzzy data. Based on the semantically well-founded concept of a fuzzy random variable, the fuzzification step is quite simple, since it only refers to an appropriate application of the extension principle. The main theoretical problem consists in finding simplifications that support the development of efficient algorithms for calculations in fuzzy statistics. It turns out that the horizontal representation of fuzzy sets and possibility distributions by using the family of their α-cuts is more appropriate than fixing on the vertical representation that attaches a membership degree or a degree of possibility to each element of the domain of the respective fuzzy set or possibility distribution.

The horizontal representation has the advantage that it reduces operations on fuzzy sets and possibility distributions to operations on α-cuts [Kruse et al. (38)]. Nevertheless, many algorithms for efficient computations in fuzzy statistics require deeper theoretical effort. For more details, see Kruse and Meyer (3).

Complexity problems may also result from nontrivial structures of probability spaces. Tractability therefore often means that one has to confine oneself to the consideration of finite probability spaces or to use appropriate approximation techniques[Kruse and Meyer (3)].

The following theorem shows that a convenient representation of the fuzzy expected value as one example for a characteristic of a fuzzy random variable $\mathscr{X}$ is derived under certain restrictions on $\mathscr{X}$.

Theorem. Let $\mathscr{X}:\Omega \to \mathscr{F}_c(\mathbb{R})$ be a finite fuzzy random variable such that $\mathscr{X}(\Omega) = \{\tilde{V}_1, \ldots, \tilde{V}_n\}$ and $p_i = P(\{\omega \in \Omega \mid \mathscr{X}_\omega = \tilde{V}_i\})$, $i = 1, \ldots, n$. Then,

$$\left\{\left[\sum_{i=1}^{n} p_i \inf(\tilde{V}_i)_\alpha, \sum_{i=1}^{n} p_i \sup(\tilde{V}_i)_\alpha\right]\right\}_{\alpha\in(0,1]}$$

is an α-cut representation of $E(\mathrm{co}(\mathscr{X}))$, where $\mathrm{co}(\mathscr{X}):\Omega \to \mathscr{F}_c(\mathbb{R})$ is defined by $\mathrm{co}(\mathscr{X})(\omega) = \mathrm{co}(\mathscr{X}_\omega)$ with $\mathrm{co}(\mathscr{X}_\omega)$ denoting the convex hull of $\mathscr{X}_\omega$.

Fuzzy Statistics. When fuzzy sets are chosen to be applied in mathematical statistics we have to consider two conceptual different approaches. The first one strictly refers to the concept of a fuzzy random variable. It assumes that, given a generic $\mathscr{X}:\Omega \to \mathscr{F}_c(\mathbb{R})$ and a fuzzy random sample $\mathscr{X}_1, \ldots, \mathscr{X}_n$ independent and identically distributed from the distribution of $\mathscr{X}$, the realization of an underlying random experiment is formalized by a tuple $(\tilde{V}_1, \ldots, \tilde{V}_n) \in [\mathscr{F}_c(\mathbb{R})]^n$ of fuzzy-valued outcomes. Kruse and Meyer (3) verified that all important limit theorems (e.g., the strong law of large numbers, the central limit theorem, and the theorem of Gliwenko–Cantelli) remain valid in the more general context of fuzzy random variables. From this it follows that the extension of mathematical statistics from crisp to fuzzy data is well-founded. As an example for the generalization of an essential theorem we present a fuzzy data version of the strong law of large numbers. More general versions can be found in Klement et al. (14), Meyer (39), Kruse and Meyer (3).

Theorem. Let $\{\mathscr{X}_i\}_{i\in\mathbb{N}}$ be an i.i.d.-sequence on the probability space $(\Omega, \mathscr{A}, P_\theta)$ with the generic fuzzy random variable $\mathscr{X}:\Omega \to \mathscr{F}_c(\mathbb{R})$. Let $E(|(\inf \mathscr{X}_i)_0|) < \infty$ and $E(|(\sup \mathscr{X}_i)_0|) < \infty$. Then there exists a null set N (i.e., a set with probability zero) such that for all $\omega \in \Omega\backslash N$

$$\lim_{n\to\infty} d_\infty\left(\frac{1}{n}\sum_{i=1}^{n}\mathscr{X}_i(\omega), \tilde{E}(\mathscr{X})\right) = 0$$

where d_∞ is the so-called generalized Hausdorff metric [first time introduced by Puri and Ralescu (40)], defined for $\tilde{V}$, $\tilde{W} \in \mathscr{F}_c(\mathbb{R})$ as follows:

$$d_\infty(\tilde{V}, \tilde{W}) = \sup_{\alpha\in(0,1]} d_H(\tilde{V}_\alpha, \tilde{W}_\alpha)$$

d_H being the Hausdorff metric on the collection of nonempty compact (and often assumed to be convex) subsets of $\mathbb{R}$, $\mathscr{K}_c(\mathbb{R})$, defined for $A, B \in \mathscr{K}_c(\mathbb{R})$ by

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\right\}$$

with $|\cdot|$ denoting the euclidean norm in $\mathbb{R}$.

The second approach to statistics with fuzzy data is not based on the concept of a fuzzy random variable, but rather on the presupposition that there is a generic random variable $U:\Omega \to \mathbb{R}$, a crisp random sample $U_1, \ldots, U_n$ (that is, $U_1, \ldots, U_n$ are i.i.d. from the distribution of U), and a corresponding realization $(u_1, \ldots, u_n)$ imperfectly specified by $(\tilde{V}_1, \ldots, \tilde{V}_n) \in [\mathscr{F}_c(\mathbb{R})]^n$.

Furthermore let $U_1, \ldots, U_n$ i.i.d. from the distribution function F_u and $T(u_1, \ldots, u_n)$ be a realization of a statistical function $T(U_1, \ldots, U_n)$. The target is to calculate the corresponding fuzzy statistical function $T(\tilde{V}_1, \ldots, \tilde{V}_n)$.

As an example consider the problem of computing fuzzy parameter tests:

Suppose that F_U depends on a parameter $\theta \in \Theta$ of a predefined parameter space $\Theta \subseteq \mathbb{R}^k$, $k \in \mathbb{N}$. Let $\mathscr{D}$ be a class of distribution functions, $F_U \in \mathscr{D}$, $D:\Theta \to \mathscr{D}$ a mapping, and $\Theta_0, \Theta_1 \subseteq \Theta$ two disjoint sets of parameters.

A function $\Phi:\mathbb{R}^n \to \{0, 1\}$ is called nonrandomized parameter test for $(\delta, \Theta_0, \Theta_1)$ with respect to $\mathscr{D}$ based on a given significance level $\delta \in (0, 1)$, null hypothesis $H_0:\theta \in \Theta_0$, and alternative hypothesis $H_1:\theta \in \Theta_1$, if and only if Φ is Borel-measurable and, for all $\theta \in \Theta_0$, $E(\Phi(U_1, \ldots, U_n)|P_\theta) \leq \delta)$ holds for $U_1, \ldots, U_n$ i.i.d. from F_U.

By application of the extension principle we obtain the corresponding fuzzy parameter test, where the calculation of $\Phi(\tilde{V}_1, \ldots, \tilde{V}_n)$ often turns out to be a time-consuming task. For this reason we present one of the simple extensions, which is the fuzzy chi-square test.

Theorem. Let $\mathscr{N}$ be the class of all normal distributions $N(\mu, \sigma^2)$ and $U:\Omega \to \mathbb{R}$ a $N(\mu_0, \hat{\sigma}^2)$-distributed random variable with given expected value μ_0, but unknown $\hat{\sigma} \in \Theta = \mathbb{R}^+$. Define $D:\Theta \to \mathscr{N}$, $D(\sigma) = N(\mu_0, \sigma^2)$, $\Theta_0 = \{\sigma_0\}$, $\Theta_1 = \Theta\backslash\Theta_0$, and choose $U_1, \ldots, U_n$ i.i.d. from F_U and $\delta \in (0, 1)$. Suppose $\Phi:\mathbb{R}^n \to \{0, 1\}$ to be the nonrandomized double-sided chi-square test for $(\delta, \Theta_0, \Theta_1)$ with respect to D. If $(\tilde{V}_1, \ldots, \tilde{V}_n) \in [\mathscr{F}_c(\mathbb{R})]^n$, then $\Phi(\tilde{V}_1, \ldots, \tilde{V}_n)$ is the realization of the corresponding fuzzy chi-square test. For $\alpha \in (0, 1]$ we obtain

$$\left(\Phi(\tilde{V}_1,\ldots,\tilde{V}_n)\right)_\alpha = \begin{cases} \{0\} & \text{iff } I_\alpha(\tilde{V}_1,\ldots,\tilde{V}_n) > \sigma_0^2\chi^2_{\delta/2}(n) \\ & \qquad \text{and } S_\alpha(\tilde{V}_1,\ldots,\tilde{V}_n) < \sigma_0^2\chi^2_{1-\delta/2}(n) \\ \{1\} & \text{iff } S_\alpha(\tilde{V}_1,\ldots,\tilde{V}_n) \leq \sigma_0^2\chi^2_{\delta/2}(n) \\ & \qquad \text{and } I_\alpha(\tilde{V}_1,\ldots,\tilde{V}_n) \geq \sigma_0^2\chi^2_{1-\delta/2}(n) \\ \{0,1\} & \text{otherwise} \end{cases}$$

where $\chi^2_{\delta/2}(n)$ denotes the $\delta/2$-quantile of the chi-square distribution with n degrees of freedom, and

$$I_\alpha(\tilde{V}_1,\ldots,\tilde{V}_n) = \sum_{i=1,\inf(\tilde{V}_i)_\alpha\geq\mu}^{n}\left(\inf(\tilde{V}_i)_\alpha-\mu\right)^2 + \sum_{i=1,\sup(\tilde{V}_i)_\alpha\leq\mu}^{n}\left(\mu-\sup(\tilde{V}_i)_\alpha\right)^2$$

$$S_\alpha(\tilde{V}_1,\ldots,\tilde{V}_n) = \sum_{i=1}^{n}\max\left\{\left(\inf(\tilde{V}_i)_\alpha-\mu\right)^2, \left(\sup(\tilde{V}_i)_\alpha-\mu\right)^2\right\}$$

The SOLD-System: An Implementation. As an example for the application of many of the concepts, methods, and results discussed here, we briefly present the software tool SOLD (Statistics on Linguistic Data) [Kruse and Gebhardt (26)], that supports the modelling and statistical analysis of linguistic data, which are representable by fuzzy sets.

An application of the SOLD system consists of two steps, which have to be considered separately with regard to their underlying concepts. In the first step (specification phase) SOLD enables its user to create an application environment (e.g., to analyze weather data), that consists of a finite set of attributes (e.g., *clouding, temperature, precipitation*) with their domains (intervals of real numbers, e.g., [0, 100] for the clouding of the sky in %). For each attribute A the user states several (possibly parameterized) elementary linguistic values (e.g., *cloudy* or *approximately* 75% as fuzzy degrees of the clouding of the sky) and defines for all of these values w the fuzzy sets $\tilde{V}_w$, that shall be associated with them. For this reason SOLD provides 15 different classes of parameterized fuzzy sets of $\mathbb{R}$ (e.g., triangular, rectangular, trapezoidal, Gaussian, and exponential functions) as well as 16 logical and arithmetical operators (*and, or, not,* +, −, *, /, **) and functions (e.g., *exp, log, min, max*), that are generalized to fuzzy sets using the extension principle.

The application of context-free generic grammars G_A permits the combination of elementary linguistic values by logic operators (*and, or, not*) and linguistic hedges (*very, considerable*) to increase or decrease the specificity of fuzzy data. By this, formal languages $L(G_A)$ are obtained, which consist of the linguistic expressions that are permitted to describe the values of the attributes A (e.g., *cloudless* or *fair* as a linguistic expression with respect to the attribute *clouding*).

In the second step (analysis phase) the application environments created in the specification phase can be applied to describe realizations of random samples by tuples of linguistic expressions. Since the random samples consist of existing numeric values, that generally cannot be observed exactly, the fuzzy sets, which are related to the particular linguistic expressions, are interpreted epistemically as possibility distributions.

The SOLD system allows one to determine convex fuzzy estimators for several characteristic parameters of the generic random variables for the considered attributes (e.g., for the expected value, variance, p-quantile, and range). In addition SOLD calculates fuzzy estimates for the unknown parameters of several classes of given distributions and also determines fuzzy tests for one- or two-sided hypotheses with regard to the parameters of normally distributed random variables.

The algorithms incorporated in this tool are based on the original results about fuzzy statistics that were presented in the monograph Kruse and Meyer (3). For reasons of efficiency, in SOLD only fuzzy sets of the classes $\mathscr{F}_{D_k}(\mathbb{R})$ are employed, namely the subclasses of $\mathscr{F}_N(\mathbb{R})$ that consist of the fuzzy sets with membership degrees out of $\{0, 1/k, \ldots 1\}$, and α-cuts that are representable as the union of a finite number of closed intervals. In this case the operations to be performed can be reduced to the α-cuts of the involved fuzzy sets [Kruse et al. (38)]. Nevertheless the simplification achieved by this restriction does not guarantee that we gain an efficient implementation, since operations on α-cuts are not equivalent to elementary interval arithmetics. The difficulties that arise can be recognized already in the following example of determining a fuzzy estimator for the variance.

Let $U:\Omega \to \mathbb{R}$ be a random variable defined with respect to a probability space $(\Omega, \mathscr{A}, P_\theta)$ and F_U its distribution function. By a realization $(u_1, \ldots, u_n) \in \mathbb{R}^n$ of a random sample $(U_1, \ldots, U_n)$ with random variables $U_n:\Omega \to \mathbb{R}$, $n \geq 2$, that are completely independent and equally distributed according to F_U, the parameter $\mathrm{Var}(U)$ can be estimated with the help of the variance of the random sample, defined as

$$S_n(U_1,\ldots,U_n) = \frac{1}{n-1}\left(\sum_{i=1}^{n}\left(U_i - \frac{1}{n}\sum_{j=1}^{n}U_j\right)^2\right)$$

$S_n(U_1, \ldots, U_n)$ is an unbiased, consistent estimator for $\mathrm{Var}(U)$.

If $(\tilde{V}_1, \ldots, \tilde{V}_n) \in [\mathscr{F}_{D_k}(\mathbb{R})]^n$ is the specification of a fuzzy observation of $(u_1, \ldots, u_n)$ for a given $k \in \mathbb{N}$, then by applying the extension principle we obtain the following fuzzy estimator for $\mathrm{Var}(U)$:

$$\hat{S}_n : [\mathscr{F}_{D_k}(\mathbb{R})]^n \to [\mathscr{F}_{D_k}(\mathbb{R})]$$
$$\hat{S}_n(\tilde{V}_1,\ldots,\tilde{V}_n)(y) = \sup\left\{\min\{\tilde{V}_1(x_1),\ldots,\tilde{V}_n(x_n)\} \mid (x_1,\ldots,x_n) \in \mathbb{R}^n \text{ and } S_n(x_1,\ldots,x_n) = y\right\}$$

For $\alpha \in (0, 1]$, this leads to the α-cuts

$$\begin{aligned}\left(\hat{S}_n(\tilde{V}_1,\ldots,\tilde{V}_n)\right)_\alpha &= S_n\left((\tilde{V}_1)_\alpha,\ldots,(\tilde{V}_n)_\alpha\right)\\ &= \left\{y \mid \exists(x_1,\ldots,x_n) \in \prod_{i=1}^{n}(\tilde{V}_i)_\alpha : S_n(x_1,\ldots,x_n) = y\right\}\\ &= \left\{y \mid \exists(x_1,\ldots,x_n) \in \prod_{i=1}^{n}(\tilde{V}_i)_\alpha : \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sum_{j=1}^{n}x_j\right)^2 = y\right\}\end{aligned}$$

It is

$$S_n\left((\tilde{V}_1)_\alpha,\ldots,(\tilde{V}_n)_\alpha\right) \subseteq \frac{1}{n-1}\sum_{i=1}^{n}\left((\tilde{V}_i)_\alpha - \frac{1}{n}\sum_{j=1}^{n}(\tilde{V}_j)_\alpha\right)^2$$

and equality does not hold in general, so that $S_n[(\tilde{V}_1)_\alpha, \ldots, (\tilde{V}_n)_\alpha]$ cannot be determined by elementary interval arithmetics.

Therefore, the creation of SOLD had to be preceded by further mathematical considerations that were helpful to the development of efficient algorithms for the calculation of fuzzy estimators. Some results can be found in Kruse and Meyer (3) and Kruse and Gebhardt (41).

The fuzzy set ν calculated during the analysis phase by statistical inference with regard to an attribute A (e.g., fuzzy estimation for the variance of the *temperature*) is not transformed back to a linguistic expression by SOLD, as might be expected at first glance. The fundamental problem consists in the fact that in general no $w \in L(G_A)$ can be found, for which $\nu \equiv \tilde{V}_w$ holds. Consequently one is left to a linguistic approximation of ν, that is, to find those linguistic expressions w of $L(G_A)$, whose interpretations $\tilde{V}_w$ approximate the fuzzy set ν under consideration as accurately as possible. The distance between two fuzzy sets is measured with the help of the generalized Hausdorff metric d_∞.

The aim of this linguistic approximation is to determine a $w_{opt} \in L(G_A)$ that satisfies for all $w \in L(G_A)$ that

$$d_\infty(\tilde{V}_{w_{opt}}, \nu) \le d_\infty(\tilde{V}_w, \nu)$$

Since this optimization problem in general is very difficult and can lead to unsatisfactory approximations, if $L(G_A)$ is chosen unfavorably (Hausdorff distance too large or linguistic expressions too complicated), SOLD uses the language $L(G_A)$ only to name the fuzzy data that appear in the random samples related to A in an expressive way. SOLD calculates the Hausdorff distance $d_\infty(\tilde{V}_w, \nu)$ between ν and a fuzzy set $\tilde{V}_w$, provided by the user as a linguistic expression $w \in L(G_A)$, that turns out to be suitable, but does not carry out a linguistic approximation by itself, since the resulting linguistic expression would not be very useful in order to make a decision making in consequence of the statistical inference.

Before concluding this section, it should be mentioned that addition and product by a real number of fuzzy numbers based on Zadeh's extension principle (in fact, intervals arithmetics) do not preserve all the properties of the real-valued case, so that in most situations

$$\frac{1}{n-1}\sum_{i=1}^{n}\left((\tilde{V}_i)_\alpha - \frac{1}{n}\sum_{j=1}^{n}(\tilde{V}_j)_\alpha\right)^2$$

is not equivalent to

$$\frac{1}{n-1}\left(\sum_{i=1}^{n}\left((\tilde{V}_i)_\alpha\right)^2 - \frac{1}{n}\left(\sum_{j=1}^{n}(\tilde{V}_j)_\alpha\right)^2\right)$$

Approach Based on Fuzzy Information

If a random experiment involving a classical (one-dimensional) random variable X is formalized by means of the induced probability space $(\mathbb{R}, \mathcal{B}_\mathbb{R}, P_\theta^X)$, $\theta \in \Theta$, in which θ means a (real or vectorial) parameter value, then in accordance with Okuda et al. (4), and Tanaka et al. (5) we have the following definition.

Definition. An element $\tilde{V} \in \mathcal{F}(\mathbb{R})$ such that $\tilde{V}$ is a Borel-measurable function from $\mathbb{R}$ to [0, 1] and supp $\tilde{V} \subseteq X(\Omega)$ is called fuzzy information associated with $(\mathbb{R}, \mathcal{B}_\mathbb{R}, P_\theta^X)$, $\theta \in \Theta$.

As we have mentioned previously, the approach based on fuzzy information considers that the available probabilities refer to the distribution of the classical random variable. Zadeh (42) suggested a probabilistic assessment to fuzzy information from the probability distribution of the original random variable, which can be described as follows:

Definition. The probability of the fuzzy information $\tilde{V}$ induced from P_θ^X is given by the Lebesgue–Stieltjes integral

$$P_\theta^X(\tilde{V}) = \int_\mathbb{R} \tilde{V}(x)\, dP_\theta^X(x)$$

(which can be considered as a particularization of LeCam's probabilistic definition (43,44), for single stage experiments).

When the induced probability space corresponds to a random sample of size k from a one-dimensional random variable X, and the perceptions or reports from X are fuzzy, the *sample fuzzy information* $(\tilde{V}_1, \ldots, \tilde{V}_k)$ is often intended as the element of $\mathcal{F}(\mathbb{R}^k)$ given by the product aggregation of $\tilde{V}_1, \ldots,$ and $\tilde{V}_k$ (that is, $(\tilde{V}_1, \ldots, \tilde{V}_k)(x_1, \ldots, x_k) = \tilde{V}_1(x_1) \cdot \ldots \cdot \tilde{V}_k(x_k)$ for all $(x_1, \ldots, x_k) \in \mathbb{R}^k$).

Eventually, the class $\mathcal{C}$ of the available fuzzy perceptions/reports should be assumed to be a *fuzzy partition* [in Ruspini's sense (45)], that is, $\sum_{\tilde{V}\in\mathcal{C}} \tilde{V}(x) = 1$ for all $x \in \mathbb{R}$, which is usually referred to as a fuzzy information system associated with the random experiment $(\mathbb{R}, \mathcal{B}_\mathbb{R}, P_\theta^X)$, $\theta \in \Theta$. Of course, if $\mathcal{C}$ is a fuzzy information system, then $\sum_{\tilde{V}\in\mathcal{C}} P_\theta^X(\tilde{V}) = 1$ for all $\theta \in \Theta$.

On the basis of the model we have just presented, several statistical problems involving fuzzy experimental data can be formulated and solved. We are now going to summarize most of the methods developed in the literature to deal with these problems, and we will describe in more detail a few of them.

Parameter Estimation from Fuzzy Information. The aim of the point parameter estimation problem on the basis of fuzzy experimental data is to make use of the information contained in these data to determine a single value to be employed as an estimate of the unknown value of the nonfuzzy parameter θ. To this purpose, the classical maximum likelihood method has been extended by using Zadeh's probabilistic definition [see Gil and Casals (46), Gil et al. (47,48)], and properties of this extension have been examined. Another technique which has been suggested to solve this problem has been introduced [Corral and Gil (49), Gil et al. (47,48)] to supply an operational approximation of the extended maximum likelihood estimates. This technique is defined as follows:

Definition. Let $(\mathbb{R}, \mathcal{B}_\mathbb{R}, P_\theta^X)$, $\theta \in \Theta$, be a random experiment in which $\{P_\theta^X, \theta \in \Theta\}$ is a parametric family of probability measures dominated by the counting or the Lebesgue measure, λ. Assume that the set $\mathbb{X}^n = \{(x_1, \ldots, x_n) | L(x_1, \ldots, x_n; \theta) > 0\}$ does not depend on θ. If we consider the sample fuzzy information $(\tilde{V}_1, \ldots, \tilde{V}_n)$ from the experiment, then the value $\theta^*(\tilde{V}_1, \ldots, \tilde{V}_n) \in \Theta$, if it exists, such that

$$\mathcal{I}(\tilde{V}_1, \ldots, \tilde{V}_n; \theta^*(\tilde{V}_1, \ldots, \tilde{V}_n)) = \inf_{\theta\in\Theta} \mathcal{I}(\tilde{V}_1, \ldots, \tilde{V}_n; \theta)$$

with

$$\mathcal{I}(\tilde{V}_1, \ldots, \tilde{V}_n; \theta) = -\int_{X^n} |(\tilde{V}_1, \ldots, \tilde{V}_n)|(x_1, \ldots, x_n) \log L(x_1, \ldots, x_n; \theta)\, d\lambda(x_1)\ldots d\lambda(x_n)$$

which is the Kerridge inaccuracy between the membership function of the "standardized form" [Saaty (50)] of $(\tilde{V}_1, \ldots, \tilde{V}_n)$, that will be denoted by $|(\tilde{V}_1, \ldots, \tilde{V}_n)|(\cdot)$, and the likelihood function of θ, $L(\cdot; \theta)$, is called minimum inaccuracy estimate of θ for the sample fuzzy information $(\tilde{V}_1, \ldots, \tilde{V}_n)$.

Some of the most valuable properties of the preceding method [see Corral and Gil (49), Gil et al. (47,48), Gebhardt et al. (51)] are those concerning the existence and uniqueness of the minimum inaccuracy solutions:

Theorem. Let $(\mathbb{R}, \mathcal{B}_\mathbb{R}, P_\theta^X)$, $\theta \in \Theta$, be a random experiment in which $\{P_\theta^X, \theta \in \Theta\}$ is a parametric family of probability measures dominated by the counting or the Lebesgue measure. Assume that the experiment satisfies the following regularity conditions: (i) Θ is a real interval which is not a singleton; (ii) the set $\mathbb{X}^n$ does not depend on θ; (iii) P_θ^X is associated with a parametric distribution function which is regular with respect to all its second θ-derivatives in Θ. Suppose that the sample fuzzy information $(\tilde{V}_1, \ldots, \tilde{V}_n)$ satisfies the following regularity conditions: (iv) $\lambda(\tilde{V}_1, \ldots, \tilde{V}_n) = \int_{\mathbb{X}^n} |(\tilde{V}_1, \ldots, \tilde{V}_n)|(x_1, \ldots, x_n)\, d\lambda(x_1) \ldots d\lambda(x_n) < \infty$ and $\mathscr{I}(\tilde{V}_1, \ldots, \tilde{V}_n; \theta) < \infty$, for all $\theta \in \Theta$; (v) the product function $|(\tilde{V}_1, \ldots, \tilde{V}_n)|(\cdot) \log L(\cdot; \theta)$ is "regular" with respect to all its first and second θ-derivatives in Θ, in the sense that

$$\frac{\partial}{\partial\theta}\mathscr{I}(\tilde{V}_1, \ldots, \tilde{V}_n; \theta) = -\int_{\mathbb{X}^n} |(\tilde{V}_1, \ldots, \tilde{V}_n)|(x_1, \ldots, x_n)\frac{\partial}{\partial\theta} \log L(x_1, \ldots, x_n; \theta)\, d\lambda(x_1) \ldots d\lambda(x_n)$$

and

$$\frac{\partial^2}{\partial\theta^2}\mathscr{I}(\tilde{V}_1, \ldots, \tilde{V}_n; \theta) = -\int_{\mathbb{X}^n} |(\tilde{V}_1, \ldots, \tilde{V}_n)|(x_1, \ldots, x_n)\frac{\partial^2}{\partial\theta^2} \log L(x_1, \ldots, x_n; \theta)\, d\lambda(x_1) \ldots d\lambda(x_n)$$

Under the regularity conditions (i)–(v), if there is an estimator of θ for the (nonfuzzy) simple random sample $X^n \equiv (\mathbb{R}^n, \mathscr{B}_{\mathbb{R}^n}, P_\theta^X)$, $\theta \in \Theta$, whose variance attains the Fréchet–Cramér–Rao bound, then the inaccuracy equation, $\partial/\partial\theta\, \mathscr{I}(\tilde{V}_1, \ldots, \tilde{V}_n; \theta) = 0$, admits a solution minimizing the inaccuracy $\mathscr{I}(\tilde{V}_1, \ldots, \tilde{V}_n; \theta)$ with respect to θ in Θ.

Moreover, under the regularity conditions (i)–(v), let $T(X^n)$ be an estimator of θ for the (non-fuzzy) simple random sample X^n, attaining the Fréchet–Cramér–Rao lower bound for the variance, and whose expected value is given by $E_\theta(T) = h(\theta)$, (h being a one-to-one real-valued function on Θ). Then, for the sample fuzzy information $(\tilde{V}_1, \ldots, \tilde{V}_n)$ the inaccuracy equation admits a unique solution minimizing the inaccuracy $\mathscr{I}(\tilde{V}_1, \ldots, \tilde{V}_n; \theta)$ and taking on the value $\theta^*(\tilde{V}_1, \ldots, \tilde{V}_n) \in \Theta$ such that

$$h\big(\theta^*(\tilde{V}_1, \ldots, \tilde{V}_n)\big) = \int_{\mathbb{X}_n} |(\tilde{V}_1, \ldots, \tilde{V}_n)|(x_1, \ldots, x_n) T(x_1, \ldots, x_n)\, d\lambda(x_1) \ldots d\lambda(x_n)$$

The aim of the interval estimation problem on the basis of fuzzy experimental data is to make use of the information contained in these data to determine an interval to be employed as an estimate of the unknown value of the nonfuzzy parameter θ. To this purpose, Corral and Gil (52) have stated a procedure to construct confidence intervals.

Testing Statistical Hypotheses from Fuzzy Information. The aim of the problem of testing a statistical hypothesis on the basis of fuzzy experimental data is to make use of the information contained in these data, either to conclude whether or not a given assumption about the experimental distribution could be accepted, or to determine how likely or unlikely the fuzzy sample information is if the hypothesis is true (depending on the fact that either we use a concrete significance level, or we compute the p-value, respectively).

To this purpose, techniques based exactly or asymptotically on the Neyman–Pearson optimality criterion have been extended. More precisely, the Neyman–Pearson test of two simple hypotheses [Casals et al. (53), Casals and Gil (54)], and the likelihood ratio test [Gil et al. (48)] for fuzzy data have been developed.

On the other hand, some significance tests, like the chi-square and the likelihood ratio test for goodness of fit, have also been extended to deal with fuzzy sample information [see Gil and Casals (46), Gil et al. (47,48)].

In particular, the last technique can be presented as follows [see Gil et al. (48)]:

Theorem. Let $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, P_\theta^X)$, $\theta \in \Theta$, be a random experiment and let $\mathscr{C}$ be a finite fuzzy information system associated with it. Consider the null hypothesis $H_0: P_\theta^X \equiv Q$. Then, the test rejecting H_0 if, and only if, the sample fuzzy information $(\tilde{V}_1, \ldots, \tilde{V}_n)$ satisfies that

$$\Gamma(\tilde{V}_1, \ldots, \tilde{V}_n) = 2\sum_{\tilde{V} \in c} \nu(\tilde{V}) \log \frac{\nu(\tilde{V})}{nQ(\tilde{V})} > c^*$$

where $\nu(\tilde{V})$ is the observed absolute frequency of $\tilde{V}$ in $(\tilde{V}_1, \ldots, \tilde{V}_n)$, $Q(\tilde{V}) = \int_{\mathbb{R}} \tilde{V}(x)\, dQ(x)$ is the (induced) expected probability of $\tilde{V}$ if Q is the experimental distribution, and c^* is the $1 - \alpha$ fractile of the chi-square distribution with $r - 1$ degrees of freedom (r being the cardinality of $\mathscr{C}$), is a test at a significance level approximately α for large n. More precisely, under H_0 the statistic Γ is asymptotically distributed as a χ^2_{r-1}.

In Gil et al. (48), the last test has been generalized to deal with composite parameter hypotheses.

Statistical Decision Making from Fuzzy Information. The aim of the problem of statistical decision making from fuzzy experimental data is to make use of the information contained in these data to make a choice from a set of possible actions, when the consequences of choosing a decision are assumed to depend on some uncertainties (states).

If a Bayesian context is considered, so that a prior distribution associated with the state space is defined, the extension of the Bayes principle of choice among actions has been developed [see Okuda et al. (4), Tanaka et al. (5), Gil et al. (55), Gil (56)].

In Gebhardt et al. (51) the extensive and normal forms of Bayesian decision analysis have been described, and conditions for their equivalence have been given. As particularizations of the Bayes principle for statistical decision making from fuzzy data, the Bayes point estimation and hypothesis testing techniques have been established [see Gil et al. (57), Gil (56), Casals et al. (53,58). In addition, studies on the Bayesian testing of fuzzy statistical hypotheses and on sequential tests from fuzzy data can be found in the literature [see Casals and Salas (59), Pardo et al. (60), Casals (61), Casals and Gil (62)].

As an instance of these studies, we can recall the Bayesian test of two simple fuzzy hypotheses, which has been stated [Casals (61)] as follows:

Theorem. Let $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, P_\theta^X)$, $\theta \in \Theta$, be a random experiment and let π be a prior distribution on a measurable space $(\Theta, \mathscr{D})$ defined on Θ. Let $\tilde{\Theta}_0$ be a fuzzy subset on Θ, and let $\tilde{\Theta}_0^c$ be its complement (in Zadeh's sense). If $A = \{a_0, a_1\}$ is the action space, with a_0 = accepting the hypothesis "θ is $\tilde{\Theta}_0$" and a_1 =

accepting the hypothesis "θ is $\tilde{\Theta}_0^c$", and we consider the real-valued loss function $L:\{\tilde{\Theta}_0, \tilde{\Theta}_0^c\} \times A$ such that $L(\tilde{\Theta}_0, a_0) = L(\tilde{\Theta}_0^c, a_1) = 0$, $L(\tilde{\Theta}_0^c, a_0) = c_0 > 0$ and $L(\tilde{\Theta}_0, a_1) = c_1 > 0$, then there exists a Bayes test with respect to the prior distribution π which chooses a_0 if, and only if, $(\tilde{V}_1, \ldots, \tilde{V}_n)$ satisfies that

$$\int_\Theta \int_{\mathbb{R}^n} \tilde{\Theta}_0^c(\theta)(\tilde{V}_1,\ldots,\tilde{V}_n)(x_1,\ldots,x_n)\, dP_\theta^X(x_1)\ldots dP_\theta^X(x_n)\, d\pi(\theta) > \frac{c_1}{c_0}\int_\Theta \int_{\mathbb{R}^n} \tilde{\Theta}_0(\theta)(\tilde{V}_1,\ldots,\tilde{V}_n)(x_1,\ldots,x_n)\, dP_\theta^X(x_1)\ldots dP_\theta^X(x_n) d\pi(\theta)$$

and a_1 otherwise.

On the other hand, and still in a Bayesian context, some criteria to compare fuzzy information systems have been developed. In this sense, we can refer to the criterion based on the extension of the Raiffa and Schlaifer EVSI [expected value of sample information (63)] [see Gil et al. (55)], and to that combining this extension with an informational measure [see Gil et al. (64)].

Quantification of the Information Contained in Fuzzy Experimental Data. The quantification of the information contained in data about the experimental distribution is commonly carried out through a measure of the amount of information associated with the experiment.

To this purpose, the expected Fisher amount of information, the Shannon information, the Jeffreys invariant of the Kullback–Leibler divergence, and the Csiszár parametric and nonparametric information, have been extended for fuzzy information systems and their properties have been examined [see Gil et al. (65,66), Pardo et al. (67), Gil and Gil (68), Gil and López (69)]. Several criteria to compare fuzzy information systems have been developed on the basis of these measures, and the suitability of these criteria, along with their agreement with the extension of Blackwell's sufficiency comparison [introduced by Pardo et al. (70)], has been analyzed.

The preceding measures and criteria have been additionally employed to discuss the loss of information due to fuzziness in experimental data [see Gil (16,56), Okuda (71)]. This discussion has been used to examine the problem of choosing the appropriate size of the sample fuzzy information to guarantee the achievement of a desirable level of information, or the increasing of the fuzzy sample size with respect to the nonfuzzy one, to compensate the loss of information when only fuzzy experimental data are available.

The main conclusion in this last study for the well-known Fisher information measure [Fisher (72,73)] is gathered in the following result [Gil and López (69)]:

Theorem. Let $X \equiv (\mathbb{R}, \mathscr{B}_{\mathbb{R}}, P_\theta^X)$, $\theta \in \Theta$, be a random experiment and let $\mathscr{C}$ be a fuzzy information system associated with it. The value of the Fisher information function associated with X, $I_X^F(\theta)$, is greater than or equal to that associated with $\mathscr{C}$, $I_C^F(\theta)$, for all $\theta \in \Theta$, where

$$I_C^F(\theta) = \sum_{\tilde{V} \in c} \left[\frac{\partial}{\partial\theta} \log P_\theta^X(\tilde{V})\right]^2 P_\theta^X(\tilde{V})$$

On the other hand, the smallest size n of the sample fuzzy information from $\mathscr{C}$ which can be guaranteed to be at least as informative on the average as a (nonfuzzy) random sample of size m from X is given by

$$n = \sup_{\theta\in\Theta} \frac{m I_X^F(\theta)}{I_C^F(\theta)} \Bigg]$$

with] denoting the greatest integer part.

FUZZY STATISTICS BASED ON EXISTING FUZZY-VALUED DATA

In this section we consider situations in which the experimental outcomes have been (directly) converted into fuzzy values, by associating with each outcome $\omega \in \Omega$ a fuzzy number or, more generally, an element of $\mathscr{F}_c(\mathbb{R}^k)$, k ≥ 1, where $\mathscr{F}_c(\mathbb{R}^k)$ will denote henceforth the class of fuzzy subsets $\tilde{V}$ of $\mathbb{R}^k$ such that for each $\alpha \in [0, 1]$ the α-cut $\tilde{V}_\alpha$ is compact (that is, $\tilde{V}$ is upper semicontinuous), $V_1 \neq \emptyset$, $\tilde{V}_0 = \text{cl}[\text{co}(\text{supp } \tilde{V})]$ is compact, and often $\tilde{V}_\alpha$ is assumed to be convex for all $\alpha \in [0, 1]$.

In these situations random fuzzy sets, as defined by Puri and Ralescu (10) [see also Klement et al. (14), Ralescu (74)] and originally and most commonly called in the literature fuzzy random variables, represent an appropriate model.

The scheme of such situations is the following:

$$\Theta \to \{P_\theta, \theta \in \Theta\} \to \Omega \to \mathscr{F}_c(\mathbb{R}^k)$$

$$\theta \xrightarrow[\text{distribution}]{\text{experimental}} P_\theta \xrightarrow{\text{experimental performance}} \omega \xrightarrow[\text{variable}]{\text{random}} \tilde{V}$$

where θ now represents a subindex.

To formalize the concept of random fuzzy set in Puri and Ralescu's sense, we have first to remark that $\mathscr{F}_c(\mathbb{R}^k)$ can be endowed with a linear structure with the fuzzy addition and product by a real number based on Zadeh's extension principle (35) (although $\mathscr{F}_c(\mathbb{R}^k)$ is not a vector space with these operations), and it can be endowed with the d_∞ metric, defined as indicated in the first approach in the previews section, and $|\cdot|$ denoting the Euclidean norm in $\mathbb{R}^k$. $(\mathscr{F}_c(\mathbb{R}^k), d_\infty)$ is a complete nonseparable metric space [see Puri and Ralescu (10), Klement et al. (14)]. Then,

Definition. Given the probability space $(\Omega, \mathscr{A}, P_\theta)$, and the metric space $(\mathscr{F}_c(\mathbb{R}^k), d_\infty)$, a random fuzzy set associated with this space is a Borel measurable function $\mathscr{X}:\Omega \to \mathscr{F}_c(\mathbb{R}^k)$. A random fuzzy set $\mathscr{X}$ is said to be integrably bounded if $\|\mathscr{X}_0\| \in L^1(\Omega, \mathscr{A}, P_\theta)$ (i.e., $\|\mathscr{X}_0\|$ is integrable with respect to $(\Omega, \mathscr{A}, P_\theta)$), where $\|\mathscr{X}_0(\omega)\| = d_H(\{0\}, X(\omega)) = \sup_{x\in X(\omega)}|x|$ for all $\omega \in \Omega$.

If $\mathscr{X}$ is a random fuzzy set in Puri and Ralescu's sense, the set-valued mapping $\mathscr{X}_\alpha:\Omega \to \mathscr{K}_c(\mathbb{R}^k)$ defined by $\mathscr{X}_\alpha(\omega) = (\mathscr{X}(\omega))_\alpha$ for all $\omega \in \Omega$ is a compact (often convex) random set for all $\alpha \in [0, 1]$ (i.e., a Borel-measurable function from Ω to $\mathscr{K}_c(\mathbb{R}^k)$).

When $\mathscr{X}$ is a random fuzzy set, an average value of $\mathscr{X}$ should be essentially fuzzy. In this sense, for an integrably bounded random fuzzy set, the fuzzy expected value has been introduced by Puri and Ralescu (10) as follows:

Definition. If $\mathscr{X}$ is an intregably bounded random fuzzy set associated with the probability space $(\Omega, \mathscr{A}, P_\theta)$, then the fuzzy expected value of $\mathscr{X}$ is the unique fuzzy subset of $\mathbb{R}^k$,

$\tilde{E}(\mathscr{X})$, satisfying that $(\tilde{E}(\mathscr{X}))_\alpha = \int_\Omega \mathscr{X}_\alpha \, dP_\theta$ for all $\alpha \in (0, 1]$, $\int_\Omega \mathscr{X}_\alpha \, dP_\theta$ being the Aumann's integral of the random set $\mathscr{X}_\alpha$ (75), that is, $\int_\Omega \mathscr{X}_\alpha \, dP_\theta = \{E(f) | f \in L^1(\Omega, \mathscr{A}, P_\theta), f \in \mathscr{X}_\alpha \; a.s.[P_\theta]\}$, where $E(f)$ is the (classical) expected value of the (real-valued) $L^1(\Omega, \mathscr{A}, P_\theta)$-random variable f.

Zhong and Zhou (76) have proven that in the case in which $k = 1$ and mappings are $\mathscr{F}_c(\mathbb{R})$-valued (although the assumption of compactness for $\mathscr{X}_0(\omega)$ is not presupposed), Puri and Ralescu's definition coincides with Kruse and Meyer's one. On the basis of the last model [whose mathematical background has been also examined in Diamond and Kloeden (77)], several studies have been developed. We are now going to summarize most of them.

Probabilistic Bases of Random Fuzzy Sets

Several studies based on Puri and Ralescu's definition have been devoted to establish proper probabilistic bases to develop statistical studies.

Among these bases, we can point out the following: the characterization of random fuzzy sets and integrably bounded random fuzzy sets, as d_H- and d_∞-limits of sequences and dominated sequences, respectively, of certain operational types of random fuzzy sets [see López-Díaz (78), López-Díaz and Gil (79,80)]. As a consequence of this characterization, two practical ways for the computation of the fuzzy expected value of an integrably bounded random fuzzy set exist.

In this way, the following characterizations of integrably bounded random fuzzy sets have been presented in detail in López-Díaz and Gil (79,80):

Theorem. Let $(\Omega, \mathscr{A}, P_\theta)$ be a probability space. A fuzzy-valued mapping $\mathscr{X}: \Omega \to \mathscr{F}_c(\mathbb{R}^k)$ is an integrably bounded random fuzzy set associated with $(\Omega, \mathscr{A}, P_\theta)$ if, and only if, there exists a sequence of simple (that is, having finite image) random fuzzy sets, $\{\mathscr{X}_m\}_m$, $\mathscr{X}_m: \Omega \to \mathscr{F}_c(\mathbb{R}^k)$, associated with $(\Omega, \mathscr{A}, P_\theta)$, and a function $h: \Omega \to \mathbb{R}$, $h \in L^1(\Omega, \mathscr{A}, P_\theta)$, such that $\|(\mathscr{X}_m)_0(\omega)\| \le h(\omega)$ for all $\omega \in \Omega$ and $m \in \mathbb{N}$, and such that

$$\lim_{m\to\infty} d_H((\mathscr{X}_m)_\alpha(\omega), \mathscr{X}_\alpha(\omega)) = 0$$

for all $\omega \in \Omega$ and for each $\alpha \in (0, 1]$.

On the other hand, $\mathscr{X}$ is an integrably bounded random fuzzy set associated with $(\Omega, \mathscr{A}, P_\theta)$ if, and only if, there exists a sequence of random fuzzy sets associated with $(\Omega, \mathscr{A}, P_\theta)$, $\{\mathscr{X}_m\}_m$, $\mathscr{X}_m: \Omega \to \mathscr{F}_c(\mathbb{R}^k)$, with simple α-cut functions, $(\mathscr{X}_m)_\alpha$, and a function $h: \Omega \to \mathbb{R}$, $h \in L^1(\Omega, \mathscr{A}, P_\theta)$, such that $\|(\mathscr{X}_m)_0(\omega)\| \le h(\omega)$ for all $\omega \in \Omega$ and $m \in \mathbb{N}$, and such that

$$\lim_{m\to\infty} d_\infty((\mathscr{X}_m)(\omega), \mathscr{X}(\omega)) = 0$$

for all $\omega \in \Omega$.

In López-Díaz and Gil (81,82), conditions are given to compute *iterated fuzzy expected values* of random fuzzy sets, irrespectively of the order of integration.

Some limit theorems, as a strong law of large numbers and a central limit theorem (in which the notion of normal random fuzzy set is introduced) have been obtained for these random fuzzy sets [see Ralescu and Ralescu (83), Klement et al. (14,84), Negoita and Ralescu (85), Ralescu (74)].

On the other hand, Li and Ogura (86–90) have studied set-valued functions and random fuzzy sets whose α-cut functions are closed rather than compact. The completeness of the space of these random fuzzy sets and the existence theorem of conditional expectations have been obtained. Furthermore, regularity theorems and convergence theorems in the Kuratowski–Mosco sense have been proven for both, closed set- and fuzzy-valued martingales, sub- and super-martingales, by using the martingale selection method instead of the embedding method, which is the usual tool in studies for compact ones.

Inferential Statistics from Random Fuzzy Sets

Several inferential problems (*point estimateion, interval estimation,* and *hypothesis testing*), concerning fuzzy parameters of random fuzzy sets, have been analyzed (see, for instance, Ralescu (11,74,91,92), Ralescu and Ralescu (83,93).

Some useful results in traditional statistics, like the Brunn–Minkowski and the Jensen inequality, have been extended for random fuzzy sets [see Ralescu (74,92)].

The one extending the well-known and valuable Jensen inequality can be presented as follows:

Theorem. Let $(\Omega, \mathscr{A}, P_\theta)$ be a probability space and let $\mathscr{X}: \Omega \to \mathscr{F}_c(\mathbb{R}^k)$ be an integrably bounded random fuzzy set associated with $(\Omega, \mathscr{A}, P_\theta)$. If $\varphi: \mathscr{F}_c(\mathbb{R}^k) \to \mathbb{R}$ is a convex function (that is, $\varphi((\lambda \odot \tilde{V}) \oplus ((1 - \lambda) \odot \tilde{W})) \le \lambda\varphi(\tilde{V}) + (1 - \lambda)\varphi(\tilde{W})$ for all $\tilde{V}, \tilde{W} \in \mathscr{F}_c(\mathbb{R}^k)$), then

$$\varphi\left(\tilde{E}(\mathscr{X})\right) \le E(\varphi \circ \mathscr{X})$$

The problem of quantifying the relative inequality associated with random fuzzy sets has been studied [see Corral et al. (94), López-García (95), Gil et al. (96)]. This study introduces some measures of the extent or magnitude of the inequality associated with fuzzy-valued variables (like some linguistic or opinion variables, and so on), which could not be quantified by means of classical indices [like those given by Gastwirth (97)]. As a consequence, the class of fields the measurement of inequality can be applied to will significantly increase (so that, not only economics and industry, but psychology, social sciences, engineering, medicine, etc., will benefit from the conclusions of this study). Main properties of the classical inequality indices, like the mean independence, the population homogeneity, the principles of transfers, the Schur-convexity, the symmetry, and the continuity (in terms of d_∞), are preserved for the fuzzy-valued indices introduced in López-García (95) and Gil et al. (96)

In López-García (95), a convenient software to compute the fuzzy expected value and the fuzzy inequality indices has been developed. This software permits an easy graphical representation of the computed values by integrating it in commercial applications.

The problem of measuring the mean dispersion of a random fuzzy set with values in $\mathscr{F}_c(\mathbb{R})$ with respect to a concrete element in $\mathscr{F}_c(\mathbb{R})$ (and, in particular, with respect to the fuzzy expected value of the random fuzzy set) has been examined in Lubiano et al. (98). The suggested measure, which will be referred to as the $\vec{\lambda}$-mean square dispersion is real-valued, since it has been introduced not only as a summary measure of the extent of the dispersion, but rather with the purpose of comparing populations or random fuzzy sets when necessary. The approach to get the extension of the variance for a random fuzzy set differs from that by Näther (36) and Körner (37), although coincides with it for a particular choices of $\vec{\lambda}$.

Another statistical problem which has been discussed is that of estimating some population characteristics associated with random fuzzy sets (like the fuzzy inequality index) in random samplings from finite populations [see López-García (95), López-García et al. (99)].

As an example of the results obtained in the last discussions, the following result has been stated [see López-García (95), López-García et al. (99):

Theorem. In the simple random sampling of size n from a population of N individuals or sampling units, an unbiased [up to additive equivalences, $\sim_\oplus$, Mareš (100)] fuzzy estimator of the fuzzy hyperbolic population index is that assessing to the sample $[\tau]$ the fuzzy value

$$\left[\widehat{\tilde{I}_H(\mathscr{X})}\right]^s([\tau]) = \frac{1}{(n-1)N} \odot \left[\left(n(N-1) \odot \tilde{I}_H(\mathscr{X}[\tau])\right) \oplus \left((n-N) \odot \tilde{I}_H^{wv}(\mathscr{X}[\tau])\right)\right]$$

where $\mathscr{X}[\tau]$ means the random fuzzy set $\mathscr{X}$ as distributed on $[\tau]$, $\tilde{I}_H(\mathscr{X}[\tau])$ is the sample fuzzy hyperbolic index in $[\tau]$, which is given by the fuzzy value such that for each $\alpha \in (0, 1]$

$$(\tilde{I}_H(\mathscr{X}[\tau])_\alpha = \left[E\left(\frac{E(\inf(\mathscr{X}[\tau])_\alpha)}{\sup(\mathscr{X}[\tau])_\alpha} - 1\right), E\left(\frac{E(\sup(\mathscr{X}[\tau])_\alpha)}{\inf(\mathscr{X}[\tau])_\alpha} - 1\right)\right]$$

and $\tilde{I}_H(\mathscr{X}[\tau])$ is the expected within-values hyperbolic inequality in sample $[\tau]$, that is,

$$\tilde{I}_H^{wv}(\mathscr{X}[\tau]) = \frac{1}{n} \odot \left[\tilde{I}_H(\mathscr{X}[U_{\tau 1}]) \oplus \ldots \oplus \tilde{I}_H(\mathscr{X}[U_{\tau n}])\right]$$

with $\mathscr{X}[U_{\tau i}]$ denoting the random fuzzy set degenerate at the fuzzy value of $\mathscr{X}$ on the i-th individual of sample $[\tau]$, $i = 1, \ldots, n$.

Statistical Decision Making with Fuzzy Utilities

A general handy model to deal with single-stage decision problems with fuzzy-valued consequences has been presented [Gil and López-Díaz (101), López-Díaz (78), Gebhardt et al. (51)], the model being based on random fuzzy set in Puri and Ralescus's sense.

This problem was previously discussed in the literature of fuzzy decision analysis [see Watson et al. (102), Freeling (103), Tong and Bonissone (104), Dubois and Prade (105), Whalen (106), Gil and Jain (107), Lamata (108)]. Gil and López-Díaz's model has a wider application than the previous ones (in the case of real-valued assessments of probabilities).

The aim of the problem of statistical decision making with fuzzy utilities on the basis of real-valued experimental data is to use the information contained in these data to help the decision maker in taking an appropriate *action* chosen from a set of possible ones, when the consequence of the choice is assumed to be the interaction of the action selected and the *state* which actually occurs.

By using the concepts of random fuzzy set and its fuzzy expected value, and the ranking of fuzzy numbers given by Campos and González (109) (the λ-average ranking method), the concept of fuzzy utility function in the fuzzy expected utility approach has been introduced [Gil and López-Díaz (101), López-Díaz (78)]. Gil and López-Díaz (101) [see also Gebhardt et al. (51)] have developed Bayesian analyses (in both, the normal and the extensive form) of the statistical decision problem with fuzzy utilities, and conditions have been given for the equivalence of these two forms of the Bayesian analysis.

An interesting conclusion obtained in this study indicates that the axiomatic developments establishing the fundamentals of the real-valued utility functions in the expected utility approach, also establishes the fundamentals of the fuzzy utility function in the fuzzy expected utility approach. Thus, [see Gil and López-Díaz (101), López-Díaz (78), Gebhardt et al. (51)]:

Theorem. Consider a decision problem with reward space $\mathscr{R}$ and space of lotteries $\mathscr{P}$. If $\mathscr{S}$ is a set of axioms guaranteeing the existence of a bounded real-valued utility on $\mathscr{R}$, which is unique up to an increasing linear transformation, then $\mathscr{S}$ also ensures the existence of a class of fuzzy utility functions on $\mathscr{R}$.

An analysis of the structure properties of the last class of fuzzy utility functions has been stated by Gil et al. (110).

Finally, a criterion to compare random experiments in the framework of a decision problem with fuzzy utilities has been developed [Gil et al. (111)].

OTHER STUDIES ON FUZZY STATISTICS

The last two sections have been devoted to univariate fuzzy statistics. Multivariate fuzzy statistics refers to descriptive and inferential problems and procedures, to manage situations including several variables and involving either fuzzy data or fuzzy dependences.

The problems of multivariate fuzzy statistics which have received a deeper attention are fuzzy data and fuzzy regression analyses. We are now going to present a brief review of some of the best known problems and methods.

Studies on fuzzy data analysis are mainly focused on cluster analysis, which is a useful tool in dealing with a large amount of data. The aim of the fuzzy cluster analysis is to group a collection of objects, each of them described by means of several variables, in a finite number of classes (*clusters*) which can overlap and allow graduate membership of objects to clusters. Fuzzy clustering supplies solutions to some problems (like bridges, strays, and undetermined objects among the clusters) which could not be solved with classical techniques.

The first approach to fuzzy clustering was developed by Ruspini (45,112), and this approach is based on the concept of fuzzy partition and an optimization problem where the objective function tends to be small as a close pair of objects have nearly equal fuzzy cluster membership. The optimal fuzzy partition is obtained by using an adapted gradient method.

Another well-known method of fuzzy clustering is the so-called fuzzy k-means, which has been developed by Dunn

(113) and Bezdek (114), and it is based on a generalization of the within-groups sum of squares and the use of a norm (usually a euclidean one) to compute distances between objects and "centres" of clusters. The solution of the optimization problem in this method is obtained by employing an algorithm, which has been recently modified (Wang et al. (115)) by considering a bi-objective function.

The classical clustering procedure based on the maximum likelihood method, has been extended to fuzzy clustering by Trauwaert et al. (116) and Yang (117). These extensions do not force clusters to have a quite similar shape.

Finally, we have to remark that a divisive fuzzy hierarchical clustering technique which does not require a previous specification of the number of clusters has been also developed [Dimitrescu (118)].

A general review of many techniques in fuzzy data analysis based on distances or similarities between objects and clusters can be found in Bandemer and Náther (15) [see also, Bandemer and Gottwald (119)].

The aim of the fuzzy regression analysis is to look for a suitable mathematical model relating a dependent variable with some independent ones, when some of the elements in the model can be fuzzy. Tanaka et al. (120–122) considered a possibilistic approach to linear regression analysis, which leads to the fuzzy linear regression in which experimental data are assumed to be real-valued, but parameters of the linear relation are assumed to be fuzzy-valued, and they are determined such that the fuzzy estimate contains the observed real value with more than a given degree, the problem being reduced to a linear programming one. Some additional studies on this problem have been also developed by Moskowitz and Kim (123). Bárdossy (124) extended the preceding study by considering the fuzzy general regression problem (the fuzzy linear regression being a special case), and also incorporating more general fuzzy numbers.

Another interesting approach to fuzzy regression is that based on extending the least squares procedure of the classical case by previously defining some suitable distances between fuzzy numbers. In this approach, we must refer to the Diamond (125,126) and the Bárdossy et al. (127) studies, which consider the fuzzy linear regression problem involving real- or vectorial-valued parameters and fuzzy set data. Salas (128) and Bertoluzza et al. (129) have studied fuzzy linear and polinomial regression based on some operational distances between fuzzy numbers [see Salas (128), Bertoluzza et al. (130)]. Näther (36) presents an attempt to develop a linear estimation theory based on the real-valued variance for random fuzzy sets mentioned in the previous two sections. Other valuable studies on fuzzy regression are due to Yager (131), Heshmaty and Kandel (132), Celminš (133), Wang and Li (134), Savic and Pedrycz (135,136), Ishibuchi and Tanaka (137,138), and Guo and Chen (139).

SOME EXAMPLES OF FUZZY STATISTICS

The models and methods of fuzzy statistics in this article can be applied to many problems. In this section we present two examples which illustrate the practical use of some of these methods.

Example. The time of attention (in minutes) to the same game of four-year old children is supposed to have an exponential distribution with unknown parameter θ. To get information and obtain conclusions about the parameter value, a psychologist considers the experiment in which the time of attention to a game chosen at random by a four-year-old child, ω, is observed. The mathematical model for this random experiment is the probability space $X \equiv (\mathbb{X}, \mathscr{B}_{\mathbb{X}}, P_\theta^X)$, $\theta \in \Theta$, where $\mathbb{X} = \mathbb{R}^+$ and P_θ is the exponential distribution $\gamma(1, \theta)$.

Assume that as the loss of interest in a game does not usually happen in an instantaneous way, the psychologist provides us with imprecise data like $\tilde{V}_1$ = *a few minutes,* $\tilde{V}_i$ = *around* 10*i minutes* ($i = 2, \ldots, 8$), and $\tilde{V}_9$ = much more than 1 hour. These data could easily be viewed as fuzzy information associated with the random experiment and can be described by means of the triangular/trapezoidal fuzzy numbers with support contained in [0, 120] in Fig. 1.

The set $\mathscr{C} = \{\tilde{V}_1, \ldots, \tilde{V}_9\}$ determines a fuzzy information system, so that we can consider methods of statistics in the approach based on fuzzy information. In this way, if the psychologist wants to estimate the unknown value of θ, and for that purpose he selects at random and independently a sample of $n = 600$ four-year-old children, and observes the time of attention to a given game, and the data reported to the statistician are $\tilde{V}_1, \ldots$, and $\tilde{V}_9$, with respective absolute frequencies $n_1 = 314$, $n_2 = 114$, $n_3 = 71$, $n_4 = 43$, $n_5 = 24$, $n_6 = 18$, $n_7 = 10$, and $n_8 = 6$, then since the experimental distribution is $\gamma(1, \theta)$, the minimum inaccuracy estimate would be given by

$$\theta^* = \frac{600}{\sum_{i=1}^{8} n_i, \int_{\mathbb{R}} x|\tilde{V}_i|(x)\,dx} = 0.05$$

Example. A neurologist has to classify his most serious patients as requiring exploratory brain surgery (action a_1), requiring a preventive treatment with drugs (action a_2), or not requiring either treatment or surgery (action a_3). From medical databases, it has been found that 50% of the people examined needed the operation (state θ_1), 30% needed the preventive treatment (state θ_2), while 20% did not need either treatment or surgery (state θ_3).

The utilities (intended as opposite to losses) of right classifications are null. The utilities of wrong classifications are diverse: an unnecessary operation means resources are wasted and the health of the patient may be prejudiced; a preventive treatment means superfluous expenses and possible side effects, if the patient does not require either preventive treat-

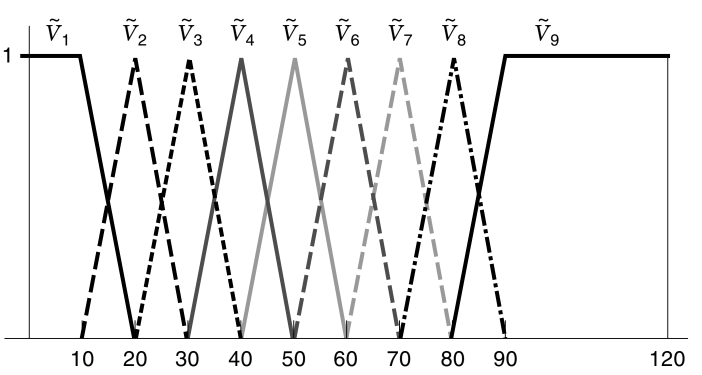

Figure 1. Time of attention to the same game of four-year-old children.

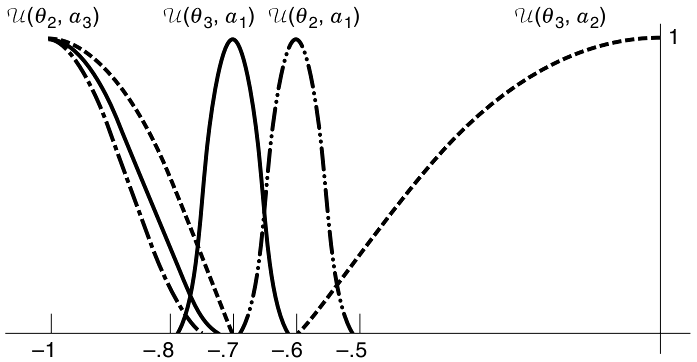

Figure 2. Fuzzy utilities of wrong classifications.

ment or surgery, and may be insufficient if the surgery is really required; if a patient requiring surgery does not get it on time and no preventive treatment is applied, the time lost until clear symptoms appear may be crucial.

The preceding problem can be regarded as a single-stage decision problem in a Bayesian context, with state space $\Theta = \{\theta_1, \theta_2, \theta_3\}$, action space $\mathcal{A} = \{a_1, a_2, a_3\}$, and prior distribution π with $\pi(\theta_1) = 0.5$, $\pi(\theta_2) = 0.3$, and $\pi(\theta_3) = 0.2$. Problems of this type usually receive in the literature a real-valued assessment of utilities [see, for instance, Wonnacott and Wonnacott (140) for a review of similar problems].

However, a real-valued assessment seems to be extremely rigid, in view of the nature of the elements in this problem, but rather a more realistic utility evaluation to describe the neurologist preferences would be the following:

$$\mathcal{U}(\theta_1, a_1) = \mathcal{U}(\theta_2, a_2) = \mathcal{U}(\theta_3, a_3) = 0$$
$$\mathcal{U}(\theta_1, a_2) = \text{very dangerous}, \mathcal{U}(\theta_1, a_3) = \text{extremely dangerous}$$
$$\mathcal{U}(\theta_2, a_1) = \text{inconvenient}, \mathcal{U}(\theta_2, a_3) = \text{dangerous}$$
$$\mathcal{U}(\theta_3, a_1) = \text{excessive}, \mathcal{U}(\theta_3, a_2) = \text{unsuitable}$$

The values assessed to the consequences of this decision problem cannot be represented on a numerical scale, but they could be expressed in terms of fuzzy numbers as, for instances, $\mathcal{U}(\theta_2, a_1) = \Pi(0.1, -0.6)$, $\mathcal{U}(\theta_3, a_1) = \Pi(0.1, -0.7)$ (Π being the well-known Pi curve, cf. Zadeh (141), Cox (142)),

$$\mathcal{U}(\theta_2, a_3)(t) = \begin{cases} 1 - 12(t+1)^2 & \text{if } t \in [-1, -0.75] \\ 20t^2 + 24t + 7 & \text{if } t \in [-0.75, -0.7] \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{U}(\theta_3, a_2)(t) = \begin{cases} 5t^2 + 8t + 3 & \text{if } t \in [-0.6, -0.5] \\ 1 - 3t^2 & \text{if } t \in [-0.5, 0] \\ 0 & \text{otherwise} \end{cases}$$

and $\mathcal{U}(\theta_1, a_2)$ and $\mathcal{U}(\theta_1, a_3)$ are both obtained from $\mathcal{U}(\theta_2, a_2)$ by applying the linguistic modifiers *very* and *extremely* [see Zadeh (141), Cox (142)], that is $\mathcal{U}(\theta_1, a_2) = [\mathcal{U}(\theta_2, a_3)]^2$ and $\mathcal{U}(\theta_1, a_3) = [\mathcal{U}(\theta_2, a_3)]^3$ (Fig. 2).

Doubtless, the situation in this problem is one of those needing a crisp choice among actions a_1, a_2, and a_3. The model and extension of the prior Bayesian analysis developed by Gil and López Díaz (101) is based on Campos and González (109) λ-average ranking criterion using the λ-average ranking function which is defined by

$$V_L^\lambda(\tilde{A}) = \int_{(0.1]} [\lambda \inf \tilde{A}_\alpha + (1-\lambda) \sup \tilde{A}_\alpha] \, d\alpha$$

for all $\tilde{A} \in \mathcal{F}_c(\mathbb{R})$, and $\lambda \in [0, 1]$ being a previously fixed optimism–pessimism parameter [see Campos and González (109) for a graphical interpretation of this function]. This model will lead us to conclude that if we apply the $V_L^{.5}$ ranking function we get that

Step 1:

$$V_L^{.5}(\mathcal{U}(\theta_1, a_1)) = V_L^{.5}(\mathcal{U}(\theta_2, a_2)) = V_L^{.5}(\mathcal{U}(\theta_3, a_3)) = 0$$
$$V_L^{.5}(\mathcal{U}(\theta_1, a_2)) = -0.922967, V_L^{.5}(\mathcal{U}(\theta_1, a_3)) = -0.930410$$
$$V_L^{.5}(\mathcal{U}(\theta_2, a_1)) = -0.6, V_L^{.5}(\mathcal{U}(\theta_2, a_3)) = -0.903334$$
$$V_L^{.5}(\mathcal{U}(\theta_3, a_1)) = -0.7, V_L^{.5}(\mathcal{U}(\theta_3, a_2)) = -0.193333$$

Step 2: The values of $V_L^{.5}$ for the prior fuzzy expected utilities of a_1, a_2, and a_3, are given by

$$V_L^{.5} \circ \tilde{E}(\mathcal{U}_{a_1}|\pi) = -0.32$$
$$V_L^{.5} \circ \tilde{E}(\mathcal{U}_{a_2}|\pi) = -0.50$$
$$V_L^{.5} \circ \tilde{E}(\mathcal{U}_{a_3}|\pi) = -0.74$$

whence a_1 is the Bayes action of the problem.

Step 3: The "value" of the decision problem in a prior Bayesian analysis is then given by $\tilde{E}(\mathcal{U}_{a_1}|\pi)$, which is the fuzzy number given by the PI curve $\Pi(0.05, -0.32)$ (Fig. 3).

ADDITIONAL REMARKS

The development of statistics involving fuzzy data or elements is often based on the extension of classical procedures from mathematical statistics. Several of these extensions do not keep their properties in the nonfuzzy case. Thus, the fuzzy chi-square test in the approach based on fuzzy random variables is in general not a test with significance level δ. In the same way, the extended maximum-likelihood methods in the approach based on fuzzy information cannot be applied to obtain a maximum-likelihood estimator for a parameter of a random fuzzy set, since maximum-likelihood methods are tied to a density of the underlying random fuzzy set, and characterization by densities does not exist for random sets/fuzzy random sets.

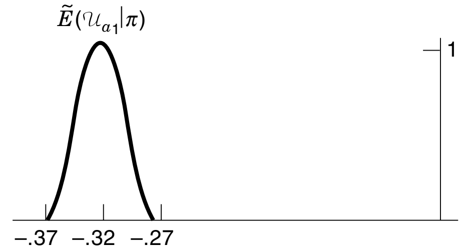

Figure 3. Value of the decision problem.

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