ANALOG COMPUTER CIRCUITS

The term *analog computer* usually refers to an electronic circuit consisting of operational amplifiers, resistors, and capacitors along with additional electronic components (e.g., diodes and transistors) connected to process analog signals (as opposed to digital signals) which are conceptually modeled as continuous functions of time. Analog computers have limited bounds namely, $E_{\max}$ and $E_{\min}$. Since the early 1960s, analog computers have used solid state components, and the signal range is typically ± 10 V. The operational amplifiers are usually made in integrated circuit form. They may be supplied as separate modules, mounted on a circuit board, or a part of a larger integrated circuit. We are here primarily interested in the operation of such electronic systems to solve ordinary differential equations, although operational amplifiers are often used in the design of signal filtering circuits and in the design of interface signal-conditioning subsystems to go between real-world signals from a wide variety of transducers and subsequently to digital signal processing systems used for data logging and analysis.

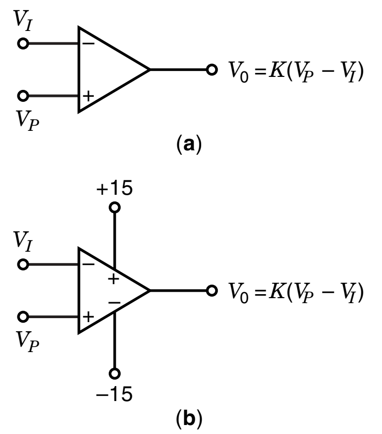

Figure 1. Operational amplifier symbol. (a) Operational amplifier. (b) Operational amplifier with power supply voltages attached.

A typical ideal operational amplifier model is shown in Fig. 1. The ideal model has an input–output description

$$V_0 = K(V_P \cdot V_I), \qquad \text{where } K \gg 1$$

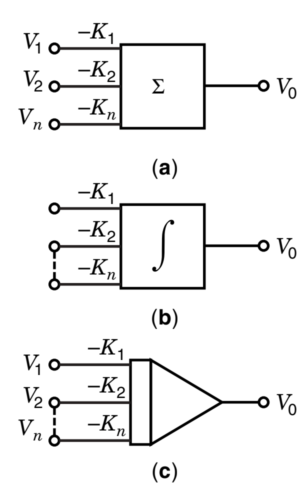

Figure 2. Conventional operational amplifier circuit block symbols. (a) Summer–inverter. (b) Inverting summer–integrator. (c) Older symbol for inverting summer–integrator.

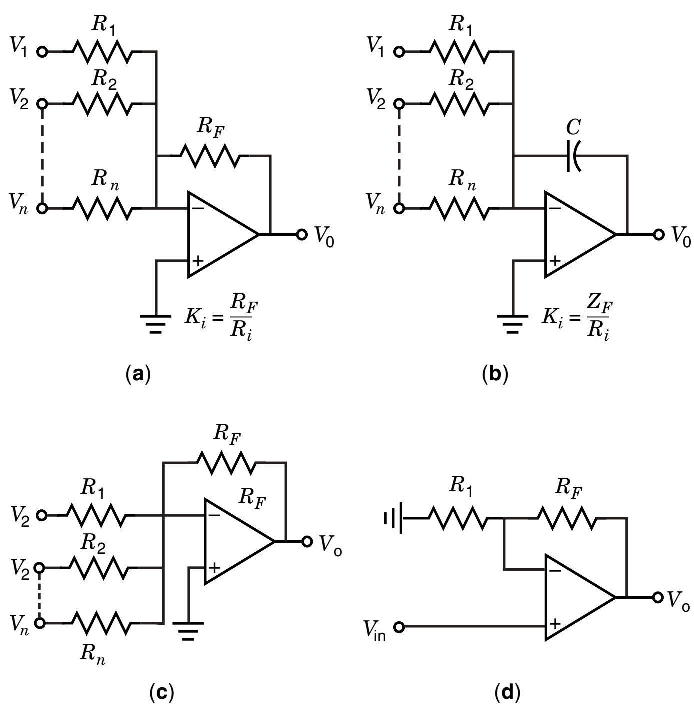

Figure 3. Operational amplifier circuits. (a) Summer–inverter. (b) Inverting summer–integrator. (c) Basic summer–inverter circuit. (d) Basic noninverting circuit.

More information on operational amplifiers is available in Refs. 1 and 2.

Primary operational amplifier circuits for analog computing are integrating and summing circuits. Figure 2(a) shows the block diagram for a summer–inverter. Figure 2(b) shows a summer–inverting integrator. A skilled analog computer programmer learns to cleverly program his or her collection of summers and integrators to implement an electronic model which "solves" a set of ordinary differential equations (ODEs)

$$F(t) = a_0X + a_1\dot{X} + a_2\ddot{X} + \cdots$$

Figure 3 shows corresponding operational amplifier circuits.

As indicated in Chapter 1 of Ref. 1, the ideal operational amplifier has a very high open loop gain and input impedance and a relatively low output impedance.

Basic input–output relations for the circuit in Fig. 3(c) are

$$V_0 = -R_F \sum_{i=1}^{n} (V/R) \quad i = 1, 2, \ldots, n$$
$$= -R_F(V_1/R_1 + V_2/R_2 + \cdots)$$
$$Z_{\text{ini}} = R_i$$

For Fig. 3(d):

$$V_0 = -(1 + R_F/R_i)V$$

Example. Consider the mechanical system of Fig. 4(a). A simple force balance equation gives

$$\sum f = 0$$

with initial conditions $X(0)$ and $X(0)$ or

$$-F + M\ddot{X} - D\dot{X} - KX = 0$$

or

$$\ddot{X} = (K/M)X + (D/M)\dot{X} + (F/M)$$

Let $M = 5$ kg, $D = 0.1$ Ns/m, and $K = 1.0$ m/N; then

$$\ddot{X} = (0.2)X + 0.02\dot{X} + 0.2F)$$

The operational amplifier circuit of Fig. 4(b) shows a basic analog computer model for this system. $F(t)$ may be a voltage generated by a variety of signal sources—for example, "function generator" types of variable-frequency sine-square wave generators, recorded signal sources, and so on. Often we may not want a "real" model of this system but will want to scale or denormalize the circuit to get a repetitive display (e.g., on an oscilloscope). We can slow down a simulation by making the capacitors larger, or we can speed it up by making them smaller by the same time scale factor.

OPERATIONAL AMPLIFIER CIRCUITS

For precise signal processing, one often uses *nonlinear* operational amplifier circuits. Important nonlinear operations include:

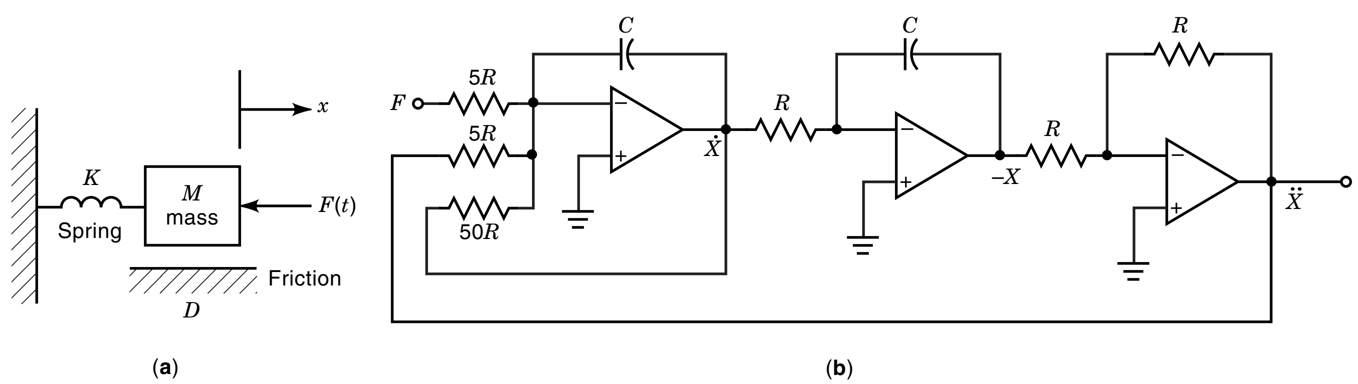

Figure 4. Spring-mass system. (a) Mechanical system. (b) Operational amplifier analog computer model. $R = 10^6\ \Omega$, $C = 10^{-6}$ F.

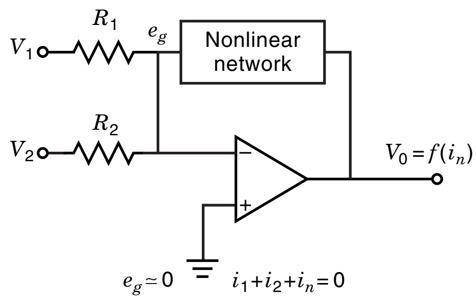

Figure 5. General nonlinear operational amplifier circuit configuration.

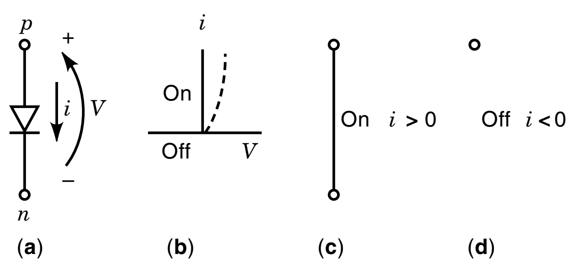

Figure 6. *P–n* junction diode. (a) Diode symbol; (b) "demon-with-a-switch" model; (c) diode ON or forward biased (d) diode OFF or reverse biased.

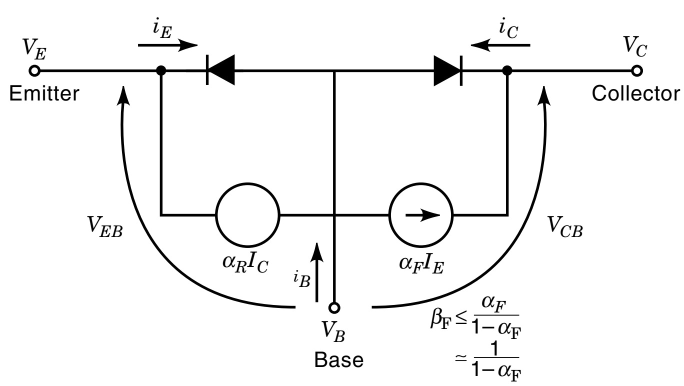

Figure 7. *Npn* bipolar junction transistor model (BJT). β_F is forward biased current gain.

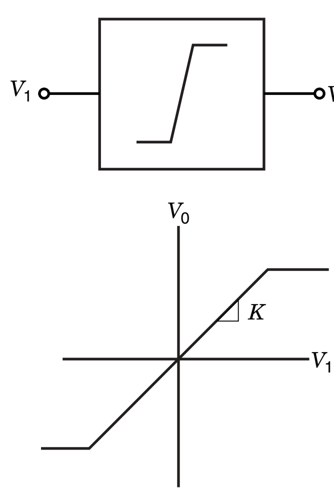

Figure 8. Limiter operator.

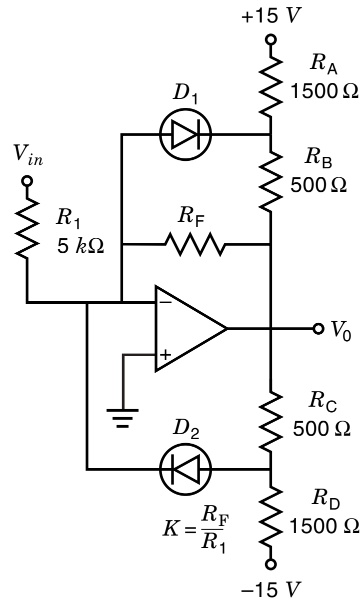

Figure 9. Operational amplifier circuit for a limiter amplifier.

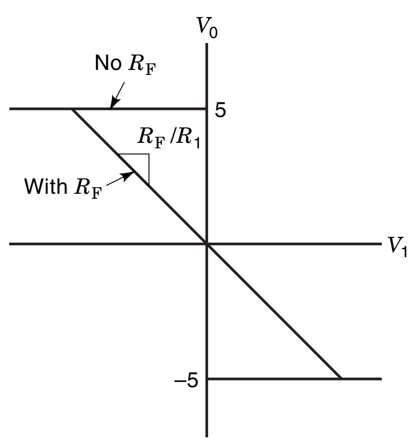

Figure 10. Direct-current transfer characteristic for circuit in Fig. 9.

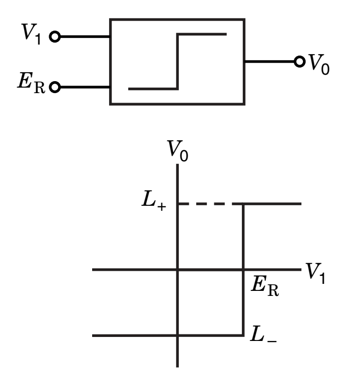

Figure 11. Comparator block symbol. E_R is reference voltage.

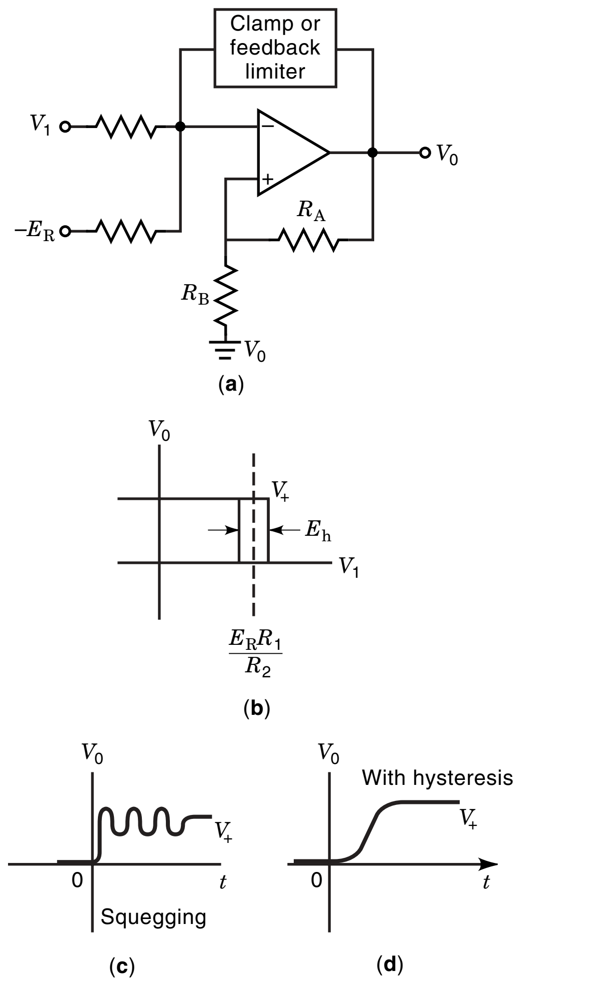

Figure 12. Comparator behavior. (a) General circuit diagram. (b) Direct-current transfer characteristic. (c) "squegging" (i.e., no hysteresis). (d) Comparator behavior with hysteresis.

Limiters (Figs. 8 to 10)

Comparators (Figs. 11 and 12)

Multipliers and dividers (Fig. 13)

Waveform generators (Figs. 14 and 15)

Circuits based upon these types of operations may be extended to circuits that precisely measure absolute value, amplitude, peak value, and logarithmic operations. (Another way of making a multiplier is by the use of antilog or exponentiation operators.) A variety of other waveform generators, including triangle waveform and very low-frequency oscillators (function generators) and frequency modulation (FM) oscillators, may be implemented. Detailed discussions of nonlinear operators appear in Refs. 1 and 2.

Figure 5 shows a general diagram for nonlinear operational amplifier circuits. The operational amplifier forces the current at the inverting terminal or "summing junction voltage" to be zero.

Figure 6 shows a "demon with a switch" model of a junction diode. When the diode is "on," the forward drop is not really zero but may be as much as 0.5 V. The bipolar junction transistor (BJT) (Fig. 7) provides a more flexible active device for nonlinear circuit design. The limiter operator (Fig. 8) is

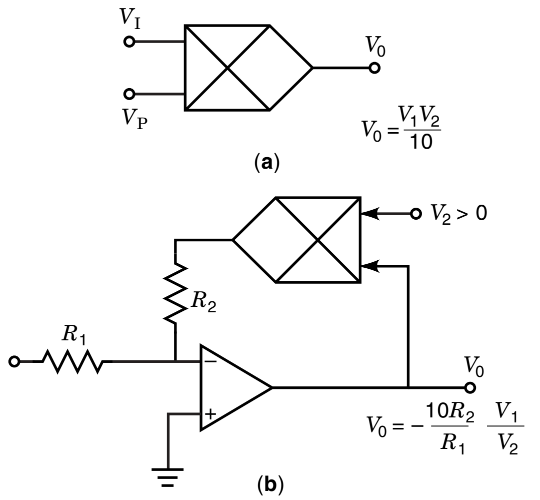

Figure 13. Block symbols: (a) Multiplier. (b) Divider.

used in instrumentation systems to prevent signal overloads. One amplifier actually provides an inverting limiter.

The one-amplifier limiter or comparator circuit of Fig. 9 actually provides the transfer characteristic of Fig. 10. The resistance string R_A–R_D sets the limiting levels.

Without R_F, the circuit of Fig. 9 provides a comparator, which may operate with a reference voltage E_R, (Figs. 11 and 12).

Useful comparator circuits should have some hysteresis E_h to prevent an ambiguous chattering or "squegging" at the comparator switch point (just as a household thermostat provides hysteresis with a small magnet associated with its contacts). Figure 12 shows a one-amplifier (inverting) comparator. The hysteresis is established by the network R_A–R_B, which yields

$$E_h = R_B(V_+)/(R_A + R_B)$$

Analog multipliers and dividers are designed in a variety of ways (Fig. 13). A popular method uses logging–antilogging circuits (see Ref. 1). Sinusoidal waveform generators may be implemented using the block diagrams of Figs. 14 and 15. The circuit of Fig. 14 generates sine waves by implementing the solution of an undamped second-order ODE. The block dia-

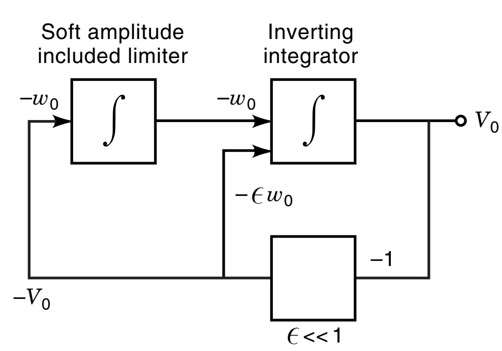

Figure 14. Sinusoidal waveform generator.

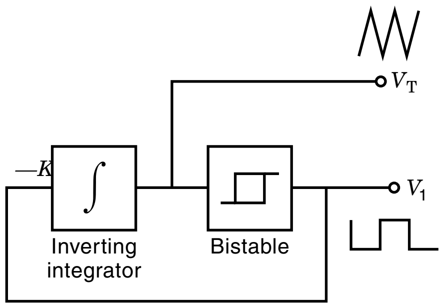

Figure 15. Function generator block diagram for generating low-frequency triangle and square waves.

gram of Fig. 15 shows a diagram of the "function generator" type of circuit to generate square and triangle waveforms (see Ref. 1).

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JOHN V. WAIT

ANALOG COMPUTERS

Computing devices capable of mapping inputs to outputs without human intervention and of providing numerical solutions to complex problems have been available in various forms for over 150 years. In many of the early devices, information was represented in mechanical form, as in the mechanical calculators that became invaluable for business data processing in the first half of the twentieth century. Others employed electric representations, as in the network analyzers that played an important role in a wide variety of engineering applications during same period.

The utilization of electronic circuits as components of automatic computers was made possible by developments and inventions stimulated by military requirements during World War II, particularly in the United States and Great Britain. One class of these computers was primarily developed as part of the Manhattan Project to help solve the complex partial differential equations that characterize various physical processes in atomic bombs. These represented extensions of mechanical calculators, but were vastly more powerful in their ability to do arithmetic. A second class of computing techniques was developed to help in the performance of integral and differential calculus as required for the simulation of dynamic mechanical and electromechanical systems, such as ships and aircraft, and for a wide variety of control tasks. The members of the first category became known as digital computers, while the second class was termed analog computers and devices.

The years immediately following World War II saw the rapid extension of electronic computers to new application areas and the formation of industrial enterprises to commercialize them. For a variety of reasons, analog computing devices emerged from military projects more ready for immediate general application than did digital computers, and in the late 1940s a number of companies were formed to market products specifically designed for the solution of the systems of nonlinear ordinary differential equations characterizing dynamic systems. These computers were termed *electronic differential analyzers* (EDAs), and they became so widely used in the 1950s that the term analog computer became largely synonymous with EDA.

As digital computers evolved during the same period, they gradually began to be used in competition with analog computers. Until well into the 1970s, however, digital computers tended to be less cost effective than analog computers in the specialized simulation application, and they were too slow to permit real-time operation. EDAs had their heydays in the 1970s as free-standing simulators or in concert with digital computers in hybrid computer systems. Companies such as Electronic Associates, Inc., Comcor, Inc., Applied Dynamics, Inc., and a number of others in the United States, Germany, and Japan grew to large size and maintained an important position in the military and industrial marketplace. In the meantime companies such as IBM, Control Data Corporation, Digital Equipment Corporation, and many others developed more and more powerful simulation hardware and software.

By the end of the 1970s, the balance began to shift in favor of digital simulation, and gradually the market for EDAs evaporated. It disappeared almost completely in the 1990s. By then, all the tasks formerly performed by electronic analog computers in the simulation of dynamic systems were handled more effectively by digital computing systems. In other application areas, however, analog devices thrived as special-purpose components embedded in a wide variety of systems. The requirements for these analog devices in communication and control systems and in a myriad of military, industrial, and commercial projects has grown almost continuously, and many prosperous companies throughout the world specialize in their manufacture.

In this article, the evolution analog computing devices is first briefly reviewed, including a discussion of the electrical network analyzers and mechanical differential analyzers that were important before World War II. Next, a survey of the EDAs that became popular during the 1960s and 1970s is presented. Finally, the rise and eventual decline of hybrid (analog/digital) computers in the 1980s and early 1990s is considered. Further details may be found in Refs. 1–5.

ANALOG AND DIGITAL PROCESSING

Modern science and engineering are based upon a quantitative description of the physical universe. A variety of so-called physical variables is measured, and inferences are drawn from the results of these measurements. In this connection, it is necessary first to distinguish between independent and dependent variables. In most system analyses, time and space constitute the independent variables. That is, measurements are distinguished from each other and ordered according to the location in the time–space continuum at which the measurements were made. The measured quantities are the dependent variables, and they may be expressed as functions of time and/or space. Some familiar dependent variables include voltage, displacement, velocity, pressure, temperature, stress, and force. The measurement of these variables requires the selection of appropriate instruments, along with a decision as to the manner in which the measurements are to be recorded and utilized. There are two major ways in which a dependent variable is treated by instrumentation and data processing systems: analog and digital. These are defined as follows:

1. A dependent variable is said to be an *analog variable* if it can assume any value between two limits.
2. A dependent variable is said to be a *digital variable* if its magnitude is limited or restricted to certain specified values or levels.

It should be recognized that this distinction does not apply to the domains of the independent variables. Thus analog computers or simulators may maintain the time and the space variables in continuous form, or they may restrict their attention to discretely spaced points in the time and space domains.

The decision as to whether to process data in analog or digital form has far-reaching consequences on the organization of the computer system and its cost, upon the accuracy of the computations, and upon their speed. In order to place the discussion of analog signal processing in its proper perspective, these considerations are briefly summarized.

A basic distinction between analog and digital data processing is that digital computations are usually performed sequentially or serially, while analog computations are performed simultaneously or in parallel. Digital data processing generally requires reference to data and programmed instructions stored in a memory unit. For technical reasons, there exists a bottleneck at the entrance to this memory, so that only one item (or a very small number of items) of information can be read into or read out of the memory at any particular instant of time. Therefore, only one arithmetic operation can be performed at a time. This implies that data processing consists of a sequence of arithmetic operations. For example, if 10 numbers are to be added, 10 successive additions are performed. No additional equipment is needed if 100 additions are required instead.

By contrast, an analog processor generally does not require a memory, which must be time-shared among the various mathematical operations. Rather, a separate electronic unit or "black box" is supplied for each mathematical operation. If a computation requires 10 additions, 10 analog operational units must be provided and interconnected; and all of these units operate simultaneously. If the number of required additions is increased to 100, the amount of electronic equipment necessary is multiplied by a factor of 10. The hardware structure and the cost of an analog data processing system is therefore determined by the types and numbers of specific mathematical operations which are to be performed. The structure of a digital processing system, on the other hand, includes standardized memory, control, and arithmetic units and is more or less independent of the types of computations that are to be performed.

The accuracy of a computation performed by a digital processor is determined by the number of bits employed to represent data. For example, if two numbers are to be multiplied in a digital processing system in which numbers are represented by 32 binary digits, the result of the multiplication must be rounded up or down to the nearest least significant bit. There is, therefore, a chance of a roundoff error corresponding to one-half of the least significant bit. In an analog processor, data are not discretized, and roundoff errors are therefore not incurred. Instead, the accuracy is limited and error is introduced by the nonideal functioning of the operational units used to carry out the computations—that is, by the quality of its components. If two variables are to be added electrically, they are each applied as continuous voltages to an adder unit. The output voltage of the adder then corresponds to the sum of the two variables. The accuracy of this addition operation is limited by the quality (tolerance) of the electronic components making up the adder and by the precision with which the output voltage can be measured and recorded. In the performance of linear mathematical operations (such as addition, subtraction, and multiplication by a constant), relative errors are usually larger than 0.01% of full scale; in the case of nonlinear operations, the best available electronic units are subject to relative errors of 0.1%.

The speed with which a sequential digital computation can be performed is determined by the complexity of the computations. The larger the number of arithmetic operations that must be performed, the longer the time required. One hundred additions require nearly 10 times as much computing time as 10 additions. By contrast, in analog data processing, the time required for computations is independent of problem complexity. One hundred additions require precisely the same time as 10 additions; approximately 10 times as much hardware is required, however. The speed with which a mathematical operation can be performed using an analog unit is determined by the characteristics of its electronic components as well as by the characteristics of the measuring or output devices.

In most modern systems utilizing analog processing, only the operational units actually required for the specific task at hand are provided. These are interconnected in a permanent or semipermanent fashion for a specific application. By contrast, the so-called general-purpose analog computers or EDAs, which have by now almost completely disappeared, were fashioned by assembling a variety of operational units and permitting the user sufficient flexibility to interconnect them as required for the solution of differential equations. Since the analog methods described in this article were applied almost exclusively to the implementation of mathematical models of real-world systems and to the experimentation with these models, the terms *analog computer* and *analog simulator* gradually became synonymous and are used in this way in this article.

CLASSIFICATION OF ANALOG METHODS

The various devices and methods comprising the general area of analog computers and simulators are best classified according to their basic principles of operation. The systems falling into the resulting categories are subdivided, in turn, according to the type of physical variables which constitute the continuous data within the computer.

One major class of analog devices depends for its operation upon the existence of a direct physical analogy between the analog and the prototype system being simulated. Such an analogy is recognized by comparing the characteristic equations describing the dynamic or static behavior of the two systems. An analogy is said to exist if the governing, characteristic equations are similar in form, term by term. For every element in the original system, there must be present in the analog system an element having mathematically similar properties—that is, an element having a similar excitation/response relationship. Furthermore, the analog elements must be joined or interconnected in a similar fashion. Members of this category of analog devices are termed *direct analogs.* Direct analogs may be of either the continuous (distributed) or the discrete variety.

Continuous direct analog simulators make use of distributed elements such as sheets or solids, made of an electrically conductive material, so that every spatial point in the analog corresponds to a specific point in the system being simulated. The conductive sheets and electrolytic tanks described below fall into that category. Stretched membrane models, in which soap films or thin rubbers sheets are supported by a mechanical framework, were also used for a time to simulated fields governed by Laplace's and Poisson's equations. Hydrodynamic models, termed *fluid mappers,* as well as direct analog simulators utilizing thermal fields, electrochemical diffusion phenomena, polarized light, and electrostatic fields, have also been successfully used for that purpose.

Discrete direct analog simulators employ lumped physical elements, such as electrical resistors and capacitors, in which case the behavior of the system being simulated is obtained only for the points in the system that correspond to the junctions in the electrical circuit. Networks of electrical resistors, resistance–capacitance networks, and inductance–capacitance networks have all been widely used to simulate fields governed by elliptic, parabolic, hyperbolic, and biharmonic partial differential equations.

The other major class of analog simulation systems includes mathematical rather than physical analogs. The behavior of the system under study is first characterized by a set of algebraic or differential equations. An assemblage of analog computing units or elements, each capable of performing some specific mathematical operation, such as addition, multiplication or integration, is provided, and these units are interconnected so as to generate numerical solutions of the problem. Such computing systems are termed *indirect analog computers.* Prior to World War II, powerful indirect analogs for the solution of differential equations were fashioned from mechanical components and termed *mechanical differential* analyzers. Electronic differential analyzers were introduced after World War II and became very important tools in the design of aerospace systems, control systems, and chemical process controllers in the United States, western Europe, Japan, and the Soviet Union.

An important distinction between direct and indirect analogs involves the significance of the physical variables within the computer. In a direct analog, an analog variable has the same significance everywhere within the analog system. For example, in the electrical analog simulation of a mechanical system, voltage everywhere in the analog may represent velocity. The time derivative of the analog voltage would then represent acceleration. In an indirect analog, on the other hand, a transient voltage at some junction in the analog may represent acceleration; this voltage is then applied to an integrator unit, and the transient voltage at the output of the integrator would represent velocity.

The general classification of analog methods is illustrated diagrammatically in Fig. 1. It should be emphasized that continuous and discrete direct analog simulators played a very significant role before World War II. By 1980 they had all been virtually completely eclipsed by digital simulation methods. Indirect analog computers enjoyed wide use in the 1960s, 1970s, and 1980s; but by the early 1990s, they too had largely been replaced by digital computers.

DIRECT ANALOG SIMULATORS

Examples of Continuous Direct Analog Simulators

One of the fundamental equations characterizing distributed parameter systems in a wide variety of areas of physics is Laplace's equation,

$$\nabla^2\phi = 0 \tag{1}$$

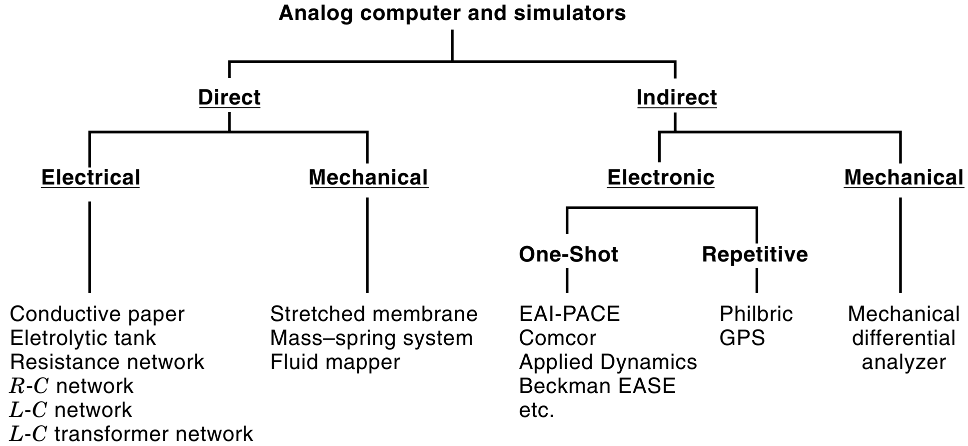


Figure 1. Classification of analog simulation methods and analog computers.

and Poisson's equation

$$\nabla^2 \phi = K \tag{2}$$

Equation (1) arises, for example, in the study of the steady-state temperature distribution in a flat plate, subject to heat sources or sinks at its boundaries. Let's apply a direct analog simulation method to such a problem:

1. A sheet made of an electrically conductive material having the same geometrical shape as the field under study is fashioned in the laboratory.
2. The boundary conditions of the original field are simulated in the analog system by appropriate voltage and current sources. For example, if one boundary of the sheet is specified to have a temperature of 100°C, and another boundary a temperature of 0°C, voltage sources 100 V and 0 V in magnitude might be applied to the corresponding locations in the analog.
3. By means of suitable sensing equipment, such as a voltmeter or an oscilloscope, lines of equal voltage in the conductive medium are detected and recorded.
4. The voltage distribution measured in the analog then constitutes the solution to the problem.

Over the years, the suitability of many different conductive materials was investigated so as to devise practical analog simulators. One technique widely used in the 1960s and 1970s involved the utilization of Teledeltos Paper developed and marketed by the Western Union Telegraph Company as a recording medium for telegrams and graphic chart instruments. This paper is formed by adding carbon black, a conductive material, to paper pulp in the pulp-beating stage of the paper-manufacturing process. This results in a high-quality paper with a fairly uniform dispersion of carbon. Because of its wide use, the paper was quite inexpensive and well-suited for "rough and dirty" simulation applications. A typical setup of this type is shown in Fig. 2(a). At times, lines of equal potential were drawn directly on the conductive paper, using a ball point pen, as illustrated in Fig. 2(b). In that case, the potentiometer is set to the voltage corresponding to the equipotential line to be sketched, and the probe is moved over the paper in such a manner that the deflection of the microameter remains zero. When a complete equipotential line has been drawn, the potentiometer is set to a different voltage, and the process is repeated until the equipotential lines of the entire field have been plotted.

For greater accuracy, an electrically conductive liquid was used in place of the resistance paper. Such so-called *electrolytic tank* analog simulators, shown in Fig. 3, were employed to simulate fields governed by Laplace's equation and were used as follows:

1. A large container (the tank), open at the top is filled with a suitable weak saline solution (the electrolyte).
2. A scale model of the boundary configuration of the two-dimensional field under study, or a conformal transformation thereof, is immersed in the container. Boundaries which are equipotential surfaces are made of metal, while streamline boundaries are fashioned from an insulating material.

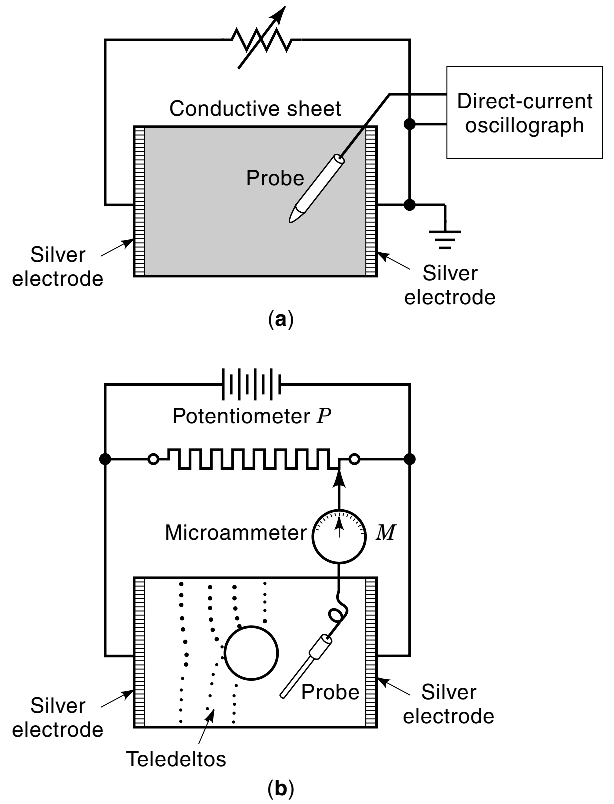


Figure 2. (a) Simple conductive sheet analog simulator for modeling fields governed by Laplace's equation in two dimensions. (b) Potentiometer plotting arrangement for drawing equipotential lines directly on the conductive paper.

3. Alternating-current (ac) voltage sources of appropriate magnitudes are applied to all equipotential boundaries.
4. The voltage distribution along the surface of the electrolyte is measured and recorded. Lines of constant voltage within the analog then correspond directly to the equipotential lines of the system being simulated.

If a field governed by Laplace's equation in three dimensions was to be simulated, the sensing probe could be extended into

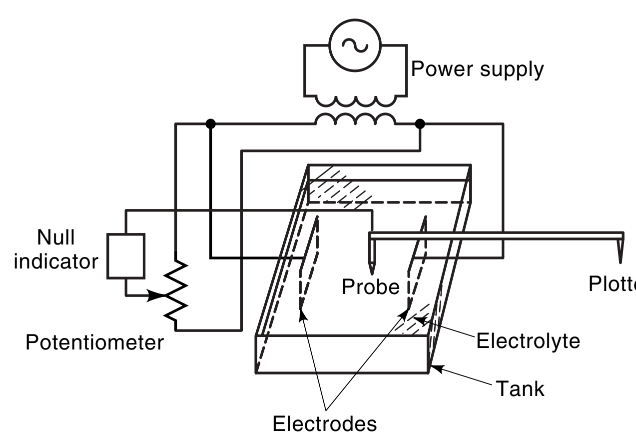


Figure 3. Typical conductive liquid analog simulation system (electrolytic tank) for modeling fields governed by Laplace's equation.

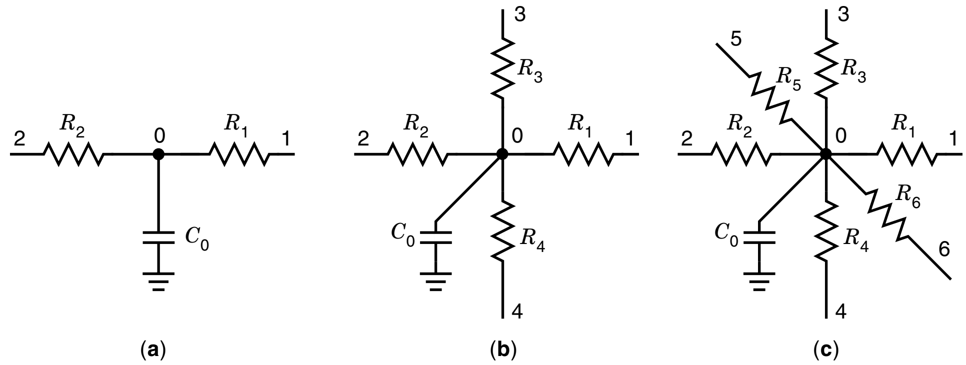


Figure 4. Typical nodes of resistance–capacitance networks used to simulated fields governed by the heat-transfer or diffusion equations. Networks may contain thousands of such node elements. (a) One dimension, (b) two dimensions, (c) three space dimensions.

the liquid and a three-dimensional record of the potential distribution within the tank obtained. Great care was taken to achieve highly accurate modeling and sensing devices, so that relative solution errors could be kept below 0.01%. Throughout the first half of the twentieth century and until the advent of digital simulators in the 1980s, electrolytic tanks remained the premier method for the accurate mapping of potential fields (see Ref. 1).

Examples of Discrete Direct Analog Simulators

Electrical network simulators are based on finite difference or finite element approximations of one-, two-, or three-dimensional partial differential equations. By far the most widely used discrete direct analog simulators were the resistance/capacitance networks for the simulation of fields governed by the diffusion equation,

$$\nabla^2 \phi = k\frac{\partial \phi}{\partial t} \tag{3}$$

in one, two, and three Cartesian coordinates. In this approach, the derivatives with respect to the space variables are replaced by finite differences, while the time variable is kept in continuous form, as

$$\frac{\phi_1 - \phi_0}{\Delta x^2} + \frac{\phi_2 - \phi_0}{\Delta x^2} \cong k\frac{\partial \phi_0}{\partial t} \tag{4a}$$

$$\frac{\phi_1 - \phi_0}{\Delta x^2} + \frac{\phi_2 - \phi_0}{\Delta x^2} + \frac{\phi_3 - \phi_0}{\Delta y^2} + \frac{\phi_4 - \phi_0}{\Delta y^2} \cong k\frac{\partial \phi_0}{\partial t} \tag{4b}$$

$$\frac{\phi_1 - \phi_0}{\Delta x^2} + \frac{\phi_2 - \phi_0}{\Delta x^2} + \frac{\phi_3 - \phi_0}{\Delta y^2} + \frac{\phi_4 - \phi_0}{\Delta y^2} + \frac{\phi_5 - \phi_0}{\Delta z^2} + \frac{\phi_6 - \phi_0}{\Delta z^2} \cong k\frac{\partial \phi_0}{\partial t} \tag{4c}$$

Electrical networks are then fashioned from resistors and capacitors, with typical nodes as shown in Fig. 4, where the magnitudes of the circuit elements are determined by the local magnitudes of the parameters in the field being simulated. Networks of this type proved extremely useful in the study of transient heat transfer (so-called *thermal analyzers*) and of the flow of fluids in porous media as in aquifers and oil reservoirs. In a number of instances, such networks contained many thousands of node elements, as well as sophisticated electronic circuitry for the application of boundary and initial conditions.

Other network simulators for the simulation of fields characterized by partial differential equations included one-, two-, and three-dimensional networks of resistors. These served to model fields governed by elliptic partial differential equations such as Eqs. (1) and (2). Networks of inductors and capacitors were occasionally used to simulate fields governed by the wave equation, particularly in the design of electromagnetic systems such as waveguides and cavity resonators.

One very sophisticated and elaborate network computer was designed at Caltech and by Computer Engineering Associates for the simulation elastic beam problems governed by the biharmonic partial differential equations,

$$\nabla^4 \phi = 0 \tag{5a}$$

$$\nabla^4 \phi = k\frac{\partial^2 \phi}{\partial t^2} \tag{5b}$$

In addition to inductors and capacitors, this simulator included high-quality transformers in every network node element. Figure 5 illustrates the simulation of the vibration of a cantilever beam using this approach. Similar networks were used to simulate the deflection of two-dimensional systems such as elastic plates. Another network analyzer including resistors, reactors and transformers was marketed by General Electric and used for the simulation of electric power distribution networks. More details are provided in Ref. 1.

INDIRECT ANALOG SIMULATORS

Mechanical Differential Analyzers

The possibility of obtaining computer solutions of ordinary differential equations by successive mechanical integrations was first suggested by Lord Kelvin in 1876. No successful machines using this method appear to have been constructed until researchers at MIT, under the leadership of Vannevar Bush, constructed a series of these computers, termed *mechanical differential analyzers,* in the 1930s. In the 1940s, General Electric marketed several such analog machines, and others were subsequently constructed and installed at a number of locations in Western Europe and in the Soviet Union.

In mechanical differential analyzers, all dependent problem variables are represented by the rotations of as many as 100 parallel shafts, rather than by voltages as in electronic

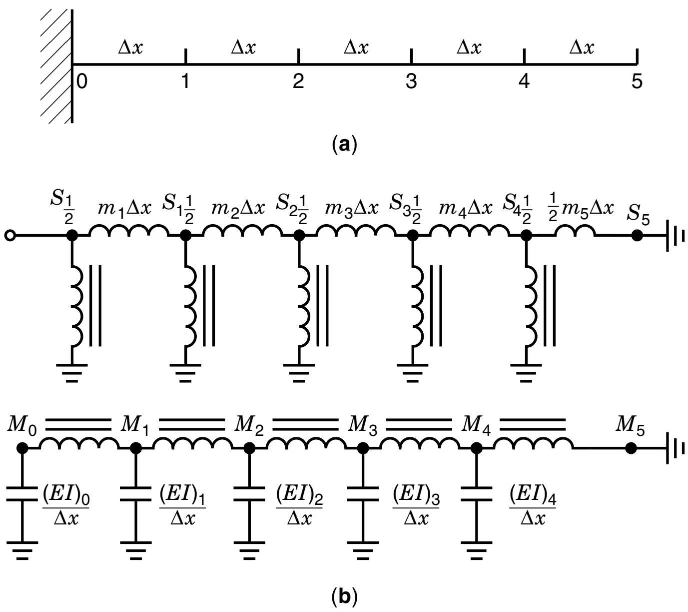


Figure 5. Network for the simulation of the vibrations of an elastic cantilever beam, governed by the biharmonic equation, which is fourth-order in x and second order in time. (a) Schematic of the beam including five finite difference or finite element sections. (b) Network containing inductors, capacitors and a transformer at each node. (See Refs. 1 and 2.)

differential analyzers. These shafts are interconnected and driven by mechanical units that accept one or more shaft rotations as inputs, and they drive another shaft the rotation of which provides the output corresponding to the desired functional input–output relationship.

The addition of two dependent variables, x and y, is accomplished with the aid of differential gears, as shown in Fig. 6(a). Multiplication by a constant is readily achieved by coupling two shafts by gears. By selecting appropriate gear ratios, one turn of one shaft can be translated into a desired multiple or fraction of a turn of the second shaft. This is illustrated in Fig. 6(b).

Integration of a dependent variable with respect to another dependent variable or with respect to an independent variable can be carried out using a *disk-and-wheel integrator* as shown schematically in Fig. 6(c). The turns of the disk, called the *turntable,* represents the differential variable x to a suitable scale factor. The distance of the wheel centerplane from the axis of the turntable represents the integrand, y, again to some suitable scale factor. These are the two inputs to the integrator. The turns of the integrating wheel represent the value z of the integral to a scale factor determined by the two input scale factors and the actual structural details of the unit. This is the output of the integrator.

A rotation of the disk through an infinitesimal fraction of a turn, dx, causes the wheel to turn through a correspondingly small part of a turn, dz. For a wheel of radius a, we obtain

$$dz = \frac{1}{a} y\,dx \tag{6}$$

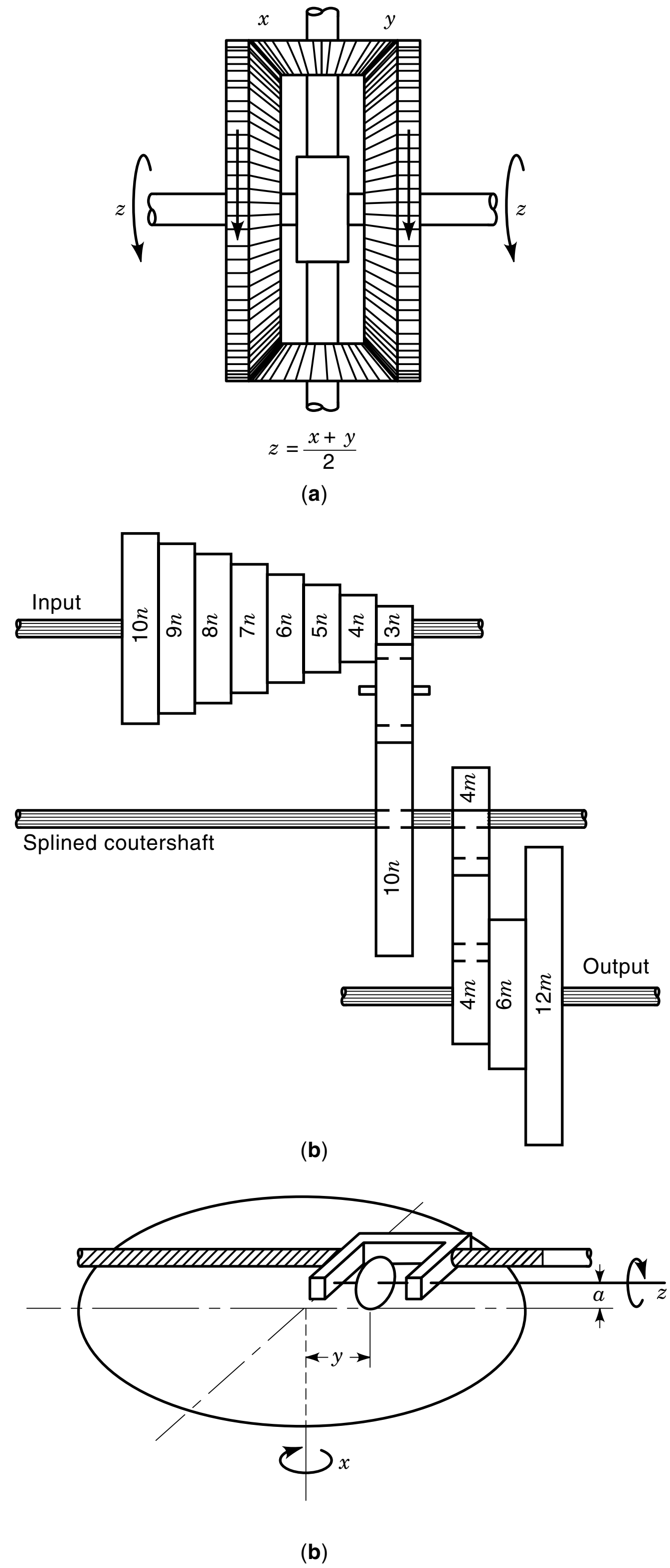


Figure 6. Mechanical computing elements employed in mechanical differential analyzers. (a) Differential gear for producing a shaft rotation z which is proportional to the sum of rotations x and y. (b) Multiplication of the rotation of a shaft using step-up or step-down gear ratios. (c) Disk-wheel integrator for generating the integral z of wheel displacements y with respect to wheel displacement x of the disk.

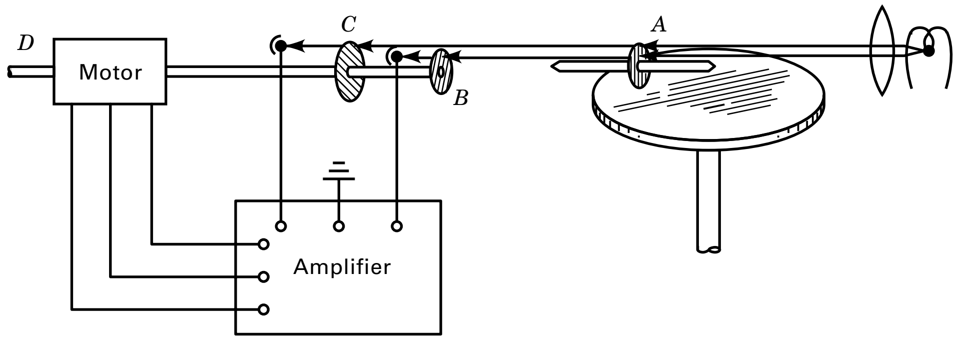


Figure 7. Polarized-light servomechanism for torque amplification in a wheel–disk integrator.

During a finite time interval, the x turntable will turn through a finite number of revolutions, and the distance y will vary, ranging through positive (on one side of center) to negative (on the other side of center) values as called for by the problem. The total number of turns registered by the integrating wheel will then be

$$z = \frac{1}{a}\int_{x_0}^{x} y\,dx \qquad (7)$$

Adequate operation of the integrator requires that the wheel roll with negligible slip, even as the rotation z is transmitted mechanically to other shafts. This calls for torque amplification, and a variety of ingenious mechanism were introduced for that purpose. The polarized light servomechanism for torque amplification is shown schematically in Fig. 7. The integrating wheel, A, consists of a polarizing disk with a steel rim and a steel hub. The direction of optical polarization is shown by the direction of the crosshatch lines on the wheel. The follow-up system consists of a pair of similar polarizing disks B and C on the motor-driven output shaft D. The two disks are mounted with their planes of polarization at right angles to each other. Two beams of light pass through polarizer A and are polarized in the same direction. One light beam passes through polarizer B, while the other passes through polarizer C. The light beams are picked off by separate phototubes, which are connected through an amplifier to a split-field series motor. Any difference in light intensity striking the two phototubes will cause the motor to turn. This will cause the output shaft D to assume an orientation with respect to wheel A so that the plane of polarization of wheel A bisects the right angle between the two planes of polarization of disks B and C. The output shaft D is thus constrained to follow the motions of the integrating wheel, with only the light beams as the coupling medium between them.

Note that the shafts representing the variables x and y can be driven by the outputs of other integrators or by a separate motor. For example, the turntable can be driven by a motor at constant speed. In that case, integration with respect to time is achieved. Multiplication of two dependent variables x and y can be effected by connecting two integrators as shown in Fig. 8, resulting in an output:

$$xy = \int x\,dy + \int y\,dx \qquad (8)$$

In Fig. 8, conventional symbols are used to represent the integrators, adder, and shafts. The initial values of the product is taken into account by providing suitable initial settings of the two integrator wheels. Note that this equation would be much more difficult to implement using an electronic analog computer, since electronic integrators are limited to integrating with respect to time.

To illustrate the application of the mechanical differential analyzer, consider first the almost-trivially simple problem of finding the area under a curve. Specifically, a curve $y = f(x)$ is shown plotted on a sheet of paper fastened to an input table I in Fig. 9(a). The curve starts at some value, x_1, of the independent variable x and ends at some other value x_2. The curve

$$z = \int_{x_1}^{x_1} y\,dx \qquad (9)$$

is to be plotted on the output table O. The differential equation corresponding to Eq. (9) is

$$\frac{dz}{dx} = y \qquad (10)$$

where y is given as a plotted function of x. The mechanical differential analyzer system for generating this solution is shown in detail in Fig. 9(a) and schematically in Fig. 9(b). The variable y displaces the integrating wheel when the hand

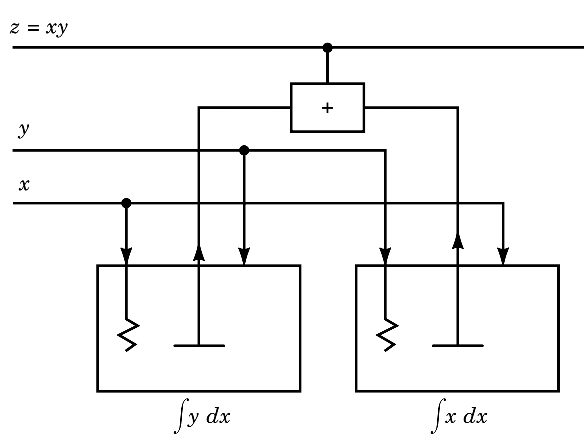
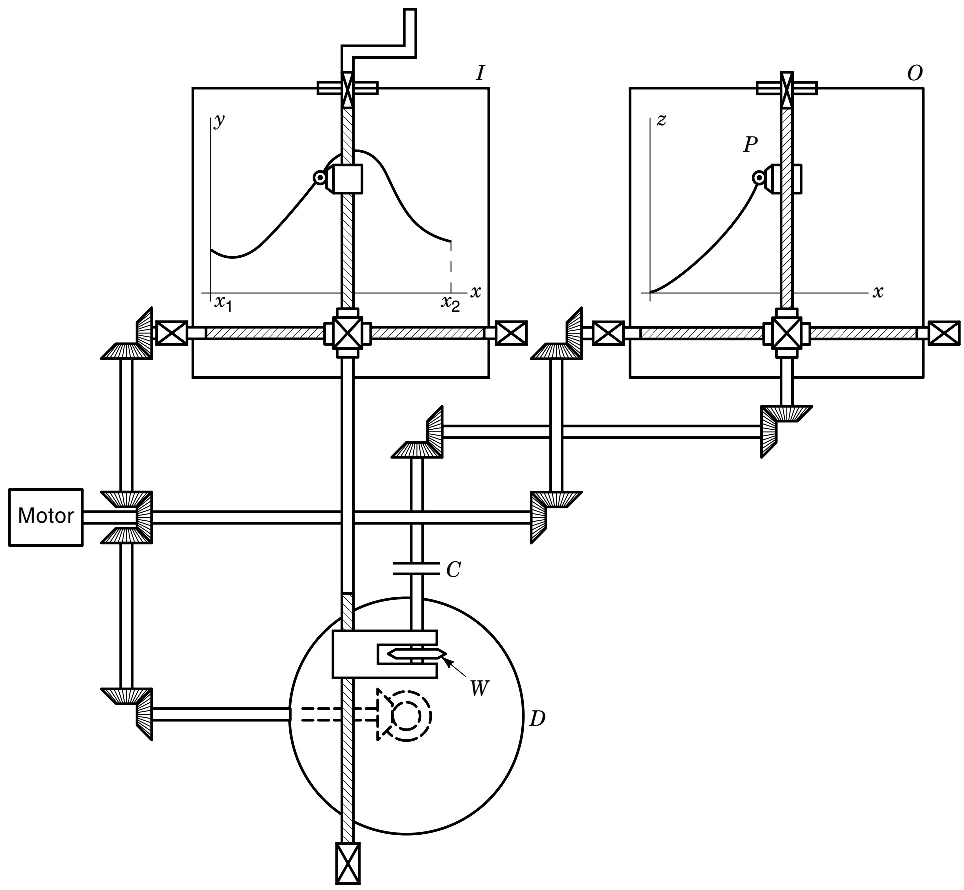
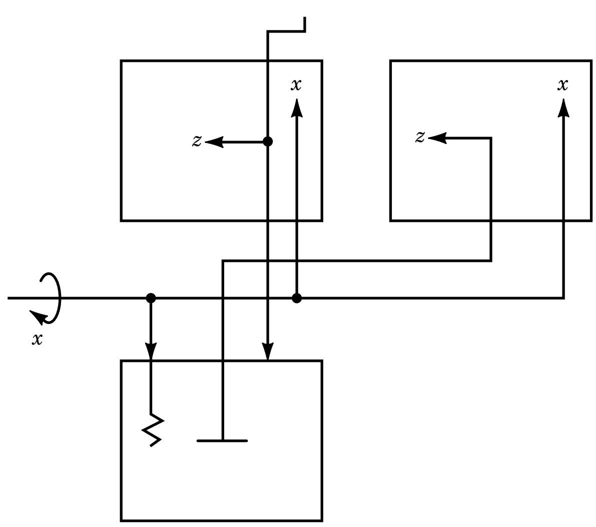


Figure 8. Schematic diagram showing the multiplication of two dependent variables x and y by implementing the formula for integration by parts, Eq. (8).



(a)



(b)

Figure 9. Mechanical differential analyzer method for generating the area under a specified curve. (a) Detailed figure, (b) schematic diagram.

crank on the input table is turned manually to keep a peephole on the given curve, while the x lead screw shifts the peephole horizontally via the independent variable motor drive. The motor also turns the integrator disk D. The integrator wheel W operates through a torque-amplifying coupling C to drive the vertical lead screw on the output table O. A nut on this lead screw carries a pen P which traces the curve $z = f(x)$, as a nut on the horizontal lead screw traverses the x range.

Consider now a simple second-order differential equation of the form

$$M \frac{d^2y}{dx^2} + b \frac{dy}{dx} + ky = 0 \qquad (11)$$

where M, b, and k are specified constants, and initial values of y and dy/dx are also given. The solution process is

$$\frac{d^2y}{dx^2} = -\left(\frac{b}{M}\frac{dy}{dx} + \frac{k}{M}y\right) \tag{12a}$$

$$\frac{dy}{dx} = \int_0^x \frac{d^2y}{dx^2}\,dx + \mathring{y}(0) \tag{12b}$$

$$y = \int_0^x \frac{dy}{dx}\,dx + y(0) \tag{12c}$$

Assume that it is desired to plot y and dy/dx as functions of x and that dy/dx is also required as a function of y and also as a function of the second derivative of y with respect to x. The mechanical differential analyzer implementation is shown schematically in Fig. 10. All variable shafts are shown as horizontal lines. Adders and the gear trains interconnecting the various shafts are shown at one end. Connections from the various shafts are carried over to the integrators and output tables by cross shafts. In a similar manner, systems of simultaneous nonlinear differential equations with variable coefficients can be solved.

Major mechanical differential analyzer facilities included 20 or more integrators and a substantial number of input and output tables. Using high-precision mechanical components, they were capable of solving complex engineering problems to a higher accuracy than were the electronic differential analyzers that eventually replaced them. At the time they were, however, very costly to construct and to maintain, and they occupied an inordinate amount of space. Additional details are discussed in Ref. 2.

Electronic Differential Analyzers (EDAs)

Electronic analog computers were first developed for military applications during World War II. Subsequently, numerous manufacturers entered into competition to provide progressively larger, more accurate, and more flexible general-purpose computers. The design of the electronic computer units and the programming of EDAs is considered in detail in other articles in this encyclopedia. General-purpose electronic differential analyzers became available in installations ranging from a modest 10 operational amplifiers to well over 2000 operational amplifiers. The accuracies of these computers in solving nonlinear equations ranged from 2% of full scale for relatively low-cost devices to better than ~0.1% for the most elegant models.

Very early in the development of electronic analog computers, it became apparent that there exist two distinct philosophies or approaches to the application of these devices. In one class of analog computers, the time necessary to obtain a solution varies from approximately 10 s to several minutes. The initial conditions and driving functions are applied at an instant of time corresponding to $t = 0$, and continuous graphical outputs are generated from selected points in the computer system. This type of EDA is termed a *long-time* or *one-shot* analog computer. The other class of differential analyzers operates on a greatly speeded-up time scale, so that solutions of problems are obtained in several milliseconds. In that case, the problem run is repeated automatically several times per second, and the result of the computation is displayed on a cathode-ray oscilloscope. Members of this second class are termed *repetitive* or *high-speed* analog computers. While both approaches had their enthusiastic adherents, the long-time computer assumed a preponderant position by a wide margin, in terms of both (a) the number of companies engaged in its production and (b) the number of computers actually in use.

Almost all commercial "long-time" installations are designed around a centrally located patch-bay housed in a control console. Wires leading to the inputs and outputs of all computer units and components are brought out to an array of several hundred or even thousands of patch-tips. Removable problem boards, made of an insulating material, are machined to fit precisely over these tips in such a manner that a clearly identified hole in the problem board lies directly over each patch-tip. Most of the programming and connecting of the computer can then be accomplished by means of patch-cords interconnecting the various holes in the problem board. Usually a considerable number of problem boards are available with each computer. Problems can be programmed on these boards, which can be stored for subsequent experimen-

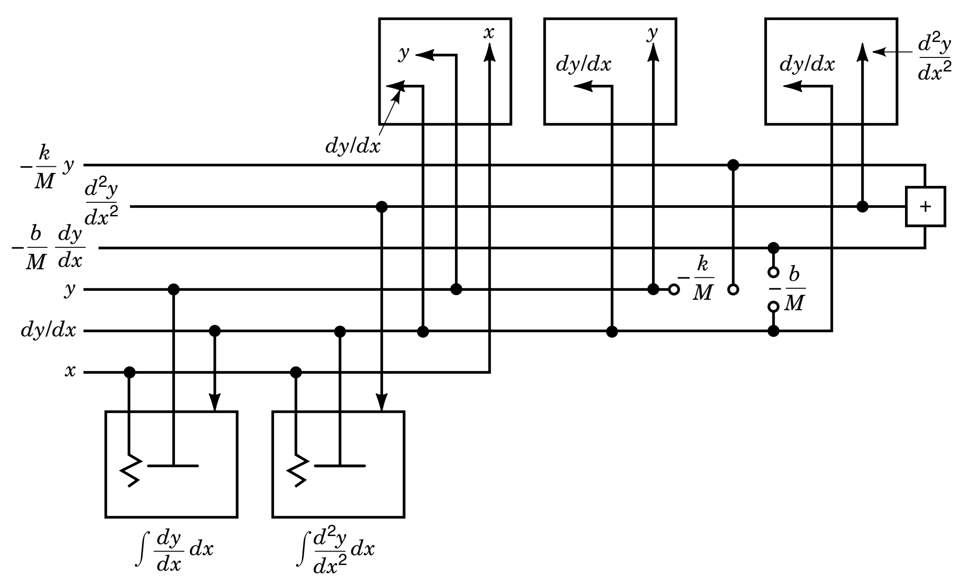


Figure 10. Mechanical differential analyzer schematic for the solution of the second-order differential equation, Eq. (11).

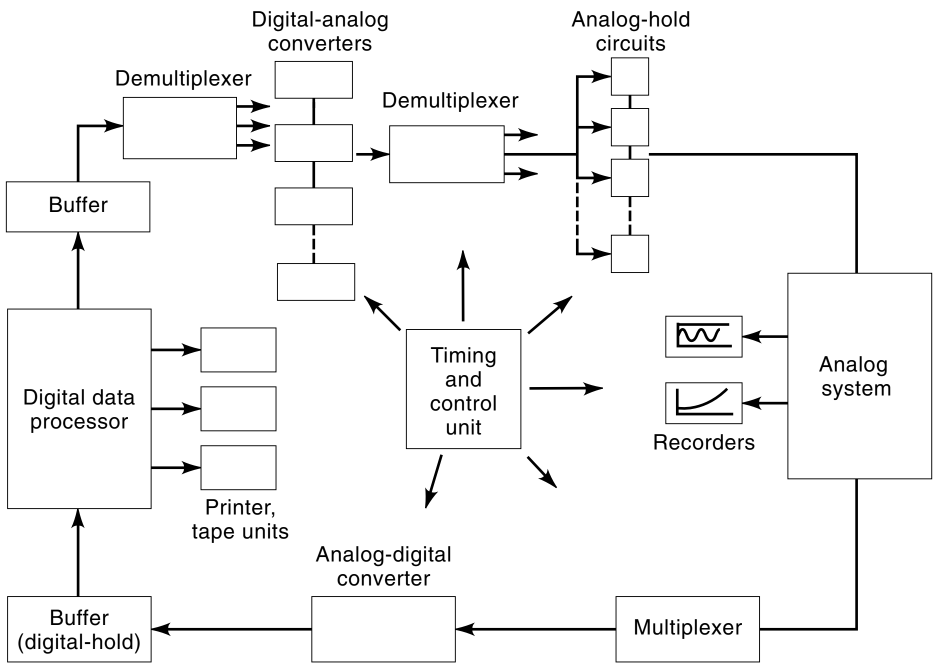


Figure 11. Major components of a hybrid (analog/digital) computer system of the type widely used in the aerospace industry in the 1970s and 1980s for the design of systems and for the training of pilots and astronauts.

tal work while the computer is being employed to solve an entirely different problem. In that manner, the computer installation is not "tied-up" by a single problem. A considerable effort has been expended in optimizing the design of problem boards to facilitate their use. Even so, the programming of reasonably complex problems results in a veritable maze of plug-in wires, a factor which not infrequently leads to errors and makes debugging very difficult. To help alleviate this situation, most manufacturers introduced color-coded plug-in connectors and multicolored problem boards, as well as special "problem-check" circuitry.

In addition to the patch-bay, the control console generally includes the principal operating relays or solid-state switches for resetting initial conditions and for commencing computer runs, as well as potentiometers and amplifier overload indicators. One set of solid-state switches facilitates the connection of direct-current (dc) power supplies to the outputs of all integrators for the setting of specified initial conditions. At the start of the computer run, at $t = 0$, all of these switches open simultaneously, and at the same instant of time, other switches connect the specified driving functions into the circuit. To repeat the computer run, the control switch is moved from the "compute" to the "reset" position, and the identical initial conditions are again applied. Frequently a control unit includes a "hold" setting. In this position, all integrator capacitors are disconnected from the input resistors, so that they are forced to maintain whatever charge they possess at the instant the control switch is turned to the "hold" position. The voltages at various points in the circuit can then be examined at leisure.

The rest of the components are mounted in standard racks in such a manner that the computer facility can readily be expanded by purchasing and installing additional racks of equipment. Precision resistors and capacitors are used throughout; and in the more refined high-accuracy installations, all resistors and capacitors actually taking part in the computing operation are kept in a temperature-controlled oven so as to minimize drift errors. All computers have variable dc power supplies for the application of initial conditions to the integrators and for the generation of excitations. The output devices are generally mounted separately and may include direct-writing oscillographs for relatively high-speed recording, servo-driven recorders, and digital voltmeters. In addition, most analog facilities possess a number of multipliers, resolvers for the generation of trigonometric functions, arbitrary function generators, Gaussian noise generators, and time-delay units for simulating transport lags. Further details are presented in Refs. 3 and 5.

Hybrid (Analog/Digital) Computers

When relatively low-cost, on-line digital computers became available in the late 1960s and 1970s, so-called *hybrid computers* became popular. Analog and digital computer units were interconnected, using analog–digital and digital–analog converters, while a single control unit controlled all computers comprising the system. In such a hybrid computer, the computing tasks were divided among the analog and digital units, taking advantage of the greater speed of the analog computer and the greater accuracy of the digital computer. For example, in simulating a space vehicle, the guidance equations were solved digitally, while the vehicle dynamics were implemented on the analog computer. Such a hybrid computer system is shown in Fig. 11. Further details are to be found in Ref. 4.

Throughout the 1970s and well into the 1980s, hybrid computers played a crucial role in the development of many military and civilian aerospace systems, including guided missiles, aircraft and space vehicles, and so on, as well as in training pilots and astronauts. By 1990, however, the development of minicomputers and microprocessors had reached a level of performance that permitted all tasks formerly assigned to the analog computer to be performed digitally at adequate speed and greatly reduced cost. This effectively spelled the end of hybrid computers as a tool for engineering design and simulation.

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