DISCRETE TIME FILTERS

A signal is defined as any physical quantity that varies with the changes of one or more independent variables. Even that independent variable can be any physical value, for example distance, it is usually refered to as "time". The independent variable may be either continuous or discrete. If the independent variable is continuous, the signal is called continuous-time signal or analog signal. Most of the signals that we encounter in nature are analog signal, such as a speech signal. The discrete-time signals are those for which the independent variable is discrete. The amplitude of both the continuous- and discrete-time signals may be continuous or discrete. Digital signals are those discrete-time signals for which the amplitude is discrete, and switched-capacitor signals are discrete-time signals with continuous amplitude. Any operation on a signal which is performed in order to obtain some more desirable properties, such as less noise or distortion, is called signal processing. A system which performs signal processing is called a filter. Signal processing depends on used technology and can be (1) analog signal processing (ASP) and (2) discrete-time signal processing (DTSP). Prior to 1960, ASP was mainly used; this means signals are processed using electrical systems with active and passive circuit elements. ASP does have some limitations such as (1) fluctuation of the component values with temperature and aging, (2) nonflexibility, (3) cost, and (4) large physical size. In order to overcome those limitations, discrete-time technologies are introduced, such as digital technology and switched-capacitor technologies. Digital technology, which gives many advantages over ASP [see Kuc (1) for a more detailed analysis], needs to convert an analog signal into a digital form. Processing the signal by digital technology is called digital signal processing (DSP), and is a special case of DTSP. In DSP, both amplitude and time are discrete, unlike switched-capacitor processing where amplitude is continuous.

DISCRETE-TIME SIGNALS AND SYSTEMS

A discrete-time signal (discrete signal) is defined as a function of an independent variable n that is an integer. In many cases, discrete signals are obtained by sampling an analog signal (taking the values of the signal only in discrete values of time). According to this, elements of the discrete signals are often called samples. But this is not always the case. Some discrete signals are not obtained from any analog signal and they are naturally discrete-time signals. There are some problems in finding a convenient notation in order to make the difference between continuous-time and discrete-time signals, and various authors use different notations [see Rorabaugh (2) for detailed analysis]. Recent practice, introduced by Oppenheim and Schafer, 1989, (3) uses parantheses () for analog signals and brackets [ ] for discrete signals. Following this practice, we denote a discrete signal as $\{x[n]\}$ or $x[n]$. Therefore $x[n]$ represents a sequence of values, (some of which can be zeros), for each value of integer n. Although the x-axis is represented as the continuous line, it is important to note that a discrete-time signal is not defined at instants between integers. Therefore, it is incorrect to think that $x[n]$ is zero at instants between integers.

Discrete signals can be classified in many different ways. If the amplitude of the discrete signal can take any value in the given continuous range, the discrete signal is continuously in amplitude, or it is a nonquantized discrete-time signal. If the amplitude takes only a countable number of discrete values, the signal is discrete in amplitude or a quantized discrete-time signal. This signal is also called a digital signal. If the signal has a finite number of elements, it is finite; otherwise, it is infinite. Therefore, the finite signal is defined for a finite number of index values n. Unlike an infinite signal, which is defined for an infinite number of index values n and can be: (1) right-sided, (2) left-sided, and (3) two-sided. The right-sided sequence is any infinite sequence that is zero for all values of n less than some integer value N_1. The left-sided sequence is equal to zero for all n more than some integer value N_2. The infinite sequence which is neither right-sided nor left-sided is a two-sided sequence. According to their nature, signals can be deterministic and random. The signals where all values can be determined without any uncertainty are deterministic. Otherwise, they are random and cannot be described by explicit mathematical relationships but by using the probability theory. We consider here deterministic signals and systems. Schwartz and Shaw (4), Hayes (5), and Candy (6) consider random discrete signals and systems. A discrete signal is periodic if the values of the sequence are repeated every N index values. The smallest value of N is called the period. A continuous periodic signal does not always result in a periodic discrete signal.

There are some basic discrete signals which are used for the description of more complicated signals. Such basic signals are (1) unit sample, (2) unit step, and (3) complex exponential sequences. Unit sample sequence is the finite sequence which has only one nonzero element at the index $n = 0$,

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

It plays the same role in the digital signal processing as the unit impulse (delta function) plays in continuous-time signal processing so that the characteristic of a discrete system can be represented as the response to the unit sample sequence. Any discrete signal can be presented as the sum of scaled delayed unit sample sequences,

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \tag{2}$$

Unit step sequence $u[n]$ is the right-sided sequence which is used to denote the start of any right-sided sequence and is defined as

$$u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Therefore, any sequence $x[n]$, which is zero for $n < N_1$, can be written as

$$x[n]u[n-N_1] = \begin{cases} x[n] & n \geq N_1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

A complex exponential sequence is defined as

$$e^{jn\omega} = \cos(n\omega) + j\sin(n\omega) \tag{5}$$

By analogy with the continuous-time case, the quantity ω is called frequency, and has a dimension in radians. We recall that the frequency for the continuous signal has the dimension radians/sec. For this difference, several notations for the frequency of the discrete signals and the continuous signals are being used. The former is usually denoted as ω and the latter as Ω. Let the time axis t is divided into intervals of the length T: $t = nT$. The axes of the discrete time signals can be understood as obtained from the axis t by dividing with T: $n = t/T$. Because the frequency and the time are inverse of each other, dividing in the time domain corresponds to multiplying in the frequency domain. Therefore, the relation between continuous and discrete frequency is the following:

$$\omega = \Omega T \tag{6}$$

Due to the different units of those two values, there are some important distinctions between them. Continuous frequency has the values $-\infty < \Omega < \infty$, and ω has only values from 0 to 2π. All other values are repeated with the period 2π. Usually, the discrete frequencies are represented in the interval

$$-\pi \leq \omega \leq \pi \tag{7}$$

As ω increases from 0 to π, oscillations become higher and have a maximum at $\omega = \pi$, and going from π to 2π, they become slower. Therefore, $\omega = \pi$ is the highest frequency, and $\omega = 0$ and $\omega = 2\pi$ are the lowest frequencies. Figure 1 shows how the sequence oscillates more rapidly with the increase of the frequency from 0 to π and more slowly with the increase of the frequency from π to 2π.

A discrete-time system (or discrete system) is defined as the transformation that maps an input sequence $x[n]$ into an output sequence $y[n]$:

$$y[n] = T\{x[n]\} \tag{8}$$

where $T\{\,\}$ presents transformations, or the set of rules for obtaining the output sequence from the given input one. Depending on transformation a discrete-time system may have different properties. The most common properties are (1) linearity, (2) time-invariance, (3) stability, (4) memoryless, and (5) invertibilty. The system is linear if the response to a scaled sum of the input sequences is equal to the sum of the responses to each of the scaled input:

$$T\left\{\sum_{i=1}^{N} a_i x_i[n]\right\} = \sum_{i=1}^{N} a_1 T\{x_i[n]\} \tag{9}$$

This relation is also known as the superposition principle. The system is time-invariant if the shift of the input sequence causes the same shift of the output sequence. In other words, the properties of the time-invariant system do not change the time:

$$T\{x[n-n_0]\} = y[n-n_0] \tag{10}$$

The systems that are in the same time linear and time-invariant are called linear time-invariant systems (LTI). The system is causal if the values of the output sequence at any index n_0 depend only on the values of the input sequence at indexes $n \leq n_0$. In other words, in a causal system the output

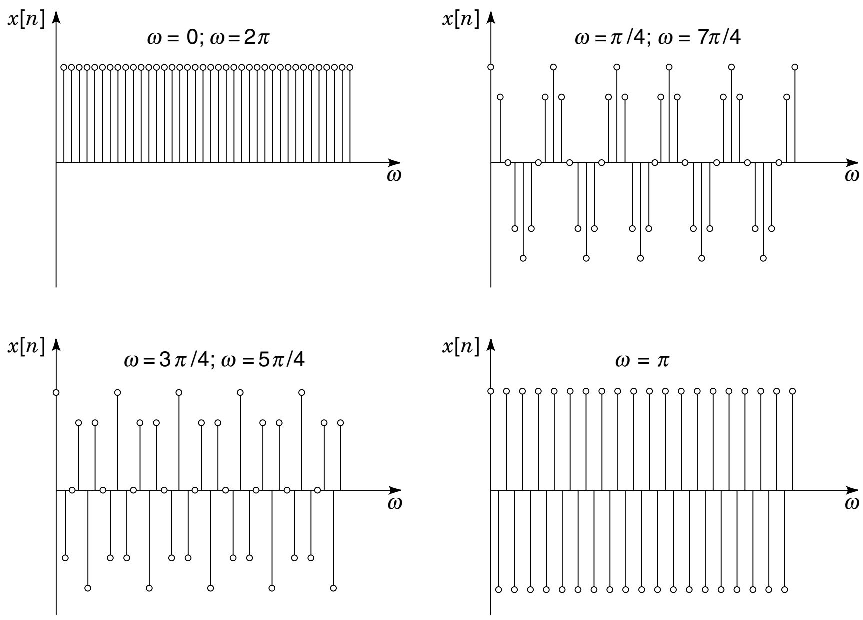

Figure 1. The interpretation of high and low frequencies for a dicrete-time sinusoisal signal. As ω increases from zero toward π the sequence oscillates more and more rapidly and as ω increases from π toward 2π, the sequence oscillates more and more slowly. Therefore the values of ω in the neighborhood of $\omega = 0$ are low frequencies (slow oscillations), and those in the vicinity of $\omega = \pi$ are high frequencies (rapid oscillations). Due to the periodicity in general the low frequencies are those in the vicinity of $\omega = 2\pi k$, $k = 0, 1, 2, \ldots$, and the high frequencies are those in the vicinity of $\omega = \pi + 2\pi k$, $k = 0, 1, 2, \ldots$.

does not precede the input (i.e., it is not possible to get an output before an input is applied to the system). Noncausal systems occur only in theory, and do not exist in this universe. A causal system can be designed by introducing corresponding amounts of delay. The system is stable if a limited input always gives a limited output. If for a limited input, the output is unlimited, the system is not stable. Therefore, the output of an unstable system is infinite with nondecaying values. The system is memoryless if the output $y[n]$ depends only on the input at the same value n. The system is invertible if the input sequence may be uniquely determined by observing the output.

Time-Domain Description

There are two main ways to describe discrete systems in the time domain. The first one considers only the relation between the input and the output of the system and is generally named the input-output analysis. The second one, besides the relation of the input and the output gives also an internal description of the system, and it is named as a state-space analysis. Both descriptions are useful in practice and are used depending on the problem under the consideration (see Ref. 7). A convenient way to present the behavior of the discrete system is to put the unit sample sequence at the input. If the system is relaxed initially,

$$y[0] = 0 \tag{11}$$

the output $y[n]$ would be the only characteristic of the system, and it is called unit sample response or shortly impulse response, and is denoted as $h[n]$:

$$h[n] = T\{\delta[n]\} \tag{12}$$

A discrete system which has the finite impulse response is called a finite impulse response (FIR) filter, and one with the infinite impulse response is known as an infinite impulse response filter (IIR). The question which arises is whether the output to any other input sequence may be related with the unit sample response. In order to answer we use relation (2), and we obtain

$$y[n] = T\{x[n]\} = T\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\} \tag{13}$$

If the system is linear, the superposition principle (9) can be used, and therefore, the Eq. (13) can be written as

$$y[n] = \sum_{k=-\infty}^{\infty} T\{x[k]\,\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} \tag{14}$$

From here, we obtain the relation for the linear system:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] \tag{15}$$

where $h_k[n]$ depends on both k and n:

$$h_k[n] = T\{\delta[n-k]\} \tag{16}$$

This relation is called the convolutional relationship. This equation can be simplified for the time-invariant system, using Eq. (10),

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{17}$$

This relation is called the convolution sum or convolution. It completely describes the output of an LTI system for the known input and for zero initial conditions. The operation convolution between sequences has its own signs *. Therefore, the convolution (17) can be written as

$$y[n] = x[n] * h[n] \tag{18}$$

This operation is commutative and distributive [see Kuc (1) for detailed analysis]. Proakis and Manolakis (7) explain the computing of the convolution step by step. From the unit sample response, we may see some important characteristics of the LTI system, such are stability and causality [see Orfanidis (8) for detailed analysis]. An LTI system is stable if and only if this condition is satisfied:

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{19}$$

FIR filter has a finite length of impulse response, and the condition (19) shall always be satisfied, which means that an FIR filter is always stable. An LTI system is causal if the next condition is satisfied:

$$h[n] = 0, \quad \text{for } n < 0 \tag{20}$$

A natural question which may arise is if we can implement digital filter by using the convolution. The answer depends on whether the system is FIR or IIR. In the case of an FIR, the convolution summation directly suggests how to implement the filter. The problem arises for an IIR filter which has an infinite impulse response since it requires an infinite number of memory locations, additions, and multiplications. The solution is given by introducing the difference equations. Such a difference equation describes an LTI system having any initial conditions unlike the discrete convolution that describes the system in which all inputs and output are initially zero (the system is initially relaxed). The difference equation is often written in the form

$$y[n] = \sum_{k=-N_f}^{N_p} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \tag{21}$$

where b_k and a_k are constant coefficients and N_f and N_p are integer values. The first summation contains past, present, and future inputs, while the second one contains only past outputs. The difference equation for FIR filter contains only the first sum where we can recognize the convolution (17). If the system is casual, it does not depend on the future values of the input, and the difference equation has $N_f = 0$. The part of the right side of the difference equation which involves past outputs is called the recoursive part, and the other part is the nonrecoursive one. The system which has only a nonrecoursive part is called the nonrecoursive filter. Otherwise, it is the recoursive filter. In general, the computation of the output $y[n]$ at the index n of a recoursive filter needs previous outputs: $y[n-1]$, $y[n-2]$, . . ., $y[0]$. Therefore in this case, the output must be computed in an order. As the difference, the output of the nonrecoursive filter can be computed in any order. An implementation of the casual LTI filter based on the difference equation (21) and which is called direct form I is presented in the Fig. 2. We see that the filter consists of an interconnection of three basic elements: (1) unit delay, (2) multiplier, and (3) adder. Direct form I is not optimal in the sense that it uses a minimum number of delaying elements. Proakis and Manolakis (7) describe different and more efficient structures of discrete systems. Signal-flow graphs are often used to describe the time-domain behavior of LTI systems [see Haykin (9) for a detailed analysis].

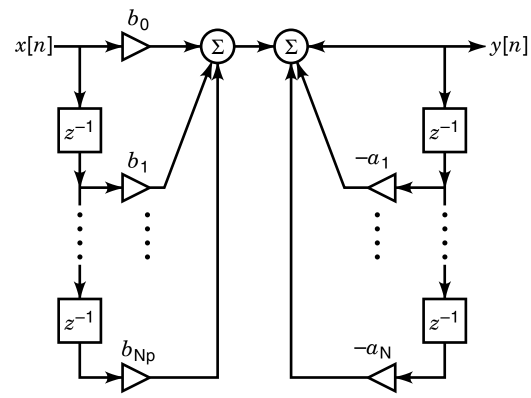

Figure 2. Direct form I realization of the causal LTI filter follows directly from the difference equation and shows explicitly the delayed values of input and output. (z^{-1} is interpreted as one-sample delay.)

A state-space approach considers that the output of the system is the result of the actual input and the set of initial conditions. This suggests that the system may be divided into two parts. One part contains memory and describes past history, and the second one describes the answer to the actual input. Following this approach, Antoniou (10) derived the state space equations for the system of an order N in the matrix-vector form

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{B}\mathbf{x}[n] \tag{22}$$

$$\mathbf{y}[n] = \mathbf{C}\mathbf{q}[n] + \mathbf{D}\mathbf{x}[n] \tag{23}$$

where $\mathbf{q}[n]$ is the n-dimensional state vector at time n, and $\mathbf{x}[n]$ and $\mathbf{y}[n]$ are the input and output sequences, respectively. The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, correspond to a particular realization of the filter.

Transform Domain Description

Frequency Domain. The sinusoidal sequences are usually used in frequency-domain description of discrete signals and systems because sinusoidal sequences have one useful characteristic which is shown in Eq. (24):

$$y[n] = H(e^{j\omega})e^{j\omega n} \tag{24}$$

Therefore, if the sinusoidal sequence is applied to the LTI system, the output is also a sinusoidal sequence with the same frequency, multiplied with the complex value:

$$H(e^{j\omega}) = \sum_{K=-\infty}^{\infty} h[k]e^{-j\omega k} \tag{25}$$

The sum in Eq. (25) presents Fourier transform of $h[n]$ and is named as frequency response, as it specifies response of the system in the frequency domain. The frequency response, being the Fourier transform of the unit sample response, is a periodic function with the period 2π. Therefore, "low frequencies" are those that are in the neighborhood of an even multiple of π, and the "high frequencies" are those that are close to an odd multiple of π. Equation (24) has also an interpretation using the eigenvalue and eigenfunction. If an input signal produces the same output signal but multiplied by a constant, this signal is called eigenfunction, and the constant is the eigenvalue of the system. Therefore, the complex sinusoidal sequence is the eigenfunction, and $H(e^{j\omega_w})$ is the corresponding eigenvalue. Fourier transform of the unit sample response $h[n]$ exists only if the sum [Eq. (25)] converges, that is if the next condition is satisfied:

$$\sum_{k=-\infty}^{\infty} |h[n]| < \infty \tag{26}$$

The magnitude of $H(e^{j\omega})$, $|H(e^{j\omega})|$, is called magnitude response, and the argument of $H(e^{j\omega})$ is called phase response and denoted as $\mathrm{Arg}\{H(e^{j\omega})\}$. Therefore, we have

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\mathrm{Arg}\{H(e^{j\omega})\}} \tag{27}$$

Frequency response can be expressed by its real and imaginary part:

$$H(e^{j\omega}) = H_\mathrm{R}(e^{j\omega}) + jH_\mathrm{I}(e^{j\omega}) \tag{28}$$

From here, the magnitude response and phase response can be expressed as follows:

$$|H(e^{j\omega})| = \sqrt{H_\mathrm{R}^2 + H_\mathrm{I}^2} = \sqrt{H(e^{j\omega}H^*(e^{j\omega})} \tag{29}$$

where $H^*(e^{j\omega})$ is the complex-conjugate of $H(e^{j\omega})$.

$$\mathrm{Arg}\{H(e^{j\omega})\} = arctg\left[\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right] \tag{30}$$

Instead, the linear scale, magnitude characteristic is usually plotted on the logarithmic scale.

$$|H(e^{j\omega})|_{db} = 10\log_{10}|H(e^{j\omega})|^2 = 20\log_{10}|H(e^{j\omega})| \tag{31}$$

In order to show better both the passband and the stopband characteristics, the log-magnitude response is plotted on two different scales: one for the passband and the second one for the stopband. For an LTI system with a real impulse response, the magnitude and phase responses have symmetry properties from which follows that the magnitude response is an even function of ω, and the phase response is an odd function of ω.

Oppenheim and Schafer (3) show that the Fourier transform of the output is the product of the Fourier transforms of the input and the impulse response:

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \tag{32}$$

This expression explains why Fourier transform is so useful in the analysis of LTI. As this expression shows, the operation of convolution is replaced by a simpler operation of multiplication in the transform domain. This equation also shows that the input spectrum is changed by the LTI system in both amplitude and the phase. The output magnitude is obtained as the product of the input magnitude spectrum and the magnitude response:

$$|Y(e^{j\omega})| = |H(e^{j\omega})|\,|X(e^{j\omega})| \tag{33}$$

The output phase is equal to the sum of the input phase and the phase response:

$$\mathrm{Arg}\{Y(e^{j\omega})\} = \mathrm{Arg}\{X(e^{j\omega})\} + \mathrm{Arg}\{H(e^{j\omega})\} \tag{34}$$

Those changes can be either desirable or undesirable when they are referred as to the magnitude and phase distortion. Generally, we may view the LTI as a filter, passing some of the frequencies of the input signal and suppressing the others. Filters are usually classified according to what frequencies pass and to what frequencies suppress as: (1) lowpass, (2) highpass, (3) bandpass, and (4) bandstop filters. The ideal filters have constant magnitude response (usually 1) in the passband and zero magnitude characteristic in the stopband. The magnitude characteristics of the different ideal filters are shown in Fig. 3. Ideal filters have the linear phase in the passband which means that the output is equal to the scaled and delayed input. Therefore, linear phase causes only delaying of the input sequence, what is not considered as the distortion and the linearity of the phase is the desirable characteristic. The group delay is introduced as the measure of the linearity of the phase

$$\tau(\omega) = -\frac{d[\mathrm{Arg}\{H(e^{j\omega})\}]}{d\omega} \tag{35}$$

The group delay can be interpreted as the time delay of the signal components of the frequency ω, introduced by the filter. Filters with symmetric impulse response have linear phase [see Oppenheim and Schafer (3) for detailed analysis]. Ideal filters are not physically realizable and serve as the mathematical approximations of physically realizable filters. As an example, we consider in Fig. 4 the magnitude characteristic of the physically realizable lowpass filter [see Ingle and Proakis (11) for a detailed analysis].

***Z*-Domain.** Z transform is a generalization of the Fourier transform that allows us to use transform techniques for signals not having Fourier transform. It plays the same role in discrete-time signals and systems as the Laplace transform does in continuous-time signals and systems. Z transform of the unit sample sequence is called system function:

$$H(z) = Z\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \tag{36}$$

The concept of a Z transform is only useful for such values of z for which the sum [Eq. (36)] is finite. Therefore, for the sequence $h[n]$ it is necessary to define the set of z values for which

$$\sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty \tag{37}$$

This set of z values is called the region of convergency (ROC). Many characteristics of a filter can be seen from ROC. For a FIR filter, the number of elements in the sum [Eq. (36)] is finite and, therefore, the problem of the existence of the Z transform does not exist, and the ROC is all z-plane, except the origin. Proakis and Manolakis (7) show that ROC for the right-sided sequence is given by $|z| > R_1$, for the left-sided is

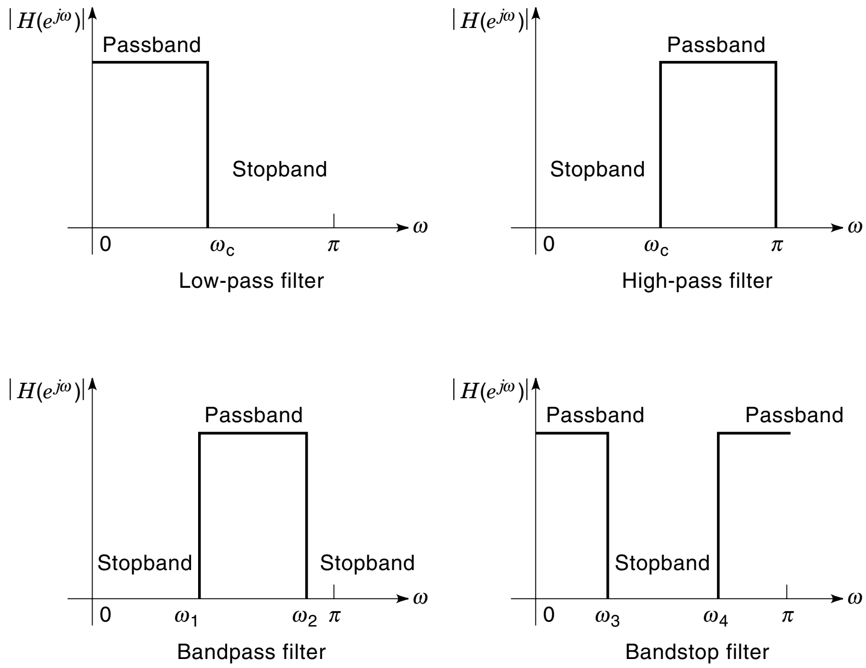

Figure 3. Magnitude characteristics of the different ideal frequency-selective filters. The ideal filters pass without any attenuation all frequencies in the passband and completely attenuate all frequencies in the stopband.

given as $|z| < R_2$, and for the two-sided sequence as $R_1 < |z| < R_2$.

The operation of convolution in the time-domain reduces to the most simple operation of multiplication in the Z-domain:

$$Y(z) = Z\{y[n]\} = Z\{x[n] * h[n]\} = X(z)H(z) \tag{38}$$

where

$$\begin{aligned} X(z) &= Z\{x[n]\} \\ Y(z) &= Z\{y[n]\} \end{aligned} \tag{39}$$

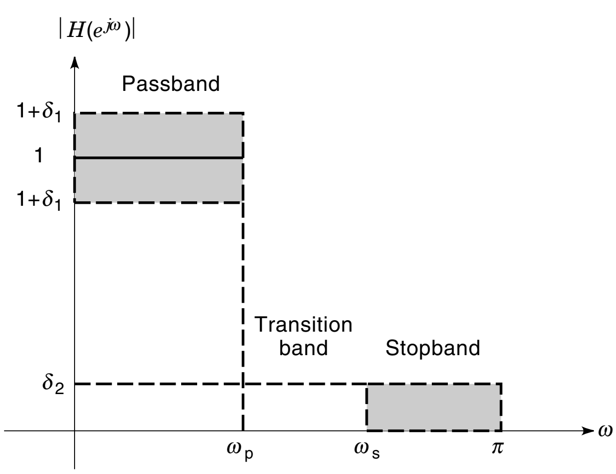

Figure 4. Magnitude specification of the physically realizable low-pass filter. Instead of sharp transition between passband and stopband, the transition band is introduced, and instead of a flat characteristic, a small amount of ripples is tolerable: In the passband: $1 - \delta_1 < |H(e^{j\omega})| < 1 + \delta_1$, where δ_1 is the passband ripple. In the stopband: $|H(e^{j\omega})| < \delta_2$, where δ_2 is the stopband ripple.

Remember that Eq. (38) is valid for LTI systems. The system function for an important class of LTI systems which are described by the constant coefficients difference equation can be expressed as the rational function and is expressed as the ratio of polynomials in z^{-1}. By taking the Z transform of Eq. (21) and using that the delay by k samples in the time-domain corresponds to the multiplication by z^{-k}, we have:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=N_f}^{N_p} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \tag{40}$$

[see Proakis and Manolakis (7) for detailed analysis]. The values of z for which $H(z)$ become zero are called zeros, and the values of z for which $H(z)$ become infinity are called poles. The zeros are roots of the numerator $N(z)$, and the poles are roots of the denominators $D(z)$. Both poles and zeros are named as the singularities of $H(z)$. The plot of zeros and poles in the z-plane is called a pole-zero pattern. Pole is usually denoted by a cross × and the zero by a circle ○. We can write the system function $H(z)$ in the factoring form:

$$H(z) = Kz^{N_f} \frac{\prod_{k=1}^{N_p+N_f}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})} \tag{41}$$

where z_k and p_k are the zeros and the poles, respectively, and K is gain. Kuc (1) shows that each factor in the numerator of Eq. (41) generates one zero at $z = z_k$ and one pole at $z = 0$; each factor in the denominator generates one pole at $z = p_k$ and one zero at $z = 0$; factor z^{N_f} generates N_f zeros at $z = 0$ and N_f poles at $z = \infty$. For the system function, the total number of poles is equal to the total number of zeros. If pole and zero are in the same location, they cancel each other. Complex singularities are always in the complex-conjugate pairs for the system presented by the difference equations with the real coefficients.

Pole-zero pattern gives much useful information about the LTI system. From the pole–zero pattern, we can see whether the filter is casual or not. For the casual filter, $N_f = 0$ and therefore there are no poles in the infinity. Besides causality, the type of the filter can also be seen from the pole–zero pattern. For an FIR filter, all singularities are only zeros (except poles at the origin and possibly in the infinity). Unlike a FIR filter, an IIR filter has zeros and poles or only poles. (Zeros are in the origin.) As the system function becomes infinity in the poles, all poles must be outside the ROC. However, for a casual right-sided sequence, ROC must be outside of the outermost pole (the pole having the largest absolute value). Another useful characteristic about LTI which can be seen from the pole–zero pattern is the stability. The problem of stability is only addressed to the IIRs and, therefore, is connected only with the position of poles. Kuc (1) shows that for a causal IIR filter, all poles must be inside the unit circle. If the pole is on the unit circle, the system is not stable.

Oppenheim and Shafer (3) shows that Z transform is equal to the Fourier transform on the unit circle:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} \tag{42}$$

In this order, the frequency response belongs to the system function evaluated on the unit circle. The magnitude response at $\omega = \omega_0$ can be presented geometrically as the ratio of the distances between the zeros and the point $z_0 = e^{j\omega_0}$ on the unit circle and the distances between poles and the point $z_0 = e^{j\omega_0}$, as it is shown in Fig. 5:

$$|H(e^{j\omega_0})| = K\frac{\prod_{k=1}^{N_p+N_f}|(z_k, z_0)|}{\prod_{k=1}^{N}|(p_k, z_0)|} \tag{43}$$

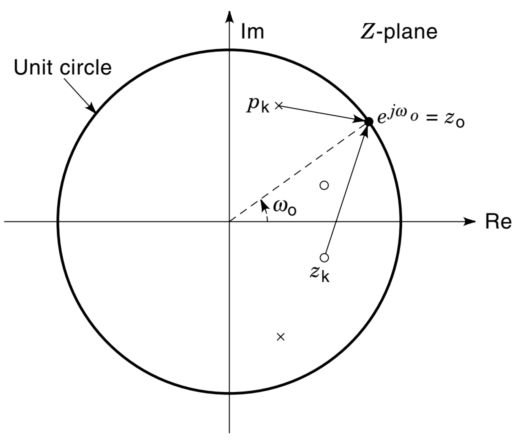

Figure 5. Geometric presentation of the Fourier transform in Z-plane along the unit circle. The magnitude response at $\omega = \omega_0$ can be presented geometrically as the ratio of the distances between the zeros z_k and the point $z_0 = e^{j\omega_0}$ on the unit circle and the distances between poles p_k and the point $z_0 = e^{j\omega_0}$. If the singularity is close to the unit circle it is called dominant singularity, because the distances from it to the neighborhood points on the unit circle are very small. Therefore the dominant zero decreases and the dominant pole increases the magnitude characteristic at the corresponding frequency. For every dominant zero on the unit circle, the magnitude characteristic is equal to the zero at the corresponding frequency.

Ingle and Proakis (11) derive geometrical presentation of the phase response as

$$\arg\{H(e^{j\omega_0})\} = C + ((N_p + N_f) - N)\omega_0 + \sum_{k=1}^{N_p+N_f}\arg\{(z_k, z_0)\} - \sum_{k=1}^{N}\arg\{(p_k, z_0)\} \tag{44}$$

where C is equal 0 or π, depending if the real frequency response is negative or not. This expression can be interpreted as the sum of the constant linear with ω term and the nonlinear term. We notice that the singularities in the origin do not affect the magnitude response but affect the phase response. As shown in Eq. (43), the magnitude response will be equal to zero at the points corresponding to the zeros on the unit circle. Similarly, the poles that are close to the unit circle (remembering that these cannot be on the unit circle for a stable LTI) give the peak value to the magnitude response. Therefore, the singularities that are close to the unit circle dominate the magnitude response, and they are called the dominant singularities.

s-PLANE TO z-PLANE TRANSFORM

The s-plane to z-plane transform depends on the characteristic of the filter we want to preserve in the process of transforming an analog to a digital filter. The most used transforms are impulse invariance transformation, where the impulse response is preserved, and bilinear transform, where the system function is preserved.

Impulse Invariance Transformation

The unit sample response of a digital filter is obtained by sampling the impulse response of the analog filter:

$$h[n] = h_A(nT) \tag{45}$$

where T is the sampling interval. Using Eq. (6) between the discrete and analog frequency, knowing that the frequency points in s-plane are

$$s = j\omega T \tag{46}$$

and that those in the z-plane are

$$z = e^{j\omega} \tag{47}$$

we obtain the relation:

$$z = e^{sT} \tag{48}$$

From Eqs. (46)–(48), it follows that the part of the frequency axis in the s-plane from 0 to π/T is mapped to the frequency points on the unit circle from $\omega = 0$ to π in the z-plane. In a similar way, the frequency points from 0 to $-\pi/T$ are mapped to the points on the unit circle from $\omega = 0$ to $-\pi$. Expressing the complex value z in the polar form

$$z = re^{j\omega} \tag{49}$$

and the complex variable s, with real value σ and imaginary value Ω

$$s = \sigma + j\Omega \tag{50}$$

we have the relation between r and the real value σ

$$r = e^{\sigma T} \tag{51}$$

We have the next observations: (1) the transform from continuous-time domain to the discrete-time domain is linear, (2) the mapping is not one-to-one, but many-to-one and (3) the frequency interval 0 to $2\pi/T$ maps into the unit circle, and the strips in the left side of the s plane of width $2\pi/T$ are mapped inside the unit circle. The entire left side of the s-plane maps into the unit circle, which means that the stable analog filter will result in a stable digital one. Due to the many-to-one mapping, the aliasing effect is present, and this is the main disadvantage of the impulse invariance transform.

Bilinear Transform

To overcome the aliasing limitation, the bilinear transform could be used, as it presents a one-to-one mapping. The system function $H(z)$ is obtained from $H_A(s)$ by replacing the s by

$$s = \frac{2}{T}\frac{z-1}{z+1} \tag{52}$$

To find the mapping of the frequencies from Ω to ω, we set $s = j\Omega$ and use Eqs. (49) and (52)

$$\begin{aligned} z = e^{j\omega} &= \frac{1 + j\Omega T/2}{1 - j\Omega T/2} = \frac{1 + (\Omega t/2)^2}{1 + (\Omega T/2)}\frac{e^{j\mathrm{arctg}(\Omega T/2)}}{e^{-j\mathrm{arctg}(\Omega T/2)}} \\ &= e^{2j\mathrm{arctg}(\Omega T/2)} \end{aligned} \tag{53}$$

From here follows

$$\omega = 2\mathrm{arctg}(\Omega T/2) \tag{54}$$

For low frequencies, the transform is approximately linear, and for higher frequencies, the transform is highly nonlinear, and frequency compression or frequency warping occurs. The effect of the frequency warping can be compensated for by prescaling or prewarping the analog filter before transform, which means to scale the analog frequency as follows:

$$\Omega' = \frac{2}{T}\,\mathrm{tg}\left(\frac{\Omega T}{2}\right) \tag{55}$$

[See Kuc (1) for a detailed analysis.] The whole left side of the s-plane is mapped into the inside of the unit circle, and the right side is mapped outside of the unit circle. Therefore, the stable analog filter will result in the stable digital filter [see Proakis and Manolakis (7) for detailed analysis].

DISCRETE-TIME ANALOG FILTERS

During the 1960s and 1970s the analog integrated filters were implemented by circuits based on resistors, capacitors, and operational amplifiers; these are denominated RC active filters. The precision of RC filters depend on RC products which rely on the precision of the absolute value of both resistors and capacitors. In integrated circuits, the tolerances of RC products can be as high as ±30%. In the past, to overcome this drawback, the resistors were adjusted by using laser trimming techniques; this approach, however, increases the cost of the system. In order to increase the precision of the filters, several techniques have been aroused; the main idea is to replace the resistor by another device like a switched-capacitor or a switched-current. Switched-capacitor techniques are discussed in this chapter.

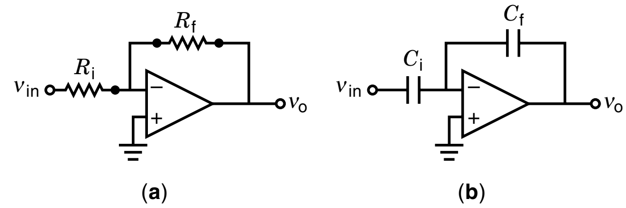

Figure 6. Continuous-time amplifiers: (a) resistor based and (b) capacitor based. The small signal voltage gain for the resistor and the capacitor based amplifiers is $-R_f/R_i$ and $-C_i/C_f$, respectively.

Basic Components of Continuous-Time Filters

Resistors, capacitors, and inductors are the main passive elements of continuous-time filters. The operational amplifiers are other important elements for the implementation of active RC filters. For the resistor, the voltage to current relationship is given by

$$i = \frac{1}{R}v \tag{56}$$

For the inductor, this relationship is

$$i = \frac{1}{sL}v \tag{57}$$

where s is the frequency variable jΩ. For the capacitor, the voltage-current relationship is given by

$$v = \frac{1}{sC}i \tag{58}$$

In the design of complex transfer functions, a basic function is the voltage amplification; two structures are depicted in Fig. 6. The inband gain of the amplifiers is given by $-R_f/R_i$ and $-C_i/C_f$, respectively. While the resistor based amplifier is stable, the circuit of Fig. 6(b) is quite sensitive to offset voltages due to the lack of dc feedback. Active filters are based on lossless integrators; the typical RC implementation of this block is shown in Fig. 7.

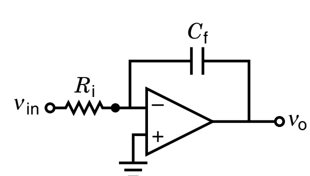

Figure 7. Continuous-time integrator. Note that the operational amplifier operates in open loop for dc signals.

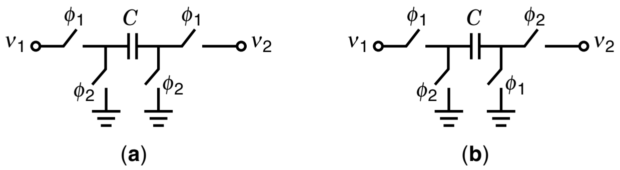

Figure 8. Switched-capacitor resistors: (a) series and (b) parallel. ϕ_1 and ϕ_2 are two nonoverlapping clock phases.

If the voltage gain of the operational amplifier, A_V, is large enough, the voltage at the inverting input, given by v_0/A_V, is very small, and this terminal can be considered as a virtual ground; hence, the current flowing through the resistor is determined by the value of both resistor and input voltage. Since the input impedance of the operational amplifier is typically very large, the resistor current is injected to the capacitor, and the output voltage becomes

$$v_o(t) = v_o(t_0) - \frac{1}{R_i C_f}\int_{t_0}^{t} v_i(t)\,dt \tag{59}$$

The minus sign appears because the current is injected from the virtual ground to the output node. As we are interested in transfer functions, it is easier to manipulate the variables in the frequency domain; therefore, the previous equation can be expressed as follows

$$\frac{v_o}{v_i} = -\frac{1}{sR_i C_f} \tag{60}$$

The differentiator can be implemented by using inductors instead of capacitors or exchanging the role of the resistor and capacitor in Fig. 7. Nevertheless, these approaches are impractical in most of the cases; the inductors are typically implemented by using integrators as will be shown in the next section. More details can be found in Refs. 12–15.

Building Blocks for Switched-Capacitor Filters

Herein after, it is assumed that the non-overlapping clock phases are defined as follows:

$$\begin{aligned}\phi_1(t) &\Rightarrow nT - T/2 < t \le nT \\ \phi_2(t) &\Rightarrow nT - T < t \le nT - T/2\end{aligned} \tag{61}$$

In switched-capacitor circuits, the resistors are implemented by a capacitor, four switches and two non-overlapping clock frequencies; Fig. 8 shows the stray insensitive switched-capacitor simulated resistors. Typically, one of the terminals is connected to a low impedance voltage source, and the other one is connected to the input of an operational amplifier a virtual ground. Hence, every clock period, in the case of the inverting switched-capacitor resistor, the capacitor extract charge equal to $-C(v_1 - v_2)$ or the charge $C(v_1 - v_2)$ is injected in the case of the non-inverting resistor. In average, the switched-capacitor simulated resistors are transferring charges proportional to the clock period leading to the following equivalent resistance

$$R_{eq} \cong \frac{1}{f_{ck}C} \tag{62}$$

where f_{ck} is the frequency of the clock. Switched-capacitor based resistors are the key elements for the design of switched-capacitor filters; with these elements, voltage amplifiers, integrators, resonators, filters, and other type of functions can be implemented; more applications can be found in Refs. 13, 15, 19–20. The simplest voltage amplifier is the circuit of Fig. 6(b); other voltage amplifiers are shown in Fig. 9. Observe that C_f and C_i are sharing several switches. In Fig. 9(a), during the clock phase ϕ_2, the operational amplifier is shortcircuited, and the capacitors are discharged. During the next clock period, the input capacitor is charged to $C_i\, v_{in}$, and the current needed for this charge flows through the feedback capacitor; therefore, the gain voltage becomes

$$\frac{v_o}{v_{in}} = -\frac{C_i}{C_f} \tag{63}$$

Note that the amplifier is available during clock phase ϕ_1. The amplifier shown in Fig. 9(b) behaves as the previous one, but the input signal is sampled at the end of the clock phase ϕ_2, and the charge is injected during the next clock period. Observe that the injected charge is inverted; hence, the voltage gain is

$$\frac{v_o}{v_{in}} = \frac{C_i}{C_f} z^{-1/2} \tag{64}$$

where: $z^{-1/2}$ represents the half delay. Other voltage amplifiers can be found in Refs. 13–15, 19–20.

Switched-Capacitor Integrators

The parasitics-insensitive integrators allow the design of high-performance analog integrated circuits. Biquadratic based filters, ladder filters, and other type of filters are based on active integrators. Switched-capacitor filters have the advantage that arranging the clock phases inverting and noninverting simulated resistors can be realized, as shown in Fig.

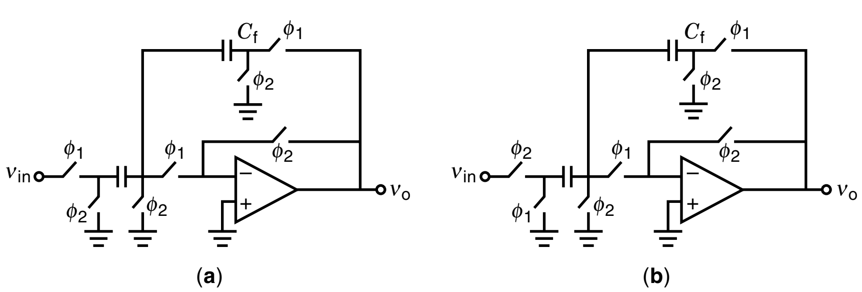

Figure 9. Switched-capacitor amplifiers available during clock phase ϕ_1 (a) inverting and (b) noninverting. Because the operational amplifier is shortcircuited during ϕ_2, the output is available only during the clock phase ϕ_1.

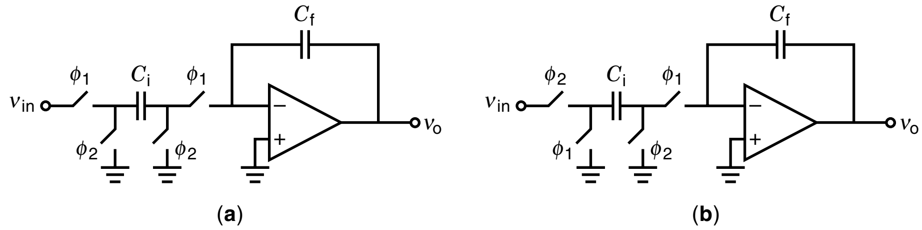

Figure 10. Switched-capacitor integrators: (a) inverting and (b) noninverting. The inverting and noninverting integrators employ the series and parallel switched-capacitor resistors, respectively.

8. As a result of this, the design of systems can be further simplified. The inverting and non-inverting integrators shown in Fig. 10 are an example of this advantage. For the implementation of an RC noninverting integrator, an additional inverter is needed.

The inverting integrator operates as follows. During the clock phase ϕ_2, C_i is discharged while the output voltage remains constant due to C_f; the output voltage is then characterized by the next equation

$$v_o(t) = v_o(nT - T) \tag{65}$$

In the next clock period, C_i extracts a charge equal to $C_i\, v_{in}$; therefore, the charge distribution can be described by the following expression

$$C_f v_o(t) = C_f v_o(nT - T/2) - C_i v_{in}(t) \tag{66}$$

If the output voltage is evaluated at the end of the clock phase ϕ_1, and considering that $v_0(nT - T/2)$ is equal to $v_0(nT - T)$, the z-domain transfer function will result in

$$\left.\frac{v_o}{v_{in}}\right|_{\phi_1} = -\frac{C_i}{C_f}\frac{1}{1 - z^{-1}} \tag{67}$$

Note that the output voltage can be sampled during the next clock period; then the output voltage is delayed by a half period, leading to the following transfer function

$$\left.\frac{v_o}{v_{in}}\right|_{\phi_2} = -\frac{C_i}{C_f}\frac{z^{-1/2}}{1 - z^{-1}} \tag{68}$$

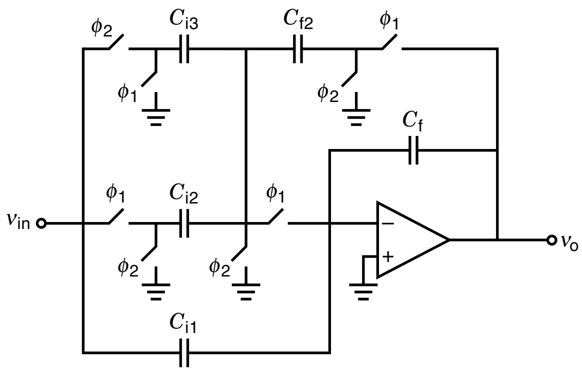

Figure 11. General first-order switched-capacitor filter. C_{i1}, C_{i2}, and C_{i3} implement an amplifier, an inverting integrator, and a noninverting integrator, respectively. C_{f2} implements a switched-capacitor resistor in parallel with C.

A similar analysis for the non-inverting integrator leads to the following transfer functions:

$$\left.\frac{v_o}{v_{in}}\right|_{\phi_2} = \frac{C_i}{C_f}\frac{z^{-1/2}}{1 - z^{-1}} \tag{69}$$

$$\left.\frac{v_o}{v_{in}}\right|_{\phi_2} = \frac{C_i}{C_f}\frac{z^{-1}}{1 - z^{-1}} \tag{70}$$

First-Order Filters

The amplifiers and integrators can be easily implemented by using switched-capacitor circuits; a general first order filter is shown in Fig. 11. By using adequate equations, we can see that the z-domain output-input relationship is described, during the clock phase ϕ_1, by the following expression:

$$(1 - z^{-1})C_f v_o = -(1 - z^{-1})C_{i1}v_{in} - C_{i2}v_{in} + z^{-1/2}C_{i3}v_{in} - C_{f2}v_o \tag{71}$$

where the left hand side term represents the charge contribution of C_f. The first right hand side term is due to capacitor C_{i1}. Note the term $1 - z^{-1}$ present in the non-switched capacitors; these terms appear as the injected or extracted charge is the difference between the actual one minus the previous clock period charge. The other terms represent the charge contribution of the C_{i2}, C_{i3}, and C_{f2}, respectively. In order to facilitate the analysis of complex circuits, it is convenient to represent the topologies with the help of flow diagrams; see, for example, Ref. 20. Note that the output voltage is feedback by the capacitor C_{f2}; this capacitor is considered in a similar way as the other capacitors. Solving the circuit, or equivalently, arranging Eq. (71), the z-domain transfer function can be found as

$$\left.\frac{v_0}{v_{in}}\right|_{\phi_1} = \frac{-(1 - z^{-1})C_{i1} - C_{i2} + z^{-1/2}C_{i3}}{(C_f + C_{f2})\left(1 - \dfrac{C_f}{C_f + C_{f2}}z^{-1}\right)} \tag{72}$$

If the output voltage is sampled during ϕ_2 and assuming that v_{in} changes neither during the transition $\phi_1 - \phi_2$ nor during the clock phase ϕ_2, we can observe that the output of the first order circuit becomes

$$v_0|_{\phi_2} = z^{-1/2}v_0|_{\phi_1} \tag{73}$$

In the first-order filter of Fig. 11, we can note that all switches connected to the inverting input of the operational amplifier have been shared by several capacitors as all of them are connected to ground during ϕ_2 and to the operational amplifier during clock phase ϕ_1.

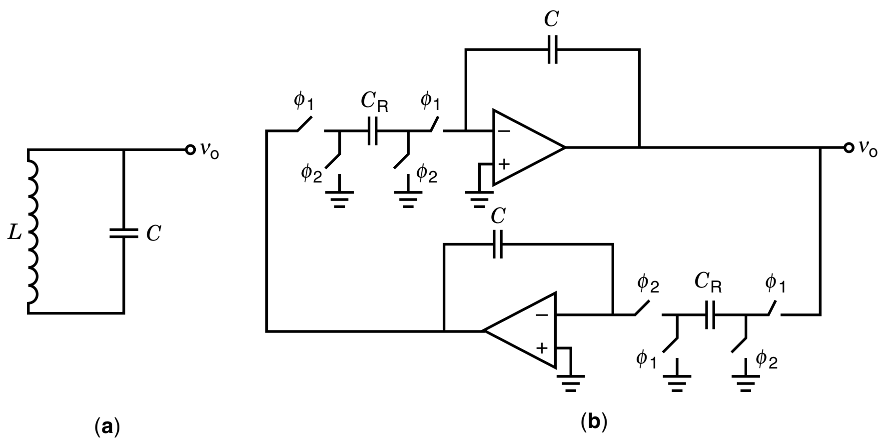

Figure 12. *LC* resonator: (a) passive and (b) switched-capacitor implementation. The inductor *L* is simulated by the switched-capacitor resistors, the bottom operational amplifier and the bottom capacitor *C*.

Active Resonator

Ladder filters are based on *LC* networks, series and parallel connected. While the capacitors are easily implemented in metal-oxide-semiconductor technologies, the inductor must be simulated by integrators and resistors. The inductor's current-voltage relationship is given by Eq. (57). For a grounded resonator as shown in Fig. 12(a), the conductor's current can be generated from the output node and an integrator. The transfer function of an *RC* integrator is given by $-1/sRC$; if an integrator's output is converted to current by a resistor, and the resulting current is fed back to the node v_0, the resulting current is then given by v_o/sR^2C. Hence, the simulated inductance results in

$$L = R^2C \tag{74}$$

Figure 12(b) shows the switched-capacitor realization of the resonator. For a high sampling rate ($f_{ck} \gg$ frequency of operation), the switched-capacitor resistors can be approximated by $R = 1/f_{ck}C_R$; for details, the reader may refer to Refs. 12–15, 19–20. According to Eq. (74), the simulated inductance is approximately given by

$$L = \frac{1}{f_{ck}^2}\frac{C}{C_R^2} \tag{75}$$

Observe that in the *LC* implementation, similar integrators have been used.

HIGH-ORDER FILTERS

High-order filters can be synthesized by using several approaches; two of the most common are based on biquads and the emulation of ladder filters. High-order filters based on biquadratic sections are more versatile, but ladder filters present low sensitivity to components' tolerances in the passband.

Second-order Filters

Second order filters are often used for the implementation of high-order filters; these filters are versatile in the sense that the filter parameters like center frequency, bandwidth, and dc or peak gain are controlled independently. A drawback of the biquadratic filters is that they present high sensitivity to component tolerances in the passband; see Refs. 14–17. A general biquadratic filter is shown in Fig. 13. The *z*-domain transfer function of the topology is given by

$$H(z) = -\frac{A_2A_5z + (z-1)(A_1A_5 + A_4z) + A_3(z-1)^2}{(1+A_8)\left[z^2 - \left(\dfrac{2 - A_5(A_6+A_7)+A_8}{1+A_8}\right)z + \dfrac{1-A_5A_6}{1+A_8}\right]} \tag{76}$$

By using this structure, several transfer functions could be implemented, namely

$A_1 = A_3 = A_4 = 0$	Low-pass filter (LDI)
$4A_2A_5 = A_3, A_1 = A_4 = 0$	Low-pass filter (Bilinear)
$A_2 = A_3 = A_4 = 0$ or	
$A_1 = A_3 = A_4 = 0$	Bandpass filter (LDI)
$A_2 = A_3 = 0, A_1A_5 = A_4$	Bandpass filter (Bilinear)
$A_1 = A_2 = A_4 = 0$	High-pass filter

In order to illustrate the design procedure, consider a second order bandpass filter with the following specifications:

Center frequency	10 kHz
Bandwidth	1 kHz
Clock frequency	60 kHz

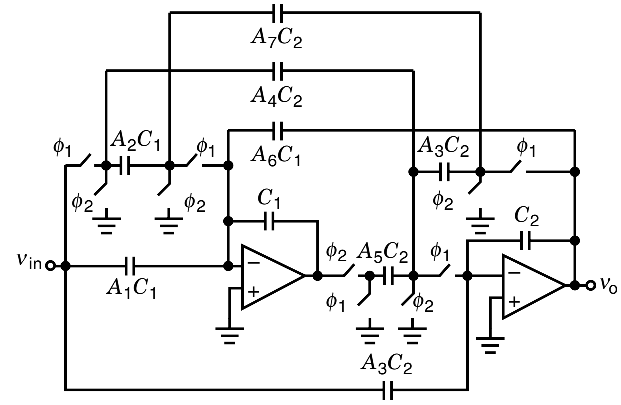

Figure 13. General second-order switched-capacitor filter. By choosing appropriated coefficients either lowpass, bandpass, highpass, or notch filters can be implemented.

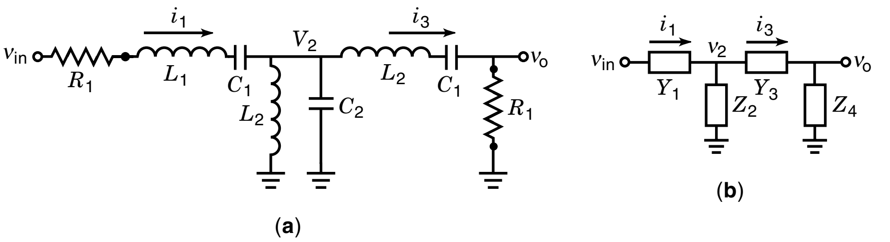

Figure 14. Passive sixth-order bandpass filter: (a) detailed *RLC* prototype and (b) simplified schematic. The element values can be obtained from well-known tables; see, for example, Ref. 12.

The continuous-time bandpass filter has the following form:

$$H(s) = \frac{BWs}{s^2 + BWs + \omega_0^2} \tag{77}$$

where ω_0 is the center frequency, equal to $2\pi f_0$, and BW the filter bandwidth. In low Q applications, it is more precise to prewarp the center frequency and the -3 dB frequencies; this prewarping scheme is discussed in (18). By using the bilinear transform, the prewarped center frequency is $f_0 = 11.0265$ kHz, and the -3 dB frequencies are mapped into 10.37 kHz and 11.7 kHz. Applying the s-domain to z-domain bilinear transform to the continuous-time bandpass filter, the following z-domain transfer function is obtained:

$$H(z) = \frac{\frac{BWT}{2}}{1 + \frac{BWT}{2} + \left(\frac{\omega_0 T}{2}\right)^2} \frac{(z-1)(z+1)}{z^2 + \left[\frac{-2 + 2\left(\frac{\omega_0 T}{2}\right)^2}{1 + \frac{BWT}{2} + \left(\frac{\omega_0 T}{2}\right)^2}\right] z + \frac{1 - \frac{BWT}{2} + \left(\frac{\omega_0 T}{2}\right)^2}{1 + \frac{BWT}{2} + \left(\frac{\omega_0 T}{2}\right)^2}} \tag{78}$$

Using the prewarped values and equating the transfer function of the bilinear bandpass filter, Eq. (72) with appropriated conditions, and the previous equation, the capacitor ratios are found: $A_1 = A_4 = 0.04965$, $A_5 = 1$, $A_6 = 0.09931$, $A_7 = 0.95034$, $A_8 = 0$.

Ladder Filters

Active ladder filters are based on the simulation of the passive components of low-sensitive passive prototypes. The active implementation of this type of filters is based on the voltage and current Kirchhoff's laws. Although this topic is treated in another chapter, here we consider an example to illustrate the design procedure of switched-capacitor ladder filters. Consider a third order, normalized, lowpass filter. For the design of a bandpass filter, the low-pass to bandpass transformation must be used; this transformation has the form

$$s_{\text{lp}} \Rightarrow \frac{s_{\text{bp}}}{BW} + \frac{\omega_0^2}{s_{\text{bp}} BW} \tag{79}$$

where BW and ω_0 are the bandwidth and center frequency of the bandpass filter, respectively. Observe that the inductor is mapped into a series of another inductor and a capacitor, while the capacitor is transformed into another capacitor in parallel with an inductor. The bandpass filter prototype and a simplified schematic are shown in Figs. 14(a) and 14(b), respectively. The transformed elements are then given by

$$\begin{aligned} L_1 &= \frac{L_{\text{LP}}}{BW} \\ C_1 &= \frac{BW}{L_{\text{LP}}\omega_0^2} \\ L_2 &= \frac{BW}{C_{\text{LP}}\omega_0^2} \\ C_2 &= \frac{C_{\text{LP}}}{BW} \end{aligned} \tag{80}$$

Current i_1 could be generated by the active circuit of Fig. 15. By using circuit analysis, i_1 could as well be obtained as

$$i_1 = \frac{1}{R_Q^2 Y_{11}}(v_{\text{in}} - v_2) \tag{81}$$

Equating i_1 of the passive realization with the simulated current, the relationship between Y_1 and Y_{11} can be determined

$$Y_1 = \frac{1}{Z_1} = \frac{1}{R_Q^2 Y_{11}} \tag{82}$$

This expression means that the impedance Z_1 is simulated by the admittance Y_{11} multiplied by the factor R_Q^2. For the bandpass filter, Z_1 is the series of a resistor, an inductor, and a capacitor; this impedance can be simulated by the parallel of similar elements. For grounded resonators, the capacitors are connected to the operational amplifier, and the inductors are simulated by using integrators and resistors, as previously discussed. In the implementation of the 6th order bandpass filter, three resonators are being employed, one for each LC resonator.

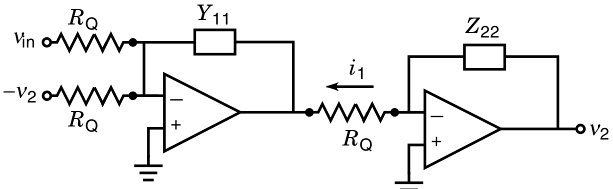

Figure 15. Active implementation of floating impedances. The admittance Y_{11} is directly associated with the floating impedance Z_1. Resistors R_Q are scaling factors.

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JOSE SILVA MARTINEZ
GORDANA JOVANOVIC-DOLECEK
Instituto Nacional de Astrofisica,
Optica y Electronica, INAOE

DISCRETE-TIME FILTERS. See DIGITAL FILTERS.