DIFFERENTIATING CIRCUITS

Since the dynamics of systems comes through derivatives, electronic circuits that perform differentiation are important components both in theory and in practice. In the following article we define differentiators, giving their transfer functions from which some properties can be inferred. Highly accurate voltage mode physical realizations in terms of op-amp circuits, as well as current mode ones suitable for very large scale integration (VLSI) realization, are given. Also included are approximate ones in terms of simple resistor-capacitor (RC) circuits appropriate for many control system applications. Since differentiators are not represented in terms of standard state variables, we conclude with a semistate representation.

DEFINITION

Here a differentiating circuit, also known as a differentiator, is defined as an electronic circuit satisfying the law

$$y(t) = D\frac{du(t)}{dt} \tag{1}$$

where $u(\cdot)$ is the input to the circuit, $y(\cdot)$ is the output, and D is the differentiator gain, usually taken to be a real constant in time t. Because the differentiator is electronic, we take u and y to be voltages or currents, though at times one may wish to consider flux linkages or charge. According to Eq. (1), ideally the differentiator is a linear system, mapping the input into the output linearly. If D is a constant, then on taking the bilateral Laplace transform, $L[\cdot]$, of Eq. (1) we obtain the system transfer function description, $H(s) = L[y]/L[u]$, in terms of the complex frequency variable $s = \sigma + j\omega$,

$$H(s) = Ds \tag{2}$$

Evaluating this in terms of real frequency, ω, we find that

$$H(j\omega) = jD\omega \tag{3}$$

which shows that the differentiator introduces a constant phase shift of 90° and amplifies the input proportionally to the frequency. This illustrates the difficulty one runs into when practically using a differentiator since high-frequency signals, which are often noise components, get greatly amplified compared to low-frequency signals, which most often are the ones carrying intelligence, according to $|H(j\omega)| = D\omega$ (for $\omega \geq 0$).

CIRCUIT IMPLEMENTATIONS

If the output is measured at the same terminals as the input, then the device is a one-port differentiator; and in the case where u is a voltage v and y is a current i, the differentiator is equivalent to a capacitor so that the gain constant becomes an equivalent capacitance, $D = C$, as illustrated in Fig. 1(a). Figure 1(b) shows the dual case of an inductor.

For voltage mode circuits, differentiators are customarily two-port devices constructed from operational amplifiers according to the circuit of Fig. 2(a) (1, p. 10). In Fig. 2(a), u is a voltage, v_{in}, as is y, v_{out}, and the gain constant is $D = -RC$. If an ideal op-amp is assumed, it is an infinite gain device with a virtual ground input. Thus,

$$v_{\text{out}}(t) = -RC\frac{dv_{\text{in}}(t)}{dt} \tag{4}$$

It should be noted that achieving a gain $D = -RC$ near unity in magnitude usually requires a large resistor; for example, if $C = 1\ \mu F$, then $R = 1\ \text{M}\Omega$.

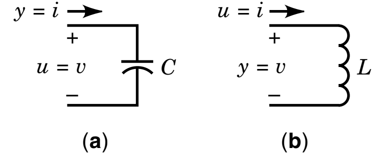

Figure 1. (a) Capacitor as a differentiator. Here the input u equals v and the output y equals i, which gives $Y(s) = I(s) = CsV(s)$ in the frequency domain. (b) Inductor as a differentiator. Here the input u equals i and the output y equals v, which gives $Y(s) = V(s) = LsI(s)$ in the frequency domain.

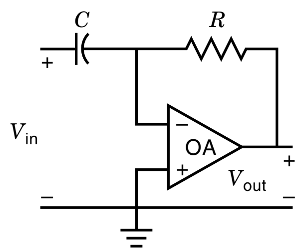

Figure 2. Op-amp differentiator.

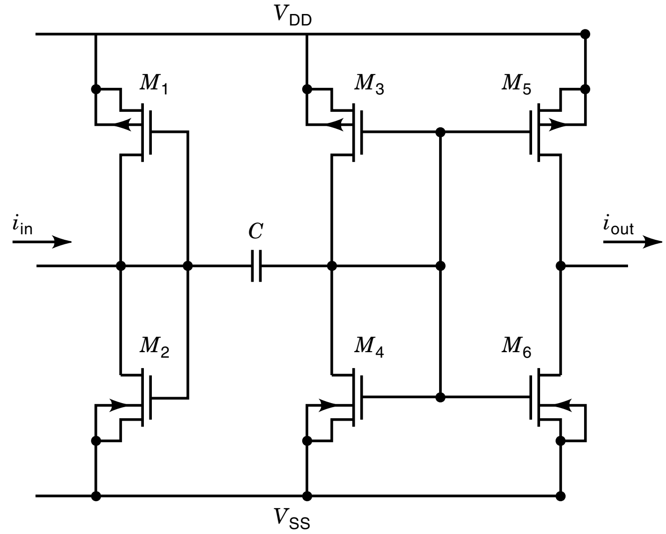

Figure 4. Current mode differentiator.

For practical op-amps there are several points to consider. One is that the op-amp gain is reasonably well approximated by a one-pole model $K(s) = K_0/(s + p_1)$ so that the differentiator transfer function becomes

$$H(s) = \frac{-sRC}{\left(1 - \dfrac{1}{K_0}\right) - \dfrac{s^2RC}{K_0} + \dfrac{s(1+RCp_1)}{K_0}} \tag{5}$$

from which we see that as the frequency gets large the gain drops toward zero according to K_0/s. Besides this sometimes favorable behavior, there are also nonlinearities due to saturation of the op-amp at the bias voltages and added noise due to the transistors and resistors used in the construction of the op-amp.

In some cases, such as derivative control, it is more economical to use the RC circuit of Fig. 3, which gives an approximate differentiator having the transfer function

$$H(s) = \frac{sC}{1+sRC} \tag{6}$$

For frequencies lower than $\omega_p = 1/RC$, this circuit approximates a differentiator reasonably well, and its gain saturates at higher frequencies so that noise is not unduly amplified.

In terms of modern very large scale integration (VLSI) implementations, it is most convenient to use current mode structures. Figure 4 shows a current mode structure which differentiates the signal component of the input using complementary metal oxide semiconductor (CMOS) transistors (2). Here the input transistors M_1 and M_2 act as resistors to convert the input current i_{in} to a voltage which is applied to the capacitor. The capacitor current, which is essentially the derivative of this voltage, is transferred through the output current mirror transistors M_5 and M_6, which can give an added gain k. The analysis of the circuit proceeds by replacing all transistors by their small-signal equivalent circuits incorporating their transconductances, g_m. Ignoring their source to drain conductances, the analysis gives the small-signal transfer function as [2, Eq. (6)]

$$H(s) = \frac{i_{out}}{i_{in}} = \left(\frac{-kC}{g_{mp}+g_{mn}}\right)\left(\frac{s}{1+\dfrac{2sC}{g_{mp}+g_{mn}}}\right) \tag{7}$$

where k is the current gain of the output current mirror transistors and g_{mn} and g_{mp} are the transconductances of the n channel metal oxide semiconductor (NMOS) and p channel metal oxide semiconductor (PMOS) transistors. Consequently, the circuit makes a reasonably good current-in current-out differentiator through the mid-megahertz frequency range. Note that since the transistor transconductances depend on the bias point of the transistors, it is important to use a regulated voltage source to keep the direct current bias stable.

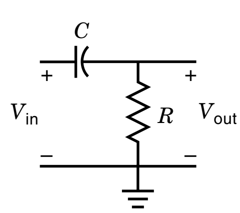

Figure 3. RC ladder circuit.

NOISE AND SIGNAL SWING CONSIDERATIONS

When working with op-amp differentiators, it is important to consider noise behavior. For this the noise sources are normally reflected into the input and then the output noise is found by applying the equivalent noiseless amplifier to the input noise sources with a good treatment given in Ref. 1, p. 141. Such an operation is automatically carried out in the PSpice noise analysis. A typical frequency response example circuit is shown in Fig. 5(a) with its noise input, Fig. 5(b, top) and noise output, Fig. 5(b, middle), along with the output, Fig. 5(b, bottom), for a 1 mV input plus noise. In this circuit a 701 op-amp is used and a reasonably small resistor is inserted in series with the differentiation capacitor to assist in damping out the noise signal.

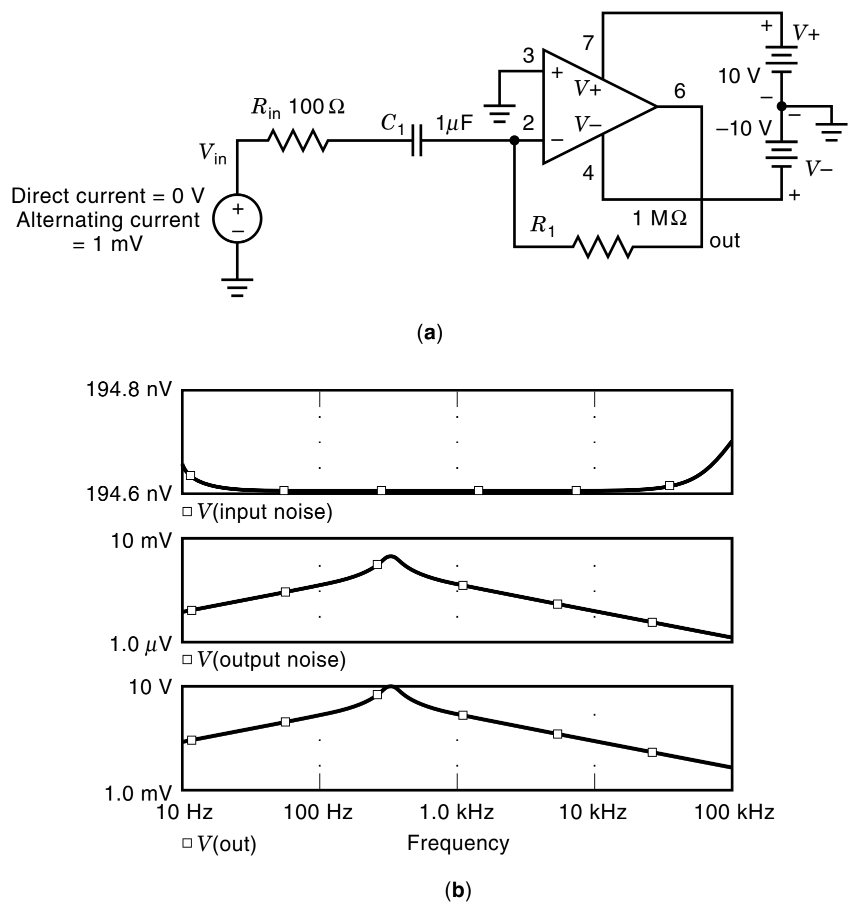

Figure 5. (a) PSpice circuit of a differentiator using 701 op-amp. (b) Response and noise behavior in the frequency domain. The top curve represents the input noise, the middle curve is the output noise, and the bottom curve is the output response to an input voltage of 1 mV plus noise.

SEMISTATE EQUATIONS

Because the standard state-space equations do not exist for a differentiator, we give a description of it in terms of semistate equations. These latter are equations of the form

$$E\frac{dx}{dt} = Ax + Bu \tag{8}$$

$$y = Cx \tag{9}$$

which have the transfer function

$$H(s) = C(sE - A)^{-1}B \tag{10}$$

On choosing the semistate vector $x = [x_1, x_2]^T = [y, u]^T$, where T denotes transpose, one can write

$$E\frac{dx}{dt} = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}\frac{dx}{dt} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}x + \begin{bmatrix} 0 \\ -1 \end{bmatrix}u = Ax + Bu \tag{11}$$

$$y = [1 \quad 0]x = Cu \tag{12}$$

which gives

$$H(s) = [1 \quad 0]\begin{bmatrix} -1 & Ds \\ 0 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = Ds \tag{13}$$

These semistate equations can be transformed via a linear transformation into a form that is useful for circuit realizations based upon integrators (3). Op-amp integrators are known to be less sensitive to noise compared to the conventional op-amp differentiator.

BIBLIOGRAPHY

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DIFFERENTIATION. See CALCULUS.
DIFFRACTION. See BACKSCATTER; ELECTROMAGNETIC WAVE SCATTERING.