NETWORK ANALYSIS, SINUSOIDAL STEADY STATE

In this article the most important aspects of electrical circuits operating in a sinusoidal steady state, also known by practitioners as alternating current circuits, will be explored. In this section a brief historical introduction is followed by a mathematical overview of sinusoids and phasors.

HISTORICAL NOTES

Today, alternating current (ac) circuits are the standard for electric power production, transmission, distribution, and consumption. The advantage of ac versus direct current (dc) systems became evident toward the end of the nineteenth century, when a number of theoretical and technical results were converted to practical machines, making long-distance power transmission feasible and economical. Most of these inventions are still in use: alternators, transformers, and asynchronous motors are the standard energy-to-energy conversion mechanisms in the modern world. At the same time the first experiments with electromagnetic waves (discovered by H. Hertz in 1887) underlined the importance of the study of ac systems and resonant circuits, and led the way to modern communications and electronics.

Important historical milestones are the invention of the transformer, asynchronous motor, and the (theoretical) definition of the rotating vector (alternatively called a phasor) formalism. The work of Faraday and Ruhmkorff provided the basis for the invention of the transformer. The first practical open-core ac transformer was introduced by Gaulard at the 1884 World's Fair in Turin, Italy. Thanks to the theoretical work of Ferraris, who defined the power factor for ac circuits, and definitely proved the high performance of the transformer, ac systems could be used for long-distance power transmission. The design of the first transformer was improved in the following year by Deri, Blathy, and Zipernowsky, with a closed-core design. The 1885 Budapest fair was lit by an array of 75 of these transformers. In the same years Ferraris and Tesla independently investigated the application of the rotating magnetic field to the design of ac asynchronous motors, patented by Tesla in 1888. A complete ac system powered by a hydroelectric plant 176 km away was demonstrated in 1891 in Frankfurt, Germany. The definitive victory for ac systems occurred in 1892, with the decision to adopt the alternators designed by Tesla and built by Westinghouse for the Niagara Falls power plant.

Finding steady state solutions in ac systems was a difficult task. J. C. Maxwell contributed by providing a general solution of his equations for an ac circuit. Even with Maxwell's simplifications, solving for a particular problem still involved the use of differential methods, not yet well known to the practical engineer. The solution to this problem came with T. Blakesley in 1885. His rotating vector method was the starting point for the subsequent theory developed by C. P. Steinmetz, which was published in 1893 (1) and 1898 (2).

SINUSOIDS AND PHASORS

Sinusoids are periodic functions known from trigonometry:

$$u_1(t) = u_{01}\cos(\omega_1 t + \phi_1) \tag{1}$$

Any sinusoid is characterized by a triplet of parameters: amplitude u_{01} and angular frequency ω_1, both positive by convention, and (initial) phase ϕ_1, defined less an integer multiple of 2π. The positiveness of u_{01} and ω_1 does not limit the generality of the definition in Eq. (1). In fact, the change in sign of u_{01} corresponds to the addition of $\pm\pi$ to ϕ_1, while the change in sign of ω_1 is equivalent to the change of sign of ϕ_1.

Two other parameters are commonly used as alternatives to ω_1: *frequency* $f_1 = \omega_1/(2\pi)$ and *period* $T_1 = 1/f_1$. Moreover, the effective value u_1^{eff} of sinusoid $u_1(t)$

$$u_1^{\text{eff}} = \lim_{(t_2-t_1)\to\infty} \sqrt{\frac{1}{t_2-t_1}\int_{t_1}^{t_2} [u_{01}\cos(\omega_1 t+\phi_1)]^2\,dt} = \frac{u_{01}}{\sqrt{2}} \quad (2)$$

may be used in place of u_{01}. Since the integrand is periodic with period $T_1/2$, this result does not change if the integration range $(t_2 - t_1)$ is coincident with any integer multiple of $T_1/2$.

Recalling trigonometry and complex number mathematics, the expression of $u_1(t)$ in Eq. (1) may assume the alternative forms:

$$u_{01}(t) = \begin{cases} u_{01}\cos(\phi_1)\cos(\omega_1 t) - u_{01}\sin(\phi_1)\sin(\omega_1 t) \\ \Re[\overline{u}_1\exp(j\omega_1 t)] \\ (1/2)\overline{u}_1\exp(j\omega_1 t) + (1/2)(\overline{u}_1)^*\exp(-j\omega_1 t) \end{cases} \quad (3)$$

where j is the imaginary unit, the complex number $\overline{u}_1 = u_{01}\exp(j\phi_1)$ is the *phasor* of the sinusoid, u_{01}, coincident with the amplitude of the sinusoid, is its modulus, and $\Re[\ ]$ denotes the real part of the complex quantity between [ ]. Analogously, $\Im[\ ]$ denotes the imaginary part. The second expression in Eq. (3) allows one to interpret sinusoids from a geometrical point of view. A sinusoid with angular frequency ω_1 and phasor $\overline{u}_1$ may be regarded as the projection on the real axis of a point moving along a circumference with angular velocity ω_1 (see Fig. 1). The circumference is centered on the origin of the complex plane: the modulus u_{01} of the phasor determines its radius, while the phase ϕ_1 determines the position of the point on the circumference in $t = 0$.

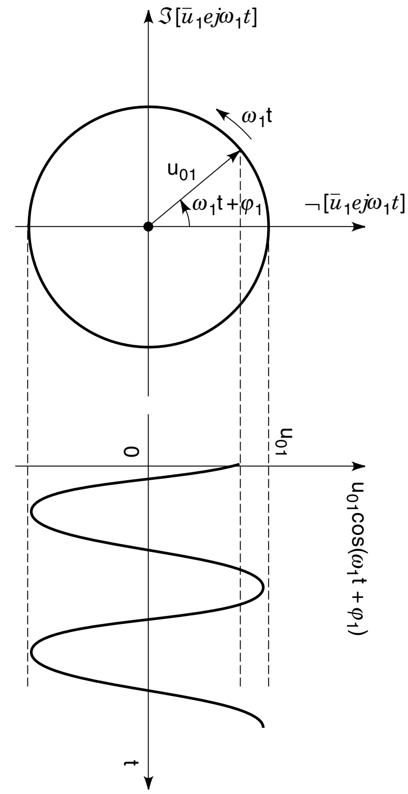

Figure 1. Geometrical relations between phasors and corresponding sinusoids in time domain.

Subclasses of Sinusoids With The Same Angular Frequency

Consider the subset of sinusoids characterized by the same angular frequency, hereinafter denoted by symbol $\hat{\omega}$, which will be called *$\hat{\omega}$-subclass.* Each sinusoid of an $\hat{\omega}$-subclass is distinguishable from other sinusoids of the same subclass by its specific phasor value. Some examples of sinusoids and corresponding phasors are shown in Table 1.

Property. According to Eqs. (4) and (5) each $\hat{\omega}$-subclass of sinusoids or, equivalently, the corresponding set of phasors, constitutes a two-dimensional linear space.

Proof. A sinusoid $u_1(t)$ of a $\hat{\omega}$-subclass and phasor $\overline{u}_1$, multiplied by any real number α, is again a sinusoid of the same subclass with phasor $\alpha\overline{u}_1$:

$$\alpha u_1(t) = \alpha\Re[\overline{u}_1\exp(j\hat{\omega}t)] = \Re[(\alpha\overline{u}_1)\exp(j\hat{\omega}t)] \quad (4)$$

while the sum of any pair of sinusoids $u_1(t)$ and $u_2(t)$ of an $\hat{\omega}$-subclass and with phasors $\overline{u}_1$ and $\overline{u}_2$ is a sinusoid of the same subclass and with phasor $\overline{u}_1 + \overline{u}_2$:

$$\begin{aligned} u_1(t)+u_2(t) &= \Re[\overline{u}_1\exp(j\hat{\omega}t)] + \Re[\overline{u}_2\exp(j\hat{\omega}t)] \\ &= \Re[(\overline{u}_1+\overline{u}_2)\exp(j\hat{\omega}t)] \end{aligned} \quad (5)$$

Property. The set of the time derivatives of all sinusoids of any $\hat{\omega}$-subclass is equivalent to the subclass itself. More exactly, the derivative of a sinusoid of an $\hat{\omega}$-subclass and with phasor $\overline{u}_1$ is again a sinusoid of the same subclass and with phasor $j\hat{\omega}\overline{u}_1$.

Proof. Consider a generic sinusoid and its time derivative:

$$\begin{aligned} u_1(t) &= \tfrac{1}{2}[\overline{u}_1\exp(j\hat{\omega}t) + \overline{u}_1^*\exp(-j\hat{\omega}t)] \Rightarrow \\ \frac{du_1(t)}{dt} &= \tfrac{1}{2}[j\hat{\omega}\overline{u}_1\exp(j\hat{\omega}t) - j\hat{\omega}\overline{u}_1^*\exp(-j\hat{\omega}t)] \end{aligned} \quad (6)$$

Comparing the derivative with the sinusoid itself proves the property.

Table 1. Some Examples of Sinusoids and Related Phasors

Sinusoid $u_k(t)$	Phasor $\overline{u}_k$
$15\cos(\hat{\omega}t + \pi/4)$	$15\exp(j\pi/4)$
$10\cos(\hat{\omega}t - \pi/2)$	$10\exp(-j\pi/2)$
$-3\sin(\hat{\omega}t)$	$3\exp(+j\pi/2)$
$-8\cos(\hat{\omega}t - \pi/6)$	$8\exp(j5\pi/6)$

Table 2. Terminology Used in Comparing Two Sinusoids and Their Phasors

$\phi_1 = \phi_2$	$\bar{u}_1$ and $\bar{u}_2$ are *in phase*	$\phi_1 - \phi_2 = \pm\pi$	$\bar{u}_1$ and $\bar{u}_2$ are *in opposition*
$\phi_2 + \pi > \phi_1 > \phi_2$	$\bar{u}_1$ *anticipates* $\bar{u}_2$	$\phi_2 - \pi < \phi_1 < \phi_2$	$\bar{u}_1$ *delays* $\bar{u}_2$
$\phi_1 - \phi_2 = +\pi/2$	$\bar{u}_1$ *anticipates in quadrate* $\bar{u}_2$	$\phi_1 - \phi_2 = -\pi/2$	$\bar{u}_1$ *delays in quadrate* $\bar{u}_2$

Analogously the integral of a sinusoid of a $\hat{\omega}$-subclass is once more a sinusoid of the same subclass and with phasor $\overline{u}_1/(j\hat{\omega})$, if the arbitrary integration constant is zero.

A set of phasors of the same $\hat{\omega}$-subclass may be represented in the complex plane. This representation, called a *phasor diagram,* is convenient in qualitative and quantitative analysis. An ad hoc terminology is commonly used when phasors (and/or the corresponding sinusoids) are compared in the complex plane; Table 2 reports such terminology for two sinusoids with phasors $\overline{u}_1 = u_{01}\exp(j\phi_1)$ and $\overline{u}_2 = u_{02}\exp(j\phi_2)$. Note that phases ϕ_1 and ϕ_2 must be defined so that $|\phi_1 - \phi_2| \leq \pi$, by choosing suitably the arbitrary integer multiples of 2π of the two phases.

PHASOR DOMAIN ANALYSIS OF CIRCUITS IN SINUSOIDAL STEADY STATE

In this section the phasor domain method will be applied to analyze a circuit operating in a sinusoidal steady state. A linear dynamic circuit operates in a *sinusoidal steady state* (SSS), that is, all voltages and currents of the circuit vary versus time as sinusoids of the same $\hat{\omega}$-subclass (4), if the following conditions are met:

1. The circuit is built using linear, resistive, and time invariant elements with any number of terminals, sinusoidal independent sources all with the same fixed angular frequency $\hat{\omega}$, linear and time invariant capacitors, inductors, and coupled inductors.
2. The circuit is asymptotically stable, that is, all the natural complex frequencies $s_k = \sigma_k + j\omega_k$ $(k = 1, 2, . . ., n)$ of the circuit are in the left side of the complex plane [i.e., $\sigma_k < 0$, $(k = 1, 2, . . ., n)$] (4).
3. The circuit has been left running with no external intervention (e.g., switch commutation) for a time interval Δt such that $\Delta t \gg 1/|\sigma_k|$, $(k = 1, 2, . . ., n)$.

Under the above circumstances the transient effects due to initial conditions vanish, because the circuit is asymptotically stable, all voltages and currents are sinusoids versus time. In conclusion, by substituting sinusoids and their derivatives with the respective phasors, the time domain linear differential equations with forcing sinusoids of the same $\hat{\omega}$-subclass are transformed into complex-domain algebraic equations.

Topological Relations in Phasor Domain

The time domain Kirchhoff's laws (see NETWORK EQUATIONS) are translated in SSS into the phasor domain (4): they are again homogeneous linear algebraic relations with the same real and constant coefficients. Table 3 shows phasor domain laws, when the incidence matrix A and a fundamental mesh matrix B are employed: vectors $\boldsymbol{v}(t)$ and $\boldsymbol{i}(t)$ group the sinusoidal branch voltages and currents, and vectors $\tilde{\boldsymbol{v}}(t)$ and $\tilde{\boldsymbol{\imath}}(t)$ group the sinusoidal node voltages and mesh currents, while vectors $\overline{\boldsymbol{v}}$, $\overline{\boldsymbol{i}}$, $\overline{\tilde{\boldsymbol{v}}}$, and $\overline{\tilde{\boldsymbol{\imath}}}$ group the respective phasors.

Constitutive Relations In Phasor Domain

The constitutive relations, also known as branch or element relations, are introduced, in the phasor domain, by using the voltage and current reference directions (defined in LINEAR NETWORK ELEMENTS) for independent voltage and current sources: the corresponding source voltage or current phasor is introduced, while the respective current or voltage remains unconstrained in the phasor domain (see Table 4): source voltages and currents are characterized by the symbol "ˆ".

Linear resistive elements are defined, in the phasor domain, by algebraic, constant coefficient relations identical to those used in the time domain (see Table 5). Table 6 shows the constitutive relations of simple dynamical elements (see LINEAR NETWORK ELEMENTS): they display the imaginary factor $j\omega$, which replaces the time domain derivative d/dt, denoted hereinafter by "˙" (see Eq. 6).

Sparse Tableau Analysis in Phasor Domain

A circuit operating in SSS is now analyzed, by using the same methods presented for general analyses (see NETWORK EQUATIONS). For the sake of brevity only the sparse tableau method will be discussed. To this end, consider Kirchhoff's laws and constitutive relations in phasor domain:

$$\begin{bmatrix} -A^{\mathrm{T}} & I_{m,m} & 0_{m,m} \\ 0_{n-1,n-1} & 0_{n-1,m} & A \\ 0_{m,n-1} & H^{v0} + j\hat{\omega}H^{v1} & H^{i0} + j\hat{\omega}H^{i1} \end{bmatrix} \begin{bmatrix} \overline{\tilde{\boldsymbol{v}}} \\ \overline{\boldsymbol{v}} \\ \overline{\boldsymbol{i}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0}_m \\ \boldsymbol{0}_{n-1} \\ \hat{\overline{\boldsymbol{u}}} \end{bmatrix} \quad (7)$$

where n and m are the number of nodes and branches in the graph; A is the $(n - 1) \times m$ incidence matrix; H^{v0}, H^{v1}, H^{i0}, and H^{i1} are $m \times m$ block diagonal matrices grouping the parameters of constitutive relations, $I_{m,m}$ is the identity $m \times m$ matrix, $\boldsymbol{0}_m$, $\boldsymbol{0}_{n-1}$ are vectors of null elements, and $0_{m,m}$, $0_{m,n-1}$, $0_{n-1,m}$, $0_{n-1,n-1}$ are matrices of null elements; subscripts denote dimensions. Vector $\hat{\overline{\boldsymbol{u}}}$ in the right-hand side groups the phasors of source voltages and currents, while the unknowns of the system are the phasors of node voltages, branch voltages, and currents. Note that the elements of matrices $H^{v0} + j\hat{\omega}H^{v1}$ and $H^{i0} + j\hat{\omega}H^{i1}$ either are adimensional or have the physical dimensions of voltage-to-current or current-to-voltage.

Table 3. Formulations of Kirchhoff's Laws in Phasor Domain

	Time Domain	Phasor Domain	Time Domain	Phasor Domain
KVL	$\boldsymbol{v}(t) = A^{\mathrm{T}}\tilde{\boldsymbol{v}}(t)$	$\overline{\boldsymbol{v}} = A^{\mathrm{T}}\overline{\tilde{\boldsymbol{v}}}$	$B\boldsymbol{v}(t) = 0$	$B\overline{\boldsymbol{v}} = 0$
KCL	$A\boldsymbol{i}(t) = 0$	$A\overline{\boldsymbol{\imath}} = 0$	$\boldsymbol{i}(t) = B^{\mathrm{T}}\tilde{\boldsymbol{\imath}}(t)$	$\overline{\boldsymbol{\imath}} = B^{\mathrm{T}}\overline{\tilde{\boldsymbol{\imath}}}$

Table 4. Constitutive Relations of Independent Sources in Phasor Domain

Voltage Source		Current Source	
Time Domain	Phasor Domain	Time Domain	Phasor Domain
$v(t) = \Re[\bar{\bar{v}} \exp(j\omega t + j\phi_v)]$	$\bar{v} = \bar{\bar{v}} = \hat{v} \exp(j\phi_v)$	$i(t) = \Re[\bar{\bar{\imath}} \exp(j\omega t + j\phi_i)]$	$\bar{\imath} = \bar{\bar{\imath}} = \hat{\imath} \exp(j\phi_i)$

Impedance and Admittance

The phasor domain representation of sinusoidal voltages and currents suggests, for any one-port element, the introduction of *impedance* and *admittance,* which have the same role as, respectively, resistance and conductance in dc circuits (see LINEAR NETWORK ELEMENTS). For the fixed value $\hat{\omega}$, *impedance* $z(j\hat{\omega})$ and *admittance* $y(j\hat{\omega})$ are complex numbers defined by the quotient of voltage-to-current and of current-to-voltage phasors, respectively:

$$z(j\hat{\omega}) = r(\hat{\omega}) + jx(\hat{\omega}) = \frac{\bar{v}}{\bar{\imath}} \quad y(j\hat{\omega}) = g(\hat{\omega}) + jb(\hat{\omega}) = \frac{\bar{\imath}}{\bar{v}} \tag{8}$$

In Eq. (8) both impedance and admittance have been decomposed into real and imaginary parts: $r(\hat{\omega})$ is called *resistance,* $x(\hat{\omega})$ is *reactance,* $g(\hat{\omega})$ is *conductance,* and $b(\hat{\omega})$ is *susceptance,* as shown in Fig. 2. Impedance and admittance are not at all phasors, since they do not represent sinusoids; they may be considered as phasor-to-phasor operators. For this reason their symbol is not barred.

Impedance and admittance of one-port subnetworks (i.e., built connecting simple one-port elements) may be calculated using the same rules given for two-terminal resistors (see TIME DOMAIN CIRCUIT ANALYSIS). For instance, the impedance $z(j\hat{\omega})$ and admittance $y(j\hat{\omega}) = 1/z(j\hat{\omega})$ of series and parallel connections of two one-port elements are:

$$\begin{aligned} &\text{Series:} \quad z(j\hat{\omega}) = z_1(j\hat{\omega}) + z_2(j\hat{\omega}) \quad y(j\hat{\omega}) = \frac{y_1(j\hat{\omega})y_2(j\hat{\omega})}{y_1(j\hat{\omega} + y_2(j\hat{\omega})} \\ &\text{Parallel:} \quad y(j\hat{\omega}) = y_1(j\hat{\omega}) + y_2(j\hat{\omega}) \quad z(j\hat{\omega}) = \frac{z_1(j\hat{\omega})z_2(j\hat{\omega})}{z_1(j\hat{\omega}) + z_2(j\hat{\omega})} \end{aligned} \tag{9}$$

where $z_1(j\hat{\omega}) = 1/y_1(j\hat{\omega})$ and $z_2(j\hat{\omega}) = 1/y_2(j\hat{\omega})$ are the impedances of the connected one-ports (see LINEAR NETWORK ELEMENTS).

For instance, consider a dynamical one-port subnetwork formed by connecting, in parallel, a resistor with value 500/3 Ω and a capacitance with value 20 μF operating in SSS characterized by $\hat{\omega} = 300\ \text{rad} \cdot \text{s}^{-1}$: admittance y and impedance z are obtained as:

$$y = \left(\frac{1}{500/3} + j\,300 \times 20 \times 10^{-6}\right) \text{S} = \left(\frac{3}{500} + j\,\frac{3}{500}\right) \text{S} \Rightarrow$$
$$z = \frac{1}{y} = \left(\frac{500}{6} - j\,\frac{500}{6}\right) \Omega$$

Representations of Dynamical Two-Port Elements in the Phasor Domain

The six representations of two-ports (see LINEAR NETWORK ELEMENTS), if they exist, are valid also for two-ports in the phasor domain. Consider, as an example, the current and voltage-controlled representations:

$$\begin{aligned} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} &= \begin{bmatrix} z_{11}(j\hat{\omega}) & z_{12}(j\hat{\omega}) \\ z_{21}(j\hat{\omega}) & z_{22}(j\hat{\omega}) \end{bmatrix} \begin{bmatrix} \bar{\imath}_1 \\ \bar{\imath}_2 \end{bmatrix} \\ \begin{bmatrix} \bar{\imath}_1 \\ \bar{\imath}_2 \end{bmatrix} &= \begin{bmatrix} y_{11}(j\hat{\omega}) & y_{12}(j\hat{\omega}) \\ y_{21}(j\hat{\omega}) & y_{22}(j\hat{\omega}) \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} \end{aligned} \tag{10}$$

The four elements of both matrices are, in general, complex because they depend on the imaginary number $j\hat{\omega}$. The impedance and admittance matrices $Z(j\hat{\omega})$ and $Y(j\hat{\omega})$ substitute the real resistance and conductance matrices R and G proper of dc circuits. The same considerations hold also for the other four representations of two-ports.

Generalization of dc Analysis Methods and Properties to ac Circuits

All the following topics, introduced for linear resistive circuits, are easily generalized to the phasor domain [see LINEAR

Table 5. Constitutive Relations of Linear Resistive Elements in Phasor Domain

Element	Time Domain	Phasor Domain	Element	Time Domain	Phasor Domain
Short circuit	$v(t) = 0$	$\bar{v} = 0$	Open circuit	$i(t) = 0$	$\bar{\imath} = 0$
Resistor	$v(t) = ri(t)$	$\bar{v} = r\bar{\imath}$	Nullor	$v_1(t) = 0$ $i_1(t) = 0$	$\bar{v}_1 = 0$ $\bar{\imath}_1 = 0$
CCVS	$v_1(t) = 0$ $v_2(t) = r_m i_1(t)$	$\bar{v}_1 = 0$ $\bar{v}_2 = r_m\bar{\imath}_1$	VCCS	$i_1(t) = 0$ $i_2(t) = g_m v_1(t)$	$\bar{\imath}_1 = 0$ $\bar{\imath}_2 = g_m\bar{v}_1$
CCCS	$v_1(t) = 0$ $i_2(t) = \beta i_1(t)$	$\bar{v}_1 = 0$ $\bar{\imath}_2 = \beta\bar{\imath}_1$	VCVS	$i_1(t) = 0$ $v_2(t) = \alpha v_1(t)$	$\bar{\imath}_1 = 0$ $\bar{v}_2 = \alpha\bar{v}_1$
Ideal transformer	$v_1(t) = nv_2(t)$ $i_1(t) = i_2(t)/n$	$\bar{v}_1 = n\bar{v}_2$ $\bar{\imath}_1 = \bar{\imath}_2/n$	Gyrator	$v_1(t) = i_2(t)/g_m$ $i_1(t) = g_m v_2(t)$	$\bar{v}_1 = \bar{\imath}_2/g_m$ $\imath_1 = g_m\bar{v}_2$

Table 6. Constitutive Relations of Dynamical Elements in Phasor Domain

Elements	Time Domain	Phasor Domain
Capacitor	$i(t) = C\dot{v}(t)$	$\bar{\imath} = j\hat{\omega}C\bar{v}$
Inductor	$v(t) = L\dot{i}(t)$	$\bar{v} = j\hat{\omega}L\bar{\imath}$
Coupled inductors	$v_1(t) = L_1\dot{i}_1(t) + M\dot{i}_2(t)$ $v_2(t) = M\dot{i}_1(t) + L_2\dot{i}_2(t)$	$\bar{v}_1 = j\hat{\omega}L_1\bar{\imath}_1 + j\hat{\omega}M\bar{\imath}_2$ $\bar{v}_2 = j\hat{\omega}M\bar{\imath}_1 + j\hat{\omega}L_2\bar{\imath}_2$

NETWORK ELEMENTS; NETWORK EQUATIONS; see also (3) or (4) for classical methods]:

- Reciprocal and nonreciprocal two-ports
- Thevenin and Norton models of one-port elements and respective theorems
- Superposition theorem
- Nodal analysis and modified nodal analysis
- Loop and cut set analysis
- Current and voltage partition rules
- First and second Millmann theorems
- $Y \rightarrow \Delta$ and $\Delta \rightarrow Y$ transformations

POWER IN SINUSOIDAL STEADY STATE

To evaluate the electrical power exchanged in linear dynamic circuits operating in SSS, the sinusoidal behavior of any branch voltage and current must be taken into account.

Instantaneous Power in One-Port Elements

Let $v(t) = v_0 \cos(\hat{\omega}t + \phi_v)$ and $i(t) = i_0 \cos(\hat{\omega}t + \phi_i)$ be the voltage and current of a one-port element or any port of a multiport element operating in SSS; the *instantaneous power* $p(t) = v(t)i(t)$ absorbed by this element is composed of a constant term plus a sinusoidal term with angular frequency $2\hat{\omega}$ (Fig. 3):

$$\begin{aligned} p(t) &= v_0 \cos(\hat{\omega}t + \phi_v) i_0 \cos(\hat{\omega}t + \phi_i) \\ &= (v_0 i_0/2)\cos(\phi_v - \phi_i) + (v_0 i_0/2)\cos(2\hat{\omega}t + \phi_v + \phi_i) \end{aligned} \quad (11)$$

The constant term $v_0 i_0 \cos(\phi_v - \phi_i)/2$ has an absolute value less than or equal to the amplitude $v_0 i_0/2$ of the sinusoidal term. In particular, the constant term coincides with this amplitude in the case of resistors and is null in the case of capacitors and inductors, according to:

$$\begin{aligned} &\text{Resistor:} && p(t) = [(ri_0^2)/2][1 + \cos(2\hat{\omega}t + 2\phi_v)] \\ &\text{Capacitor:} && p(t) = [(\hat{\omega}Cv_0^2)/2]\cos(2\hat{\omega}t + 2\phi_v + \pi/2) \\ &\text{Inductor:} && p(t) = [(\hat{\omega}Li_0^2)/2]\cos(2\hat{\omega}t + 2\phi_v - \pi/2) \end{aligned} \quad (12)$$

Recall that the sum of instantaneous powers absorbed by all the K elements (including possible multiport elements) of a circuit is zero.

$$\sum_{k=1}^{K} p_k(t) = 0 \quad (13)$$

Note that in Eq. (13), $p_k(t)$ is negative for independent sources delivering power.

Active Power and Power Factor

Instantaneous power $p(t)$ is somewhat inconvenient and does not have much practical use. Other definitions dealing with power are often preferred.

Definition. In generic dynamic situations, *active power* P is defined as the average of instantaneous power $p(t)$ over a time

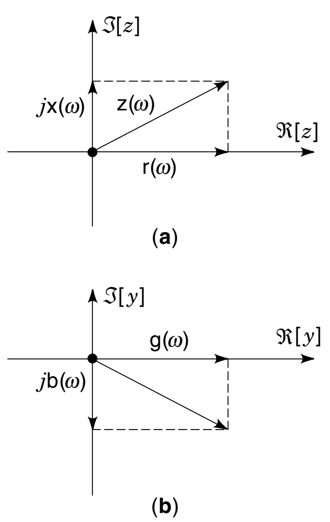

Figure 2. Real and imaginary parts of (a) impedance and (b) admittance.

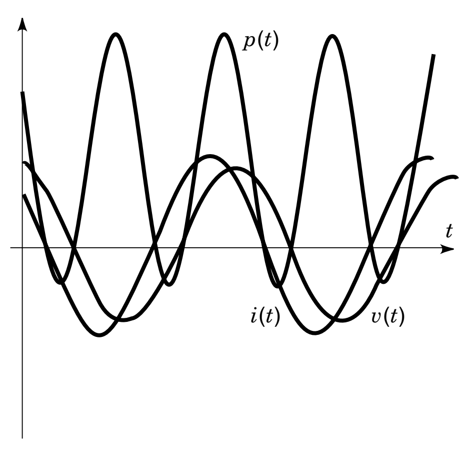

Figure 3. Instantaneous power of a generic one-port element.

interval long enough:

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t)\mathrm{d}t \quad \text{for } (t_2 - t_1) \to \infty \tag{14}$$

In SSS active power P assumes a compact and popular form since it coincides with the constant term appearing in Eq. (11):

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t)\,\mathrm{d}t = (v_0 i_0/2)\cos(\phi_v - \phi_i) \tag{15}$$

with $(t_2 - t_1) \gg \hat{T}/2$ or $(t_2 - t_1) = \mu\hat{T}/2$ (where μ is an arbitrary integer) because $p(t)$ is periodic with period $\hat{T}/2$.

Active Power Theorem. Recalling the power theorem in Eq. (13), and averaging both sides of the equation that fixes at zero the sum of instantaneous powers, one obtains that the sum of active powers over all the K elements of a circuit is zero:

$$\sum_{k=1}^{K} P_k = 0 \tag{16}$$

Definition. For a given one-port element, the factor $\cos(\phi) = \cos(\phi_v - \phi_i)$, appearing in Eq. (15), is called *power factor,* where $\phi = \phi_v - \phi_i$ coincides with the phase of the impedance of the one-port defined in Eq. (8). By using the definition in Eq. (2) and Eq. (15), active power P exchanged through a port is equal to the popular formula:

$$P = v^{\mathrm{eff}} i^{\mathrm{eff}} \cos(\phi) \tag{17}$$

Other equivalent expressions for active power are often used: the first is equal to one-half the voltage amplitude v times the component $i_c = i_0 \cos(\phi)$ of current $\bar{i}$ in phase with $\bar{v}$, while the second is equal to one-half the current amplitude i times the component $v_c = v_0 \cos(\phi)$ of voltage $\bar{v}$ in phase with $\bar{i}$: $P = v_0 i_c/2$ and $P = v_c i_0/2$.

In practice, active power is a measure of the absorbed or delivered electrical energy in a unit time interval (see POWER MEASUREMENT).

Complex Power

A definition, whose significance will be clarified later, is now introduced, relating to a quantity that depends upon the product of the voltage phasor with the conjugate of that of the current. A priori this product does not have a physical meaning, since it is an unspecified operation in phasor theory [see Eqs. (4–6)].

Definition. The *complex power* $\overline{P}$ in one-port elements is defined as one-half the product of the voltage phasor $\bar{v}$ and the conjugate of the current phasor $\bar{i}^*$:

$$\begin{aligned} \overline{P} &= (1/2)\bar{v}\bar{i}^* = \Re[\overline{P}] + j\Im[\overline{P}] \\ &= (v_0 i_0/2)\cos(\phi_v - \phi_i) + j(v_0 i_0/2)\sin(\phi_v - \phi_i) \end{aligned} \tag{18}$$

Note that the real part of complex power $\Re[\overline{P}]$ coincides with the active power in Eq. (15), while the imaginary part $\Im[\overline{P}]$ will be discussed later on.

Introducing the impedance z or the admittance y of the one-port, and their real and imaginary parts in Eq. (8), two different and popular expressions of complex power are obtained:

$$\begin{aligned} \overline{P} &= \tfrac{1}{2} z\bar{i}\bar{i}^* = P + j\Im[\overline{P}] = (1/2)ri_0^2 + j(1/2)xi_0^2 \\ \overline{P} &= \tfrac{1}{2} y^*\bar{v}\bar{v}^* = P + j\Im[\overline{P}] = (1/2)gv_0^2 - j(1/2)bv_0^2 \end{aligned} \tag{19}$$

The introduction of the complex power is justified by the following theorem:

Complex Power Theorem. The sum of complex powers over all K elements of a circuit is null:

$$\sum_{k=1}^{K} \overline{P}_k = 0 \tag{20}$$

Proof. With reference to Table 3, Kirchhoff's laws may be written as: $A\bar{\boldsymbol{i}} = 0$ or, equivalently, $A\bar{\boldsymbol{i}}^* = 0$ and $\bar{\boldsymbol{v}} = A^{\mathrm{T}}\bar{\boldsymbol{v}}$. Computing the scalar product of $\bar{\boldsymbol{v}}$ and $\bar{\boldsymbol{i}}^*$, one obtains:

$$\sum_{k=1}^{K} \overline{P}_k = \tfrac{1}{2}\bar{\boldsymbol{v}}_0^{\mathrm{T}}\bar{\boldsymbol{i}}_0^* = \tfrac{1}{2}[A^{\mathrm{T}}\bar{\boldsymbol{v}}_0]^{\mathrm{T}}\bar{\boldsymbol{i}}_0^* = \tfrac{1}{2}\bar{\boldsymbol{v}}_0^{\mathrm{T}}[A\bar{\boldsymbol{i}}_0^*] = 0 \tag{21}$$

From the above proof, since the sum of active powers over all elements of a circuit coincides with the real part of the sum of complex powers, it is again proved that the sum of active powers equals zero [see Eq. (16)].

Definition. The modulus of complex power is called *apparent power* and is symbolized as A:

$$A = |\overline{P}| = \sqrt{P^2 + Q^2} = (v_0 i_0/2) \tag{22}$$

In general, the sum of all apparent powers extended to all elements of a circuit is not null.

Reactive Power

The focus will now be on the imaginary part of complex power $\Im[\overline{P}]$ in Eqs. (18) and (19).

Definition. The imaginary part of complex power $\Im[\overline{P}]$, denoted by Q

$$Q = \Im[\overline{P}] = (v_0 i_0/2)\sin(\phi_v - \phi_i) \tag{23}$$

is called *reactive power.*

By observing the factor $\sin(\phi_v - \phi_i)$ in Eq. (23), reactive power Q appears to be positive, if the voltage anticipates the current (resistive-inductive one-port), and negative otherwise (obviously $Q = 0$ if voltage and current are in phase). The resulting sign of Q is only a convention, universally accepted, due to the introduction of the conjugate of the current phasor in Eq. (18). If complex power were defined as $(1/2)\bar{v}^*\bar{i}$, its imaginary part would change sign. Reactive power Q in a capacitor or in an inductor has a strong relation with the maximum value of the instantaneous energy $w(t)$ stored in the element during one whole period.

Property. Reactive power Q and the maximum w_M of $w(t)$ are related by:

$$\begin{aligned}&\text{Capacitor:}\begin{cases}Q=-\frac{1}{2}bv_0^2=-\frac{1}{2}\hat{\omega}Cv_0^2\\ w_M=\max\limits_{\langle t\rangle}[w(t)]=\frac{1}{2}Cv_0^2\end{cases}\Rightarrow Q=-\hat{\omega}w_M\\ &\text{Inductor:}\begin{cases}Q=\frac{1}{2}xi_0^2=\frac{1}{2}\hat{\omega}Li_0^2\\ w_M=\max\limits_{\langle t\rangle}[w(t)]=\frac{1}{2}Li_0^2\end{cases}\Rightarrow Q=\hat{\omega}w_M\end{aligned}\tag{24}$$

The above physical interpretation of reactive power does not at all hold for other elements. Reactive power in a resistor is always zero, because $\sin(\phi_v - \phi_i) = 0$, while in an independent voltage or current source Q may be different from zero, even if these elements are resistive. This result is explained by considering that in independent sources the current or the voltage is unconstrained.

Reactive Power Theorem. The sum of reactive powers over all K elements of a SSS circuit is zero.

$$\sum_{k=1}^{K} Q_k = 0 \tag{25}$$

Proof. The sum of all reactive powers coincides with the imaginary part of the sum of all complex powers. The latter is zero because of Eq. (20).

Active, Reactive, and Complex Power in Two-Ports

Consider now the complex power absorbed by a two-port in the case that representation matrix $\underline{Z}$ exists. The complex power $\overline{P}$ absorbed by a two-port element has the complex quadratic form:

$$\begin{aligned}\overline{P}&=\tfrac{1}{2}\begin{bmatrix}\bar{i}_1^*\\ \bar{i}_2^*\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix}=\tfrac{1}{2}\begin{bmatrix}\bar{i}_1^*\\ \bar{i}_2^*\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}z_{11}&z_{12}\\ z_{21}&z_{22}\end{bmatrix}\begin{bmatrix}\bar{i}_1\\ \bar{i}_2\end{bmatrix}\\ &=\tfrac{1}{2}[\bar{i}_1^*\bar{i}_1z_{11}+\bar{i}_1^*\bar{i}_2z_{12}+\bar{i}_2^*\bar{i}_1z_{21}+\bar{i}_2^*\bar{i}_2z_{22}]\end{aligned}\tag{26}$$

where "T" denotes transposition.

Equating the real and imaginary parts of the two sides of Eq. (26), the active and reactive power are obtained.

$$\begin{aligned}P&=\tfrac{1}{2}\{r_{11}i_{01}^2+r_{22}i_{02}^2+(r_{12}+r_{21})\Re[\bar{i}_1^*\bar{i}_2]+\underbrace{(x_{21}-x_{12})\Im[\bar{i}_1^*\bar{i}_2]}\}\\ Q&=\tfrac{1}{2}\{x_{11}i_{01}^2+x_{22}i_{02}^2+\underbrace{(r_{12}-r_{21})\Im[\bar{i}_1^*\bar{i}_2]}+(x_{21}+x_{12})\Re[\bar{i}_1^*\bar{i}_2]\}\end{aligned}\tag{27}$$

Similar formulas hold for the other representations of two-ports.

The form of the underbraced terms $(r_{12} - r_{21})\Im[\bar{i}_1^*\bar{i}_2]$ and $(x_{21} - x_{12})\Im[\bar{i}_1^*\bar{i}_2]$ in Eq. (27) denotes the following properties.

Property. A two-port with a pure imaginary impedance and/or admittance matrix does not absorb or deliver active power if and only if it is reciprocal (i.e., $x_{12} = x_{21}$).

Property. A two-port with a pure real impedance and/or admittance matrix does not absorb or deliver reactive power if and only if it is reciprocal (i.e., $r_{12} = r_{21}$).

Both active power P and reactive power Q in the four controlled sources and in the nullor may have any value in $(-\infty, +\infty)$ (see LINEAR NETWORK ELEMENTS and Table 5). Indeed, these five two-port elements are characterized by having the voltage and/or current of the output port unconstrained.

In an ideal transformer P and Q are both zero independently of the rest of the circuit. In fact, the instantaneous power is null and the ideal transformer is reciprocal.

Compute the reactive power absorbed by a gyrator (see Table 5). In general, it may have any value, even if the instantaneous power absorbed is always zero, since the gyrator is antireciprocal. This apparent paradox may be verified by applying Eq. (27) to the impedance matrix of a gyrator.

$$P = \tfrac{1}{2}\{(r_m - r_m)\Re[\bar{i}_1^*\bar{i}_2]\} = 0 \quad Q = -r_m\Im[\bar{i}_1^*\bar{i}_2] \tag{28}$$

where r_m is the gyration transresistance. The result in Eq. (28) shows why the first port of a gyrator with the second port closed on a capacitor (which absorbs negative reactive power) is equivalent to an inductor (which absorbs positive reactive power).

NETWORK FUNCTIONS

The previous sections considered circuits operating in SSS with a fixed angular frequency $\hat{\omega}$. Now consider the properties of these circuits by considering the angular frequency ω as an arbitrary variable of the problem. Toward this aim it is necessary to introduce the network functions of a circuit in SSS (4).

Definition of Network Functions

Set at zero the value of any sinusoidal source voltage and current in the circuit except the source voltage or current, generically denoted by $\hat{u}(t)$, which is considered to be an *input variable* of the circuit. Any sinusoidal branch voltage or current, generically denoted by $y(t)$, may be chosen as *output variable.*

Definition. A *network function* is the quotient of the output phasor $\bar{y}$ of $y(t)$ by the input phasor $\bar{\hat{u}}$ of $\hat{u}(t)$.

Circuit linearity causes the above quotient to depend only on the angular frequency ω of the source, and not on its phasor $\bar{\hat{u}}$; so $j\omega$ is the argument of the network function since it appears in the constitutive relations of any dynamical element. Obviously since, in general, a circuit may have more independent sources (inputs) and many branch voltages or currents that may be considered as output, several different network functions may be defined in any dynamic circuit. In a general situation, it is possible to define network functions as the quotient of the generalized phasors $\bar{y}$ and $\bar{\hat{u}}$ of the complex exponential functions $y(t) = \Re[\bar{y}\exp(st)]$ and $\hat{u}(t) = \Re[\bar{\hat{u}}\exp(st)]$, characterized by complex frequency $s = \sigma + j\omega$. Equivalently, network functions may be defined as the quotient of the Laplace transforms of the same quantities (see FREQUENCY-DOMAIN CIRCUIT ANALYSIS). In these cases the network function is a complex valued function $F(s)$ of the complex variable s: $F(s)$ results to be the quotient of two polynomials with real coefficients. This property can be shown by considering the solution of the linear system in Eq. (7) obtained using the Kramer rule: the denominator of $F(s)$ coincides with the determinant of the matrix of the system, while the numerator coincides with the determinant of a suitable submatrix, less possible common factors that cancel out. The roots of the numerator polynomial are the *zeroes* of the network function, while the roots of the denominator are the *poles.* Zeroes and poles may be real or complex conjugate pairs: their values characterize, less a constant factor, the network function and, in particular, its behavior along the imaginary axis (see the section titled Logarithmic Scales and Bode Plots and the subsection titled Factorization of Network Functions, later in this article). If $F(s)$ is evaluated along the imaginary axis, that is, for $s = j\omega$, the network function $F(j\omega)$ defined in SSS is obtained.

Classes of Network Functions

In any network function in SSS the output phasor $\bar{y}$ may be any branch current or voltage, and the input phasor $\hat{u}$ may be any source voltage or current. Any network function may then be seen either as the admittance or impedance of a composite one-port element, or as an off-diagonal element of the impedance, admittance, or hybrid matrix [see Eq. (10)] of a two-port subnetwork extracted from the circuit (see LINEAR NETWORK ELEMENTS). One may then define the following classes of network functions:

- *Impedance Function.* The quotient of the voltage phasor $\bar{v}$ of a current source by the current phasor $\hat{i}$ of the source itself (see Fig. 4a)
- *Admittance Function.* The quotient of the current phasor $\bar{i}$ of a voltage source by the voltage phasor $\hat{v}$ of the source itself (see Fig. 4b)
- *Transimpedance Function.* The quotient of any voltage phasor by any source current phasor
- *Transadmittance Function.* The quotient of any current phasor by any source voltage phasor
- *Voltage Gain Function.* The quotient of any voltage phasor by any source voltage phasor
- *Current Gain Function.* The quotient of any current phasor by any source current phasor

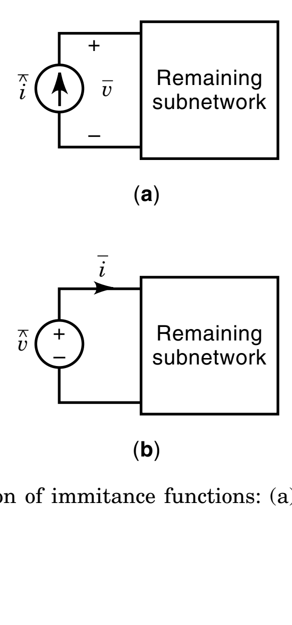

Figure 4. Definition of immitance functions: (a) impedance and (b) admittance.

Impedance and admittance functions are jointly called *immittance* functions, from the contraction of the terms *im*pedance and ad*mittance*. The last four network functions in the above list are called *transfer* functions, because the input and output are related to two different branches of the circuit.

Magnitude and Phase of Network Functions

In SSS, network functions are, in general, complex valued functions of the imaginary variable $j\omega$, that is, any network function may be written as $F(j\omega)$. Using complex number mathematics, it is possible to derive from network function $F(j\omega)$, two real-valued functions: magnitude $|F(j\omega)|$ and phase $\beta^F(\omega) = \angle F(j\omega)$. It is possible to better examine these real-valued functions by splitting the numerator $N(s)$ and the denominator $D(s)$ of $F(s)$ into even and odd parts:

$$N(s) = N_2(s^2) + sN_1(s^2) \quad D(s) = D_2(s^2) + sD_1(s^2) \tag{29}$$

where $N_2(s^2)$ and $D_2(s^2)$ contain the even terms of polynomials $N(s)$ and $D(s)$, while $sN_1(s^2)$ and $sD_1(s^2)$ contain the odd terms. By substituting $s \to j\omega$, one can decompose both numerator and denominator into real and imaginary parts:

$$F(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{N_2(-\omega^2) + j\omega N_1(-\omega^2)}{D_2(-\omega^2) + j\omega D_1(-\omega^2)} \tag{30}$$

The magnitude $|F(j\omega)|$ is an even function of ω:

$$|F(j\omega)| = \sqrt{\frac{[N_2(-\omega^2)]^2 + [\omega N_1(-\omega^2)]^2}{[D_2(-\omega^2)]^2 + [\omega D_1(-\omega^2)]^2}} \tag{31}$$

It is often preferable to use the *squared magnitude* $|F(j\omega)|^2$ because it is rational in ω^2. For this reason it is used to solve approximation problems in filter design (see FILTER APPROXIMATION METHODS; ANALOG FILTERS).

The phase function $\beta^F(\omega)$ may be computed using the numerator and denominator of the network function:

$$\beta^F(\omega) = \angle F(j\omega) = \angle N(j\omega) - \angle D(j\omega) \tag{32}$$

defined less an arbitrary integer multiple of 2π. The phase function is odd symmetric, $\angle F(j\omega) = -\angle F(-j\omega)$, with respect to ω, because the substitution $j\omega \to -j\omega$ causes the change of sign of the imaginary parts of $N(j\omega)$ and $D(j\omega)$. In general, the phase function $\beta^F(\omega)$ is continuous in ω, except in correspondence of pure imaginary conjugates zeroes or poles, where phase has a $\pm k\pi$ discontinuity, where k is the multiplicity of the zeroes or poles, including possible poles or zeroes in the origin.

Phase of the Immittance of One-Port Elements

The phase of an immittance function is important in classifying one-port elements; to this end the terminology reported in Table 7 and illustrated in Fig. 5 is used.

Properties. It may be easily seen that any one-port subnetwork containing only resistors and inductors is resistive-inductive for any value of ω, because this subnetwork absorbs nonnegative reactive power for any ω. Analogously, any one-port will be resistive-capacitive if it is built using only capacitors and resistors. Subnetworks containing resistors, capaci-

Table 7. Phases of Immittances and Related Classification

One-Port Element Class	Phase	Comment
Inductive	$\angle z(j\omega) = -\angle y(j\omega) = \pi/2$	$\bar{v}$ anticipates in quadrate $\bar{\imath}$
Resistive-inductive	$0 < \angle z(j\omega) = -\angle y(j\omega) < \pi/2$	$\bar{v}$ anticipates $\bar{\imath}$
Resistive	$\angle z(j\omega) = -\angle y(j\omega) = 0$	$\bar{v}$ and $\bar{\imath}$ are in phase
Resistive-capacitive	$0 > \angle z(j\omega) = -\angle y(j\omega) > -\pi/2$	$\bar{\imath}$ anticipates $\bar{v}$
Capacitive	$\angle z(j\omega) = -\angle y(j\omega) = -\pi/2$	$\bar{\imath}$ anticipates in quadrate $\bar{v}$

tors, and inductors will be, in general, resistive-capacitive in some frequency intervals and resistive-inductive in others. The series and parallel resonators are of this type (see the section titled Resonance, later in this article).

Factorization of Network Functions

Since any network function is rational, it may be factorized in order to evidence poles and zeroes.

$$F(j\omega) = h \times (j\omega)^{\mu_0} \times \frac{\overbrace{\prod_{\nu=1}^{K_{zr}}[1 + j\omega/\sigma_{z\nu}]}^{\text{Real zeroes}} \overbrace{\prod_{\nu=1}^{k_{zc}}[1 + j\omega/(q_{z\nu}\omega_{z\nu}) + (j\omega/\omega_{z\nu})^2]}^{\text{Complex conjugate zeroes}}}{\underbrace{\prod_{\nu=1}^{K_{pr}}[1 + j\omega/\sigma_{p\nu}]}_{\text{Real poles}} \underbrace{\prod_{\nu=1}^{k_{pc}}[1 + j\omega/(q_{p\nu}\omega_{p\nu}) + (j\omega/\omega_{p\nu})^2]}_{\text{Complex conjugate poles}}} \quad (33)$$

Where h is the real constant factor and $|\mu_0|$ is the number of zeroes in the origin if $\mu_0 > 0$ or the number of poles in the origin if $\mu_0 < 0$. K_{zr} and K_{pr} are the number of real zeroes and poles, excluding those in the origin, while K_{zc} and K_{pc} are the number of complex conjugate zero and pole pairs. Parameters $-\sigma_{z\nu}$ and $-\sigma_{p\nu}$ are the νth real zero and pole, $\omega_{z\nu}$ and $\omega_{p\nu}$ are the modulus of the νth pair of complex conjugate zeroes and poles. Parameters $q_{z\nu}$ and $q_{p\nu}$ of the νth pair of complex conjugate zeroes and poles are strictly related to the phase ϕ of complex conjugate poles or zeroes: $q = 1/[2\,\sin(\phi - \pi/2)]$, where ϕ is the phase of the complex zero or pole with positive imaginary part. This formula shows that the q parameter has, for complex conjugate pairs of poles or zeroes, an absolute value greater than 0.5.

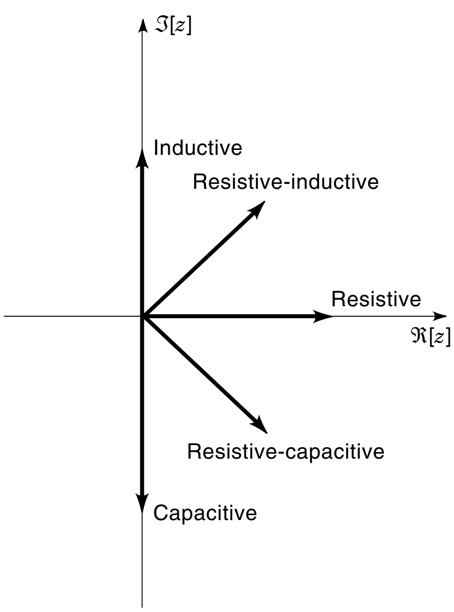

Figure 5. Impedances in complex plane.

Definition. When the degree of numerator and the degree of denominator are not equal, the network function has a zero or pole at infinity. Introducing the integer parameter μ_∞ as the difference in degree,

$$\mu_\infty = -\mu_0 + K_{pr} + 2K_{pc} - K_{zr} - 2K_{zc} \quad (34)$$

it may be easily seen that

$$\mu_\infty > 0{:}\ F(j\omega) \to 0 \text{ of order } \mu_\infty \text{ if } \omega \to \infty$$
$$\mu_\infty < 0{:}\ F(j\omega) \to \infty \text{ of order } |\mu_\infty \text{ if } \omega \to \infty$$
$$\mu_\infty = 0{:}\ \text{no zeroes or poles of } F(j\omega) \text{ at infinity}$$

LOGARITHMIC SCALES AND BODE PLOTS

Often the magnitude and phase of a network function are most easily analyzed if logarithmic scales and logarithmic quantities are adopted. In particular, plots are usually more readable, the numbers involved in practical calculations are more manageable, and the magnitude function may be easily decomposed in simple addends.

Logarithmic Scale for Angular Frequency

In the practical analysis of network functions it is often necessary to evaluate the magnitude or phase of the function in many different values of ω, differing by several orders of magnitude. In this case, if a linear scale for ω is used to represent magnitude and phase of a function, the resulting plot may be quite unreadable—too compressed for small values of ω, and too expanded for high values. To avoid the problems mentioned above, a logarithmic transform of the ω axis is adopted: the angular frequency ω is normalized with respect to $\omega_0 = 2\pi f_0$ and the base 10 logarithm is introduced:

$$\omega \to \log(\omega/\omega_0) = \log(f/f_0) \quad (35)$$

With the above scale a *decade* is a unit length interval of the logarithmic quantity just defined, that is, an interval where ω, and analogously f, vary by a factor of 10.

Logarithmic Scale for Magnitude Functions

The magnitude $|F(j\omega)|$ or the squared magnitude of a network function may have values differing by several orders of magnitude, even for small variations of ω. If a linear scale was used, the magnitude plots of some network functions would be poorly readable. To overcome this problem a logarithmic transform is used: the magnitude of the network function $|F(j\omega)|$ is substituted by the *attenuation* $\alpha^F(\omega)$. When (trans)-impedance or (trans)admittance functions are considered, they must be normalized with respect to a conventional resistance. The base unit is called "decibel" (dB):

$$\alpha^F(\omega) = 20\log(1/|F(j\omega)|) = -20\log(|F(j\omega)|) \tag{36}$$

Depending on the application, it is possible to use, instead of the attenuation $\alpha^F(\omega)$ in dB, the *gain* defined as the negative of the attenuation.

Definitions. The diagram obtained by representing the attenuation $\alpha^F(\omega)$ (or the gain in dB) on the y-axis and $\log(\omega/\omega_0)$ on the x-axis is called the *magnitude Bode plot.* The *phase Bode plot* $\beta^F(\omega)$ is obtained by representing the phase on the y-axis with the usual linear scale and $\log(\omega/\omega_0)$ on the x-axis.

Decomposition of Attenuation (Gain) and Phase Functions

From complex number mathematics it is known that the modulus of the product or quotient of two complex numbers is equal to the product or quotient of their moduli. For this reason the factorization of a network function, shown in Eq. (33), is appropriate also for the corresponding magnitude $|F(j\omega)|$.

If the attenuation or gain of $|F(j\omega)|$ is considered and logarithmic scales are introduced, the factorization of the magnitude of a network function is transformed into a sum or difference of terms. Each term is the attenuation or gain of a factor of the numerator or denominator polynomials of the network function in Eq. (33), and carries information regarding a single real zero or pole, or a complex conjugate pair of zeroes or poles, respectively. Thus it is possible to obtain the Bode plot of the attenuation or gain as the addition of the simple plots relating to each single term. For the attenuation:

$$\begin{aligned}\alpha^F(\omega) = {} & -10\log[|F(j\omega)|^2] = -10\log(h^2) - \mu_0 10\log(\omega^2) + \\ & -10\sum_{\nu=1}^{K_{zr}}\log[1+(\omega/\sigma_{z\nu})^2] + \\ & -10\sum_{\nu=1}^{K_{zc}}\log[(1-(\omega/\omega_{z\nu})^2)^2 + (\omega/(q_{z\nu}w_{z\nu}))^2] + \\ & +10\sum_{\nu=1}^{K_{pr}}\log[1+(\omega/\sigma_{p\nu})^2] + \\ & +10\sum_{\nu=1}^{K_{pc}}\log[(1-(\omega/\omega_{p\nu})^2)^2 + (\omega/(q_{p\nu}\omega_{p\nu}))^2]\end{aligned} \tag{37}$$

The plots of a single first- or second-degree factor, both of the numerator and denominator of the network function, are called *elementary Bode plots.*

The phase of the product or quotient of two complex numbers is equal, respectively, to the sum or difference of the phase of the single factors. In this case it is not necessary to use a logarithmic scale, as for magnitude, to expand the phase as a sum of terms. If the numerator and denominator polynomials of a network function $F(j\omega)$ are decomposed into the first- and second-degree factors, the following property is obtained:

Property. The phase $\angle F(j\omega)$ of a network function is equal to the sum (for the numerator factors) and the difference (for the denominator factors) of the phases of the single first- and second-degree factors:

$$\begin{aligned}\beta^F(\omega) = \angle F(j\omega) = {} & \mu_0\pi/2 + \sum_{\nu=1}^{K_{zr}}\angle(1+j\omega/\sigma_{z\nu}) + \\ & + \sum_{\nu=1}^{K_{zc}}\angle\left[1 + j\frac{\omega/(q_{z\nu}\omega_{z\nu})}{1-(\omega/\omega_{z\nu})^2}\right] - \sum_{\nu=1}^{K_{pr}}\angle(1+j\omega/\sigma_{p\nu}) + \\ & - \sum_{\nu=1}^{K_{pc}}\angle\left[1 + j\frac{\omega/(q_{p\nu}\omega_{p\nu})}{1-j(\omega/\omega_{p\nu})^2}\right]\end{aligned} \tag{38}$$

If in Eq. (33) h is negative, a constant contribute of $\pm\pi$ must be added to phase in Eq. (38).

RESONANCE

Resonance is a very important phenomenon in many fields of physics. Resonant circuits have played a relevant role in communication systems since their origin. They are of series and parallel type and may be divided into ideal and nonideal types (4).

Ideal Resonators

Ideal resonators are composed by the series or parallel connection of a capacitor and an inductor. Their immitance functions are:

$$\begin{aligned}&\text{Ideal series resonator}\\ &z(j\omega) = 0 + jx(\omega) = j\left[\omega L - \frac{1}{\omega C}\right]\\ &\text{Ideal parallel resonator}\\ &y(j\omega) = 0 + jb(\omega) = j\left[\omega C - \frac{1}{\omega L}\right]\end{aligned} \tag{39}$$

Both reactance $x(\omega)$ and susceptance $b(\omega)$ are monotone increasing with respect to ω in $(-\infty, \infty)$. When $\omega = \omega_0 = 1/\sqrt{LC}$, called *resonance angular frequency,* both $x(\omega)$ and $b(\omega)$ are null because the reactance and susceptance of capacitor and inductor cancel out. In other words, at resonance the series resonator is equivalent to a short circuit and the parallel resonator to an open circuit. Analogously, frequency $f_0 = \omega_0/(2\pi)$ is called *resonance frequency.*

For $\omega \gg \omega_0$ and $\omega \ll \omega_0$ the resonators are equivalent to a single element:

$$\begin{aligned}&z(j\omega) = jx(\omega) \simeq -j\frac{1}{\omega C} \quad \text{for} \ll \omega_0\\ &z(j\omega) = jx(\omega) \simeq j\omega L \quad \text{for} \gg \omega_0\\ &y(j\omega) = jb(\omega) \simeq -j\frac{1}{\omega L} \quad \text{for}\ \omega \ll \omega_0\\ &y(j\omega) = jb(\omega) \simeq j\omega C \quad \text{for} \gg \omega_0\end{aligned} \tag{40}$$

The previous results can be revisited in time domain.

In the ideal series resonator, the voltage over the capacitor $v_c(t)$ and the voltage over the inductor $v_l(t)$ coincide instant-by-instant less the sign: $v_c(t) = -v_l(t)$, while the corresponding currents $i_c(t)$ and $i_l(t)$ are coincident.

In the ideal parallel resonator, the current through the capacitor $i_c(t)$ and the current through the inductor $i_l(t)$ coincide instant-by-instant less the sign: $i_c(t) = -i_l(t)$, while the corresponding voltages $v_c(t)$ and $v_l(t)$ are coincident.

So, voltage over a series resonator and current in a parallel resonator are zero, and instantaneous power $p(t)$ exchanged with the rest of the circuit is zero. Consequently, the sum of the energies stored in the capacitor and in the inductor is constant. Therefore, the exchange of instantaneous power takes place only between the inductor and the capacitor inside the ideal resonator.

Lossy Resonators

Since the model of an ideal resonator is equivalent, at the resonance frequency, to an ideal short circuit or open circuit, a more realistic model might be needed in many situations. For instance, if a sinusoidal voltage source, with angular frequency ω_0, is connected to an ideal series resonator with resonance frequency equal to ω_0, the model of the circuit is inconsistent in SSS. This model becomes consistent if the resonator is assumed to be nonideal, that is, with a very small, but nonzero impedance at $\omega = \omega_0$.

The nonideal model of a series/parallel resonator may be characterized by a series/parallel resistor added to the corresponding ideal model (Fig. 6) and it is called a *lossy series/parallel resonator.* In the series case a very small resistance r value is chosen, while in the parallel case a small conductance $1/r$ is chosen, and so a large resistance value is used. For any ω the nonideal model is not equivalent to a short or open circuit. The admittance of lossy series resonator and the impedance of lossy parallel resonator may be easily analyzed:

$$\begin{aligned} y(j\omega) &= \frac{1}{r + j\omega L + 1/(j\omega C)} \\ z(j\omega) &= \frac{1}{1/r + j\omega C + 1/(j\omega L)} \end{aligned} \tag{41}$$

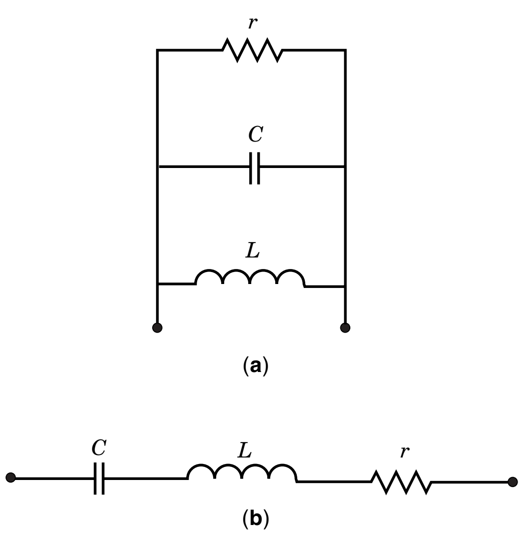

Figure 6. Lossy (a) series and (b) parallel resonators.

In lossy resonators the resonance angular frequency $\omega_0 = 1/\sqrt{LC}$ is again introduced as in the ideal case, although the physical meaning is somewhat different: ω_0 is the angular frequency at which the admittance or impedance are pure real and coincide with that due to the embedded resistor: $y(j\omega_0) = 1/r$ for the series resonator and $z(j\omega_0) = r$ for the parallel one. For ω distant from ω_0 the approximated formulas in Eq. (40) also hold for lossy resonators. So, the effect of the added resistor is relevant only when ω is close to ω_0.

Rewrite $y(j\omega)$ and $z(j\omega)$ in Eq. (41) by introducing the normalized angular frequency $\Omega = \omega/\omega_0$ and by normalizing them with respect to resistance r; one obtains the normalized immittances $F_s(j\Omega)$ and $F_p(j\Omega)$:

$$\begin{aligned} F_s(j\Omega\omega_0) &= ry(j\Omega\omega_0) = \frac{1}{1 + jQ^s(\Omega - 1/\Omega)} \\ F_p(j\Omega\omega_0) &= z(j\Omega\omega_0)/r = \frac{1}{1 + jQ^p(\Omega - 1/\Omega)} \end{aligned} \tag{42}$$

Factors Q^s and Q^p in Eq. (42) are defined as $Q^s = r_0/r$ and $Q^p = r/r_0$ with $r_0 = \sqrt{L/C}$. For the series resonators Q^s is also equal to the absolute value of the quotient of inductor or capacitor impedance, at ω_0, by the resistance of the resistor: $Q^s = (\omega_0 L)/r = 1/(\omega_0 Cr)$. For the parallel resonator Q^p is also equal to the absolute value of the quotient of the capacitor or inductor admittance, at ω_0, by the conductance of the resistor: $Q^p = \omega_0 Cr = r/(\omega_0 L)$.

The energy exchange of a lossy resonator with the remaining part of the circuit is not zero as in the ideal case. However, if the Q factor is high, the energy dissipated inside the resonator during each whole period $2\pi/\omega_0$ is a small fraction of the total energy stored in the capacitor and inductor.

Normalized Immittance of Lossy Resonators

The expressions in Eq. (42) of the normalized admittance $F_s(j\Omega)$ and the normalized impedance $F_p(j\Omega)$ are equivalent. Then, for both resonators, the unique normalized immittance function $F(j\Omega)$ is introduced:

$$F(j\Omega) = \frac{1}{1 + jQ(\Omega - 1/\Omega)} \tag{43}$$

where Q may be either Q^s or Q^p.

The maximum value of the magnitude $|F(j\Omega)|$ of $F(j\Omega)$ in Eq. (43) occurs for $\Omega = 1$, where the imaginary part $jQ(\Omega - 1/\Omega)$ is zero. So, the magnitude function is bell-shaped.

Property. By substituting $\Omega \to 1/\Omega$ in Eq. (43), note that any pair of values $F(j\Omega)$ and $F(j/\Omega)$ satisfies the relation

$$F(j\Omega) = F(-j/\Omega) = [F(j/\Omega)]^* \quad \forall\Omega \tag{44}$$

So, in complex plane each pair of points $F(j\Omega)$ and $F(j/\Omega)$ is symmetric with respect to the real axis, since they have the same real part, but opposite imaginary part. This property is, in general, regarded as *geometric symmetry.*

The Nyquist plot of $F(j\Omega)$ (see NYQUIST CRITERION, DIAGRAMS, AND STABILITY) is a complete circle (Fig. 7) with the segment $0 \leftrightarrow 1$ on the real axis as a diameter; for Ω increasing from 0 to $+\infty$ point $F(j\Omega)$ describes the circle clockwise, starting and ending in the origin.

Magnitude and Phase Functions of Lossy Resonators

Consider the magnitude $|F(j\Omega)|$ and phase $\angle F(j\Omega)$ of lossy resonators. The geometric symmetry of $F(j\Omega)$ implies that $|F(j\Omega)|$ is geometrically *even symmetric,* while $\angle F(j\Omega)$ is geometrically *odd symmetric,* with respect to resonance frequency $\Omega = 1$. For increasing values of Q factor this geometric even or odd symmetry tends, respectively, to arithmetic even or odd symmetry for values of Ω close to resonance. If the Bode plots are drawn by adopting a logarithmic scale for Ω on the abscissa, the previous geometric symmetries become arithmetic symmetries.

Consider now the normalized frequencies Ω_1 and Ω_2 marked in Fig. 7. The geometric symmetry states that $\Omega_1\Omega_2 = 1$. By means of a simple inspection of Nyquist plot, both Ω_1 and Ω_2 satisfy the relations $\Re[F(j\Omega)] = \pm\Im[F(j\Omega)]$ and $|F(j\Omega)| = 1/\sqrt{2}$, that is,

$$\Omega_1 - 1/\Omega_1 = -1/Q \quad \Omega_2 - 1/\Omega_2 = 1/Q \tag{45}$$

By subtracting the first equation from the second one, one obtains:

Property. The normalized frequencies Ω_1 and $\Omega_2 = 1/\Omega_1$ satisfy the following relations:

$$\Omega_2 + 1/\Omega_1 - 1/\Omega_2 - \Omega_1 = 2/Q \Rightarrow \Omega_2 - \Omega_1 = 1/Q \tag{46}$$

The difference $\Omega_2 - \Omega_1 = (\omega_2 - \omega_1)/\omega_0$ is the so-called *relative bandwidth* of lossy resonators and so Eq. (46) shows that factor Q is a measure of the selectivity of the immittance magnitude of lossy resonators. Higher Q factors correspond to a narrower relative band $\Omega_2 - \Omega_1$, and to resonators closer to the ideal case. The magnitude of the immittance function of lossy resonators is bell-shaped and it is called *band-pass* (Fig. 8).

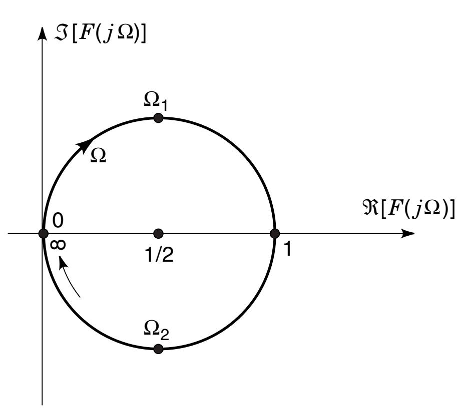

Figure 7. Nyquist plot of normalized immittance of lossy resonators.

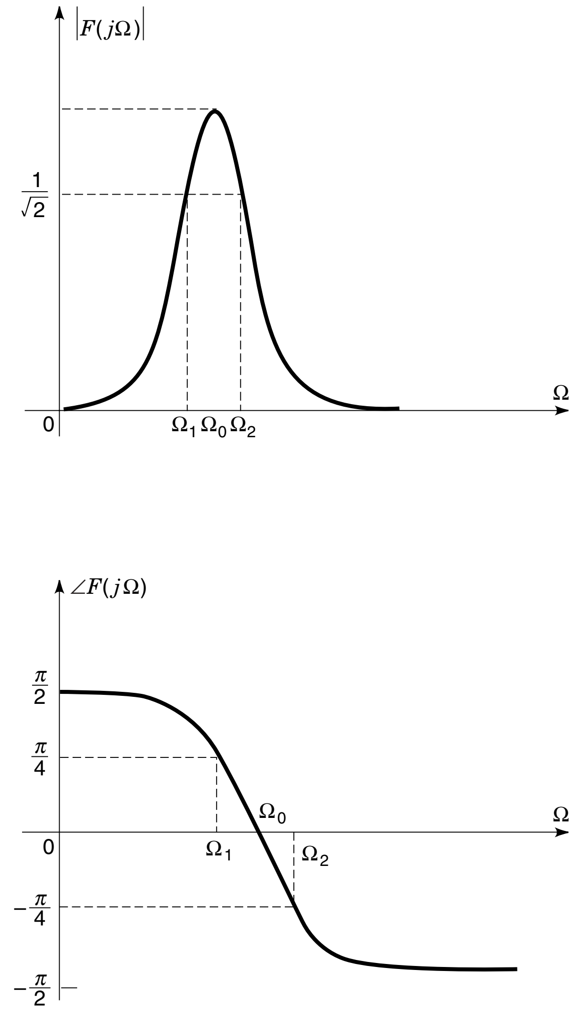

Figure 8. Plots of magnitude and phase of normalized immittance of lossy resonators.

Property. The phase Bode plot of $F(j\Omega)$ in Fig. 8 depends on Q factor of resonator:

$$\angle F(j\Omega) = -\arctan[Q(\Omega - 1/\Omega)] \Rightarrow$$
$$\left[\frac{d\angle F(j\Omega)}{d\Omega}\right]_{\Omega=1} = -2Q \Rightarrow Q = -\frac{1}{2}\left[\frac{d\angle F(j\Omega)}{d\Omega}\right]_{\Omega=1} \tag{47}$$

The phase decreases from $\pi/2$ to $-\pi/2$, it is null in $\Omega = 1$, and it has a derivative in $\Omega = 1$ with absolute value $\rightarrow \infty$ for $Q \rightarrow \infty$. A higher selectivity of magnitude function corresponds to a phase function with higher slope. Note that the Q factor coincides with the parameter q introduced in the second-order terms derived from the factorization of network functions in Eq. (33).

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NETWORK ANALYSIS, TIME-DOMAIN.

See TIME-DOMAIN NETWORK ANALYSIS.