STOCHASTIC SYSTEMS

Many control systems in practice are subject to imperfectly known disturbances which may be taken as random. Such disturbances have been ignored in the study of deterministic control problems. In many control systems some unknown parameters occur. The system behavior depends on the parameters and the fact that the value of the parameters is unknown makes the system unknown. Some crucial information concerning the system control is not available to the controller, and this information should be learned during the system's performance. The problem described is the problem of adaptive control. The adaptive control problem can be considered the identification problem and the control problem. A solution to an adaptive control problem will be understood as a solution to the identification problem and to the control problem. We focus on control and adaptive control of continuous-time linear stochastic systems. The theory of adaptive control of continuous-time linear stochastic systems has been recently developed as an important application of the stochastic control theory in engineering, biology, economics, finance, and manufacturing systems. Continuous-time linear systems are an important and commonly used class of systems (1,2). It is assumed that the models evolve in continuous time rather than discrete time because this assumption is natural for many models and it is important for studying discrete time models when the sampling rates are large and for analyzing numerical round-off errors. Stochastic systems are described by linear stochastic differential equations.

The general approach to adaptive control described here exhibits a splitting or separation of the problems of identification of the unknown parameters and adaptive control. Maximum likelihood, least squares, or weighted least squares estimators are used to identify the unknown constant parameters (3,4,5). These estimates are given recursively and are strongly consistent, which means that the family of estimates converges to the true value of the parameter with the probability one. It turns out that for some cases the weighted least squares estimator is strongly consistent whereas the least squares estimator is not. The adaptive control constructed by the so-called certainty equivalence principle, that is, the optimal stationary control, is computed by replacing the unknown true parameter values by the current estimates of these values. Because the optimal stationary controls are continuous functions of the unknown parameters, the self-tuning property is verified, which means that asymptotically adaptive control using the estimate of the unknown parameter is as good as optimal control if we know the system. It is also shown that the family of average costs using the control from the certainty equivalence principle converges to the optimal average cost. This verifies the self-optimizing property.

In describing a stochastic control model, the kind of information available to the controller at each instant plays an important role. Several situations are possible:

(1) The controller has no information during the system's operation. In this case a function of time is chosen as a control. These controls are often called "open loop" as distinct from closed-loop or feedback controls, in which the actual value of control input at time t is a function of the observation at time t (6). The distinction between open-loop and feedback control is fundamental. An open-loop control is deterministic whereas a feedback law determines a random control process. An open-loop control is formally a special case of a closed-loop control. For a deterministic system the reverse also holds. The important point is that this is not true for stochastic systems.

(2) The controller knows the state of the system at each time t. We call this the case of complete observations.

(3) The controller has partial knowledge of the system states. This is called the case of partial observations.

For a deterministic control model, the state of the system at any time t can be deduced from the initial data and the control used up to time t by solving the corresponding differential equation. Thus observing the current state of the system at each time t does not really give more information than knowing the initial data. For stochastic control systems, there are many paths which the system states follow given the control and the initial data. In the stochastic case, the best system performance depends on the information available to the controller at each time t.

When we consider continuous-time stochastic control problems, the system states are described by the stochastic process that changes according to a stochastic differential equation. We need to be aware that to deal with more realistic mathematical models of the situation we allow for some randomness in some of the coefficients of an ordinary differential equation (7). Randomness arises when we consider real life situations with some disturbances or noise. These disturbances or noise are modeled by a stochastic process, often by the so-called Brownian motion or Wiener process (8,9), which we define later.

Stochastic control problems have been difficult to solve explicitly (10,11,12). One of the fundamental results of the stochastic control problem, where the optimal control is obtained explicitly, is the linear quadratic Gaussian (LQG) problem (5,6,13). It is called linear because the system is described by a linear difference equation (in the discrete-time case) or by a linear differential equation (in the continuous-time case). It is called quadratic because the cost criterion to be minimized or maximized is quadratic. It is quadratic in a state and in a control. It is called Gaussian because the noise is modeled by a Gaussian process.

Another stochastic control problem that has been solved explicitly is the portfolio selection and consumption model known as Merton's model (14) and described in Example 2 below.

A linear quadratic Gaussian problem and a portfolio selection and consumption model were the two of a very few stochastic control problems that have been solved explicitly. Recently Duncan and Upmeir (15) provided a pretty wide class of stochastic control problems that are solved explicitly. The stochastic control problems are the control of Brownian motion in noncompact symmetric spaces by a drift vector field.

In solving stochastic control problems the basic difficulty lies in solving the so-called Hamilton–Jacobi–Bellman (HJB) equation (11). To solve this equation it is assumed that the solution, the value function, is smooth. Typically the solution is not smooth enough to satisfy the HJB equation in a classical sense. The theory of viscosity solutions, first introduced by M. G. Grandall and P. L. Lions, provides a convenient framework in which to study HJB equations. See Ref. 16 on the theory of viscosity solutions.

Stochastic calculus provides powerful mathematical tools for solving stochastic control problems and stochastic adaptive control problems. The mathematical tools of stochastic calculus most commonly used are stochastic integral (17), Itô's differential rule, and martingales (7,8). These elements of stochastic calculus together with the limit theorems of the probability theory, the theory of stochastic processes (18,19,20,21,22), and the theory of stochastic differential equations are extremely useful in proving the results of stochastic control and stochastic adaptive control of continuous time. This is another motivation for considering stochastic control problems in continuous time rather than in discrete time. There has been always a greater chance of stronger results because those mathematical techniques are more sophisticated and because of that, more powerful.

Stochastic adaptive control problems are applications of stochastic control theory (23,24,25,26). Because we use the certainty equivalence control as an adaptive control, it means that we need the optimal control given explicitly or an almost optimal control (27).

The stochastic adaptive control problems with discrete time have been under investigation for a long time. Many references are in the books in Refs. 5, 6, 23, and 28, and in a few others (17,24,29). Continuous-time stochastic adaptive control problems have been investigated relatively recently. Many results have been obtained under conditions difficult to verify. The goal of further investigation was to provide results which could be obtained under the same assumptions as stochastic control problems, that is, controllability and observability which have been always considered the most natural conditions for optimal control (30).

The results presented here are obtained under these kinds of conditions for linear systems. Corresponding results for nonlinear systems do not yet exist. Stochastic adaptive control problems with continuous time are under intensive investigation.

It is important to mention that the problems of stochastic adaptive control previously described are problems of stochastic adaptive control with complete observation of the state.

Let us describe the adaptive control scheme used here. The control as feedback gains is obtained from solving an infinite-time, quadratic cost control problem by replacing the correct values of parameters in this solution by the estimates of parameters at time t to obtain the feedback gain at time t. The estimates of the parameters are the least squares estimates or the weighted least squares estimates. The least squares estimate based on the observations of the state until time t is used as the correct value of the parameter to solve the infinite-time control problem by solving the algebraic Riccati equation. This method gives a feedback gain at each time t and therefore a control policy.

Problems of continuous-time stochastic adaptive control with partial observations in a general setting remain still open. Partial results have been provided by Bertrand (31). For discrete time see Ref. 32 and references therein.

Many problems with partial observations of a state can be solved by the filtering theory (33). To recognize the differences among filtering problems, control problems, and adaptive control problems, let us consider some simple examples.

Example 1. Filtering problem (7,11,34). Suppose that we would like to improve our knowledge about the solution of a differential equation so we observe $X(s)$ at times $s \leqslant t$. Let us imagine that the $X(t)$ is the charge at time t at a fixed point in an electric circuit. However, due to inaccuracies in our measurements we do not really measure $X(s)$ but a perturbed version of it: $Y(s) = X(s) +$ noise.

The filtering problem is, what is the best estimate of $X(t)$ satisfying $Y(s) = X(s) +$ noise based on the observations $Y(s)$ where $s \leqslant t$?

Initially the problem is to "filter" the noise away from the observations optimally.

In 1961 Kalman and Bucy (33) proved what is now known as the Kalman-Bucy filter. Basically the filter gives a procedure for estimating the state of a system which satisfies a linear differential equation, based on a series of observations with a noise.

Example 2. Stochastic control problem. Portfolio Selection and Consumption Model. Let $X(t)$, $t \geqslant 0$ be the wealth of an individual at time t who invests his wealth in two types of assets: the safe asset has the return rate r and the risky asset has the average return rate α. The wealth $X(t)$ at time t changes according to the stochastic differential equation

$$\begin{aligned} dX(t) = {} & r[1 - U_1(t)]X(t)\,dt \\ & + U_1(t)X(t)[\alpha\,dt + \tau\,dW(t)] - U_2(t)\,dt \end{aligned}$$

where $U_1(t)$ is the fraction of the wealth invested in the risky asset at time t and $U_2(t)$ is the consumption rate at time t; r, α, τ are constants with $r < \alpha$ and $\tau > 0$; and $(W(t), t \geqslant 0)$ is a real-valued standard Wiener process.

The controls are naturally constrained as $0 \leqslant U_1(t) \leqslant 1$ and $U_2(t) \geqslant 0$. The stochastic control problem is to maximize the expected discounted total utility

$$\mathscr{I}(U) = E_y \int_s^\infty \exp(-\rho t) F[U_2(t)]\,dt, \qquad y = W(s)$$

where $F(u) = u^\gamma$ with $0 < \gamma < 1$ is the utility function and $\rho > 0$ is the discount rate,

Example 3. Stochastic adaptive control problem. Portfolio Selection and Consumption Model. Consider the situation described in Example 2. For the adaptive control problem, it is assumed that α is an unknown parameter such that $\alpha \in [a_1, a_2]$ with $r < a_1$. The adaptive control procedure in this adaptive setting is to define the control at a time t, that is, the portfolio selection and the consumption rate, using the optimal infinite-time control, where the estimate of the unknown parameter at time t is used for the unknown parameter.

Example 4. Stochastic adaptive control problem. Manufacturing Model (35). Consider a manufacturing system that produces n distinct part types using m identical machines. Let $U(t) \in \mathbb{R}^n$ denote the vector of productions rates, $X(t) \in \mathbb{R}^n$ the vector of total inventories/backlogs, and $Z(t) \in \mathbb{R}^n$ the vector demands. These processes are related by the following stochastic differential equation:

$$dX(t) = U(t)\,dt - dZ(t), \qquad X(0) = x_0 \in \mathbb{R}^n$$

The demand $Z(\cdot)$ is given by the following stochastic differential equation:

$$dZ(t) = \overline{Z}\,dt + \sqrt{\epsilon}\sigma\,dW(t) \qquad Z(0) = z_0$$

where $\overline{Z}$ is a vector of unknown constants, $\epsilon > 0$ is a small parameter, σ is a given $n \times n$ matrix, and $[W(t), t \geq 0]$ is a standard $\mathbb{R}^n$-valued Wiener process defined on a complete probability space $(\Omega, \mathscr{F}, P)$.

The manufacturing system under consideration consists of machines subject to breakdown and repair. Let $\mathscr{M} = \{0, 1, \ldots, m\}$ denote the set of machine total capacity states, and let the process $[\alpha(t), t \geq 0]$ where $\alpha(t) \in \mathscr{M}$ denotes the total capacity process for the manufacturing system. Because only a finite amount of production capacity is available at any given time t, an upper bound is imposed on the production rate $U(t)$. For example, in the one-dimensional case ($n = 1$), the production constraint is $0 \leq U(t) \leq \alpha(t)$.

The cost function J is defined by

$$J[U(\cdot)] = E\int_0^\infty e^{-\rho t} G[X(t), U(t)]\,dt$$

where G is the running cost of inventory/backlog and production and $\rho > 0$ is the discount rate. The problem is to find an admissible control $U(\cdot)$ that minimizes $J[U(\cdot)]$.

A noise that appeared in the above example is described by a Wiener process. A Wiener process plays a very important and exceptional role in stochastic control theory. It is useful for modeling disturbances which corrupt transmitted information and also for modeling controlled system disturbances. A Wiener process also occurs in the Kalman–Bucy filter (33). It became as a model of Brownian motion, very popular in physics since Einstein's and Smoluchowski's work, but it is not well known that it appeared for the first time in the scientific literature as a model of a price process in Bachelier's Ph.D. dissertation in 1900.

PRELIMINARY RESULTS

We shall assume that $(\Omega, \mathscr{F}, P)$ is a probability space and that $\{\mathscr{F}_t, t \geq 0\}$ is a nondecreasing, right-continuous family of sub-σ-algebras of $\mathscr{F}$. We assume that the filtration $(\mathscr{F}_t)$ is complete with respect to P, that is, each $\mathscr{F}_t$ contains all the P-null sets of $\mathscr{F}$.

Martingales as Important Tools in Proofs (8,9)

A stochastic process is a parametrized collection of random variables. We consider collections (and so processes) for which the parameter t takes values in positive real numbers. It is common to call this kind of process a continuous-time stochastic process. Let us start by introducing the concept of martingale. The intuitive idea of martingale is related to the description of a fair game: we are in fact in a situation in which the fortune expected after a certain time is exactly equal to what we have. Then we define this *fair game*.

Definition 1 A stochastic process $(X(t), t \geq 0)$ is a *martingale* with respect to the increasing family of σ-algebras $(\mathscr{F}_t, t \geq 0)$ if

(1) $X(t)$ is $\mathscr{F}_t$ measurable;

(2) $E\{|X(t)|\} < \infty$ for every $t \geq 0$;

(3) if $s < t$, then

$$E[X(t)|\mathscr{F}_s] = X(s) \quad \text{a.s.} \tag{1}$$

Let us define now the *unfair games* in which we expect to lose (supermartingale) and to win (submartingale)

Definition 2 A stochastic process $(T(t), t \geq 0)$ is a *supermartingale* with respect to the increasing family of σ-algebras $(\mathscr{F}_t, t \geq 0)$ if it satisfies properties (1) and (2) of Definition 1 and

(3)′ if $s < t$, then

$$E[X(t)|\mathscr{F}_s] \leq X(s) \quad \text{a.s.} \tag{2}$$

$(X(t), t \geq 0)$ is a *submartingale* with respect to the increasing family of σ-algebras $(\mathscr{F}_t, t \geq 0)$ if it satisfies properties (1) and (2) of Definition 1 and

(3)″ if $s < t$, then

$$E[X(t)|\mathscr{F}_s] \geq X(s) \quad \text{a.s.} \tag{3}$$

where a.s. stands for almost surely.

We are interested in the martingale theory because of its strong connection with stochastic calculus and because of the properties that the processes defined previously have. The first property that we illustrate is the Doob–Kolmogorov inequality that tells how (and when) one can get information about a family of ordered, random variables based on the knowledge of the last one:

Theorem 3. Let $(Y(t), \mathscr{F}_t, t \in [0, T])$ be a submartingale (with right continuous sample paths). Then

$$P\left[\sup_{t\in[0,T]} Y(t) \geq x\right] \leq \frac{1}{x}E[Y^+(T)] \tag{4}$$

where $x > 0$ and $Y^+(T) = \max(Y(T), 0)$.

One of the consequences of this result is the martingale convergence theorem which is commonly used in proving convergence results for stochastic adaptive control problems.

Theorem 4. Let $(Y(t), \mathscr{F}_t, t \in [0, T])$ be a nonnegative submartingale. If $\sup_{t\geq 0} E[Y(t)] = \gamma < \infty$, then

$$\lim_{t\to\infty} Y(t) = Y_\infty \quad \text{a.s.} \tag{5}$$

and $E[Y_\infty] \leq \gamma$.

The Wiener Process Used for Modeling the Noise

Definition 5. The stochastic process $(W(t), t \geq 0)$ is a *Brownian motion* (or a *Wiener process*) if

(1) $W(0) = 0$;

(2) for $t \geq 0$, $W(t + s) - W(t)$ is a random variable that is normally distributed with mean zero and variance $\sigma^2 s$ where σ^2 is a constant; and

(3) the process has independent increments (i.e., if $0 \leq t_1 < t_2 < \cdots < t_n$, then $W(t_2) - W(t_1)$, $W(t_3) - W(t_2), \ldots, W(t_n) - W(t_{n-1})$ are mutually independent random variables).

If $\sigma^2 = 1$, then the process is called *standard Brownian motion.*

The following result provides important information about the quadratic variation of the Wiener process:

Theorem 6. Consider a family of partitions $\{P_n\}_{n=1}^{\infty}$ of $[0, T]$ where $P_n = \{t_i^n | 0 \leqslant i \leqslant n,\ 0 = t_0^n < t_1^n < \cdot \cdot \cdot < t_n^n = T\}$ with $P_n \subset P_{n+1}$ for all n and such that $\{P_n\}_{n=1}^{\infty}$ is dense in $[0, T]$ (i.e., $\lim_{n\to\infty} [\sup_i (t_{i+1}^n - t_i^n)] = 0$). Let $W(t)$, $t \geqslant 0]$ be a standard Brownian motion process. Then

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} [W(t_{i+1}^n) - W(t_i^n)]^2 = T \quad \text{a.s.} \tag{6}$$

and

$$\lim_{n\to\infty} E \left\{ \sum_{i=0}^{n-1} [W(t_{i+1}^n) - W(t_i^n)]^2 - T \right\}^2 = 0 \tag{7}$$

that is $\Sigma_{i=0}^{n-1}$ $(W(t_{i+1}^n) - (W(t_i^n))^2$ converges to t both almost surely and in L^2.

It follows from this result that almost every sample path of a Brownian motion process has infinite arc length on every nonempty closed interval. Moreover it follows that there is not a closed interval in which Brownian motion is differentiable (otherwise by the mean value theorem we would have convergence to zero). Much more than this is true:

Theorem 7. Almost all sample paths of the Wiener process are nowhere differentiable.

Let us give another important result about the sample path behavior of the Wiener process:

Theorem 8. The strong law of large numbers for Brownian motion:

$$\lim_{t\to\infty} \frac{W(t)}{t} = 0 \quad \text{a.s.} \tag{8}$$

If $(W(t), t \geqslant 0)$ is a Brownian motion process with continuous sample paths and we let $\mathscr{F}_t$ be the σ-algebras generated by $[W(s), s \geqslant 0]$, it follows from the independence of increments property of Brownian motion that $\{W(t), \mathscr{F}_t t \in [0, T]\}$ is a square integrable martingale with continuous sample paths. The converse of this statement is true:

Theorem 9. If $\{W(t), \mathscr{F}_t, t \in [0, T]\}$ is a square integrable martingale with continuous sample paths such that for $t > s$

$$E\{[W(t) - W(s)]^2 \,|\, \mathscr{F}_s\} = t - s \tag{9}$$

then $\{W(t), t \in [0, T]\}$ is a Brownian motion process.

Stochastic Integrals

In this section we define the stochastic integral

$$\int_0^t \phi(s)\, dW(s) \tag{10}$$

The unbounded variation property of Brownian motion sample paths described in the previous section does not allow us to use the definition of the Riemann–Stieljes integral. Nevertheless by proceeding similarly we can define stochastic integration (in Itô's sense) satisfactorily. We start by defining the integral for simple processes:

Definition 10 Let $\mathscr{F}_t$ be the σ-algebra generated by $[W(s), s \leqslant t]$. We call $\{\phi(t), t \in [0, T]\}$ a simple process, if there is a partition $0 = t_0 < t_1 < \cdot \cdot \cdot < t_n = T$ such that

$$\phi(t) = \varphi_j \qquad t \in [t_j, t_{j+1}) \tag{11}$$

and φ_j is measurable with respect to $\mathscr{F}_{t_j}$.

Define the stochastic integral for simple processes when $t \in [t_j, t_{j+1})$ as

$$\int_0^t \phi(s)\, dW(s) = \sum_{i=0}^{j-1} \varphi_i [W(t_{i+1}) - W(t_j)] + \varphi_j [W(t) - W(t_j)] \tag{12}$$

Note that the integral is linear and continuous (almost surely) in t. We want to preserve these properties and at the same time we want to extend the class of functions that we can integrate. To do this, we need the following result:

Proposition 11. Let $\{\phi(t), t \in [0, T]\}$ be a process adapted to $\mathscr{F}_t, t \in [0, T]\}$ such that

$$\int_0^T \phi^2(s)\, ds < \infty \quad \text{a.s.} \tag{13}$$

Then there is a sequence of simple processes $\{{}^n\phi(t), t \in [0, T]\}$ such that for every $\epsilon > 0$

$$\lim_{n\to\infty} P \left[\int_0^T |{}^n\phi(t) - \phi(t)|^2\, dt > \epsilon \right] = 0 \tag{14}$$

Now if we define Y_n as the integral of the simple process ${}^n\phi$, that is,

$$Y_n(t) = \int_0^t {}^n\phi(s)\, dW(s) \tag{15}$$

then using Proposition 11, the family Y_n converges to a process Y uniformly in probability. Note that the uniform convergence and the continuity of sample paths for Y_n imply the continuity of the sample paths of the process Y. Now we can define the stochastic integral as follows:

$$\int_0^t \phi(s)\, dW(s) = Y(t) \tag{16}$$

The construction does not depend on the sequence of simple processes used, and the integral is unique up to indistinguishability, that is, if

$$\int_0^t \phi(s)\, dW(s) = Y(t) \tag{17}$$

and

$$\int_0^t \phi(s)\, dW(s) = Z(t) \tag{18}$$

then $P\{Y(t) = Z(t)$ for all $t \in [0, T]\} = 1$. The following result is the strong law of large numbers for stochastic integrals (22).

Theorem 12. Let $(W(t), t \geqslant 0)$ be a standard Wiener process, and let $\mathscr{F}_t$ be the σ-algebra generated by $(W(s), s \leqslant t)$. Let $(f(t), t \geqslant 0)$ be a random process adapted to $(\mathscr{F}_t, t \geqslant 0)$ such that for $0 < T < \infty$

$$\int_0^T f^2(t)\,dt < \infty \quad \text{a.s.} \tag{19}$$

and

$$\int_0^\infty f^2(t)\,dt = \infty \quad \text{a.s.} \tag{20}$$

Then

$$\lim_{T\to\infty} \frac{\displaystyle\int_0^T f(t)\,dW(t)}{\displaystyle\int_0^T f^2(t)\,dt} = 0 \quad \text{a.s.} \tag{21}$$

An interesting feature of the stochastic integral is that it does not satisfy the ordinary calculus rules of integration. A standard example of this situation is the fact that

$$\int_0^t W(s)\,dW(s) = \frac{W^2(t)}{2} - \frac{t}{2} \tag{22}$$

This is a consequence of the fact that Brownian motion has nonzero quadratic variation (Theorem 6). We see in the next section the differential rule that is to be used in stochastic calculus.

Stochastic Differential Equations (7)

In the previous section we introduced the concept of stochastic integration. Now we can use it to define stochastic differential equations. In fact we say that $\{X(t), t \in [0, T]\}$ is a solution for the stochastic differential equation

$$dX(t) = a[t, X(t)]\,dt + \sigma[t, X(t)]\,dW(t), \qquad X(0) = X_0 \tag{23}$$

if for every $t \in [0, T]$ it satisfies the following equation

$$X(t) = X_0 + \int_0^t a[u, X(u)]\,du + \int_0^t \sigma[u, X(u)]\,dW(u) \tag{24}$$

We have the following result about the existence and uniqueness of the solution:

Theorem 13. If $a(t, x)$ and $\sigma(t, x)$ are continuous in t and satisfy the Lipschitz condition in x uniformly in t, that is, for every $t \in [0, T]$

$$|a(t,x) - a(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leqslant K|x-y|, \qquad K \in \mathbb{R} \tag{25}$$

then the solution to the stochastic differential Eq. (23) exists and it is unique (up to indistinguishability).

We mentioned previously that we shall present the differential rule that we must use in stochastic calculus. This rule was introduced by Itô.

Theorem 14. (Itô's Lemma). For $i = 1, 2, \ldots, k$ let $(f_i(t), t \in [0, T])$ and $(g_i(t), t \in [0, T])$ be two processes adapted to $(\mathscr{F}_t, t \in [0 < t])$ such that

$$\int_0^T f^2(s)\,ds < \infty \quad \text{a.s.} \tag{26}$$

$$\int_0^T g^2(s)\,ds < \infty \quad \text{a.s.} \tag{27}$$

and let $F: \mathbb{R}^{k+1} \to \mathbb{R}$ be twice continuously differentiable in the first k variables and continuously differentiable in the last variable. Let

$$dX_i(t) = g_j(t)\,dt + f_i(t)\,dW(t) \tag{28}$$

and

$$Y(t) = F[X_1(t), \ldots, X_k(t), t] \tag{29}$$

Then

$$dY(t) = \sum_{i=1}^k F_i\,dX_i(t) + \left[\frac{1}{2}\sum_{i,j=1}^k F_{ij} f_i(t) f_j(t) + F_{k+1}\right] dt \tag{30}$$

where

$$F_i = \frac{\partial F}{\partial X_i}[X_1(t), \ldots, X_k(t), t]$$

$$F_{ij} = \frac{\partial^2 F}{\partial X_i \partial X_j}[X_1(t), \ldots, X_k(t), t]$$

and

$$F_{k+1} = \frac{\partial F}{\partial t}[X_1(t), \ldots, X_k(t), t]$$

$i, j = 1, \ldots, k$.

THE FILTERING PROBLEM (7)

Example 1 previously given is a special case of the following general filtering problem:

Suppose that the state $X(t) \in \mathbb{R}^n$ of a system at time t is given by a stochastic differential Eq. (23).

We assume that the observations $H(t) \in \mathbb{R}^m$ are performed continuously and are of the form

$$H(t) = c[t, X(t)] + \gamma[t, X(t)]\dot{W}(t) \tag{31}$$

where $c: \mathbb{R}^{n+1} \to \mathbb{R}^m$, $\gamma: \mathbb{R}^{n+1} \to \mathbb{R}^{m+r}$ are functions satisfying Eq. (25) and $\dot{W}(t)$ denotes r-dimensional white noise. If we introduce

$$Z(t) = \int_0^t H(s)\,ds \tag{32}$$

then

$$dZ = c[t, X(t)]\,dt + \gamma[t, X(t)]\,dW(t)$$
$$Z(0) = 0 \tag{33}$$

where $W(t)$ is an r-dimensional Wiener process.

Note that if $H(s)$ is also known for $0 \leqslant s \leqslant t$, then $Z(s)$ is also known for $0 \leqslant s \leqslant t$ and vice versa. So no information is lost or gained by considering $Z(t)$ as our observations instead of $H(t)$, but this allows us to obtain a well-defined mathematical model of the situation.

The filtering problem is the following:

Given the observations $Z(s)$ satisfying Eq. (33) for $0 \leqslant s \leqslant t$, what is the best estimate $\hat{X}(t)$ of the state $X(t)$ of the system Eq. (23) based on these observations?

By saying that the estimate $\hat{X}(t)$ is based on the observations $\{Z(s): s \leqslant t\}$, we mean that

$$\hat{X}(t) \text{ is } \mathscr{F}_t\text{-measurable,}$$
$$\text{where } \mathscr{F}_t \text{ is the } \sigma\text{-algebra generated by } \{Z(s),\ s \leqslant t\} \tag{34}$$

By saying that $\hat{X}(t)$ is the best such estimate, we mean that

$$\int_\Omega |X(t) - \hat{X}(t)|^2\,dP = E[|X(t) - \hat{X}(t)|^2]$$
$$= \inf\{E[|X(t) = Y|^2] : Y \in \mathscr{Y}\} \tag{35}$$

where

$$\mathscr{Y} := \{Y : \Omega \to \mathbb{R}^n;\ Y \in L^2(P) \text{ and } Y \text{ is } \mathscr{F}_t\text{-measurable}\}, \tag{36}$$

where $L^2(P) = L^2(\Omega, P)$ and $(\Omega, \mathscr{F}, P)$ is a probability space.

One of the most important results is the following:

$$\hat{X}(t) = E[X(t)|\mathscr{F}_t]$$

This is the basis for the general Fujisaki–Kallianpur-Kunita equation of filtering theory. See, for example, Ref. 34.

Here we concentrate on the linear case, which allows an explicit solution in terms of a stochastic differential equation for $\hat{X}(t)$ (the Kalman–Bucy filter):

In the linear filtering problem the system and observation equations have the following form:

$$\text{(linear system)} \quad dX(t) = F(t)X(t)\,dt + C(t)\,dW(t);$$
$$F(t) \in \mathbb{R}^{n\times n},\ C(t) \in \mathbb{R}^{n\times p} \tag{37}$$

$$\text{(linear observations)} \quad dX(t) = G(t)X(t)\,dt + D(t)\,dV(t);$$
$$G(t) \in \mathbb{R}^{m\times n},\ D(t) \in \mathbb{R}^{m\times r} \tag{38}$$

Theorem 15. The One-Dimensional Kalman–Bucy Filter. The solution $\hat{X}(t) = E[X(t)|\mathscr{F}_t]$ of the one-dimensional linear filtering problem

$$\text{(linear system)} \quad dX(t) = F(t)X(t)\,dt + C(t)\,dW(t);$$
$$F(t), C(t) \in \mathbb{R}$$

$$\text{(linear observations)} \quad dZ(t) = G(t)X(t)\,dt + D(t)\,dV(t);$$
$$G(t), D(t) \in \mathbb{R}$$

satisfies the stochastic differential equation

$$d\hat{X}(t) = \left[F(t) - \frac{G^2(t)S(t)}{D^2(t)}\right]\hat{X}(t)\,dt + \frac{G(t)S(t)}{D^2(t)}\,dZ(t)$$
$$\hat{X}_0 = E[X_0] \tag{39}$$

where

$$S(t) = E\{[X(t) - \hat{X}(t)]^2\}$$

satisfies the (deterministic) Riccati equation

$$\frac{dS}{dt} = 2F(t)S(t) - \frac{G^2(t)}{D^2(t)}S^2(t) + C^2(t),$$
$$S(0) = E\{(X_0 - E[X_0])^2\} \tag{40}$$

Example 5. Estimation of a parameter. Suppose we want to estimate the value of a (constant) parameter α based on observations $Z(t)$ satisfying the model

$$dZ(t) = \alpha X(t)\,dt + \sigma(t)\,dW(t)$$

where $X(t)$, $\sigma(t)$ are known functions. In this case the stochastic differential equation for α is of course

$$d\alpha = 0$$

so the Riccati equation for $S(t) = E\{[\alpha - \hat{\alpha}(t)]^2\}$ is

$$\frac{dS}{t} = -\left[\frac{X(t)S(t)}{\sigma(t)}\right]^2$$

which gives

$$S(t) = \left[S_0^{-1} + \int_0^t X(s)^2\sigma(s)^{-2}\,ds\right]^{-1}$$

and the Kalman–Bucy filter is given by

$$d\hat{\alpha}(t) = \frac{X(t)S(t)}{\sigma(t)^2}[dZ(t) = X(t)\hat{\alpha}(t)]$$

This can be written as

$$S_0^{-1} + \int_0^t X(s)^2\sigma(t)^{-2}\,d\hat{\alpha}(t) + X(t)^2\sigma(t)^{-2}\hat{\alpha}\,dt$$
$$= X(t)\sigma(t)^{-2}\,dZ(t)$$

We recognize the left-hand side as

$$d\left\{\left[S_0^{-1} + \int_0^t X(s)^2\sigma(t)^{-2}\,ds\right]\hat{\alpha}(t)\right\}$$

so

$$\hat{\alpha} = \frac{\hat{\alpha}_0 S_0^{-1} + \int_0^t X(s)\sigma(s)^{-2}\,dZ(s)}{S_0^{-1} + \int_0^t X(s)^2\sigma(t)^{-2}\,ds}$$

This estimate coincides with the maximum likelihood estimate in the classical estimation theory if $S_0^{-1} = 0$. See (22).

For more information about estimates of drift parameters in diffusion, see the next section.

Theorem 16. The Multidimensional Kalman–Bucy Filter. The solution

$$\hat{X}(t) = E[X(t)|\mathscr{F}_t]$$

of the multidimensional linear filtering problem

$$(\textit{linear system})\qquad dX(t) = F(t)X(t)\,dt + C(t)\,dW(t);\quad F(t)\in\mathbb{R}^{n\times n},\ C(t)\in\mathbb{R}^{n\times p}$$

$$(\textit{linear observations})\ dZ(t) = G(t)X(t)\,dt + D(t)\,dV(t);\quad G(t)\in\mathbb{R}^{m\times n},\ D(t)\in\mathbb{R}^{m\times r}$$

satisfies the stochastic differential equation

$$d\hat{X}(t) = [F - SG^T(DD^T)^{-1}G]\hat{X}(t)\,dt + SG^T(DD^T)^{-1}dZ(t):$$
$$\hat{X}(0) = E[X_0]$$

where

$$S(t) := E\{[X(t)-\hat{X}(t)][X(t)-\hat{X}(t)]^T\} \in \mathbb{R}^{n\times n}$$

satisfies the matrix Riccati equation

$$\frac{dS}{dt} = FS + SF^T - DG^T(DD^T)^{-1}GS + CC^T$$
$$S(0) = E(\{X(0)-E[X(0)]\}\{X(0)-E[X(0)]\}^T)$$

The condition on $D(t)\in\mathbb{R}^{m\times r}$ is now that $D(t)D(t)^T$ is invertible for all t and that $[D(t)D(t)^T]^{-1}$ is bounded on every bounded t-interval.

A similar solution can be found for the more general situation

$$\begin{aligned}(\text{system})\qquad dX(t) &= [F_0(t)+F_1(t)X(t)+F_2(t)Z(t)]\,dt + C(t)\,dW(t)\\ (\text{observations})\ dZ(t) &= [G_0(t)+G_1(t)X(t)+G_2(t)Z(t)]\,dt + D(t)\,dV(t)\end{aligned}$$

where $X(t)\in\mathbb{R}^n$, $Z(t)\in\mathbb{R}^m$ and $B(t) = [W(t), V(t)]$ is $(n+m)$-dimensional Brownian motion with appropriate dimensions on the matrix coefficients. See (34), which also treats the nonlinear case.

For various applications of filtering theory see Ref. 33.

STOCHASTIC CONTROL (7,10,11,36)

Let $(X(t),\ t\geqslant 0)$ be the solution of the stochastic differential Eq. (23) with $X(0)=x$. Let f be a bounded, continuous function, and let T be the operator defined by

$$(T_t f)(x) = E_x\{f[t, X(t)]\} \tag{41}$$

This operator satisfies the semigroup property

$$T_t(T_s f)(x) = (T_{t+s}f)(x) \tag{42}$$

If we *differentiate* this operator,

$$\lim_{h\downarrow 0}\frac{T_{t+h}-T_t}{h} = \mathscr{A}_t \tag{43}$$

where $\mathscr{A}_t$ is an operator that acts on f in the following way:

$$\mathscr{A}_t f(x) = a(t,x)\frac{\partial f}{\partial x}(x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}(x) \tag{44}$$

$(\mathscr{A}_t,\ t\geqslant 0)$ is called the *infinitesimal generator of the semigroup* $(T_t,\ t\geqslant 0)$. Consider the problem of controlling the solution of the following stochastic differential equation:

$$\begin{aligned}dX(t) &= a[t, X(t), U(t)]\,dt + \sigma[t, X(t), U(t)]\,dB(t)\\ X(0) &= x\end{aligned} \tag{45}$$

to minimize the objective function (also called the cost functional)

$$J(x,U) = E_x\left\{\int_0^T L[s, X(s), U(s)]\,ds + \psi[X(T)]\right\} \tag{46}$$

where the admissible choices of the controls are all of the smooth functions of the observations of $X(t)$. We define the Hamilton–Jacobi–Bellman equation of dynamic programming as

$$\frac{\partial W}{\partial s} + \min\{\mathscr{A}_s^U W + L(s,X,U)\} = 0 \tag{47}$$

where $W(T,x) = E_{s,x}\psi[X(T)]$ is the value function and $(\mathscr{A}_t,\ t\geqslant 0)$ is the infinitesimal generator of the semigroup associated with the stochastic differential Eq. (45). The following result is known as the verification theorem

Theorem 17. Let $W(s,y)$ be a solution of Eq. (47) such that $W\in\mathscr{C}^2(Q)$ and it is continuous on the closure of Q, (we say W is a smooth function). Then

(1) $W(s,y)\leqslant J(s, y, U)$ for any admissible control U and any initial condition $(s,y)\in Q$; and

(2) if U^* is an admissible control such that

$$\mathscr{A}_s^{U^*}W + L(s,y,U^*) = \min_{v\in\mathscr{U}}\{\mathscr{A}_s^U W + L(s,y,v)\} \tag{48}$$

for all $(s,y)\in Q$, then

$$W(s,y) = J(s,y,U^*) \tag{49}$$

Remark. U^* is an optimal control.

The Linear Stochastic Gaussian Problem (7,11)

Suppose that the state $X(t)$ of the system at time t is given by the following linear stochastic differential equation:

$$\begin{aligned}dX(t) &= [H(t)X(t) + M(t)U(t)]\,dt + \sigma(t)\,dB(t)\\ &t\geqslant s;\ X(s) = x\end{aligned} \tag{50}$$

and the cost has the form

$$J^u(s,x) = E^{s,x}\left\{\int_s^{t_1}[X(t)^TC(t)X(t) + U(t)^TD(t)U(t)]\,dt + X(t_1)^TRX(t_1)\right\}\quad s\leqslant t_1 \tag{51}$$

where all the coefficients $H(t)\in\mathbb{R}^{n\times n}$, $M(t)\in\mathbb{R}^{n\times k}$, $\sigma(t)\in\mathbb{R}^{n\times m}$, $C(t)\in\mathbb{R}^{n\times n}$, $D(t)\in\mathbb{R}^{k\times k}$, and $R\in\mathbb{R}^{n\times n}$, are t-continuous and deterministic. We assume that $C(t)$ and R are symmetric, nonnegative-definite and $D(t)$ is symmetric, positive-definite, for all t. We also assume that t_1 is a deterministic time. $[B(t),\ t\geqslant 0]$ is a standard Brownian motion.

Then the problem is to choose the control $u = u[t, X(t)]$ so that it minimizes $J^u(s,x)$. We may interpret this as follows: The aim is to find a control u which makes $|X(t)|$ ($|\ |$ denotes the norm) small quickly and so that the energy used ($\sim u^T Du$) is small. The sizes of $C(t)$ reflects the cost of large values of $|X(t)|$, whereas the size of $D(t)$ reflects the cost (energy) of applying large values of $|u(t)|$.

In this case the HJB equation for $W(s, x)$ becomes

$$0 = \inf_v \{F^v(s,x) + (L^v W)(s,x)\}$$
$$= \frac{\partial W}{\partial s} + \inf_v \left\{ x^T C(t)x + v^T D(t)v + \sum_{i=1}^{n} (H(t)x + M(t)v)i \frac{\partial W}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} (\sigma\sigma^T)_{ij} \frac{\partial^2 W}{\partial x_i \partial x_j} \right\} \qquad \text{for } s < t_1 \tag{52}$$

and

$$W(t_1, x) = x^T R x \tag{53}$$

We find a solution W of Eqs. (52)–(53) of the form

$$W(t,x) = x^T S(t)x + a(t) \tag{54}$$

where $S(t) = S_t \in \mathbb{R}^{n\times n}$ is symmetric, nonnegative-definite, $a(t) \in \mathbb{R}$ are both $a(t)$, and $S(t)$ are continuously differentiable w.r.t. t (and deterministic).

The following

$$u^*(t,x) = -D(t)^{-1}M(t)^T S(t)x, \qquad t < t_1 \tag{55}$$

is an optimal control, and the minimum cost is given by

$$W(s,x) = x^T S(s)x + \int_s^{t_1} \operatorname{tr}(\sigma\sigma^T S)_t\, dt, \qquad s < t_1 \tag{56}$$

where $S(t)$ satisfies the corresponding Riccati type equation from linear filtering theory. This formula shows that the extra cost due to the noise in the system is given by

$$a(s) = \int_s^{t_1} \operatorname{tr}(\sigma\sigma^T S)_t\, dt$$

The Separation Principle

The separation principle (5,18) states that if we only have partial knowledge of the state $X(t)$ of the system, that is, if we have only noisy observations

$$dZ(t) = g(t)X(t)\,dt + \gamma(t)\,d\tilde{B}(t) \tag{57}$$

at our disposal, then the optimal control $u^*(t, \omega)$ (required to be $\mathscr{F}_t$-adapted, where $\mathscr{F}_t$ is the σ-algebra generated by $\{Z(r) : r \leqslant t\}$), is given by

$$u^*(t,\omega) = -D(t)^{-1}M(t)^T S(t)\hat{X}(t)(\omega) \tag{58}$$

where $\hat{X}(t)$ is the filtering estimate of $X(t)$ based on the observations $\{Z(r) : r \leqslant t\}$, given by the Kalman–Bucy filter. Comparing with Eq. (55), we see that the stochastic control problem in this case splits into a linear filtering problem and a deterministic control problem.

LEAST SQUARES AND CONTINUOUS-TIME STOCHASTIC ADAPTIVE CONTROL (2)

The general approach to adaptive control described here exhibits a splitting or separation of the problems of identification of the unknown parameters and adaptive control. Maximum likelihood (or equivalently least squares) estimates are used to identify unknown constant parameters. These estimates are given recursively and are strongly consistent. The adaptive control is usually constructed by the so-called certainty equivalence principle, that is, the optimal stationary controls are computed by replacing the unknown true parameter values by the current estimates of these values. Because the optimal stationary controls are continuous functions of the unknown parameters, the self-tuning property is verified. The family of average costs using the control from the certainty equivalence principle converges to the optimal average cost. This verifies the self-optimizing property.

A model for the adaptive control of continuous-time linear stochastic systems with complete observations of the state is described by the following stochastic differential equation:

$$dX(t) = \left[A(\alpha)X(t) + BU(t)\right]dt + dW(t) \tag{59}$$

where $X(t) \in \mathbb{R}^n$, $U(t) \in \mathbb{R}^m$.

$$A(\alpha) = A_0 + \sum_{i=1}^{p} \alpha^i A_i \tag{60}$$

$A_i \in \mathscr{L}(\mathbb{R}^n)$ $i = 0, \ldots, p$, $B \in \mathscr{L}(\mathbb{R}^m, \mathbb{R}^n)$, $(W(t), t \in \mathbb{R}_+)$ is a standard $\mathbb{R}^n$-valued Wiener process and $X_0 \equiv a \in \mathbb{R}^n$. It is assumed that

(5.A1) $\mathscr{A} \subset \mathbb{R}^p$ is compact and $\alpha \in \mathscr{A}$.

(5.A2) $(A(\alpha), B)$ is reachable for each $\alpha \in \mathscr{A}$.

(5.A3) The family $(A_i,\ i = 1, \ldots, p)$ is linearly independent.

Let $(\mathscr{F}_t,\ t \in \mathbb{R}_+)$ be a filtration such that X_t is measurable with respect to $\mathscr{F}_t$ for all $t \in \mathbb{R}_+$ and $(W(t), \mathscr{F}_t, t \in \mathbb{R}_+)$ is a Brownian martingale. The ergodic, quadratic control problem for Eq. (59) is to minimize the ergodic cost functional

$$\limsup_{t\to\infty} \frac{1}{t} J(X_0, U, \alpha, t) \tag{61}$$

where

$$J(X_0, U, \alpha, t) = \int_0^t \left[\langle QX(s), X(s)\rangle + \langle PU(s), U(s)\rangle\right] ds \tag{62}$$

and $t \in (0, \infty)$, $X(0) = X_0$, $Q \in \mathscr{L}(\mathbb{R}^n)$; $P \in \mathscr{L}(\mathbb{R}^m)$ are self-adjoint and P^{-1} exists; $[X(t), t \in \mathbb{R}_+]$ satisfies Eq. (59); and $[U(t), t \in \mathbb{R}_+]$ is adapted to $(\mathscr{F}_t, t \in \mathbb{R}_+)$. It is well known (33) that, if α is known, then there is an optimal linear feedback control such that

$$U^*(t) = KX(t) \tag{63}$$

where $K = -P^{-1}B^*V$ and V is the unique, symmetric, nonnegative-definite solution of the algebraic Riccati equation

$$VA + A^*V - VB^*P^{-1}BV + Q = 0 \tag{64}$$

For an unknown α the admissible adaptive control policies $[U(t), t \in \mathbb{R}_+]$ are linear feedback controls

$$U(t) = K(t)X(t) = \tilde{K}\left[t, X(u),\ u \leqslant t - \Delta\right]X(t) \tag{65}$$

where $[K(t), t \geqslant 0]$ is an $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$-valued process that is uniformly bounded and there is a fixed $\Delta > 0$ such that $(K(t), t \geqslant 0)$ is measurable with respect to $\sigma(X_u, u \leqslant t - \Delta)$ for each $t \geqslant \Delta$ and $[K(t), t \in [0, \Delta)]$ is a deterministic function. For such an adaptive control, it is elementary to verify that there is a unique strong solution of Eq. (59). The delay $\Delta > 0$ accounts for some time required to compute the adaptive control law from the observation of the solution of Eq. (59).

Let $(U(t), t \geqslant 0)$ be an admissible adaptive control and let $[X(t), t \geqslant 0]$ be the associated solution of Eq. (59). Let $\mathscr{A}(t) = [a_{ij}(t)]$ and $\tilde{\mathscr{A}}(t) = [\tilde{a}_{ij}(t)]$ be $\mathscr{L}(\mathbb{R}^p)$-valued processes such that

$$a_{ij}(t) = \int_0^t \langle A_i X(s), A_j X(s)\rangle\, ds$$
$$\tilde{a}_{ij}(t) = \frac{a_{ij}(t)}{a_{ii}(t)}$$

To verify the strong consistency of a family of least squares estimates, it is assumed that

$$\liminf_{t\to\infty} |\det \tilde{\mathscr{A}}(t)| > 0 \quad \text{a.s.}$$

The estimate of the unknown parameter vector at time t, $\hat{\alpha}(t)$, for $t > 0$ is the minimizer for the quadratic functional of α, $L(t, \alpha)$, given by

$$L(t,\alpha) = -\int_0^t \langle [A(\alpha) + BK(s)]X(s), dX(s)\rangle + \frac{1}{2}\int_0^t \left|[A(\alpha) + BK(s)]X(s)\right|^2 ds \tag{66}$$

where $U(s) = K(s)X(s)$ is an admissible adaptive control. The following result (31) gives the strong consistency of these least squares estimators.

Theorem 18. Let $(K(t), t \geqslant 0)$ be an admissible adaptive feedback control law. If (5.A1–5.A4) are satisfied and $\alpha_0 \in \mathscr{A}^\circ$, the interior of $\mathscr{A}$, then the family of least squares estimates $[\hat{\alpha}(t), t > 0]$ where $\hat{\alpha}(t)$ is the minimizer of Eq. (66), is strongly consistent, that is,

$$P_{\alpha_0}\left[\lim_{t\to\infty} \hat{\alpha}(t) = \alpha_0\right] = 1 \tag{67}$$

where α_0 is the true parameter vector.

The family of estimates $[\hat{\alpha}(t), t > 0]$ can be computed recursively because this process satisfies the following equation

$$d\hat{\alpha}(t) = \mathscr{A}^{-1}(t)\langle \mathbb{A}(t)X(t),\ dX(t) - A[\hat{\alpha}(t)]X(t)\,dt - BU(t)\,dt\rangle \tag{68}$$

where $\langle \mathbb{A}(t)x, y\rangle = (\langle A_i x, y\rangle)\ i = 1, \ldots, p$.

Now the performance of some admissible adaptive controls is described.

Proposition 19. Assume that (5.A1–5.A4) are satisfied and that

$$\lim_{t\to\infty} \frac{1}{t}\langle VX(t), X(t)\rangle = 0 \quad \text{a.s.} \tag{69}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t |X(t)|^2\, ds < \infty \quad \text{a.s.} \tag{70}$$

where $(X(t), t \geqslant 0)$ is the solution of Eq. (59) with the admissible adaptive control $[U(t), t \geqslant 0]$ and $\alpha = \alpha_0 \in \mathscr{K}$ and V is the solution of the algebraic Riccati Eq. (64) with $\alpha = \alpha_0$. Then

$$\liminf_{T\to\infty} \frac{1}{T} J(X_0, U, \alpha_0, T) \geqslant \operatorname{tr} V \quad \text{a.s.} \tag{71}$$

If U is an admissible adaptive control $U(t) = K(t)X(t)$ such that

$$\lim_{t\to\infty} K(t) = k_0 \quad \text{a.s.} \tag{72}$$

where $k_0 = -P^{-1}B^*V$, then

$$\lim_{T\to\infty} \frac{1}{T} J(X_0, U, \alpha_0, T) = \operatorname{tr} V \quad \text{a.s.} \tag{73}$$

Corollary. Under the assumptions of the Proposition, if Eq. (72) is satisfied, then Eq. (69), and (70) are satisfied.

The previous results can be combined for a complete solution to the stochastic adaptive control problem Eqs. (59, 61).

Theorem 20. Assume that (5.A1–5.A4) are satisfied. Let $[\hat{\alpha}(t), t > 0]$ be the family of least squares estimates where $\hat{\alpha}(t)$ is the minimizer of Eq. (66). Let $[K(t), t \geqslant 0]$ be an admissible adaptive control law such that

$$K(t) = -P^{-1}B^*V[\hat{\alpha}(t-\Delta)]$$

where $V(\alpha)$ is the solution of Eq. (64) for $\alpha \in \mathscr{A}$. Then the family of estimates $(\hat{\alpha}(t), t > 0)$ is strongly consistent.

$$\lim_{t\to\infty} K(t) = k_0 \quad \text{a.s.} \tag{74}$$

where $k_0 = -P^{-1}B^*V(\alpha_0)$, and

$$\lim_{T\to\infty} \frac{1}{T} J(X_0, U, \alpha_0, T) = \operatorname{tr} V \quad \text{a.s.} \tag{75}$$

WEIGHTED LEAST SQUARES AND CONTINUOUS-TIME ADAPTIVE LQG CONTROL (30)

Introduction

The linear, Gaussian control problem with an ergodic, quadratic cost functional is probably the most well-known ergodic control problem. Because the optimal control is easily computed and the existence of an invariant measure follows directly from the stability of the optimal system, it is a basic problem to solve for stochastic adaptive control. For discrete-time linear systems it has been studied extensively, especially for ARMAX models (see Refs. 6, 28, 37–40 for many references). Although this adaptive control problem has been less studied for continuous-time linear systems, it is nonetheless an important problem as a model for physical systems that naturally evolve in continuous time and as an approximation for discrete-time sampled systems when the sampling rate is high.

The adaptive control problem is solved using only the natural assumptions of controllability and observability. The weighted least squares scheme is used to obtain the convergence of the family of estimates (self convergence), and the scheme is modified by a random regularization to obtain the uniform controllability and observability of the family of estimates. A diminishing excitative white noise is used to obtain strong consistency. The excitation is sufficient to include the identification of unknown deterministic linear systems.

This approach eliminates some other assumptions previously used that are unnecessary for the control problem for a known system and are often difficult to verify. Furthermore, this approach eliminates the need for random switchings or resettings which often occur in previous work.

Weighted Least Squares Identification

Let $[X(t), t \geqslant 0]$ be the process satisfying the stochastic differential equation

$$dX(t) = AX(t)\,dt + BU(t)\,dt + D\,dW(t) \tag{76}$$

where $X(0) = X_0$, $X(t) \in \mathbb{R}^n$, $U(t) \in \mathbb{R}^m$, $[W(t), t \geqslant 0]$ is an $\mathbb{R}^p$-valued standard Wiener process, and $[U(t), t \geqslant 0]$ is a control from a family specified subsequently. The random variables are defined on a fixed, complete probability space $(\Omega, \mathscr{F}, P)$, and there is a filtration $(\mathscr{F}_t, t \geqslant 0)$ defined on this space that is specified subsequently. It is assumed that A and B are unknown.

The following assumption is made:

(A1) (A, B) is controllable.

To describe the identification problem in a standard form, let

$$\theta^{\mathbb{T}} = [A \quad B] \tag{77}$$

and

$$\varphi(t) = \begin{bmatrix} X(t) \\ U(t) \end{bmatrix} \tag{78}$$

so that Eq. (76) is rewritten as

$$dX(t) = \theta^{\mathbb{T}}\varphi(t)\,dt + D\,dW(t) \tag{79}$$

A family of weighted least squares (WLS) estimates $[\theta(t), t \geqslant 0]$ is given by

$$d\theta(t) = a(t)P(t)\varphi(t)[dX^{\mathbb{T}}(t) - \varphi^{\mathbb{T}}(t)\theta(t)\,dt] \tag{80}$$

$$dP(t) = -a(t)P(t)\varphi(t)\varphi^{\mathbb{T}}(t)P(t)\,dt \tag{81}$$

where $\theta(0)$ is arbitrary, $P(0) > 0$ is arbitrary

$$a(t) = \frac{1}{f[r(t)]} \tag{82}$$

$$r(t) = \mathrm{e} + \int_0^t |\varphi(s)|^2\,ds \tag{83}$$

and $f \in \mathbb{F}$.

$$\mathbb{F} = \left\{ f \mid f : \mathbb{R}_+ \to \mathbb{R}_+,\ f \text{ is slowly increasing and } \int_c^\infty \frac{dx}{xf(x)} < \infty \text{ for some } c \geqslant 0 \right\} \tag{84}$$

A function f is slowly increasing if it is positive, increasing, and $f(x^2) = 0[f(x)]$ as $x \to \infty$.

Applying the Itô formula to $[X^{\mathrm{T}}(t)X(t), t \geqslant 0]$, it is elementary to verify (31) that for $D \neq 0$

$$\lim_{t\to\infty} r(t) = \infty \quad \text{a.s.}$$

If $D = 0$, then the diminishing excitation used subsequently implies this result.

The dependence of $[\theta(t), t \geqslant 0]$ on $[a(t), t \geqslant 0]$ and the dependence of $[a(t), t \geqslant 0]$ on f are suppressed for notational convenience.

The following result describes some basic properties of the WLS algorithm.

Proposition 21. Let $[\theta(t), P(t)\ t \geqslant 0]$ satisfy Eqs. (80)–(81). The following properties are satisfied:

(1) $$\sup_{t\geqslant 0} |P^{-1/2}(t)\tilde{\theta}(t)|^2 < \infty \quad \text{a.s.} \tag{85}$$

(2) $$\int_0^\infty a(t)|\tilde{\theta}^{\mathbb{T}}(t)\varphi(t)|^2\,dt < \infty \quad \text{a.s.} \tag{86}$$

(3) $$\lim_{t\to\infty} \theta(t) = \bar{\theta} \quad \text{a.s.} \tag{87}$$

where $\tilde{\theta}(t) = \theta(t) - \theta$ and $\bar{\theta}$ is a matrix-valued random variable.

Define $\bar{\theta}(t, x)$ as

$$\overline{\theta}(t, x) = \theta(t) - P^{1/2}(t)x$$

and

$$\overline{\theta}^{\mathbb{T}}(t, x) = [A(t, x) \quad B(t, x)]$$

A well-known test for controllability of (A, B) is the positivity of F where

$$F(t, x) = \det\left[\sum_{i=0}^{n-1} A^i(t, x)B(t, x)B^{\mathbb{T}}(t, x)A^{i\mathbb{T}}(t, x)\right]$$

and similarly a test for observability of (A, C) is the positivity of G where

$$G(t, x) = \det\left[\sum_{i=0}^{n-1} A^{i\mathrm{T}}(t, x)C^{\mathrm{T}}CA^i(t, x)\right]$$

The linear transformation C is known. For the adaptive control problem, $C = Q_1^{1/2}$ where Q_1 determines the quadratic form of the state in the cost functional.

A random search method ensures uniform controllability and observability of a family of estimates. Let $(\eta_n, n \in \mathbb{N})$ be a sequence of independent, identically distributed $\mathscr{M}(n + m,n)$-valued random variables that is independent of $[W(t), t \geqslant 0]$ so that each random variable η_n is uniformly distributed in the unit ball or the unit sphere for a norm of the matrices. Define a sequence of $\mathscr{M}(n + m,n)$-valued random variables by induction as follows:

$$\beta_0 = \eta_0$$

$$\beta_k = \begin{cases} \eta_k & \text{if } f(k, \eta_k) \geqslant (1+\gamma) f(k, \beta_{k-1}) \\ \beta_{k-1} & \text{otherwise} \end{cases}$$

where $\gamma \in (0, \sqrt{2} - 1)$ is fixed and $f(k, x) = F(k, x)G(k, x)$. By the compactness of the unit ball or the unit sphere and the continuity of $F(k, \cdot)$ and $G(k, \cdot)$ for $k \in \mathbb{N}$, it follows that the sequence $(\beta_k, k \in \mathbb{N})$ terminates after some random integer so there is a β_∞ such that

$$\beta_\infty = \lim_{k\to\infty} \beta_k \quad \text{a.s.}$$

Define a family of estimates $[\hat{\theta}(t), t \geqslant 0]$ as

$$\hat{\theta}(t) = \bar{\theta}_k \quad t \in (k, k+1] \tag{88}$$

where $k \in \mathbb{N}$ and

$$\bar{\theta}_k = \theta(k) - P^{1/2}(k)\beta_k \tag{89}$$

It is clear from Eqs. (81) and (87) that $[\hat{\theta}(t), t \geqslant 0]$ converges a.s. For notational simplicity the dependence on $(\beta_k, k \in \mathbb{N})$ in Eq. (88) is suppressed so that

$$\hat{\theta}^{\mathbb{T}}(t) = [A(t) \quad B(t)]$$

Theorem 22. If $(\hat{\theta}(t), t \geqslant 0)$ is the family of estimates given by Eq. (88), (A, B) in Eq. (76) is controllable, and $(A, Q_1^{1/2})$ is observable, then for any admissible control $[U(t), t \geqslant 0]$, $[\hat{\theta}(t), t \geqslant 0]$ has the following properties a.s.:

(1) self-convergence,
(2) uniform controllability and observability, and
(3) semiconsistency.

Adaptive Control

Consider the following ergodic cost functional for the system Eq. (76):

$$J(U) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T [X^T(t)Q_1X(t) + U^T(t)Q_2U(t)]\,dt \tag{90}$$

where $[U(t), t \geqslant 0]$ is an admissible control, $Q_2 > 0$, and $Q_1 \geqslant 0$.

The following assumption is made:

(A2) $(A, Q_1^{1/2})$ is observable.

The stochastic algebraic Riccati equation

$$A^T(t)R(t) + R(t)A(t) - R(t)B(t)Q_2^{-1}B^T(t)R(t) + Q_1 = 0$$

has a unique, random, symmetric, positive solution $R(t)$ a.s. where $[A(t)\ B(t)] = \hat{\theta}^{\mathbb{T}}(t)$. Because the solution of the algebraic Riccati equation is a smooth function of the parameters of the equation, there is a symmetric, positive $\mathscr{L}(\mathbb{R}^n)$-valued random variable $R(\infty)$ such that

$$\lim_{t\to\infty} R(t) = R(\infty) \quad \text{a.s.}$$

If $\Phi(t)$ is given by

$$\Phi(t) = A(t) - B(t)Q_2^{-1}B^T(t)R(t)$$

and

$$\Phi(\infty) = A(\infty) - B(\infty)Q_2^{-1}B^T(\infty)R(\infty)$$

then

$$\lim_{t\to\infty} \Phi(t) = \Phi(\infty)$$

and $\Phi(t)$ and $\Phi(\infty)$ are stable a.s.

The lagged certainty equivalence control is

$$U(t) = -Q_2^{-1}B^T(t)R(t)X(t) \tag{91}$$

It is called "lagged" because $B(t)$ and $R(t)$ depend on $X(s)$ and η_j for $s < t$ and $j < t$. This follows easily by induction recalling the construction of $(\hat{\theta}(t), t \geqslant 0)$, that is,

$$[B(t), R(t)] = [B(0), R(0)]$$

for $t \in [0, 1]$,

$$[B(t), R(t)] = (\overline{B}_k, \overline{R}_k)$$

for $t \in (k, k+1]$, and $\bar{\theta}_k = [\overline{A}_k, \overline{B}_k]^{\mathbb{T}}$ is given by Eq. (89).

To obtain strong consistency for the family of estimates $(\hat{\theta}(t), t \geqslant 0)$, a diminishing excitation is added to the adaptive control Eq. (91), that is,

$$dU(t) = L_k\,dX(t) + \gamma_k\,dV(t) \tag{92}$$

for $t \in (k, k+1]$ and $k \in \mathbb{N}$ where $U(0)$ is arbitrary,

$$L_k = -Q_2^{-1}B(k)R(k)$$

and

$$\gamma_k^2 = \frac{\log k}{\sqrt{k}}$$

for $k \geqslant 1$. The process $[V(t), t \geqslant 0]$ is an $\mathbb{R}^m$-valued standard Wiener process that is independent of $[W(t), t \geqslant 0]$ and $(\eta_k, k \in \mathbb{N})$. As noted before, it easily follows that there is a matrix-valued random variable L such that

$$\lim_{k\to\infty} L_k = L \quad \text{a.s.}$$

The sub-σ-algebra $\mathscr{F}_t$ is the P-completion of $\sigma[X_0, W(s), \eta_j, V(s); s \leqslant t, j \leqslant t]$.

The state $X(t)$ is augmented with the control $U(t)$ as follows:

$$\varphi(t) = \begin{bmatrix} X(t) \\ U(t) \end{bmatrix}$$

$$F_k = \begin{bmatrix} A & B \\ L_kA & L_kB \end{bmatrix}$$

$$G_k = \begin{bmatrix} D & 0 \\ L_kD & \gamma_kI \end{bmatrix}$$

$$\xi(t) = \begin{bmatrix} W(t) \\ V(t) \end{bmatrix}$$

The stochastic differential equation for the augmented state process ($\varphi(t)$, $t \geqslant 0$) is

$$d\varphi(t) = F_k\varphi(t)\,dt + G_k d\xi(t)$$

for $t \in (k, k+1]$.

The following result shows that the family of regularized, weighted, least squares estimates is strongly consistent using the lagged certainty equivalence control with diminishing excitation.

Proposition 23. Let [$\hat{\theta}(t)$, $t \geqslant 0$] be the family of estimates given by Eq. (88) using the control Eq. (92) in (Eq. 76). Then

$$\lim_{t\to\infty} \hat{\theta}(t) = \theta \quad \text{a.s.}$$

where

$$\theta^{\mathbb{T}} = [A \quad B]$$

Remark. It should be noted that the conditions of this proposition are satisfied if $D \equiv 0$ so that the identification of deterministic systems is included in this result.

Because the family of estimates [$\hat{\theta}(t)$, $t \geqslant 0$] is strongly consistent, the self-optimality of the diminishingly excited lagged certainty equivalence control Eq. (92) can be verified.

Theorem 24. Let (A1) and (A2) be satisfied for the stochastic system

$$dX(t) = AX(t)\,dt + BU(t)\,dt + D\,dW(t)$$

with the cost functional Eq. (90) where A and B are unknown. If the admissible adaptive control

$$dU^*(t) = L_k\,dX(t) + \gamma_k\,dV(t)$$
$$L_k = -Q_2^{-1}B^{\mathbb{T}}(k)R(k)$$

is used, then

$$J(U^*) = \inf_U J(U) = \mathrm{tr}(D^{\mathbb{T}}RD) \quad \text{a.s.}$$

where R is the unique, positive, symmetric solution of the algebraic Riccati equation

$$A^{\mathbb{T}}R + RA - RBQ_2^{-1}B^{\mathbb{T}}R + Q_1 = 0$$

and

$$J(U^*) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T [X^{\mathbb{T}}(t)Q_1X(t) + U^{*\mathbb{T}}(t)Q_2U^*(t)]\,dt \quad \text{a.s.}$$

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