J. Webster (ed.), *Wiley Encyclopedia of Electrical and Electronics Engineering*

STABILITY OF NONLINEAR SYSTEMS

Stability of Nonlinear Systems

A nonlinear system refers to a set of nolinear equations (algebraic, difference, differential, integral, functional, or abstract operator equations or a combination of some of these) used to describe a physical device or process that otherwise cannot be well defined by a set of linear equations of any kind. Dynamical system is used as a synonym for a mathematical or physical system when the describing equations represent evolution of a solution with time and, sometimes, with control inputs and/or other varying parameters.

The theory of nonlinear dynamical systems, or nonlinear control systems if control inputs are involved, has been greatly aanced since the nineteenth century. Today, nonlinear control systems are used to describe a great variety of scientific and engineering phenomena ranging from the social, life, and physical sciences to engineering and technology. This theory has been applied to a broad spectrum of problems in physics, chemistry, mathematics, biology, medicine, economics, and various engineering disciplines.

Stability theory plays a central role in system engineering, especially in the field of control systems and automation, with regard to both dynamics and control.

Stability of a dynamical system, with or without control and disturbance inputs, is a fundamental requirement for its practical value, particularly in real-world applications. Roughly speaking, stability means that the system outputs and its internal signals are bounded within admissible limits (the so-called bounded-input bounded-output stability), or, sometimes more strictly, the system outputs tend to an equilibrium state of interest (the so-called asymptotic stability). Conceptually, there are different kinds of stabilities, among which three basic notions are the main concerns in nonlinear dynamics and control systems: the stability of a system with respect to its equilibria, the orbital stability of a system output trajectory, and the structural stability of a system itself.

The basic concept of stability emerged from the study of an equilibrium state of a mechanical system, dating back to as early as 1644 when E. Torricelli studied the equilibrium of a rigid body under the natural force of gravity. The classical stability theorem of G. Lagrange, formulated in 1788, is perhaps the best known result about stability of conservative mechanical systems. It states that if the potential energy of a conservative system, currently at the position of an isolated equilibrium and perhaps subject to some simple constraints, has a minimum, then this equilibrium position of the system is stable (1). The evolution of the fundamental concepts of system and trajectory stabilities then went through a long history, with many fruitful advances and developments, until the celebrated Ph.D. Thesis of A. M. Lyapunov, "The General Problem of Motion Stability," summarized it all in 1892 (2). This monograph is so fundamental that its ideas and techniques are virtually leading all basic research and applications regarding stabilities of dynamical systems today. In fact, not only dynamical behavior analysis in modern physics but also controller design in engineering systems depend upon the principles of Lyapunov's stability theory. This article is devoted to a brief description of the basic stability theory, criteria, and methodologies of Lyapunov, as well as a few related important stability concepts, for nonlinear dynamical systems.

Nonlinear System Preliminaries

Nonlinear Control Systems. A continuous-time nonlinear control system is generally described by a differential equation of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t; \boldsymbol{u}), \qquad t \in [t_0, \infty) \tag{1}$$

where $\boldsymbol{x} = \boldsymbol{x}(t)$ is the state of the system belonging to a (usually bounded) region $\Omega_x \subset \boldsymbol{R}^n$, $\boldsymbol{u}$ is the control input vector belonging to another (usually bounded) region $\Omega_u \subset \boldsymbol{R}^m$ (usually, $m \le n$), and $\boldsymbol{f}$ is a Lipschitz or continuously differentiable nonlinear function, so that the system has a unique solution for each admissible control input and suitable initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \Omega_x$. To indicate the time evolution and the dependence on the initial state $\boldsymbol{x}_0$, the trajectory (or orbit) of a system state, $\boldsymbol{x}(t)$, is sometimes denoted as $\varphi_t(\boldsymbol{x}_0)$.

In the control system (1), the initial time used is $t_0 \ge 0$, unless otherwise indicated. The entire space $\boldsymbol{R}^n$, to which the system states belong, is called the state space. Associated with the control system (1), there usually is an observation or measurement equation

$$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}, t; \boldsymbol{u}), \tag{2}$$

where $\mathbf{y} = \mathbf{y}(t) \in \mathbf{R}^l$ is the output of the system, $1 \le l \le n$, and $\boldsymbol{g}$ is a continuous or smooth nonlinear function. When both $n, l > 1$, the system is called a multi-input multi-output (*MIMO*) system, whereas if $n = l = 1$, it is called a single-input single-output (*SISO*) system. The MISO and SIMO systems are similarly defined.

In a discrete-time setting, a nonlinear control system is described by a difference equation of the form

$$\begin{cases} \boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k, k; \boldsymbol{u}_k) \\ \boldsymbol{y}_k = \boldsymbol{g}(\boldsymbol{x}_k, k; \boldsymbol{u}_k), \end{cases} \qquad k = 0, 1, \ldots \tag{3}$$

where all notation is similarly defined.

Most of the time in this article, only the control system in Eq. (1), or the first equation of Eq. (3), is discussed. In this case, the system state $\boldsymbol{x}$ is also considered the system output for simplicity.

A special case of the system in Eq. (1), with or without control, is said to be autonomous if the time variable t does not appear separately (independently) from the state vector in the system function $\boldsymbol{f}$. For example, with a state-feedback control $\boldsymbol{u}(t) = \boldsymbol{h}(\boldsymbol{x}(t))$, this is often the situation. In this case, the system is usually written as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \boldsymbol{R}^n \tag{4}$$

Otherwise, as Eq. (1) stands, the system is said to be nonautonomous. The same terminology may be applied in the same way to discrete-time systems, although they may have different characteristics.

An equilibrium, or fixed point, of system (4), if it exists, is a solution $\boldsymbol{x}*$ of Eq. (4) that satisfies the algebraic equation

$$\boldsymbol{f}(\boldsymbol{x}^*) = 0 \tag{5}$$

It then follows from Eqs. (4) and (5) that $\dot{\boldsymbol{x}}^* = 0$, which means that an equilibrium of a system must be a constant state. For the discrete-time case, an equilibrium of system

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k), \qquad k = 0, 1, \ldots \tag{6}$$

is a solution, if it exists, of equation

$$x^* = f(x^*) \tag{7}$$

An equilibrium is stable, if all the nearby trajectories of the system states, starting from various initial states, approach it; it is unstable, if nearby trajectories move away from it. The concept of system stability with respect to an equilibrium will be precisely introduced in section 3, entitled "Lyapunov, Orbital, and Structural Stabilities."

A control system is deterministic, if there is a unique consequence to every change of the system parameters or initial states. It is random or stochastic, if there is more than one possible consequence for a change in its parameters or initial states according to some probability distribution (3). This article only deals with deterministic systems.

Hyperbolic Equilibria and Their Manifolds. Consider the autonomous system in Eq. (4). The Jacobian of this system is defined by

$$J(x) = \frac{\partial f}{\partial x} \tag{8}$$

Clearly, this is a matrix-valued function of time. If the Jacobian is evaluated at a constant state, say x^* or x_0, then it becomes a constant matrix determined by f and x^* or x_0 .

An equilibrium, x^*, of system in Eq. (4) is said to be hyperbolic, if all eigenvalues of the system Jacobian, evaluated at this equilibrium, have nonzero real parts.

For a p-periodic solution of the system in Eq. (4), $\tilde{x}(t)$, with a fundamental period $p > 0$, let $J(\tilde{x}(t))$ be its Jacobian evaluated at $\tilde{x}(t)$. Then this Jacobian is also p-periodic:

$$J(\tilde{x}(t+p)) = J(\tilde{x}(t)) \qquad \text{for all} \quad t \in [t_0, \infty)$$

In this case, there always exist a p-periodic nonsingular matrix $M(t)$ and a constant matrix Q such that the fundamental solution matrix associated with the Jacobian $J(\tilde{x}(t))$ is given by (4)

$$\Phi(t) = M(t)e^{tQ}$$

Here, the fundamental matrix $\Phi(t)$ consists of, as its columns, n linearly independent solution vectors of the linear equation $\dot{x} = J(\tilde{x}(t))\, x$, with $x(t_0) = x_0$.

In the preceding discussion, the eigenvalues of the constant matrix e^{pQ} are called the Floquet multipliers of the Jacobian. The p-periodic solution $\tilde{x}(t)$ is called a hyperbolic periodic orbit of the system if all its corresponding Floquet multipliers have nonzero real parts.

Next, let D be a neighborhood of an equilibrium, x^*, of the autonomous system in Eq. (4). A local stable and local unstable manifold of x^* is defined by

$$W^{s}_{\rm loc}(x^*) = \{x \in \mathcal{D} | \varphi_t(x) \in \mathcal{D} \forall t \geq t_0 \text{ and } \varphi_t(x) \to x^* \text{ as } t \to \infty\} \tag{9}$$

and

$$W^{u}_{\rm loc}(x^*) = \{x \in \mathcal{D} | \varphi_t(x) \in \mathcal{D} \forall t \leq t_0 \text{ and } \varphi_t(x) \to x^* \text{ as } t \to -\infty\} \tag{10}$$

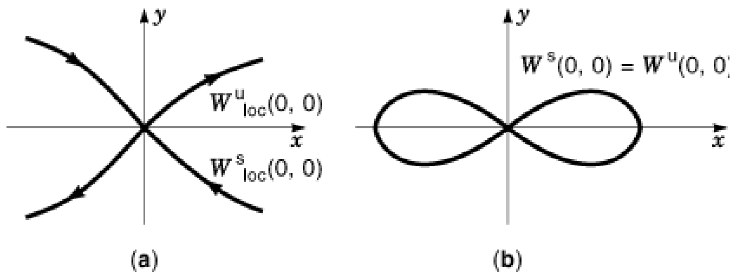

Fig. 1. Stable and unstable manifolds.

respectively. Furthermore, a stable and unstable manifold of $\boldsymbol{x}^*$ is defined by

$$W^{\mathrm{s}}(\boldsymbol{x}^*) = \{\boldsymbol{x} \in \mathcal{D} | \varphi_t(\boldsymbol{x}) \cap W^{\mathrm{s}}_{\mathrm{loc}}(\boldsymbol{x}^*) \neq \phi\} \qquad (11)$$

and

$$W^{\mathrm{u}}(\boldsymbol{x}^*) = \{\boldsymbol{x} \in \mathcal{D} | \varphi_t(\boldsymbol{x}) \cap W^{\mathrm{u}}_{\mathrm{loc}}(\boldsymbol{x}^*) \neq \phi\} \qquad (12)$$

respectively, where φ denotes the empty set. For example, the autonomous system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x(1 - x^2) \end{cases}$$

has a hyperbolic equilibrium $(x^*, y^*) = (0, 0)$. The local stable and unstable manifolds of this equilibrium are illustrated by Fig. 1(a), which is enlarged, and the corresponding stable and unstable manifolds are visualized by Fig. 1(b).

A hyperbolic equilibrium only has stable and/or unstable manifolds because its associated Jacobian only has stable and/or unstable eigenvalues. The dynamics of an autonomous system in a neighborhood of a hyperbolic equilibrium is quite simple: it has either stable (convergent) or unstable (divergent) properties. Therefore, complex dynamical behaviors such as chaos are usually not associated with isolated hyperbolic equilibria or isolated hyperbolic periodic orbits (5 6,7) (also see Theorem 16); they generally are confined within the so-called center manifold $W^{\mathrm{c}}(\boldsymbol{x}^*)$, where $\dim(W^{\mathrm{s}}) + \dim(W^{\mathrm{c}}) + \dim(W^{\mathrm{u}}) = n$.

Open-Loop and Closed-Loop Systems. Let S be an *MIMO* system, which can be linear or nonlinear, continuous-time or discrete-time, deterministic or stochastic, or any well-defined input-output map. Let U and Y be the sets (sometimes, spaces) of the admissible input and corresponding output signals, respectively, both defined on the time domain $D = [a, b]$, $-\infty \leq a < b \leq \infty$ (for control systems, usually $a = t_0 = 0$ and $b = \infty$). This simple relation is described by an open-loop map

$$S : \boldsymbol{u} \rightarrow \boldsymbol{y} \qquad \text{or} \qquad \boldsymbol{y}(t) = S(\boldsymbol{u}(t)) \qquad (13)$$

and its block diagram is shown in Fig. 2. Actually, every control system described by a differential or difference equation can be viewed as a map in this form. But, in such a situation, the map S can only be defined implicitly via the equation and initial conditions.

In the control system in Eq. (1), or (3), if the control inputs are functions of the state vectors, $\boldsymbol{u} = \boldsymbol{h}(\boldsymbol{x}; t)$, then the control system can be implemented via a closed-loop configuration. A typical closed-loop system is

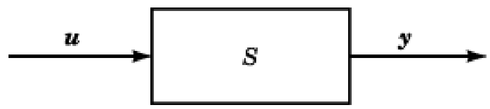

Fig. 2. The block diagram of an open-loop system.

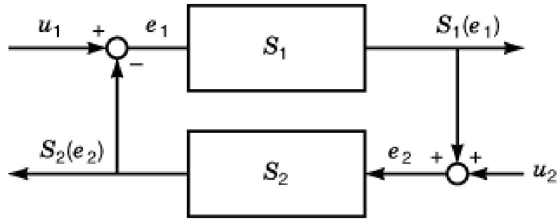

Fig. 3. A typical closed-loop control system.

shown in Fig. 3, where usually S_1 is the plant (described by $\boldsymbol{f}$) and S_2 is the controller (described by $\boldsymbol{h}$); yet they can be reversed.

Norms of Functions and Operators. This article only deals with finite-dimensional systems. For an n-dimensional vector-valued function of the form $\boldsymbol{x}(t) = [x_1(t) \cdots x_n(t)]^{\mathrm{T}}$, let $\|\cdot\|$ and $\|\cdot\|_p$ denote its Euclidean norm and L_p-norm, defined respectively by the "length"

$$\|\boldsymbol{x}(t)\| = \sqrt{x_1^2(t) + \cdots + x_n^2(t)}$$

and by

$$\|\boldsymbol{x}\|_p = \left(\int_a^b \|\boldsymbol{x}(t)\|^p dt\right)^{1/p}, \qquad 1 \le p < \infty$$
$$\|\boldsymbol{x}\|_\infty = \operatorname{ess} \sup_{\substack{a \le t \le b \\ 1 \le i \le n}} |x_i(t)|$$

Here, a few remarks are in order:

(1) "ess sup" means essential supremum (i.e., the supremum except perhaps over a set of measure zero). For a piecewise continuous function $f(t)$, they are actually the same:

$$\operatorname{ess} \sup_{a \le t \le b} |f(t)| = \sup_{a \le t \le b} |f(t)|$$

(2) The main difference between the "sup" and the "max" is that max $|f(t)|$ is attainable but sup $|f(t)|$ may not be. For example, $\max_{0 \le t < \infty} |\sin(t)| = 1$ but $\sup_{0 \le t < \infty} |1 - e^{-t}| = 1$.

(3) A difference between the Euclidean norm and the L_p-norms is that the former is a function of time but the latter are all constants.

(4) For a finite-dimensional vector $\boldsymbol{x}(t)$, with $n < \infty$, all the L_p-norms are equivalent in the sense that for any $p, q \in [1, \infty]$, there exist two positive constants α and β such that

$$\alpha\|\boldsymbol{x}\|_p \leq \|\boldsymbol{x}\|_q \leq \beta\|\boldsymbol{x}\|_p$$

For the input-output map in Eq. (13), the so-called operator norm of the map S is defined to be the maximum gain from all possible inputs over the domain of the map to their corresponding outputs. More precisely, the *operator norm* of the map S in Eq. (13) is defined by

$$|||S||| = \sup_{\substack{\boldsymbol{u}_1, \boldsymbol{u}_2 \in U \\ \boldsymbol{u}_1 \neq \boldsymbol{u}_2}} \frac{\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_Y}{\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_U} \tag{14}$$

where $\boldsymbol{y}_i = S(\boldsymbol{u}_i) \in Y$, $i = 1, 2$, and $\|\cdot\|_U$ and $\|\cdot\|_Y$ are the norms of the functions defined on the input-output sets (or spaces) U and Y, respectively.

Lyapunov, Orbital, and Structural Stabilities

Three different types of stabilities, namely, the Lyapunov stability of a system with respect to its equilibria, the orbital stability of a system output trajectory, and the structural stability of a system itself, are of fundamental importance in the studies of nonlinear dynamics and control systems.

Roughly speaking, the Lyapunov stability of a system with respect to its equilibrium of interest is about the behavior of the system outputs toward the equilibrium state—wandering nearby and around the equilibrium (stability in the sense of Lyapunov) or gradually approaching it (asymptotic stability); the orbital stability of a system output is the resistance of the trajectory to small perturbations; the structural stability of a system is the resistance of the system structure against small perturbations (1,8 9 10 11 12 13 14 15 16,17). These three basic types of stabilities are introduced in this section for dynamical systems without explicitly involving control inputs.

Consider the general nonautonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \boldsymbol{R}^n \tag{15}$$

where the control input $\boldsymbol{u}(t) = \boldsymbol{h}(\boldsymbol{x}(t),t)$, if it exists [see the system in Eq. (1)], has been combined into the system function $\boldsymbol{f}$ for simplicity of discussion. Without loss of generality, assume that the origin $\boldsymbol{x} = 0$ is the system equilibrium of interest. Lyapunov stability theory concerns various stabilities of the system orbits with respect to this equilibrium. When another equilibrium is discussed, the new equilibrium is first shifted to zero by a change of variables, and then the transformed system is studied in the same way.

Stability in the Sense of Lyapunov. System in Eq. (15) is said to be stable in the sense of Lyapunov with respect to the equilibrium $\boldsymbol{x}^* = 0$, if for any $\epsilon > 0$ and any initial time $t_0 \geq 0$, there is a constant, $\delta = \delta(\epsilon, t_0) > 0$, such that

$$\|\boldsymbol{x}(t_0)\| < \delta \Rightarrow \|\boldsymbol{x}(t)\| < \epsilon \qquad \text{for all } t \geq t_0 \tag{16}$$

This stability is illustrated by Fig. 4.

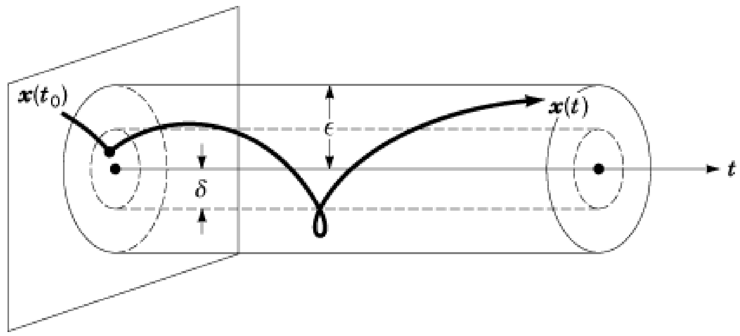

Fig. 4. Geometric meaning of stability in the sense of Lyapunov.

It should be emphasized that the constant δ generally depends on both ϵ and t_0 . It is particularly important to point out that, unlike autonomous systems, one cannot simply fix the initial time $t_0 = 0$ for a nonautonomous system in a general discussion of its stability. For example, consider the following linear time-varying system with a discontinuous coefficient:

$$\dot{x}(t) = \frac{1}{1-t}x(t), \qquad x(t_0) = x_0$$

It has an explicit solution

$$x(t) = x_0 \frac{1-t_0}{1-t}, \qquad 0 \leq t_0 \leq t < \infty$$

which is stable in the sense of Lyapunov about the equilibrium $x^* = 0$ over the entire time domain $[\,0, \infty\,)$ if and only if $t_0 \geq 1$. This shows that the initial time t_0 does play an important role in the stability of a nonautonomous system.

The previously defined stability, in the sense of Lyapunov, is said to be uniform with respect to the initial time, if the existing constant $\delta = \delta(\epsilon)$ is indeed independent of t_0 over the entire time interval $[\,0, \infty\,)$. According to this discussion, uniform stability is defined only for nonautonomous systems since it is not needed for autonomous systems (for which it is always uniform with respect to the initial time).

Asymptotic and Exponential Stabilities. System in Eq. (15) is said to be asymptotically stable about its equilibrium $\boldsymbol{x}^* = 0$, if it is stable in the sense of Lyapunov and, furthermore, there exists a constant, $\delta = \delta(t_0) > 0$, such that

$$\|\boldsymbol{x}(t_0)\| < \delta \Rightarrow \|\boldsymbol{x}(t)\| \to 0 \qquad \text{as } t \to \infty \qquad (17)$$

This stability is visualized by Fig. 5.

The asymptotic stability is said to be uniform, if the existing constant δ is independent of t_0 over $[\,0, \infty\,)$, and is said to be global, if the convergence, $\|\boldsymbol{x}\| \to 0$, is independent of the initial state $\boldsymbol{x}(t_0)$ over the entire spatial domain on which the system is defined (e.g., when $\delta = \infty$). If, furthermore,

$$\|\boldsymbol{x}(t_0)\| < \delta \Rightarrow \|\boldsymbol{x}(t)\| \leq ce^{-\sigma t} \qquad (18)$$

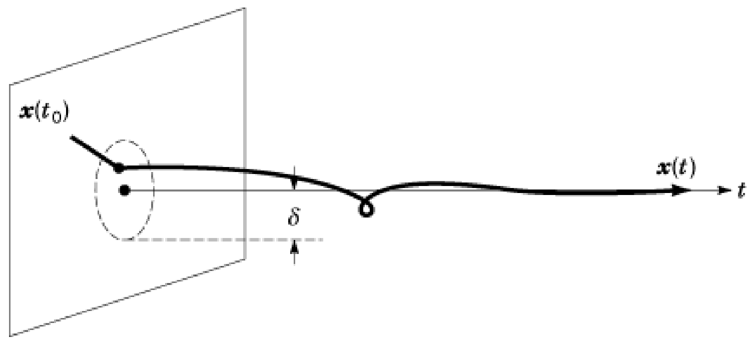

Fig. 5. Geometric meaning of the asymptotic stability.

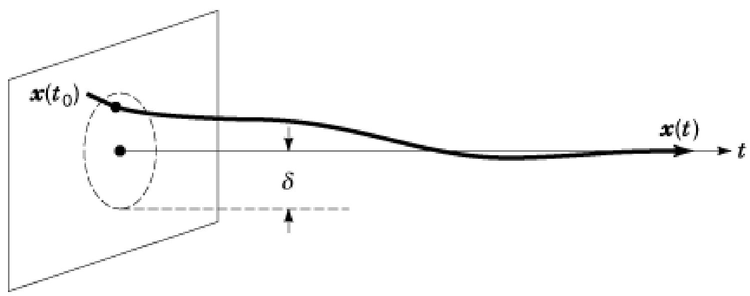

Fig. 6. Geometric meaning of the exponential stability.

for two positive constants c and σ, then the equilibrium is said to be exponentially stable. The exponential stability is visualized by Fig. 6.

Clearly, exponential stability implies asymptotic stability, and asymptotic stability implies the stability in the sense of Lyapunov, but the reverse need not be true. For illustration, if a system has output trajectory $x_1(t) = x_0 \sin(t)$, then it is stable in the sense of Lyapunov about 0, but it is not asymptotically stable. A system with output trajectory $x_2(t) = x_0(1 + t - t_0)^{-1}$ is asymptotically stable (so it is also stable in the sense of Lyapunov), but it is not exponentially stable about 0. A system with output $x_3(t) = x_0 e^{-t}$ is exponentially stable (hence, it is both asymptotically stable and stable in the sense of Lyapunov).

Orbital Stability. The orbital stability differs from the Lyapunov stabilities in that it is concerned with the stability of a system output (or state) trajectory under small external perturbations.

Let $\varphi_t(\boldsymbol{x}_0)$ be a p-periodic solution, $p > 0$, of the autonomous system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \boldsymbol{R}^n \tag{19}$$

and let Γ represent the closed orbit of $\varphi_t(\boldsymbol{x}_0)$ in the state space, namely,

$$\Gamma = \{\boldsymbol{y} | \boldsymbol{y} = \varphi_t(\boldsymbol{x}_0), 0 \le t < p\}$$

If, for any $\epsilon > 0$, there exists a constant $\delta = \delta(\epsilon) > 0$ such that for any $\boldsymbol{x}_0$ satisfying

$$d(\boldsymbol{x}_0, \Gamma) := \inf_{\boldsymbol{y} \in \Gamma} \|\boldsymbol{x}_0 - \boldsymbol{y}\| < \delta$$

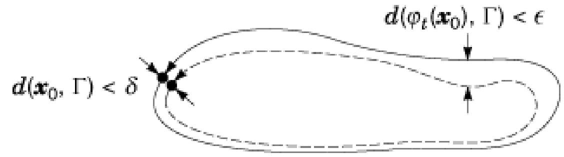

Fig. 7. Geometric meaning of the orbital stability.

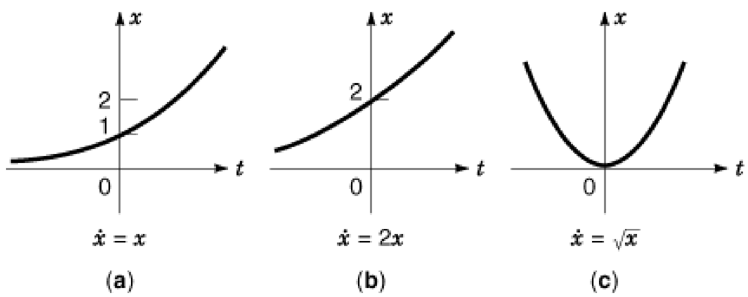

Fig. 8. Trajectories of three systems for comparison.

the solution of the system, $\varphi_t(\boldsymbol{x}_0)$, satisfies

$$d(\varphi_t(\boldsymbol{x}_0), \Gamma) < \epsilon, \qquad \text{for all } t \geq t_0$$

then this p-periodic solution trajectory, $\varphi_t(\boldsymbol{x}_0)$, is said to be orbitally stable.

Orbital stability is visualized by Fig. 7. For a simple example, a stable periodic solution, particularly a stable equilibrium of a system is orbitally stable. This is because all nearby trajectories approach it, and, as such, it becomes a nearby orbit after a small perturbation and so will move back to its original position (or stay nearby). On the contrary, unstable and semistable (saddle-type of) periodic orbits are orbitally unstable. Orbital stability for nonperiodic solutions may also be defined.

Structural Stability. Two systems are said to be topologically orbitally equivalent, if there exists a homeomorphism (i.e., a continuous map whose inverse exists and is also continuous) that transforms the family of trajectories of the first system to that of the second while preserving their motion directions. Roughly, this means that the geometrical pictures of the orbit families of the two systems are similar (no one has extra knots, sharp corners, bifurcating branches, etc.). For instance, systems $\dot{x} = x$ and $\dot{x} = 2x$ are topologically orbitally equivalent, but they are not so between $\dot{x} = x$ and $\dot{x} = \sqrt{x}$. These three system trajectories are shown in Fig. 8.

Return to the autonomous system in Eq. (19). If the dynamics of the system in the state space changes radically, for example by the appearance of a new equilibrium or a new periodic orbit, due to small external perturbations, then the system is considered to be structurally unstable.

To be more precise, consider the following set of functions:

$$\mathcal{S} = \left\{ \boldsymbol{g}(\boldsymbol{x}) \middle| \|\boldsymbol{g}(\boldsymbol{x})\| < \infty, \left\| \frac{\partial \boldsymbol{g}(x)}{\partial \boldsymbol{x}} \right\| < \infty \quad \text{for all } \boldsymbol{x} \in \boldsymbol{R}^n \right\}$$

If, for any $\boldsymbol{g} \in S$, there exists an $\epsilon > 0$ such that the orbits of the two systems

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(x) \qquad \text{and} \qquad \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \epsilon \boldsymbol{g}(\boldsymbol{x})$$

are topologically orbitally equivalent, then the autonomous system in Eq. (19), namely, the first (unperturbed) system above, is said to be structurally stable.

For example, $\dot{x} = x$ is structurally stable, but $\dot{x} = x^2$ is not, in a neighborhood of the origin. This is because, when the second system is slightly perturbed, to become say $\dot{x} = x^2 + \epsilon$, where $\epsilon > 0$, then the resulting system has two equilibria, $x^*{}_1 = \sqrt{\varepsilon}$ and $x^*{}_2 = -\sqrt{\varepsilon}$, which has more numbers of equilibria than the original system that possesses only one, $x^* = 0$.

Various Stability Theorems

Consider the general nonautonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \boldsymbol{R}^n \tag{20}$$

where $\boldsymbol{f}$: $D \times [0, \infty) \to \boldsymbol{R}^n$ is continuously differentiable in a neighborhood of the origin, $D \subseteq \boldsymbol{R}^n$, with a given initial state $\boldsymbol{x}_0 \in D$. Again, without loss of generality, assume that $\boldsymbol{x}* = 0$ is a system equilibrium of interest.

Lyapunov Stability Theorems. First, for the autonomous system in Eq. (19), an important special case of Eq. (20), with a continuously differentiable $\boldsymbol{f}$: $D \to \boldsymbol{R}^{\mathrm{n}}$, the following criterion of stability, called the first (or indirect) method of Lyapunov, is very convenient to use.

Theorem 1 (First Method of Lyapunov, for Continuous-Time Autonomous Systems). In system Eq. (19), let $J = [\delta f / \delta x]_{x = x^* = 0}$ be the system Jacobian evaluated at the zero equilibrium. If all the eigenvalues of J have negative real parts, then the system is asymptotically stable about $\boldsymbol{x}^* = 0$.

First, note that this and the following Lyapunov theorems also apply to linear systems because linear systems are merely a special case of nonlinear systems. When $\boldsymbol{f}(\boldsymbol{x}) = A\,\boldsymbol{x}$, the linear time-invariant system $\boldsymbol{x} = A\,\boldsymbol{x}$ has the only equilibrium $\boldsymbol{x}^* = 0$. If A has all eigenvalues with negative real parts, Theorem 1 implies that the system is asymptotically stable about its equilibrium since the system Jacobian is simply $J = A$. This is consistent with the familiar linear stability results.

Note also that the region of asymptotic stability given by Theorem 1 is local, which can be quite large for some nonlinear systems but may be very small for some others. However, there is no general criterion for determining the boundaries of such local stability regions when this and the following Lyapunov methods are applied.

Moreover, it is important to note that this theorem cannot be applied to a general nonautonomous system, since for general nonautonomous systems this theorem is neither necessary nor sufficient (18). A simple counterexample is the following linear time-varying system (11,19):

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -1 + 1.5\cos^2(t) & 1 - 1.5\sin(t)\cos(t) \\ -1 - 1.5\sin(t)\cos(t) & -1 + 1.5\sin^2(t) \end{bmatrix} \boldsymbol{x}(t)$$

This system has eigenvalues $\lambda_{1,2} = -0.25 \pm j\,0.25\sqrt{7}$, both having negative real parts and being independent of the time variable t . If Theorem 1 is used to judge this system, the conclusion would be that the system is asymptotically stable about its equilibrium 0. However, the solution of this system is

$$\boldsymbol{x}(t) = \begin{bmatrix} e^{0.5t}\cos(t) & e^{-t}\sin(t) \\ -e^{0.5t}\sin(t) & e^{-t}\cos(t) \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}$$

which is unstable, for any initial conditions with a bounded and nonzero value of x_1 (t_0), no matter how small this initial value is. This example shows that by using the Lyapunov first method alone to determine the stability of a general time-varying system, the conclusion can be wrong.

This type of counterexamples can be easily found (13). On the one hand, this demonstrates the necessity of other general criteria for asymptotic stability of nonautonomous systems. On the other hand, however, a word of caution is that these types of counterexamples do not completely rule out the possibility of applying the first method of Lyapunov to some special nonautonomous systems in case studies. The reason is that there is no theorem saying that "the Lyapunov first method cannot be applied to all nonautonomous systems." Due to the complexity of nonlinear dynamical systems, they often have to be studied class by class, or even case by case. It has been widely experienced that the first method of Lyapunov does work for some, perhaps not too many, specific nonautonomous systems in case studies (e.g., in the study of some chaotic systems (5); see also Theorem 18). The point is that one has to be very careful when this method is applied to a particular nonautonomous system; the stability conclusion must be verified by some other means at the same time.

Here, it is emphasized that a rigorous approach for asymptotic stability analysis of general nonautonomous systems is provided by the second method of Lyapunov, for which the following set of class-κ functions are useful:

$$\mathcal{K} = \{g(t)\colon\ g(t_0) = 0,\ g(t) > 0 \quad \text{if} \quad t > t_0, \quad \text{and} \quad g(t) \\ \text{is continuous and nondecreasing on } [t_0, \infty)\}$$

Theorem 2 (Second Method of Lyapunov, for Continuous-Time Nonautonomous Systems). The system (20) is globally (over the entire domain D), uniformly (with respect to the initial time over the entire time interval $[t_0, \infty)$), and asymptotically stable about its zero equilibrium, if there exist a scalar-valued function, $V(\boldsymbol{x}, t)$, defined on $D \times [t_0, \infty]$, and three functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot) \in \kappa$, such that

(1) $V(0, t) = 0$ for all $t \geq t_0$;
(2) $V(\boldsymbol{x}, t) > 0$ for all $\boldsymbol{x} \neq 0$ in D and all $t \geq t_0$;
(3) $\alpha(\|\boldsymbol{x}\|) \leq V(\boldsymbol{x}, t) \leq \beta(\|\boldsymbol{x}\|)$ for all $t \geq t_0$;
(4) $\dot{V}(\boldsymbol{x}, t) \leq -\gamma(\|\boldsymbol{x}\|) < 0$ for all $t \geq t_0$.

In Theorem 2, the function V is called a Lyapunov function. The method of constructing a Lyapunov function for stability determination is called the second (or direct) method of Lyapunov.

The geometric meaning of a Lyapunov function used for determining the system stability about the zero equilibrium may be illustrated by Fig. 9. In this figure, assuming that a Lyapunov function V $(\boldsymbol{x})$ has been found, which has a bowl-shape as shown based on conditions i and ii. Then, condition iv is

$$\dot{V}(\boldsymbol{x}) = \left[\frac{\partial V}{\partial \boldsymbol{x}}\right] \dot{\boldsymbol{x}} < 0 \qquad (21)$$

where $[\partial v/\partial \boldsymbol{x}]$ is the gradient of V along the trajectory $\boldsymbol{x}$. It is known, from calculus, that if the inner product of this gradient and the tangent vector $\dot{x}$ is constantly negative, as guaranteed by the condition in Eq. (21), then the angle between these two vectors is larger than 90°, so that the surface of $V(\boldsymbol{x})$ is monotonically decreasing

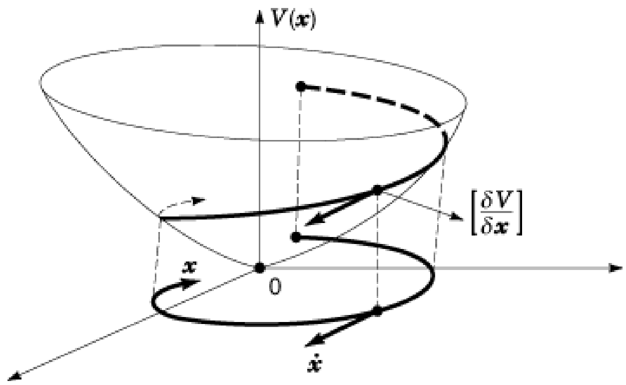

Fig. 9. Geometric meaning of the Lyapunov function.

to zero (this is visualized in Fig. 9). Consequently, the system trajectory $\boldsymbol{x}$, the projection on the domain as shown in Fig. 9, converges to zero as time evolves.

As an example, consider the following nonautonomous system:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}, t)$$

where A is a stable constant matrix and $\boldsymbol{g}$ is a nonlinear function satisfying $\boldsymbol{g}(0, t) = 0$ and $\|\boldsymbol{g}(\boldsymbol{x}, t)\| \leq c\, \|\boldsymbol{x}\|$ for a constant $c > 0$ for all $t \in [t_0, \infty)$. Since A is stable, the following Lyapunov equation

$$PA + A^\top P + I = 0$$

has a unique positive definite and symmetric matrix solution P . Using the Lyapunov function $V(\boldsymbol{x}, t) = \boldsymbol{x}^T P \boldsymbol{x}$, it can be easily verified that

$$\begin{aligned}\dot{V}(\boldsymbol{x}, t) &= \boldsymbol{x}^\top [PA + A^\top P]\boldsymbol{x} + 2\boldsymbol{x}^\top P\boldsymbol{g}(\boldsymbol{x}, t) \\ &\leq -\boldsymbol{x}^\top \boldsymbol{x} + 2\lambda_{\max}(P)c\|\boldsymbol{x}\|^2\end{aligned}$$

where $\lambda_{\max}(P)$ is the largest eigenvalue of P . Therefore, if the constant $c < 1/(2\lambda_{\max}(P))$ and if the class-κ functions

$$\alpha(\zeta) = \lambda_{\min}(P)\zeta^2, \qquad \beta(\zeta) = \lambda_{\max}(P)\zeta^2,$$
$$\gamma(\zeta) = [1 - 2c\lambda_{\max}(P)]\zeta^2$$

are used, then conditions iii and iv of Theorem 2 are satisfied. As a result, the system given here is globally, uniformly, and asymptotically stable about its zero equilibrium. This example shows that the linear part of a weakly nonlinear nonautonomous system can indeed dominate the stability.

Note that in Theorem 2, the uniform stability is guaranteed by the class-κ functions α, β, γ stated in conditions iii and iv, which are necessary since the solution of a nonautonomous system may sensitively depend on the initial time, as seen from the numerical example discussed earlier in the section entitled "stability in the sense of Lyapunov." For autonomous systems, these class-κ functions (hence, condition iii) are not needed. In this case, Theorem 2 reduces to the following simple form.

Theorem 3 (Second Method of Lyapunov, for Continuous-Time Autonomous Systems). The autonomous system (19) is globally (over the entire domain D) and asymptotically stable about its zero equilibrium, if there exists a scalar-valued function, $V(\boldsymbol{x})$, defined on D, such that

(1) $V(0) = 0$;
(2) $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq 0$ in D ;
(3) $\dot{V}(\boldsymbol{x}) < 0$ for all $\boldsymbol{x} \neq 0$ in D .

Note that if condition iv in Theorem 3 is replaced by

(1) $\dot{V}(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in D$;

then the resulting stability is only in the sense of Lyapunov but may not be asymptotic. For example, consider a simple model of an undamped pendulum of length ℓ described by

$$\begin{cases} \dot{x} = -\frac{g}{\ell}\sin(y) \\ \dot{y} = x, \end{cases}$$

where $y = \theta$ is the angular variable defined on $-\pi < \theta < \pi$, with the vertical axis as its reference, and g is the gravity constant. Since the system Jacobian at the zero equilibrium has a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm\sqrt{-g/\ell}$, Theorem 1 is not conclusive. However, if one uses the Lyapunov function

$$V = \frac{g}{\ell}(1 - \cos(y)) + \frac{1}{2}x^2$$

then it can be easily verified that $\dot{V} = 0$ over the entire domain. Thus, the conclusion is that the undamped pendulum is stable in the sense of Lyapunov but not asymptotically, consistent with the physics of the undamped pendulum.

Theorem 4 (Krasovskii Theorem, for Continuous-Time Autonomous Systems). For the autonomous system in Eq. (19), let $J\,\boldsymbol{x} = [\partial \boldsymbol{f}/\partial \boldsymbol{x}]$ be its Jacobian evaluated at $\boldsymbol{x}(t)$. A sufficient condition for the system to be asymptotically stable about its zero equilibrium is that there exist two real positive definite and symmetric constant matrices, P and Q, such that the matrix

$$J^{\top}(\boldsymbol{x})P + PJ(\boldsymbol{x}) + Q$$

is seminegative definite for all $\boldsymbol{x} \neq 0$ in a neighborhood D of the origin. For this case, a Lyapunov function is given by

$$V(\boldsymbol{x}) = \boldsymbol{f}^{\top}(\boldsymbol{x})P\boldsymbol{f}(\boldsymbol{x})$$

Furthermore, if $D = \boldsymbol{R}^n$ and $V(x) \to \infty$ as $\|\boldsymbol{x}\| \to \infty$, then this asymptotic stability is also global.

Similar stability criteria can be established for discrete-time systems. Two main results are summarized as follows.

Theorem 5 (First Method of Lyapunov, for Discrete-Time "Autonomous" Systems). Let $\boldsymbol{x}^* = 0$ be an equilibrium of the discrete-time "autonomous" system

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k) \tag{22}$$

where $\boldsymbol{f}: D \to \boldsymbol{R}^n$ is continuously differentiable in a neighborhood of the origin, $D \subseteq \boldsymbol{R}^n$, and let $J = [\partial f / \partial x_k]_{x_k = x^* = 0}$ be the Jacobian of the system evaluated at this equilibrium. If all the eigenvalues of J are strictly less than one in absolute value, then the system is asymptotically stable about its zero equilibrium.

Theorem 6 (Second Method of Lyapunov, for Discrete-Time "Nonautonomous" Systems). Let $\boldsymbol{x}^* = 0$ be an equilibrium of the "nonautonomous" system

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_k(\boldsymbol{x}_k), \tag{23}$$

where $\boldsymbol{f}_k: D \to \boldsymbol{R}^n$ is continuously differentiable in a neighborhood of the origin, $D \subseteq \boldsymbol{R}^n$. Then the system in Eq. (22) is globally (over the entire domain D) and asymptotically stable about its zero equilibrium, if there exists a scalar-valued function, $V(\boldsymbol{x}_\mathrm{k}, k)$, defined on D and continuous in $\boldsymbol{x}_\mathrm{k}$, such that

(1) $V(0, k) = 0$ for all $k \geq k_0$;
(2) $V(\boldsymbol{x}_\mathrm{k}) > 0$ for all $\boldsymbol{x}_\mathrm{k} \neq 0$ in D and for all $k \geq k_0$;
(3) $\Delta V(\boldsymbol{x}_k, k) := V(\boldsymbol{x}_k, k) - V(\boldsymbol{x}_{k-1}, k-1) < 0$ for all $\boldsymbol{x}_k \neq 0$ in D and all $k \geq k_0 + 1$;
(4) $0 < W(\|\boldsymbol{x}_k\|) < V(\boldsymbol{x}_k, k)$ for all $k \geq k_0 + 1$, where $W(\tau)$ is a positive continuous function defined on D, satisfying $W(0) = 0$ and $\lim_{\tau\to\infty} W(\tau) = \infty$ monotonically.

As a special case for discrete-time "autonomous" systems, Theorem 6 reduces to the following simple form.

Theorem 7 (Second Method of Lyapunov, for Discrete Time "Autonomous" Systems). Let $\boldsymbol{x}^* = 0$ be an equilibrium for the "autonomous" systems in Eq. (22). Then the system is globally (over the entire domain D) and asymptotically stable about this zero equilibrium if there exists a scalar-valued function $V(\boldsymbol{x}_\mathrm{k})$, defined on D and continuous in $\boldsymbol{x}_\mathrm{k}$, such that

(1) $V(0) = 0$;
(2) $V(\boldsymbol{x}_k) > 0$ for all $\boldsymbol{x}_k \neq 0$ in D;
(3) $\Delta V(\boldsymbol{x}_\mathrm{k}) := V(\boldsymbol{x}_\mathrm{k}) - V(\boldsymbol{x}_{\mathrm{k}-1}) < 0$ for all $\boldsymbol{x}_\mathrm{k} \neq 0$ in $*$;
(4) $V(\boldsymbol{x}) \to \infty$ as $\|\boldsymbol{x}\| \to \infty$.

To this end, it is important to emphasize that all the Lyapunov theorems stated earlier offer only sufficient conditions for asymptotic stability. On the other hand, usually more than one Lyapunov function may be constructed for the same system. For a given system, one choice of a Lyapunov function may yield a less conservative result (e.g., with a larger stability region) than other choices. However, no conclusion regarding stability may be drawn if, for technical reasons, a satisfactory Lyapunov function cannot be found. Nevertheless, there is a necessary condition in theory about the existence of a Lyapunov function (7).

Theorem 8 (Massera Inverse Theorem). Suppose that the autonomous system in Eq. (19) is asymptotically stable about its equilibrium $\boldsymbol{x}^*$ and $\boldsymbol{f}$ is continuously differentiable with respect to $\boldsymbol{x}$ for all $t \in [t_0, \infty)$. Then a Lyapunov function exists for this system.

Some Instability Theorems. Once again, consider a general autonomous system,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \boldsymbol{R}^n \tag{24}$$

with an equilibrium $\boldsymbol{x}^* = 0$. To disprove the stability, the following instability theorems may be used.

Theorem 9 (A Linear Instability Theorem). For system in Eq. (24), let $J = [\partial f/\partial x]_{x=x^*=0}$ be the system Jacobian evaluated at $\boldsymbol{x}^* = 0$. If at least one of the eigenvalues of J has a positive real part, then $\boldsymbol{x}^* = 0$ is unstable.

For discrete-time systems, there is a similar result: A discrete-time "autonomous" system

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k), \quad k = 0, 1, 2, \cdots$$

is unstable about its equilibrium $\boldsymbol{x}^* = 0$ if at least one of the eigenvalues of the system Jacobian is larger than 1 in absolute value.

The following two negative theorems can be easily extended to nonautonomous systems in an obvious way.

Theorem 10 (A General Instability Theorem). For system in Eq. (24), let $V(\boldsymbol{x})$ be a positive and continuously differentiable function defined on a neighborhood D of the origin, satisfying $V(0) = 0$. Assume that in any subset, containing the origin, of D, there is an $\tilde{x}$ such that $V(\tilde{x}) > 0$. If, moreover,

$$\dot{V}(\boldsymbol{x}) > 0 \qquad \text{for all} \quad \boldsymbol{x} \neq 0 \text{ in } \mathcal{D}$$

then the system is unstable about the equilibrium $\boldsymbol{x}^* = 0$.

One example is the system

$$\begin{cases} \dot{x} = y + x(x^2 + y^4) \\ \dot{y} = -x + y(x^2 + y^4) \end{cases}$$

which has equilibrium $(x^*, y^*) = (0, 0)$. The system Jacobian at the equilibrium has a pair of imaginary eigenvalues, $\lambda_{1,2} = \pm\sqrt{-1}$, so Theorem 1 is not conclusive. On the contrary, the Lyapunov function

$$V = \frac{1}{2}(x^2 + y^2)$$

leads to $\dot{V} = (x^2 + y^2)(x^2 + y^4) > 0$ for all $(x, y) \neq (0, 0)$. Therefore, the conclusion is that this system is unstable about its zero equilibrium.

Theorem 11 (Chetaev Instability Theorem). For system in Eq. (24), let $V(\boldsymbol{x})$ be a positive and continuously differentiable function defined on D, and let Ω be a subset, containing the origin, of D, (i.e., $0 \in D \cap \Omega$). If

(1) $V(\boldsymbol{x}) > 0$ and $\dot{V}(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq 0$ in D ,

(2) $V(\boldsymbol{x}) = 0$ for all $\boldsymbol{x}$ on the boundary of Ω ,

then the system is unstable about the equilibrium $\boldsymbol{x}^* = 0$.

This instability theorem is illustrated by Fig. 10, which graphically shows that if the theorem conditions are satisfied, then there is a gap within any neighborhood of the origin, so that a system trajectory can escape from the neighborhood of the origin along a path in this gap (1).

As an example, consider the system

$$\begin{cases} \dot{x} = x^2 + 2y^5 \\ \dot{y} = xy^2 \end{cases}$$

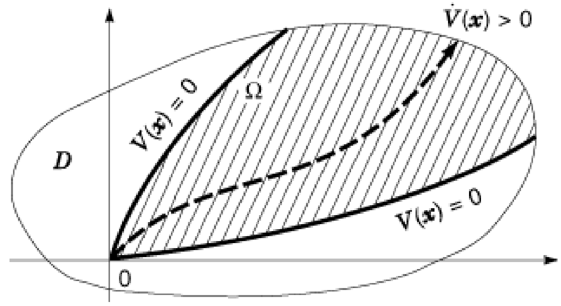

Fig. 10. Illustration of the Chetaev theorem.

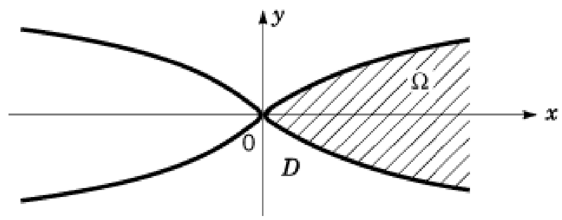

Fig. 11. The defining region of a Lyapunov function.

with the Lyapunov function

$$V = x^2 - y^4$$

which is positive inside the region defined by

$$x = y^2 \qquad \text{and} \qquad x = -y^2$$

Let D be the right-half plane and Ω be the shaded area shown in Fig. 11. Clearly, $V = 0$ on the boundary of Ω and $V > 0$ and $\dot{V} = 2x^3 > 0$ for all $(x, y) \in D$. According to the Chetaev theorem, this system is unstable about its zero equilibrium.

LaSalle Invariance Principle. Consider again the autonomous system in Eq. (24) with an equilibrium $\boldsymbol{x}^* = 0$. Let $V(\boldsymbol{x})$ be a Lyapunov function defined on a neighborhood D of the origin. Let also $\varphi_t(\boldsymbol{x}_0)$ be a bounded solution orbit of the system, with the initial state $\boldsymbol{x}_0$ and all its limit states being confined in D . Moreover, let

$$E = \{\boldsymbol{x} \in \mathcal{D} | \dot{V}(\boldsymbol{x}) = 0\} \tag{25}$$

and $M \subset E$ be the largest invariant subset of E in the sense that if the initial state $\boldsymbol{x}_0 \in M$ then the entire orbit $\varphi_t(\boldsymbol{x}_0) \subset M$ for all $t \geq t_0$.

Theorem 12 (LaSalle Invariance Principle). Under the preceding assumptions, for any initial state $\boldsymbol{x}_0 \in D$, the solution orbit satisfies

$$\varphi_t(\boldsymbol{x}_0) \to M \qquad \text{as } t \to \infty$$

This invariance principle is consistent with the Lyapunov theorems when they are applicable to a problem (6,12). Sometimes when $\dot{V} = 0$ over a subset of the domain of V, a Lyapunov theorem is not easy to apply directly, but the LaSalle invariance principle may be convenient to use. For instance, consider the system

$$\begin{cases} \dot{x} = -x + \frac{1}{3}x^3 + y \\ \dot{y} = -x \end{cases}$$

The Lyapunov function $V = x^2 + y^2$ yields

$$\dot{V} = \frac{1}{2}x^2\left(\frac{1}{3}x^2 - 1\right)$$

which is negative for $x^2 < 3$ but is zero for $x = 0$ and $x^2 = 3$, regardless of variable y . Thus, Lyapunov theorems do not seem to be applicable, at least not directly. However, observe that the set E defined earlier has only three straight lines: $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$, and that all trajectories which intersect the line $x = 0$ will remain on the line only if $y = 0$. This means that the largest invariant subset M containing the points with $x = 0$ is the only point (0, 0). It then follows from the LaSalle invariance principle that starting from any initial state located in a neighborhood of the origin bounded within the two stripes $x = \pm\sqrt{3}$, say located inside the disk

$$\mathcal{D} = \{(x, y) | x^2 + y^2 < 3\}$$

the solution orbit will always be attracted to the point (0, 0). This means that the system is (locally) asymptotically stable about its zero equilibrium.

Comparison Principle and Vector Lyapunov Functions. For large-scale and interconnected nonlinear (control) systems, or systems described by differential inequalities rather than differential equations, the preceding stability criteria may not be directly applicable. In many such cases, the comparison principle and vector Lyapunov function methods turn out to be advantageous (20 21,22).

To introduce the comparison principle, consider the general nonautonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{26}$$

where $\boldsymbol{f}(0, t) = 0$ is continuous on a neighborhood D of the origin, $t_0 \leq t < \infty$.

In this case, since $\boldsymbol{f}$ is only continuous (but not necessarily satisfying the Lipschitz condition), this differential equation may have more than one solution (23). Let $\boldsymbol{x}_{\max}(t)$ and $\boldsymbol{x}_{\min}(t)$ be its maximum and minimum solutions, respectively, in the sense that

$$\boldsymbol{x}_{\min}(t) \leq \boldsymbol{x}(t) \leq \boldsymbol{x}_{\max}(t) \quad \text{componentwise}, \quad \text{for all } t \in [t_0, \infty)$$

where $\boldsymbol{x}(t)$ is any solution of the equation, and $\boldsymbol{x}_{\min}(t_0) = \boldsymbol{x}(t_0) = \boldsymbol{x}_{\max}(t_0) = \boldsymbol{x}_0$.

Theorem 13 (The Comparison Principle). Let $\boldsymbol{y}(t)$ be a solution of the following differential inequality:

$$\dot{\boldsymbol{y}}(t) \leq \boldsymbol{f}(\boldsymbol{y}, t) \qquad \text{with } \boldsymbol{y}(t_0) \leq \boldsymbol{x}_0 \qquad \text{componentwise}$$

If $\boldsymbol{x}_{\max}(t)$ is the maximum solution of the system in Eq. (26), then

$$\boldsymbol{y}(t) \leq \boldsymbol{x}_{\max}(t) \qquad \text{componentwise} \quad \text{for all} \quad t \in [t_0, \infty)$$

The next theorem is established based on this comparison principle.

A vector-valued function, $\boldsymbol{g}\ (\boldsymbol{x}, t) = [g_1(\boldsymbol{x}, t) \cdots g_n(\boldsymbol{x}, t)]^{\mathrm{T}}$ is said to be quasi-monotonic, if

$$x_i = \tilde{x}_i \text{ and } x_j \geq \tilde{x}_j \quad (j \neq i) \quad \Rightarrow \quad g_i(\boldsymbol{x}, t) \geq g_i(\tilde{\boldsymbol{x}}, t),$$
$$i = 1, \cdots, n$$

Theorem 14 (Vector Lyapunov Function Theorem). Let $\boldsymbol{v}\ (\boldsymbol{x},\ t)$ be a vector Lyapunov function associated with the nonautonomous system in Eq. (26), with $\boldsymbol{v}(\boldsymbol{x}, t) = [V_1(\boldsymbol{x}, t) \cdots V_n(\boldsymbol{x}, t)]^{\mathrm{T}}$ in which each V_i is a continuous Lyapunov function for the system, $i = 1, \cdots, n$, satisfying $\|\boldsymbol{v}\ (\boldsymbol{x}, t)\| > 0$ for $\boldsymbol{x} \neq 0$. Assume that

$$\dot{\boldsymbol{v}}(\boldsymbol{x}, t) \leq \boldsymbol{g}(\boldsymbol{v}(\boldsymbol{x}, t), t) \qquad \text{componentwise}$$

for a continuous and quasi-monotonic function $\boldsymbol{g}$ defined on D . Then

(1) if the system

$$\dot{\boldsymbol{y}}(t) = \boldsymbol{g}(\boldsymbol{y}, t)$$

is stable in the sense of Lyapunov (or asymptotically stable) about its zero equilibrium $\boldsymbol{y}^* = 0$, then so is the nonautonomous system in Eq. (26);

(2) if, moreover, $\|\boldsymbol{v}(\boldsymbol{x}, t)\|$ is monotonically descreasing with respect to t and the preceding stability (or asymptotic stability) is uniform, then so is the nonautonomous system (26);

(3) if, furthermore, $\|\boldsymbol{v}(\boldsymbol{x}, t)\| \geq c\|\boldsymbol{x}\|^{\sigma}$ for two positive constants c and σ, and the preceding stability (or asymptotic stability) is exponential, then so is the nonautonomous system (26).

A simple and frequently used comparison function is

$$\boldsymbol{g}(\boldsymbol{y}, t) = A\boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}, t), \qquad \lim_{||\boldsymbol{y}|| \to 0} \frac{||\boldsymbol{h}(\boldsymbol{y}, t)||}{||\boldsymbol{y}||} = 0$$

where A is a stable M matrix (Metzler matrix). Here, $A = [a_{ij}]$ is an M matrix if

$$a_{ii} < 0 \quad \text{and} \quad a_{ij} \geq 0 \quad (i \neq j), \quad i, j = 1, \ldots, n$$

Theorem 14 (Vector Lyapunov Function Theorem).

Theorem 15 (Orbital Stability Theorem). Let $\tilde{x}(\boldsymbol{t})$ be a p-periodic solution of an autonomous system. Suppose that the system has Floquet multipliers λ_i, with $\lambda_1 = 0$ and $|\lambda_i| < 1$ for $i = 2, \cdots, n$. Then this periodic solution $\tilde{x}(t)$ is orbitally stable. Note that Floquet multipliers are defined preceding Eq. (9).

Theorem 16 (Peixoto Structural Stability Theorem). Consider a two-dimensional autonomous system. Suppose that $\boldsymbol{f}$ is twice differentiable on a compact and connected subset D bounded by a simple closed curve, Γ, with an outward normal vector, $\boldsymbol{n}$. Assume that $\boldsymbol{f}{\cdot}\boldsymbol{n} \neq 0$ on Γ . Then the system is structural stable on D if and only if

(1) all equilibria are hyperbolic;
(2) all periodic orbits are hyperbolic;
(3) if x and y are hyperbolic saddles (probably, $x = y$), then $W^s(x) \cap W^u(y) = \phi$.

Linear Stability of Nonlinear Systems

The first method of Lyapunov provides a linear stability analysis for nonlinear autonomous systems. In this section, the following general nonautonomous system is considered:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \epsilon \boldsymbol{R}^n \tag{27}$$

which is assumed to have an equilibrium $\boldsymbol{x}^* = 0$.

Linear Stability of Nonautonomous Systems. For the system in Eq. (27), Taylor-expanding the function $\boldsymbol{f}$ about $\boldsymbol{x}^* = 0$ gives

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) = \boldsymbol{J}(t)\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}, t) \tag{28}$$

where $J(t) = [\partial \boldsymbol{f}/\partial \boldsymbol{x}]_{x=0}$ is the Jacobian and $\boldsymbol{g}(\boldsymbol{x}, t)$ is the residual of the expansion, which is assumed to satisfy

$$||\boldsymbol{g}(\boldsymbol{x}, t)|| \le a||\boldsymbol{x}||^2 \qquad \text{for all} \quad t \in [t_0, \infty)$$

in a neighborhood of zero, where $a > 0$ is a constant. It is known, from the theory of elementary ordinary differential equations (16), that the solution of Eq. (28) is given by

$$\boldsymbol{x}(t) = \Phi(t, t_0)\boldsymbol{x}_0 + \int_{t_0}^{t} \Phi(t, \tau)\boldsymbol{g}(\boldsymbol{x}(\tau), \tau)\, d\tau \tag{29}$$

where $\Phi(t, \tau)$ is the fundamental matrix of the system associated with matrix $J(t)$.

Theorem 17 (A General Linear Stability Theorem). For the nonlinear nonautonomous system in Eq. (28), if there are two positive constants, c and σ, such that

$$||\Phi(t, \tau)|| \le ce^{-\sigma(t-\tau)} \qquad \text{for all} \quad t_0 \le \tau \le t < \infty$$

and if

$$\lim_{||\boldsymbol{x}|| \to 0} \frac{||\boldsymbol{g}(\boldsymbol{x}, t)||}{||\boldsymbol{x}||} = 0$$

uniformly with respect to $t \in [t_0, \infty)$, then there are two positive constants, γ and δ, such that

$$||\boldsymbol{x}(t)|| \le c||\boldsymbol{x}_0||e^{-\gamma(t-t_0)}$$

for all $\|\boldsymbol{x}_0\| \le \delta$ and all $t \in [t_0, \infty)$.

This result implies that under the theorem conditions, the system is locally, uniformly, and exponentially stable about its equilibrium $\boldsymbol{x}^* = 0$.

In particular, if the system matrix $J(t) = J$ is a stable constant matrix, then the following simple criterion is convenient to use.

Theorem 18 (A Special Linear Stability Theorem). Suppose that in system (28), the matrix $J(t) = J$ is a stable constant matrix (all its eigenvalues have a negative real part), and $\boldsymbol{g}(0, t) = 0$. Let P be a positive definite and symmetric matrix solution of the Lyapunov equation

$$PJ + J^{\top}P + Q = 0$$

where Q is a positive definite and symmetric constant matrix. If

$$||\boldsymbol{g}(\boldsymbol{x}, t)|| \le a||\boldsymbol{x}||$$

for a constant $a < \frac{1}{2}\lambda_{\max}(P)$ uniformly on $[t_0, \infty)$, where $\lambda_{\max}(P)$ is the maximum eigenvalue of P, then system in Eq. (28) is globally, uniformly, and asymptotically stable about its equilibrium $\boldsymbol{x}^* = 0$.

This actually is the example used previously for illustration of Theorem 2.

Linear Stability of Nonlinear Systems with Periodic Linearity. Consider a nonlinear nonautonomous system of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) = J(t)\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \boldsymbol{R}^n \qquad (30)$$

where $\boldsymbol{g}(0, t) = 0$ and $J(t)$ is a p-periodic matrix ($p > 0$):

$$J(t + p) = J(t) \qquad \text{for all} \quad t \in [t_0, \infty)$$

Theorem 19 (Floquet Theorem). For system in Eq. (30), assume that $\boldsymbol{g}(\boldsymbol{x}, t)$ and $\partial\boldsymbol{g}(\boldsymbol{x}, t)/\partial\boldsymbol{x}$ are both continuous in a bounded region D containing the origin. Assume, moreover, that

$$\lim_{||\boldsymbol{x}|| \to 0} \frac{||\boldsymbol{g}(\boldsymbol{x}, t)||}{||\boldsymbol{x}||} = 0$$

uniformly on $[t_0, \infty)$. If the system Floquet multipliers satisfy

$$|\lambda_i| < 1, \qquad i = 1, \ldots, n, \qquad \text{for all} \ \ t \in [t_0, \infty) \qquad (31)$$

then system in Eq. (30) is globally, uniformly, and asymptotically stable about its equilibrium $\boldsymbol{x}^* = 0$.

Total Stability: Stability Underpersistent Disturbances

Consider a nonautonomous system of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) + \boldsymbol{h}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \boldsymbol{R}^n \qquad (32)$$

where $\boldsymbol{f}$ is continuously differentiable, with $\boldsymbol{f}(0, t) = 0$, and $\boldsymbol{h}$ is a persistent perturbation in the sense that for any $\epsilon > 0$, there are two positive constants, δ_1 and δ_2, such that if $\|\boldsymbol{h}(\tilde{x}, t)\| < \delta_1$ for all $t \in [t_0, \infty)$ and if $\|\tilde{x}(t_0)\| < \delta_2$ then $\|\tilde{x}(t)\| < \epsilon$.

The equilibrium $\boldsymbol{x}^* = 0$ of the unperturbed system [system in Eq. (32) with $\boldsymbol{h} = 0$ therein] is said to be totally stable, if the persistently perturbed system in Eq. (32) remains to be stable in the sense of Lyapunov.

As the next theorem states, all uniformly and asymptotically stable systems with persistent perturbations are totally stable, namely, a stable orbit starting from a neighborhood of another orbit will stay nearby (7,9).

Theorem 20 (Malkin Theorem). If the unperturbed system in Eq. (32) (i.e., with $\boldsymbol{h} = 0$ therein) is uniformly and asymptotically stable about its equilibrium $\boldsymbol{x}^* = 0$, then it is totally stable, namely, the persistently perturbed system in Eq. (32) remains to be stable in the sense of Lyapunov.

Next, consider an autonomous system with persistent perturbations:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{h}(\boldsymbol{x}, t), \qquad \boldsymbol{x} \in \boldsymbol{R}^n \tag{33}$$

Theorem 21 (Perturbed Orbital Stability Theorem). If $\varphi_t(\mathbf{x}_0)$ is an orbitally stable solution of the unperturbed autonomous system in Eq. (33) (with $\boldsymbol{h} = 0$ therein), then it is totally stable, that is, the perturbed system remains to be orbitally stable under persistent perturbations.

Absolute Stability and Frequency Domain Criteria

Consider a feedback system in the Lur'e form:

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{h}(\boldsymbol{y}) \\ \boldsymbol{y} = C\boldsymbol{x} \end{cases} \tag{34}$$

where A, B, C are constant matrices, in which A is nonsingular but B and C are not necessarily square (yet, probably, $B = C = I$), and $\boldsymbol{h}$ is a vector-valued nonlinear function. By taking the Laplace transform with zero initial conditions and denoting the transform by $\tilde{x} = \mathcal{L}\{\boldsymbol{x}\}$, the state vector is obtained as

$$\hat{\boldsymbol{x}} = [sI - A]^{-1} B\mathcal{L}\{\boldsymbol{h}(\boldsymbol{y})\} \tag{35}$$

so that the output is given by

$$\hat{\boldsymbol{y}} = CG(s)\mathcal{L}\{\boldsymbol{h}(\boldsymbol{y})\} \tag{36}$$

with the system transfer matrix

$$G(s) = [sI - A]^{-1} B \tag{37}$$

This can be implemented via the block diagram shown in Fig. 12, where, for notational convenience, both time- and frequency-domain symbols are mixed.

The Lur'e system shown in Fig. 12 is a closed-loop configuration, where the block in the feedback loop is usually considered as a "controller." Thus, this system is sometimes written in the following equivalent form:

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} = C\boldsymbol{x} \\ \boldsymbol{u} = \boldsymbol{h}(\boldsymbol{y}) \end{cases} \tag{38}$$

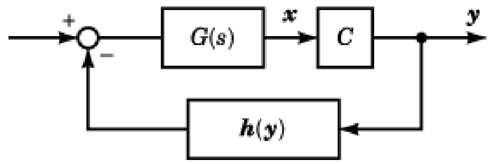

Fig. 12. Configuration of the Lur'e system.

***SISO* Lur'e Systems.** First, single-input single-output Lur'e systems are discussed, where $u = h(y)$ and $y = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}$ are both scalar-valued functions:

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}u \\ y = \boldsymbol{c}^{\top}\boldsymbol{x} \\ u = h(y) \end{cases} \tag{39}$$

Assume that $h(0) = 0$, so that $\boldsymbol{x}^* = 0$ is an equilibrium of the system.

The sector condition. The Lur'e system in Eq. (39) is said to satisfy the local (global) sector condition on the nonlinear function $h(\cdot)$, if there exist two constants, $\alpha < \beta$, such that

(1) local sector condition:

$$\alpha y^2(t) \le y(t)h(y(t)) \le \beta y^2(t) \quad \text{for all} \quad a \le y(t) \le b \quad \text{and} \quad t \in [t_0, \infty) \tag{40}$$

(2) global sector condition:

$$\alpha y^2(t) \le y(t)h(y(t)) \le \beta y^2(t) \quad \text{for all} \quad -\infty < y(t) < \infty \quad \text{and} \quad t \in [t_0, \infty) \tag{41}$$

Here, $[\alpha, \beta]$ is called the sector for the nonlinear function $h(\cdot)$. Moreover, the system in Eq. (39) is said to be absolutely stable within the sector $[\alpha, \beta]$ if the system is globally asymptotically stable about its equilibrium $\boldsymbol{x}^* = 0$ for any nonlinear function $h(\cdot)$ satisfying the global sector condition. These local and global sector conditions are visualized by Fig. 13(a, b), respectively.

Theorem 22 (Popov Criterion). Suppose that the *SISO* Lur'e system in Eq. (39) satisfies the following conditions:

(1) A is stable and $\{ A, \boldsymbol{b} \}$ is controllable;
(2) the system satisfies the global sector condition with $\alpha = 0$ therein;
(3) for any $\epsilon > 0$, there is a constant $\gamma > 0$ such that

$$\mathrm{Re}\{(1 + j\gamma\omega)G(j\omega)\} + \frac{1}{\beta} \ge \epsilon \quad \text{for all} \quad \omega \ge 0 \tag{42}$$

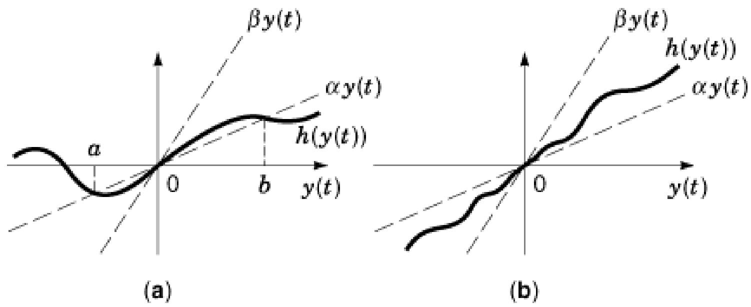

Fig. 13. Local and global sector conditions.

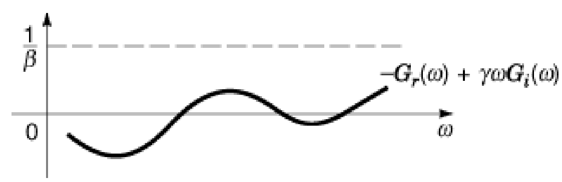

Fig. 14. Geometric meaning of the Popov criterion.

where $G(s)$ is the transfer function defined by Eq. (37), and Re$\{\cdot\}$ denotes the real part of a complex number (or function). Then, the system is globally asymptotically stable about its equilibrium $\boldsymbol{x}^* = 0$ within the sector.

The Popov criterion has the following geometric meaning: Separate the complex function $G(s)$ into its real and imaginary parts, namely,

$$G(j\omega) = G_r(\omega) + jG_i(\omega)$$

and so rewrite condition iii as

$$\frac{1}{\beta} > -G_r(\omega) + \gamma\omega G_i(\omega) \qquad \text{for all} \quad \omega \geq 0$$

Then any graphical situation of the Popov criterion shown in Fig. 14 implies the global asymptotic stability of the system about its zero equilibrium.

The Popov criterion has a natural connection to the linear Nyquist criterion (11,15,16,24). A more direct generalization of the Nyquist criterion to nonlinear systems is the following.

Theorem 23 (Circle Criterion). Suppose that the *SISO* Lur'e system in Eq. (39) satisfies the following conditions:

(1) A has no purely imaginary eigenvalues and has κ eigenvalues with positive real parts;
(2) the system satisfies the global sector condition;
(3) one of the following situation holds:
  $0 < \alpha < \beta$: the Nyquist plot of $G(j\omega)$ encircles the disk $D(-1/\alpha, -1/\beta)$ counterclockwise κ times but does not enter it;
  $0 = \alpha < \beta$: the Nyquist plot of $G(j\omega)$ stays within the open half-plane Re$\{s\} > -1/\beta$;

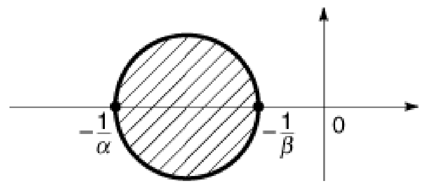

Fig. 15. The disk $D(-1/\alpha, -1/\beta)$.

$\alpha < 0 < \beta$: the Nyquist plot of $G(j\omega)$ stays within the open disk $D(-1/\beta, -1/\alpha)$;
$\alpha < \beta < 0$: the Nyquist plot of $-G(j\omega)$ encircles the disk $D(1/\alpha, 1/\beta)$ counterclockwise κ times but does not enter it.

Then, the system is globally asymptotically stable about its equilibrium $\boldsymbol{x}^* = 0$.

Here, the disk $D(-1/\alpha, -1/\beta)$, for the case of $0 < \alpha < \beta$, is shown in Fig. 15.

***MIMO* Lur'e Systems.** Consider a multi-input multi-output Lur'e system, as shown in Fig. 12, namely,

$$\begin{cases} \boldsymbol{x}(s) = G(s)\boldsymbol{u}(s) \\ \boldsymbol{u}(t) = -\boldsymbol{h}(\boldsymbol{y}(t)) \end{cases} \tag{43}$$

with $G(s)$ as defined in Eq. (37). If this system satisfies the following Popov inequality:

$$\int_{t_0}^{t_1} \boldsymbol{y}^{\top}(\tau)\boldsymbol{x}(\tau)\,d\tau \geq -\gamma \qquad \text{for all} \quad t_1 \geq t_0 \tag{44}$$

for a constant $\gamma \geq 0$ independent of t, then it is said to be hyperstable.

The linear part of this *MIMO* system is described by the transfer matrix $G(s)$, which is said to be positive real if

(1) there are no poles of $G(s)$ located inside the open half-plane $\text{Re}\{s\} > 0$;
(2) poles of $G(s)$ on the imaginary axis are simple, and the residues form a semi-positive definite matrix;
(3) the matrix $[\, G(j\omega) + G^{\text{T}}(j\omega)\,]$ is a semi-positive definite matrix for all real values of ω that are not poles of $G(s)$.

Theorem 24 (Hyperstability Theorem). The *MIMO* Lur'e system in Eq. (43) is hyperstable if and only if its transfer matrix $G(s)$ is positive real.

Describing Function Method. Return to the *SISO* Lur'e system in Eq. (39) and consider its periodic output $y(t)$. Assume that the nonlinear function $h(\cdot)$ therein is a time-invariant odd function and satisfies the property that for $y(t) = \alpha \sin(\omega t)$, with real constants ω and $\alpha \neq 0$, only the first-order harmonic of $-h(y)$ in its Fourier series expansion is significant. Under this setup, the specially defined function

$$\Psi(\alpha) = -\frac{2\omega}{\alpha\pi}\int_0^{\pi/\omega} h(\alpha \sin(\omega t))\sin(\omega t)\,dt \tag{45}$$

is called the describing function of the nonlinearity $-h(\cdot)$, or of the system (15,24,25).

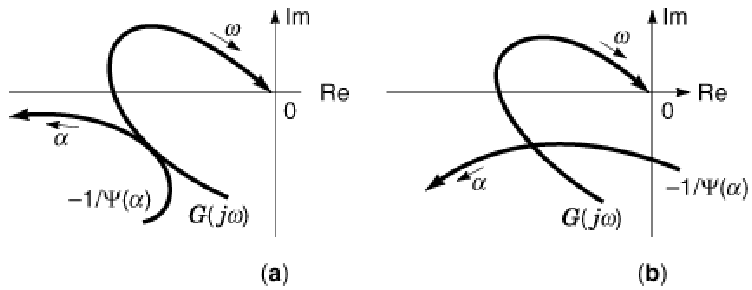

Fig. 16. Graphical describing function analysis.

Theorem 25 (First-Order Harmonic Balance Approximation). Under the preceding conditions, if furthermore the first-order harmonic balance equations

$$G_r(j\omega)\Psi(\alpha) = 1 \qquad \text{and} \qquad G_i(j\omega) = 0$$

have solutions ω and $\alpha \neq 0$, then

$$y^{\langle 1 \rangle}(t) = \frac{j\alpha}{2} e^{-j\omega t} - \frac{j\alpha}{2} e^{j\omega t}$$

is the first-order approximation of a possible periodic orbit of the output of system in Eq. (39). However, if these harmonic balance equations have no solution, then likely the system will not have any periodic output.

When solving the equation $G_r(j\omega)\ \Psi\ (\alpha) = 1$ graphically, one can sketch two curves in the complex plane: $G_r(j\ \omega)$ and $-\ 1/\Psi\ (\alpha)$ by increasing gradually ω and α, respectively, to find their crossing points:

(1) If the two curves are (almost) tangent, as illustrated by Fig. 16(a), then a conclusion drawn from the describing function method will not be satisfactory in general.
(2) If the two curves are (almost) transversal, as illustrated by Fig. 16(b), then a conclusion drawn from the describing function analysis will generally be reliable.

Theorem 26 (Graphical Stability Criterion for a Periodic Orbit). Each intersection point of the two curves, $G_r(j\omega)$ and $-1/\Psi(\alpha)$, in Fig. 16 corresponds to a periodic orbit, $y^{\langle 1 \rangle}\ (t)$, of the output of system (39). If the points, near the intersection and on one side of the curve $-1/\Psi(\alpha)$ where $-\alpha$ is increasing, are not encircled by the curve $G_r(j\ \omega)$, then the corresponding periodic output is stable; otherwise, it is unstable.

Bibo Stability

A relatively simple, and also relatively weak notion of stability is discussed in this section. This is the bounded-input bounded-output (*BIBO*) stability, which refers to the property of a system that any bounded input to the system produces a bounded output through the system (11,26,27).

Return to the input-output map in Eq. (13) and its configuration Fig. 2.

Definition 1. The system S is said to be *BIBO* stable from the input set U to the output set Y, if for each admissible input $\boldsymbol{u} \in U$ and the corresponding output $\boldsymbol{y} \in Y$, there exist two nonnegative constants, b_i and b_o, such that

$$\|\boldsymbol{u}\|_U \leq b_i \Rightarrow \|\boldsymbol{y}\|_Y \leq b_o \qquad (46)$$

Note that since all norms are equivalent for a finite-dimensional vector, it is generally insignificant to distinguish under what kind of norms for the input and output signals the *BIBO* stability is defined and achieved. Moreover, it is important to note that in this definition, even if b_{i} is small and b_{o} is large, the system is still considered to be *BIBO* stable. Therefore, this stability may not be very practical for some systems in certain applications.

Small Gain Theorem. A convenient criterion for verifying the *BIBO* stability of a closed-loop control system is the small gain theorem (11,26,27), which applies to almost all kinds of systems (linear and nonlinear, continuous-time and discrete-time, deterministic and stochastic, time-delayed, of any dimensions), as long as the mathematical setup is appropriately formulated to meet the theorem conditions. The main disadvantage of this criterion is its over-conservativity.

Return to the typical closed-loop system shown in Fig. 3, where the inputs, outputs, and internal signals are related via the following equations:

$$\begin{cases} S_1(e_1) = e_2 - u_2 \\ S_2(e_2) = u_1 - e_1 \end{cases} \qquad (47)$$

It is important to note that the individual *BIBO* stability of S_1 and S_2 is not sufficient for the *BIBO* stability of the connected closed-loop system. For instance, in the discrete-time setting of Fig. 3, suppose that $S_1 \equiv 1$ and $S_2 \equiv -1$, with $u_1(k) \equiv 1$ for all $k = 0, 1, \cdots$. Then S_1 and S_2 are *BIBO* stable individually, but it can be easily verified that $y_1(k) = k \to \infty$ as the discrete-time variable k evolves. Therefore, a stronger condition describing the interaction of S_1 and S_2 is necessary.

Theorem 27 (Small Gain Theorem). If there exist four constants, L_1, L_2, M_1, M_2, with $L_1 L_2 < 1$, such that

$$\begin{cases} ||S_1(e_1)|| \leq M_1 + L_1||e_1|| \\ ||S_2(e_2)|| \leq M_2 + L_2||e_2|| \end{cases} \qquad (48)$$

then

$$\begin{cases} ||e_1|| \leq (1 - L_1L_2)^{-1}(||u_1|| + L_2||u_2|| + M_2 + L_2M_1) \\ ||e_2|| \leq (1 - L_1L_2)^{-1}(||u_2|| + L_1||u_1|| + M_1 + L_1M_2) \end{cases} \qquad (49)$$

where the norms $\|\cdot\|$ are defined over the spaces that the signals belong. Consequently, Eqs. (48) and (24) together imply that if the system inputs (u_1 and u_2) are bounded then the corresponding outputs [$S_1(e_1)$ and $S_2(e_2)$] are bounded.

Note that the four constants, L_1, L_2, M_1, M_2, can be somewhat arbitrary (e.g., either L_1 or L_2 can be large) provided that $L_1 L_2 < 1$, which is the key condition for the theorem to hold [and is used to obtain ($1 - L_1 L_2)^{-1}$ in the bounds in Eq. (49)].

In the special case where the input-output spaces, U and Y, are both the L_2-space, a similar criterion based on the system passivity property can be obtained (11,27). In this case, an inner product between any two vectors in the space is defined by

$$\langle \xi, \eta \rangle = \int_{t_0}^{\infty} \xi^{\mathsf{T}}(\tau)\eta(\tau)\, d\tau$$

Theorem 28 (Passivity Stability Theorem). If there exist four constants, L_1, L_2, M_1, M_2, with $L_1 + L_2 > 0$, such that

$$\begin{cases} \langle e_1, S_1(e_1)\rangle \geq L_1\|e_1\|_2^2 + M_1 \\ \langle e_2, S_2(e_2)\rangle \geq L_2\|S_2(e_2)\|_2^2 + M_2 \end{cases} \tag{50}$$

then the closed-loop system in Eq. (47) is *BIBO* stable.

As mentioned earlier, the main disadvantage of this criterion is its over-conservativity in providing the sufficient conditions for the *BIBO* stability. One resolution is to transform the system into the Lur'e structure, and then apply the circle or Popov criterion under the sector condition (if it can be satisfied), which can usually lead to less-conservative stability conditions.

Contraction Mapping Theorem. The small gain theorem by nature is a kind of contraction mapping theorem. The contraction mapping theorem can be used to determine the *BIBO* stability property of a system described by a map in various forms, provided that the system (or the map) is appropriately formulated. The following is a typical (global) contraction mapping theorem.

Theorem 29 (Contraction Mapping Theorem). If the operator norm of the input-output map S, defined by Eq. (14) on $\boldsymbol{R}^n$, satisfies $|||S||| < 1$, then the system equation

$$\boldsymbol{y}(t) = S(\boldsymbol{y}(t)) + \boldsymbol{c}$$

has a unique solution for any constant vector $\mathbf{c} \in \boldsymbol{R}^n$. This solution satisfies

$$\|\boldsymbol{y}\| \leq (1 - |||S|||)^{-1}\|\boldsymbol{c}\|$$

In the discrete-time setting, the solution of the equation

$$\boldsymbol{y}_{k+1} = S(\boldsymbol{y}_k), \quad \boldsymbol{y}_0 \in \boldsymbol{R}^n, \quad k = 0, 1, \cdots$$

satisfies

$$\|\boldsymbol{y}_k\| \to 0 \qquad \text{as} \qquad \boldsymbol{k} \to \infty$$

Concluding Remarks

This article has offered a brief introduction and description of the basic theory and methodology of the Lyapunov stability, orbital stability, structural stability, and input-output stability for nonlinear dynamical systems. More subtle details for stability analysis of general dynamical systems can be found in, for example, Refs. 1,6,8 9 10 12 12,13,15,16,23,24, and 27 28,29. When control is explicitly involved, stability and stabilization issues are studied in Refs. 11,14,26,30,31, and 35 to name just a few.

Several important classes of nonlinear (control) systems have been left out in the preceding discussion of various stability issues: some general functional systems such as systems with time delays (32), measure ordinary differential equations such as systems with impulses (33,34), and some weakly nonlinear systems like piecewise linear and switching (non)linear systems. Moreover, discussion on more advanced nonlinear systems such as singular nonlinear systems (perhaps with time delays), infinite-dimensional (non)linear systems, spatiotemporal systems described by nonlinear partial differential equations, and nonlinear stochastic (control) systems are all beyond the scope of this elementary expository article.

BIBLIOGRAPHY

1. D. R. Merkin, *Introduction to the Theory of Stability*, New York: Springer-Verlag, 1997.
2. A. M. Lyapunov, *The General Problem of Stability of Motion*, 100 Anniversary, London: Taylor & Francis, 1992.
3. G. Chen, G. Chen, S. H. Hsu, *Linear Stochastic Control Systems*, Boca Raton, FL: CRC Press, 1995.
4. F. Brauer, J. A. Nohel, *The Qualitative Theory of Ordinary Differential Equations: An Introduction*, New York: Dover, 1989.
5. G. Chen, X. Dong, *From Chaos to Order: Methodologies, Perspectives and Applications*, Singapore: World Scientific, 1998.
6. P. Glendinning, *Stability, Instability and Chaos*, New York: Cambridge Univ. Press, 1994.
7. F. C. Hoppensteadt, *Analysis and Simulation of Chaotic Systems*, New York: Springer-Verlag, 1993.
8. N. P. Bhatia, G. P. Szegö, *Stability Theory of Dynamical Systems*, Berlin: Springer-Verlag, 1970.
9. W. Hahn, *Stability of Motion*, Berlin: Springer-Verlag, 1967.
10. D. W. Jordan, P. Smith, *Nonlinear Ordinary Differential Equations*, 2nd ed., New York: Oxford Univ. Press, 1987.
11. H. K. Khalil, *Nonlinear Systems*, 2nd ed., Upper Saddle River, NJ: Prentice-Hall, 1996.
12. J. P. LaSalle, *The Stability of Dynamical Systems*, Philadelphia: SIAM, 1976.
13. S. Lefschetz, *Differential Equations: Geometric Theory*, 2nd ed., New York: Dover, 1977.
14. N. Minorsky, *Theory of Nonlinear Control Systems*, New York: McGraw-Hill, 1969.
15. R. R. Mohler, *Nonlinear Systems: Vol. 1, Dynamics and Control*, Englewood Cliffs, NJ: Prentice-Hall, 1991.
16. P. C. Parks, V. Hahn, *Stability Theory*, New York: Prentice-Hall, 1981.
17. M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed., Englewood Cliffs, NJ: Prentice-Hall, 1993.
18. M. Y. Wu, A note on stability of linear time-varying systems, *IEEE Trans. Autom. Control*, **AC-19**: 162, 1974.
19. F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, 2nd ed., Berlin: Springer-Verlag, 1996.
20. V. Lakshmikantham, V. M. Matrosov, S. Sivasundaram, *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems*, Boston: Kluwer, 1991.
21. A. N. Michel, R. K. Miller, *Qualitative Analysis of Large Scale Dynamical Systems*, New York: Academic Press, 1977.
22. D. D. Siljack, *Large Scale Dynamic Systems: Stability and Structure*, Amsterdam: North-Holland, 1978.
23. R. Grimshaw, *Nonlinear Ordinary Differential Equations*, Boca Raton, FL: CRC Press, 1993.
24. K. S. Narendra, J. H. Taylor, *Frequency Domain Criteria for Absolute Stability*, New York: Academic Press, 1973.
25. D. P. Atherton, *Nonlinear Control Engineering*, New York: Van Nostrand Reinhold, 1982.
26. R. J.P. de Figueiredo, G. Chen, *Nonlinear Feedback Control Systems: An Operator Theory Approach*, San Diego: Academic Press, 1993.
27. C. A. Desoer, M. Vidyasagar, *Feedback Systems: Input-Output Properties*, New York: Academic Press, 1975.
28. L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, New York: Springer-Verlag, 1971.
29. S. Yu. Pilyugin, *Introduction to Structurally Stable Systems of Differential Equations*, Boston: Birkhäser, 1992.
30. H. Nijmeijer, A. J. van der Schaft, *Nonlinear Dynamical Control Systems*, New York: Springer-Verlag, 1990.
31. E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, 2nd ed., New York: Springer-Verlag, 1998.
32. J. Hale, *Theory of Functional Differential Equations*, New York: Springer-Verlag, 1977.
33. D. D. Bainov, and P. S. Simeonov, *Systems with Impulse Effect: Stability, Theory and Applications*, Chichester: Ellis Horwood, 1989.
34. A. M. Samoilenko, N. A. Perestyuk, *Impulsive Differential Equations*, Singapore: World Scientific, 1995.
35. S. Sastry, *Nonlinear Systems: Analysis, Stability, and Control*, New York: Springer, 1999.

GUANRONG CHEN
University of Houston