FUZZY MODEL FUNDAMENTALS

The concept of the fuzzy set was introduced in 1965 by Zadeh (1). After this important event, a large number of theoretical contributions were proposed and the formal framework of fuzzy set theory grew fast. For several years, fuzzy models were mainly devoted to specific problems in the areas of pattern recognition and decision-making (2, 3). In the mid-1980s, the successful development of fuzzy controllers opened up new vistas in the application of fuzzy models to engineering problems. Rule-based approaches emerged, in particular, as a powerful and general methodology for information processing. As a result, fuzzy systems became very attractive and the number of applications increased very rapidly in different fields (4–6). During the first half of the 1990s, important relationships with artificial neural networks were established. Fuzzy and neural techniques were presented from a common perspective (7), and new structures able to combine the advantages of fuzzy and neural paradigms were proposed (8–16).

Fuzzy set computing is now a well-established problem-solving technology which aims at replacing (or improving) classical methods in a growing number of research and application areas including control systems, pattern recognition, data classification, signal processing, and low-level and high-level computer vision (17–28).

The aim of this article is not to provide a thorough description of all concepts of fuzzy models. There is a large body of fuzzy literature devoted to this purpose. This article rather aims at presenting an up-to-date selection of most useful concepts from an electronic engineering perspective. For this reason, theoretical aspects and mathematical formalism will be kept to a minimum.

FUZZINESS AND UNCERTAINTY

One of the key features of fuzzy models is their ability to deal with the uncertainty which typically affects physical systems and human activities. Unlike classical methods which resort to a crisp Yes/No approach, fuzzy models adopt a gradual approach which deals with degrees (or grades) of certainty. Let us focus on a simple example. If we observe the object A depicted in Fig. 1, we can easily see that it represents a square. How do we describe the object B in the same figure? It is *more or less* a square. It does not belong to the (crisp) class of squares, because it possesses round corners. However, it may partially belong to a *fuzzy class* of squares. Its degree of membership could be, for example, 0.8 (where unity denotes full membership). Conversely, object D is more or less a circle. It does not belong to the (crisp) class of circles, because it possesses straight lines. However, it may belong to a fuzzy class of circles to a certain extent. Depending on their shapes, all objects in Fig. 1 possess degrees of membership to both fuzzy classes. This simple example also highlights the difference between fuzziness and probability. This important subject has been addressed by different authors in the literature (3–17). It suffices here to observe that probability is related to the occurrence of events, whereas fuzziness is not. Again, let us focus on the object B in Fig. 1. A sentence like "Is it probably a square?" is quite inappropriate to address the uncertainty which affects our process of characterizing the object. The object is not exactly a square. It is *more or less* a square.

Fuzzy concepts represent the basis of human thinking and decision-making. Sentences are very often characterized by vagueness and linguistic imprecision. As an example, if we are driving a car, we could act according to the following statement: "If the speed is low and the vehicle ahead is more or less far away, then moderately increase the speed." Despite their vague appearance, fuzzy concepts represent a powerful way to condense information about real life. The great success of fuzzy models is the result of combination of the following key features:

1. Effectiveness in representing the knowledge about a problem
2. Effectiveness in processing this knowledge by adopting a numerical framework

FUZZY SETS

A *fuzzy set* can be considered a generalization of a classical ("crisp") set. In classical set theory, the degree of membership of an element to a set is either *zero* (no membership) or *unity* (full membership). The membership of an element to a crisp set, say A, is described by the *characteristic function* χ_A:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \tag{1}$$

No partial membership is allowed. Fuzzy set theory permits us to deal with partial membership. A fuzzy set F is indeed represented as a set of ordered pairs (2):

$$F = \{(x, \mu_F(x)) | x \in U\} \tag{2}$$

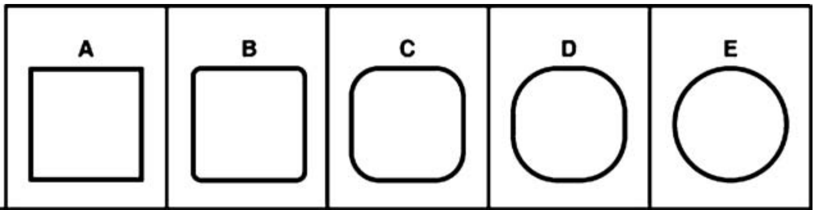
where U is the *universe of discourse* (i.e., the collection of objects where the fuzzy set is defined) and $\mu_F(x)$ is the *membership function* that maps U to the real interval [0, 1]:

$$\mu_F(x) : U \to [0, 1] \tag{3}$$

For each element $x \in U$, the function $\mu_F(x)$ yields a real number which represents the degree (or grade) of membership of x to the fuzzy set F ($0 \leq \mu_F(x) \leq 1$). As an example, let us consider the fuzzy set: F = numbers close to 4. A possible membership function μ_F describing this fuzzy set is represented in Fig. 2.

It can be observed that the maximum degree of membership is obtained for $x = 4$: $\mu_F(4) = 1$. The closer the number to 4 the more is the membership to F. On the contrary, a number very different from 4 is assigned a low (or zero) membership degree, as it should be. The difference between fuzzy and crisp sets is graphically highlighted in the same figure which shows the characteristic function χ_A of the crisp set A = real numbers between 3 and 5. According to the "crisp" nature of set A, we observe a hard transition from full membership to no membership and vice versa.

As a second example, let $U = \{0, 1, 2, \ldots, 255\}$ be the set of integers ranging from 0 to 255. Such a universe may



	A	B	C	D	E
Membership to the CRISP class of squares	1	0	0	0	0
Membership to the FUZZY class of squares	1	0.8	0.5	0.2	0
Membership to the CRISP class of circles	0	0	0	0	1
Membership to the FUZZY class of circles	0	0.2	0.5	0.8	1

Figure 1. Example of crisp and fuzzy classes.

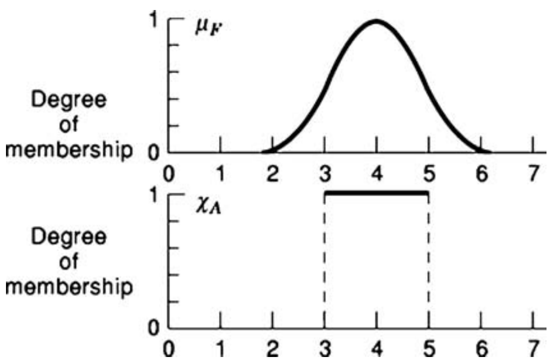


Figure 2. Example of membership and characteristic functions.

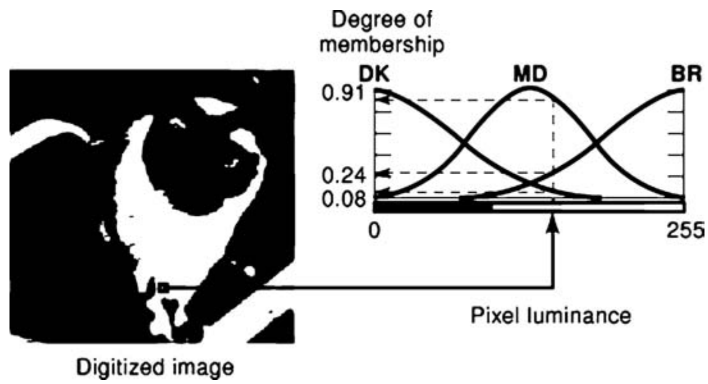


Figure 3. Example of fuzzy sets dark (DK), medium (MD), and bright (BR).

represent the set of possible gray levels (or luminances) of a digitized image, as represented in Fig. 3 (0 = black, 255 = white). Let us define three fuzzy sets labeled *dark* (DK), *medium* (MD), and *bright* (BR) by means of the membership functions μ_{DK}, μ_{MD}, and μ_{BR} depicted in the same figure. It is worth pointing out that almost all pixel luminances possess a nonzero degree of membership to more than one fuzzy set. For example, if we choose a pixel luminance $x = 135$ as shown in Fig. 3, we have $\mu_{DK}(x) = 0.08$, $\mu_{MD}(x) = 0.91$ and $\mu_{BR}(x) = 0.24$.

The concept of membership function plays a key role in fuzzy modeling. Indeed, properties and operators dealing with fuzzy sets can be easily defined in terms of membership functions. The use of linguistic labels to identify fuzzy sets is also quite common. Linguistic labels are often associated with simple operations which change or modify the "shape" of a fuzzy set.

Complement of a Fuzzy Set. The *complement* $\bar{F}$ of fuzzy set F is described by the membership function:

$$\mu_{\overline{F}}(x) = 1 - \mu_F(x) \tag{4}$$

The linguistic label which is usually adopted is "NOT." As an example, the membership function of fuzzy set NOT DARK is represented in Fig. 4.

Union of Fuzzy Sets. The *union* $F_{un} = F_1 \cup F_2$ of fuzzy sets F_1 and F_2 is described by the membership function:

$$\mu_{F_{un}}(x) = \mu_{F_1}(x) \vee \mu_{F_2}(x) = \max_{x \in U} \{\mu_{F_1}(x), \mu_{F_2}(x)\} \tag{5}$$

The commonly used linguistic label is "OR." The membership function of fuzzy set DARK OR MEDIUM is shown in Fig. 5.

Intersection of Fuzzy Sets. The *intersection* $F_{int} = F_1 \cap F_2$ of fuzzy sets F_1 and F_2 is described by the membership function:

$$\mu_{F_{int}}(x) = \mu_{F_1}(x) \wedge \mu_{F_2}(x) = \min_{x \in U} \{\mu_{F_1}(x), \mu_{F_2}(x)\} \tag{6}$$

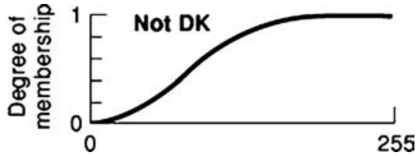


Figure 4. Complement of fuzzy set DARK.

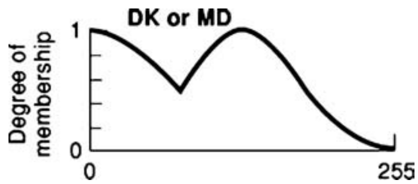


Figure 5. Union of fuzzy sets DK and MD.

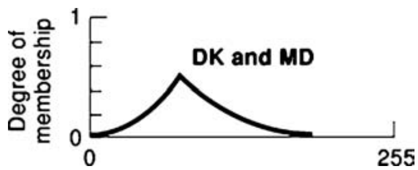


Figure 6. Intersection of fuzzy sets DK and MD.

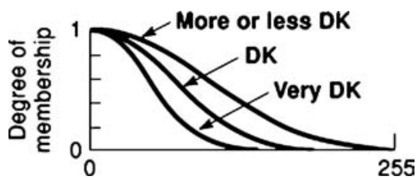


Figure 7. Examples of concentration and dilation.

The associated label is "AND." The membership function of fuzzy set DARK AND MEDIUM is represented in Fig. 6. It should be noted that the above definitions are generalizations of the corresponding definitions for crisp sets.

Linguistic Modifiers. *Linguistic modifiers* (also called *linguistic hedges*) operate on membership functions in order to modify the meaning of the corresponding fuzzy set. Two popular modifiers are described here.

Concentration is a modifier that operates on the membership function of a fuzzy set F in order to decrease values smaller than unity. A commonly used definition (2) is:

$$\mu_{\mathrm{conc(F)}}(x) = \left(\mu_{\mathrm{F}}(x)\right)^2 \tag{7}$$

(Remember that $0 \le \mu_{\mathrm{F}}(x) \le 1$.) The typical linguistic label is "VERY." The membership function of fuzzy set VERY DARK is depicted in Fig. 7.

Dilation is a modifier that operates on the membership function of a fuzzy set F in order to increase values smaller than unity. A typical definition is yielded by the following relationship:

$$\mu_{\mathrm{dil}(F)}(x) = \sqrt{\mu_F(x)} \tag{8}$$

The associated label is "MORE OR LESS." The membership function of fuzzy set MORE OR LESS DARK is represented in Fig. 7 too. Other fuzzy modifiers can be found in Refs. 2 and 17.

We previously used Eq. (2) to generically represent a fuzzy set F. When the universe U is continuous, the following expression is also adopted in the fuzzy literature (17):

$$F = \int_U \mu_F(x)/x \tag{9}$$

On the contrary, when U is discrete, fuzzy set F is often expressed in the following form:

$$F = \sum_U \mu_F(x)/x \tag{10}$$

Of course, integral and summation symbols in the above expressions do not mean integration and arithmetic addition. They are used to denote the collection of all elements $x \in U$. The slash symbol is also typically adopted to associate x with the corresponding degree of membership.

Let us introduce some specific terminology.

Support. The *support* of a fuzzy set F on the universe U is the crisp set $S(F)$ formed by the elements having nonzero degree of membership:

$$S(F) = \{x \in U | \mu_F(x) > 0\} \tag{11}$$

Crossover Point. The *crossover point* of a fuzzy set F is an element x_c with membership degree $\mu_F(x_c) = 0.5$.

Fuzzy Singleton. A fuzzy singleton is a fuzzy set whose support is a single element x with $\mu_F(x) = 1$.

Normal Fuzzy Set. A fuzzy set F is said to be normal if $\max_{x \in U}\{\mu_F(x)\} = 1$.

α-Level Set. The *α-level set* (*α-cut*) of fuzzy set F is the crisp set defined by the following relationship (2):

$$F_\alpha = \{x \in U | \mu_F(x) \ge \alpha\} \tag{12}$$

The *strong* α-level set is defined as:

$$F'_\alpha = \{x \in U | \mu_F(x) > \alpha\} \tag{13}$$

A more general definition resorts to the concept of α-level set. A fuzzy set is convex if all its α-level sets are convex (as crisp sets).

Convex Fuzzy Set. A fuzzy set F is said to be *convex* (4) if its support is a set of real numbers and the following relation applies for all $x \in [x_1, x_2]$ over any interval $[x_1, x_2]$:

$$\mu_F(x) \ge \mu_F(x_1) \wedge \mu_F(x_2) \tag{14}$$

Extension Principle. The extension principle is commonly used to generalize crisp mathematical concepts to fuzzy sets (2). Let F be a fuzzy set on U and let $y = f(x)$ denote a function from U to V (f:$U \to V$). By extending the function f, the fuzzy set $f(F)$ of V is defined as follows (4):

$$\mu_{f(F)}(y) = \sup_{y=f(x)} \mu_F(x) \tag{15}$$

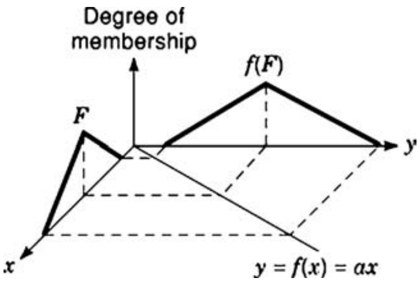


Figure 8. Example of a fuzzy relation.

The fuzzy set $f(F)$ is also expressed by

$$f(F) = \int_V \mu_F(x)/f(x) \tag{16}$$

A simple example is depicted in Fig. 8.

ARITHMETIC OF FUZZY NUMBERS

Fuzzy Numbers. A fuzzy number is a normal and convex fuzzy set such that (2):

1. Only one element (called the mean value) has membership degree equal to unity.
2. Its membership function is piecewise continuous.

In practice the above definition is often modified in order to include trapezoid-shaped fuzzy sets.

Fuzzy arithmetic resorts to the extension principle in order to extend algebraic operations from crisp to fuzzy numbers. Since computational efficiency is an element of paramount importance for many applications, a simplified representation of a fuzzy number, called "*LR* representation," is often adopted.

A fuzzy number is of *LR* type (2) if its membership function is defined by means of two reference functions L (left) and R (right):

$$\mu_F(x) = \begin{cases} L\left(\dfrac{x_m - x}{\alpha}\right) & \text{if } x \le x_m \\ R\left(\dfrac{x - x_m}{\beta}\right) & \text{if } x \ge x_m \end{cases} \tag{17}$$

where x_m is the mean value and $\alpha (\alpha > 0)$ and $\beta (\beta > 0)$ are called the left and right spreads, respectively. A fuzzy number of *LR* type is symbolically denoted by $(x_m, \alpha, \beta)_{LR}$. The choice of functions $L(u)$ and $R(u)$ depends on the context.

A fuzzy interval of *LR* type is very similarly defined by the membership function:

$$\mu_F(x) = \begin{cases} L\left(\dfrac{x'_m - x}{\alpha}\right) & \text{if } x \le x'_m \\ 1 & \text{if } x'_m \le x \le x''_m \\ R\left(\dfrac{x - x''_m}{\beta}\right) & \text{if } x \ge x''_m \end{cases} \tag{18}$$

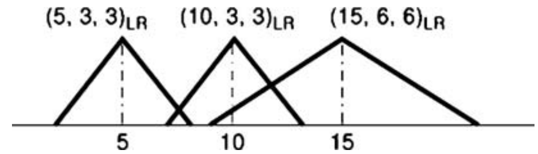


Figure 9. Example of extended addition.

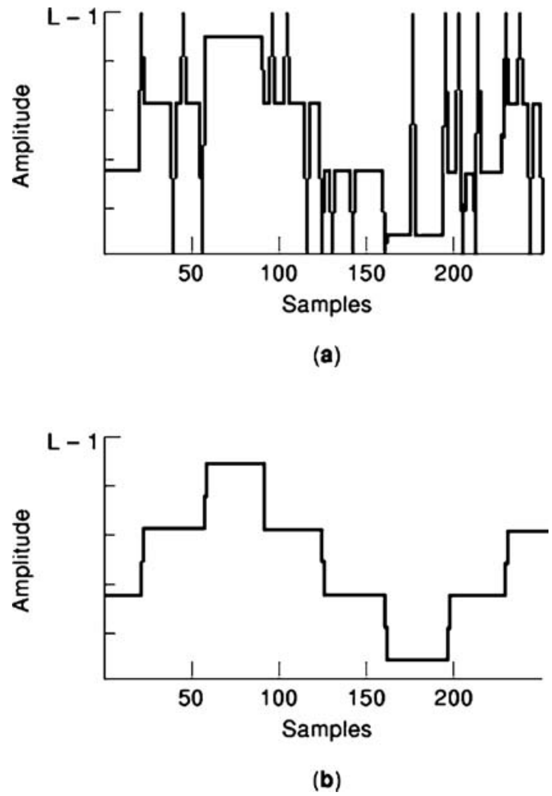


Figure 10. (a) Signal corrupted by impulse noise. (b) Result of fuzzy filtering.

A fuzzy interval is symbolically denoted by $(x'_m, x''_m, \alpha, \beta)_{LR}$. As mentioned above, the extension principle is used to extend some algebraic operations to fuzzy numbers.

Let F_1 and F_2 be two fuzzy numbers of *LR* type: $F_1 = (x_{m1}, \alpha_1, \beta_1)_{LR}$, $F_2 = (x_{m2}, \alpha_2, \beta_2)_{LR}$. The following relations can be used to define extended addition and subtraction (2)

$$(x_{m1}, \alpha_1, \beta_1)_{LR} + (x_{m2}, \alpha_2, \beta_2)_{LR} = (x_{m1} + x_{m2}, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR} \tag{19}$$

$$(x_{m1}, \alpha_1, \beta_1)_{LR} - (x_{m2}, \alpha_2, \beta_2)_{LR} = (x_{m1} - x_{m2}, \alpha_1 + \beta_2, \alpha_2 + \beta_1)_{LR} \tag{20}$$

As an example, let us consider the fuzzy numbers: "about 5" = $(5, 3, 3)_{LR}$ and "about 10" = $(10, 3, 3)_{LR}$. In order to give fuzzy numbers a triangular shape, we adopt the following reference functions: $L(u) = R(u) = \max\{0, 1 - u\}$.

The result yielded by the extended addition $(5, 3, 3)_{LR} + (10, 3, 3)_{LR} = (15, 6, 6)_{LR}$ is depicted in Fig. 9.

FUZZY RELATIONS

A fuzzy relation R between sets U and V is a fuzzy set characterized by a membership function μ_R:$U \times V \to [0, 1]$ and is expressed by (2):

$$R = \{((x,y), \mu_R(x,y)) | (x,y) \in U \times V\} \quad (21)$$

As an example, let $U = V$ be a set of real numbers. The relation: "x is much larger than y" can be described by the membership function:

$$\mu_R(x,y) = \begin{cases} \dfrac{1}{1+\dfrac{5}{x-y}} & \text{if } x > y \\ 0 & \text{if } x \le y \end{cases} \quad (22)$$

If sets U and V represent finite sets $U = \{x_1, x_2, \ldots, x_m\}$ and $V = \{y_1, y_2, \ldots, y_n\}$, a fuzzy relation R can be described by an $m \times n$ matrix (4):

$$R = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ & & \cdots & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \quad (23)$$

where $a_{i,j} = \mu_R(x_i, y_j)$ represents the strength of association between a pair of elements.

Composition. Let R_1 and R_2 be two fuzzy relations defined in different product spaces:

$$\mu_{R1}(x,y): U \times V \to [0, 1] \quad (24)$$

$$\mu_{R2}(y,z): V \times W \to [0, 1] \quad (25)$$

The above relations can be combined by means of the operation "composition." A variety of methods have been proposed in the literature (2). For example, we could be interested in combining the relations R_1 (patients, symptoms) and R_2 (symptoms, diseases) in order to discover relationships between patients and diseases. The so-called max–min composition yields a resulting fuzzy relation described as follows:

$$\mu_{R_1 \circ R_2}(x,z) = \max_y \{\mu_{R_1}(x,y) \wedge \mu_{R_2}(y,z)\} \quad (26)$$

The max–$*$ composition is a more general definition of composition (2). It is defined by the following membership function:

$$\mu_{R_1 * R_2}(x,z) = \max_y \{\mu_{R_1}(x,y) * \mu_{R_2}(y,z)\} \quad (27)$$

FUZZY AGGREGATION CONNECTIVES

Minimum and maximum operators represent the simplest way to aggregate different degrees of membership. More sophisticated choices are available in the literature. They resort to *fuzzy aggregation connectives.* Fuzzy aggregation connectives are (possibly nonlinear) functions that map a set of membership (or certainty) values $\mu_1, \mu_2, \ldots, \mu_N$ to the real interval [0, 1]. Fuzzy aggregation connectives can be grouped into the following classes (13,16,29):

1. Union connectives
2. Intersection connectives
3. Compensative connectives

Union Connectives

The simplest aggregation connective of union type is the mentioned "Max" operator. A useful generalization is represented by the family of union aggregators defined by Yager (30):

$$y_U(\mu_1, \mu_2, \ldots, \mu_n) = \min\left\{1, \left(\sum_{i=1}^{n} \mu_i^p\right)^{1/p}\right\} \quad (28)$$

It can be observed that $\lim_{p\to\infty} y_U(\mu_1, \mu_2, \ldots, \mu_n) = \max(\mu_1, \mu_2, \ldots, \mu_n)$. Thus, the range of this connective is between max and unity. In this respect, this aggregation connective is more optimistic than the MAX operator (13). By varying the value of parameter p from zero to $+\infty$, different aggregation strategies can be realized.

Intersection Connectives

The simplest aggregation connective of intersection type is the very popular "min" operator. A useful generalization is represented by the family of intersection aggregators defined by (30)

$$y_I(\mu_1, \mu_2, \ldots, \mu_n) = 1 - \min\left\{1, \left(\sum_{i=1}^{n} (1-\mu_i)^p\right)^{1/p}\right\} \quad (29)$$

It can be observed that $\lim_{p\to\infty} y_I(\mu_1, \mu_2, \ldots, \mu_n) = \min(\mu_1, \mu_2, \ldots, \mu_n)$. Thus, the range of this connective is between min and zero. This aggregation connective is more pessimistic than the min operator. As in the previous case, different aggregation strategies can be realized by suitably varying the value of parameter p.

Compensative Connectives

Compensative connectives can be categorized into the following classes depending on their aggregation structure:

1. Mean operators
2. Hybrid operators

Mean Connectives. A mean connective is a mapping m: $[0, 1] \times [0, 1] \to [0, 1]$ such that

1. $m(\mu_1, \mu_2) \ge m(\mu_3, \mu_4)$ if $\mu_1 \ge \mu_3$ and $\mu_2 \ge \mu_4$
2. $\min(\mu_1, \mu_2) \le m(\mu_1, \mu_2) \le \max(\mu_1, \mu_2)$

A useful mean connective is the *generalized mean* (31). By using this connective, different degrees of certainty (or criteria) can be suitably weighted in order to take care of their relative importance:

$$m(\mu_1, \mu_2, \ldots, \mu_n : w_1, w_2, \ldots, w_n) = \left(\sum_{i=1}^{n} w_i \mu_i^p \right)^{1/p} \tag{30}$$

where $\Sigma^n{}_{i=1}\, w_i = 1$. It is worth pointing out that this connective yields all values between min and max by varying the parameter p between $p \to -\infty$ and $p \to +\infty$.

Hybrid Connectives. Hybrid connectives combine outputs of union and intesection operators (29). This combination is generally performed using a multiplicative or an additive model as follows:

$$y_H = (y_I)^{1-\gamma}(y_U)^{\gamma} \qquad \text{multiplicative model} \tag{31}$$

$$y_H = (1-\gamma)(y_I) + \gamma(y_U) \qquad \text{additive model} \tag{32}$$

where y_U and y_I denote the outputs of union and intersection operators. The degree of compensation between these components depends on the value of the parameter γ.

The multiplicative γ-model proposed by Zimmermann and Zysno (32) adopts union and intersection components based on products:

$$y_H(\mu_1, \mu_2, \ldots, \mu_n : w_1, w_2, \ldots, w_n) = \left(\prod_{i=1}^{n} \mu_i^{w_i} \right)^{1-\gamma} \left(1 - \prod_{i=1}^{n} (1-\mu_i)^{w_i} \right)^{\gamma} \tag{33}$$

where $\Sigma^n{}_{i=1}\, w_i = n$ and $0 \le \gamma \le 1$. The additive γ-model is defined by

$$y_H(\mu_1, \mu_2, \ldots, \mu_n : w_1, w_2, \ldots, w_n) = (1-\gamma) \left(\prod_{i=1}^{n} \mu_i^{w_i} \right) + \gamma \left(1 - \prod_{i=1}^{n} (1-\mu_i)^{w_i} \right) \tag{34}$$

The additive γ-model adopting Yager's union and intersection is defined by

$$y_H = (1-\gamma) \left(1 - \min\left\{ 1, \left(\sum_{i=1}^{n} (1-\mu_i^{w_i})^p \right)^{1/p} \right\} \right) + \gamma \min\left\{ 1, \left(\sum (\mu_i^{w_i})^p \right)^{1/p} \right\} \tag{35}$$

LINGUISTIC VARIABLES AND FUZZY SYSTEMS

As mentioned in the section entitled "Fuzzy Set," fuzzy models permit us to express concepts in a way that is very close to human thinking. In fact, linguistic labels can be associated with fuzzy sets in order to form sentences like "the pixel luminance is very bright," "the voltage is low," "the temperature is high," and so on. In this respect, quantities such as pixel luminance, voltage, and temperature can be interpreted as *linguistic variables*—that is, variables whose values are words or sentences (17). For example, the linguistic variable *pixel luminance* can be decomposed into a set of terms such as *dark, medium, and bright* (Fig. 3) which correspond to fuzzy sets in its universe of discourse. *Fuzzy rules* permit us to express a processing strategy in a form that mimics human decision making. For example: *If x is low and y is medium, then z is large.* The typical IF–THEN structure of a fuzzy rule includes a group of *antecedent clauses* which define conditions and a consequent clause which identifies the corresponding action. In general, fuzzy systems adopt rules to map fuzzy sets to fuzzy sets (7). Many engineering applications, however, require techniques which map scalar inputs to scalar outputs. We can address this issue by adding an input fuzzifier and an output defuzzifier to the classical model (17). The result is a very important class of fuzzy systems which are able to map scalar inputs to one (or more) scalar output(s). Since the successful application of these systems is playing a key role in the widespread diffusion of fuzzy techniques, we shall decribe their structure in details. Let us consider a fuzzy system which maps M input variables $x_1, x_2, \ldots, x_M$ to one output variable y by means of N fuzzy rules R1, R2, ..., RN. Such a system can be expressed in the following form:

$$\begin{aligned}
\mathbf{R1}: \quad & \mathbf{IF}\,(x_1, A_{1,1})\ \mathbf{AND}\,(x_2, A_{2,1})\ \mathbf{AND}\ldots \\
& \qquad \mathbf{AND}\,(x_M, A_{M,1}),\ \mathbf{THEN}\,(y, B_1) \\
\mathbf{R2}: \quad & \mathbf{IF}\,(x_1, A_{1,2})\ \mathbf{AND}\,(x_2, A_{2,2})\ \mathbf{AND}\ldots \\
& \qquad \mathbf{AND}\,(x_M, A_{M,2}),\ \mathbf{THEN}\,(y, B_2) \\
\mathbf{R}j: \quad & \mathbf{IF}\,(x_1, A_{1,j})\ \mathbf{AND}\,(x_2, A_{2,j})\ \mathbf{AND}\ldots \\
& \qquad \mathbf{AND}\,(x_M, A_{M,j}),\ \mathbf{THEN}\,(y, B_j) \\
\mathbf{R}N: \quad & \mathbf{IF}\,(x_1, A_{1,N})\ \mathbf{AND}\,(x_2, A_{2,N})\ \mathbf{AND}\ldots \\
& \qquad \mathbf{AND}\,(x_M, A_{M,N}),\ \mathbf{THEN}\,(y, B_N)
\end{aligned}$$

where $A_{i,j} (1 \le i \le M, 1 \le j \le N)$ is the fuzzy set associated with the ith input variable in the jth rule and B_j is the fuzzy set associated with the output variable in the same rule. The set of fuzzy rules as a whole is called a *rulebase.* Since the fuzzy rulebase contains the necessary information to process the data, it represents the *knowledge base* of the system.

The knowledge base is numerically processed by the *fuzzy inference mechanism.* For a given set of input data, the inference mechanism evaluates the degrees of activation of the component rules and then combines their resulting effects. More precisely, let λ_j be the degree of activation (or satisfaction) of the jth rule. This degree can be evaluated by using the following relation:

$$\lambda_j = \min_{i=1,\ldots,M} \{\mu_{A_{i,j}}(x_i)\} \tag{36}$$

where $\mu_{A_{i,j}}$ denotes the membership function of fuzzy set $A_{i,j}$. It should be noticed that the choice of an intersection connective to aggregate membership degrees depends on the presence of the "AND" for combining the antecedent clauses in each fuzzy rule. Of course, different aggregation connectives (see the section entitled "Fuzzy Aggregation Connectives") can be adopted depending on the specific problem.

The degree of activation λ_j yields the following effect on fuzzy set B_j which identifies the consequent action of the jth rule. A new fuzzy set B'_j is generated, whose membership function is defined by

$$\mu_{B'_j}(u) = \lambda_j * \mu_{B'_j}(u) = \begin{cases} \lambda_j \cdot \mu_{B_j}(u) & \text{correlation product} \\ \lambda_j \wedge \mu_{B_j}(u) & \text{correlation minimum} \end{cases} \tag{37}$$

Two different inference schemes are commonly used. If the *correlation-product* inference is adopted (7), symbol "$*$" denotes the product operator. If, on the other hand, the *correlation-minimum* inference is chosen, symbol "$*$" denotes the minimum operator.

Fuzzy sets B'_j ($j = 1, \ldots, N$) are then combined in order to obtain a resulting fuzzy set B. If we resort to the union, the corresponding membership function $\mu_B(u)$ is yielded by

$$\mu_B(u) = \max_{j=1,\ldots,N}\{\mu_{B'_j}(u)\} \tag{38}$$

If we adopt the additive model (7), on the contrary, we obtain

$$\mu_B(u) = K\sum_{j=1}^{N}\mu_{B'_j}(u) \tag{39}$$

where K is a scaling factor that limits the degree of membership to unity. As a final step, we want to derive a scalar value from the fuzzy set B. A very popular technique is the so-called "centroid" or "center of gravity" method which yields the output y as follows:

$$y = \frac{\int_V u\mu_B(u)\,du}{\int_V \mu_B(u)\,du} \tag{40}$$

where V denotes the support of fuzzy set B. (If this support is discrete, summation should replace the integral symbol. Of course, integral and summation symbols here denote integration and arithmetic addition.) Notice that if we adopt the additive scheme, we can evaluate the output y by means of the centroids y'_j of the component fuzzy sets B'_j:

$$y = \frac{\sum_{j=1}^{N} w'_j y'_j}{\sum_{j=1}^{N} w'_j} \tag{41}$$

where

$$w'_j = \int_V \mu_{B'_j}(u)\,du \tag{42}$$

$$y'_j = \frac{\int_V u\mu_{B'_j}(u)\,du}{w'_j} \tag{43}$$

Let us adopt correlation-product inference. Relations (42) and (43) become:

$$w'_j = \lambda_j w_j; \qquad w_j = \int_V \mu_{B_j}(u)\,du \tag{44}$$

$$y'_j = y_j; \qquad y_j = \frac{\int_V u\mu_{B_j}(u)\,du}{w_j} \tag{45}$$

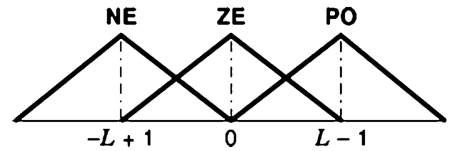


Figure 11. Fuzzy sets positive (PO), zero (ZE), and negative (NE).

Thus, we can express relation (41) as follows:

$$y = \frac{\sum_{j=1}^{N} \lambda_j w_j y_j}{\sum_{j=1}^{N} \lambda_j w_j} \tag{46}$$

Relation (46) is very attractive from the point of view of computational efficiency. In fact, the component terms w_j and y_j do not depend on λ_j. If all consequent fuzzy sets B_j have the same shape, i.e., $w_j = w (j = 1, \ldots, N)$, we finally obtain

$$y = \frac{\sum_{j=1}^{N} \lambda_j y_j}{\sum_{j=1}^{N} \lambda_j} \tag{47}$$

In this case, the final output only depends on the degrees of activation of fuzzy rules and on the centroids of the original consequent fuzzy sets.

Let us consider a simple example. Let $\{s_k\}$ be the digitized signal in the range $[0, L - 1]$ depicted in Fig. 10(a). This signal represents a staircase waveform corrupted by impulse noise. Suppose we want to design a filter able to reduce (or possibly cancel) the noise pulses (33). Let s_k be the sample to be processed at the time k. Let $\Delta_{k-1} = s_k - s_{k-1}$ and $\Delta_{k+1} = s_k - s_{k+1}$ be the amplitude differences between this element and the neighboring samples s_{k-1} and s_{k+1}, respectively. In order to estimate the noise amplitude n_k, we may use the following fuzzy system:

R1: IF (Δ_{k-1} is PO) AND (Δ_{k+1} is PO), THEN n_k is PO

R2: IF (Δ_{k-1} is NE) AND (Δ_{k+1} is NE), THEN n_k is NE

R3: IF (Δ_{k-1} is ZE) OR (Δ_{k+1} is ZE), THEN n_k is ZE

where PO (positive), ZE (zero), and NE (negative) are triangular fuzzy sets represented in Fig. 11. The first fuzzy rule (R1) aims at detecting a positive noise pulse (i.e., a noise pulse whose amplitude is higher than the one of the neighborhood). The second fuzzy rule (R2) aims at detecting a negative noise pulse (i.e., a noise pulse whose amplitude is lower than the one of the neighborhood). The third fuzzy rule (R3) deals with the absence of any noise pulse (i.e., with the case of an uncorrupted sample). Formally, we have $A_{1,1}$ = PO, $A_{2,1}$ = PO, $A_{1,2}$ = NE, $A_{2,2}$ = NE, $A_{1,3}$ = ZE, $A_{2,3}$ = ZE, B_1 = PO, B_2 = NE, B_3 = ZE.

The degrees of activation $\lambda^{(k)}{}_1$, $\lambda^{(k)}{}_2$, $\lambda^{(k)}{}_3$ of three rules at the time k are evaluated by

$$\lambda_1^{(k)} = \min\{\mu_{PO}(\Delta_{k-1}), \mu_{PO}(\Delta_{k+1})\} \tag{48}$$

$$\lambda_2^{(k)} = \min\{\mu_{NE}(\Delta_{k-1}), \mu_{NE}(\Delta_{k+1})\} \tag{49}$$

$$\lambda_3^{(k)} = \max\{\mu_{ZE}(\Delta_{k-1}), \mu_{ZE}(\Delta_{k+1})\} \tag{50}$$

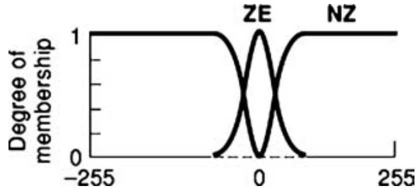


Figure 12. Fuzzy sets zero (ZE) and nonzero (NZ).

Suppose we adopt correlation-product inference and the additive model. Since all fuzzy sets have the same shape, the output is yielded by relation (47). We observe, in particular, that the centroids have the following values (Fig. 11): $y_{PO} = L - 1$, $y_{ZE} = 0$ and $y_{NE} = -L + 1$. Thus, we have

$$n_k = (L-1)\frac{\lambda_1^{(k)} - \lambda_2^{(k)}}{\lambda_1^{(k)} + \lambda_2^{(k)} + \lambda_3^{(k)}} \tag{51}$$

Let $s'_k = s_k - n_k$ be the output of the filter. The result of the application is shown in Fig. 10(b).

As a second example, let us consider the digitized image in Fig. 3. Let $x_{i,j}$ be the pixel luminance at location (i, j). Let $\Delta_{i,j-1} = x_{i,j} - x_{i,j-1}$ and $\Delta_{i-1,j} = x_{i,j} - x_{i-1,j}$ be the luminance differences between this element and the neighboring pixels at locations $(i, j-1)$ and $(i-1, j)$, respectively. Let us suppose we want to detect edges in the image—that is, possible object borders (34). Our goal is to produce another image (called "edge map") where dark pixels denote uniform regions and bright pixels denote possible object contours. In order to perform this task, we define a pair of fuzzy rules as follows:

$$\text{R1}: \quad \text{IF } (\Delta_{i,j-1} \text{ is ZE}) \text{ AND } (\Delta_{i-1,j} \text{ is ZE}), \text{ THEN } y_{i,j} \text{ is BL}$$
$$\text{R2}: \quad \text{IF } (\Delta_{i,j-1} \text{ is NZ}) \text{ OR } (\Delta_{i-1,j} \text{ is NZ}), \text{ THEN } y_{i,j} \text{ is WH}$$

where $y_{i,j}$ is the luminance of the pixel at location (i, j) in the edge map. *Zero* (ZE) and *nonzero* (NZ) are fuzzy sets in the interval $[-L+1, L-1]$ (Fig. 12). White (WH) and black (BL) are fuzzy singletons centered on $L-1$ and zero. We can evaluate the degrees of activation $\lambda^{(i,j)}{}_1$ and $\lambda^{(i,j)}{}_2$ by using simple intersection and union aggregators:

$$\lambda_1^{(i,j)} = \min\{\mu_{\mathbf{ZE}}(\Delta_{i,j-1}), \mu_{\mathbf{ZE}}(\Delta_{i-1,j})\} \tag{52}$$

$$\lambda_2^{(i,j)} = \max\{\mu_{\mathbf{NZ}}(\Delta_{i,j-1}), \mu_{\mathbf{NZ}}(\Delta_{i-1,j})\} \tag{53}$$

The output $y_{i,j}$ is yielded by

$$y_{i,j} = \frac{(L-1)\lambda_2^{(i,j)}}{\lambda_1^{(i,j)} + \lambda_2^{(i,j)}} \tag{54}$$

The result is shown in Fig. 13.

Fuzzy inference schemes different from that described above are also possible. As an example, the well-known Takagi–Sugeno Model (4, 35) found wide application in the design of fuzzy controllers. More sophisticated approaches are also available in the literature (36–38). In any case an appropriate choice of fuzzy sets and rules plays a key role in determining the desired behavior of a fuzzy system. If we adopt parameterized membership functions, we can try to acquire the optimal fuzzy set shapes from a set of training data. In general, neuro-fuzzy models can be successfully adopted to find the most appropriate rulebase for a given application.

Figure 13. Resulting edge map.

PARAMETERIZED MEMBERSHIP FUNCTIONS

Fuzzy systems are powerful tools for data processing. However, it is not always necessary to express fuzzy reasoning in form of rules. Sometime one (ore more) parameterized fuzzy sets suffice. As an example, let us consider the filtering of Gaussian noise in digital images. It is known that noise having Gaussian-like distribution is very often encountered during image acquisition. Our goal is to reduce the noise without (significantly) blurring the image details. A simple idea is to adopt a *fuzzy weighted mean filter* for this purpose (39, 40). Again, let us suppose we deal with digitized images having L gray levels (typically L = 256). Let $x_{i,j}$ be the pixel luminance at location (i, j) in the noisy image and let $\Delta_{i+m,j+n} = x_{i,j} - x_{i+m,j+n}$ be the luminance difference between this element and the neighboring pixel at location (i+m, j+n). The output $y_{i,j}$ of the fuzzy weighted mean filter is defined by the following relationships:

$$y_{i,j} = \sum_{m=-N}^{N}\sum_{n=-N}^{N} w_{i+m,\ j+n}\, x_{i+m,\ j+n} \tag{55}$$

$$w_{i+m,\ j+n} = \frac{\mu_{SM}(\Delta_{i+m,\ j+n})}{\sum_{m=-N}^{N}\sum_{n=-N}^{N}\mu_{SM}(\Delta_{i+m,\ j+n})} \tag{56}$$

where μ_{SM}(u) is the membership function of fuzzy set *small*. Let us define this set by resorting to a bell-shaped parameterized function:

$$\mu_{SM}(u) = \exp\{-(\frac{u}{c})^2\} \tag{57}$$

A graphical representation of μ_{SM}(u) is depicted in Fig. 14 for three different values of the parameter c ($u \geq 0$). According to (58–59), the algorithm performs a weighted mean of the luminance values in a (2N+1) × (2N+1) window around $x_{i,j}$. The weights are chosen according to a simple fuzzy model: small luminance differences (possibly) denote noise, while large luminance differences denote object contours. Thus, when $\Delta_{i+m,j+n}$ is small, the corresponding $w_{i+m,j+n}$ is large and vice versa. As a result, the processing gradually excludes pixel luminances that are different from $x_{i,j}$ in order to preserve image details. The value of the parameter c mainly depends upon the variance of the Gaussian noise. Typically, this value is chosen so that a suitable per-

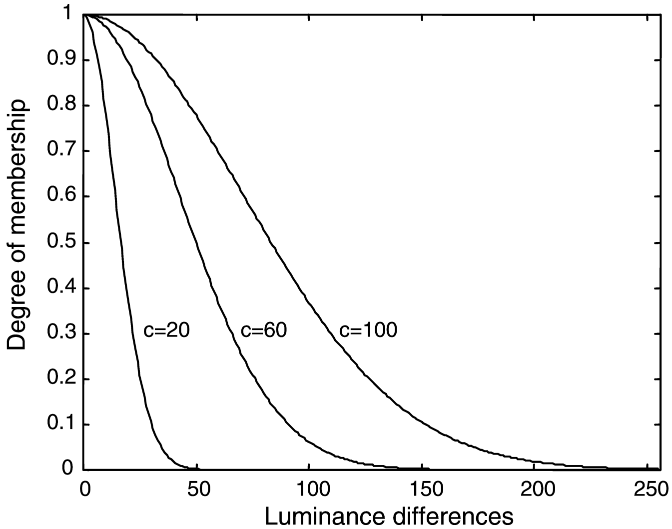


Figure 14. Graphical representation of the membership function $\mu_{SM}(u)$ for three different values of the parameter c.

formance index is maximized, for example, the well-known peak signal-to-noise ratio (PSNR), which is defined as:

$$PSNR = 10\log_{10}\left(\frac{\sum_i \sum_j (L-1)^2}{\sum_i \sum_j (y_{i,j} - s_{i,j})^2}\right) \quad (58)$$

where $s_{i,j}$ and $y_{i,j}$ denote the pixel luminances of the original noise-free image and the filtered image, respectively, at location (i,j). This procedure is briefly depicted in Fig. 15. An example of processed data is also reported in Fig. 16. We generated the picture in Fig. 16a by adding Gaussian noise with variance σ^2=100 to the original noise-free image. The result of the application of the fuzzy filter is reported in Fig. 16b (N=2). Details of the noisy and the processed images are respectively depicted in Fig 16c and 16d for visual inspection. The noise reduction is apparent, especially in the uniform regions of the image. According to our previous observation, we chose the parameter value that gives the maximum PSNR (Fig. 17). Larger values would increase the image blur, smaller values would leave some noise unprocessed.

It is worth pointing out that we can define the same filtering operation by resorting to the concept of fuzzy relation. In an equivalent way, we can formally define the weights of the filter as follows:

$$w_{i+m,\ j+n} = \frac{\mu_{EQ}(x_{i,\ j}, x_{i+m,\ j+n})}{\sum_{m=-N}^{N} \sum_{n=-N}^{N} \mu_{EQ}(x_{i,\ j}, x_{i+m,\ j+n})} \quad (59)$$

where $m_{EQ}(u,v)$ is the parameterized membership function that describes the fuzzy relation "u is equal to v":

$$\mu_{EQ}(u, v) = \exp\{-(\frac{u-v}{c})^2\} \quad (60)$$

A graphical representation of $m_{EQ}(u,v)$ is shown in Fig. 18 (c=40).

The fuzzy weighted mean filter is not the only available scheme for reducing Gaussian noise. Other approaches are possible (40). For example, we can adopt fuzzy models to estimate the noise amplitude $g_{i,j}$ and then subtract it from

Figure 16. (a) Image corrupted by Gaussian noise, (b) filtered image, (c) detail of the noisy image, (d) detail of the filtered image.

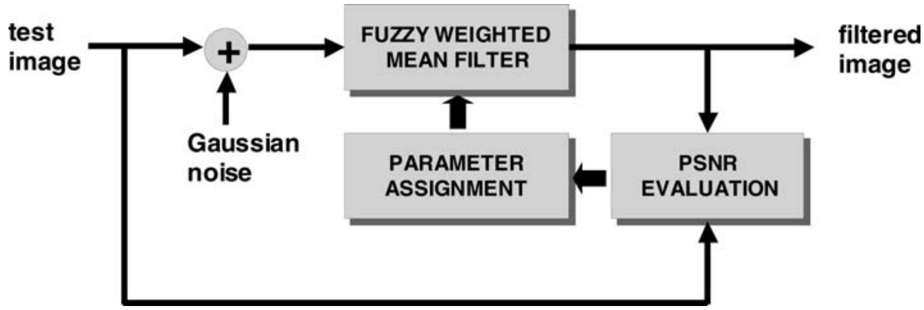


Figure 15. Block diagram of the procedure for parameter tuning.

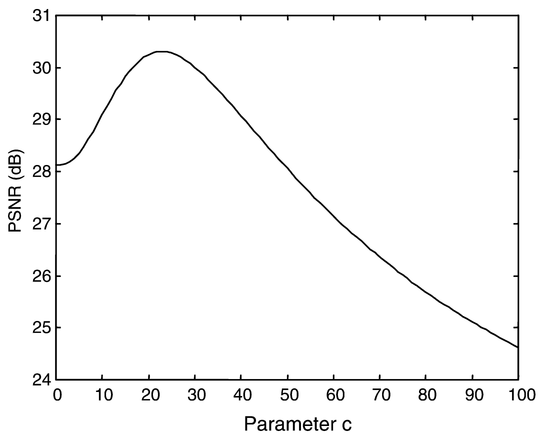


Figure 17. Filtering performance.

the pixel luminance $x_{i,j}$, as follows:

$$g_{i,\,j} = \frac{1}{8}\sum_{m=-1}^{1}\sum_{n=-1}^{1}(x_{i,\,j} - x_{i+m,\,j+n})\,\mu_{SI}(x_{i,\,j},\ x_{i+m,\,j+n}) \quad (61)$$

$$y_{i,\,j} = x_{i,\,j} - g_{i,\,j} \quad (62)$$

where μ_{SI}(u,v) is the parameterized membership function of fuzzy relation "u is similar to v". A possible definition is given by the following relationship (Fig. 19):

$$\mu_{SI}(u,\ v) = \begin{cases} 1 & |u-v|<c \\ \dfrac{5c - |u-v|}{4|u-v|} & c \le |u-v|<5c \\ 0 & |u-v| \ge 5c \end{cases} \quad (63)$$

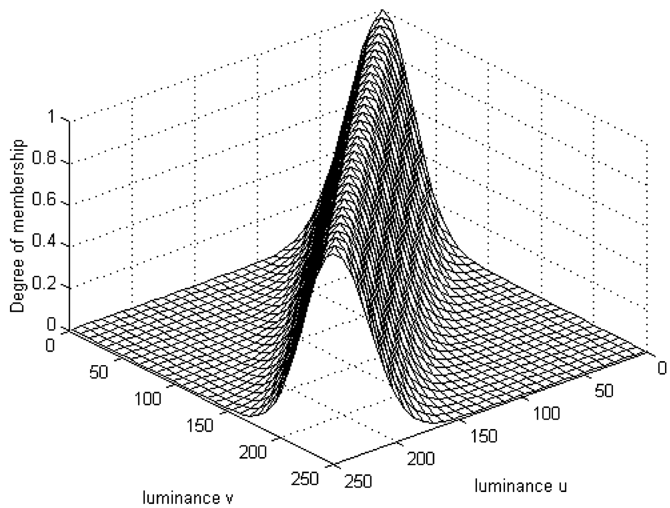
Often, more parameters can increase the effectiveness of the fuzzy processing. For example, let us define the fuzzy relation "u is different from v" by adopting the following two-parameters function:

$$\mu_{DI}(u,v) = \begin{cases} 0 & |u-v|<b \\ 1 - \exp\{-(\dfrac{|u-v|-b}{c})^2\} & |u-v| \ge b \end{cases} \quad (64)$$

This relation is graphically depicted in Fig. 20 (b=30, c=60). Again, let us focus on the problem of edge detection. We can exploit relation (67) to design a simple operator whose behavior can easily be controlled by two parameters b and c. For example (41):

$$y_{i,\,j} = (L-1)\,MAX\{\mu_{DI}(x_{i,\,j}, x_{i,\,j-1}),\ \mu_{DI}(x_{i,\,j}, x_{i-1,\,j})\} \quad (65)$$

The operation is very simple. In the presence of an object border, at least one of the following inequalities occurs: $x_{i,j} > x_{i,j-1}, x_{i,j} > x_{i-1,j}, x_{i,j} < x_{i,j-1}, x_{i,j} < x_{i-1,j}$. As a result, $\mu_{DI}(x_{i,j}, x_{i,j-1}) \approx 1$ and/or $\mu_{DI}(x_{i,j}, x_{i-1,j}) \approx 1$, and the output of the edge detector is $y_{i,j} \approx L-1$. Conversely, in the presence of an uniform region, we have $x_{i,j} \approx x_{i,j-1}$ and $x_{i,j} \approx x_{i-1,j}$. Thus the output becomes $y_{i,j}=0$, as it should be. The parameters b and c control the actual behavior of the edge detector. Large values of these parameters can be chosen to decrease its sensitivity to fine details and to noise.



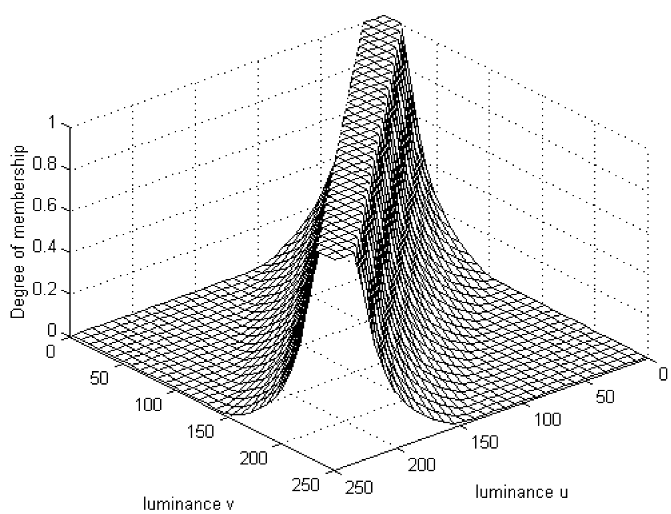


Figure 19. Graphical representation of the membership function $\mu_{SI}(u,v)$ describing the fuzzy relation "u is similar to v".

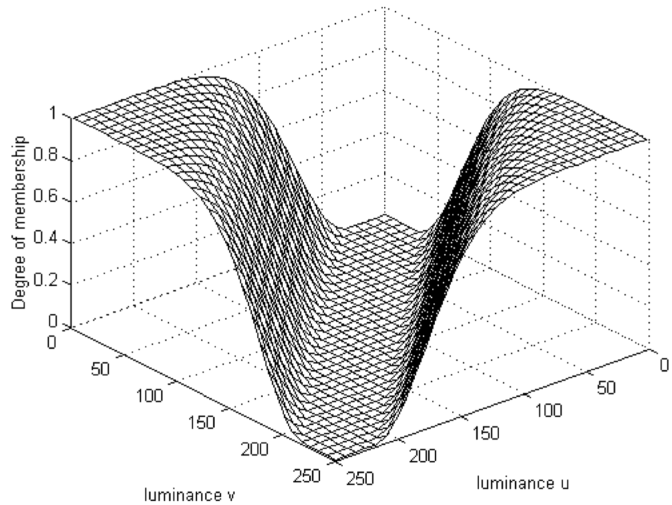


Figure 20. Graphical representation of the membership function $\mu_{DI}(u,v)$ describing the fuzzy relation "u is different from v".

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