NETWORK ANALYSIS USING LINEARIZATION

Generally speaking, the term *linearization* indicates the substitution of a *nonlinear* relationship by a *linear* one which, according to some criterion, is approximately equivalent to it. However, the subject considered in this contribution should not be confused with the concept of *piecewise linearization* (see PIECEWISE LINEAR TECHNIQUES). In fact, *linearization* is intended here to be equivalent to *small-signal linearization,* which is a useful procedure in the analysis of physical systems and which is, in particular, widely employed for the analysis and the design of *lumped electronic circuits.*

As is well known from Circuit Theory (1–3) (see also NETWORK EQUATIONS), a lumped circuit is a model of a physical circuit composed by a suitable interconnection of electric and electronic devices, which applies whenever the size of the physical circuit is sufficiently smaller than the wavelength of the electromagnetic field associated with the electric phenomena. Nevertheless, the term *circuit* is also practically employed to indicate a circuit schematic and any graphical representation of a circuit mathematical model. However, notice that these two ways of using the same term may be considered equivalent whenever the circuit schematic is completed by sets of equations representing the models associated with the device symbols. In the following, the term *circuit* will be used with both meanings.

A circuit can thus be considered as composed of (ideal) circuit elements such as resistors, capacitors, inductors, independent voltage and current sources, voltage-controlled voltage sources (VCVS), voltage-controlled current sources (VCCS), current-controlled voltage sources (CCVS), and current-controlled current sources (CCCS). In fact, the model of more complex elements such as bipolar junction or MOS transistors can be suitably expressed by using the former elements. Therefore, the mathematical model of a circuit consists of a set of time-dependent variables (voltages, currents, electric charges, and magnetic fluxes) linked by a set of equations composed of:

(a) Kirchhoff's equations, namely linear algebraic equations deriving either from Kirchhoff's current law (KCL), stating that the sum of branch currents entering a circuit node is zero, or from Kirchhoff's voltage law (KVL), stating the vanishing of the sum of branch voltages along a closed node sequence;

(b) linear or nonlinear nondifferential equations which model the relationships among the variables associated to the physical phenomena taking place in the devices and which are referred to as the device's constitutive laws;

(c) linear first-order differential equations which express some currents as time derivatives of electric charges and some voltages as time derivatives of magnetic fluxes.

In particular, circuit topology determines equations (a), while the structure of the constitutive laws (b) may considerably vary, depending on the kind of the device. Typical examples of *independent branches* constitutive laws are $V = \overline{V}$ for an independent voltage source, $I = \overline{I}$ for an independent current source, $V = RI$ for a linear resistor, and $I = I_S \exp(V/V_T - 1)$ for a nonlinear resistor representing a junction diode model. Examples of *controlled branches* constitutive laws are $V_j = \alpha \cdot V_i$ for a linear VCVS, the voltage at branch j being controlled by the voltage at branch i, or $I_j = \beta(V_i)$ for a nonlinear VCCS. In order to model the storage of energy in the electric and the magnetic field, *reactive branches,* namely capacitors and inductors, have also to be considered. They are described by constitutive laws of type (b) like $\Phi = LI$ (which links magnetic flux and current in a linear inductor) or $f(Q, V) = 0$ (which links electric charge and voltage in a nonlinear capacitor) together with, respectively, the auxiliary equations of type (c) $V = d\Phi/dt$ and $I = dQ/dt$. Although not completely general, we will preliminarily assume that a circuit may include only the above-mentioned elements and that the corresponding constitutive laws link only two variables. These assumptions will be relaxed in the section entitled "A More General Approach to Linearization."

Because the constitutive laws of electronic devices are usually nonlinear, electronic circuits perform, in the most general case, nonlinear signal transformations. On the other hand, since such circuits are often employed to achieve *linear* information processing, like for instance in amplifiers or active filters (4,5), a question naturally arises about the way of obtaining such a goal.

To answer this question, first remember that in any information processing device, the information support is the variation of some physical quantity with respect to a reference value, which, although potentially variable in time, is often assumed to be constant [with the exception of the so-called *parametric* circuits (6) or parametric amplifiers]. In the following, we shall therefore consider the latter case only, and we shall define as *quiescent values* the constant reference values of the physical quantities used to carry information. In this context, a *signal* is defined as the difference between the physical quantity and its quiescent value. An electronic circuit intended for linear signal processing must therefore, first of all, ensure that, in the absence of signals, all physical quantities have an appropriate constant value. For this purpose any nontrivial circuit comprises a set of devices which is specifically devoted to establish suitable quiescent values and which is usually called a *bias circuit.* In particular, a bias circuit includes one or more *bias sources* which are independent voltage or current sources constant with time.

A set of quiescent values which satisfy the circuit equations—when all capacitors have been replaced by open circuits, all inductors have been replaced by short circuits, and all independent sources have been set to their quiescent value—is called a *quiescent operating point* or *dc operating point* (DCOP). A circuit may possess several DCOPs, some stable and others unstable, but, in order to allow linear signal processing, the existence of a stable and usually unique DCOP must be ensured by a proper design. Moreover, as shown in the following section, the use of sufficiently small signals around a DCOP allows us to replace the circuit nonlinear equations by linear ones which are approximately equivalent to them as far as the relationships among small signals are concerned. These linear equations can be interpreted as referring to a *small-signal equivalent circuit* composed by linear elements only, which may then be analyzed by using the well-known powerful techniques applicable to linear circuits (3). In particular, linear differential equations theory from Calculus or Laplace transform-based methods can be used for transient analyses, while complex numbers representation of sinusoidal functions or Fourier transform can be used for alternating current (ac) analyses and lead to very useful concepts as driving-point immittances and transfer functions (see LINEAR SYSTEMS).

Finally, it is important to stress that linearization is additionally involved in other important topics such as stability analysis of a DCOP, sensitivity analysis, and determining the DCOP(s) of a circuit. In fact, this amounts to solving a nonlinear system of equations, which is often numerically accomplished by using circuit analysis programs like SPICE (7,8). The numerical algorithm typically employed is iterative and gradually converges toward the solution by repeatedly solving linear equations systems obtained by suitable linearizations of the circuit equations.

A SIMPLE APPROACH TO LINEARIZATION

Consider a circuit element characterized by the constitutive law

$$Y = f(X) \tag{1}$$

where $f: \mathbb{R} \mapsto \mathbb{R}$ is a differentiable function. The two variables X and Y may stand, for instance, for the voltage across a voltage-controlled resistor and the current flowing in it, or for the electric charge of a charge-controlled capacitor and the voltage across it, or even for the input and output voltages of an amplifier.

Let X_0 be a reference value of the independent variable and consider the class of first-degree polynomials $Y = l(X)$ such that

$$l(X_0) = f(X_0) = Y_0 \tag{2}$$

namely

$$l(X) = f(X_0) + p(X - X_0) \tag{3}$$

with $p \in \mathbb{R}$. Since from Eq. (2) it follows that $f(X) - l(X)$ is at least first-order infinitesimal for $X \to X_0$, one could wonder if a suitable choice of p allows the above difference to be infinitesimal of order greater than one with $|X - X_0|$, namely

$$f(X) - l(X) = O(|X - X_0|^2) \quad \text{for } X \to X_0 \tag{4}$$

In other terms, the problem is to determine whether a "best choice" exists between all the possible linear functions (3) approximating the device constitutive law (1) in a neighborhood of X_0. Consider a small displacement x of X with respect to X_0. Since f is differentiable, the dependent variable value corresponding to $X = X_0 + x$ can be expressed as

$$Y = f(X) = f(X_0) + \left.\frac{df}{dX}\right|_0 x + O(|x|^2) \tag{5}$$

where the subscript 0 in df/dX means that this quantity must be evaluated at $X = X_0$. By employing Eqs. (3) and (5), one obtains that Eq. (4) is satisfied if and only if

$$p = \left.\frac{df}{dX}\right|_0 \tag{6}$$

On the basis of the above considerations, the linearization procedure of a nonlinear circuit may be summarized as follows:

1. The DCOP of the circuit, and therefore of all its devices, is suitably chosen during the design or computed during an analysis step.
2. For each nonlinear device, the constitutive law (1) is substituted by Eq. (3) satisfying Eq. (6). This corresponds to changing each element into its *small-signal equivalent device* whose constitutive law can be recast in terms of small signals only as

$$y = px \tag{7}$$

   The parameter p is called *differential* parameter and is indicated by several different terms according to the physical dimensions of the variables X and Y and to the nature of the circuit element to which it refers, as shown in Table 1.

It is important to stress that the whole linearization procedure applies whenever the signal $x = X - X_0$ (and hence $y = Y - Y_0$) is "small" in the sense that it can be considered as infinitesimal without introducing unacceptable errors. In order to verify this condition, circuit topology, devices characteristics, and DCOP have to be considered, as well as the maximum tolerable distortion in the circuit. Typically, this amounts to subsequently performing a suitable (nonlinear) *distortion analysis* (9).

Figure 1 shows a simple geometrical interpretation of the linearization process. Given the diagram of the function $Y = f(X)$ and the operating point $P_0 = (X_0, Y_0)$, then the diagram of $Y = l(X)$ with p chosen according to Eq. (6) is represented by the straight line tangent to f in P_0. If the coordinate axes are shifted so that the new origin coincides with P_0, then the tangent equation in the new reference system is expressed by Eq. (7).

It is worthwhile to note that the above considerations remain valid, under suitable assumptions, even when the circuit element constitutive law is provided in implicit form as

$$g(X, Y) = 0 \tag{8}$$

Table 1. Different Terms Used to Indicate the Differential Parameter p in Eq. (7), Depending on the Physical Dimension of X and Y in the Circuit Element Constitutive Law Expressed by Eq. (1)

Circuit element	X, x	Y, y	p
Resistor	Current	Voltage	Differential resistance
Resistor	Voltage	Current	Differential conductance
Capacitor	Voltage	Charge	Differential capacitance
Inductor	Current	Magnetic flux	Differential inductance
VCVS	Voltage	Voltage	Voltage gain
VCCS	Voltage	Current	Transconductance
CCVS	Current	Voltage	Transresistance
CCCS	Current	Current	Current gain

with $g: \mathbb{R}^2 \mapsto \mathbb{R}$. In fact, consider a DCOP P_0 satisfying Eq. (8) and suppose that a neighborhood U of P_0 exists such that g is differentiable in U with nonvanishing partial derivative $\partial g/\partial Y|_0$ in P_0. Then, by the *implicit function* theorem (10), a neighborhood V of X_0 and a differentiable function $f: V \mapsto \mathbb{R}$ exist such that $g(X, f(X)) = 0$ and $df/dX = -(\partial g/\partial X)/(\partial g/\partial Y)$. In this way the previously considered procedure still holds in V with $p = -(\partial g/\partial X)|_0/(\partial g/\partial Y)|_0$.

Examples

1. Consider a p–n junction diode whose dc voltage–current relationship in the forward bias region ($I \geq 0$) is expressed as (11)

$$V = nV_{\mathrm{T}} \ln\left(1 + \frac{I}{I_{\mathrm{S}}}\right)$$

where I_{S} is the reverse saturation current, V_{T} is the thermal voltage, and n the emission coefficient. If one indicates with $P_0 = (I_0, V_0 = nV_{\mathrm{T}} \ln(1 + I_0/I_{\mathrm{S}}))$ the diode operating point, the differential resistance of the diode in P_0 is given by

$$r_{\mathrm{d}} = \left.\frac{dV}{dI}\right|_0 = \frac{nV_{\mathrm{T}}}{I_0 + I_{\mathrm{S}}}$$

and the small signals $v = V - V_0$, $i = I - I_0$ are linked by the simple relationship

$$v = r_{\mathrm{d}} i$$

Since the last equation represents Ohm's law, as far as dc, small-signal relationships are concerned, the diode can be substituted by its *small-signal equivalent one-port*—that is, a linear resistor whose resistance is r_{d}.

2. Figure 2(a) shows a two-port representing a bipolar junction transistor (BJT), whose elementary model when operating in the forward normal region can be formulated as (11)

$$I_{\mathrm{C}} = I_{\mathrm{S}}\left[\exp\left(\frac{V_{\mathrm{BE}}}{V_{\mathrm{T}}}\right) - 1\right], \qquad I_{\mathrm{B}} = \frac{I_{\mathrm{S}}}{\beta_{\mathrm{F}}}\left[\exp\left(\frac{V_{\mathrm{BE}}}{V_{\mathrm{T}}}\right) - 1\right]$$

where I_{C} and I_{B} are the transistor collector and base currents, I_{S} is the transport saturation current, V_{BE} is the base-emitter voltage, and β_{F} is the large signal forward current gain of the common emitter configuration.

The associated small-signal relationships are expressed by

$$i_{\mathrm{C}} = g_{\mathrm{m}} v_{\mathrm{BE}}, \qquad i_{\mathrm{B}} = \frac{v_{\mathrm{BE}}}{r_{\mathrm{BE}}} \tag{9}$$

where $g_{\mathrm{m}} = \partial I_{\mathrm{C}}/\partial V_{\mathrm{BE}}|_0 = (I_{\mathrm{C0}} + I_{\mathrm{S}})/V_{\mathrm{T}}$ is the BJT transconductance and where $r_{\mathrm{BE}} = \beta_{\mathrm{F}}/g_{\mathrm{m}}$. By using a VCCS, Eqs. (9) can be considered as referring to the *small-signal equivalent two-port* shown in Fig. 2(b).

3. As a last example, consider a reverse-biased p/n junction diode. If one neglects the reverse current, its model reduces to a nonlinear capacitor having a voltage–charge relationship given by

$$Q = Q_0\left(1 + \frac{V_{\mathrm{R}}}{V_{\mathrm{J}}}\right)^{1-m}$$

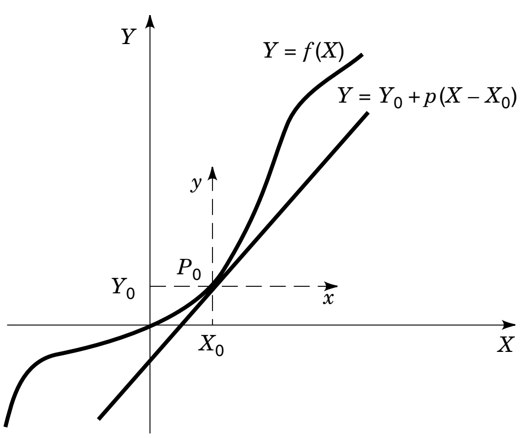

Figure 1. Geometrical interpretation of the basic linearization procedure: in a small neighborhood of a DCOP P_0, the curve $Y = f(X)$ is approximated by its tangent straight line $Y = Y_0 + p(X - X_0)$ whose slope is $p = (df/dX)|_0$. The "small signals" x and $y = px$ are measured in the coordinate system with origin in P_0.

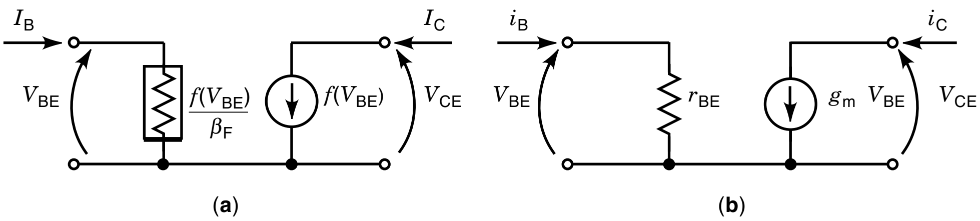

Figure 2. Basic BJT model, where $f(V_{BE}) = I_S[\exp(V_{BE}/V_T) - 1]$: (a) A linearization procedure applied to (a) gives rise to the linear constitutive laws of the small signal equivalent two-port represented in (b).

where V_R is the reverse voltage across the junction, Q_0 is the charge at $V_R = 0$, V_J is the built-in potential, and $m < 1$ is the junction grading coefficient (11). The small-signal equivalent one-port is readily found to be a linear capacitor whose capacitance is

$$C_j = \left.\frac{dQ}{dV_R}\right|_0 = \frac{C_{j0}}{\left(1 + \frac{V_{R0}}{V_J}\right)^m}$$

where V_{R0} is the dc operating voltage and $C_{j0} = (1 - m)Q_0/V_J$. The small-signal charge $q = Q - Q_0$ and the small-signal voltage $v_R = V_R - V_{R0}$ are related by $q = C_j v_R$.

The material developed so far shows that circuit elements characterized by means of two-variable constitutive laws have linear counterparts which are described in terms of the corresponding two-variable increments. Moreover, substituting increments for variables also in Kirchhoff's and auxiliary differential equations yields a linear mathematical model whose graphical representations are called *small-signal equivalent circuits* of the original nonlinear circuit.

A MORE GENERAL APPROACH TO LINEARIZATION

Consider a lumped circuit and express Kirchhoff's equations in vector form as (1,2)

$$\mathbf{AI} = \mathbf{0}, \qquad \mathbf{BV} = \mathbf{0} \tag{10}$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrices whose elements are 0, +1, and −1 and whose structure depends on the network topology, $\mathbf{I}$ is the n_b-dimensional vector of branch currents, and $\mathbf{V}$ is the n_b-dimensional vector of branch voltages. If the circuits possess n_c capacitor and n_l inductors, let $\mathbf{V}_c$ ($\mathbf{V}_l$) and $\mathbf{I}_c$ ($\mathbf{I}_l$) be the $n_c(n_l)$-dimensional vectors of voltages and currents at capacitive (inductive) branches and let $\mathbf{V}_r$ and $\mathbf{I}_r$ be the voltage and current vectors for the $n_r = n_b - n_c - n_l$ resistive branches, which include also dependent and independent sources. Then, by suitably ordering the vector components, one has $\mathbf{V} = [\mathbf{V}_c^t|\mathbf{V}_l^t|\mathbf{V}_r^t]^t$ and $\mathbf{I} = [\mathbf{I}_c^t|\mathbf{I}_l^t|\mathbf{I}_r^t]^t$, where $\cdot^t$ denotes transposition. The n_b (generally nonlinear) constitutive laws of the circuit elements may be expressed in the form

$$\mathbf{F}_c(\mathbf{Q}_c, \mathbf{V}_c) = \mathbf{0} \tag{11}$$

$$\mathbf{F}_l(\mathbf{\Phi}_l, \mathbf{I}_l) = \mathbf{0} \tag{12}$$

$$\mathbf{F}_r(\mathbf{V}_r, \mathbf{I}_r, \mathbf{E}) = \mathbf{0} \tag{13}$$

where $\mathbf{E}$ is the vector of circuit excitations, while $\mathbf{Q}_c$ and $\mathbf{\Phi}_t$ are the vectors of capacitors charges and inductors fluxes, respectively, which also satisfy the auxiliary differential equations

$$\frac{d\mathbf{Q}_c}{dt} = \mathbf{I}_c, \qquad \frac{d\mathbf{\Phi}_l}{dt} = \mathbf{V}_l \tag{14}$$

Notice also that with the above notation, a DCOP is defined as a set of time-independent variables ($\mathbf{V}_0$, $\mathbf{I}_0$, $\mathbf{Q}_{c0}$, $\mathbf{\Phi}_{l0}$) such that Eqs. (10)–(13) are satisfied with the dc values $\mathbf{E}_0$ replacing the original excitations $\mathbf{E}(t)$ and with $\mathbf{I}_c = 0$ and $\mathbf{V}_l = 0$. The small-signal linearization procedure relies on assuming the existence of the total differential of the functions $\mathbf{F}_c$, $\mathbf{F}_l$, $\mathbf{F}_r$ in Eqs. (11)–(13) and on substituting, for each vector $\mathbf{X}$, its differential $\mathbf{dX}$ by the small increment $\mathbf{x}$, while the linear equations (10) and (14) hold for $\mathbf{x}$ as for $\mathbf{X}$. By applying this procedure one gets

$$\mathbf{Ai} = \mathbf{0}, \qquad \mathbf{Bv} = \mathbf{0} \tag{15}$$

$$\left.\frac{\partial \mathbf{F}_c}{\partial \mathbf{Q}_c}\right|_0 \mathbf{q}_c + \left.\frac{\partial \mathbf{F}_c}{\partial \mathbf{V}_c}\right|_0 \mathbf{v}_c = \mathbf{0} \tag{16}$$

$$\left.\frac{\partial \mathbf{F}_l}{\partial \mathbf{\Phi}_l}\right|_0 \boldsymbol{\varphi}_l + \left.\frac{\partial \mathbf{F}_l}{\partial \mathbf{I}_l}\right|_0 \mathbf{i}_l = \mathbf{0} \tag{17}$$

$$\left.\frac{\partial \mathbf{F}_r}{\partial \mathbf{V}_r}\right|_0 \mathbf{v}_r + \left.\frac{\partial \mathbf{F}_r}{\partial \mathbf{I}_r}\right|_0 \mathbf{i}_r + \left.\frac{\partial \mathbf{F}_r}{\partial \mathbf{E}}\right|_0 \mathbf{e} = \mathbf{0} \tag{18}$$

$$\frac{d\mathbf{q}_c}{dt} = \mathbf{i}_c \tag{19}$$

$$\frac{d\boldsymbol{\varphi}_l}{dt} = \mathbf{v}_l \tag{20}$$

where $\partial\mathbf{F}_c/\partial\mathbf{Q}_c|_0$, $\partial\mathbf{F}_c/\partial\mathbf{V}_c|_0$, . . ., $\partial\mathbf{F}_r/\partial\mathbf{E}|_0$ are the Jacoby matrices of the functions $\mathbf{F}_c$, . . ., $\mathbf{F}_r$ with respect to $\mathbf{Q}_c$, $\mathbf{V}_c$, . . ., $\mathbf{E}$ at the DCOP.

Small-Signal Equivalent Circuits

A small-signal equivalent circuit can be defined as a graphical representation of Eqs. (15)–(20) constructed by means of linear ideal circuit components which, in addition to those considered in the section entitled "A Simple Approach to Linearization," may include, for instance, dependent voltage and current sources controlled by any number of variables. Moreover, if the derivative with respect to time of Eqs. (16) and (17) is considered and Eqs. (19) and (20) are used, dependent branches controlled by time derivatives of voltages and currents may also appear. By simple inspection of Eqs. (10)–(14) and (15)–(20) it can be observed that, since Kirchhoff's equations sets (10) and (15) have an identical structure, a small-signal equivalent circuit having the same topology as the orig-

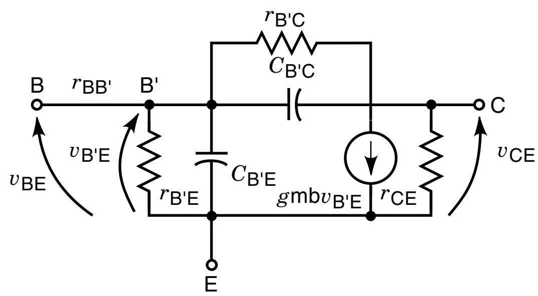

Figure 3. Hybrid-π equivalent circuit of a BJT which is widely used for small-signal analysis of bipolar and BiCMOS circuits.

inal nonlinear one may be obtained by substituting small-signal increments for the corresponding circuit variables and a linear component for each corresponding nonlinear one. In practice, however, the topological correspondence between the nonlinear and the linearized circuit models is often not perfect. This is due, on one hand, to the common practice of rearranging the equations to simplify the associated equivalent circuit or, conversely, of performing transformations which modify the topology of the small-signal circuit to simplify the associated equations and, on the other hand, to the presence of bias sources. In fact, a voltage bias source is, by its very definition, an independent voltage source whose value is unaffected by the signals. Therefore, its small-signal equivalent one-port is an independent voltage source of zero voltage—that is, a short-circuit. Dually, the small-signal equivalent one-port of a current bias source is an open-circuit. So, the small signal equivalent sources corresponding to bias sources do not explicitly appear in equivalent circuits.

As simple but important examples, commonly used equivalent circuits of a bipolar junction transistor operating in normal region and of a MOS field-effect transistor are reported in Figs. 3 and 4, respectively (11,12). In addition to the components in Fig. 2(b), the well-known hybrid-π equivalent circuit for a BJT shown in Fig. 3 accounts for the base resistance ($r_{BB'}$), the junction capacitances ($C_{B'E}$ and $C_{B'C}$), an internal resistive feedback ($r_{B'C}$), and the Early effect (r_{CE}). Notice that in a BJT nonlinear model the Early effect is accounted for by a dependence of the transport current on the base-collector voltage and the corresponding equivalent circuit should have a current source controlled by $v_{B'C}$ connected between the collector node and the emitter node. However, simple algebraic transformations allow us to use $v_{B'E}$ and v_{CE} as controlling voltages so that the considered VCCS may be equivalently replaced by adding a resistor of resistance r_{CE} and by considering a slightly different expression for the transconductance (11). In this way one obtains a circuit characterized by a different topology, but described by an equivalent equations set.

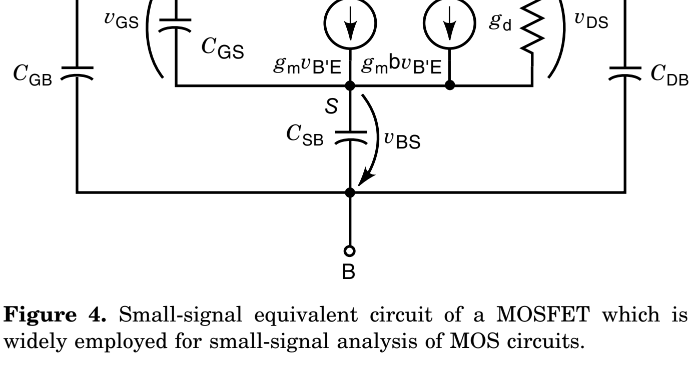

Figure 4. Small-signal equivalent circuit of a MOSFET which is widely employed for small-signal analysis of MOS circuits.

Finally, it should also be noted that the knowledge of a nonlinear model is not compulsory in order to obtain a small-signal equivalent circuit. In fact, this one can also arise, especially when high-frequency behavior is of interest, from an empirical or semiempirical procedure, namely by using a set of experimental values possibly integrated by physical considerations. For instance, let a three-terminal model of an electronic device like a BJT be represented as a two-port having one of the terminals shared by the ports. Its small-signal ac behavior around a fixed DCOP and at a fixed frequency can be described by the results of a suitable set of measurements which allow the four complex parameters of the 2×2 admittance or scattering matrix to be identified (see MULTIPOLE AND MULTIPORT ANALYSIS). By repeating the measurements at different frequencies, an approximate characterization of the device in a limited frequency range may be obtained. For an ac analysis of the circuit, one could employ look-up tables or functions obtained by interpolating the measured data, but computational efficiency may often be improved if a linear two-port circuit having the same matrix and a relatively small number of parameters, compared to the data set, can be devised. This corresponds to considering a small-signal equivalent circuit of the device in the considered frequency range.

While a nonlinear model would still be necessary for dc and for (nonlinear) transient analyses, this empirically identified linear equivalent circuit does not have to be strictly related to it; so more so as the desired approximations for the different types of analyses may be somewhat different, and the ease of linear analyses allows us to use equivalent circuits with many more parameters than desirable in a nonlinear model. Therefore, it may be more practical to consider both nonlinear and linear models as independently associated to the physical device or circuit, with not too tight a relationship each to the other.

An Example of Small-Signal ac Analysis

As is well known from basic Circuit Theory (3), a stable linear circuit excited by a sinusoidal signal reaches, after a transient phase, a steady state characterized by sinusoidal signals having the same frequency, but in general different amplitudes and nonzero phase shifts, with respect to the excitation. These signals represent the circuit ac response, whose derivation is defined as *ac analysis.* In nonlinear circuits with ac sources of a given frequency, the same effect arises in practice when excitation amplitudes are small enough to obtain a negligibly nonlinear response of circuit elements in a neighborhood of the bias point. In this case one has the so-called *small-signal ac response.* Hence, performing a small-signal ac analysis requires the solution in the frequency domain of the small-signal equivalent circuit equations.

As an application example, consider the MOSFET common source amplifier M_1 with active load M_2 shown in Fig. 5. The

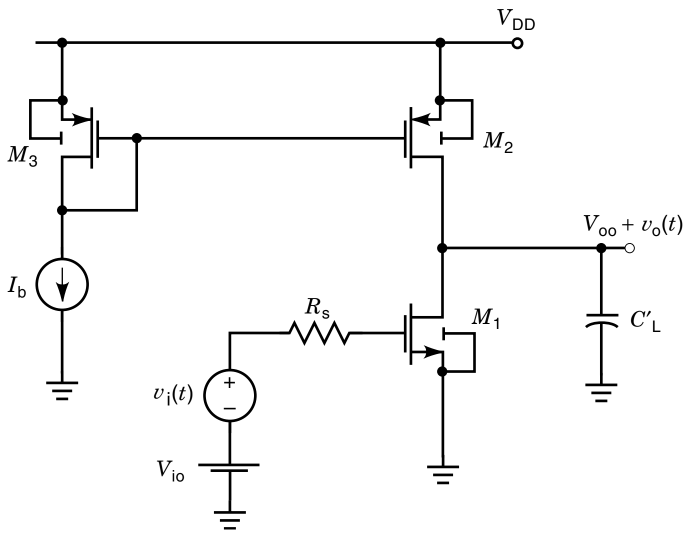

Figure 5. A MOSFET common source amplifier with active load. The transistor M_3 and the current bias source I_b are employed to bias the gate of transistor M_2 to a suitable voltage.

circuit including the supply voltage generator V_{DD}, the transistor M_3, and the constant current source I_b is used to bias the gate of M_2 to a suitable voltage, thus establishing, together with the bias source V_{io}, the circuit DCOP. Moreover, $v_i(t)$ represents a small-signal input source with internal resistance R_s, and C'_L is a load capacitance. Substituting the equivalent circuit shown in Fig. 4 for M_1 and M_2 yields the small-signal equivalent circuit reported in Fig. 6, where $C_g = C_{GS1} + C_{GB1}$, $C_L \simeq C'_L + C_{DB1} + C_{DB2} + C_{GD2}$, and $G_L = g_{d1} + g_{d2}$. In order to characterize the circuit behavior, a very meaningful quantity to be computed is the *amplifier voltage gain,* namely the *transfer function* $A_v = V_0/V_i$, where V_0 and V_i represent the Laplace or Fourier transform of the small signals $v_0(t)$ and $v_i(t)$. By applying the KCL to the nodes M and N of the circuit in Fig. 6, one gets

$$(V_{GS1} - V_i)G_S + j\omega C_g V_{GS1} + j\omega C_{GD1}(V_{GS1} - V_0) = 0 \tag{21}$$

$$j\omega C_{GD1}(V_0 - V_{GS1}) + g_{m1}V_{GS1} + V_0(G_L + j\omega C_L) = 0 \tag{22}$$

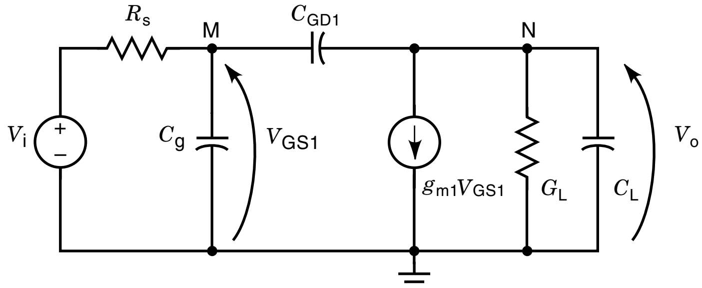

Figure 6. Small-signal equivalent circuit of the amplifier shown in Fig. 5, obtained by substituting the equivalent circuit of Fig. 4 for transistors M_1 and M_2.

where $G_s = 1/R_s$ and where V_{GS1}, V_i and V_0 indicate the Fourier transform of the voltages $v_{GS1}(t)$, $v_i(t)$ and $v_0(t)$, respectively. By solving Eqs. (21) and (22) for $V_i(j\omega)$ and $V_0(j\omega)$, one easily obtains

$$A_v(j\omega) = \frac{V_0(j\omega)}{V_i(j\omega)} = A_{v0}\frac{1 - j\omega\dfrac{C_{GD1}}{g_{m1}}}{1 + j\omega\beta - \omega^2\alpha}, \tag{23}$$

where $A_{v0} = -g_{m1}/G_L$ is the dc voltage gain and

$$\alpha = \frac{R_s}{G_L}[C_L(C_{GD1} + C_g) + C_g C_{GD1}]$$

$$\beta = \frac{R_s}{G_L}[G_L(C_{GD1} + C_g) + G_s(C_L + C_{GD1}) + g_{m1}C_{GD1}]$$

The voltage gain (23) is then characterized by a real positive zero at the angular frequency $\omega_z = g_{m1}/C_{GD1}$ and by two real negative poles corresponding to angular frequencies ω_{p1} and ω_{p2}. By assuming, as verified in practice, that $\omega_{p1} \ll \omega_{p2}$, Eq. (23) can be recast in a more useful form as

$$A_v(j\omega) \approx A_{v0}\frac{1 - j\dfrac{\omega}{\omega_z}}{1 + j\dfrac{\omega}{\omega_{p1}} - \dfrac{\omega^2}{\omega_{p1}\omega_{p2}}} \tag{24}$$

where $\omega_{p1} = G_L/\{C_L + C_{GD1} + R_s G_L[C_g + C_{GD1}(1 - A_{v0})]\}$ and $\omega_{p2} = [G_L(C_{GD1} + C_g) + G_s(C_L + C_{GD1}) + g_{m1}C_{GD1}]/[C_L(C_{GD1} + C_g) + C_g C_{GD1}]$. Since the voltage gain (24) is a complex function, it is usually represented in terms of magnitude response $|A_v(j\omega)|$ and phase response $\phi(j\omega) = \arg A_v(j\omega)$, which are commonly represented as Bode diagrams and from which several meaningful quantities, like the amplifier *gain-bandwidth* product or *phase margin,* can be computed (4,5) (see also SIGNAL AMPLIFIERS).

Linearization and Sensitivity

By recalling that the (relative, small-change) sensitivity of a function H with respect to a parameter γ is defined as (1)

$$S_\gamma^H = \frac{\partial H}{\partial \gamma}\frac{\gamma}{H} = \frac{\partial H/H}{\partial \gamma/\gamma}$$

and observing that its expression may be interpreted as the ratio of the fractional change in H due to a unit fractional change in γ provided that all variations are sufficiently small, it is not surprising that sensitivity can be related to the concepts of linearization and small-signal equivalent circuit. This quantity is of course a valuable information for any electronic circuit designer. For instance, if the output voltage of a filter is very sensitive to the resistance value of a resistor, a circuit VLSI implementation would probably fail to meet one or more constraints, due to the unavoidable spreading introduced by the devices physical realization or to temperature changes and aging.

In the following, we restrict our considerations to the case of a purely resistive circuit, which is formally simpler because its model includes only nondifferential equations (for the

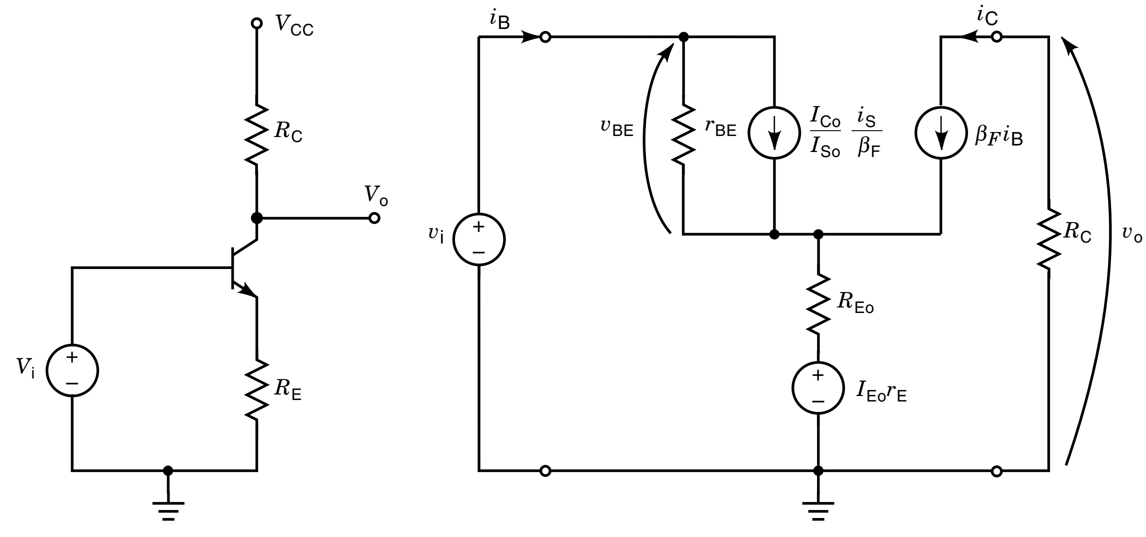

Figure 7. Sensitivity calculations may be included in a general linearization procedure. Parameters changes in the circuit (a) are accounted for in the small-signal equivalent circuit (b) by suitable independent sources.

more general case of nonlinear reactive circuits see Ref. 13). In this case, the circuit is described by the system

$$\mathbf{A}\mathbf{I}_r = \mathbf{0}, \qquad \mathbf{B}\mathbf{V}_r = \mathbf{0}, \qquad \mathbf{F}_r(\mathbf{V}_r, \mathbf{I}_r, \mathbf{E}, \boldsymbol{\Gamma}) = \mathbf{0} \tag{25}$$

where, with respect to Eq. (13), the dependence on the parameters vector $\boldsymbol{\Gamma}$ has been accounted for. By applying a linearization procedure to Eqs. (25), one obtains

$$\mathbf{A}\mathbf{i}_r = \mathbf{0}, \qquad \mathbf{B}\mathbf{v}_r = \mathbf{0},$$

$$\left.\frac{\partial \mathbf{F}_r}{\partial \mathbf{V}_r}\right|_0 \mathbf{v}_r + \left.\frac{\partial \mathbf{F}_r}{\partial \mathbf{I}_r}\right|_0 \mathbf{i}_r + \left.\frac{\partial \mathbf{F}_r}{\partial \mathbf{E}}\right|_0 \mathbf{e} + \left.\frac{\partial \mathbf{F}_r}{\partial \boldsymbol{\Gamma}}\right|_0 \boldsymbol{\gamma} = \mathbf{0}$$

where $\boldsymbol{\gamma}$ indicates the small changes parameters vector with respect to the nominal parameter values $\boldsymbol{\Gamma}_0$. Note that only $\mathbf{v}_r$ and $\mathbf{i}_r$ are unknown variables and therefore $\boldsymbol{\gamma}$ may be dealt with as $\mathbf{e}$; that is, the effects of small parameter changes may be accounted for by suitable independent sources. Once $\mathbf{v}_r$, $\mathbf{i}_r$ are expressed as functions of $\mathbf{e}$ and $\boldsymbol{\gamma}$, any desired sensitivity is readily obtained.

As a simple example, consider the amplifier stage of Fig. 7(a) and suppose that small spreads or changes of the BJT transport saturation current I_S and of the resistance R_E must be considered. Assume that I_S has nominal value I_{S0} and variation i_S, while R_E has nominal value R_{E0} and variation r_E. The equations describing the circuit behavior may be written as

$$\begin{aligned}
V_i &= V_{BE} + R_E I_E & I_C &= I_S\left[\exp\left(\frac{V_{BE}}{V_T}\right) - 1\right] \\
V_{CC} &= R_C I_C + V_{CE} + R_E I_E & I_E &= I_C + I_B \\
V_{CE} &= V_{CB} + V_{BE} & I_B &= \frac{I_S}{\beta_F}\left[\exp\left(\frac{V_{BE}}{V_T}\right) - 1\right] \\
V_o &= V_{CC} - R_C I_C
\end{aligned}$$

The corresponding linearized equations may be recast in the form

$$\begin{aligned}
v_i &= v_{BE} + R_{E0} i_E + I_{E0} r_E & i_C &= \beta_F i_B \\
0 &= R_C i_C + v_{CE} + R_{E0} i_E + I_{E0} r_E & i_E &= i_C + i_B \\
v_{CE} &= v_{CB} + v_{BE} & i_B &= \frac{v_{BE}}{r_{BE}} + \frac{I_{C0}}{\beta_F I_{S0}} i_S \\
v_o &= -R_C i_C
\end{aligned}$$

and may be interpreted by the small-signal equivalent circuit of Fig. 7(b), where the changes of R_E and of I_S are accounted for by a voltage source $I_{E0}r_E$ and a current source $(I_{C0}/\beta_F I_{S0})i_S$, respectively. In this way, sensitivities such as

$$S_{I_S}^{V_0} = \frac{I_{S0}}{V_{00}} \left.\frac{v_0}{i_S}\right|_{\substack{v_i=0 \\ r_E=0}} = -\frac{R_C I_{C0}}{V_{00}} \frac{r_{BE}}{r_{BE} + R_{E0}(\beta_F + 1)}$$

can then be computed from the equivalent circuit by means of standard linear circuit analysis.

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NETWORK ANALYZERS. See STANDING WAVE METERS AND NETWORK ANALYZERS.

NETWORK AVAILABILITY. See NETWORK RELIABILITY AND FAULT-TOLERANCE.