CHEBYSHEV FILTERS

Any signal can be considered to be composed of several sinusoidal components with different frequencies, amplitudes, and phases. Filtering is one of the fundamental methods for signal processing, where the signal is processed by a linear system that changes the amplitudes and phases of these components, but not their frequencies. In a simple form, this processing can be used to let pass or to reject selected frequency bands, ideally with no attenuation at the passbands and infinite attenuation at the stopbands. This article discusses a class of approximations to this kind of ideal filter, known as an Chebyshev filter. It starts with a discussion on a technique for the derivation of optimal magnitude filters, then discusses the direct and inverse Chebyshev approximations for the ideal filtering operator, ending with comments on extensions of the technique. Tables with example filters are included.

The magnitude approximation problem in filter design consists essentially of finding a convenient transfer function with the magnitude satisfying given attenuation specifications. Other restrictions can exist, such as structure for implementation, maximum order, and maximum Q of the poles, but in most cases the problem can be reduced to the design of a normalized continuous-time low-pass filter, which can be described by a transfer function in Laplace transform. This filter must present a given maximum passband attenuation ($A_{\max}$), between $\omega = 0$ and $\omega = \omega_p = 1$ rad/s, and a given minimum stopband attenuation ($A_{\min}$) in frequencies above a given limit ω_r rad/s. From this prototype filter, the final filter can be obtained by frequency transformations and by continuous-time to discrete-time transformations in the case of a digital filter (1).

A convenient procedure for the derivation of optimal magnitude filters is to start with the transducer function $H(s)$ and the characteristic function $K(s)$. $H(s)$, which can also be called the attenuation function, is the inverse of the filter transfer function, scaled such that min $|H(j\omega) = 1|$. $K(s)$is related to $H(s)$ by the equation

$$|H(j\omega)|^2 = 1 + |K(j\omega)|^2 \tag{1}$$

This greatly simplifies the problem, because $K(j\omega)$ can be a ratio of two real polynomials in ω, both with roots located symmetrically on both sides of the real axis, while $H(j\omega)$ is a complex function. $K(s)$ is obtained by replacing ω by s/j in $K(j\omega)$ and ignoring possible $\pm j$ or -1 multiplying terms resulting from the operation. The complex frequencies where $K(s) = 0$ are the attenuation zeros, and where $K(s) = \infty$ correspond to the transmission zeros. If $K(s)$ is a ratio of real polynomials, $K(s) = F(s)/P(s)$, $H(s)$ is also a ratio of real polynomials in s, with the same denominator, $H(s) = E(s)/P(s)$, and $E(s)$ can be obtained by observing that for $s = j\omega$ Eq. (1) is equivalent to

$$\begin{aligned} H(s)H(-s) = 1 + K(s)K(-s) \therefore E(s)E(-s) \\ = P(s)P(-s) + F(s)F(-s) \end{aligned} \tag{2}$$

Because $E(s)$ is the denominator of the filter transfer function, which must be stable, $E(s)$ is constructed from the roots of the polynomial $P(s)P(-s) + F(s)F(-s)$ with negative real parts. The desired transfer function is then $T(s) = P(s)/E(s)$.

CHEBYSHEV POLYNOMIALS

Two important classes of approximations, the direct and inverse Chebyshev approximations, can be derived from a class of polynomials known as Chebyshev polynomials. These poly-

Table 1. Chebyshev Polynomials

n	Polynomial
0	1
1	x
2	$2x - 1$
3	$4x^3 - 3x$
4	$8x^4 - 8x^2 + 1$
5	$16x^5 - 20x^3 + 5x$
6	$32x^6 - 48x^4 + 18x^2 - 1$
7	$64x^7 - 112x^5 + 56x^3 - 7x$
8	$128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$
9	$256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$
10	$512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$
11	$1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x$
12	$2048x^{12} - 6144x^{10} + 6912x^8 - 3584x^6 + 840x^4 - 72x^2 + 1$

nomials were first described by P. L. Chebyshev (2). The Chebyshev polynomial of order n can be obtained from the expression

$$C_n(x) = \cos(n \cos^{-1} x) \tag{3}$$

It is simple to verify that this expression corresponds, for $-1 \leq x \leq 1$, to a polynomial in x. Using the trigonometric identity $\cos(a + b) = \cos a \cos b - \sin a \sin b$, we obtain

$$\begin{aligned} C_{n+1}(x) &= \cos[(n+1)\cos^{-1} x] \\ &= xC_n(x) - \sin(n \cos^{-1} x)\sin(\cos^{-1} x) \end{aligned} \tag{4}$$

Applying now the identity $\sin a \sin b = \frac{1}{2}[\cos(a - b) - \cos(a + b)]$ and rearranging, a recursion formula is obtained:

$$C_{n+1}(x) = 2xC_n(x) - C_{n-1}(x) \tag{5}$$

For $n = 0$ and $n = 1$, we have $C_0(x) = 1$ and $C_1(x) = x$. Using Eq. (5), the series of Chebyshev polynomials shown in Table 1 is obtained.

The values of these polynomials oscillate between -1 and $+1$ for x between -1 and $+1$, in a pattern identical to a stationary Lissajous figure (3). For x out of this range, $\cos^{-1} x = j \cosh^{-1} x$, an imaginary value, but Eq. (3) is still real, in the form

$$C_n(x) = \cos(nj \cosh^{-1} x) = \cosh(n \cosh^{-1} x) \tag{6}$$

For high values of x, looking at the polynomials in Table 1, we see that $C_n(x) \approx 2^{n-1}x^n$ and grows monotonically. The plots of some Chebyshev polynomials for $-1 \leq x \leq 1$ are shown in Fig. 1.

THE CHEBYSHEV LOW-PASS APPROXIMATION

This normalized Chebyshev low-pass approximation is obtained by using

$$K(j\omega) = \epsilon C_n(\omega) \tag{7}$$

The result is a transducer function with the magnitude given by [from Eq. (1)]

$$|H(j\omega)| = \sqrt{1 + [\epsilon C_n(\omega)]^2} \tag{8}$$

Or, the attenuation in decibels is

$$A(\omega) = 10 \log\{1 + [\epsilon C_n(\omega)]^2\} \tag{9}$$

The parameter ϵ controls the maximum passband attenuation, or the passband ripple. Considering that when $C_n(\omega) = \pm 1$ the attenuation $A(\omega) = A_{\max}$ Eq. (9) gives

$$\epsilon = \sqrt{10^{0.1A_{\max}} - 1} \tag{10}$$

Figure 2 shows examples of the magnitude function $|T(j\omega)|$ in the passband and in the stopband obtained for some normalized Chebyshev low-pass approximations, with $A_{\max} = 1$ dB. The magnitude of the Chebyshev approximations presents uniform ripples in the passband, with the gain departing from 0 dB at $\omega = 0$ for odd orders and from $-A_{\max}$ dB for even orders.

The stopband attenuation is the maximum possible among filters derived from polynomial characteristic functions and with the same $A_{\max}$ and degree (4). This can be proved by assuming that there exists a polynomial $P_n(x)$ that is also bounded between -1 and 1 for $-1 \leq x \leq 1$, with $P_n(x) = \pm P_n(-x)$ and $P_n(+\infty) = +\infty$, which exceeds the value of $C_n(x)$ for some value of $x > 1$. An approximation using this polynomial instead of $C_n(x)$ in Eq. (7) would be more selective. The curves of $P_n(x)$ and $C_n(x)$ will always cross x times for $-1 \leq x \leq 1$, due to the maximum oscillations of $C_n(x)$, but if $P_n(x)$ grows faster, they will cross another two times for $x \geq 1$ and $x \leq -1$. This makes $P_n(x) - C_n(x)$ a polynomial of degree $n + 2$, because it has $n + 2$ roots, which is impossible since both are of degree n.

The required approximation degree for given $A_{\max}$ and $A_{\min}$ can be obtained by substituting Eq. (6) in Eq. (9), with $A(\omega_r) = A_{\min}$ and solving for n. The result, including a denormalization for any ω_p, is

$$n \geq \frac{\cosh^{-1} \gamma}{\cosh^{-1}(\omega_r/\omega_p)} \tag{11}$$

where we define the constant γ as

$$\gamma = \sqrt{\frac{10^{0.1A_{\min}} - 1}{10^{0.1A_{\max}} - 1}} \tag{12}$$

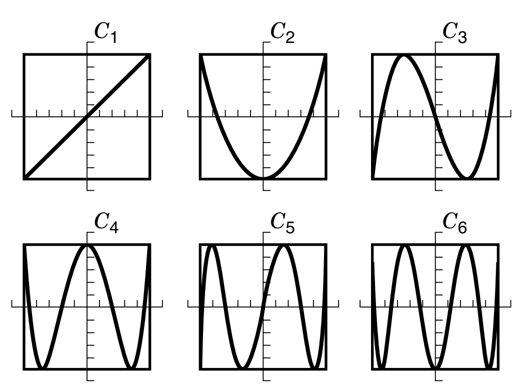

Figure 1. Plots of the first six Chebyshev polynomials $C_n(x)$. The squares limit the region $-1 \leq x \leq 1$, $-1 \leq C_n(x) \leq 1$, where the polynomial value oscillates.

The transfer functions for the normalized Chebyshev filters can be obtained by solving Eq. (2). For a polynomial approximation, using $P(s) = 1$, from Eq. (7) it follows that

$$E(s)E(-s) = 1 + \left[\epsilon C_n\left(\frac{s}{j}\right)\right]^2 \tag{13}$$

The roots of this polynomial are the solutions for s in

$$C_n\left(\frac{s}{j}\right) = \cos\left(n\cos^{-1}\frac{s}{j}\right) = \pm\frac{j}{\epsilon} \tag{14}$$

Identifying

$$n\cos^{-1}\frac{s}{j} = a + jb \tag{15}$$

it follows that $\pm j/\epsilon = \cos(a + jb) = \cos a \cos jb - \sin a \sin jb = \cos a \cosh b - j \sin a \sinh b$. Equating real and imaginary parts, we have $\cos a \cosh b = 0$ and $\sin a \sinh b = \mp 1/\epsilon$. Since $\cosh x \geq 1$, the equation of the real parts gives:

$$a = \frac{\pi}{2}(1 + 2k),\ k = 0,\ 1,\ \ldots,\ 2n - 1 \tag{16}$$

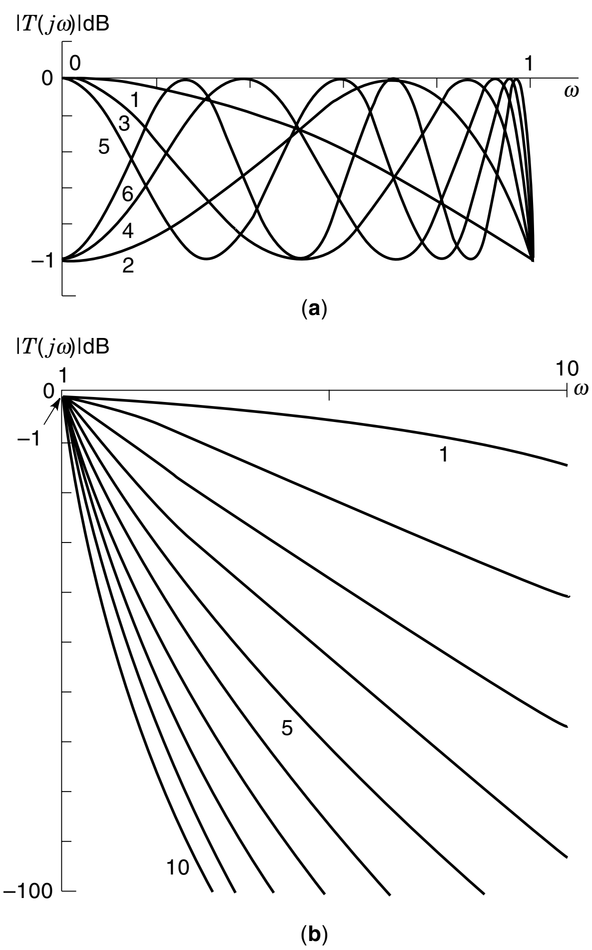

Figure 2. Passband gain (a) and stopband gain (b) for the first normalized Chebyshev approximations with 1 dB passband ripple. Observe the uniform passband ripple and the monotonic stopband gain decrease.

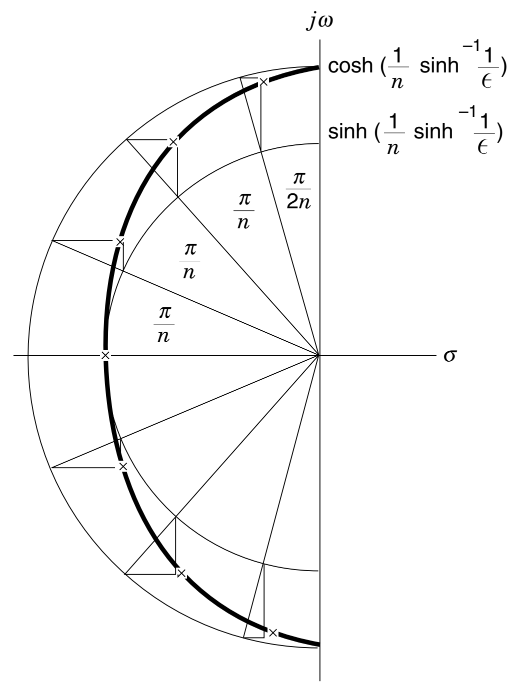

Figure 3. Localization of the poles in a normalized Chebyshev lowpass approximation (seventh order, in this case). The pole locations can be obtained as shown.

and as for these values of a, $\sin a = \pm 1$, the equation of the imaginary parts gives

$$b = \mp \sinh^{-1}\frac{1}{\epsilon} \tag{17}$$

By applying these results in Eq. (15), it follows that the roots of $E(s)E(-s)$ are

$$\begin{aligned} s_k &= \sigma_k + j\omega_k \quad k = 0,\ 1,\ \ldots,\ 2n - 1 \\ \sigma_k &= \sin\left(\frac{\pi}{2}\frac{1+2k}{n}\right)\sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\epsilon}\right) \\ \omega_k &= \cos\left(\frac{\pi}{2}\frac{1+2k}{n}\right)\cosh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\epsilon}\right) \end{aligned} \tag{18}$$

The roots s_k with negative real parts ($k \geq n$) are the roots of $E(s)$. By the expressions in Eq. (18), it is easy to see that the roots s_k are located on an ellipse with vertical semi-axis cosh $(1/n \sinh^{-1} 1/\epsilon)$, horizontal semi-axis sinh $(1/n \sinh^{-1} 1/\epsilon)$, and foci at $\pm j$. The location of the roots can be best visualized with the diagram shown in Fig. 3 (3).

REALIZATION OF CHEBYSHEV FILTERS

These approximations were originally developed for realization in passive form, and the best realizations were obtained as LC doubly terminated structures designed for maximum power transfer at the passband gain maxima. These structures are still important today as prototypes for active and digital realizations, due to the low sensitivity to errors in element values. At each attenuation zero, and the Chebyshev approximations have the maximum possible number of them distributed in the passband, maximum power transfer occurs between the terminations. In this condition, errors in the capacitors and inductors can only decrease the gain (5). This causes zeros in the derivative $\partial|T(j\omega)|/\partial L, C$ at all the attenuation zeros, and keeps low the error in all the passband. Table 2 lists polynomials, poles, frequency, and Q of the poles, and values for LC doubly terminated ladder structures, with the structure shown in Fig. 4(a), for some normalized Chebyshev low-pass filters. Note in the realizations that odd-order filters have identical terminations, but even-order filters require different terminations, because there is no maximum power transfer at $\omega = 0$, since the gain is not maximum there. With the impedance normalization shown, it is clear that the even-order realizations have antimetrical structure (one side is the dual of the other). The odd-order structures are symmetrical.

THE INVERSE CHEBYSHEV LOW-PASS APPROXIMATION

This normalized inverse Chebyshev approximation is the most important member of the inverse polynomial class of approximations. It is conveniently obtained by using the characteristic function obtained from

$$K(j\omega) = \frac{F(j\omega)}{P(j\omega)} = \frac{\epsilon\gamma}{C_n(1/\omega)} = \frac{\epsilon\gamma\omega^n}{\omega^n C_n(1/\omega)} \tag{19}$$

where ϵ and γ are given by Eqs. (10) and (11). The polynomials $F(s)$ and $P(s)$ are then

$$\begin{aligned} F(s) &= \epsilon\gamma(s/j)^n \\ P(s) &= (s/j)^n C_n(j/s) \end{aligned} \tag{20}$$

Ignoring $\pm j$ or -1 multiplying factors in Eq. (20) and renormalizations, $F(s)$ reduces to $\epsilon\gamma\ s^n$, and $P(s)$ to a Chebyshev polynomial with all the terms positive and the coefficients in reverse order. The magnitude characteristic of this approximation is maximally flat at $\omega = 0$, due to the n attenuation zeros at $s = 0$, and so is similar in the passband to a Butterworth approximation. In the stopband, it presents a series of transmission zeros at frequencies that are the inverse of the roots of the corresponding Chebyshev polynomial. Between adjacent transmission zeros, there are gain maxima reaching the magnitude of $-A_{\min}$ dB. Without a renormalization, the stopband starts at 1 rad/s, and the passband ends where the magnitude of the characteristic function, Eq. (19), reaches ϵ:

$$\omega_p = \frac{1}{C_n^{-1}(\gamma)} = \frac{1}{\cosh\left(\frac{1}{n}\cosh^{-1}\gamma\right)} \tag{21}$$

Odd-order filters present a single transmission zero at infinity, and even-order filters end up with a constant gain $-A_{\min}$ at $\omega = \infty$. From Eqs. (1) and (19), the attenuation in decibels for a normalized inverse Chebyshev approximation is

$$A(\omega) = 10\log\left\{1 + \left[\frac{\epsilon\gamma}{C_n(1/\omega)}\right]^2\right\} \tag{22}$$

The gains for some normalized inverse Chebyshev approximations are plotted in Fig. 5. A frequency scaling by the inverse of the factor given by Eq. (21) was applied to make the passband end at $\omega = 1$.

The selectivity of the inverse Chebyshev approximation is the same as the corresponding Chebyshev approximation, for the same $A_{\max}$ and $A_{\min}$. This can be verified by calculating the ratio ω_p/ω_r for both approximations. For the normalized Chebyshev approximation, $\omega_p = 1$, and ω_r occurs when $\epsilon C_n(\omega_r) = \gamma$. For the normalized inverse Chebyshev approximation, $\omega_r = 1$, and ω_p occurs when $(\epsilon\gamma)/C_n(1/\omega_p) = \epsilon$. In both cases, the resulting ratio is $\omega_r/\omega_p = C_n^{-1}(\gamma)$. Equation (11) can be used to compute the required degree.

The transmission zero frequencies are the frequencies that make Eq. (19) infinite:

$$C_n(1/\omega_k) = \cos(n\cos^{-1}(1/\omega_k)) = 0 \therefore$$
$$\omega_k = \frac{1}{\cos\left(\frac{\pi}{2}\frac{1+2k}{n}\right)}, \quad k = 0, 1, \ldots, n-1 \tag{23}$$

The pole frequencies are found by solving Eq. (2) with $F(s)$ and $P(s)$ given by Eq. (20):

$$E(s)E(-s) = (\epsilon\gamma)^2\left(\frac{s}{j}\right)^{2n} + \left(\frac{s}{j}\right)^{2n} C_n\left(\frac{j}{s}\right)^2 \tag{24}$$

The roots of this equation are the solutions of

$$C_n\left(\frac{j}{s}\right) = \pm j\epsilon\gamma \tag{25}$$

By observing the similarity of this equation to Eq. (14), the roots of $E(s)E(-s)$ can be obtained as the complex inverses of the values given by Eq. (18), with ϵ replaced by $1/(\epsilon\gamma)$. They lie on a curve that is not an ellipse. $E(s)$ is constructed from the roots with negative real parts, which are distributed in a pattern that resembles a circle shifted to the left side of the origin.

The similarity of the passband response to the Butterworth response makes the phase characteristics of the inverse Chebyshev filters much closer to linear than those of the Chebyshev filters. The Qs of the poles are also significantly lower for the same gain specifications.

REALIZATION OF INVERSE CHEBYSHEV FILTERS

The realization based on LC doubly terminated ladder structures is also convenient for inverse Chebyshev filters for the same reasons mentioned for the direct approximation. In this case, the passband sensitivities are low due to the nth-order attenuation zero at $s = 0$, which results in nullification of the first n derivatives of the filter gain in relation to all the reactive elements at $s = 0$ and keeps the gain errors small in all the passband. Stopband errors are also small, because the transmission zero frequencies depend only on simple LC series or parallel resonant circuits. The usual structures used are shown in Fig. 4(b).

Those realizations are possible only for the odd-order cases, because those structures cannot realize the constant gain at infinity that occurs in the even-order approximations (realizations with transformers or with negative elements are possible). Even-order modified approximations can be obtained by using, instead of the Chebyshev polynomials, poly-

Table 2. Normalized Chebyshev Filters with A_{max} = 1 dB

Polynomials $E(s)$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
1	1.96523	1.00000									
2	1.10251	1.09773	1.00000								
3	0.49131	1.23841	0.98834	1.00000							
4	0.27563	0.74262	1.45392	0.95281	1.00000						
5	0.12283	0.58053	0.97440	1.68882	0.93682	1.00000					
6	0.06891	0.30708	0.93935	1.20214	1.93082	0.92825	1.00000				
7	0.03071	0.21367	0.54862	1.35754	1.42879	2.17608	0.92312	1.00000			
8	0.01723	0.10734	0.44783	0.84682	1.83690	1.65516	2.42303	0.91981	1.00000		
9	0.00768	0.07060	0.24419	0.78631	1.20161	2.37812	1.88148	2.67095	0.91755	1.00000	
10	0.00431	0.03450	0.18245	0.45539	1.24449	1.61299	2.98151	2.10785	2.91947	0.91593	1.00000

Poles

n	re/im 1	ω/Q 1	re/im 2	ω/Q 2	re/im 3	ω/Q 3	re/im 4	ω/Q 4	re/im 5	ω/Q 5
1	−1.96523									
2	−0.54887	1.05000								
	0.89513	0.95652								
3	−0.24709	0.99710	−0.49417							
	0.96600	2.01772								
4	−0.13954	0.99323	−0.33687	0.52858						
	0.98338	3.55904	0.40733	0.78455						
5	−0.08946	0.99414	−0.23421	0.65521	−0.28949					
	0.99011	5.55644	0.61192	1.39879						
6	−0.06218	0.99536	−0.16988	0.74681	−0.23206	0.35314				
	0.99341	8.00369	0.72723	2.19802	0.26618	0.76087				
7	−0.04571	0.99633	−0.12807	0.80837	−0.18507	0.48005	−0.20541			
	0.99528	10.89866	0.79816	3.15586	0.44294	1.29693				
8	−0.03501	0.99707	−0.09970	0.85061	−0.14920	0.58383	−0.17600	0.26507		
	0.99645	14.24045	0.84475	4.26608	0.56444	1.95649	0.19821	0.75304		
9	−0.02767	0.99761	−0.07967	0.88056	−0.12205	0.66224	−0.14972	0.37731	−0.15933	
	0.99723	18.02865	0.87695	5.52663	0.65090	2.71289	0.34633	1.26004		
10	−0.02241	0.99803	−0.06505	0.90245	−0.10132	0.72148	−0.12767	0.47606	−0.14152	0.21214
	0.99778	22.26303	0.90011	6.93669	0.71433	3.56051	0.45863	1.86449	0.15803	0.74950

Polynomials $P(s)$

n	Multiplier	a_0
1	1.96523	1.00000
2	0.98261	1.00000
3	0.49131	1.00000
4	0.24565	1.00000
5	0.12283	1.00000
6	0.06141	1.00000
7	0.03071	1.00000
8	0.01535	1.00000
9	0.00768	1.00000
10	0.00384	1.00000

Doubly terminated LC ladder realizations

n	Rg/R_l	L/C_1	L/C_2	L/C_3	L/C_4	L/C_5	L/C_6	L/C_7	L/C_8	L/C_9	L/C_{10}
1	1.00000										
	1.00000	1.01769									
2	1.63087		1.11716								
	0.61317	1.11716									
3	1.00000		0.99410								
	1.00000	2.02359		2.02359							
4	1.63087		1.73596								
	0.61317	1.28708		1.73596	1.28708						
5	1.00000		1.09111								
	1.00000	2.13488		3.00092	1.09111	2.13488					
6	1.63087		1.80069		1.87840		1.32113				
	0.61317	1.32113		1.87840		1.80069					
7	1.00000		1.11151		1.17352		1.11151				
	1.00000	2.16656		3.09364		3.09364		2.16656			
8	1.63087		1.82022		1.93073		1.90742		1.33325		
	0.61317	1.33325		1.90742		1.93073		1.82022			
9	1.00000		1.11918		1.18967		1.18967		1.11918		
	1.00000	2.17972		3.12143		3.17463		3.12143		2.17972	
10	1.63087		1.82874		1.94609		1.95541		1.91837		1.33890
	0.61317	1.33890		1.91837		1.95541		1.94609		1.82874	

nomials obtained by the application, to the Chebyshev polynomials, of the Moebius transformation (4,6):

$$x^2 \rightarrow \frac{x^2 - x_{z1}^2}{1 - x_{z1}^2};\ x_{z1} = \cos\frac{k_{\max}\pi}{2n} \tag{26}$$

where $k_{\max}$ is the greatest odd integer that is less than the filter order n. This transformation moves the pair of roots closest to the origin of an even-order Chebyshev polynomial to the origin. If the resulting polynomials are used to generate polynomial approximations, starting from Eq. (7), the results are filters with two attenuation zeros at the origin, which are realizable as doubly terminated ladder filters with equal terminations, a convenience in passive realizations. If the same polynomials are used in inverse polynomial approximations, starting from Eq. (19), the results are filters with two transmission zeros at infinity, which now are realizable by doubly terminated *LC* structures. The direct and inverse approximations obtained in this way have the same selectivity, slightly smaller than in the original case.

Table 3 lists polynomials, poles, zeros, frequency and *Q* of the poles, and *LC* doubly terminated realizations for some inverse Chebyshev filters. The filters were scaled in frequency to make the passband end at 1 rad/s. The even-order realizations are obtained from modified approximations and are listed separately in Table 4. The structures are a mix of the two forms in Fig. 5(b). Note that some realizations are missing. These are cases where the zero-shifting technique for the realization of *LC* doubly terminated ladder filters fails. For inverse Chebyshev filters, and other inverse polynomial filters, there is a minimum value of $A_{\min}$ for each order that makes the realization in this form possible (7).

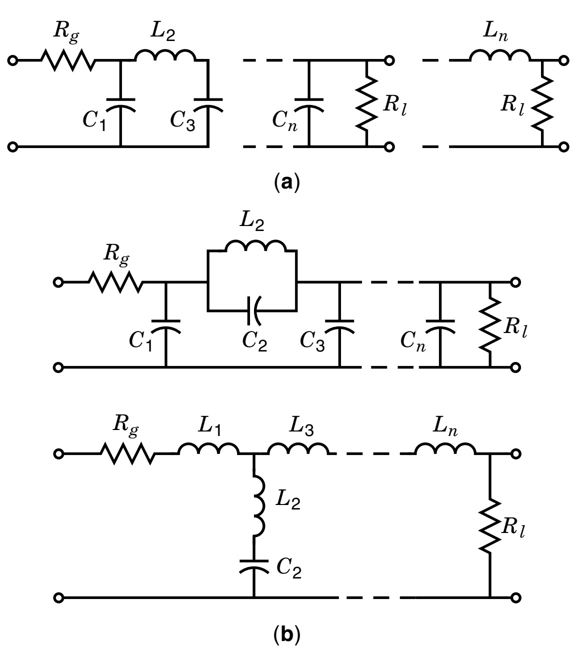

Figure 4. *LC* doubly terminated ladder realizations for Chebyshev filters, in the direct form (a), and in the inverse form (b). These classical realizations continue to be the best prototypes for active realizations, due to their low sensitivity to errors in element values.

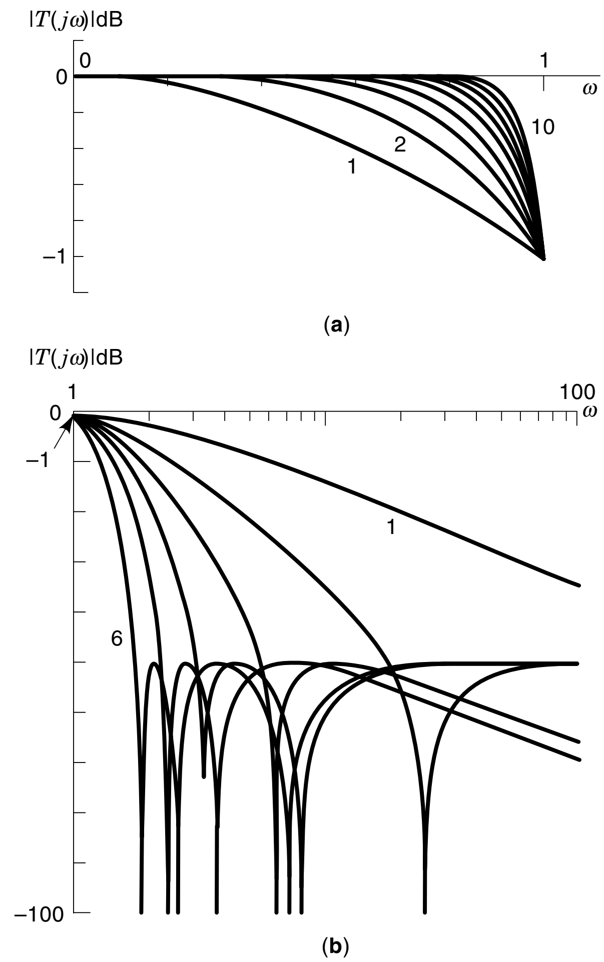

Figure 5. Passband gain (a) and stopband gain (b) for the first normalized inverse Chebyshev approximations with $A_{\max}$ = 1 dB and $A_{\min}$ = 50 dB. Observe the maximally flat passband and the uniform stopband ripple.

OTHER SIMILAR APPROXIMATIONS

Different approximations with uniform passband or stopband ripple, somewhat less selective, can be generated by reducing the number or the amplitude of the oscillations in a Chebyshev-like polynomial and generating the approximations starting from Eqs. (7) or (19) numerically (8).

A particularly interesting case results if the last oscillations of the polynomial value end in 0 instead of ±1. This creates double roots close to $x = \pm 1$ in the polynomial. In a polynomial approximation, the higher-frequency passband minimum disappears, replaced by a second-order maximum close to the passband border. In an *LC* doubly terminated realization, the maximum power transfer at this frequency causes the nullification of the first two derivatives of the gain in relation to the reactive elements, substantially reducing the gain error at the passband border. In an inverse polynomial approximation, this causes the joining of the first two transmission zeros, as a double transmission zero, which increases the attenuation and reduces the error at the beginning of the stopband, allowing also symmetrical realizations for orders 5 and 7.

Table 3. Normalized Inverse Chebyshev Filters with A_{max} = 1 dB and A_{min} = 50 dB

Polynomials $E(s)$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
1	1.96523	1.00000									
2	1.96838	1.98099	1.00000								
3	2.01667	3.14909	2.51015	1.00000							
4	2.19786	4.52937	4.90289	3.13118	1.00000						
5	2.60322	6.42983	8.61345	7.26320	3.81151	1.00000					
6	3.35081	9.35051	14.61162	14.91369	10.30744	4.54023	1.00000				
7	4.64002	14.09440	24.72451	29.03373	24.18372	14.09633	5.30979	1.00000			
8	6.82650	22.03426	42.29782	55.31092	52.89124	37.20009	18.68307	6.11268	1.00000		
9	10.54882	35.60372	73.49954	104.6829	111.4815	90.07839	54.81844	24.10445	6.94337	1.00000	
10	16.95789	59.19226	129.8094	198.2422	230.3472	207.4480	145.4877	77.89699	30.39330	7.79647	1.00000

Poles

n	re/im 1	ω/Q 1	re/im 2	ω/Q 2	re/im 3	ω/Q 3	re/im 4	ω/Q 4	re/im 5	ω/Q 5
1	−1.96523									
2	−0.99049	1.40299								
	0.99363	0.70823								
3	−0.61468	1.25481	−1.28079							
	1.09395	1.02071								
4	−0.42297	1.18385	−1.14262	1.25229						
	1.10571	1.39945	0.51249	0.54799						
5	−0.30648	1.13993	−0.94418	1.23656	−1.31018					
	1.09795	1.85969	0.79849	0.65483						
6	−0.23016	1.10962	−0.75398	1.21506	−1.28598	1.35770				
	1.08549	2.41056	0.95283	0.80576	0.43545	0.52789				
7	−0.17794	1.08768	−0.59638	1.18959	−1.14085	1.36871	−1.47946			
	1.07303	3.05632	1.02930	0.99735	0.75619	0.59986				
8	−0.47425	1.16431	−0.14101	1.07137	−0.95398	1.34983	−1.48710	1.55173		
	1.06334	1.22752	1.06205	3.79891	0.95496	0.70747	0.44316	0.52173		
9	−0.38185	1.14152	−0.77805	1.31643	−0.11413	1.05899	−1.34453	1.56247	−1.70623	
	1.07575	1.49471	1.06189	0.84597	1.05282	4.63922	0.79596	0.58105		
10	−0.63221	1.27960	−0.31203	1.12193	−0.09407	1.04944	−1.13939	1.53032	−1.72054	1.78611
	1.11252	1.01201	1.07766	1.79777	1.04521	5.57772	1.02161	0.67155	0.47954	0.51906

Polynomials $P(s)$

n	Multiplier	a_0	a_2	a_4	a_6	a_8	a_{10}
1	1.96523	1.00000					
2	0.00316	622.4562	1.00000				
3	0.05144	39.20309	1.00000				
4	0.00316	695.0228	74.56663	1.00000			
5	0.03477	74.86195	19.34709	1.00000			
6	0.00316	1059.620	494.9652	57.80151	1.00000		
7	0.03463	133.9940	95.81988	19.57753	1.00000		
8	0.00316	2158.727	2130.497	657.0734	64.84805	1.00000	
9	0.03786	278.6600	354.3952	150.2380	23.58892	1.00000	
10	0.00316	5362.556	8380.916	4584.365	1023.530	79.98165	1.00000

Zeros

n	ω_1	ω_2	ω_3	ω_4	ω_5
1					
2	24.94907				
3	6.26124				
4	7.97788	3.30455			
5	3.74162	2.31245			
6	6.92368	2.53424	1.85520		
7	3.60546	2.00088	1.60458		
8	7.29689	2.56233	1.71209	1.45144	
9	3.88896	2.06927	1.53587	1.35062	
10	8.08496	2.78589	1.78865	1.41948	1.28053

LC doubly terminated realizations

n	Rg/Rl	L/C 1	L/C 2	L/C 3	L/C 4	L/C 5	L/C 6	L/C 7	L/C 8	L/C 9	L/C 10
1	1.00000										
	1.00000	1.01769									
3	1.00000		1.56153								
	1.00000	0.78077	0.01634	0.78077							
5	1.00000		1.16364		1.30631						
	1.00000	0.37813	0.16071	1.62010	0.05468	0.47172					
7	1.00000		0.72897		1.34370		0.96491				
	1.00000	0.09574	0.34265	1.32044	0.28905	1.32059	0.07972	0.30081			

Table 4. Normalized Even-order Modified Inverse Chebyshev Filters with Two Transmission Zeros at Infinity, with A_{max} = 1 dB and A_{min} = 50 dB

Polynomials $E(s)$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
2	1.96523	1.98254	1.00000								
4	2.12934	4.47598	4.86847	3.12041	1.00000						
6	3.14547	9.02141	14.23655	14.65051	10.18872	4.51414	1.00000				
8	6.32795	20.98707	40.68275	53.69811	51.69862	36.58972	18.48009	6.07949	1.00000		
10	15.69992	56.17036	124.1801	191.3464	223.6989	202.6490	142.8395	76.84994	30.12162	7.76165	1.00000

Poles

n	*re*/*im* 1	ω/Q 1	re/*im* 2	ω/Q 2	*re*/*im* 3	ω/Q 3	*re*/*im* 4	ω/Q 4	*re*/*im* 5	ω/Q 5
2	−0.99127	1.40187								
	0.99127	0.70711								
4	−0.43134	1.18419	−1.12886	1.23225						
	1.10284	1.37268	0.49409	0.54580						
6	−0.23626	1.11107	−0.76275	1.20632	−1.25806	1.32324				
	1.08566	2.35141	0.93457	0.79077	0.41016	0.52590				
8	−0.14421	1.07247	−0.48399	1.16315	−0.96075	1.33672	−1.45079	1.50858		
	1.06273	3.71848	1.05767	1.20162	0.92940	0.69566	0.41357	0.51992		
10	−0.64341	1.27784	−0.31781	1.12245	−0.09573	1.05011	−1.14584	1.51514	−1.67803	1.73625
	1.10404	0.99303	1.07652	1.76590	1.04574	5.48452	0.99132	0.66115	0.44586	0.51735

Polynomials $P(s)$

n	Multiplier	a_0	a_2	a_4	a_6	a_8
2	1.96523	1.00000				
4	0.16412	12.97454	1.00000			
6	0.11931	26.36278	10.89186	1.00000		
8	0.13145	48.13911	44.73326	12.54437	1.00000	
10	0.16119	97.39855	147.0191	76.50032	15.68797	1.00000

Zeros

n	ω_1	ω_2	ω_3	ω_4
2				
4	3.60202			
6	2.69467	1.90542		
8	2.71078	1.74464	1.46706	
10	2.94484	1.82001	1.43081	1.28694

LC doubly terminated ladder realizations

n	*Rg*/*Rl*	*L*/*C* 1	*L*/*C* 2	*L*/*C* 3	*L*/*C* 4	*L*/*C* 5	*L*/*C* 6	*L*/*C* 7	*L*/*C* 8
2	1.00000		1.00881						
	1.00000	1.00881							
4	1.00000		1.51207	0.05275	0.58997				
	1.00000	0.64094		1.46110					
6	1.00000		0.87386		1.67233	0.13065	0.32187		
	1.00000	0.17880	0.31519	1.63514		1.05413			
8	1.00000		0.67581	0.32023	1.34317	0.38303	1.12998	0.18178	0.16760
	1.00000	0.32898		1.02594		1.21303		0.74862	

Other variations arise from the shifting of roots to the origin. This is also best done numerically. Odd- (even-) order polynomial approximations with any odd (even) number of attenuation zeros at $\omega = 0$, up to the approximation's order (in the last case resulting in a Butterworth approximation), can be generated. The same polynomials generate inverse polynomial approximations with any odd (even) number of transmission zeros at infinity.

In all cases, the Q of the poles is reduced and the phase is closer to linear. Similar techniques can also be applied to elliptic approximations. For example, a low-pass elliptic approximation can be transformed into a Chebyshev approximation by the shifting of all the transmission zeros to infinity, or into an inverse Chebyshev approximation by shifting all the attenuation zeros to the origin. There are many variations between these extremes.

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