BUTTERWORTH FILTERS

Electrical filters made by linear, lumped, and finite components present a real rational transfer function $T(s) = V_{out}(s)/V_{in}(s)$. For real frequencies, $T(j\omega) = |T(j\omega)|e^{j\Theta(\omega)}$, where $|T(j\omega)|$ is the filter linear gain. Alternatively we may use $G(\omega) = 20 \log_{10} |T(j\omega)|$, the filter gain in dB, or $A(\omega) = -G(\omega)$, the filter attenuation. $\Theta(\omega)$ is the filter phase, but we may advantageously use the group delay $\tau(\omega) = -\partial\Theta(\omega)/\partial\omega$.

Ideal filters should have constant gain and constant group delay in frequency bands called pass bands and infinite attenuation in frequency bands called stop bands. Real filters only approximate these characteristics. In general, no attention is paid to phase when approximating gain characteristics, for satisfying gain and phase in tandem is a problem that usually does not admit closed-form solution and requires an iterative optimization procedure. If phase equalization is necessary, the usual practice is to perform it later using other circuits. Classical approximation methods are usually developed for normalized low-pass filters with a pass band between 0 rad/s and 1 rad/s, where the attenuation variation—the pass-band ripple—must not exceed A_p, and with a stop band from $\omega_s >$ 1 rad/s to infinity, where attenuation must exceed the minimum attenuation in the pass band by at least A_s.

Other low-pass, high-pass, band-pass, and band-stop filters can be designed by trivial frequency transformations applied onto the normalized low-pass-filter prototype, and will not be discussed here.

HISTORICAL BACKGROUND

The concept of electrical filters was independently developed by Campbell and Wagner during World War I. These early electrical wave filters easily accomplished the stop-band requirements, but a reasonably constant gain in the pass band required heuristically tuning of some filter resistors.

In his seminal work, Butterworth (1) attacked the problem of designing linear intervalve resonating circuits in such a way that the overall circuit combined amplification and filtering, matching the desired characteristics in both stop band and pass band without a tuning procedure. For the normalized low-pass filter, Butterworth proposed a "filter factor" F, nowadays corresponding to the filter gain $|T(j\omega)|$, such that

$$|T(j\omega)|^2 = 1/(1+\omega^{2n}) = 1/L(\omega^2) \tag{1}$$

where n is the filter order. Figure 1 shows $|T(j\omega)|$ vs. ω for n = 2, 4, 7, and 10. It is clear that $|T(j\omega)|$ approaches the ideal low-pass-filter characteristics when n is increased, satisfying pass-band and stop-band requirements. Moreover, Butterworth cleverly realized that for n even $L(\omega^2)$ could be decomposed as a product of $n/2$ second-order polynomials $P_i(\omega)$,

$$L(\omega^2) = \prod_{i=1}^{n/2} P_i(\omega) = \prod_{i=1}^{n/2} [1 + 2\omega\cos(2i-1)\pi/2n + \omega^2]$$

and showed that (1) the product of conveniently selected pairs of polynomials comprised a function $L_i(\omega^2) = P_i(\omega)P_{n/2-i}(\omega)$, which could be associated with a second-order filter, and (2) the product of four polynomials selected as described previously comprised a function $L_i(\omega^2)L_j(\omega^2)$, which could be associated with a fourth-order filter. All components of these low-order filters could be easily calculated, and the filters could be used as coupling stages between amplifier tubes. For the first time it was possible to construct a filter amplifier that required no heuristic tuning (2).

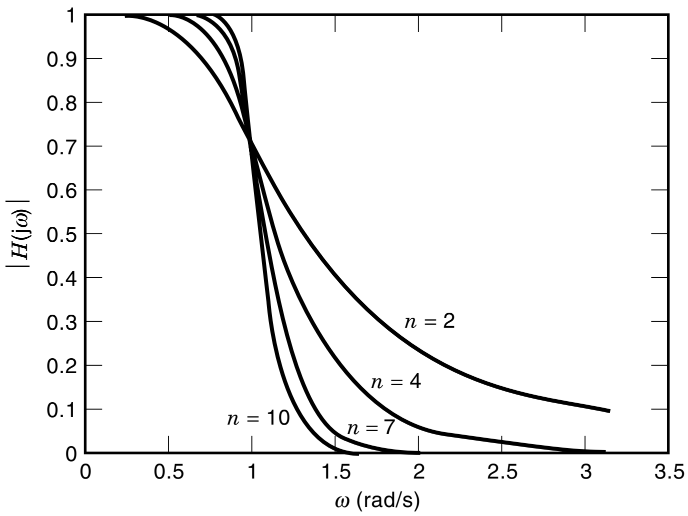

Figure 1. Magnitude of the Butterworth transfer function for $n = 2$, 4, 7, and 10.

SOME COMMENTS ON BUTTERWORTH FILTERS

Butterworth's contribution for approximating and synthesizing filters that are well behaved both in the pass band and stopband was immediately recognized and still stands for its historical and didactic merits. Even nowadays, compared with others, Butterworth's approximation is simple, allows simple and complete analytical treatment, and leads to a simple network synthesis procedure. For these reasons, it is an invaluable tool to give students insight on filter design and to introduce the subject to the novice. From now on, we intend to present the Butterworth filter with emphasis on this didactic approach.

The Butterworth approximation is an all-pole approximation, that is, its transfer function has the form

$$T(s) = k_0/D(s) \tag{2}$$

and its n transmission zeros are located at infinity.

The Butterworth approximation is sometimes called the maximally flat approximation, but this denomination, first used by Landon (3), also includes other approximations. The idea of the maximally flat gain is to preserve filter gain as constant as possible around the frequency at which the most important signal components appear, by zeroing the largest possible number of derivatives of the gain at that frequency. The Butterworth approximation is just the all-pole approximation with maximally flat gain at $\omega = 0$.

MAXIMALLY FLAT APPROXIMATION

Let us consider the family of normalized low-pass all-pole filters of order n. We search for the approximation with maximally flat gain at $\omega = 0$. For all-pole approximations, $L(\omega^2)$ is a polynomial in ω^2. Comparing the polynomial $L(\omega^2)$ and its MacLaurin series

$$L(\omega^2) = \sum_{i=0}^{n} a_{2i}\omega^{2i} = \sum_{j=0}^{2n} \frac{1}{j!} \frac{\partial^j L(\omega^2)}{\partial \omega^j}\bigg|_{\omega=0} \omega^j \tag{3}$$

we verify that $\partial^j L(\omega^2)/\partial\omega^j$ at $\omega = 0$ is zero for j odd and equals $j!a_j$ for j even. For the moment, let us normalize $|T(j0)| = 1$. This leads to $a_0 = 1$ in Eq. (3). For a filter to be of order n, $a_{2n} = \epsilon^2 \neq 0$ in Eq. (3). From these considerations, to cancel the maximum number of derivatives of $L(\omega^2)$—and consequently those of $|T(j\omega)|$—at $\omega = 0$, we must choose $a_j = 0$ for $j = 1$ to $2n - 1$, leading to

$$L(\omega^2) = 1 + \epsilon^2\omega^{2n} \tag{4}$$

which is similar to $L(\omega^2)$ in Eq. (1) if we choose $\epsilon = 1$. The Butterworth approximation is the answer we were looking for. Two important considerations must be given on this new parameter ϵ.

1. As $A(0) = 0$ and $A(\omega)$ is clearly a monotonically increasing function, ϵ controls the attenuation variation, or gain ripple, in the pass band:

$$A_p = A(1) - A(0) = 10\log_{10}(1 + \epsilon^2) \tag{5}$$

It is interesting to notice that the pass-band ripple is independent of the filter order n, depending only on ϵ. On the other hand,

$$A_s = A(\omega_s) - A(0) = 10\log_{10}(1 + \epsilon^2\omega_s^{2n}) \tag{6}$$

From Eqs. (5) and (6) we verify that the requirements for the pass band and stop band are satisfied if

$$\epsilon \le (10^{0.1A_p} - 1)^{1/2} \tag{7}$$

and

$$n \ge \frac{\log_{10}[(10^{0.1A_s} - 1)/\epsilon^2]}{2\log_{10}\omega_s} \tag{8}$$

2. As $\epsilon^2\omega^{2n} = (\epsilon^{1/n}\omega)^{2n}$, Eq. (4) and consequently the filter gain may be written as a function of $\epsilon^{1/n}\omega$ instead of ω. So $\epsilon^{1/n}$ is merely a frequency scaling factor that may be normalized to 1 without loss of generality in the study of the maximally flat approximation, yielding the original Butterworth approximation given by Eq. (1).

BUTTERWORTH TRANSFER FUNCTION

We are now interested in determining the filter transfer function $T(s)$, which corresponds to Eq. (1). Using the analytical continuation $\omega = s/j$ (or $\omega^2 = -s^2$)

$$\begin{aligned} |T(j\omega)|^2 = T(s)T(-s)|_{s=j\omega} &= \frac{1}{D(s)}\frac{1}{D(-s)}\bigg|_{s=j\omega} \\ &= \frac{1}{L(-s^2)}\bigg|_{-s^2=\omega^2} \end{aligned} \tag{9}$$

and

$$L(-s^2) = D(s)D(-s) = 1 + (-s^2)^n = 1 + (-1)^n s^{2n}$$

we verify that the roots of $L(-s^2)$ are the filter poles and their symmetrical points with respect to the origin. The roots s_i of

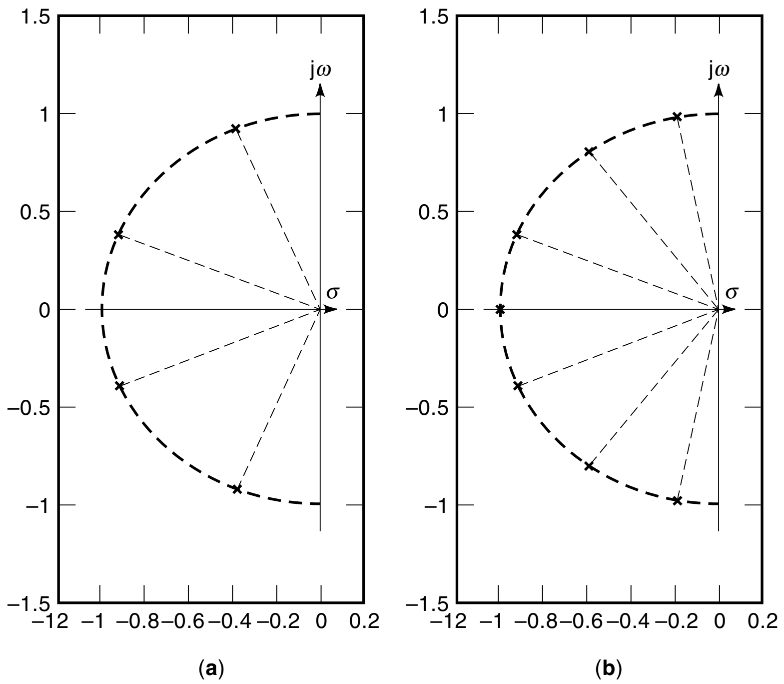

Figure 2. Butterworth filter poles for (a) $n = 4$ and (b) $n = 7$.

$L(-s^2)$ are

$$s_i = e^{j(2i+n-1)\pi/2n}, \qquad i = 1, 2, \ldots, 2n$$

where $j = +\sqrt{(-1)}$. All roots s_i have unitary modulus and are equally spaced over the unitary-radius circle. As the filter must be stable, we take the roots s_i in the left-half s plane as the roots p_i of $D(s)$ (the poles of the filter), that is,

$$p_i = e^{j(2i+n-1)\pi/2n}, \qquad i = 1, 2, \ldots, n$$

For n odd there exists a real pole at $s = -1$; for n even all poles are complex. Figure 2 shows the filter poles for $n = 4$ and $n = 7$.

Knowledge of the roots of $D(s)$ allows us to calculate its coefficients d_i, or to factor $D(s)$ in second-order polynomials $(s^2 + s/Q_i + 1)$ where

$$Q_i = -\frac{1}{2\cos[(2i+n-1)\pi/2n]}$$

for $i = 1, \ldots, n/2$ if n is even or $i = 1, \ldots, (n-1)/2$ if n is odd, when a factor $s + 1$ is also added.

$$D(s) = \sum_{i=0}^{n} d_i s^i = \begin{cases} \prod_{i=1}^{n/2} (s^2 + s/Q_i + 1) & \text{for } n \text{ even} \\ (s+1) \prod_{i=1}^{(n-1)/2} (s^2 + s/Q_i + 1) & \text{for } n \text{ odd} \end{cases} \tag{10}$$

A curious property is that, as $D(s)$ is real and its roots are on the unitary-radius circle, its coefficients are “symmetrical,” that is, $d_i = d_{n-i}$ for $i = 0, \ldots, n$. Obviously $d_0 = d_n = 1$.

Table 1 shows the coefficients d_i and the Q_i factors of the poles of $D(s)$ for $n = 1$ to 7.

PHASE AND GROUP DELAY

Group delay gives more practical information than phase characteristics and is easier to calculate. From Eqs. (9) and (10) we verify that $T(s)$ may be written as the product of first- and second-order functions, and so the group delay is the addition of the group delay of each of these functions. Using

$$\tau(\omega) = E_v\left(\frac{1}{T(s)}\frac{\partial T(s)}{\partial s}\right)\bigg|_{s=j\omega}$$

in these low-order functions, where $E_v(\cdot)$ is the even part of $(\cdot)$, it is easy to show that the group delay is given by

$$\tau(\omega) = \frac{1}{1+\omega^2} + \sum_i \frac{1}{Q_i}\frac{1+\omega^2}{1-(2-1/Q_i^2)\omega^2+\omega^4}$$

where the first term comes from the first-order term of $T(s)$, if it exists, and the summation is done over all second-order

Table 1. Coefficients and *Q* Factors for the Butterworth Filters

n	d_1	d_2	d_3	Q_1	Q_2	Q_3
1	1.0000					
2	1.4142	1.0000		0.7071		
3	2.0000	2.0000	1.0000	1.0000		
4	2.6131	3.4142	2.6131	1.3066	0.5412	
5	3.2361	5.2361	5.2361	1.6180	0.6180	
6	3.8637	7.4641	9.1416	1.9319	0.7071	0.5176
7	4.4940	10.0978	14.5918	2.2470	0.8019	0.5550

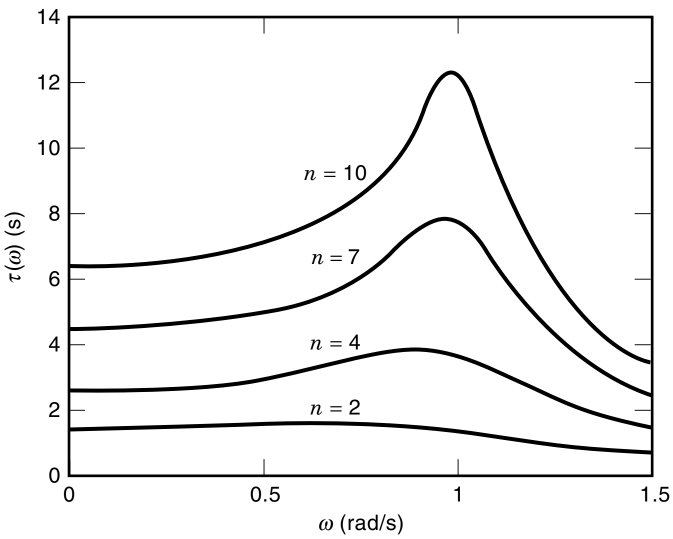

Figure 3. Group delay of Butterworth filters for $n = 2, 4, 7$, and 10.

terms of $T(s)$. The first term monotonically decreases from 1 to 0 as ω increases. The other terms begin as $1/Q_i$ at $\omega = 0$, peak to $\approx 2Q_i$ at $\omega \approx 1 - 1/8Q_i^2 \approx 1$, and then decrease to zero as ω further increases. The approximations hold for $4Q_i^2 \gg 1$. The result is that for the cases of interest, with A_p not too large, say $A_p < 1$ dB, and n not too small, say $n > 4$, the group delay of Butterworth filters is practically monotonically increasing in the pass band and is easily compensated by simple all-pass filters. Figure 3 shows the group delay for $n = 2$, 4, 7, and 10. Remember that, due to the frequency scaling used to reduce the pass-band ripple, the pass-band edge corresponds to $\epsilon^{1/n} < 1$.

SYNTHESIZING BUTTERWORTH FILTERS

The filter is implemented as a linear network synthesized to exhibit the desired frequency behavior. Implementation is related to the technology chosen for the filter assemblage. Although progress has brought new technologies for synthesizing and implementing filters, the doubly loaded LC ladder network with maximum power gain still remains the basis of most synthesis procedures, due to the its low gain sensitivity to the network components. This synthesis method is usually very complex, but the peculiarities of Butterworth equations allow simplifications that make this filter very adequate to introduce the doubly loaded LC ladder network synthesis method to students.

First, we already know a structure to be used with all-pole filters. As the transmission zeros of a ladder network are the poles of the series-branch impedances and the shunt-branch admittances, a possible network topology is shown in Fig. 4 for n odd and even.

If we find a suitable $Z_1(s)$, the synthesis problem reduces to the easy realization of a one-port network with impedance $Z_1(s)$ through the extraction of poles at infinity, alternating from the residual admittance and impedance (the "chop-chop" method) until it remains only a constant, implemented by the load resistor R_L. The final topology will be given by Fig. 4.

As the LC network is lossless, the active power $P_1(\omega)$ going into $Z_1(j\omega)$ will be dissipated as active power $P_0(\omega)$ at R_L. For a given source resistor R_s, $P_1(\omega)$ reaches its maximum P_m when $Z_1(j\omega)$ matches R_s, that is, at frequencies where $Z_1(j\omega) = R_s$.

$$P_1(\omega) = \left|\frac{V_{in}(\omega)}{Z_1(j\omega)}\right|^2 \mathrm{Re}\{Z_1(j\omega)\}$$

$$P_o(\omega) = \frac{|V_{out}(\omega)|^2}{R_L}$$

$$P_m = \frac{|V_{in}(\omega)|^2}{4R_s}$$

where Re$\{\cdot\}$ means the real part of $\{\cdot\}$. Butterworth filters present maximum gain at $\omega = 0$, and at this frequency the filter must transmit the maximum possible power to the load R_L. At $\omega = 0$ inductors act as short circuits, capacitors as open circuits, and by inspection of Fig. 4 we verify that $Z_1(j0) = R_L$. For maximum power gain at $\omega = 0$ we choose $R_L = R_s$. Still by inspection of Fig. 4, we verify that $|T(j0)| = \frac{1}{2}$. As $d_0 = 1$, we must choose $k_0 = \frac{1}{2}$ in Eq. (2) and consequently Eq. (1) becomes

$$|T(j\omega)|^2 = 1/4L(\omega^2)$$

We will use here the simple synthesis procedure described in Ref. 4 and will make use of all simplifications allowed by the Butterworth approximation. Let us introduce the filter transducer function $H(j\omega)$ such that

$$|H(j\omega)|^2 = \frac{P_m}{P_1(\omega)} = \frac{R_L}{4R_s}\frac{1}{|T(j\omega)|^2} = L(\omega^2) = 1 + \omega^{2n}$$

measures the power rejection by the filter at frequency ω. As $P_m \geq P_1(\omega)$, $|H(j\omega)|^2 \geq 1$, and the equality is reached only at the frequencies of maximum power gain. Let us also introduce the filter characteristic function $K(j\omega)$ such that

$$|K(j\omega)|^2 = |H(j\omega)|^2 - 1 = \omega^{2n}$$

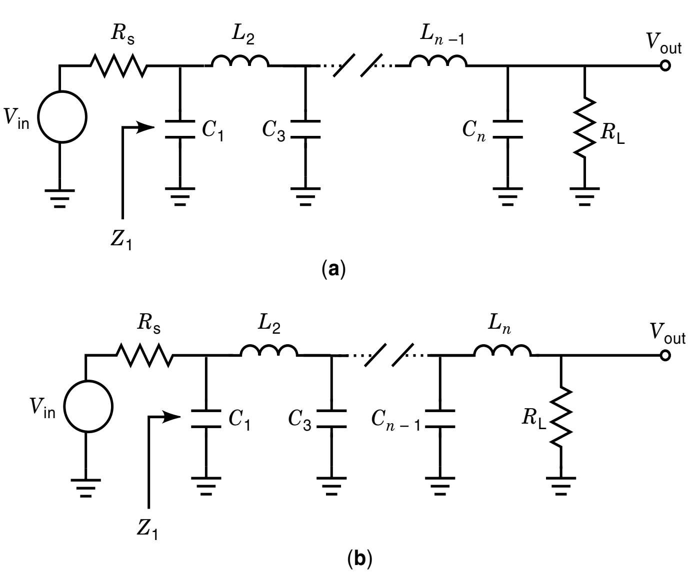

Figure 4. Low-pass all-pole doubly loaded LC ladder network for (a) n odd and (b) n even.

Table 2. Element Values for the Butterworth Filter

n	A_1	A_2	A_3	A_4
1	2.0000			
2	1.4141	1.4142		
3	1.0000	2.0000	1.0000	
4	0.7653	1.8478	1.8478	0.7653
5	0.6180	1.6180	2.0000	1.6180
6	0.5176	1.4142	1.9319	1.9319
7	0.4450	1.2470	1.8019	2.0000

is also a measurement of the power rejection by the filter at frequency ω. As $|H(j\omega)|^2 \geq 1$, $|K(j\omega)|^2 \geq 0$, and the equality is reached only at the frequencies of maximum power gain.

After some algebraic manipulation and using analytical continuation to obtain

$$\frac{K(s)}{H(s)} = \frac{s^n}{D(s)}$$

it is possible to show that

$$Z_1(s) = R_s \frac{1 - \dfrac{K(s)}{H(s)}}{1 + \dfrac{K(s)}{H(s)}} = R_s \frac{D(s) - s^n}{D(s) + s^n}$$

and $Z_1(s)$ may be easily synthesized by the chop-chop method, concluding the synthesis procedure.

For Butterworth filters the number of frequency points with maximum power transfer is maximum (all concentrated at the origin). All-pole filters with this characteristic are symmetrical if n is odd and antisymmetrical if n is even, that is, A_i is equal to A_{n+1-i} for all i, where A_i denotes the numerical value of either C_i or L_i. Table 2 presents the values of A_i for $n = 1$ to 7 and for $R_s = R_L = 1\ \Omega$.

There exists a simple equation that directly provides the component values, but its deduction is rather complicated. For $R_s = R_L = 1\ \Omega$,

$$A_i = 2\sin(2i-1)\pi/2n \tag{11}$$

For a more complete discussion on the synthesis of Butterworth ladder network design, even with arbitrary resistors R_s and R_L, as well as the historical facts surrounding the discovery of those formulas, the reader is referred to Refs. 5 to 7.

DESIGNING BUTTERWORTH FILTERS

Only few steps are necessary to design a Butterworth filter:

1. Obtain the normalized low-pass filter and its restrictions A_p, A_s, and ω_s.
2. Determine ϵ and n using Eqs. (7) and (8).
3. Synthesize, use tables, or use Eq. (11) to obtain the standard filter of order n.
4. Obtain the normalized low-pass prototype by frequency-scaling the standard filter, multiplying all reactive elements by $\epsilon^{1/n}$.
5. Invert step 1, that is, denormalize the low-pass prototype obtained in step 4.

CONCLUSIONS

Butterworth's paper is the touchstone of modern filter-approximation theory. However, its low selectivity when compared to that of other approximations restricts its use to situations that require low sensitivity at the center of the pass band and/or a good group delay flatness. Probably the major importance of Butterworth filter nowadays is for didactic purposes. Its simplicity and complete analytical formulation make it the best option to introduce filter synthesis to students.

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