LINEAR NETWORK ELEMENTS

Within the limits of circuit model approximation the operating conditions for each component of an electrical network are identified univocally by the currents and the voltages at their terminals. The voltages and currents of each component are linked by relations that depend only on their physical constitution. Such relations are not, however, capable of determining the actual operating conditions of the individual components—that is, the values the voltages and currents assume when a component is put in a given circuit. This indetermination disappears if one takes into account the fact that, again within the circuit model approximation, each component of the electrical network interacts with the other components through the currents and the voltages at their terminals. More precisely, one may say that the operating conditions of each component, and thus of the whole circuit, are the result of two distinct requirements: that the component should behave in a manner compatible with the nature of the other components of the network, as they are connected, and that its behavior, in turn, should be compatible with its specific nature.

While the interaction of the single component with the rest of the circuit is regulated by Kirchhoff's laws and is dealt with in TIME-DOMAIN CIRCUIT ANALYSIS and NETWORK EQUATIONS, the specific nature of each single component is manifested through particular relations imposed between the voltages and the currents at their terminals. These are the *constitutive relations* and are the subject of this article.

This distinction, which is fully justified at the conceptual level and is also used in practice, cannot, of course, be interpreted in an absolute manner. One cannot speak of electrical networks without referring to the nature of their components, just as one cannot deal with circuit components without considering that they are inserted in a network. Thus a certain overlapping of topics in this article and the others mentioned is inevitable. What is substantially different in them is the viewpoint from which these topics are considered.

As long as there is more than one, there can be any number of terminals in a circuit component. For the sake of simplicity, we will begin by speaking of components with only two terminals, also called one-ports. The one-port operation is characterized by a single current and a single voltage, and its constitutive relation expresses the link between them. The extension to components with more than two terminals does not raise any question of principle.

Even if the definitions of voltage and current for one-ports are deemed to be known by the reader of this article, it is useful to recall them briefly so as to be clear as to the limits within which they can be introduced. The limits are in fact those of the validity of the circuit model. When one speaks of electric currents in the theory of circuits, one is, in effect, committing a venial linguistic sin. One should more rightly use the term *intensity of electric current.* To be clear, let us suppose that within a region of space there are carriers of electric charges with assigned number of charge particles per unit volume N (numerical density) and velocities for each type of particle. For simplicity's sake, let us assume that each carrier has the same charge q, the numerical density N is uniform, and all the charges have the same average velocity $\boldsymbol{u}$. Now let us consider a plane surface S_0 and ask ourselves what amount of charge ΔQ flows through it in a given direction during the time interval Δt. One may easily recognize that ΔQ is equal to the amount of charge in the volume enclosed by the oblique cylinder with base S_0 and height $u\Delta t\cos\alpha$ (see Fig. 1), that is,

$$\Delta Q = (qNu)\Delta t S_0 \cos\alpha \tag{1}$$

where α is the angle that the direction of motion of the charges forms with the normal $\boldsymbol{n}$ to the surface S_0, oriented in one of the two possible directions, and u is the module of the velocity $\boldsymbol{u}$. The amount of charge passing through the surface per unit time, which is known as the electric current intensity, is thus equal to

$$i = \Delta Q/\Delta t = (qNu)S_0 \cos\alpha \tag{2}$$

In the SI system the unit of the electric charge is the *coulomb* (C) and that of electric current intensity is the *ampere* (A): 1 C = 1 A · 1 s.

Introducing the current density vector field $\boldsymbol{J} = qN\boldsymbol{u}$, which describes the motion of the charges and the vector $\boldsymbol{S}_0 = S_0\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is the unit vector of the normal $\boldsymbol{n}$, we may write $i = \boldsymbol{J}\cdot\boldsymbol{S}_0$ (with the dot we indicate the scalar product). Thus the intensity of the electric current appears as the flux of the current density vector field through a given surface in a given direction. This definition can be easily extended to the

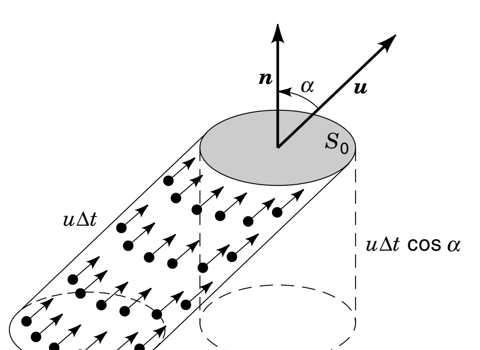


Figure 1. The particles flowing through S_0 during the time Δt are all those contained in the oblique cylinder of height $u\Delta t\cos\alpha$ and base S_0.

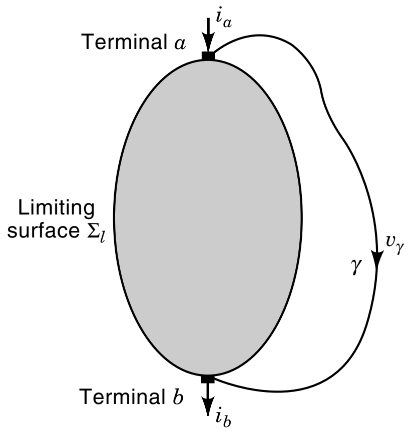


Figure 2. A one-port may be schematized as a box enclosed by the limiting surface Σ_l interacting with the rest of the circuit only through the two terminals a and b.

case where all the variables change in the time, the numerical density is not uniform, the average velocity is not the same for all the carriers, which themselves may not carry the same amount of charge, and the surface S may be of any kind. In general we have the expression

$$i_S(t) = \iint_S \boldsymbol{J} \cdot d\boldsymbol{S} \tag{3}$$

and hence the electric current intensity $i_S(t)$ flowing through a given surface S in an assigned direction—or more simply the electric current—is the flux, instant by instant, of the current density field $\boldsymbol{J}$ through the surface S according to the preselected direction.

In the light of these considerations, the fact that the operation of a circuit component with two terminals may be characterized by a single electric current requires a more precise explanation. We can consider a generic component with two terminals as a box enclosed by a limiting surface Σ_l. The component can interact with the "outside" only through two perfectly conducting regions of Σ_l that we will suppose to be tiny and call terminals (see Fig. 2). The rest of the limiting surface consists of a perfectly insulating material. Therefore it is always possible to define at least two current intensities involving the component itself: that flowing through the surface at terminal a with the normal oriented on the inlet side of Σ_l, for example, i_a; and that flowing through the surface at terminal b with the normal oriented on the outlet side, i_b. The component we are analyzing can be considered a one-port only if $i_a = i_b$.

This seemingly banal property is never, in fact, truly verified in a component, unless it is not operating under strictly steady-state conditions—that is, when the electrical variables are constant in time. In such conditions, in fact, $i_a = i_b$ is assured by Maxwell equations (for more details see Ref. 1), which establish that in steady state the flux of the electric current density field $\boldsymbol{J}$ through any closed surface must be zero. In a dynamic operation, that is, when the electrical variables vary with the time, such a condition can only be approximate: The slower the current variation in the time, the closer the approximation. This problem will be examined in greater detail later in the article.

Analogous considerations can be developed for the voltage between the two terminals of the component. In this case too, it is necessary to choose a direction, from a to b, for example, and a particular path along which to integrate the tangent component of electric field $\boldsymbol{E}$. By definition, the voltage v_γ is the line integral of $\boldsymbol{E}$ along a path γ (Fig. 2), going from terminal a to terminal b lying outside the limiting surface of the component,

$$v_\gamma = \int_\gamma \boldsymbol{E} \cdot d\boldsymbol{l} \tag{4}$$

In the SI system the unit of the electrical voltage is the *volt* (V). Thus v_γ represents the work that would be done by the electric field to carry a positive unit charge from a to b along the path γ considered. For such work to be unique the line integral of the field $\boldsymbol{E}$ will obviously have to be independent of the path; that is, the field $\boldsymbol{E}$ must be conservative. This is rigorously ensured by Maxwell's equations only in steady-state (for details see again Ref. 1). In this case the line integral of the electric field does not depend on the path chosen but only on the ends, and it may be represented by the difference between the values that a new function, the electrical potential, assumes at a and b. Thus we should use different terminology for stationary and dynamic operations. It would be better to speak of potential difference between the terminals in the case of steady-states and would be better to speak of voltage, stating along which path, for dynamic operations. Once again, in a dynamic operation, the fact that the behavior of the component may be characterized by a single voltage is merely an approximation, the more so when the electrical variables change slowly. From such considerations it is evident that one can speak of one-ports only if the voltage across the terminals may be expressed as a difference of potential or may be approximated through it. This problem, too, will be dealt with more fully later in this article. For the sake of linguistic simplicity, we will use the terms voltage and current without further specification, with the considerations that have been outlined being implicitly assumed.

There still remains the problem of choosing the reference directions for the currents and voltages: It is not possible to assign unambiguously the constitutive relations of a one-port without first identifying them.

The voltage reference direction is usually indicated by the signs + and − near the terminals: The + sign indicates the starting point and the − sign the ending of the path for the line integral of the electric field. The reference direction for the current is usually indicated with an arrow placed on one of the two terminals. Referring to the generic one-port we have already spoken of, one may see that two alternative choices exist, which are represented in Fig. 3. Suppose that a

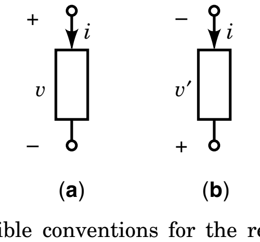


Figure 3. Two possible conventions for the reference directions of the current and voltage of a one-port: (a) "normal" convention and (b) "source" convention, $v' = -v$.

current reference direction is chosen. Then the voltage reference direction can be chosen with either the + sign or the − sign at the terminal where the current arrow enters. In the first case [Fig. 3(a)], one speaks of the *normal convention* and this is the one to which we shall implicitly refer, unless otherwise specified.

In technical language the same term is used to indicate both the physical object concretely characterized by a certain constitutive law and the ideal component that we find in circuit layouts. Naturally, in the first case the constitutive law is intended as an approximation sufficient to describe the behavior of the component in a certain set of conditions. In the second case, it is an exact law that fully describes the behavior of an ideal component extrapolated from the real one. Even if this ambiguity of language does not cause any confusion, since the limits within which the model is intended to operate are always very clear, in general one prefers to call the ideal component the *circuit element.*

This article focuses essentially on the ideal components of an electrical network. Nonetheless, we shall not fail to refer to the "physical objects" to whose characteristic they approximate, both when they are "elementary" components (e.g., as for example resistors) and when they are the result of a more complex aggregation of other elementary components (e.g., operational amplifiers or controlled sources).

The relationship between real and ideal components may be interpreted from two opposite viewpoints. One may think of starting from a real component by identifying the approximate constitutive law—perhaps experimentally—and then, by extrapolation, "building" the corresponding ideal component. Alternatively, it is possible to imagine a determined constitutive law—of which perhaps one feels the need for a particular application—and attribute an ideal component to it and then, if possible, build a real component suitably approximate to it in behavior. From the "historical" viewpoint, one may clearly say that both paths have been trodden: the former for the components we have previously called "elementary" and the latter for more complex ones. We, too, for didactic reason will adopt the possibility of introducing diverse components from the two distinct viewpoints, beginning naturally with elementary components.

A first and fundamental classification of one-ports divides them into linear and nonlinear. Obviously a linear one-port is one whose constitutive relation is of the linear type. In this article we will limit ourselves to linear elements only, while nonlinear ones are dealt with in NONLINEAR NETWORK ELEMENTS.

A second classification of one-ports that it is convenient to introduce is that distinguishing "nondynamic" or *resistive* one-ports from *dynamic* one-ports. The former are those characterized by an "algebraic" link between the voltage and the current. If, for example, the current is the independent variable (the control variable), the constitutive relation of the one-port will be of type $v = \hat{v}(i)$. If $\hat{v}(i)$ is single-valued, one speaks of current controlled one-ports. However, one can also express the constitutive relation as $\hat{\imath} = \hat{\imath}(v)$. If $\hat{\imath}(v)$ is single-valued, one speaks of voltage-controlled one-ports. In general, a one-port is both voltage- and current-controlled if its constitutive relation is invertible.

The linear resistors are nondynamic one-ports that are characterized by a linear algebraic constitutive relation between the voltage and current. They can both be described by

$$v = Ri \quad \text{and} \quad i = Gv \tag{5}$$

where R and G are two constant parameters called, respectively, resistance and conductance. They are linked by the relation $R = 1/G$. The symbol of the linear resistor is illustrated in Fig. 4(a). In the SI system, electrical resistance is measured in *ohms* (Ω; 1 Ω = 1 V/1 A) and conductance is measured in *siemens* (S; 1 S = 1/1 Ω). There are, however, cases where inversion is not possible—for example, if the constitutive relation $v = \hat{v}(i)$ is nonlinear and $\hat{v}(i)$ is not strictly increasing. In such cases the nature of the one-port imposes one of two descriptions and one then says that one-ports are intrinsically current- or voltage-controlled.

There are nondynamic one-ports that impose the time behavior of the voltage at their terminals independent of the current circulating therein, $v = e(t)$. These are the independent voltage sources. Likewise, current-independent sources impose the time behavior of current circulating therein independent of voltage, $i = j(t)$. The symbol for voltage- and current-independent sources are, respectively, shown in Figs. 4(b) and 4(c).

Dynamic one-ports, instead, are characterized by a more complex relation between voltage and current: The derivative of one of the two electric variables is present. Such one-ports, when present, introduce ordinary differential equations in the circuit equations, notably enlarging the behavioral complexity of the electric network (for more details see NETWORK EQUATIONS).

Elementary dynamic one-ports are the capacitor, whose operation is described by the differential equation

$$i = C\frac{dv}{dt} \tag{6}$$

and the inductor, whose operation is described by the differential equation

$$v = L\frac{di}{dt} \tag{7}$$

The parameters C and L are two constants that are, respectively, known as capacity and inductance. In the SI system, capacity is measured in *farads* (F) and inductance is measured in *henries* (H). The symbols for linear capacitors and linear inductors are illustrated, respectively, in Figs. 5(a) and 5(b).

A third classification divides one-ports into *active* and *passive.* One-ports unable to supply more electric energy than they have previously absorbed belong to the second category. Let us first consider nondynamic one-ports. From the definitions of voltage and current previously given, it is obvious

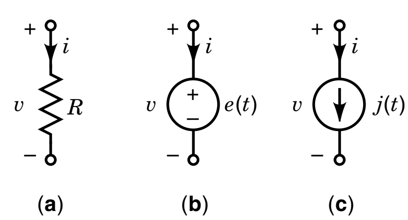


Figure 4. Symbols for (a) linear resistor, (b) independent voltage source, and (c) independent current source.

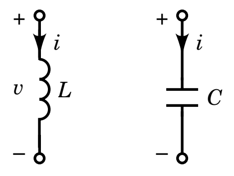


Figure 5. Symbols for (a) linear time-invariant inductors and (b) linear time-invariant capacitors.

that the product $p = iv$ is, instant by instant, the electrical power flowing in the one-port. In fact, v is the work needed to carry a positive charge from one terminal to the other inside the element, while i is the amount of electric charge transported per unit time through the one-port. In particular, if the normal convention is assumed, $p = iv$ represents the work per unit time done by the electric field on the charges: It is equal to the electrical energy per unit time absorbed by the one-port. Thus we call it *electric power absorbed* by the one-port. If this power is always positive, obviously the one-port is only able to absorb electric energy and thus, as previously defined, is passive. If operating conditions exist wherein the power absorbed is negative, the nondynamic one-port is said to be active. In the SI system, electrical power is measured in *watts* (W) and the electrical energy is measured in *joules* (J), $1\ \text{J} = 1\ \text{W} \cdot 1\ \text{s}$.

In the case of nondynamic one-ports the property of passivity has a simple verification on the characteristic curve. Since the constitutive relation of a resistive one-port is described by an algebraic relation between voltage and current, it can be represented on a cartesian plane (i,v). In particular, for linear resistors the representative curve will be a straight line passing through the origin, inclined with respect to the axis of the currents at an angle ϕ so that $R = \tan\phi$ [Fig. 6(a)]. Thus in the plane (i,v) the characteristic curve always develops in the first and third quadrants if the resistance is positive. In the first quadrant a positive voltage corresponds to a positive current, while in the third quadrant a negative voltage corresponds to a negative current, so that the power absorbed is always positive or at most equal to zero. If the one-port characteristic curve has points in the second or fourth quadrants (recall that the convention assumed is the normal one), operating conditions exist in which the power absorbed is negative and thus corresponds to positive supplied energy and the one-port is said to be active. This happens, for example, for voltage- and current-independent sources [Fig. 6(b)].

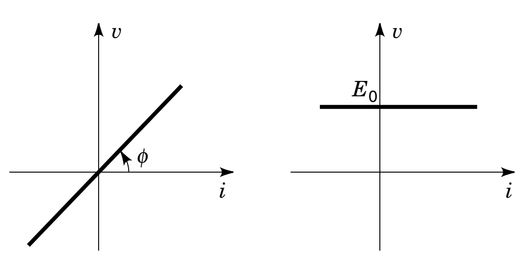


Figure 6. (a) Characteristic curve of a linear resistor; (b) characteristic curve of an independent voltage source.

In conclusion a resistor is passive if we have $vi \geq 0$ for any pair (v, i) (we are assuming the normal convention for the reference directions of the current and the voltage). If such is not the case, the resistor is active. The resistor is said to be *strictly passive* if $vi = 0$ only for the case $i = 0$ and $v = 0$, and so a linear resistor with $R > 0$ is strictly passive. As we will see later there are also one-ports that absorb zero-power for any pair of (v, i). Unlike voltage and current sources, a linear resistor with negative resistance always absorbs a negative power, except for the case $v = 0$ and $i = 0$ where the absorbed power is zero. To stress this property we could introduce the concept of *strictly active one-port*. However, we have to note that physical devices whose characteristic curve always lies in the second and fourth quadrant of the (v, i) plane do not exist.

For dynamic one-ports, too, $p = iv$ represents the electrical power absorbed. In this case, however, the definition of passivity—which remains the same as previously stated—has more complex implications. The fact is that a dynamic one-port is generally able to store electrical energy for a certain interval of time and supply it later. Consider, for example, a capacitor of positive capacity C [its characteristic is represented by Eq. (6)] and integrate the power absorbed p on the interval (t_0,t). In this way we calculate the electric energy absorbed by the capacitor in the interval (t_0,t), which is given by

$$\begin{aligned} w(t_0, t) &= \int_{t_0}^{t} p(\tau)\, d\tau = \int_{t_0}^{t} \frac{d}{d\tau}\left(\frac{Cv^2}{2}\right) d\tau \\ &= \frac{1}{2} Cv^2(t) - \frac{1}{2} Cv^2(t_0) \end{aligned} \tag{8}$$

Such energy is positive (and therefore effectively absorbed) if $v^2(t) > v^2(t_0)$, whereas it is negative (and thus corresponds to energy effectively supplied by the one-port) if $v^2(t) < v^2(t_0)$. If at the initial instant $v(t_0) = 0$, the energy $w(t_0,t)$ is certainly positive. All these considerations lead to the following conclusions: The capacitor is able to absorb or supply electrical energy; at each instant the level of energy stored is equal to $Cv^2(t)/2$; the energy that can be supplied at any given instant is never greater than that stored previously if $C > 0$. In other words, the capacitor with $C > 0$ is passive because

$$\int_{-\infty}^{t} p(\tau)\, d\tau \geq 0 \tag{9}$$

for each instant of time t.

Analogous considerations can be developed for the inductor with positive L. Then we have

$$w(t_0, t) = \int_{t_0}^{t} p(\tau)\, d\tau = \int_{t_0}^{t} \frac{d}{d\tau}\left(\frac{Li^2}{2}\right) d\tau = \frac{1}{2} Li^2(t) - \frac{1}{2} Li^2(t_0) \tag{10}$$

The amount of energy stored by an inductor at any instant is proportional to the square of the current circulating therein at that instant. Here too passivity is assured by the condition $L > 0$ and so Eq. (9) is valid.

An immediate consequence of what has been said is that the constitutive laws of the dynamic one-ports are not sufficient in themselves to univocally predict their behavior starting from an instant t_0: It is necessary to know the level of energy stored at that instant. In other words, one needs to know an initial condition, which for the capacitor is the voltage value at $t = t_0$ and for the inductor is the current value at $t = t_0$.

In conclusion, as regards classification we should recall that a one-port is said to be time-invariant, whether resistive or dynamic, if the parameters of its constitutive relation do not vary in the time.

In this article first we will describe the linear resistive one-ports and then the main properties of linear resistive elements with more than two terminals that are of notable importance in the circuit theory and applications. Then we will study in detail the capacitor and inductor. We shall conclude by describing the behavior of mutually coupled circuits.

RESISTIVE ONE-PORTS

In this section we will introduce different resistive one-ports by briefly describing their concrete structure and discussing the properties implicit in their characteristic laws. Let us begin with linear resistor one-ports operating in the steady-state condition, which in a certain way can be considered typical. As we shall see many of the things we will say are also true when the resistors operate in nonstationary conditions. We will then illustrate the ideal voltage source, the ideal current source, the short circuit, the open circuit, the nullator, and the norator.

The Linear Resistor

Materials—usually metals—exist in which, for a suitable range of parameters, the current density field $\boldsymbol{J}$ is, at every instant and throughout the material, directly proportional to the electrical field $\boldsymbol{E}$ according to the equation

$$\boldsymbol{J} = \sigma \boldsymbol{E} \tag{11}$$

where σ is the electric conducibility of the material, which is measured in *siemens / meter* (S/m) in the SI system. Equation (11), which is called the local Ohm's law, can also be rewritten as $\boldsymbol{E} = \eta \boldsymbol{J}$, where η is the electrical resistivity of the material that in the SI system is measured in *ohm · meter* ($\Omega \cdot$ m).

The fact that some materials, called in fact ohmic materials, verify Ohm's law has a very subtle significance that we will try and examine, even if only from the qualitative point of view. From its definition it is evident that current density is directly proportional to the average velocity of the charge carriers. On the other hand, the electric field is directly proportional to the force exercised on the carriers themselves. Therefore, Ohm's law affirms that the velocity is directly proportional to the force, in apparent contradiction with the laws of dynamics that require that the force is directly proportional to the acceleration. In effect, the contradiction is only apparent since in Newton's law the body is assumed to be completely free to move in the surrounding space under the action of the force. Obviously the charge carriers in an ohmic conductor are not completely free to move! The crystal lattice which constitutes the material body in which the carriers are forced to move offers some obstacle to the charge movement. Ohm's law, indeed, allows us to determine what type of obstacle there is. Let us, in fact, suppose that the overall effect of the motionless charges constituting the lattice is equivalent to a "friction" directly proportional to the velocity. The overall force acting on the charges will then be $q\boldsymbol{E} - k\boldsymbol{v}$, given that the friction is opposed to the action of the electric field. If a steady state is achieved, the charge velocity remains constant and the acceleration is therefore nil. Then we will have

$$0 = q\boldsymbol{E} - k\boldsymbol{v} \tag{12}$$

and so $\boldsymbol{v} = q\boldsymbol{E}/k$ as prescribed by Ohm's law. This model of conduction in ohmic materials is known as Drude's model, and what we have explained only qualitatively can be further considered at a quantitative level. We are principally interested in underlining the fact that the verification of Ohm's law requires rather particular conditions. It is not surprising therefore that there are conducting materials that do not satisfy this law, and that ohmic materials themselves are such only in determined conditions. For example, the resistivity of a material does not remain constant when the material temperature varies, as we will see later.

Let us now consider a cylindrical conductor of uniform cross section S and length l [Fig. 7(a)], crossed by a steady-state current density field $\boldsymbol{J}$ that we will suppose to be uniform, directed parallel to the cylinder axis. A uniform electric field will be associated to it. Integrating Eq. (11) between two points a and b on the extreme surfaces of the cylinder along any path γ connecting them within the cylinder, we obtain

$$v_{ab} = \int_\gamma \boldsymbol{E} \cdot d\boldsymbol{l} = \eta l J = \eta l \frac{i}{S} = Ri \tag{13}$$

where v_{ab} is the voltage and $i = JS$ is the intensity of the electric current flowing across any section of the conductor cylinder. The orientation of the normal to the generic section, indispensable in calculating the current, has been chosen in agreement with the orientation of path γ. Factor $R = \eta l/S$ is called the *electric resistance of the conductor tract,* and Eq. (13) is Ohm's law applied globally.

This result may easily be generalized to any conductor form involving a charge flux constant in time, in conditions where all the charge carriers start from one end and arrive at the other end of the conductor tract. Two perfectly conducting electrodes—that is, $\eta = 0$—constrain the electric potential to be uniform at the tract ends [Fig. 7(b)]; we will call them terminal a and terminal b. Even without knowing the actual distribution of the current density $\boldsymbol{J}$ inside the body, we can certainly affirm that all the field lines must start from and meet

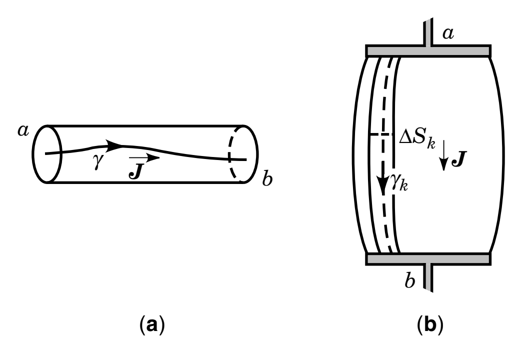


Figure 7. (a) Cylindrical resistor with uniform cross section; (b) cylindrical resistor with a varying cross section in which an elementary flux tube is represented. The terminals are perfect conducting electrodes.

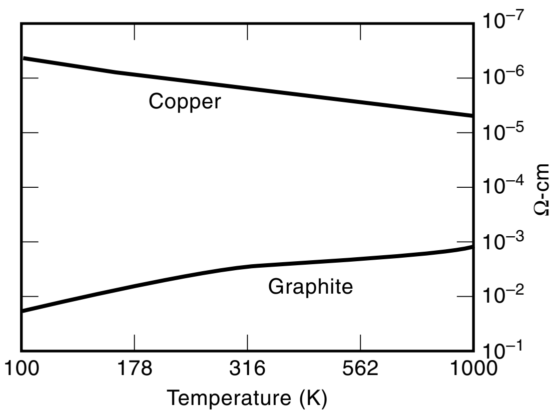


Figure 8. Resistivity versus temperature for two conducting materials (in both axes, logarithmic scales have been used).

at the terminals a and b. Let us consider a flux tube of $\boldsymbol{J}$ having an elementary cross section ΔS_k—that is, an elementary tube on whose side vector field $\boldsymbol{J}$ is tangent. Integrating Ohm's law [Eq. (11)], as before, along the median of the elementary flux tube, one has

$$v_{ab} = \int_{\gamma_k} \boldsymbol{E} \cdot d\boldsymbol{l} = \int_{\gamma_k} \frac{\Delta S_k J_k \eta}{\Delta S_k} \, dl = \Delta i_k \int_{\gamma_k} \frac{\eta}{\Delta S_k} \, dl \qquad (14)$$

because the intensity of the current Δi_k along the elementary flux tube is, by definition, constant. Taking Δi_k from Eq. (14) and summing on all elementary flux tubes of the conductor tract—that is, on k—one obtains

$$i = \sum_k \Delta i_k = \frac{v_{ab}}{\sum_k \int_{\gamma_k} \dfrac{\eta}{\Delta S_k} \, dl} = \frac{v_{ab}}{R} \qquad (15)$$

which is the constitutive relation [Eq. (5)] of a linear resistor with $R = \Sigma_k \int_{\gamma k} (\eta/\Delta S_k)\, dl$.

As we have already observed, the fact that a one-port has a type (5) constitutive relation requires the verification of quite strict conditions and that the movement of the charges in the component itself is regulated by particular laws. Therefore, it is not surprising that a real resistor behaves as such only in a determined range of parameters that characterize its physical conditions.

A physical condition that regulates the behavior of a resistor, for example, is the temperature. In fact, the resistivity of a material generally depends on the temperature. Figure 8 shows the typical resistivity performance as a function of the temperature for two materials, copper and graphite. As can be seen, the resistivity can increase or decrease with the change in temperature. Even with the same material, these two behaviors can be encountered at different temperature ranges.

For changes in temperature that are not too wide, it is possible to take into account such dependence by developing the function $\eta(T)$, where T indicates the temperature of the metal, in Taylor series with initial point T_0 and stopping it at the second term. So one obtains

$$\eta(T) \cong \eta(T_0) \left[1 + \frac{1}{\eta(T_0)} \frac{d\eta}{dT}\bigg|_{T_0} (T - T_0) \right] \qquad (16)$$

Coefficient $\chi(T_0) = [1/\eta(T_0)]\ (d\eta/dT)|_{T_0}$ is called the material temperature coefficient at $T = T_0$, or resistor temperature coefficient (RTC), because it is also found in the dependence on the temperature of the resistance R,

$$R(T) = R(T_0)[1 + \kappa(T_0)(T - T_0)] \qquad (17)$$

The fact that R depends on the temperature has an important consequence that we wish to examine in greater detail. As we know, a resistor with resistance R crossed by an electric current i for a time interval Δt absorbs an electric energy equal to $Ri^2\Delta t$. This energy is all transformed into heat, according to Joule's well-known law. As a consequence of this phenomenon, the temperature of the resistor tends to increase and so the resistance tends to vary. The result, therefore, is an indirect dependence of R on i that modifies the characteristic of the one-port itself. In effect, however, the resistor soon reaches a steady-state temperature that can easily be determined by a simple energy balance. The temperature reached will be that at which the power dissipated in the resistor is exactly equal to the quantity of heat per unit time transferred by the resistor to the surrounding ambient, which depends on the temperature difference between the resistor body and the environment. Once the temperature is stabilized, the value of R stabilizes, even if at a different value from that initially held. It follows that for every resistor, in addition to the value of its resistance and the precision with which it is guaranteed, the maximum value of the current—or the power Ri^2—for which the resistance value is guaranteed must also be given. For these reasons, resistors are generally classified on the basis of the power they are capable of dissipating without the resistance value exceeding the limits of precision guaranteed, or the limit at which the resistor itself deteriorates irreversibly. Naturally, to allow a resistor to dissipate greater power, while maintaining its temperature within acceptable limits, the simplest way is to increase the exchange surface with the surrounding ambient, so as to increase the quantity of heat lost per unit time. On the other hand, larger surfaces involve greater volumes and so, generally, the size of a resistor is an index to its capacity to dissipate power.

Another factor that can affect the size of the resistor is the operating voltage for which it is built. This is particularly significant with resistors designed for high voltages. So far we have implicitly assumed that the resistor is embedded in an insulating medium so that the motion of the electric charges must develop only within the resistor itself. In effect, any insulating medium behaves as such only if the force due to the electric field acting on the charges present therein (we are thinking about the molecular and atomic structures) does not exceed determined limits. With very high values of the electric field the insulator loses its characteristics, the passage of the charges is no longer impeded, and a "discharge" develops inside it. The breakdown field above which the discharge develops depends significantly on the physicochemical conditions in the insulating medium (composition, temperature, etc.); for example, the breakdown electric field for the air is about 25 kV/cm in normal conditions. Consequently, the dimensions of the resistor must be such as to guarantee that, for the voltages for which it is designed, the breakdown field cannot be exceeded at any point in the insulator. As we have said, this factor is particularly important for high-voltage resistors.

In general, all these characteristics depend on the way in which the resistor is built. From this viewpoint, resistors can be broadly classified as wirewound, foil, thin film, thick film, and bulk resistors. The specific characteristics for each of these classes depend, of course, on the different technologies used in their production and will not be dealt with further in this article. There are numerous handbooks that deal with these matters in detail (see, for example, Ref. 2).

Another useful classification is that which divides resistors into two classes: fixed resistors and potentiometers. In the former there are low-power resistors (typically from 0.05 W to 2 W) high-power resistors, high-ohmic resistors, and chip resistors. For some types of sufficiently low power resistors, and thus small size, it is usual to indicate the resistance and its relative tolerance with a colored band code whose key is given in the manuals. However, the second class includes all variable resistors that, according to the way in which their variability is obtained, can be divided into rotary control, slide control and preset potentiometers. For further details one should again consult Ref. 2.

On the other hand, the same words used to identify the different classes of resistors already give an idea of how they are built. In each of them, then, one varies between three available parameters—the resistivity η (and hence the material), the length l, and the cross section S—to obtain the resistance values desired.

In conclusion, it is worth recalling that in this section we have described the operation of resistors when the voltage and current are in steady state, that is, we have described the so-called direct current (dc) operation. As we shall see, many of the things we have said are also true when the resistor does not operate in steady state.

Short-Circuit and Open-Circuit One-Ports

As we have seen, resistor constitutive relation (5) is represented on the plane (i,v) by a straight line passing through the origin inclined to the current axis at an angle ϕ so that $R = \tan\phi$ [Fig. 6(a)]. Thus, as R varies, the straight line will be more or less inclined on the i axis. Two limit cases immediately come to mind: one in which the angle ϕ is zero and the other in which it is equal to $\pi/2$.

In the first case, $R = 0$ and the voltage of the one-port will always be zero for any current flowing in it, that is,

$$v = 0 \qquad \text{for any } i \tag{18}$$

Such a one-port is called *short circuit.* Its symbol is illustrated in Fig. 9(a); and, in theory, it can be realized with a perfect conductor, namely, a conductor with infinite conducibility (that is, with zero resistivity). We may imagine the connections between different one-ports in a circuit as being of this nature. In reality, of course, a real conductor can at most approximate such behavior, and the shorter the conductor tract, the closer the approximation will be. This also justifies its being called a "short-circuit" one-port.

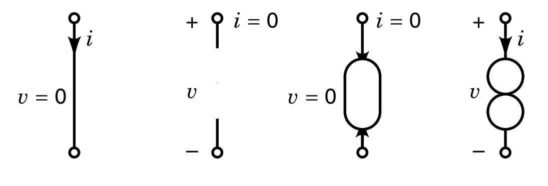


Figure 9. Symbols for (a) short circuits, (b) open circuits, (c) nullators, and (d) norators.

The other case, $R = \infty$, corresponds to that in which the electric conducibility of the conductor is equal to zero. In such an event, in contrast, for any voltage at the one-port terminals, the current crossing the one-port is always zero, that is,

$$i = 0 \qquad \text{for any } v \tag{19}$$

Such a one-port can be realized by placing an ideal insulator between the terminals. For this reason the one-port is called *open circuit* and its symbol is represented in Fig. 9(b). The electrical power absorbed by the short-circuit and open-circuit one-ports is always zero. For this reason we call them *zero-power* one-ports.

Ideal Voltage and Current Sources

Starting from the characteristic of a short circuit, let us imagine moving its characteristic curve parallel to itself, for example, in the direction of the positive voltages. Let E_0 be the value at which the straight line parallel to the axis of the currents so obtained intersects the axis of the voltages. The one-port having such a characteristic could be described as follows: For any value of the current flowing in it, the voltage at the terminals is always equal to E_0. Such a characteristic must develop in the first and second quadrants of the plane (i,v) [Fig. 6(b)]. Because we have assumed the normal convention on the one-port, when the operating point lies in the second quadrant, the electric power absorbed by the one-port is negative and therefore the one-port supplies electric energy. For this reason the one-port so identified is called a *source.* It is to be noted that, contrary to that which happens in the resistor with positive resistance, where the positive charge movement is always from the higher potential terminal to the lower (so the power absorbed is always positive), in this one-port when the operating point lies in the second or fourth quadrant the positive charges move from the lower potential terminal to the higher, against the force exercised by the electric field. Thus, it is evident that such a one-port must be the site of more complex phenomena, wherein forces of a nature other than that produced by the electric field come into play (for example, such forces in a battery are of a chemical nature and of an electrodynamic nature in a dynamo).

More generally, an ideal voltage source can impose a voltage variable in the time with a known waveform and independent of the current that circulates within; that is,

$$v = e(t) \qquad \text{for any } i \tag{20}$$

The symbol used to indicate an ideal voltage source is shown in Fig. 4(b).

The characteristic of an ideal voltage source can only approximate that of a real source. In fact, it is implicit in the characteristic of an ideal voltage source that it can supply as great a power as one wants, to infinite limit when the current is infinite. Naturally a real source cannot possess such a property. If the current circulating in the source becomes too high, the voltage will not remain equal to the value assumed when the current is zero (open-circuit voltage), but will fall until it tends to zero and change sign for a finite current value (short-circuit current). The simplest way to represent a real voltage source is to consider an ideal voltage source, with the voltage intensity equal to the open-circuit voltage of the real source, in series with a resistor that takes into account the effects due to the "internal resistance" of the real source. This real voltage source model tends to the ideal one when the internal resistance tends to zero.

Similarly, starting from an open-circuit one-port, one can deduce a new ideal one-port in which a current with an assigned waveform circulates irrespective of the voltage between the terminals,

$$i = j(t) \qquad \text{for any } v \tag{21}$$

Such a one-port, for which considerations analogous to those relative to the ideal voltage source can be developed, is called an ideal current source and its symbol is shown in Fig. 4(c). Naturally it may be better not to use the normal convention for source one-ports, but rather the other [Fig. 2(b)], which for this reason we call "source convention."

Nullators and Norators

Let us complete this overall view of resistive one-ports by introducing another two ideal one-ports whose characteristics are in fact "pathological." Their use, which may not be clear at first sight, lies in the fact that they allow models of complex components to be built, as the following examples illustrate.

The first is the nullator which is an ideal one-port, whose symbol is given in Fig. 9(c), defined by the constitutive equations

$$i = 0, \qquad v = 0 \tag{22}$$

This, unlike the short-circuit one-port, imposes a zero voltage with a zero current. With such a one-port, one can make two nodes of a circuit have the same potential without changing the distribution of the currents. The electric power absorbed by a nullator is zero; in fact, in the plane (i, v) the characteristic of the nullator is reduced to a point, the origin of the axes. This is another example of zero-power one-port.

The other one-port, whose symbol is shown in Fig. 9(d), is the norator. It is an ideal one-port that on the contrary imposes no constraint on the voltage and the current: The voltage and the current can assume any value whatsoever. In other words, the characteristic of norators is the whole (v, i) plane and hence its pathology is complementary to that of nullators. The norator, unlike an open circuit, allows any current to flow, whatever the voltage. With a similar one-port it is possible to connect different parts of a circuit without altering the voltage distribution. The electric power absorbed by a norator is not usually zero and can even be negative. Thus the norator is an active element.

LINEAR RESISTIVE ELEMENTS WITH MORE THAN TWO TERMINALS

As we have already said, the components of an electrical circuit can have more than two terminals. An element with terminals is called an n-pole and is characterized by n currents and $n(n - 1)$ voltages if the order in each pair of nodes is taken into account. In agreement with Kirchhoff's laws, only $(n - 1)$ currents and $(n - 1)$ voltages are independent (see NETWORK EQUATIONS). The operation of an n-pole can be conditioned by the topology of the circuit into which it is inserted; the most significant example is that of a $2m$-pole connected to m circuits, each of which can be considered as a one-port. In this case the current entering from a given terminal is equal to the current exiting from the other terminal. Let us call each pair of terminals having this property a *port* and call the element operating in this way *m-port.* There are also $2m$-poles that operate intrinsically as m-ports because of their physical constitution. As for the one-ports, we could classify n-poles and m-ports into linear and nonlinear, into "nondynamic" or resistive and dynamic, into active and passive, and into time-invariant and time-variant.

If an n-pole does not function as an $n/2$-port, either because of its internal physical structure or because n is odd, it can still be described by an $(n - 1)$-port through its descriptive variables after having chosen an arbitrary common terminal (see NETWORK EQUATIONS). For example, let us consider a 3-pole. To identify a set of independent voltages and currents one may choose an arbitrary reference terminal, for instance the terminal labeled "3," and consider the currents of the other two terminals i_1 and i_2 and the voltages v_1 and v_2 between these two terminals and the reference terminal "3." To current i_1 and voltage v_1 we can associate a port whose terminals are those labeled "1" and "3," and to current i_2 and voltage v_2 we can associate a port whose terminals are those labeled "2" and "3": the terminal labeled "3" is in common to the two ports. In this way it is possible to characterize the 3-pole as a 2-port. However, the general theory of linear n-poles and linear m-ports is dealt with in MULTIPOLE AND MULTIPORT ANALYSIS, whereas NONLINEAR NETWORK ELEMENTS considers nonlinear resistive elements with more than two terminals that are of special interest in circuit applications. Here we will limit ourselves to describing the basic linear resistive two-ports that are of notable importance in the theory of circuits and in the building of complex models, beginning with linear controlled sources.

Linear Controlled Sources

Linear controlled sources are two-ports in which one of the variables—voltage or current—at one of the two ports is directly proportional to one of the variables of the other port. Considering all the possible combinations one has the elements which follows.

The voltage-controlled voltage source is a linear two-port defined by the constitutive equations [the symbol is shown in Fig. 10(a)]

$$i_1 = 0, \qquad v_2 = \alpha v_1 \tag{23}$$

where α is a constant called the voltage transfer ratio. Port 1 is equivalent to an open circuit and port 2 is equivalent to a

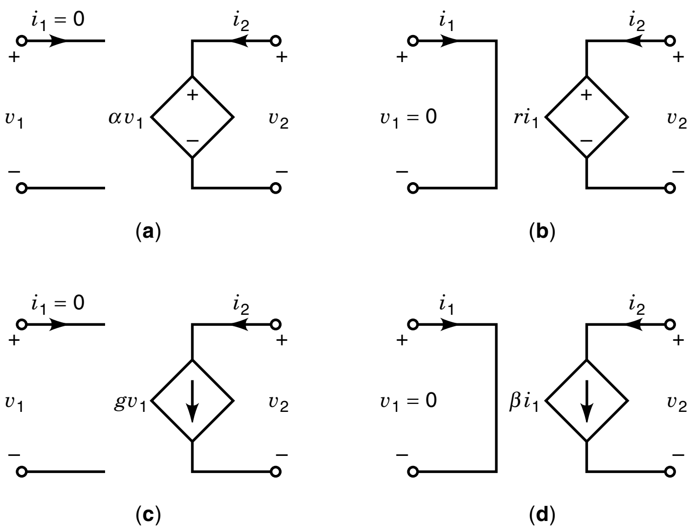


Figure 10. Symbols for the linear controlled sources.

voltage source that imposes a voltage linearly dependent on the voltage at port 1 and independent of the current i_2.

The current-controlled voltage source is a two-port defined by the constitutive equation [whose symbol is shown in Fig. 10(b)]

$$v_1 = 0, \qquad v_2 = ri_1 \tag{24}$$

where r is a constant called transresistance. Port 1 is equivalent to a short circuit and port 2 is equivalent to a voltage source that imposes a voltage linearly dependent on the current circulating at port 1 and independent of the current i_2.

The voltage-controlled current source is a two-port defined by the constitutive equations [whose symbol is given in Fig. 10(c)]

$$i_1 = 0, \qquad i_2 = gv_1 \tag{25}$$

where g is a constant called transconductance. Port 1 is equivalent to an open circuit and port 2 is equivalent to a current source that imposes a current linearly dependent on the voltage at port 1 and independent of the voltage v_2.

Finally, the current-controlled current source is a linear two-port defined by the constitutive equations [whose symbol is illustrated in Fig. 10(d)]

$$v_1 = 0, \qquad i_2 = \beta i_1 \tag{26}$$

where β is a constant called the current transfer ratio. Port 1 is equivalent to a short circuit and port 2 is equivalent to a current source that imposes a current linearly dependent on the current circulating at port 1 and independent of the voltage v_2.

One should note that the electric power absorbed by the controlled sources is always equal to $p = v_2 i_2$, because port 1 never absorbs power. This power can be negative, so the controlled sources are active two-ports. Controlled sources are also nonreciprocal two-ports, that is, the two ports considered, one as an input and the other as an output, do not behave in the same manner. For fuller consideration of this question refer to NETWORK THEOREMS.

It can easily be shown that, by connecting a current-controlled voltage source in cascade to a voltage-controlled current source, we obtain a current-controlled current source with a transfer ratio given by $\beta = rg$. However, if one connects a voltage-controlled current source to a current-controlled voltage source, we obtain a voltage-controlled voltage source with a transfer ratio given by $\alpha = rg$.

Even if the controlled sources are to be imagined as ideal components, so as to simplify the circuit representation of more complex components, such as linear models of transistors (see NONLINEAR NETWORKS ELEMENTS), it is also true that by using the operational amplifier itself it is possible to realize components whose constitutive relations approximate satisfactorily to the various controlled sources. Thus these elements are often used in real circuits to obtain particular effects. For example, it is possible to connect two two-ports by means of a voltage-controlled source so that the operating of the first is not affected by the presence of the second. This separation technique is an important tool in the designing of electronic circuits.

To complete the picture of linear and resistive two-ports, let us introduce another three two-ports: the gyrator, the ideal transformer, and the operational amplifier. Thus we will have all the elements needed to realize two-ports with any link between their electrical variables.

The Gyrator

The gyrator is a linear resistive two-port, whose operation is described by the equations

$$i_1 = Gv_2, \qquad i_2 = -Gv_1 \tag{27}$$

where the constant G is called the gyration conductance. The symbol is shown in Fig. 11(a). The electric power absorbed by the gyrator is zero in any operating condition, so it is a globally passive two-port that neither dissipates nor stores energy. Even if globally passive, the nonamplification of voltages and currents is not verified for this two-port. For example, if one considers a gyrator with port 1 connected to an ideal voltage source and port 2 connected to a short circuit, the current in the voltage source is zero, while the current in the short circuit is different from zero. Consequently, even if the gyrator is globally passive, it must be constituted by active elements.

The gyrator is an antireciprocal two-port, which is a particular case of nonreciprocity. This strong property is the basis of the most important property of a gyrator, which can be

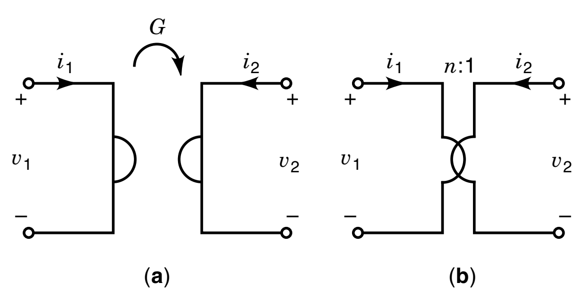


Figure 11. Symbols for (a) gyrators and (b) ideal transformers.

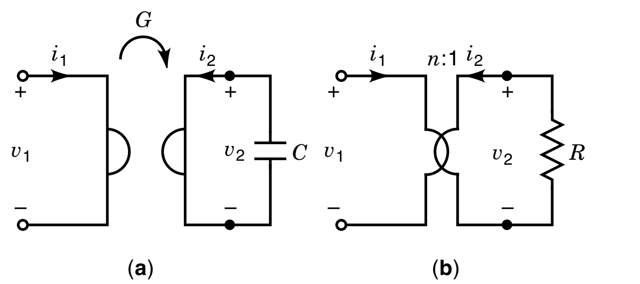


Figure 12. (a) A gyrator terminated at the output with a capacitor; (b) an ideal transformer terminated at the output with a resistor.

illustrated by means of the circuit in Fig. 12(a). Port 2 is connected to a linear time-invariant capacitor C. In this case, one has

$$v_1 = -\frac{i_2}{G} = \frac{C}{G}\frac{dv_2}{dt} = \frac{C}{G^2}\frac{di_1}{dt} \tag{28}$$

Therefore when a gyrator is connected to a time-invariant linear capacitor of capacity C, the inlet port behaves as a linear time-invariant inductor of inductance C/G^2. Thus the gyrator allows inductor one-ports to be realized starting from capacitors. It is also possible to realize a capacitor from an inductor by using a gyrator.

The following properties may also easily be demonstrated: If a gyrator is connected to a linear resistor of resistance R, the equivalent one-port behaves like a linear resistor of resistance $1/(RG^2)$; if the gyrator is connected to a voltage (current)-controlled resistor—for example, a diode tunnel—the equivalent one-port then behaves as if it were a current (voltage)-controlled resistor. Because of these characteristics, it is called a gyrator.

There are on the market components that approximate to the operation of a gyrator. A gyrator can also be made from two voltage-controlled current sources with transconductance G, as illustrated in Fig. 13(a).

The Ideal Transformer

The ideal transformer is a linear resistive two-port whose operation is defined by the equations

$$v_1 = nv_2, \qquad i_2 = -ni_1 \tag{29}$$

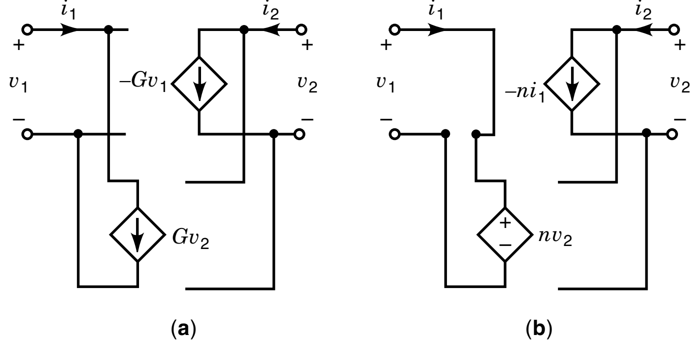


Figure 13. Realization (a) of a gyrator and (b) of an ideal transformer by using linear controlled sources.

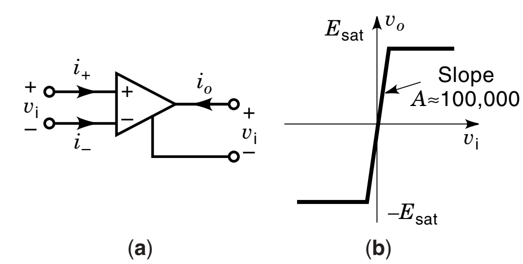


Figure 14. (a) Symbols for the biased operational amplifier; (b) transfer characteristic curve.

where the constant n is the transformation ratio. The symbol for the ideal transformer is illustrated in Fig. 11(b). The electric power absorbed by the ideal transformer is equal to zero whatever the operating condition. Therefore like the gyrator it is a globally passive two-port that neither dissipates nor stores energy. For this two-port too, the nonamplification of the voltages and currents does not apply, even if it is globally passive. Unlike controlled linear sources and gyrators, the ideal transformer is a reciprocal two-port.

One of the most important properties of the transformer can be illustrated by considering the circuit shown in Fig. 12(b), where a transformer port is connected to a linear resistor with resistance R. In this case we have

$$v_1 = nv_2 = -nRi_2 = n^2Ri_1 \tag{30}$$

Thus, when a transformer is connected to a linear resistor of resistance R, the equivalent one-port behaves like a linear resistor of resistance n^2R and therefore the transformer allows the resistance of the resistor to be "changed." It is also easy to show the following properties: When a transformer is connected to a linear inductor with inductance L (a linear capacitor of capacity C), the equivalent one-port behaves like an inductor of inductance n^2L (a capacitor with capacity C/n^2).

An ideal transformer can be realized by means of a current-controlled current source and a voltage-controlled voltage source as shown in Fig. 13(b). It, too, is the fundamental element in representing a real transformer realized with two coupled circuits; and, in turn, a real transformer, under certain operating conditions, approximates to the operation of an ideal one.

The Ideal Operational Amplifier

The ideal operational amplifier is an extremely complex semiconductor component. In use at low frequencies it behaves like a nonlinear resistive element with four terminals, whose operation may be described by the approximated relations [the symbol is shown in Fig. 14(a)]

$$\begin{aligned} &i_- = I_-, \qquad i_+ = I_+ \\ &v_o = f(v_i) = \begin{cases} E_{\text{sat}}, & v_i \ge (E_{\text{sat}}/A) \\ Av_i, & -(E_{\text{sat}}/A) \le v_i \le (E_{\text{sat}}/A) \\ -E_{\text{sat}}, & v_i \le -(E_{\text{sat}}/A) \end{cases} \end{aligned} \tag{31}$$

where I_- and I_+ are the so-called input polarization currents, E_{sat} is the maximum absolute value of the output voltage v_u, and A is the so-called open-loop voltage gain. It should be

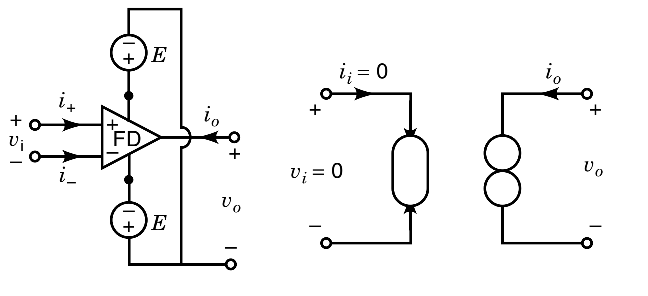


Figure 15. (a) Typical biasing scheme for the operational amplifier; (b) equivalent circuit of an ideal operational amplifier in the linear region.

noted that the transfer characteristic $f(v_i)$ of this approximate model is piecewise linear and has three segments: a linear tract and two saturation tracts [see Fig. 14(b)].

The device known as an operational amplifier that we can buy in any electronics shop has at least five terminals [Fig. 15(a)]. To function as an operational amplifier, as described by characteristic Eq. (31), it must be "biased" with two equal constant voltage sources, as indicated in Fig. 15(a). The component, thus polarized, is shown by means of the symbol illustrated in Fig. 14(a) and its operation is described by Eq. (31). The polarization voltage is typically 15 V. Besides the five terminals shown in Fig. 15(a), other terminals are added to the device to allow it to be controlled.

Currents I_- and I_+ do not normally exceed values of the order of 0.1 mA. For example, for $\mu A741$, I_- and I_+ are of the order of 0.1 mA, while for $\mu A740$, they are of the order of 0.1 nA. Typically the open-loop voltage gain A is 10^5 and the saturation voltage E_{sat} is 2 V less than the polarization voltage of the operational amplifier.

Many operational amplifiers are produced with integrated circuits using bipolar transistors as base elements (bipolar technology). For example, the $\mu A741$ contains about a dozen bipolar transistors. In very large scale integration (VLSI) circuits, operational amplifiers are made with a different technology, known as CMOS technology. For further information the reader is referred to OPERATIONAL AMPLIFIERS and Ref. 3.

Because of the typical values of I_-, I_+, and A, the precision is only slightly diminished if one assumes $I_- = I_+ = 0$ and $A = \infty$. This simplified assumption is the basis of the ideal operational amplifier model defined by the characteristic equations

$$\begin{aligned} &i_- = 0 \\ &i_+ = 0 \\ &v_o = E_{sat}\,\mathrm{sgn}(v_i), \qquad v_i \neq 0 \quad \text{(saturation region)} \\ &v_i = 0, \qquad -E_{sat} < v_o < +E_{sat} \quad \text{(linear region)} \end{aligned} \tag{32}$$

Therefore, the ideal operational amplifier is an element that functions intrinsically as a two-port. The current in the input port $i_i = i_+ = i_-$ is zero, and that at the output port is independent of the input voltage v_i: The output port behaves as if it were a voltage source "controlled" by the input voltage. The electric power absorbed by the ideal operational amplifier is given by $p = i_o v_o$. It may be positive or negative, according to the circuit into which the amplifier is inserted. Thus, the operational amplifier is an active two-port.

If the operational amplifier in the circuit into which it is inserted functions in the linear region—that is, the output voltage satisfies the relation $-E_{sat} < v_o < +E_{sat}$—then the characteristic relations are

$$i_i = 0, \qquad v_i = 0 \tag{33}$$

Consequently, the linear model of the ideal amplifier may be thought of as a two-port consisting of a nullator and a norator, as illustrated in Fig. 15(b): The input port of the linear operational amplifier behaves as if it were a nullator—that is, both current and voltage are zero—while the output port behaves as if it were a norator, since the current and voltage can be of any value whatsoever. The linear model of the operational amplifier is a non reciprocal and active two-port.

By using linear operational amplifiers, one can realize all possible kinds of controlled sources. Let us consider the two-port shown in Fig. 16 consisting of an ideal operational amplifier and two linear resistors. One assumes that the outlet voltage of the operational amplifier is, in absolute value, lower than the saturation voltage. In this operating condition it is possible to use the linear model described by Eqs. (33). Applying Kirchhoff's laws and using the characteristic equations of the elements, one obtains

$$R_a i_1 + v_2 = 0, \qquad R_b i_2 + v_1 = 0 \tag{34}$$

If the resistor of resistance R_b is a short circuit—that is, $R_b = 0$—then from Eqs. (34) one obtains

$$v_1 = 0, \qquad v_2 = -R_a i_1 \tag{35}$$

Characteristic equations [Eq. (35)] are those for a current-controlled voltage source: $r = -R_a$ is the transresistance of the source (the sign for the transresistance may be changed by inverting the terminals of a port).

If the resistor of conductance $G_a = 1/R_a$ is an open circuit—that is, $G_a = 0$—then from Eqs. (34) one obtains

$$i_1 = 0, \qquad i_2 = -v_1/R_b \tag{36}$$

Characteristic equations [Eq. (36)] are those for a voltage-controlled current source and the transconductance is $g = -1/R_b$ (as for the transresistance, the sign of the transconductance can be changed by inverting the terminals of a port).

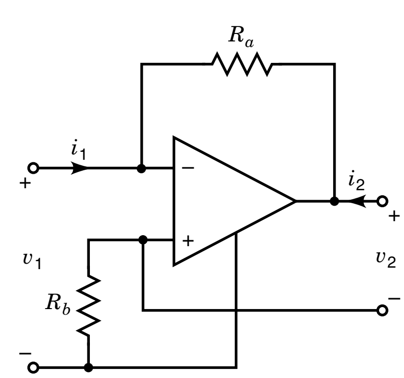


Figure 16. Basic circuit to realize controlled sources.

DYNAMIC ONE-PORTS

Previously we have introduced the two fundamental dynamic one-ports: the capacitor and the inductor and we have done so by hypothesising the existence of a mathematical model type (6) for the capacitor and type (7) for the inductor. In effect, however, a deeper examination shows that dynamic one-ports present different problems and their existence will now be demonstrated.

For the electric field in a dynamic regime we have

$$\oint_\Gamma \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt}\iint_{S_\Gamma} \boldsymbol{B} \cdot d\boldsymbol{S} = -\frac{d\Phi_\Gamma}{dt} \tag{37}$$

for every closed line Γ, where $\boldsymbol{B}$ is the magnetic field and S_Γ is any open surface that has Γ as contour, oriented in agreement with the right-hand rule. Equation (37) is Faraday's induction law. For fuller information consult Ref. 1. The surface integral represents the flux of the magnetic field through the surface S_Γ. This flux does not depend on the particular surface S_Γ we are considering because the flux of the magnetic field through any closed surface is always equal to zero. It depends only on the magnetic field and on the contour Γ. For these reasons the quantity Φ_Γ is called the magnetic flux linked with Γ.

Therefore, as the electric field in a dynamic regime is not conservative, the voltage between the two one-port terminals must no longer be independent of the path γ chosen to calculate it. One should remember that the voltage of a one-port is the line integral of the electric field along a determined path that connects the two terminals. Because of the presence of the time-varying magnetic field the voltage depends on the predetermined path γ. In such conditions it is the idea of the constitutive relation itself that loses significance, since it is no longer possible to identify a single voltage to be associated to the two terminals. Naturally, one can always define the voltage at the terminals by making a particular choice of the integration path. However, such a choice cannot be satisfactory since it would give a much more limited meaning to Kirchhoff's second law.

In the same way, in a dynamic regime, one cannot associate a single current to the two terminals of the component. Even if we admit that the element is embedded in a nonconductor medium, as in fact it is, and that it may interact with the rest of the network only through its terminals, nothing will assure that the current entering one of them will be equal to that exiting from the other, instant by instant. In fact, we have

$$\oiint_\Sigma \boldsymbol{J} \cdot d\boldsymbol{S} = -\frac{dQ_\Sigma}{dt} \quad \text{or} \quad \oiint_\Sigma \boldsymbol{J} \cdot d\boldsymbol{S} = -\oiint_\Sigma \epsilon\frac{\partial \boldsymbol{E}}{\partial t} \cdot d\boldsymbol{S} \tag{38}$$

where Σ is any closed surface whatsoever, ϵ is the dielectric constant of the medium (which is assumed to be linear, isotropic, and time-invariant), and $Q_\Sigma(t)$ is the electric charge that is within Σ at time t; the generic surface element $d\boldsymbol{S}$ is oriented to the outside. The first equation in Eq. (38) is the well-known electric charge conservation law and the second is derived by using the Gauss law (for fuller information consult Ref. 1). Therefore the flux of the current density field $\boldsymbol{J}$ through any closed surface is not zero, but is equal to the minus of the time derivative of the electric charge contained in it. As a consequence the current entering the terminal can be different from that exiting from the other terminal at every instant, because there can be an increase or reduction of the total charge stored in the component. In the dynamic operation, in fact, it is no longer the vector field $\boldsymbol{J}$ but the vector field $(\boldsymbol{J} + \epsilon\partial\boldsymbol{E}/\partial t)$ that has to be conservative with respect to the flux.

It would seem that in a dynamic operation the very basis on which the theory of circuits is founded collapses, and indeed it does. Of course to limit ourselves to this simple observation is to be superficial. It is better to ask whether conditions exist under which a single voltage and a single current can be associated to a component with an acceptable error. To answer this it is necessary to consider the problem more fully.

Referring to Fig. 17(a) we can affirm that the path integrals of the electric field $\boldsymbol{E}$ between points a and b along two lines γ_1 and γ_2 differ by the flux of field $\partial\boldsymbol{B}/\partial t$ through any surface that has the closed line $\Gamma_{12} = \gamma_1 \cup \gamma_2$ as its contour:

$$v_1 - v_2 = \int_{\gamma_1} \boldsymbol{E} \cdot \boldsymbol{t}\, dl - \int_{\gamma_2} \boldsymbol{E} \cdot \boldsymbol{t}\, dl = -\frac{d\Phi_{\Gamma_{12}}}{dt} \tag{39}$$

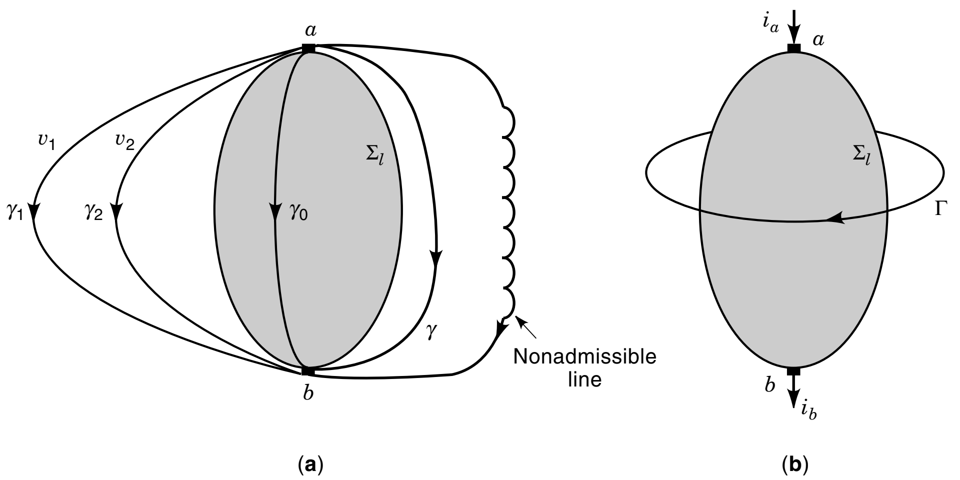


Figure 17. (a) The voltages v_1 and v_2 differ by the time derivative of the magnetic flux linked with $\gamma_1 \cup \gamma_2$; (b) the currents i_a and i_b differ by the time derivative of the total electric charge contained in the limiting surface Σ_l.

As long as the voltages v_1 and v_2 differ by a negligible amount, there must be

$$|v| = \left|\int_\gamma \boldsymbol{E}\cdot\boldsymbol{t}\,dl\right| \gg \left|\frac{d}{dt}\iint_S \boldsymbol{B}\cdot\boldsymbol{n}\,dS\right| \tag{40}$$

where γ is any line going from a to b that does not pierce the limit surface Σ_l. The surface S is open and it has as its contour a closed line given by the union of γ and any line γ_0 lying on the limit surface of the element and joining a to b [Fig. 17(a)]. In effect, it would be exaggerated to require that this condition be verified for any line γ, and it would not correspond to real needs. In fact, our real aim is to define univocally a voltage at the terminals of the one-port so as to be able to write the Kirchhoff equations for the network into which the one-port is inserted. The lines γ that we require to satisfy Eq. (40) are therefore lines that have the same length as the longest characteristic length of the one-port itself. For simplicity of language we will henceforth call these lines "admissible lines" to distinguish them from "nonadmissible lines" of the type shown qualitatively in Fig. 17(a) and that form a great number of turns.

As regards the currents, current i_a entering at terminal a can differ from that exiting at terminal b by an amount equal to the flux of field $\epsilon\partial\boldsymbol{E}/\partial t$ through a closed surface, which, at most, can be the same as the limit surface Σ_l enclosing the entire component and thereby cutting the terminals at two distinct points a and b [Fig. 17(b)]:

$$i_a - i_b = \frac{d}{dt}\oiint_{\Sigma_l}(\epsilon\boldsymbol{E})\cdot d\boldsymbol{S} \tag{41}$$

It follows that for our needs it must be

$$|i| = \left|\iint_{S_a}\boldsymbol{J}\cdot d\boldsymbol{S}\right| \gg \left|\frac{d}{dt}\oiint_{\Sigma_l}(\epsilon\boldsymbol{E})\cdot d\boldsymbol{S}\right| = \left|\frac{dQ_{\Sigma_l}}{dt}\right| \tag{42}$$

where i is the current flowing through terminal a and S_a is any open surface cutting only this terminal.

To render condition (42) in a form analogous to condition (40), let us consider the other fundamental Maxwell law, the Maxwell–Ampere law (for fuller information consult Ref. 1)

$$\oint_\Gamma(\boldsymbol{B}/\mu)\cdot d\boldsymbol{l} = \iint_{S_\Gamma}\boldsymbol{J}\cdot d\boldsymbol{S} + \frac{d}{dt}\iint_{S_\Gamma}(\epsilon\boldsymbol{E})\cdot d\boldsymbol{S} \tag{43}$$

where μ is the magnetic permeability of the medium (which is assumed to be linear, isotropic, and time-invariant) and S_Γ is any open surface that has Γ as its contour. If we have

$$\left|\oint_\Gamma(\boldsymbol{B}/\mu)\cdot d\boldsymbol{l}\right| \gg \left|\frac{d}{dt}\iint_{S_\Gamma}(\epsilon\boldsymbol{E})\cdot d\boldsymbol{S}\right| \tag{44}$$

for each closed line Γ which links the component, then

$$i_{S_\Gamma} = \iint_{S_\Gamma}\boldsymbol{J}\cdot d\boldsymbol{S} \cong \oint_\Gamma(\boldsymbol{B}/\mu)\cdot d\boldsymbol{l} \tag{45}$$

and Eq. (42) is satisfied.

In this way we must investigate the conditions given by (40) and (44). These relations require a comparison of the order of magnitude, so it is opportune, first, to render the variables concerned dimensionless. This is easily done by introducing the new dimensionless variables:

$$\boldsymbol{e} = \frac{\boldsymbol{E}}{E_c}, \qquad \boldsymbol{b} = \frac{\boldsymbol{B}}{B_c}, \qquad x = \frac{\boldsymbol{r}}{L_c}, \qquad \tau = \frac{t}{T_c} \tag{46}$$

where E_c, B_c, L_c, and T_c are the respective reference variables, for the moment all arbitrary, for the electric field, the magnetic field, position vector r, and the time t. With the introduction of these new variables, Eqs. (40) and (44) assume the forms

$$\left|\int_{\gamma_x}\boldsymbol{e}\cdot d\boldsymbol{x}\right| \gg \alpha\beta\left|\frac{d}{d\tau}\iint_{S_{\Gamma_x}}\boldsymbol{b}\cdot d\boldsymbol{S}_x\right| \tag{47}$$

$$\left|\oint_{\Gamma_x}\boldsymbol{b}\cdot d\boldsymbol{x}\right| \gg \frac{\beta}{\alpha}\left|\frac{d}{d\tau}\iint_{S_{\Gamma_x}}\boldsymbol{e}\cdot d\boldsymbol{S}_x\right| \tag{48}$$

where the dimensionless parameters α and β are defined as

$$\alpha = \frac{cB_c}{E_c}, \qquad \beta = \frac{L_c}{cT_c} \tag{49}$$

and $c = 1/\sqrt{\epsilon\mu}$ is the propagation velocity of the light in the medium.

At this point it is convenient to transform the dimensional analysis in a proper scaling, not by choosing reference variables arbitrarily but by choosing them so that we have

$$\left|\int_{\gamma_x}\boldsymbol{e}\cdot d\boldsymbol{x}\right| \approx \left|\frac{d}{d\tau}\iint_{S_{\Gamma_x}}\boldsymbol{b}\cdot d\boldsymbol{S}_x\right| \tag{50}$$

$$\left|\oint_{\Gamma_x}\boldsymbol{b}\cdot d\boldsymbol{x}\right| \approx \left|\frac{d}{d\tau}\iint_{S_{\Gamma_x}}\boldsymbol{e}\cdot d\boldsymbol{S}_x\right| \tag{51}$$

where the symbol "≈" indicates "equal in order of magnitude." In this way, the conditions in Eqs. (47) and (48) are respectively reduced to

$$\alpha\beta \ll 1 \tag{52}$$

$$\frac{\beta}{\alpha} \ll 1 \tag{53}$$

Naturally this result is always possible if the reference variables are chosen equal to the orders of magnitude of the relative variables in the regions of the space concerned. For example, to have the same order of magnitude for $\boldsymbol{e}$ and $\partial\boldsymbol{e}/\partial\tau$ it is sufficient to choose T_c equal to the characteristic time of the dynamics of the system. Thus, for dynamics of sinusoidal type with frequency f it is sufficient to set $T = 1/f$. From what has been said previously, concerning "admissible lines," it is clear that for L_c we need to choose the characteristic length of the element being examined.

At this point, parameters α and β, which the adimenstionalization has very naturally underlined, assume a particular significance. It is noted that time and space enter only into parameter β, together with velocity c, which is a characteristic constant of the electromagnetic propagation phenomena. By defining the characteristic wavelength of the electromagnetic field, $\lambda_c = cT_c$, we can say that β is the ratio between the characteristic length of the element and the wavelength,

$\beta = L_c/\lambda_c$. Parameter α can tell us whether the effects of the electric field or those of the magnetic field prevail. More precisely, introducing the energy densities associated to the reference fields E_c and B_c and to the reference dielectric constant ϵ_c and magnetic permeability μ_c, we have

$$W_m = \frac{1}{2}\frac{B_c^2}{\mu_c}, \qquad W_e = \frac{1}{2}\epsilon_c E_c^2 \tag{54}$$

and we can say that

$$\alpha^2 = \frac{W_m}{W_e} \tag{55}$$

If we could retain α independent of β, we could immediately conclude that for a sufficiently small β, Eqs. (52) and (53) are both verified and that the error made by associating a single voltage and a single current to a two-terminal element is, in fact, on the order of β. In reality, however, α depends on β; and when the latter tends to zero, it varies in a way that requires investigation. If we imagine of varying β by varying the characteristic time of the dynamics T_c, we will have $\beta \to 0$ for $T_c \to \infty$ and $\beta \to \infty$ for $T_c \to 0$. In other words, $\beta = 0$ corresponds to the steady-state.

It is evident that, for the same element, the reference fields E_c and B_c must of necessity be different if the rapidity of the dynamics under consideration is different. Thus we can distinguish between three fundamental cases: (1) when β tends to zero α tends to zero like β; (2) when β tends to zero α diverges like $1/\beta$; and (3) when β tends to zero α tends to a finite value $\alpha_0 \neq 0$.

The Capacitor (α Tends to Zero as β for $\beta \to 0$). In these conditions, Eq. (52) and hence (40) certainly hold for sufficiently small β. This means that if we limit ourselves to considering only admissible lines, the line integral of the electric field is with good approximation (the error goes to zero as β for $\beta \to 0$) independent of the integration path. In other words, the electric field is conservative with a good approximation. We note the fact that α tends to zero like β implies that in the steady-state limit ($\beta = 0$) the reference magnetic field and so also the field $\boldsymbol{B}$ is zero, while the corresponding electric field is not zero. The annulment of $\boldsymbol{B}$ in the steady-state limit leads to the annulment of the current density $\boldsymbol{J}$. Thus, one concludes that the component under examination must contain a material that blocks the passage of the electric current. In other words, a layer of insulating material must have been "interposed" between the two terminals a and b, as illustrated in Fig. 18. Evidently this component corresponds to the idea we have of a capacitor. A capacitor, in fact, consists of a pair of conducting plates from which two wires, the terminals, are brought out. The plates may be of any shape whatsoever, and they are separated by some dielectric material. Furthermore, we assume that the plates and the wires are perfect conductors and that the dielectric is a perfect insulator.

Apparently we can say nothing about condition (42) because α is proportional to β for small values. In effect, however, in this case the total charge stored in the component is zero, instant by instant, on the order of β. In fact, the electric field, which we have already said can be considered conservative on the order of β, can be represented by a scalar electric

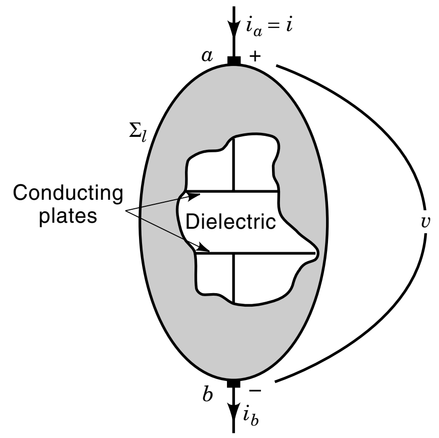


Figure 18. Sketch of a capacitor: It consists of two conducting plates separated by a dielectric.

potential. In this case the electric potential is the solution of the Laplace equation within the dielectric because in an insulating material it is not possible to have free electric charge (see, for example, Ref. 1). The boundary conditions for such a problem are given by the electric potentials on the two conducting plates of the capacitor and, thus, at each instant, by the differences between the potentials themselves and by the regularity conditions at infinity. Here we are assuming that there is no direct interaction with any other elements that might be present. In this hypothesis, as we know, the solution to the Laplace problem is, at each instant, unique (see again Ref. 1) and so is the same as that of the steady-state problem. In other words the dynamics of the electric field can be seen, at each instant, as a succession of steady-state fields and the time enters into the equation only as a parameter. In this situation the charges on the two conducting plates can only be equal in absolute value and opposite in sign, at each instant, and so the total charge must be zero (since the system is insulated, it is assumed that it is initially not charged). Thus, from Eq. (41) we have $i_a = i_b$. In conclusion, the condition $i_a = i_b = i$ must be considered to be verified with the same approximation as that by which the electric field can be held to be conservative.

Once we have defined the limits within which it is possible to associate a single voltage and a single current to the component being examined, we may ask ourselves which constitutive relation the element itself must respect. According to the charge conservation law, the current is equal to the time derivative of the total charge Q deposited on the electrode toward which the reference arrow for the current direction points:

$$i = \frac{dQ}{dt} \tag{56}$$

Charge Q is, in turn, directly proportional to the electric field, which is itself directly proportional to the voltage and so

$$Q = Cv \tag{57}$$

where the constant C is the capacitance of the capacitor (positive if the normal convention is assumed). Substituting Eq. (57) into Eq. (56) we get Eq. (6).

As examples we recall that the capacitance of a parallel-plate capacitor is $C = \epsilon S/d$, where S is the surface area of the capacitor plates, d is the distance between them, and the capacitance of a cylindrical capacitor is $C = 2\pi l\epsilon/ln(r_2/r_1)$, where l is the length and r_1 and r_2 are the radii of the internal and external cylinders.

The Inductor (α Diverges Like $1/\beta$ for $\beta \to 0$)

The case of the inductor can be dealt with in similar way. In this case, Eq. (53) and hence (42) are verified for a sufficiently small β, and so it is possible to identify a single current i for the element we are examining. It is to be noted that, as long as α diverges like $1/\beta$ for $\beta \to 0$, it is necessary for the electric field to be zero in the steady-state limit, while the magnetic field and hence the current density field are not zero. Moreover, as the electric field in the steady-state limit regime must be zero, the material has to be a perfect conductor, otherwise it would not be possible to have an electric current without an electric field. This means that in the component we are examining there is a perfect conducting wire joining the two terminals, as illustrated in Fig. 19. Thus, the element we are examining corresponds to our idea of an inductor.

However, Eq. (52) and hence (40) are not necessarily satisfied in these conditions. In effect, nothing more can be said unless one fixes how the conducting wire develops inside the element. An inductor is made by winding many turns of conducting wire in the form of a coil and bringing the two ends out at some distance from the coil. Let us consider line Γ, obtained by closing line γ_e shown in Fig. 19, which is completely outside the element, with a line γ_i developing wholly within the conducting wire. The line integral of the electric field along Γ coincides with the voltage along γ_e, since the electric field in the conductor is zero. On the other hand, from Eq. (37) we get

$$v_{\gamma_e} = -\frac{d}{dt}\iint_{S_\Gamma} \boldsymbol{B}\cdot\boldsymbol{n}\,dS = -\frac{d\Phi_\Gamma}{dt} \tag{58}$$

where Φ_Γ is the flux linked with Γ. But Φ_Γ can be decomposed into the sum of two terms, which corresponds to a suitable choice for surface S_Γ. The first contribution, which we will call Φ_i, is the flux linked with a line obtained by closing γ_i

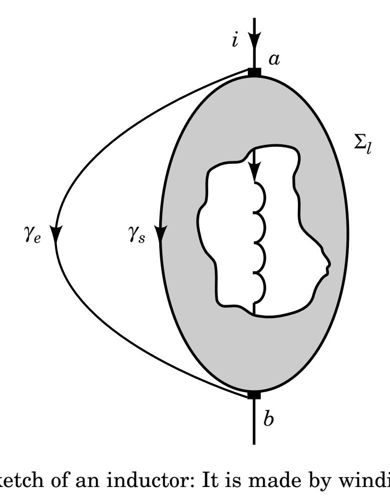


Figure 19. Sketch of an inductor: It is made by winding many turns of conducting wire.

with a line that γ_s that develops on the limit surface of the element and joins the two terminals. The second, which we will call Φ_e, is the flux linked with the line obtained by closing γ_e again with line γ_s. Then we have

$$v_{\gamma_e} = -\left(\frac{d\Phi_e}{dt} + \frac{d\Phi_i}{dt}\right) \tag{59}$$

By varying γ_e, the voltage v_{γ_e} varies because Φ_e varies, whereas Φ_i does not vary. If at this point we hypothesize that the coil ending in a and b within the component develops by forming a large number of turns, we have

$$\Phi_i \gg \Phi_e \tag{60}$$

and thus v_{γ_e} will in practice be independent of the external line γ_e. In this way we have achieved our purpose of defining a single voltage for the element we are examining, the approximation being closer as the number of turns is increased. This voltage coincides with the line integral of the electric field along γ_e, provided that such a line is wholly outside the limit surface of the component itself and that, naturally, it is "admissible" in the way we have previously specified.

We must now determine the constitutive equation of this one-port. The flux Φ_i is directly proportional to the magnetic field, which is, at each instant, directly proportional to the current i flowing in the coil; that is,

$$\Phi_i = -Li \tag{61}$$

where the constant L is the self-induction coefficient of the inductor (positive if the normal convention is assumed). At this point, by utilizing Eq. (60) from Eqs. (59) and (61) we deduce Eq. (7).

An an example we recall that the self-induction of a long solenoid of length l, cross section S, and with N turns is given by $L = (\mu N^2 S)/l$.

The Resistor in Dynamic Operation (α Tends to $\alpha_0 \neq 0$ for $\beta \to 0$)

In this case, Eqs. (52) and (53) are satisfied for sufficiently small β, and so it is possible to associate a single voltage and a single current to the component being examined. It is to be noted that as long as α tends to $\alpha_0 \neq 0$ for $\beta \to 0$, both the electric and the magnetic fields do not go to zero in the steady-state limit ($\beta = 0$). This means that there is a "link" between a and b within the one-port that develops entirely within a conductor material ($\boldsymbol{J} \neq 0$) with a finite conducibility ($\boldsymbol{E} \neq 0$), as shown in Fig. 20. Two wires which we take to be perfect conductors go from the terminals a and b to the ends of a bar of ohmic material with infinite conducibility. Thus this component is a resistor.

At this point one must deduce the constitutive relation of the element under examination. Since the current i is directly proportional to $\boldsymbol{J}$, which in turn is directly proportional to $\boldsymbol{E}$ according to Ohm's local law (11), which in turn is directly proportional to the voltage v, one deduces Eq. (5), where R is the resistance of the resistor.

The value of R is not necessarily the same that one would obtain in the steady-state limit. One can demonstrate that if (see, for example, Ref. 1)

$$\tau_d = \mu\sigma D^2 \ll T_c \tag{62}$$

where D is the characteristic transverse length of the conductor material, then the field $\boldsymbol{J}$ has with good approximation the same spatial distribution as in the steady state and so R assumes the same value given by Eq. (15). Otherwise R depends on T_c or, likewise, on the characteristic frequency of the dynamics. Parameter τ_d is called the characteristic diffusion time. By introducing the penetration depth $\delta = \sqrt{T_c/(\pi\sigma\mu)}$, Eq. (62) can be written as

$$\delta \gg D \tag{63}$$

If condition (63) is not verified, most of the current flows in the vicinity of the surface of the bar in a layer of thickness δ. For example, the penetration depth for copper is equal to 2 mm at $f = 100$ Hz.

The three limit cases that we have analyzed have led us to consider three ideal one-ports: the capacitor, the inductor, and the resistor. Naturally, in practice, such components are never ideal, so it follows that the behavior that we have described can be present in the same component at the same time. If, for example, the inductor wire is not a perfect conductor, as is the case in reality, Eq. (7) is modified. Because of the finite conducibility, the behavior of the resistor is added to that of the inductor and the constitutive equation becomes

$$v = R_p i + L\frac{di}{dt} \tag{64}$$

where R_p is the resistance of the wire in the steady-state limit. Equation (64) can also be thought of as the constitutive equation of a more complex one-port consisting of the series of an ideal inductor with an ideal resistor.

In general, an inductor consists of a large number of turns concentrated in a limited volume, and so every turn will be in close contact with other turns. Electrical contact between the coils is avoided by means of an insulating varnish covering the wire. The arrangement just described reminds us that of

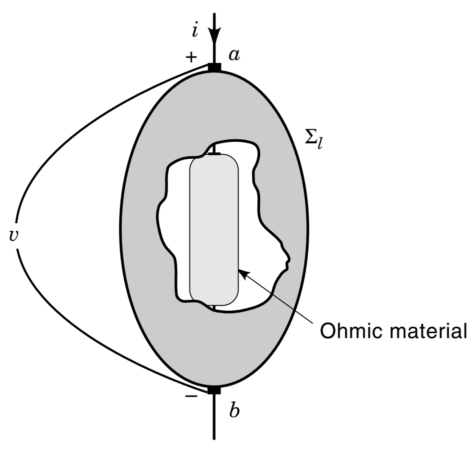


Figure 20. Sketch of a resistor: Two wires of perfect conductor connect the two terminals a and b to the ends of an ohmic material.

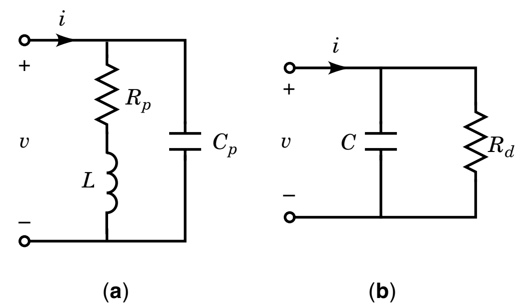


Figure 21. (a) Equivalent circuit of a "real" inductor; (b) equivalent circuit of a "real" capacitor.

the capacitor (the conducting plates are the turns, and the dielectric is the insulating varnish) and leads us to consider an overall capacity of the entire element, equivalent to the effect of all the capacities between the individual turns. One concludes that an equivalent and more refined circuit for a real inductor is that shown in Fig. 21(a), where, of course, both R_p and C_p are very small if the inductor is well-built.

A similar situation may be found in a wire-wound resistor. Here too, the numerous turns needed to obtain the required resistance cause the one-port to behave like an inductor. When such an effect is not wanted, particular arrangements are adopted to reduce the flux linked with the many windings in the resistor. For example, one may wind not a single wire, but one bent back on itself so that linked flux is practically zero. Such resistors are called anti-inductive. In the same way we have to consider that in the case of the capacitor the dielectric cannot be perfect. In such cases a conduction current density field is added to the current density field $\epsilon\partial\boldsymbol{E}/\partial t$ in the dielectric. Consequently, a more realistic equivalent circuit of a real capacitor will be that shown in Fig. 21(b), where R_d (dispersion resistance) will normally have to be very high. Although it may seem strange, for capacitors too it is sometimes necessary to consider an inductor in series with it. This is due to the fact that in some types of construction, the plates are made by rolling two layers of conducting material with the dielectric sandwiched between. The "turns" so formed make it necessary to introduce a parasite inductance in the equivalent circuit.

MUTUALLY COUPLED CIRCUITS

If a coil of the type described in Fig. 19—an inductor therefore—is placed in the immediate vicinity of another analogous element, the flux linked with each of them will depend on both the current that circulates in the first coil and that which circulates in the second. We are, therefore in the presence of an intrinsic two-port that we will call mutually coupled circuits. Mutually coupled circuits are widely used in communication circuits, in measuring instruments, and in power systems. Transformers used in power networks that transmit and distribute electric energy are coupled circuits. Electric motors and generators can also be modeled by time-varying coupled circuits. We will limit ourselves to describing the more simple, but nonetheless significant, case in which there are two coils and the reciprocal coupling does not vary in time.

Let us consider two coils wound on a toroidal-shaped magnetic iron (typically ferrite or special steel thin plates), as shown in Fig. 22(a). The coils are constituted, respectively, of N_1 and N_2 turns of wire coated with insulating varnish. If approximation (60) is valid for both the coils, one can write

$$v_1 = \frac{d\phi_1}{dt}, \qquad v_2 = \frac{d\phi_2}{dt} \tag{65}$$

where ϕ_1 and ϕ_2 are, respectively, the magnetic field fluxes linked with coils 1 and 2 produced by currents i_1 and i_2 circulating in the two coils (the electrical conducibility of the two wires is considered infinite).

To determine the relation between the two fluxes and the two currents it is necessary to make some approximations. In the limit $\beta \ll 1$ it is possible to ignore the effects due to the displacement current. Let us assume also that the toroidal loop is made of an ideal magnetic material, in which we can ignore the effects due to nonlinear phenomena, such as saturation and magnetic hysteresis. Finally, we can also assume that the effects of the eddy currents induced in the toroidal loop because of the time variation of the magnetic field are negligible (a magnetic iron material is usually an electric conductor). With these hypotheses, since the superimposition of the effects is valid and the only sources of the magnetic field are the currents circulating in the two coils, we may affirm that the relation between fluxes and the currents must be algebraic and, moreover, linear:

$$\phi_1 = L_1 i_1 + M_{12} i_2, \qquad \phi_2 = M_{21} i_1 + L_2 i_2 \tag{66}$$

where L_1, L_2, M_{12}, and M_{21} are four constants in time, independent of currents i_1 and i_2. The term $L_1 i_1$ is the flux linked with the first coil when the current i_2 in the second coil is zero, and $L_2 i_2$ is the flux linked with the second coil when current i_1 in the first coil is zero. Therefore the coefficients L_1 and L_2 are, respectively, the self-induction coefficients of coils 1 and 2. The coefficients M_{12} and M_{21} are called the mutual induction coefficients: M_{12} represents the flux of the magnetic field linked with coil 1 produced by a unitary current circulating in coil 2 when $i_1 = 0$, while M_{21} represents the flux of the magnetic field linked with coil 2 produced by a unitary current circulating in coil 1 when $i_2 = 0$.

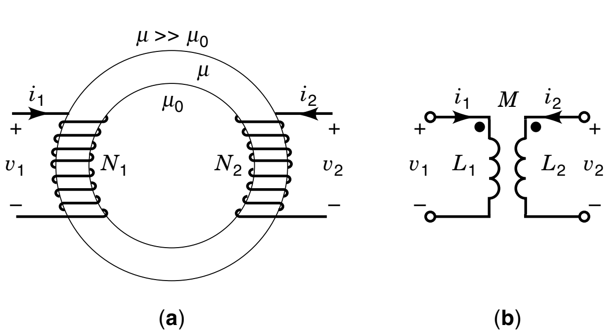


Figure 22. (a) Sketch of two coupled circuits; (b) circuital symbol for two coupled circuits.

If we define the average fluxes of self- and mutual induction

$$\begin{aligned} \phi_{11m} &= \frac{L_1 i_1}{N_1}, \qquad \phi_{12m} = \frac{M_{12} i_2}{N_1}, \\ \phi_{21m} &= \frac{M_{21} i_1}{N_2}, \qquad \phi_{22m} = \frac{L_2 i_2}{N_2} \end{aligned} \tag{67}$$

we can affirm that

$$\phi_{1d} = \phi_{11m} - \phi_{21m}, \qquad \phi_{2d} = \phi_{22m} - \phi_{12m} \tag{68}$$

are the average dispersion fluxes at coils 1 and 2, respectively. In practice if the coupling is "perfect," one can expect ϕ_{1d} and ϕ_{2d} to be zero. In other words, one may expect a current circulating in the first coil to produce, on average, the same linked flux per coil in both the first and the second coils. It can be easily demonstrated that this condition gives

$$L_1 L_2 = M_{12} M_{21} \tag{69}$$

For the mutual fluxes of magnetic fields and currents, one can demonstrate a property of reciprocity analogous to that valid for voltages and currents in resistive circuits. Consider the case where $i_1 \neq 0$ and $i_2 = 0$: Current i_1 in coil 1 may be considered as the "cause," and flux $M_{21} i_1$ linked with coil 2 may be considered as the effect. In the same way, let us consider the case in which $i_1 = 0$ and $i_2 \neq 0$: Current i_2 may be considered as the cause, and the flux $M_{12} i_2$ linked with coil 1 may be considered the effect. It is possible to show (see, for example, Ref. 1), by using the equations for the steady-state magnetic field, that the ratio between cause and effect in the two coupled circuits with $i_1 = 0$ is equal to the ratio between cause and effect in the two coupled circuits with $i_2 = 0$ and so

$$M_{12} = M_{21} = M \tag{70}$$

Combining Eqs. (65), (66) and (70), we obtain the constitutive relations of mutually coupled circuits:

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}, \qquad v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \tag{71}$$

(These equations are not valid if the self- and mutual inductances are time-varying.) Two coupled circuits constitute a dynamic two-port: The values of the two voltages, v_1 and v_2, at a generic instant do not depend only on the values of the two currents at that instant, but also on the values that they assume in the neighborhood of that instant.

The self-induction coefficients are positive if we assume the normal convention on both the ports. Instead, the mutual induction coefficient can be positive or negative, according to the reference chosen for the direction of the currents. For example, with the choice made in Fig. 22(a), the sign for M is positive. Figure 22(b) reports the circuit symbol for two coupled circuits. The two terminals are countersigned for the reference direction of the currents that make M positive. If the references for the direction of the two currents are both in agreement or both in disagreement with the countersigns, then M must be considered positive.

The Energy Properties of Coupled Circuits

The electric power absorbed by two coupled circuits is given by

$$p(t) = i_1 v_1 + i_2 v_2 = \frac{dW_m}{dt} \qquad (72)$$

where

$$W_m(i_1, i_2) = \tfrac{1}{2} L_1 i_1^2 + M i_1 i_2 + \tfrac{1}{2} L_2 i_2^2 \qquad (73)$$

On the other hand it can be demonstrated that

$$W_m(i_1, i_2) = \iiint (\boldsymbol{B}^2/2\mu)\, dv \geq 0 \qquad (74)$$

where μ is the permeability of the medium. Therefore, $W_m(i_1,i_2) = L_1 i_1^2/2 + M i_1 i_2 + L_2 i_2^2/2$ represents the energy stored in the component and it is a quadratic form, which is positive defined. Energy $W(t_0,t)$ which the coupled circuits absorb in the time interval (t_0,t) is given by

$$W(t_0, t) = W_m[i_1(t), i_2(t)] - W_m[i_1(t_0), i_2(t_0)] \qquad (75)$$

As in the case of the inductor, the energy absorbed in the time interval (t_0,t) depends only on the values that the stored energy $W_m(i_1,i_2)$ assumes at the extremities of the interval and therefore depends only on the initial and final values of the two currents i_1 and i_2, and not on their history. For example, if the values of currents at instant t are equal to the values they assume at instant t_0, then the energy absorbed by the component in the interval considered is zero, irrespective of the waveform of the currents in the interval (t_0,t). One notes that if $M_{12} = M_{21}$ were not true, it would not be possible to express the power absorbed as the time derivative of a quadratic function of currents only and so the energy absorbed would also depend on the time history of the currents.

Coupled circuits store the electric energy that they absorb in the form of magnetic field energy. The energy stored can be recovered, even completely so, in the form of electric energy in the circuit into which they are inserted. However, the electric energy that can be supplied cannot be greater than that previously absorbed due to the fact that the energy stored is positive-defined. Therefore coupled circuits are passive and conservative two-ports.

Perfect Coupling

The mutual induction coefficient is often expressed by the coupling coefficient k as a function of the self-induction coefficients with the relation

$$k = \frac{M}{\sqrt{L_1 L_2}} \qquad (76)$$

Because the energy stored in the two coupled circuits is positive-defined and the two self-induction coefficients are both positive, the coupling coefficient must verify the inequality

$$|k| \leq 1 \qquad (77)$$

Thus it is impossible to obtain a coupling coefficient greater than one. When $k = 0$, $M = 0$, and there is no interaction between the two inductors.

Consider the other limit case—that is, $k = \pm 1$. In this case, as we have already seen, the coupling is perfect. It is evident that a transformer, which in general is required to furnish the most efficient energy transfer between the two coils, must be designed and built as near as possible to the perfect coupling conditions. Let us observe that when the coupling is perfect the energy stored is given by

$$W_m(i_1, i_2) = \frac{1}{2} L_1 \left(i_1 + \frac{M}{L_1} i_2 \right)^2 \geq 0 \qquad (78)$$

and can therefore be annulled, even with $i_1 \neq 0$, $i_2 \neq 0$ if the condition $i_1 = -(M/L_1)i_2$ holds. As long as this happens, the magnetic field produced by the two currents must be zero at every point in the space; that is, the field produced by current i_1 must cancel the field due to current i_2 at every point, which is a further justification for the expression "perfect coupling."

Condition $M^2 = L_1 L_2$ is, naturally, a limit condition that can be approached by using, for example, a torus of ferromagnetic material with very high permeability ($\mu \gg \mu_0$). When this condition holds, the lines of the magnetic field are practically confined in the magnetic material. The torus behaves as if it was a flux tube for the magnetic field because the normal component of $\boldsymbol{B}$ at the limit surface of the toroidal core is practically zero and so the field in the surrounding medium is much weaker. (The analogy with the current field that flows in a conductor with electric conducibility much greater than that of the surrounding space, in which it is embedded, springs to mind.)

If the two coils are made so as to be described as two long solenoids of length l—and thus to be schematized as tracts of length of two infinite solenoid—for the coefficients L_1 and L_2 we have the following approximate expressions:

$$L_1 = \mu \frac{N_1^2 S}{l}, \qquad L_2 = \mu \frac{N_2^2 S}{l} \qquad (79)$$

It is also assumed that the two cylindrical solenoids have the same cross-section S. When the coupling is perfect the mutual induction coefficient M is given by

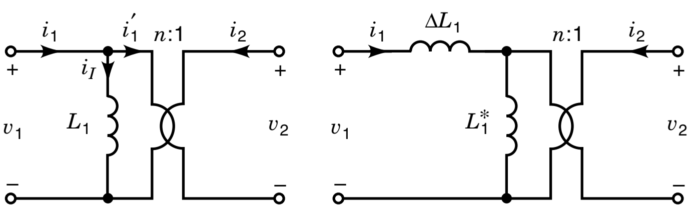
$$M = \mu \frac{N_1 N_2 S}{l} \qquad (80)$$

From characteristic Eqs. (71), in the case of perfect coupling, we obtain

$$\frac{v_1}{v_2} = \frac{L_1}{M} \qquad (81)$$

which is, in fact, the relation between the voltages of an ideal transformer with transformation ratio

$$n = \frac{L_1}{M} \qquad (82)$$



(a) (b)

Figure 23. (a) Equivalent circuit of a perfect coupling; (b) equivalent circuit of a nonperfect coupling.

From this it is easy to show that a perfect coupling is equivalent to a two-port consisting of an ideal transformer and an inductor as illustrated in Fig. 23(a). In fact we have

$$v_1 = L_1 \frac{di_L}{dt} = L_1 \frac{d}{dt}\left(i_1 + \frac{i_2}{n}\right) = L_1 \frac{di_1}{dt} + M\frac{di_2}{dt} \tag{83}$$

From the relations in Eqs. (79), (80) and (82) we obtain that for a transformer with perfect coupling the transformation ratio is approximately given by

$$n = \frac{N_1}{N_2} \tag{84}$$

Equivalent Circuit of a Nonperfect Coupling

The condition of the perfect coupling, as we have seen, is only an ideal limit condition to which one may approach. In reality, toroidal magnetic material does not provide a perfect flux tube and so the coupling coefficient, in absolute value, is less than one, even if a little less. We can show, however, that even with a nonperfect coupling it is possible to have an equivalent circuit that uses the ideal transformer. In fact, for any L_1, L_2, and M, with $M^2 < L_1L_2$, it is always possible to decompose L_1 (or L_2) into the form

$$L_1 = L_1^* + \Delta L_1 \tag{85}$$

where

$$L_1^* L_2 = M^2 \tag{86}$$

and

$$\Delta L_1 = L_1 - \frac{M^2}{L_2} > 0 \tag{87}$$

These considerations justify the equivalent circuit of a nonperfect coupling illustrated in Fig. 23(b). The inductance ΔL_1 is related to the dispersed fluxes. It describes the contribution of the flux linked with the first coil due to the lines of magnetic field that are not linked with the other coil; for $k^2 \to 1$, $\Delta L_1 \to 0$. L_1^* is said to be the magnetization inductance, and it takes account of the common flux at both the coils.

It is interesting to observe that a transformer designed and produced to obtain the best performances possible tends to be an "ideal transformer." In fact, for the coupling to be perfect it is necessary for the two coils to be strictly wound on a nucleus of ferromagnetic material with a high relative permeability, $\mu_r = (\mu/\mu_0) \gg 1$. In such condition, indeed, $\Delta L_1 \to 0$ and $k^2 \to 1$. Moreover, within the limit $\mu_r \to \infty$ we have $L_1^* \to \infty$ and so the magnetized current circulating in the inductor of inductance L_1^* must tend to zero and in consequence the equivalent circuit in Fig. 23(b) is reduced to the single ideal transformer.

FINAL CONSIDERATIONS

In the introductory paragraphs to this article we stressed the fact that the currents and voltages that concern a circuit in a given operating condition are the result of two distinct requirements: that each component in the network should behave in a manner compatible with its own nature—the constitutive relation—and that such behavior should be compatible with the interaction imposed by the rest of the circuit. In this article we have been concerned with the former aspect and have shown that within the limit $\beta \to 0$, not only for resistive elements, but also for dynamic ones, such constitutive relations are reduced to relations between the voltages and currents at the terminals of the elements in question.

Interaction with the remaining part of the network is subject to two very simple laws, Kirchhoff's law for currents and Kirchhoff's law for voltages. These laws are discussed in detail in NETWORK EQUATIONS and TIME DOMAIN CIRCUIT ANALYSIS. In concluding this article we may show that, always in relation to the hypothesis $\beta \ll 1$, Kirchhoff's laws too are deducible from the Maxwell equations for the electromagnetic field.

Kirchhoff's law for voltages, which states that in a mesh the algebraic sum of the voltages is equal to zero, is in reality a direct consequence of the fact that a one-port, or more generally a couple of terminals of an n-pole, is inserted in every branch of the mesh. The voltage between two terminals of any circuit component is in fact, in the limit $\beta \ll 1$, independent of the path (obviously we refer only to "admissible" paths) and is therefore the same, whether if it is calculated along the line γ_a or along $\gamma_b \cup \gamma_c$ as in the example shown in Fig. 24). In this light Kirchhoff's second law simply expresses

$$v_a + v_b + v_c = \oint_\Gamma \boldsymbol{E}\cdot d\boldsymbol{l} = -\frac{d}{dt}\iint_{S_\Gamma} \boldsymbol{B}\cdot d\boldsymbol{S} \cong 0 \tag{88}$$

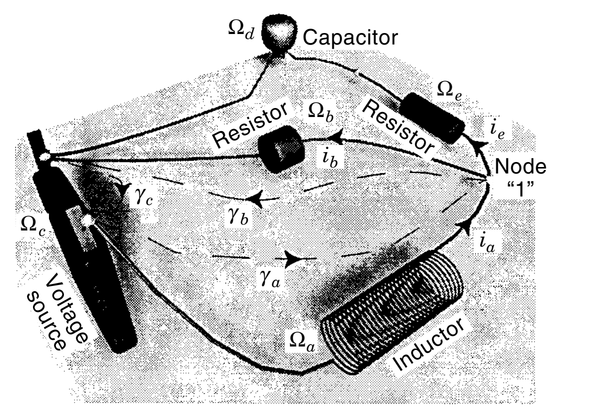


Figure 24. Sketch of an electrical circuit composed of two resistors, a generator, an inductor, and a capacitor.

where $\Gamma = \gamma_a \cup \gamma_b \cup \gamma_c$ and v_a, v_b, and v_c are, respectively, the voltages across the inductor, the resistor (Ω_b), and the voltage source with the reference directions in agreement with the orientation of γ_a, γ_b, and γ_c. In fact the time derivative of the magnetic field flux linked with the mesh Γ can always be ignored provided that Γ does not pierce the limit surfaces and the currents vary slowly in the time, which certainly holds true in the limit $\beta \ll 1$.

Kirchhoff's law for currents is also deducible from Maxwell equations, in the limit $\beta \ll 1$. In fact the algebraic sum of the currents in a node—for example, the node labeled "1" in Fig. 24—can differ from zero only if, in accordance with the charge conservation law, on the node itself there is an increase or a reduction of the electric charge Q_1:

$$-i_a + i_b + i_e = \oiint_{\Sigma_1} \boldsymbol{J} \cdot d\boldsymbol{S} = -\frac{dQ_1}{dt} \tag{89}$$

where Σ_1 is the node surface. Outside the one-ports, the electric field is quasi-conservative and thus can be expressed by a scalar potential. In these conditions the behavior of the field is the same as that which there would be in rigorously static conditions, so the charge accumulated on the node is negligible, given its smallness. In consequence we still have $dQ_1/dt \cong 0$, provided that the voltages vary very slowly in time, which is certainly so in the limit $\beta \ll 1$.

Referring to Fig. 24, where a simple circuit is illustrated by putting the spaces occupied by the single components in evidence, we note that in the circuit model the space can always be subdivided into parts, in each of which a "simplified" model of the electromagnetic field can be considered. In the region Ω_a, where there is an inductor, the model is that of the quasi-stationary magnetic field: In Maxwell equations the density of the displacement current ($\epsilon\partial\boldsymbol{E}/\partial t$) is ignored but not $\partial\boldsymbol{B}/\partial t$ ($\alpha \to \infty$ as $1/\beta$ for $\beta \to 0$). In the region Ω_d, where there is a capacitor, the model is that of the quasi-stationary electric field: In Maxwell's equations, $\partial\boldsymbol{B}/\partial t$ is ignored but not $\epsilon\partial\boldsymbol{E}/\partial t$ ($\alpha \to 0$ as β for $\beta \to 0$). Finally, in the regions Ω_b, Ω_c, and Ω_e where there are, respectively, a generator and two resistors, both the terms $\partial\boldsymbol{B}/\partial t$ and ($\epsilon\partial\boldsymbol{E}/\partial t$) are ignored and the model is that of the quasi-stationary current field ($\alpha \to \alpha_0 \neq 0$ for $\beta \to 0$). In all these models the two fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are separated, and so, if the boundary conditions—that is, the voltages or currents at the terminals of the one-ports—are assigned, then the equations for each of them can be resolved independently and univocally. It is this which makes it possible to express the constitutive equations as relations between voltage and current. Then outside the components the electric field must verify the condition $\oint_\gamma \boldsymbol{E} \cdot d\boldsymbol{l} \cong 0$ for each admissible closed line γ that does not pierce the limit surfaces, and the current density field must verify the condition $\oiint_\Sigma \boldsymbol{J} \cdot d\boldsymbol{S} \cong 0$ for every closed surface Σ that does not cut the limit surfaces. These equations express, respectively, the Kirchhoff law for the voltages and the Kirchhoff law for the currents. As a consequence, the boundary conditions of single one-ports—that is, the voltages and currents at their terminals—are subject to the two Kirchhoff laws. Note that the Kirchhoff laws are rigorously exact in steady state.

This way of interpreting circuit models allows us on the one hand to recognize the limits—which today we are approaching nearer and nearer as studied in electromagnetic compatibility—and on the other to observe its enormous simplifying possibilities: The solution of a circuit, even the simplest (such as that shown in Fig. 24), in terms of the electromagnetic field would be practically impossible.

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Giovanni Miano
University of Naples "Federico II"

LINE CODING. See Information theory of modulation codes and waveforms.

LINE ECHO CANCELLATION. See Echo cancellation for speech signals.

LINKED LISTS. See List processing.