HARMONIC OSCILLATORS, CIRCUITS

In electronics a harmonic oscillator is an electronic circuit that generates a sinusoidal signal. This signal can either be a voltage, a current, or both. Harmonic oscillators are not restricted to electronics. They can be found in many other disciplines. However, they always can be described by similar mathematical equations. A very familiar harmonic oscillator is the harmonic pendulum, which is found in many high school physics textbooks. It is a mechanical system consisting of a mass suspended by a fixed length thread. Figure 1 illustrates this. When mass m is slightly separated from its equilibrium point (so that angle θ in Fig. 1 is sufficiently small) and set free, the earth's gravitational force will make it move toward its resting point. When the mass reaches the resting point it has gained some speed that will make it keep running toward the other side of the equilibrium point, until it stops and comes back. And so it will oscillate from one side of the equilibrium point to the other. What happens is that by initially departing the mass from its equilibrium point, an external agent is increasing its potential energy. When it is set free the action of the earth's gravitational force, together with the constraint imposed by the fixed length thread, will gradually change this initial increase of potential energy into kinetic energy. At the equilibrium point all potential energy supplied initially by the external agent is in form of kinetic energy and speed is maximum. At the points of maximum elongation the kinetic energy (and speed) is zero and the original potential energy is recovered. The pendulum oscillates at constant frequency and, if there is no friction, it keeps oscillating indefinitely with constant maximum elongation or amplitude. However, in practice friction cannot be completely suppressed. Consequently, in order to have a pendulum oscillating permanently there must be a way of supplying the energy lost by friction.

In an electronic oscillator there is also a mechanism by which energy of one type is changed into another type (energy can also be of the same type but interchanged between differ-

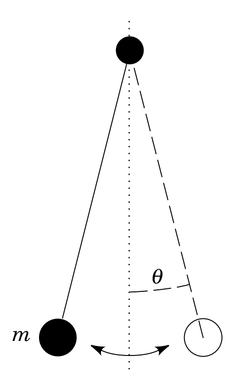

Figure 1. The mechanical pendulum behaves as a harmonic oscillator in the limit of very small maximum angle deviations.

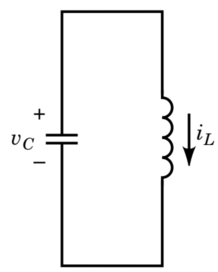

Figure 2. An ideal capacitor connected in parallel with an ideal inductor form a harmonic oscillator.

ent devices). Figure 2 shows a capacitor connected in parallel with an inductor. At equilibrium there is no voltage across the capacitor and no current through the inductor. However, if by some means, an initial voltage (or equivalently, charge) is supplied to the capacitor, its stored energy increases. The inductor provides a path to discharge the capacitor so that a current builds up through the inductor. However, by the time the capacitor has zero charge the current flowing through the inductor is maximum and the inductor stores all the original capacitor energy in the form of magnetic flux energy. The consequence is that the current keeps flowing through the inductor, charging now the capacitor oppositely, until the current is zero. If there are no resistive losses this process will continue indefinitely: capacitor and inductor keep interchanging their stored energies. The voltage across the capacitor will be sinusoidal in time, and so will be the current through the inductor. The amplitude (or maximum elongation) of the voltage oscillations is equal to the initial voltage supplied to the capacitor. In practice both capacitor and inductor have resistive losses, so that in order to keep the system oscillating indefinitely there must be a way of supplying the energy being lost.

IDEAL RESONATOR MATHEMATICAL MODEL

In Fig. 2 the capacitor voltage v_C and its current i_C are related mathematically by the expression

$$i_C = C\frac{dv_C}{dt} \tag{1}$$

where C is the capacitor's capacitance. For the inductor, its voltage v_L and current i_L are related by

$$v_L = L\frac{di_L}{dt} \tag{2}$$

where L is the inductor's inductance. Besides this, the circuit of Fig. 2 imposes the following topological constraints

$$\begin{aligned} v_C &= v_L \\ i_C &= -i_L \end{aligned} \tag{3}$$

Solving Eqs. (1–3) yields

$$\frac{d^2v_C}{dt^2} + \frac{1}{LC}v_C = 0 \tag{4}$$

The solution to this second order time domain differential equation is

$$v_C(t) = v_C(0)\cos(\omega t) - i_L(0)\sqrt{L/C}\sin(\omega t) \tag{5}$$

where $v_C(0)$ is the capacitor voltage at time zero, $i_L(0)$ is the inductor current at time zero and ω is the angular frequency of the resulting oscillation whose value is

$$\omega = \frac{1}{\sqrt{LC}} \tag{6}$$

Using Eqs. (3), (5), and (6) in Eq. (1) results in

$$i_L(t) = i_L(0)\cos(\omega t) + v_C(0)\sqrt{C/L}\sin(\omega t) \tag{7}$$

By means of basic trigonometric manipulations Eqs. (5) and (7) can be rewritten as

$$\begin{aligned} v_C(t) &= V_{\max}\cos(\omega t + \varphi) \\ i_L(t) &= V_{\max}\sqrt{C/L}\sin(\omega t + \varphi) \end{aligned} \tag{8}$$

where,

$$\begin{aligned} V_{\max} &= \sqrt{v_C^2(0) + i_L^2(0)L/C} \\ \varphi &= \arctan\left(\frac{i_L(0)}{v_C(0)}\sqrt{L/C}\right) \end{aligned} \tag{9}$$

Equation (8) reveals that $v_C(t)$ and $i_L(t)$ have a phase shift of $\pi/2$ radians. This is usually referred to as $v_C(t)$ and $i_L(t)$ being in quadrature, and the resonator in Fig. 2 as being a quadrature resonator or oscillator. Note that the maximum oscillation amplitudes ($V_{\max}$ or $V_{\max}\sqrt{C/L}$, respectively) depend on the initial conditions $v_C(0)$ and $i_L(0) = -C\,\dot{v}_C(0)$.

Usually, differential equations like Eq. (4) are not solved directly in the time domain but in the frequency domain. For this, let us take the Laplace transform of Eq. (4)

$$s^2V_C(s) - sv_C(0) - \dot{v}_C(0) + \frac{V_C(s)}{LC} = 0 \tag{10}$$

where $V_C(s)$ is the Laplace transform of $v_C(t)$. Since $i_L(0) = -C\,\dot{v}_C(0)$, Eq. (10) can be rewritten as

$$V_C(s) = \frac{s}{s^2 + 1/LC}v_C(0) - \frac{1}{s^2 + 1/LC}\frac{i_L(0)}{C} \tag{11}$$

Taking the inverse Laplace transform of Eq. (11) results in Eq. (5). Usually in circuits, the initial conditions involved in the Laplace transform are ignored and Eq. (10) is simplified to

$$s^2 + \frac{1}{LC} = 0 \tag{12}$$

which has the following solutions

$$\begin{aligned} s_1 = j\omega&,\ s_2 = -j\omega \\ \omega &= \frac{1}{\sqrt{LC}} \end{aligned} \tag{13}$$

Solutions s_1 and s_2 are called the *poles* of the system, and in this case the two poles are complex conjugate and are purely imaginary (their real part is zero). In circuits, people don't take the inverse Laplace transform to know the solution. They know that if a system has a pair of purely imaginary poles the signals have a sinusoidal steady state whose amplitude depends on the initial conditions.

REAL RESONATOR MATHEMATICAL MODEL

As mentioned earlier, the circuit of Fig. 2 is ideal. In practice there will always be resistive losses in the capacitor, in the inductor, or in both. Either introducing a small resistance R_L in series with the inductor, a large resistance R_C in parallel with the capacitor, or both can model this. Solving such a circuit yields the following time domain differential equation

$$\frac{d^2 v_C(t)}{dt^2} + b\frac{dv_C(t)}{dt} + \omega^2 v_C(t) = 0 \tag{14}$$

with

$$\begin{aligned} b &= \frac{1}{R_C C} + \frac{R_L}{L} \\ \omega^2 &= \frac{1 + R_L/R_C}{LC} \end{aligned} \tag{15}$$

The solution to Eq. (14) is

$$v_C(t) = V_{\max} e^{-bt/2} \cos(\omega_o t + \rho) \tag{16}$$

where $\omega_o^2 = \omega^2 - (b/2)^2$. Parameters $V_{\max}$ and ρ can be found from the initial conditions $v_C(0)$ and $\dot{v}_C(0)$,

$$\begin{aligned} V_{\max} &= \frac{v_C(0)}{\cos\rho} \\ \rho &= -\arctan\left(\frac{\dot{v}_C(0) + \frac{b}{2} v_C(0)}{\omega_o v_C(0)}\right) \end{aligned} \tag{17}$$

However, circuit people prefer to solve Eq. (14) in the frequency domain by taking its Laplace transform,

$$s^2 + bs + \omega^2 = 0 \tag{18}$$

The solution to this equation provides the following poles

$$\begin{aligned} s_1 &= -\frac{b}{2} + j\omega\sqrt{1 - \left(\frac{b}{2\omega}\right)^2} = -\frac{b}{2} + j\omega_o \\ s_2 &= -\frac{b}{2} - j\omega\sqrt{1 - \left(\frac{b}{2\omega}\right)^2} = -\frac{b}{2} - j\omega_o \end{aligned} \tag{19}$$

which are two complex conjugate poles with a negative real part. Circuit people know that when a system has a pair of complex conjugate poles with a negative real part, the system oscillates in a sinusoidal fashion with an amplitude that vanishes after some time. This is what Eq. (16) shows. The amplitude of the oscillations $A(t) = V_{\max}e^{-bt/2}$ decreases exponentially with time. After a few time constants $2/b$ the amplitude is negligible and one can consider that the system has stopped oscillating. Consequently, in practice, the circuit of Fig. 2 is not useful for building an oscillator.

Imagine that somehow we could make R_L or R_C (or both) negative, so that $b < 0$. A negative resistance behaves as an energy source that replaces the energy dissipated by positive resistances. In this case the poles would have a positive real part and the amplitude of the oscillations

$$A(t) = V_{\max} e^{-bt/2} \tag{20}$$

would increase exponentially in time, assuming $V_{\max} \neq 0$ (due to noise $V_{\max}$ cannot be exactly zero all the time). This would yield an oscillator with initial startup but that would be unstable, because its amplitude would be "out of control." What circuit designers do to build oscillators with stable amplitude is to make the term b in Eq. (18) depend on the instantaneous oscillation amplitude $A(t)$,

$$b(t) = b(A(t)) \tag{21}$$

and in such a way that b increases with A, b is negative for $A = 0$ (to ensure initial startup), and b becomes positive above a certain amplitude. This is called amplitude control. For instance, assume that by adding some special circuitry to Fig. 2 we are able to make

$$b(A) = -b_0 + b_1 A \tag{22}$$

where b_0 and b_1 are positive constants (note that A is always positive). Initially, if $A = 0$, $b = -b_0$ is negative and the real part of the poles is positive: amplitude $A(t)$ increases exponentially with time. As $A(t)$ increases b will eventually become positive (poles with negative real part) and this will decrease the amplitude $A(t)$. The consequence of these two tendencies is that a steady state will be reached for which $b = 0$ and the amplitude is constant. Solving Eq. (22) for $b = 0$ yields the value of the steady state oscillation amplitude A_o,

$$A_o = \frac{b_0}{b_1} \tag{23}$$

Note that, as opposed to the ideal resonator, the steady state amplitude is independent of any initial conditions.

In general, a harmonic oscillator does not have to be a second order system like the case of Eq. (14) or Eq. (18). It can have any order. What is important is that it has a pair of complex conjugate poles whose real part can be controlled by the oscillation amplitude (so that the real part becomes zero in the steady state), and that the rest of the poles (either complex conjugate or not) have negative real parts. The way b depends on A does not have to be like in Eq. (22). Strictly speaking, the conditions are

$$\begin{aligned} &b(A=0) = -b_0 < 0 && \text{for initial startup} \\ &b > 0 && \text{for some A} \\ &\frac{db(A)}{dA} > 0 && \text{for stable amplitude control} \end{aligned} \tag{24}$$

This will ensure stable oscillator operation. In what follows we will concentrate on second order systems and will provide two different ways of performing amplitude control.

AMPLITUDE CONTROL BY LIMITATION

A very widely used method for oscillator amplitude control is by limitation. This method usually is simple to implement, so simple that many times it is implicit in the components used to build the resonator with initial startup. This makes practical circuits easy to build, although many times people don't understand the underlying amplitude control mechanism.

Let us consider the ideal resonator of Fig. 2 with an additional resistor R_p in parallel to make it real. In order to assure initial startup, let us put a negative resistor in parallel also. Figure 3(a) shows a very simple way to implement one using a real (positive) resistor and a voltage amplifier of gain larger than one (for example, two). Current I_{in} will be

$$I_{\text{in}} = -\frac{V_{\text{in}}}{R_{\text{n}}} \tag{25}$$

and the structure Resistor–Amplifier behaves as a grounded negative resistor of value $-R_n$. Figure 3(b) shows how to build the amplifier using an operational amplifier and resistors. Connecting this negative resistor in parallel with a positive one of value $R_p = R_n + R_\epsilon$ (with $R_\epsilon \ll R_n$), the equivalent parallel resistance would be

$$R_{\text{eq}} = -\frac{R_{\text{n}}^2}{R_\epsilon} \tag{26}$$

which is a very high but negative resistance. Connecting this equivalent resistor in parallel with the ideal resonator of Fig. 2 provides an oscillator with initial startup. This is shown in Fig. 3(c).

Due to the fact that the operational amplifier's output voltage cannot go above its positive power supply V_{DD} or below its negative one V_{SS}, the negative resistance emulator circuit just described works as long as the amplifier output is below V_{DD} and above V_{SS}, or equivalently voltage V_{in} is between $V_{DD}/2$ and $V_{SS}/2$. It is easy to compute the current through R_{eq} as a function of V_{in} taking into account this saturation effect. Figure 3(d) shows the resulting curve. If $V_{SS}/2 \le V_{in} \le V_{DD}/2$ resistor R_{eq} behaves as a negative resistance of high value, but if V_{in} is outside this range the slope of I_{in} versus V_{in} is that of a positive resistance with much smaller value. In order to analyze what happens to the circuit of Fig. 3(c) when the oscillating amplitude increases beyond $V_{DD}/2$ or $V_{SS}/2$ (whichever is smaller) the concept of describing function can be used.

Describing Function

Figure 4 shows a sinusoidal signal $x(t)$ applied to a nonlinear element $f(x)$ that outputs a distorted signal $y(t)$. Signal $y(t)$ is no longer sinusoidal, but it is periodic. Consequently, a Fourier series can describe it. The first (or fundamental) harmonic has the same frequency as the input sinusoid, while the others have frequencies which are integer multiples of the first one. If the block of Fig. 4 is used in a system such that the end signals will be approximately sinusoidal (like in a harmonic oscillator) then one can neglect all higher harmonics of the Fourier expansion of $y(t)$ and approximate it using

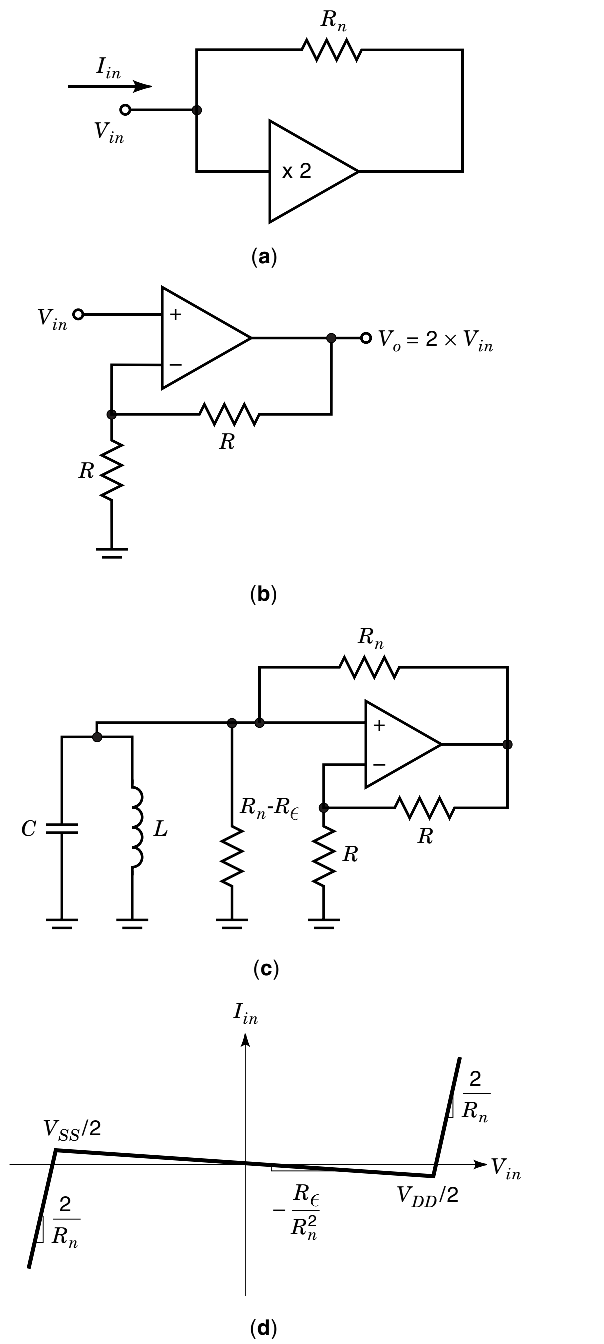

Figure 3. A real harmonic oscillator can be made by adding a negative resistor to a real resonator. (a) A negative resistor can be emulated using a resistor and a voltage amplifier of gain greater than unity. (b) Implementation of negative resistor using an operational amplifier and resistors. (c) Oscillator composed of capacitor inductor and negative resistor. (d) Transfer characteristics of the negative resistor implementation of (b).

the first or fundamental harmonic only,

$$\begin{aligned} y(t) &\approx N(A)x(t) \\ x(t) &= A\sin(\omega t) \\ N(A) &= \frac{\omega}{\pi A}\int_0^{2\pi/\omega} f(x(t))\sin(\omega t)\,dt \end{aligned} \tag{27}$$

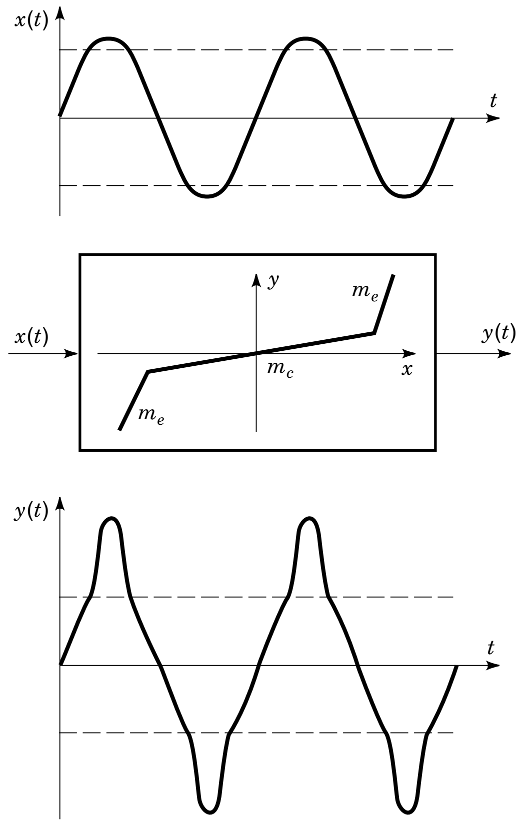

Figure 4. A sinusoidal signal applied to a nonlinear element results, in general, in a distorted output signal.

Note that this approximation makes $y(t)$ to be linear with $x(t)$ so that the nonlinear block in Fig. 4 can be modeled by a linear amplifier of gain $N(A)$. Function $N(A)$ is called the describing function of the nonlinear element $f(\cdot)$.

This approach is valid for any nonlinear function $f(\cdot)$, but let us consider only piece-wise linear functions, like in Figs. 3 and 4, with three pieces: a central linear piece of slope m_c and two external linear pieces of slope m_e. When amplitude A is small enough so that $x(t)$ is always within the central piece then $N(A) = m_c$. When A increases beyond the central piece $N(A)$ will change gradually towards value m_e. In the limit of $A = \infty$ the describing function will be $N(A) = m_e$. Computing the first Fourier term provides the exact expression (let us assume $V_{SS} = -V_{DD}$ for simplicity). If $A \geq V_{DD}/2$

$$N(A) = m_e - 2\frac{m_e - m_c}{\pi}\left[\sin^{-1}\left(\frac{V_{DD}}{2A}\right) + \frac{V_{DD}}{2A}\sqrt{1-\left(\frac{V_{DD}}{2A}\right)^2}\right] \tag{28}$$

and if $A \leq V_{DD}/2$

$$N(A) = m_c \tag{29}$$

Applying the describing function method to the nonlinearity of Fig. 3(d) results in

$$I_{in} = N(A)V_{in} \tag{30}$$

where $N(A) = -R_\epsilon/R_n^2$ for $A \leq V_{DD}/2$ and $N(A)$ tends towards $2/R_n$ as it increases beyond $V_{DD}/2$. Since $N(A)$ is continuous and monotonic, there will be a value of A (and only one) for which $N(A) = 0$. Let us call this value A_o. Note that A_o depends only on the shape of the nonlinear function $f(\cdot)$ of Fig. 3(d). Equation (29) is the equation of a resistor of value $R_{eq} = 1/N(A)$. For small values of A, $R_{eq} = -R_n^2/R_\epsilon$ (high resistance but negative) and the oscillator possesses exponentially increasing amplitude. When A increases beyond $V_{DD}/2$, R_{eq} will become more and more negative until $N(A) = 0$. At this point $A = A_o$, $R_{eq} = \infty$ and we have the ideal resonator. If A increases further, R_{eq} becomes positive and the oscillator presents exponentially decreasing amplitude. This represents a stable amplitude control mechanism such that in the steady state $A = A_o$ and $R_{eq} = \infty$.

General Formulation

In general, the block diagram of Fig. 5 describes a harmonic oscillator with amplitude control by limitation, where $H(s)$ is a linear block (or filter) and $f(x)$ is the nonlinear element responsible for the amplitude control. Applying the describing function method to the nonlinear block results in

$$y(t) = N(A)x(t) \tag{31}$$

for a time domain description. For a frequency domain description it would be

$$Y(s) = N(A)X(s) \tag{32}$$

On the other hand, input and output of the linear block or filter are related in the frequency domain by

$$X(s) = H(s)Y(s) \tag{33}$$

Equation (32) and (33) result in

$$H(s)N(A) = 1 \tag{34}$$

If $H(s)$ is a second order block it can be described by

$$H(s) = \frac{a_1s^2 + a_2s + a_3}{s^2 + a_4s + a_5} \tag{35}$$

which together with Eq. (34) yields an equation of the form

$$\begin{aligned} s^2 + sb + \omega^2 &= 0 \\ b &= \frac{a_1 - a_2N(A)}{1 - a_1N(A)} \\ \omega^2 &= \frac{a_5 - a_3N(A)}{1 - a_1N(A)} \end{aligned} \tag{36}$$

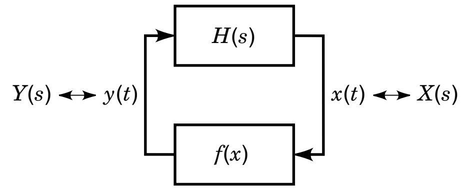

Figure 5. A general block diagram of an oscillator with amplitude control by limitation consists of a linear filter and a nonlinear amplitude controlling element connected in a loop.

For small amplitudes $N(A)$ is equal to some constant (for example n_0) and Eq. (36) is called the characteristics equation. It must be assured that $b(A = 0) < 0$. This is usually referred to as the oscillation condition. For stable amplitude control it should be

$$\frac{db(A)}{dA} > 0 \tag{37}$$

and ω^2 must be kept always positive for all possible values of A. In practice, it is desirable to make in Eq. (35) $a_1 = a_3 = 0$, which will make ω^2 and b not be coupled through a common parameter. This way the oscillation amplitude and frequency can be controlled independently.

A Practical Example: the Wien-Bridge Oscillator

In a practical circuit it is not convenient to rely on inductors because of their limited range of inductance values, high price, and, in VLSI (very large scale integration) design, they are not available unless one operates in the GHz frequency range. But it is possible to implement the filter function of Fig. 5 without inductors. The Wien-Bridge oscillator of Fig. 6 is such an example. Figure 6(a) shows its components: two resistors, two capacitors, and a voltage amplifier of gain k_0. Figure 6(b) illustrates an implementation using an opamp and resistors for the voltage amplifier, and Fig. 6(c) shows its piece-wise linear transfer characteristics. Using the describing function, the effective gain of the amplifier $k(A)$ can be expressed as a function of the sinusoidal amplitude A at node v_1,

$$k(A) = k_0 N(A) \tag{38}$$

where $k_0N(A)$ is the describing function for the function in Fig. 6(c) and is given by Eq. (28) with $m_c = k_0$, $m_e = 0$, and the breakpoint changes from $V_{DD}/2$ to V_{DD}/k_0. Consequently, the frequency domain description of the circuit in Fig. 6(a) is

$$\begin{aligned} s^2 + sb + \omega^2 &= 0 \\ b &= \frac{1}{R_2C_2} + \frac{1}{R_1C_1} - \frac{k_0N(A) - 1}{R_1C_2} \\ \omega^2 &= \frac{1}{R_1C_1}\frac{1}{R_2C_2} \end{aligned} \tag{39}$$

For initial startup it must be $b < 0$ for $A = 0$. The final amplitude A_0 is obtained by solving $b(A_0) = 0$, and the frequency of the oscillation is ω (in radians per second) or $f = \omega/2\pi$ (in hertz). Optionally, the diodes in Fig. 6(b), connected to voltage sources v^+ and v^-, can be added to control the oscillation amplitude. These diodes change the saturation voltage V_{DD} of Fig. 6(c), and hence will modify the describing function $N(A)$.

In general, when using amplitude control by limitation, a practical advice is to make $-b_0$ as close as possible to zero but without endangering its sign. This way the nonlinear element will distort very little the final sinusoid, because it needs to use only a small portion of its nonlinear nature to make $b(A)$ become zero. If $-b_0$ is too large the resulting waveform will probably look more like a triangular signal than a sinuoidal one.

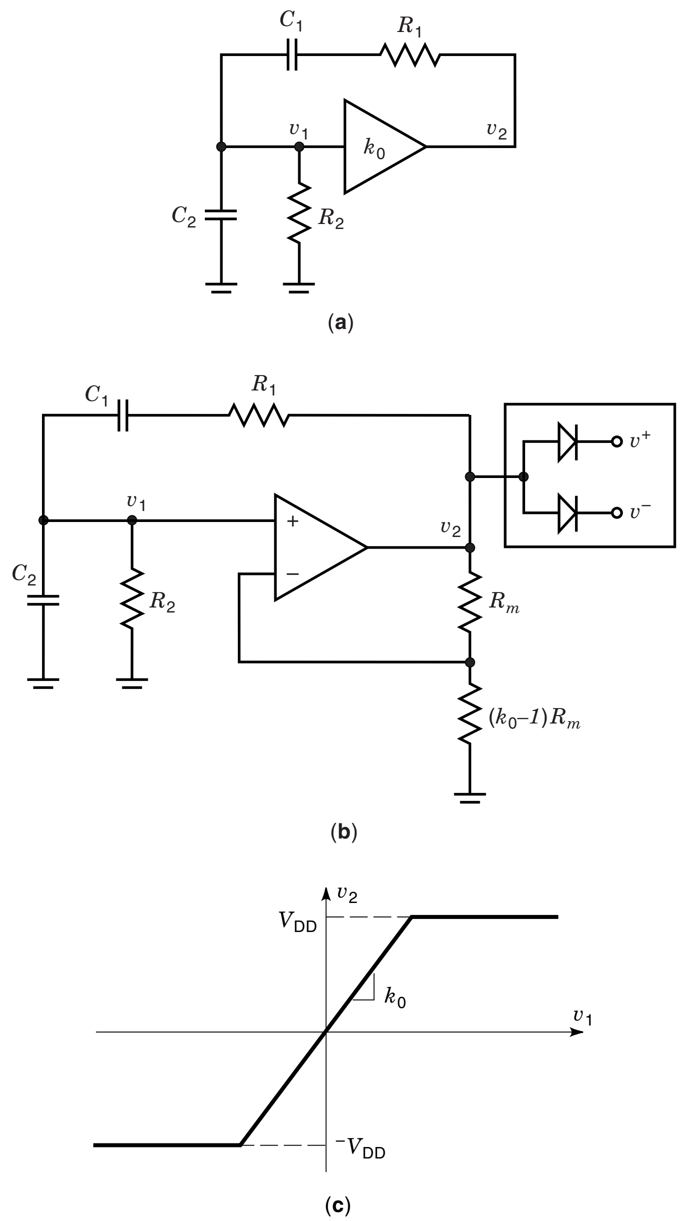

Figure 6. The Wien-Bridge oscillator is an example of an oscillator that does not require an inductor. (a) It consists of two resistors, two capacitors, and a voltage amplifier circuit. (b) The voltage amplifier can be assembled using an opamp and two resistors. (c) The resulting voltage amplifier has nonlinear transfer characteristics.

AMPLITUDE CONTROL BY AUTOMATIC GAIN CONTROL

Let us illustrate the amplitude control by AGC (automatic gain control) using an OTA-C oscillator. An OTA (operational transconductance amplifier) is a device that delivers an output current I_0 proportional to its differential input voltage V_{in}. Figure 7(a) shows its symbol and Fig. 7(b) its transfer characteristics. The gain (slope g_m in Fig. 7(b)) is called the transconductance. This gain is electronically tunable through voltage V_{bias} (depending on the technology and the design, the tuning signal can also be a current). Using these devices, the self-starting oscillator of Fig. 7(c) can be assembled. Note that g_{m1}, g_{m2}, and C_1 emulate an inductance of value $L =$

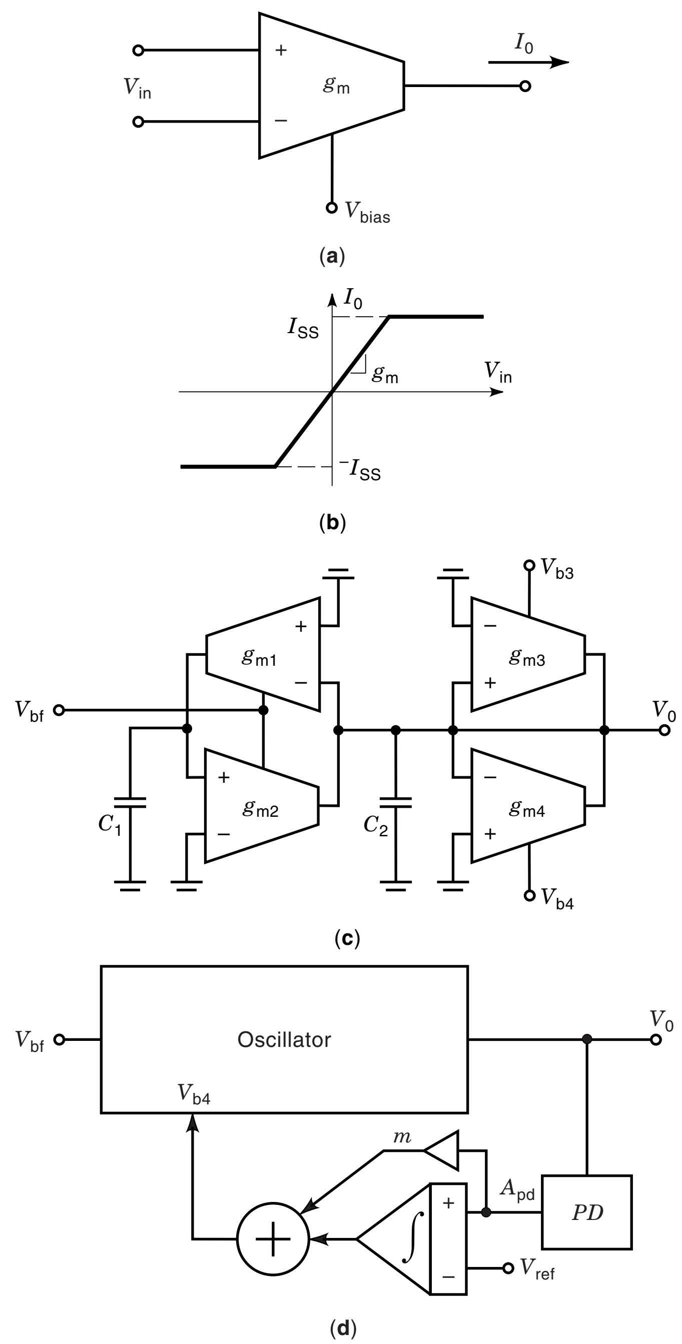

Figure 7. An oscillator with amplitude control by AGC can be made easily with OTAs and capacitors (OTA-C). (a) An OTA delivers an output current proportional to its differential input voltage. (b) It has nonlinear transfer characteristics. (c) A self-starting OTA-C oscillator can be made with four OTAs and two capacitors. (d) The amplitude control by AGC requires an additional peak detector and integrator.

$C_1/(g_{m1}g_{m2})$, g_{m3} emulates a negative resistance of value $R_3 = -1/g_{m3}$, and g_{m4} emulates a positive one of value $R_4 = 1/g_{m4}$. The characteristics equation of the OTA-C oscillator is

$$
\begin{aligned}
s^2 + bs + \omega^2 &= 0 \\
b &= \frac{g_{m4} - g_{m3}}{C_2} \\
\omega^2 &= \frac{g_{m1}}{C_1}\frac{g_{m2}}{C_2}
\end{aligned}
\tag{40}
$$

To assure initial startup V_{b3} and V_{b4} must be such that $g_{m3} > g_{m4}$. By making g_{m3} (or g_{m4}) depend on the oscillation amplitude A, an AGC for amplitude control can be realized. This is illustrated in Fig. 7(d) where the box labeled oscillator is the circuit in Fig. 7(c), the box labeled PD is a peak detector, the large triangle represents a differential input integrator of time constant τ_{AGC}, the small triangle is an amplifier of gain m necessary for stability of the AGC loop, and the circle is a summing circuit. The output of the peak detector $A_{pd}(t)$ follows (with a little delay) $A(t)$, the amplitude of the sinusoid at V_0. The error signal resulting from subtracting A_{pd} and V_{ref} is integrated and used to control g_{m4}. If $A_{pd} > V_{ref}$ gain g_{m4} will increase (making b positive, thus decreasing A), and if $A_{pd} < V_{ref}$ gain g_{m4} will decrease (making b negative, thus increasing A). In the steady state $A = A_{pd} = V_{ref}$ and g_{m4} will automatically be adjusted to make $b = 0$. Note that V_{ref} must be such that the node voltages are kept within the linear range of all OTAs, otherwise amplitude control by limitation may be taking place.

OTA-C oscillators are convenient for AGC because their gain can be adjusted electronically. In order to do this for the Wien-Bridge oscillator of Fig. 6, either a capacitor or a resistor must be made electronically tunable (using a varicap or a JFET). Also, OTA-C oscillators are interesting because they do not need resistors, and this is very attractive for VLSI in CMOS technology where resistors have very bad electrical characteristics and a limited range of values.

Stability of Automatic Gain Control Loop

An AGC loop for amplitude control, like the one in Fig. 7(d), presents a certain dynamic behavior which can be analyzed in order to (1) make sure it is a stable control loop and (2) optimize its time response.

In Fig. 7(d) the peak detector output $A_{pd}(s)$ can be modeled as a delayed version of $A(s)$,

$$A_{pd}(s) = A(s)(1 - s\tau_{pd}) \tag{41}$$

where $A_{pd}(s)$ and $A(s)$ are the Laplace transforms of the small signal components of $A_{pd}(t)$ and $A(t)$, respectively. Signal $V_{b4}(s)$ (the Laplace transform of small signal component of $V_{b4}(t)$), according to Fig. 7(d) satisfies

$$V_{b4}(s) = \frac{1}{s\tau_{AGC}}\left[(1 + sm\tau_{AGC})A_{pd}(s) - V_{ref}(s)\right] \tag{42}$$

and controls parameter b in Eq. (40). Let us assume that $b(t)$ follows instantaneously $V_{b4}(t)$ so that

$$b(s) = \alpha V_{b4}(s) \tag{43}$$

Now what is left in order to close the control loop is to know how the amplitude $A(t)$ (or $A(s)$ in the frequency domain) at node V_0 depends on b.

This dependence can easily be obtained from the time-domain differential equation (like Eq. (14)) in the following way: assume $b(t)$ is a time dependent signal that has small changes around $b = 0$ and keeps $A(t)$ approximately constant around A_0. Then the solution to $V_0(t)$ (or $v_C(t)$ in Eq. (14)) can be written as

$$V_o(t) = A(t)\cos(\omega_o t + \varphi) \tag{44}$$

where $A(t) = A_0 + a(t)$ and $|a(t)| \ll A_0$. Substituting Eq. (44) into $v_C(t)$ of Eq. (14) yields the following coefficients for the $\cos(\cdot)$ and $\sin(\cdot)$ terms, respectively, which must be identically zero [if Eq. (44) is indeed a solution for Eq. (14)],

$$\begin{aligned}\frac{d^2A(t)}{dt^2} + b(t)\frac{dA(t)}{dt} + b^2(t)\frac{A(t)}{4} &= 0 \\ 2\frac{dA(t)}{dt} + A(t)b(t) &= 0\end{aligned} \tag{45}$$

The first equation is not of much use, but from the second it follows that

$$A(t) = A(t_0)e^{-\frac{1}{2}\int_{t_0}^{t} b(t)dt} \tag{46}$$

When the AGC loop is in its steady state $A(t) = A_0 + a(t)$ and the integral is a function that moves above and below zero but is always close to zero. Consequently, the exponential can be approximated by its first order Taylor expansion resulting in

$$\begin{aligned}A(t) &\approx A(t_0)\left[1 - \frac{1}{2}\int_{t_0}^{t} b(t)\,dt\right] \\ \Rightarrow a(t) &\approx -\frac{A(t_0)}{2}\int_{t_0}^{t} b(t)\,dt \approx -\frac{A_0}{2}\int_{t_0}^{t} b(t)\,dt\end{aligned} \tag{47}$$

In the frequency domain this is

$$A(s) \approx -\frac{A_0}{2s}b(s) \tag{48}$$

From Eqs. (41–43) and (48) a loop equation for the AGC control can be written

$$\begin{aligned}A(s) &= \frac{V_{\mathrm{ref}}(s)}{s^2k_1 + sk_2 + 1} \\ k_1 &= \frac{2\tau_{\mathrm{AGC}}}{\alpha A_0} - m\tau_{\mathrm{AGC}}\tau_{\mathrm{pd}} \\ k_2 &= m\tau_{\mathrm{AGC}} - \tau_{\mathrm{pd}}\end{aligned} \tag{49}$$

This equation represents a stable control system if the poles have negative real part. This is achieved if $k_1 \geq 0$ and $k_2 > 0$. Parameters k_1 and k_2 can also be optimized for optimum amplitude transient response (for example, after a step response in $V_{\mathrm{ref}}(t)$).

VOLTAGE CONTROLLED HARMONIC OSCILLATORS

An oscillator whose frequency can be electronically controlled is required in many applications. Such an oscillator is called a voltage controlled oscillator or VCO, although sometimes the control parameter can also be a current.

In the case of the Wien-Bridge oscillator of Fig. 6 the frequency of oscillation ω is controlled by R_1, R_2, C_1, and C_2. Changing one or more of these parameters would enable external control of the frequency. In order to have an electronic control there are two options: (1) continuous or analog control, and (2) discrete or digital control.

For analog control of the Wien-Bridge oscilator of Fig. 6 either a voltage controlled resistor (JFET) or a voltage controlled capacitor (varicap) is needed. Digital control can easily be implemented by using a binary weighted array of resistors or capacitors that are switched in and out of the circuit by means of a digital bus. This is exemplified in Fig. 8 for resistor R_2. Signals s_i are either 0 when the switch is open or 1 when it is closed. This yields

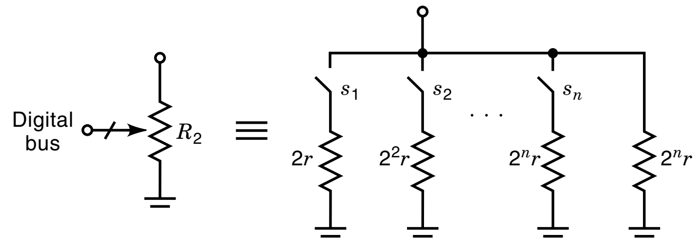

Figure 8. The frequency of the Wien-Bridge oscillator can be digitally controlled by replacing one of the resistors by a binarily weighted resistor array controlled through a digital data bus.

$$\frac{1}{R_2} = \frac{1}{r}\left[\frac{1}{2^n} + \sum_{i=1}^{n}\frac{s_i}{2^i}\right] = \frac{1}{r}d_n \tag{50}$$

where d_n is a number that ranges from $1/2^n$ to 1 in steps of $1/2^n$. Number d_n is represented in binary format by the bits $\{s_ns_{n-1}\ .\ .\ .\ s_2s_1\}$.

The OTA-C oscillator of Fig. 7 is much better suited for analog or continuous control of frequency. If $g_{\mathrm{m1}} = g_{\mathrm{m2}}$ and $C_1 = C_2$, the frequency is equal to $\omega = 2\pi f = g_{\mathrm{m1}}/C_1$. Since voltage V_{bf} in Fig. 7(d) controls simultaneously g_{m1} and g_{m2} (making them equal), this voltage can be used directly to control the frequency of the VCO.

Whether a VCO is made with OTAs and capacitors, or with resistors, capacitors, and opamps, or uses some other technique, in general it turns out that the frequency does not have a linear dependence on the control voltage. In practical circuits it also happens that if the control voltage is maintained constant, the frequency may change over long periods of time due to temperature changes which cause device and circuit parameters (such as transconductance and resistance) to drift. Both problems can be overcome by introducing a frequency control loop.

FREQUENCY CONTROL LOOP

Figure 9(a) shows the basic concept of a frequency control loop for VCOs. It consists of a VCO (for example, the one in Fig. 7(d)), a differential input voltage integrator, and a frequency to voltage converter (FVC) circuit. Voltage V_{CO} is now the external control of the VCO frequency. The FVC circuit delivers an output voltage V_{FVC} that depends linearly on the frequency f of its input signal V_{OSC},

$$V_{\mathrm{FVC}} = \rho f + V_{\mathrm{F0}} \tag{51}$$

Parameters ρ and V_{F0} must be constants and should not depend on temperature or technological parameters that change from one prototype to another. If such an FVC is available, the circuit in Fig. 9(a) would stabilize at $V_{\mathrm{FVC}} = V_{\mathrm{CO}}$. According to Eq. (51), this means that the resulting oscillation fre-

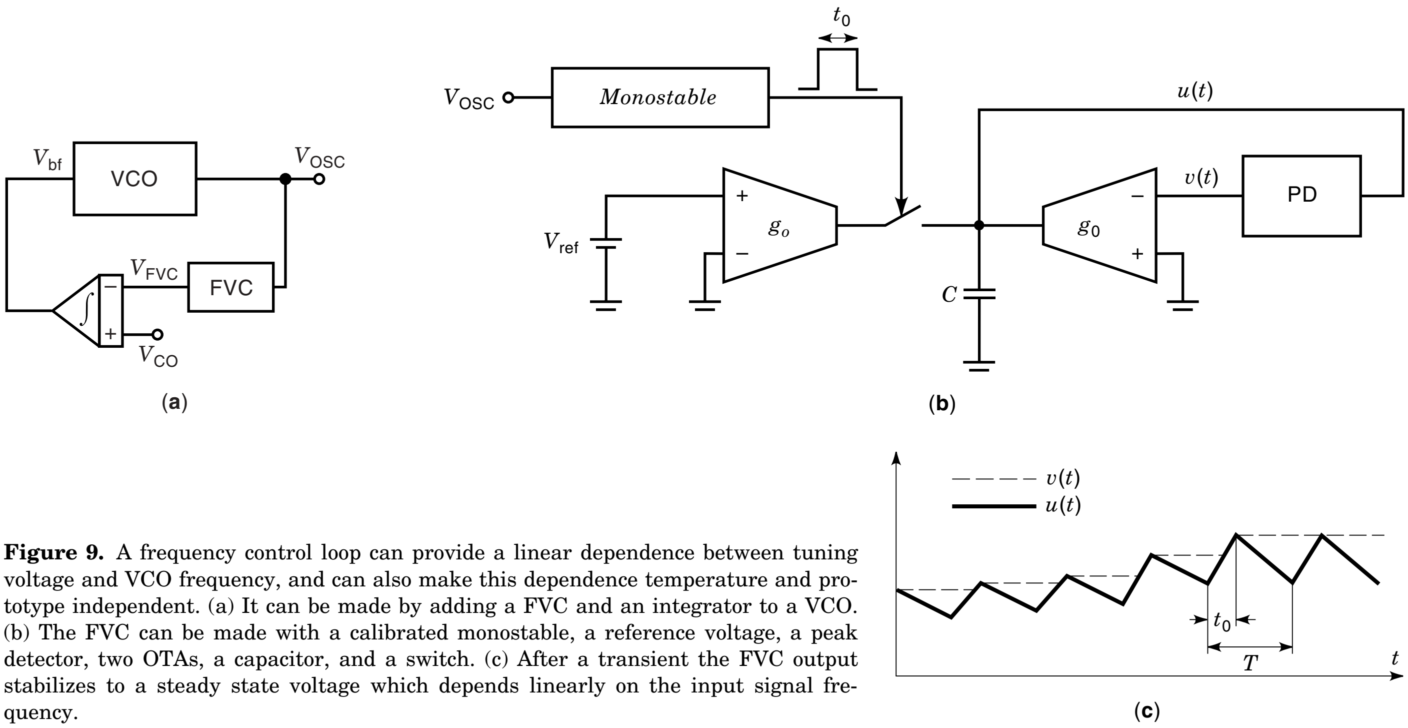

Figure 9. A frequency control loop can provide a linear dependence between tuning voltage and VCO frequency, and can also make this dependence temperature and prototype independent. (a) It can be made by adding a FVC and an integrator to a VCO. (b) The FVC can be made with a calibrated monostable, a reference voltage, a peak detector, two OTAs, a capacitor, and a switch. (c) After a transient the FVC output stabilizes to a steady state voltage which depends linearly on the input signal frequency.

quency f_0 depends on V_{CO} as

$$f_o = \frac{V_{CO} - V_{F0}}{\rho} \tag{52}$$

which is linear and temperature independent.

A possible implementation with OTAs of the FVC is shown in Fig. 9(b). It uses two OTAs of transconductance g_0, a capacitor C, a peak detector, a switch, a temperature independent voltage reference V_{ref}, and a monostable triggered by the oscillating signal V_{OSC}. During each period $T = 1/f$ of signal V_{OSC} the monostable delivers a pulse of constant width t_0, which must be temperature independent and well calibrated. Many times it is convenient to add a sine-to-square wave converter (and even a frequency divider) between V_{OSC} and the monostable. The circuit of Fig. 9(b) uses three components that are not temperature independent and may vary over time and from one prototype to another: the two OTAs and the capacitor. However, provided that both OTAs have the same transconductance (which is a reasonable assumption for VLSI implementations), the resulting parameters ρ and V_{F0} do not depend on C nor g_0. An example of the time waveforms of $u(t)$ and $v(t)$ of Fig. 9(b) is shown in Fig. 9(c). During each period T of V_{OSC} the monostable is triggered once, turning the switch ON during a time t_0. While the switch is ON capacitor C is charged by a constant current $g_0(V_{ref} - v(t))$, and when the switch is OFF a constant current of value $-g_0v(t)$ discharges it. The output of the FVC $v(t)$ changes from cycle to cycle but is constant during each cycle. If v_m is its value during one cycle and v_{m+1} for the next one, it follows that

$$u(t+T) = u(t) + \frac{g_0(V_{ref} - v_m)}{C}t_o - \frac{g_o v_{m+1}}{C}(T - t_0) \tag{53}$$

where $u(t)$ is taken at one of the peaks: $u(t) = v_m$ and $u(t + T) = v_{m+1}$. Consequently,

$$v_{m+1} = \frac{v_m(C - g_0t_0) + V_{ref}g_0t_0}{C + g_0(T - t_0)} \tag{54}$$

In the steady state of the FVC $v_{m+1} = v_m$. Applying this condition to Eq. (54) and calling V_{FVC} the stabilized value of v_m, yields

$$V_{FVC} = \frac{V_{ref}t_0}{T} = V_{ref}t_0 f \tag{55}$$

Consequently, the circuit of Fig. 9(b) implements a FVC with $\rho = V_{ref}t_0$ and $V_{F0} = 0$, which both are temperature and prototype independent.

FURTHER CONSIDERATIONS

The different concepts and considerations mentioned so far have been illustrated with practical circuits using either resistors, capacitors, and opamps, or using OTAs and capacitors. There are many other circuit techniques available that can be used to implement the different blocks and equations needed for stable harmonic oscillator circuits. Some of these techniques could be continuous current mode, switched capacitor, switched current, digital circuit techniques, or even any combination of these techniques.

Depending on the frequency range of the oscillator it may be necessary to consider circuit parasitics that have not been mentioned so far. For example, opams and OTAs both have nonideal input and output impedances, leakage currents, and most importantly gains which are frequency dependent. All these parasitics result in modified characteristics equations.

A very sensitive parameter to parasitics is the oscillation condition $-b_0$ required for initial startup. Since in practice it is desirable to have $-b_0$ very close to zero but still guarantee its negative sign, it is apparent that parasitics can result in either very negative (resulting in very distorted sinusoids) or positive (resulting in no oscillation) values. Each circuit technique has its own parasitics, and depending upon the intended frequency range, they will have a different impact on the final oscillator performance. Consequently, for good oscillator design the dominant parasitics need to be well known and taken into account.

Another interesting and advanced issue when designing oscillators is distortion. Both amplitude control mechanisms, limitation and AGC, are nonlinear and will introduce some

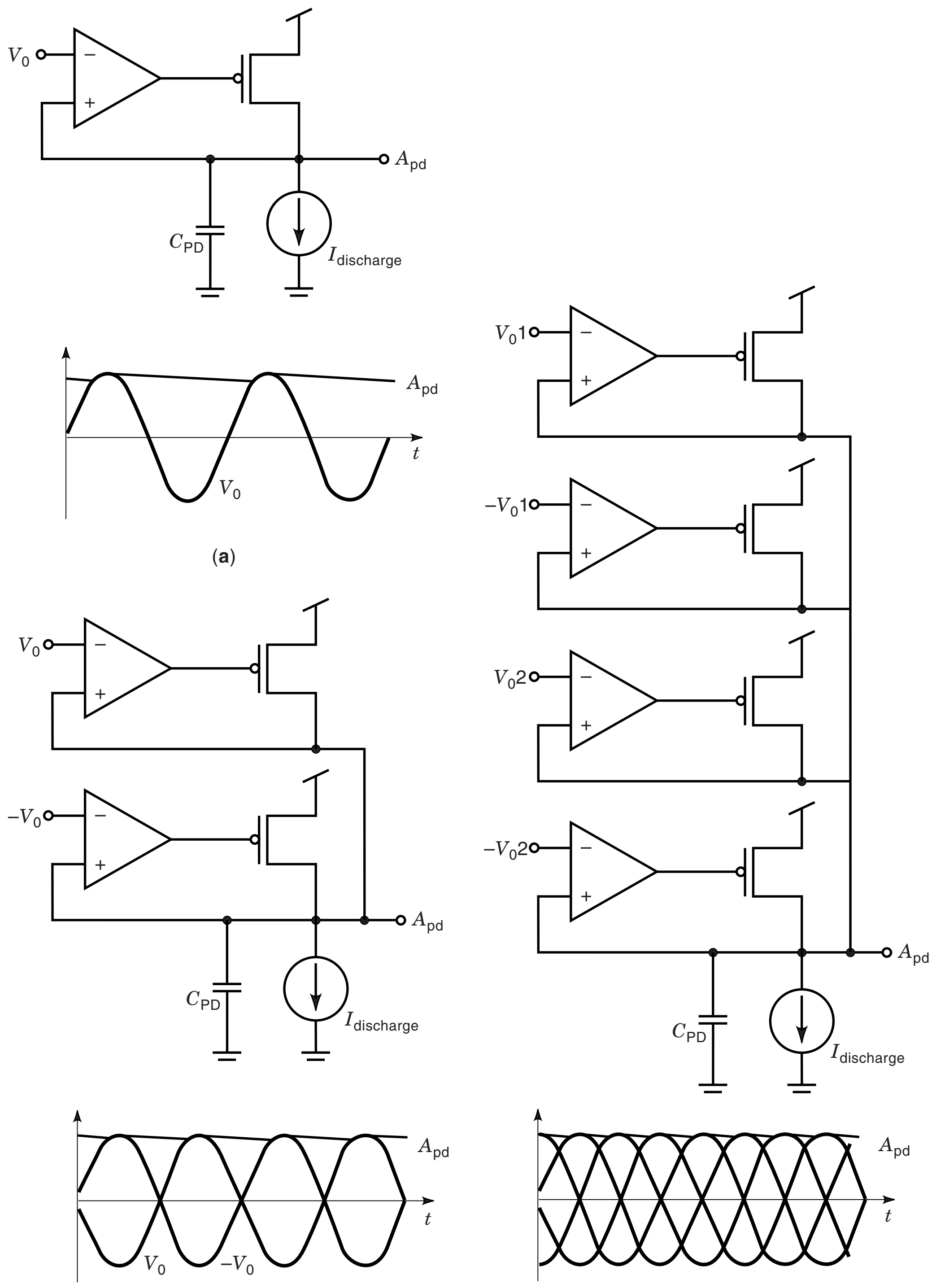

Figure 10. Possible implementations for a peak detector: (a) One phase based (or half wave rectifier based), (b) Two phase based (or full wave rectifier based), and (c) Four phase based peak detector.

degree of distortion. Is there a way to predict how much distortion will render an oscillator?

Distortion for Amplitude Control by Automatic Gain Control

In an oscillator with AGC for amplitude control, like in Fig. 7(d), the element that introduces most of the distortion is the peak detector. Figure 10 shows examples of peak detectors based on one-phase or half-wave [Fig. 10(a)], two-phase or full-wave [Fig. 10(b)], and four-phase [Fig. 10(c)] rectifying principles. For the four-phase case, either the oscillator should provide two phases with $\pi/2$ shift, or an additional integrator is needed. Peak detectors with more phases can be implemented by linearly combining previous phases. Increasing the number of phases in the peak detector results in faster response [delay τ_{pd} in Eq. (41) is smaller for more phases] and less distortion. However, all phases have to present the same amplitude, otherwise distortion will increase. In practice, as the number of phases increases it becomes more difficult (due to offsets and component mismatch) to keep the amplitude of the phases sufficiently equal.

In the peak detectors of Fig. 10, whenever one of the phases becomes larger than A_{pd} it slightly turns ON its corresponding P transistor injecting a current into C_{PD} until A_{pd} increases sufficiently to turn OFF the P transistor. The discharge current ensures that A_{pd} will follow the amplitude of the oscillations if it decreases. Increasing $I_{discharge}$ results in faster response but higher distortion. Whatever peak detector is used, waveform $A_{pd}(t)$ is not constant nor sinusoidal. It has a shape similar to those shown in Fig. 10. Since $A_{pd}(t)$ is periodic its Fourier series expansion can be computed,

$$A_{pd}(t) = A_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t + \varphi_n) \tag{56}$$

The high-order harmonic components $\{a_2, a_3, \ldots\}$ are those which contribute to distortion at node V_0 [in Fig. 7(d)]. Applying the filtering functions that go from node A_{pd}, through V_{b4}, to V_0 [Eqs. (41–43) and (48)] to these higher-order harmonics provides their amplitudes at node V_0,

$$|A_{0n}| = \frac{\alpha A_o}{2\tau_{AGC} n^2 \omega_o^2} |1 + jn\omega_0 m\tau_{AGC}||1 - jn\omega_0 \tau_{pd}||a_n| \tag{57}$$

The total harmonic distortion at the output of the oscillator is then defined as

$$\mathrm{THD}(V_0) = \sqrt{\sum_{n=2}^{\infty} \left(\frac{A_{0n}}{A_0}\right)^2} \tag{58}$$

Distortion for Amplitude Control by Limitation

This problem is computationally complicated but can be solved by harmonic balance. Consider the general block diagram of Fig. 5. In the steady state, periodic waveforms $x(t)$ and $y(t)$ can be expressed by their respective Fourier series expansions

$$\begin{aligned} x(t) &= x_1 \cos(\omega_0 t) + \sum_{n=2}^{\infty} x_n \cos(n\omega_0 t + \varphi_n) \\ y(t) &= \sum_{n=1}^{\infty} y_n \cos(n\omega_0 t + \phi_n) \end{aligned} \tag{59}$$

where,

$$\begin{aligned} |x_n| &\ll |x_1| \\ |y_n| &\ll |y_1| \end{aligned} \tag{60}$$

In general, this problem is solved numerically for a finite set of harmonics. Let N be the highest harmonic to be computed. Since $f(x(t))$ is also periodic, computing its Fourier series expansion yields that of $y(t)$. Therefore, given a set of parameters for $x(t)$ $\{x_1, \ldots x_N, \varphi_2, \ldots \varphi_N\}$ the set of parameters $\{y_1, \ldots y_N, \phi_1, \ldots \phi_N\}$ for $y(t)$ can be obtained this way. By applying each component of $y(t)$ (characterized by y_n and ϕ_n) to filter $H(s)$ yields the corresponding component for $x(t)$,

$$\begin{aligned} x_n &= y_n |H(jn\omega_0)| \\ \varphi_n &= \phi_n + phase(H(jn\omega_0)) \end{aligned} \tag{61}$$

By iterating this procedure until all values x_n, y_n, φ_n, and ϕ_n converge, the distortion of $x(t)$

$$\mathrm{THD}(x) = \sqrt{\sum_{n=2}^{N} \left(\frac{x_n}{x_1}\right)^2} \tag{62}$$

or $y(t)$

$$\mathrm{THD}(y) = \sqrt{\sum_{n=2}^{N} \left(\frac{y_n}{y_1}\right)^2} \tag{63}$$

can be predicted.

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HARMONICS. See POWER SYSTEM HARMONICS.