PERIODIC CONTROL

The fact that a periodic operation may be advantageous has been well known to humankind since time immemorial. All farmers know that it is not advisable to grow the same product repeatedly in the same field because the yield can be improved by rotating crops. So, cycling is good.

More recently, similar concepts have been applied to industrial problems. Traditionally, almost every continuous industrial process was set and kept, in the presence of disturbances, at a suitable steady state. However, there are circumstances under which a periodic time-varying action proves to be better. This observation germinated in the field of chemical engineering where it was seen that the performance of a number of catalytic reactors improved by cycling; see the pioneering contributions in Refs. 1–3. Unfortunately, as pointed out in Ref. 4, periodic control was still considered "too advanced" in the industrial control scenario, in that "the steady-state operation is the norm and unsteady process behaviour is taboo." Its use was therefore confined to advanced (aerospace or classified) applications, such as those treated in Refs. 5 and 6. Today, however, the new possibilities offered by current control technology, together with the theoretical developments of the field, have opened the way for using periodic controllers in place of the traditional stationary ones. In fact, the term periodic control takes a wider significance in the contemporary literature. In addition to the control problems that arise when operating a plant periodically, periodic control also includes all situations where either the controller or the plant is a proper periodic system. One of the reasons behind such an extension is the possible improvement of the performances, in terms of stability and robustness, of plants described by time-invariant models, when using a periodic controller (see Ref. 7).

The diffusion of digital apparatuses in control has also contributed to the increasing importance of periodic control because computer-controlled systems are often based on sample-and-hold devices for output measurements and input updating. In multivariable control, it may also be necessary, for technological or economical reasons, to adopt different sampling and/or hold intervals for the various actuators or transducers. For example, certain variables may exhibit a much slower dynamic than others so that different sampling intervals must be adopted. In such situations, we usually resort to an internal clock for setting a synchronization mode; the digital equipment complementing the plant performs the selection of the output sampling instants and the control updating instants out of the clock time points. It turns out that these sampling selection mechanisms are described by periodic models in discrete time, with period equal to the least common factor of the ratios between the sampling and updating intervals over the basic clock period. The overall control system obtained in this way is known as a *multirate sampled-data system* (8,9).

Finally, resorting to control laws that are subject to periodic time variations is natural to govern phenomena that are intrinsically periodic. An important field where we encounter such dynamics is helicopter modeling and control, as witnessed by the fact that a full chapter of the classical reference book in the field (10) is devoted to periodic systems. The main interest in this framework is rotor dynamics modeling. Indeed, consider the case of *level forward flight,* when the velocity vector of the flying machine is constant and parallel to its body. Those flight conditions are achieved by imposing a periodic pattern on the main control variables of the helicopter (i.e., the pitch angles of each blade). Consequently, the aerodynamic loads present a cyclic pattern, with period determined by the rotor revolution period, and any model of the rotor dynamics is periodic (see Refs. 11–13). The interest for periodic systems goes far beyond these situations. Periodicity arises in the study of nonlinear time-invariant systems dealing with closed orbit operations. Classical examples of such situations are relay-operated control plants, hysteretic oscillators, and processes subject to seasonal-load effects. For the study of system behavior against small perturbations, a linearized approximated model is often used. And, although the original system is time-invariant, the linearization procedure generates periodicity in the approximated linear model (see Refs. 14–16).

In this article, the most important techniques of periodic control will be outlined, avoiding, however, overly technical details. The article is organized as follows. The first part deals with the analysis of periodic systems. Initially, it is shown how state-space periodic models arise from multirate sampling or linearization around closed orbits. The periodic input/output representation is also introduced as an alternative to state-space modelization. Then, the possibility of analyzing a periodic system via time-invariant models is investigated and a number of techniques are introduced. Further, the frequency-response concept for periodic systems is outlined. The fundamental concept of stability comes next. It calls for the definition of the monodromy matrix and involves the theory of Floquet and Lyapunov. In passing, the notion of cyclostationary stochastic process is touched on and briefly discussed.

The second part is devoted to periodic control, and discusses three main problems: (1) choice of the control signal in order to force a periodic regime with better performance than any possible steady state operation, (2) periodic control of time-invariant plants, and (3) periodic control of periodic systems.

The literature on the subject is so vast that it is impossible to cover all aspects of theoretical and application interest. The interested reader will find a rather detailed list of references in the bibliography, including survey papers (17–22) and early reference books (23,24).

BASICS IN PERIODIC SYSTEMS ANALYSIS

A basic classification of linear periodic systems depends on the nature of the time variable t. We focus our attention herein on *continuous-time* or *discrete-time* periodic systems. In the former case t is a real variable, whereas in the latter t is an integer.

State-Sampled Representation

Nowadays, the most widely used mathematical modelization of dynamical systems hinges on the concept of *state-variable.* The state variables are latent variables that establish a bridge between the input variables $u(t)$ and the output variables $y(t)$. They are collected in a vector denoted by $x(t)$, and the basic *state-space model* is the set of difference equations

$$x(t+1) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

in discrete-time, or the set of differential equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

in continuous time.

Matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$ are real matrices, of appropriate dimensions, that depend periodically on t:

$$A(t+T) = A(t), \qquad B(t+T) = B(t)$$
$$C(t+T) = C(t), \qquad D(t+T) = D(t)$$

The smallest T for which these periodicity conditions are met is called the system period.

These state-space models may be generalized and extended in various ways, among which are the class of descriptor models (25).

Periodicity Induced by Linearization. As mentioned earlier, a linear periodic system can be used to describe the small perturbation behavior along a periodic regime. For example, consider the continuous-time nonlinear system

$$\dot{\xi}(t) = f(\xi(t), v(t))$$
$$\eta(t) = h(\xi(t), v(t))$$

and let $\tilde{v}(\cdot)$, $\tilde{\xi}(\cdot)$, $\tilde{\eta}(\cdot)$ be an associated periodic regime of period T. This means that $\tilde{\xi}(\cdot)$ is a periodic solution of period T associated with the periodic input $\tilde{v}(\cdot)$ and that $\tilde{\eta}(\cdot)$ is the corresponding periodic output. The linearized equations result in a linear continuous-time system with

$$u(t) = v(t) - \tilde{v}(t), \quad x(t) = \xi(t) - \tilde{\xi}(t), \quad y(t) = \eta(t) - \tilde{\eta}(t)$$

and matrices

$$A(t) = \left.\frac{\partial f(\xi, v)}{\partial \xi}\right|_{\xi=\tilde{\xi}, v=\tilde{v}}, \qquad B(t) = \left.\frac{\partial f(\xi, v)}{\partial v}\right|_{\xi=\tilde{\xi}, v=\tilde{v}}$$

$$C(t) = \left.\frac{\partial h(\xi, v)}{\partial \xi}\right|_{\xi=\tilde{\xi}, v=\tilde{v}}, \qquad D(t) = \left.\frac{\partial h(\xi, v)}{\partial v}\right|_{\xi=\tilde{\xi}, v=\tilde{v}}$$

These matrices are obviously periodic of period T. Mutatis mutandis, the same reasoning applies in discrete-time as well. The linearization rationale is illustrated in Fig. 1.

Periodicity Induced by Multirate Sampling. Multirate schemes arise in digital control and digital signal processing whenever it is necessary to sample the outputs and/or update the inputs with different rates. To explain in simple terms how multirate sampled-data mechanisms generate periodicity, consider a system with two inputs and two outputs described by the time-invariant differential equations

$$\dot{x}(t) = Ax(t) + B\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Cx(t) + D\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

The two outputs $y_i(\cdot)$, $i = 1, 2$, are sampled with sampling intervals τ_{y_i}, $i = 1, 2$, whereas the two inputs $u_j(\cdot)$, $j = 1, 2$, are updated at the end of intervals of length τ_{u_j}, $j = 1, 2$, and kept constant in between. The sampling and updating instants are denoted by $t_{y_i}(k)$, $i = 1, 2$ and $t_{u_j}(k)$, $j = 1, 2$, k integer, respectively. Typically, these instants are taken as multiples of the basic clock period Δ. Moreover, for simplicity, assume that

$$t_{y_i}(k) = k\tau_{y_i}, \quad t_{u_j}(k) = k\tau_{uj}$$

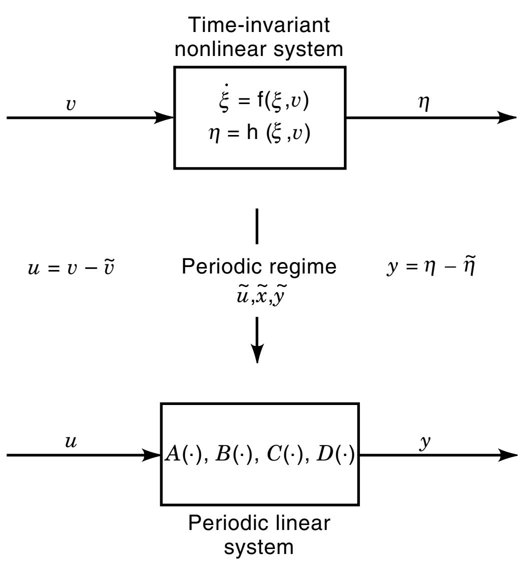

Figure 1. Linearization around a periodic orbit—the dynamics of the nonlinear system in the orbit vicinity is governed by a linear periodic system.

The overall behavior of the obtained system is ruled by the discrete-time output variables $\tilde{y}_i(k)$, $i = 1, 2$ and the discrete-time input variables $\tilde{u}_j(k)$, $j = 1, 2$ defined as

$$\tilde{y}_i(k) = y(k\tau_{y_i}), \quad i = 1, 2$$
$$u_j(t) = \tilde{u}_j(k), \quad t \in [k\tau_{u_j}, k\tau_{u_j} + \tau_{u_j}), \quad j = 1, 2$$

For the modelization, however, it is advisable to introduce the "fast-rate" signals

$$\hat{y}(k) = y(k\Delta), \quad \hat{u}(k) = u(k\Delta)$$

In contrast, $\tilde{y}(k)$ and $\tilde{u}(k)$ can be considered as the "slow-rate" samples. The *sampling selector,* namely the device operating the passage from the fast sampled-output to the slow sampled-output, is described by the linear equation

$$\tilde{y}(k) = N(k)\hat{y}(k)$$

where

$$N(k) = \begin{bmatrix} n_1(k) & 0 \\ 0 & n_2(k) \end{bmatrix}$$

with

$$n_i(k) = \begin{cases} 1 & \text{if } k \text{ is a multiple of } \tau_{y_i}/\Delta \\ 0 & \text{otherwise} \end{cases}$$

Note that matrix $N(\cdot)$ is periodic with the period given by the integer T_y defined as the least common multiple of τ_{y_1}/Δ and τ_{y_2}/Δ.

As for the hold device, introduce the *holding selector* matrix

$$S(k) = \begin{bmatrix} s_1(k) & 0 \\ 0 & s_2(k) \end{bmatrix}$$

with

$$s_j(k) = \begin{cases} 0 & \text{if } k \text{ is a multiple of } \tau_{u_j}/\Delta \\ 1 & \text{otherwise} \end{cases}$$

Then, the analog input signal $u(\cdot)$ is given by

$$u(t) = \hat{u}(k), \quad t \in [k\Delta, k\Delta + \Delta)$$

where the fast updated signal $\hat{u}(k)$ is obtained from the slow one $\tilde{u}(k)$ according to the holding selector mechanism

$$v(k+1) = S(k)v(k) + (I - S(k))\tilde{u}(k)$$
$$\hat{u}(k) = S(k)v(k) + (I - S(k))\tilde{u}(k)$$

Matrix $S(k)$ is periodic of period T_u given by the least common multiple of τ_{u_1}/Δ and τ_{u_2}/Δ. This situation is schematically illustrated in Fig. 2, where $\tau_{u_1} = 3\Delta$, $\tau_{u_2} = 4\Delta$, $\tau_{y_1} = 2\Delta$, $\tau_{y_2} = 5\Delta$, so that $T_u = 12$ and $T_y = 10$.

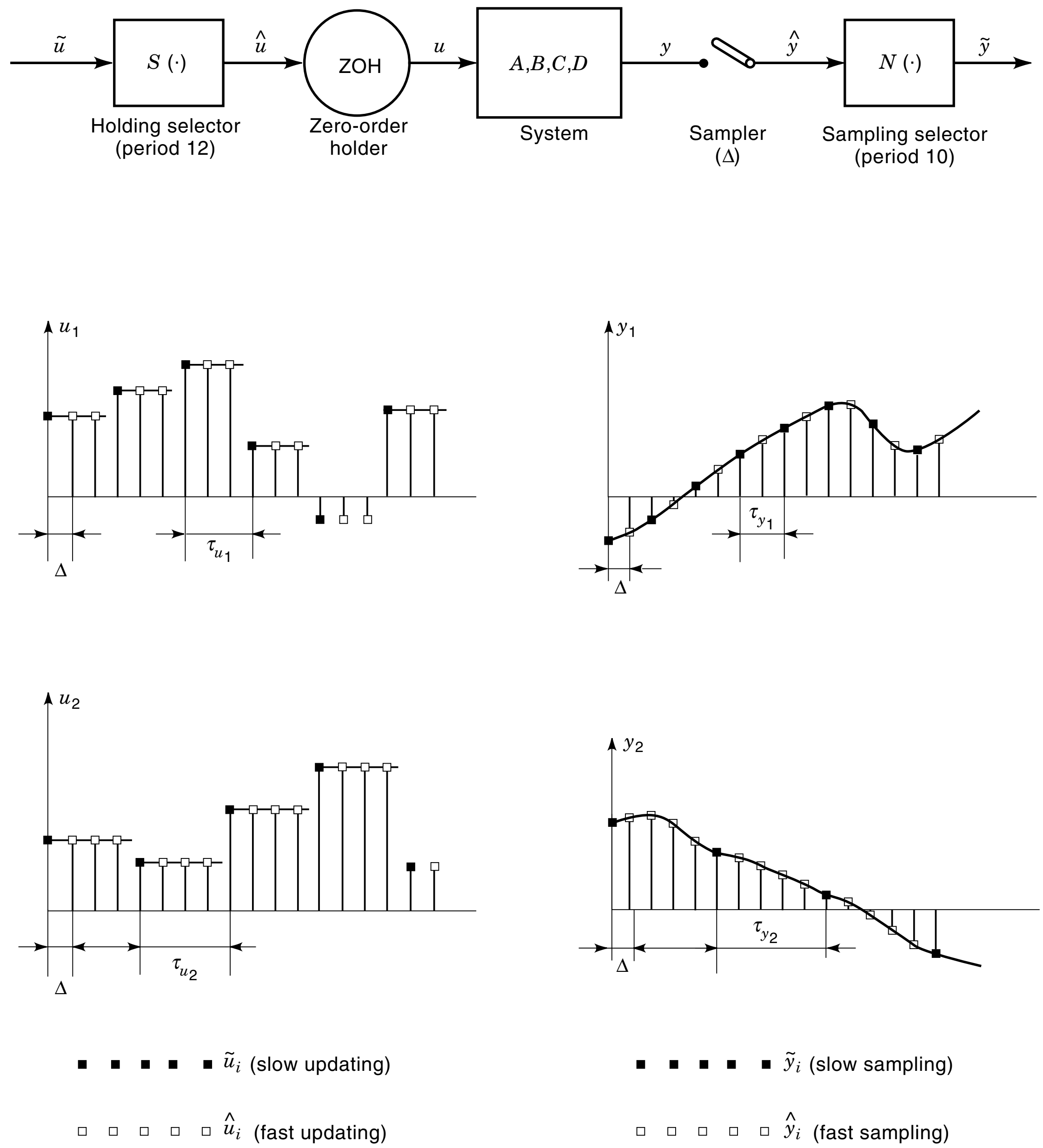

Figure 2. A multirate sampled-data system with two inputs and two outputs. The symbol Δ denotes the clock period. The first output signal y_1 is sampled at rate $\tau_{y_1} = 2\Delta$ and the second y_2 at rate $\tau_{y_2} = 5\Delta$. Hence, the sampling selector is a periodic discrete-time system with period $T_y = 10$. Moreover, the first input signal u_1 is updated at rate $\tau_{u_1} = 3\Delta$ and the second u_2 at rate $\tau_{u_2} = 4\Delta$. The holding selector is a periodic discrete-time system with period $T_u = 12$. The period of the global system is therefore $T = 60$.

The overall multirate sampled-data system is a discrete-time periodic system with state

$$\hat{x}(k) = \begin{bmatrix} x(k\Delta) \\ v(k) \end{bmatrix}$$

and equations

$$\hat{x}(k+1) = \hat{A}(k)\hat{x}(t) + \hat{B}(k)\tilde{u}(k)$$
$$\tilde{y}(k) = \hat{C}(k)\hat{x}(k) + \hat{D}(k)\tilde{u}(k)$$

where

$$\hat{A}(k) = \begin{bmatrix} e^{A\Delta} & \int_0^{\Delta} e^{A\sigma}\, d\sigma B S(k) \\ 0 & S(k) \end{bmatrix}$$

$$\hat{B}(k) = \begin{bmatrix} \int_0^{\Delta} e^{A\sigma}\, d\sigma B(I - S(k)) \\ I - S(k) \end{bmatrix}$$

$$\hat{C}(k) = N(k)[C\; DS(k)]$$

$$\hat{D}(k) = N(k)D(I - S(k))$$

The so-obtained system is periodic with period T given by the least common multiple of T_u and T_y ($T = 60$ in the example of Fig. 2).

Lagrange Formula. The free motion of the periodic system, i.e., the solution of the homogeneous equation

$$\begin{cases} \dot{x}(t) = A(t)x(t) & \text{in continuous time} \\ x(t+1) = A(t)x(t) & \text{in discrete time} \end{cases}$$

starting from state $x(\tau)$ at time τ is obtained as

$$x(t) = \Phi_A(t,\tau)x(\tau)$$

where the *transition matrix* $\Phi_A(t, \tau)$ is given by

$$\Phi_A(t,\tau) = \begin{cases} I & t = \tau \\ A(t-1)A(t-2)\dots A(\tau) & t > \tau \end{cases}$$

in discrete time and by the solution of the differential matrix equation

$$\frac{\partial}{\partial t}\Phi_A(t,\tau) = A(t)\Phi_A(t,\tau), \quad \Psi(\tau,\tau) = I$$

in continuous time. Therefore, the state solution with a generic initial state $x(\tau)$ and input function $u(\cdot)$ is

$$x(t) = \Phi_A(t,\tau)x(\tau) + \sum_{j=\tau+1}^{t} \Phi_A(t,j)B(j-1)u(j-1)$$

in discrete time and

$$x(t) = \Phi_A(t,\tau)x(\tau) + \int_\tau^t \Phi_A(t,\sigma)B(\sigma)u(\sigma)$$

in continuous time. These expressions are known as *Lagrange formulas* (also called variations-of-constants formulas).

We can easily see that the periodicity of the system entails the "biperiodicity" of matrix $\Phi_A(t, \tau)$, namely that

$$\Phi_A(t+T,\tau+T) = \Phi_A(t,\tau)$$

The transition matrix over one period

$$\Psi_A(\tau) = \Phi_A(\tau+T,\tau)$$

plays a major role in the analysis of periodic systems and is known as *monodromy matrix* at time τ.

Reversibility. In continuous time, there is no analytic expression for the transition matrix. However, its determinant can be worked out from the so-called *Jacobi formula.* In other words,

$$\det[\Phi_A(t,\tau)] = \exp\left[\int_\tau^t \text{trace}[A(\sigma)]\,d\sigma\right]$$

Therefore, for any choice of t and τ, the transition matrix is invertible. This means that the system is *reversible,* in that the state $x(\tau)$ can be uniquely recovered from $x(t)$, $t > \tau$ [assuming that input $u(\cdot)$ over the interval $[\tau, t)$ is known]. This is not true, in general, in discrete-time since the transition matrix is singular when $A(i)$ is singular for some i.

Input-Output Representation

Another mathematical representation of periodic systems is based on a direct time-domain relationship between the input and the output variables without using any intermediate latent variables. In discrete time, this leads to a model of the form

$$\begin{aligned} y(t) = {} & F_1(t)y(t-1) + F_2(t)y(t-2) + \cdots + F_r(t)y(t-r) \\ & + G_1(t)u(t-1) + G_2(t)u(t-2) + \cdots + G_s(t)u(t-s) \end{aligned}$$

where $F_i(\cdot)$ and $G_j(\cdot)$ are periodic real matrices. Such a representation is frequently used whenever a cyclic model must be estimated from data, as happens in model identification, data analysis, and signal processing. For the passage from a state-space periodic system to an input-output periodic model and vice versa, see, for example, Refs. 26 and 27. Input-output periodic models can also be introduced in continuous time by means of differential equations with time-varying coefficients.

Note that state-space or input-output periodic models are used in the stochastic modeling of *cyclostationary processes,* a type of stochastic nonstationary processes with periodic characteristics. In such a context, the input $u(\cdot)$ is typically a remote signal described as a white noise. Then, the input-output models are known as PARMA models, where PARMA means periodic auto-regressive moving average.

In the following sections, the main attention will be focused on state-space models.

TIME-INVARIANT REPRESENTATIONS

As seen before, periodicity is often the result of ad hoc operations over time-invariant systems. On the other hand, in periodic systems analysis and control, a major point is to address the "backward" problem of finding a way to "transform" a periodic system into a time-invariant one. In such a way, we can resort to the results already available in the time-invariant realm.

Sample and Hold

The simplest way to achieve stationarity is to resort to a sample-and-hold procedure. Indeed, with reference to a continuous or a discrete-time periodic system, suppose that the input is kept constant over a period, starting from an initial time point τ. That is,

$$u(t) = \tilde{u}(k), \quad t \in [kT+\tau, kT+T+\tau)$$

Then the evolution of the system state sampled at $\tau + kT$ [i.e., $x_\tau(k) = x(kT + \tau)$] is governed by a *time-invariant* equation in discrete time. Precisely,

$$x_\tau(k+1) = \Phi_A(T+\tau,\tau)x_\tau(k) + \Gamma(\tau)\tilde{u}(k)$$

where

$$\Gamma(\tau) = \begin{cases} \int_{\tau}^{T+\tau} \Phi_A(T+\tau,\sigma)B(\sigma)\,d\sigma & \text{in continuous time} \\ \sum_{i=\tau}^{T+\tau-1} \Phi_A(T+\tau,i+1)B(i) & \text{in discrete time} \end{cases}$$

Generalized Sample and Hold. It is possible to generalize the sample-and-hold representation by equipping the holding mechanism with a time-varying periodic modulating function $H(\cdot)$ acting over the period as follows:

$$u(t) = H(t)\tilde{u}(k), \quad t \in [kT+\tau, kT+T+\tau)$$

In this way, the evolution of the sampled state $\tilde{x}(k)$ is still governed by the previous equations provided that $B(t)$ is replaced by $B(t)H(t)$. Such a *generalized sample-and-hold representation* allows for a further degree of freedom in the design of periodic controllers. Indeed, function $H(\cdot)$ is a free parameter to be chosen by the designer; see Ref. 28.

Lifted and Cyclic Reformulations

Despite its interest, the sample-and-hold representation is by no means an *equivalent* reformulation of the original periodic system because the input function is constrained into the class of piecewise constant signals. Truly equivalent reformulations can be pursued in a number of ways, depending on the transformations allowed, in frequency or in time, on the input, state, and output signals.

For ease of explanation, it is advisable to focus on discrete time, where the most important reformulations are time-lifted, cyclic, and frequency-lifted representations.

The *time-lifted reformulation* goes back to early papers (29, 30). The underlying rationale is to sample the system state with a sampling interval coincident with the system period T and to organize the input and output signals in packed segments of subsequent intervals of length T, so as to form input and output vectors of enlarged dimensions. That is, let τ be a sampling tag and introduce the "packed input" and "packed output" segments as follows:

$$\tilde{u}_\tau(k) = [u(kT+\tau)'\,u(kT+\tau+1)'\ldots u(kT+\tau+T-1)']'$$
$$\tilde{y}_\tau(k) = [y(kT+\tau)'\,y(kT+\tau+1)'\ldots y(kT+\tau+T-1)']'$$

The vectors $\tilde{u}_\tau(\cdot)$ and $\tilde{y}_\tau(\cdot)$ are known as *lifted* input and *lifted* output signals. The introduction of the lifting concept enables us to determine $x_\tau(k+1) = x(kT+T+\tau)$ from $x_\tau(k) = x(kT+\tau)$ and then to work out $\tilde{y}_\tau(k)$ from $x_\tau(k)$. More precisely, define $F_\tau \in R^{n\times n}$, $G_\tau \in R^{n\times mT}$, $H_\tau \in R^{pT\times n}$, and $E_\tau \in R^{pT\times mT}$ as

$$F_\tau = \Psi_A(\tau)$$
$$G_\tau = [\Phi_A(\tau+T,\tau+1)B(\tau) \quad \Phi_A(\tau+T,\tau+2)B(\tau+1)\ldots B(\tau+T-1)]$$
$$H_\tau = [C(\tau)' \quad \Phi_A(\tau+1,\tau)'C(\tau+1)'\ldots \Phi_A(\tau+T-1,\tau)'C(\tau+T-1)']'$$
$$E_\tau = \{(E_\tau)_{ij}\}, \qquad i,j = 1,2,\ldots,T$$
$$(E_\tau)_{ij} = \begin{cases} 0 & i<j \\ D(\tau+i-1) & i=j \\ C(\tau+i-1)\Phi_A(\tau+i-1,\tau+j)B(\tau+j-1) & i>j \end{cases}$$

The time-lifted reformulation can then be introduced:

$$x_\tau(k+1) = F_\tau x_\tau(k) + G_\tau \tilde{u}_\tau(k)$$
$$\tilde{y}_\tau(k) = H_\tau x_\tau(k) + E_\tau \tilde{u}_\tau(k)$$

Note that if $u(\cdot)$ is kept constant over the period, then the state equation of this lifted reformulation boils down to the sample-and-hold state equation.

In this reformulation, only the output and input vectors were enlarged, whereas the dimension of the state-space was preserved. In the *cyclic reformulation,* (31,32), every system's signal, say $v(t)$ of dimension q, is transformed into an enlarged signal $\overline{v}_\tau(t)$ of dimension qT. This transformation takes place according to the following rule (given over one period starting from a given initial instant τ):

$$\overline{v}_\tau(\tau) = \begin{bmatrix} v(\tau) \\ \square \\ \vdots \\ \square \\ \square \end{bmatrix}, \quad \overline{v}_\tau(\tau+1) = \begin{bmatrix} \square \\ v(\tau+1) \\ \square \\ \vdots \\ \square \end{bmatrix}, \ldots,$$

$$\overline{v}_\tau(\tau+T-2) = \begin{bmatrix} \square \\ \square \\ \vdots \\ v(\tau+T-2) \\ \square \end{bmatrix},$$

$$\overline{v}_\tau(\tau+T-1) = \begin{bmatrix} \square \\ \square \\ \vdots \\ \square \\ v(\tau+T-1) \end{bmatrix}$$

where $\square$ is any element, typically set to zero. Obviously, the previous pattern repeats periodically for the other periods. This signal transformation is used for the input, output, and state of the system. Then, we can relate the cyclic input to the cyclic state by means of a time-invariant state-equation and the cyclic output to the cyclic state via a time-invariant transformation. In this way, we obtain an nT-dimensional time-invariant system with mT inputs and pT outputs.

Finally, the *frequency-lifted reformulation* is based on the following considerations. For a discrete-time (vector) signal $\boldsymbol{v}(t)$, let $V(z)$ be its z-transform. Now, one can associate with $V(z)$ the frequency augmented vector $\boldsymbol{V}_f(z)$ as follows:

$$\boldsymbol{V}_f(z) = \begin{bmatrix} V(z) \\ V(z\phi) \\ V(z\phi^2) \\ \vdots \\ V(z\phi^{T-1}) \end{bmatrix}$$

where $\phi = e^{2j\pi/T}$. By applying this procedure to the z-transforms of the input and output signals of the periodic system, it is possible to establish an input-output correspondence described by a matrix transfer function; see Ref. 33. Such a transfer function is referred to as the frequency-lifted representation.

The three reformulations are input-output equivalents of each other. Indeed, for any pair of them it is possible to work out a one-to-one correspondence between the input-output signals. For the correspondence between the cyclic and the time-lifted reformulations, see Ref. 22.

Lifting and Cycling in Continuous Time

In continuous time, the frequency-lifted reformulation can be appropriately worked out as well leading to infinite-dimensional time-invariant systems. For example, the time-lifted reformulation appears as in discrete time, but now G_τ, H_τ, and E_τ are linear operators on/from Hilbert spaces. On this topic, the interested reader is referred to Refs. 34 and 35.

PERIODIC SYSTEMS IN FREQUENCY DOMAIN

The frequency domain representation is a fundamental tool in the analysis and control of time-invariant linear systems. It is related to the well-known property that, for this class of systems, sinusoidal inputs result into sinusoidal outputs at the same frequency and different amplitude and phase.

A similar tool can be worked out for periodic systems by making reference to their response to the so-called *exponentially modulated periodic* (EMP) signals. Herein, we limit our attention to continuous-time systems. Then, given any complex number s, a (complex) signal $u(t)$ is said to be EMP of period T and modulation s if

$$u(t) = \sum_{k\in Z} u_k e^{s_k t}$$

where

$$s_k = s + jk\Omega$$

The quantity $T = 2\pi/\Omega$ is the named period of the EMP signal. The class of EMP signals is a generalization of the class of T-periodic signals. As a matter of fact, an EMP signal with $s = 0$ is just an ordinary time-periodic signal. Indeed, as it is easy to verify, an EMP signal is such that

$$u(t+T) = \lambda u(t), \quad \lambda = e^{sT}$$

In much the same way as a time-invariant system subject to a (complex) exponential input admits an exponential regime, a periodic system of period T subject to an EMP input of the same period admits an EMP regime. In such a regime, all signals of interest can be expanded as EMP signals as follows:

$$\begin{aligned} x(t) &= \sum_{k\in Z} x_k e^{s_k t} \\ \dot{x}(t) &= \sum_{k\in Z} s_k x_k e^{s_k t} \\ y(t) &= \sum_{k\in Z} y_k e^{s_k t} \end{aligned}$$

Consider now the Fourier series for the periodic matrix coefficients. That is,

$$A(t) = \sum_{k\in Z} A_k e^{jk\Omega t}$$

and similarly for $B(t)$, $C(t)$, and $D(t)$, and plug the expansions of the signals $x(t)$, $u(t)$, $\dot{x}(t)$ and the matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$ into the system equations. By equating all terms at the same frequency, we obtain an infinite-dimensional matrix equation of the following kind:

$$\begin{aligned} s\mathcal{X} &= (\mathcal{A} - \mathcal{N})\mathcal{X} + \mathcal{B}\mathcal{U} \\ \mathcal{Y} &= \mathcal{C}\mathcal{X} + \mathcal{D}\mathcal{U} \end{aligned}$$

where $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$, are doubly infinite vectors found with the harmonics of x, u and y respectively, organized in the following fashion:

$$\mathcal{X}^T = [\ldots, x_{-2}^T, x_{-1}^T, x_0^T, x_1^T, x_2^T, \ldots]$$

and similarly for $\mathcal{U}$ and $\mathcal{Y}$. $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are doubly infinite Toeplitz matrices formed with the harmonics of $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$, respectively, as

$$\mathcal{A} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & A_0 & A_{-1} & A_{-2} & A_{-3} & A_{-4} & \cdots \\ \cdots & A_1 & A_0 & A_{-1} & A_{-2} & A_{-3} & \cdots \\ \cdots & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & \cdots \\ \cdots & A_3 & A_2 & A_1 & A_0 & A_{-1} & \cdots \\ \cdots & A_4 & A_3 & A_2 & A_1 & A_0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and similarly for $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. As for matrix $\mathcal{N}$, it is the block diagonal matrix

$$\mathcal{N} = \text{blkdiag}\{jk\Omega I\}, \quad k \in Z$$

Then, we can define the *harmonic transfer function* as the operator

$$\hat{\mathcal{G}}(s) = \mathcal{C}[s\mathcal{I} - (\mathcal{A} - \mathcal{N})]^{-1}\mathcal{B} + \mathcal{D}$$

Such an operator provides a most useful connection between the input harmonics and the output harmonics (organized in the infinite vectors $\mathcal{U}$ and $\mathcal{Y}$, respectively). In particular, if we take $s = 0$ (so considering the truly periodic regimes), the appropriate input/output operator is

$$\hat{\mathcal{G}}(0) = \mathcal{C}[\mathcal{N} - \mathcal{A}]^{-1}\mathcal{B} + \mathcal{D}$$

If $u(\cdot)$ is a sinusoid, this expression enables us to compute the amplitudes and phases of the harmonics constituting the output signal $y(\cdot)$ in a periodic regime.

In general, the input/output operator representation of a periodic system may be somewhat impractical, given that it is infinite-dimensional. From an engineering viewpoint, anyway, this model can be satisfactorily replaced by a finite-dimensional approximation obtained by truncation of the Fourier series of the system matrices, which in turn implies that matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{N}$ also are truncated and have therefore finite dimensions.

Analyzing the frequency domain behavior of a continuous-time periodic system in terms of the Fourier expansion of its coefficients is a long-standing idea in the field; see Ref. 36 for a classical reference and a more recent paper (37).

Interestingly enough, it can be shown that the discrete-time version of this rationale leads to a finite-dimensional time-invariant system whose transfer function coincides with that of the frequency-lifted reformulation.

MONODROMY MATRIX AND STABILITY

The monodromy matrix $\Psi_A(\tau)$ relates the value of the state in free motion at a given time-point τ to the value after one period $\tau + T$. Precisely, if $u(\cdot) = 0$ over the considered interval of time,

$$x(\tau + T) = \Psi_A(\tau)x(\tau)$$

Therefore, the sampled state $x_\tau(k) = x(\tau + kT)$ is governed in the free motion by the time-invariant discrete-time equation

$$x_\tau(k+1) = \Psi_A(\tau)x_\tau(k)$$

This is why the eigenvalues of $\Psi_A(\tau)$ play a major role in the modal analysis of periodic systems. In the literature, such eigenvalues are referred to as the *characteristic multipliers* of $A(\cdot)$. Note that, although the monodromy matrix may depend upon τ, the characteristic multipliers are constant (21). Moreover, in continuous time, all characteristic multipliers are different from *zero* as can be easily seen from the Jacobi formula. Conversely, a discrete-time system may exhibit null characteristic multipliers. This happens when at least one among matrices $A(i)$, $i = 0, 1, \ldots, T - 1$ is singular, so that the system is nonreversible.

Obviously the family of periodic systems includes that of time-invariant ones, in which case the monodromy matrix takes the expression

$$\Psi_A(\tau) = \begin{cases} e^{AT} & \text{in continuous time} \\ A^T & \text{in discrete time} \end{cases}$$

Therefore, denoting by λ an eigenvalue of A, the characteristic multipliers of a time-invariant system seen as a periodic system of period T are given by $e^{\lambda T}$ and λ^T in continuous time and discrete time, respectively.

In general, the monodromy matrix is the basic tool in the stability analysis of periodic systems. Indeed, any free motion goes to zero asymptotically if and only if all characteristic multipliers have modulus lower than one. Hence, a periodic system (in continuous or discrete time) is stable if and only if its characteristic multipliers belong to the open unit disk.

To be more precise, this stability concept is usually referred to as asymptotic stability. However, there is no need for this article to introduce all possible notions of stability, so the attribute asymptotic is omitted for the sake of conciseness.

Notice that there is no direct relation between the eigenvalues of $A(t)$ and the system stability. In particular, it may well happen that all eigenvalues of $A(t)$ belong to the stable region (i.e., the left half plane in continuous time and the unit disk in discrete time) and nevertheless the system is unstable. Notable exceptions in continuous time are slowing-varying matrices or high-frequency perturbed matrices, see Refs. 38 and 39, respectively.

A celebrated stability condition can be formulated in terms of the so-called *Lyapunov equation.*

There are two possible formulations of such an equation, known as *filtering Lyapunov equations* and *control Lyapunov equation,* reflecting the fact that Lyapunov equations may arise in the analysis of both filtering and control problems. In continuous time, the filtering Lyapunov equation takes the form

$$\dot{P}(t) = P(t)A(t)' + A(t)P(t) + Q(t)$$

and the control Lyapunov equation is

$$-\dot{P}(t) = A(t)'P(t) + P(t)A(t) + Q(t)$$

where $Q(\cdot)$ is a periodic $[Q(t + T) = Q(t), \forall t]$ and positive definite $[x'Q(t)x > 0, \forall t, \forall x \neq 0]$ matrix.

It turns out that the continuous-time periodic system is stable if and only if the Lyapunov equation (in any of the two forms above) admits a (unique) periodic positive-definite solution $P(\cdot)$.

An analogous result holds in discrete time, by making reference to

$$P(t+1) = A(t)P(t)A(t)' + Q(t)$$
$$P(t) = A(t)'P(t+1)A(t) + Q(t)$$

as filtering and control Lyapunov equations, respectively. As before, $Q(\cdot)$ is periodic and positive definite.

The Lyapunov stability theorem can be expressed in a more general form by referring to positive semidefinite matrices $Q(\cdot)$ provided that further technical assumptions on the pair $(A(\cdot), Q(\cdot))$ are met with; see Ref. 40 for more details on the theoretical aspects and Ref. 41 for the numerical issues.

It is useful to point out that the above *Lyapunov stability condition* can also be stated in a variety of different forms. In particular, it is worth mentioning that one can resort to the *Lyapunov inequality,* i.e., in continuous time

$$\dot{P}(t) > A(t)P(t) + P(t)A(t)' \quad \text{(filtering)}$$
$$-\dot{P}(t) > A(t)'P(t) + P(t)A(t) \quad \text{(control)}$$

and in discrete time

$$P(t+1) > A(t)P(t)A(t)' \quad \text{(filtering)}$$
$$P(t) > A(t)'P(t+1)A(t) \quad \text{(control)}$$

Here it is meant that, given two square matrices M and N, $M > N$ is equivalent to saying that $M - N$ is positive definite. Then, an equivalent stability condition is that the system is stable if and only if the Lyapunov inequality admits a periodic positive definite solution. The advantage of expressing the condition in this form is that no auxiliary matrix $Q(\cdot)$ is required.

Cyclostationary Processes

In the stochastic realm, the so-called *cyclostationary processes* are well suited to deal with pulsatile random phenomena and are the subject of intense investigation in signal processing; see Ref. 42. Specifically, a stochastic process with periodic mean and covariance function $\gamma(t, \tau)$ satisfying the biperiodicity condition $\gamma(t + T, \tau + T) = \gamma(t, \tau)$ is said to be a cyclostationary process. In particular, its variance $\gamma(t, t)$ is T-periodic.

The periodic Lyapunov equation serves as a fundamental tool in the analysis of these processes. Assume that the initial state $x(0)$ of the system is a random variable with zero mean and covariance matrix P_0, and assume also that the input of the system is a white-noise process, independent of $x(0)$, with zero mean and unitary intensity. Then (21), under the stability assumption, the state of the periodic system asymptotically converges to a zero mean cyclostationary process with variance $\gamma(t, t)$, which can be computed via the periodic filtering Lyapunov equation by letting $Q(t) = B(t)B(t)'$ and $P(0) = P_0$. It turns out that

$$\lim_{t\to\infty}\{\gamma(t,t) - P(t)\} = 0$$

Floquet Theory

One of the long-standing issues in periodic systems is whether it is possible to find a state-coordinate transformation leading to a periodic system with *constant* dynamic matrix. In this way, the eigenvalues of such a dynamic matrix would determine the modes of the system. With reference to linear differential equations, this issue was considered by various mathematicians of the nineteenth century. Among them, a prominent role was played by the French scientist Gaston Floquet (1847–1920) who worked out a theory to solve linear homogeneous periodic systems, which is now named after him (43).

This theory can be outlined in a simple form as follows. If $S(\cdot)$ is a T-periodic invertible state-space transformation, $\hat{x}(t) = S(t)x(t)$, then, in the new coordinates, the dynamic $\hat{A}(t)$ is given by

$$\hat{A}(t) = \begin{cases} S(t)A(t)S(t)^{-1} + \dot{S}(t)S(t)^{-1} & \text{in continuous time} \\ S(t+1)A(t)S(t)^{-1} & \text{in discrete time} \end{cases}$$

The Floquet problem is then to find $S(t)$ (if any) in order to obtain a constant dynamic matrix $\hat{A}(t) = \hat{A}$.

In continuous time, it can be shown that such a transformation $S(\cdot)$ does exist, and the Floquet problem can be solved. Indeed, $\hat{A}$ can be obtained by solving $e^{\hat{A}T} = \Psi_A(\tau)$, where τ is any given time point. The appropriate transformation $S(\cdot)$ is simply given by

$$S(t) = e^{\hat{A}(t-\tau)}\Phi_A(\tau, t)$$

Such a matrix is indeed periodic of period T and satisfies the linear differential equation

$$\dot{S}(t) = \hat{A}S(t) - S(t)A(t)$$

with initial condition $S(\tau) = I$.

The discrete-time case is rather involved. Indeed, certain nonreversible systems do not admit any Floquet representation, as can easily be seen in the simple case $T = 2$, $A(0) = 0$, $A(1) = 1$, for which the equation $S(t + 1)A(t)S(t)^{-1}$ = constant does not admit any solution.

In the reversible case, such a representation always exists, and matrix $\hat{A}$ can be obtained by solving $\hat{A}^T = \Psi_A(\tau)$ and the transformation $S(\cdot)$ is given by

$$S(t) = \hat{A}^{t-\tau}\Phi(t, \tau)^{-1}$$

Again, it can be seen that such $S(\cdot)$ is periodic of period T and satisfies the linear difference equation

$$S(t+1) = \hat{A}S(t)A(t)^{-1}$$

with initial condition $S(\tau) = I$.

Whenever a Floquet representation exists, the eigenvalues of $\hat{A}$ are named *characteristic exponents.* In continuous time, the correspondence between a characteristic multiplier z and a characteristic exponent s is $z = e^{sT}$, whereas in discrete time, such a correspondence is $z = s^T$.

Main references for the Floquet theory and stability issues are Refs. 24, 36, 44, and 45. It should be emphasized that Floquet theory does not consider systems driven by external inputs. This nontrivial extension is touched upon in the sequel.

PERIODIC CONTROL

The early developments of periodic control were concentrated on the problem of forcing a periodic regime in order to improve the performance of an industrial plant (*periodic optimization*). At present, the term *periodic control* has taken a wider sense, so as to include the design of control systems where the controller and/or the plant are described by periodic models.

Periodic Optimization

In the 1970s, it was observed that "there is evidence that periodic operation [of catalytic reactors] can produce more reaction products or more valuable distribution of products, [and that] the production of wastes can perhaps be suppressed by cycling" (4). Ever since, the same idea has been further elaborated in other application fields, such as aeronautics (5,46,47), solar energy control (48), and social and economic sciences (49). This list of papers is largely incomplete, but in the bibliography the interested reader can find many more useful references.

In continuous time, the basic periodic optimization problem can be stated as follows. Consider the system

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t))$$

subject to the periodicity constant

$$x(T) = x(0)$$

and to further constraints of integral or pathwise type. The performance index to be maximized is

$$J = \frac{1}{T}\int_0^T g(x(t), u(t))\,dt$$

If we limit the desired operation to steady-state conditions, then an algebraic optimization problem arises, which can be tackled with mathematical programming techniques. Indeed, letting $u(t) = \text{const} = \overline{u}$ and $x(t) = \text{const} = \bar{x}$, the problem becomes that of maximizing $J = g(\bar{x}, \overline{u})$ under the constraint $f(\bar{x}, \overline{u}) = 0$. When passing from steady-state to periodic operations, an important preliminary question is whether the optimal steady-state regime can be improved by cycling or not. Denoting by $\overline{u}^0$ the optimal input at the steady state, consider the perturbed signal

$$u(t) = \overline{u}^0 + \delta u$$

where $\delta u(t)$ is a periodic perturbation. A problem for which there exists a (nonzero) periodic perturbation with a better performance is said to be *proper.* The issue of proper periodicity was originally dealt with in Refs. 50 and 51, by means of calculus of variation concepts. The underlying rationale is to express $\delta u(\cdot)$ in its Fourier expansion:

$$\delta u = \sum_{k=-\infty}^{\infty} U_k e^{jk\Omega t}, \quad \Omega = \frac{2\pi}{T}$$

By means of variational techniques, it is possible to work out a quadratic expression for the second variation of the performance index

$$\delta^2 J = \sum_{k=-\infty}^{\infty} U_k^* \Pi(k\Omega) U_k$$

where U_k^* is the conjugate transpose of U_k. Matrix $\Pi(\omega)$ is a complex square matrix defined on the basis of the system equations linearized around the optimal steady-state regime and on the basis of the second derivatives of the so-called Hamiltonian function associated with the optimal control problem, again evaluated at the optimal steady-state regime. Notice that $\Pi(\omega)$ turns out to be a Hermitian matrix, namely it coincides with its conjugate transpose [$\Pi^*(\omega) = \Pi(\omega)$]. Thanks to the preceding expression of $\delta^2 J$, it is possible to work out a proper periodicity condition in the frequency domain, known as Π-*test.* Basically, this test says that the optimal control problem is proper if, for some $\omega \neq 0$, $\Pi(\omega)$ is "partially positive" [i.e., there exists a vector $x \neq 0$ such that $x^*\Pi(\omega)x > 0$]. The partial positivity of $\Pi(\omega)$ is also a necessary condition for proper periodicity if we consider the weak variations $\delta u(\cdot)$. In the single-input single-output case, the test can be given a graphical interpretation in the form of a "circle criterion."

Variational tools have been used in periodic control by many authors (see Refs. 52–57). Moreover, along a similar line, it is worth mentioning the area of *vibrational control* (see Ref. 58), dealing with the problem of forcing a time-invariant system to undertake a periodic movement in order to achieve a better stabilization property or a better performance specification.

In general, if we leave the area of weak variations, periodic optimization problems do not admit closed-form solutions. There is, however, a notable exception, as pointed out in Refs. 59 and 60. With reference to a linear system, if the problem is that of maximizing the output power

$$J = \frac{1}{T}\int_0^T y(t)'y(t)\,dt$$

under the periodicity state constraint

$$x(0) = x(T)$$

and the input power constraint

$$\frac{1}{T}\int_0^T u(t)'u(t)\,dt \leq 1$$

then the optimal input function is given by a sinusoidal signal of suitable frequency. In the single-input single-output case, denoting by $G(s)$ the system transfer function, the optimal frequency $\overline{\omega}$ is that associated with the peak value of the Bode diagram

$$|G(j\overline{\omega})| \geq |G(j\omega)|, \forall\omega$$

(In modern jargon, $\overline{\omega}$ is the value of the frequency associated with the H_∞ norm of the system.) In particular, the problem is proper if $\overline{\omega} > 0$. Otherwise, the optimal steady-state operation cannot be improved by cycling.

Periodic Control of Time-Invariant Systems

The application of periodic controllers to time-invariant *linear plants* has been treated in an extensive literature. Again, the basic concern is to solve problems otherwise unsolvable with time-invariant controllers or to improve the achievable control performances.

A typical line of reasoning adopted in this context can be explained by referring to the classical *output stabilization problem,* namely the problem of finding an algebraic feedback control law *based on the measurements of the output signal* in order to stabilize the overall control system. If the system is

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with the control law

$$u(t) = Fy(t)$$

the problem is to find a matrix F (if any) such that the closed-loop system

$$\dot{x}(t) = (A + BFC)x(t)$$

is stable. Although a number of necessary and sufficient conditions concerning the existence of a stabilizing matrix F have been provided in the literature, no effective algorithms are available for its determination, as discussed in Ref. 61. Moreover, it may be difficult or impossible to stabilize three linear time-invariant plants (62). Periodic sampled control seems to offer a practical way to tackle this problem. Indeed, consider the time-varying control law based on the sampled measurements of $y(\cdot)$

$$u(t) = F(t)y(kT), \qquad t \in [kT, kT + T)$$

The modulating function $F(\cdot)$ and the sampling period T have to be selected in order to stabilize the closed-loop system, now governed by the equation

$$x(kT + T) = A_c x(kT)$$

where

$$A_c = \left[e^{AT} + \int_0^T e^{A(T-\sigma)} BF(\sigma)C\, d\sigma \right]$$

The crucial point is the selection of matrix $F(\cdot)$ for a given period T. A possibility, originally proposed in Ref. 28, is to consider an $F(\cdot)$ given by the following expression

$$F(t) = B' e^{A'(T-t)} \left[\int_0^T e^{A(T-\sigma)} BB' e^{A'(T-\sigma)}\, d\sigma \right]^{-1} Z$$

with matrix Z still to be specified. Note that this formula is valid provided that the matrix inversion can be performed (this is indeed the case under the so-called reachability condition). In this way, the closed-loop matrix A_c takes the form

$$A_c = e^{AT} + ZC$$

Then, provided that some weak condition on the pair (A, C) is met, period T and matrix Z can be selected so as to stabilize A_c, (or, even, to assign its eigenvalues). The generalized sample-and-hold philosophy outlined previously in the simple problem of stabilization has been pursued in many other contexts, ranging from the problem of simultaneous stabilization of a finite number of plants (28) to that of fixed poles removal in decentralized control (63), from the issue of pole and/or zero-assignment (64–69), to that of gain margin or robustness improvement (7,70), from adaptive control (71) to model matching (28), and so on.

When using generalized sample-data control, however, the intersample behavior can present some critical aspects, as pointed out in several papers, such as Refs. 72 and 73. Indeed, the action of the generalized sample-and-hold function is a sort of amplitude modulation, which, in the frequency domain, may lead to additional high-frequency components centered on multiples of the sampling frequency. Consequently, there are nonnegligible high-frequency components both in the output and control signals. To smooth out these ripples, remedies have been studied, see Refs. 9 and 74. An obvious possibility is to *continuously* monitor the output signal and to adopt the feedback control strategy $u(t) = F(t)y(t)$, with a periodic gain $F(t)$, in place of the sampled strategy before seen. This point of view is adopted in Ref. in 75, where a pole-assignment problem in discrete time is considered.

In the control of time-invariant systems, linear-quadratic optimal control theory represents a cornerstone achievement of the second half of the twentieth century. We can wonder whether, by enlarging the family of controllers from the time-invariant class to the class of periodic controllers, the achievable performance can be improved. To this question, the reply may be negative, even in the presence of bounded disturbances, as argued in Ref. 76.

Periodic Control of Periodic Systems

A typical way to control a plant described by a linear periodic model is to impose

$$u(t) = K(t)x(t) + S(t)v(t)$$

where $K(\cdot)$ is a periodic feedback gain [$K(t + T) = K(t)$, $\forall t$], $S(\cdot)$ is a periodic feedforward gain [$S(t + T) = S(t)$, $\forall t$], and $v(t)$ is a new exogenous signal. The associated closed-loop system is then

$$\dot{x}(t) = (A(t) + B(t)K(t))x(t) + B(t)S(t)v(t)$$

in continuous time and

$$x(t+1) = (A(t) + B(t)K(t))x(t) + B(t)S(t)v(t)$$

in discrete time. In particular, the closed-loop dynamic matrix is the periodic matrix $A(t) + B(t)K(t)$.

The main problems considered in the literature follow:

1. *Stabilization.* Find a periodic feedback gain in such a way that the closed-loop system is stable [any $K(\cdot)$ meeting such a requirement is named stabilizing gain].
2. *Pole Assignment.* Find a periodic feedback gain so as to position the closed-loop characteristic multipliers in given locations in the complex plane.
3. *Optimal Control.* Set $v(\cdot) = 0$ and find a periodic feedback gain so as to minimize the quadratic performance index

$$J = \begin{cases} \int_0^\infty [x(t)'Q(t)x(t) + u(t)'R(t)u(t)]\, dt & \text{in continuous time} \\ \sum_{k=0}^\infty x(k)'Q(k)x(k) + u(k)'R(k)u(k) & \text{in discrete time} \end{cases}$$

4. *Invariantization.* Find a feedback control law such that the closed-loop system is time-invariant up to a periodic state-space coordinate change.
5. *Exact Model Matching.* Let $y(t) = C(t)x(t)$ be a system output variable. Find a feedback control law such that the closed loop input-output relation [from $v(\cdot)$ to $y(\cdot)$] matches the input-output behavior of a given periodic system.
6. *Tracking and Regulation.* Find a periodic controller in order to guarantee closed-loop stability and robust zeroing of the tracking errors for a given class of reference signals.

We now briefly elaborate on these problems by reviewing the main results available in the literature.

As a paradigm problem in control, the *stabilization* issue is the starting point of further performance requirement problems. A general parametrization of all periodic stabilizing gains can be worked out by means of a suitable matrix inequality. Specifically, by making reference to discrete time, the *filtering Lyapunov inequality* seen in the section devoted to the monodromy matrix and stability enables us to conclude that the closed-loop system associated with a periodic gain $K(\cdot)$ is stable if and only if there exists a positive definite periodic matrix $Q(\cdot)$ satisfying the inequality:

$$Q(t+1) > (A(t)+B(t)K(t))Q(t)(A(t)+B(t)K(t))', \forall t$$

Then, it is possible to show that a periodic gain is stabilizing if and only if it can be written in the form:

$$K(t) = W(t)'Q(t)^{-1}$$

where $W(\cdot)$ and $Q(\cdot) > 0$ are periodic matrices (of dimensions $m \times n$ and $n \times n$, respectively), solving the matrix inequality

$$\begin{aligned}Q(t+1) > {}& A(t)Q(t)A(t)' + B(t)W(t)'A(t)' + A(t)W(t)B(t)' \\ &+ B(t)W(t)'Q(t)^{-1}W(t)B(t)', \forall t\end{aligned}$$

that can be equivalently given in a linear matrix inequality (LMI) form. The *pole assignment problem* (by state feedback) is somehow strictly related to the *invariantization problem.* Both problems have been considered in an early paper (77), where continuous-time systems are treated, and subsequently in Refs. 78–80. The basic idea is to render the system algebraically equivalent to a time-invariant one by means of a first periodic state feedback (invariantization) and then to resort to the pole assignment theory for time-invariant systems in order to locate the characteristic multipliers. Thus, the control scheme comprises two feedback loops, the inner for invariantization and the outer for pole placement.

Analogous considerations can be applied in the discrete-time case (81), with some care for the possible nonreversibility of the system.

The *model matching* and the *tracking* problems are dealt with in Refs. 82 and 83, respectively.

Finally, the optimal control approach to periodic control deserves an extensive presentation and therefore is treated in the next section.

PERIODIC OPTIMAL CONTROL

As for the vast area of optimal control, attention focuses herein on two main design methodologies, namely (i) linear quadratic control and (ii) receding horizon control. Both will be presented by making reference to continuous time.

Linear Quadratic Periodic Control

For a continuous-time periodic system, the classical *finite horizon* optimal control problem is that of minimizing the quadratic performance index over the time interval (t, t_f):

$$J(t,t_f,x_t) = x(t_f)'P_{t_f}x(t_f) + \int_t^{t_f} z(\tau)'z(\tau)\,d\tau$$

where x_t is the system initial state at time t, $P_{t_f} \geq 0$ is the matrix weighting the final state $x(t_f)$, and $z(\cdot)$ is a "performance evaluation variable." Considering that the second term of $J(t, t_f, x_t)$ is the "energy" of $z(\cdot)$, the definition of such a variable reflects a main design specification. A common choice is to select $z(t)$ as a linear combination of $x(t)$ and $u(t)$, such that

$$z(t) = \tilde{C}(t)x(t) + \tilde{D}(t)u(t)$$

where $\tilde{C}(t)$ and $\tilde{D}(t)$ are to be tuned by the designer. In this way, the performance index can also be written in the (perhaps more popular) form

$$\begin{aligned}J(t,t_f,x_t) = {}& x(t_f)'P_{t_f}x(t_f) + \int_t^{t_f}\{x(\tau)'Q(\tau)x(\tau) \\ &+ 2u(\tau)'S(\tau)x(\tau) + u(\tau)'R(\tau)u(\tau)\}\,d\tau\end{aligned}$$

where $Q(\tau) = \tilde{C}(\tau)'\tilde{C}(\tau)$, $S(\tau) = \tilde{D}(\tau)'\tilde{C}(\tau)$, and $R(\tau) = \tilde{D}(\tau)'\tilde{D}(\tau)$. We will assume for simplicity that the problem is nonsingular [i.e., $R(\tau) > 0$, $\forall\tau$].

This problem is known as the *linear quadratic* (LQ) optimal control problem. To solve it, the auxiliary matrix equation

$$-\dot{P}(t) = \tilde{A}(t)'P(t) + P(t)\tilde{A}(t) - P(t)B(t)R(t)^{-1}B(t)'P(t) + \tilde{Q}(t)$$

where

$$\tilde{A}(t) = A(t) - B(t)R(t)^{-1}S(t), \quad \tilde{Q}(t) = Q(t) - S(t)'R(t)^{-1}S(t)$$

is introduced. This is the well-known *differential Riccati equation,* in one of its many equivalent forms. More precisely, because the coefficients are periodic, the equation is referred to as the *periodic differential Riccati equation.*

Let $\Pi(\cdot, t_f)$ be the backward solution of the periodic Riccati equation with terminal condition $\Pi(t_f, t_f) = P_{t_f}$. Assuming that the state $x(\cdot)$ can be measured, the solution to the minimization problem can be easily written in terms of $\Pi(\cdot, t_f)$ as follows

$$u(\tau) = \Lambda_o(\tau, t_f)x(\tau)$$

where

$$\Lambda_o(\tau,t_f) = -R(\tau)^{-1}[B(\tau)'\Pi(\tau,t_f) + S(\tau)]$$

Moreover, the value of the performance index associated with the optimal solution is

$$J^o(t,t_f,x_t) = x_t'\Pi(t,t_f)x_t$$

The passage from the finite horizon to the *infinite horizon* problem ($t_f \to \infty$) can be performed provided that $\Pi(t, t_f)$ remains bounded for each $t_f > t$ and converges as $t_f \to \infty$: In other words, if there exists $P(t)$ such that

$$\lim_{t_f\to\infty} \Pi(t,t_f) = P(t), \forall t$$

Under suitable assumptions concerning the matrices [$A(\cdot)$, $B(\cdot)$, $Q(\cdot)$, $S(\cdot)$], the limit matrix $P(\cdot)$ exists and is the unique positive semidefinite and T-periodic solution of the periodic differential Riccati equation. The optimal control action is given by

$$u(\tau) = K_o(\tau)x(\tau)$$

where $K_o(\tau)$ is the periodic matrix obtained from $\Lambda_o(\tau, t_f)$ by letting $t_f \to \infty$. Finally, the optimal infinite horizon performance index takes on the value

$$\lim_{t_f\to\infty} J^o(t,t_f,x_t) = x_t'P(t)x_t$$

If the state is not accessible, we must rely instead on the measurable output

$$y(t) = C(t)x(t) + D(t)u(t)$$

A first task of the controller is then to infer the actual value of the state $x(t)$ from the past observation of $y(\cdot)$ and $u(\cdot)$ up to time t. This leads to the problem of finding an estimate $\hat{x}(t)$ of $x(t)$ as the output of a linear system (filter) fed by the available measurements. The design of such a filter can be carried out in a variety of ways, among which it is worth mentioning the celebrated *Kalman filter,* the implementation of which requires the solution of another matrix Riccati equation with periodic coefficients. When $\hat{x}(t)$ is available, the control action is typically obtained as

$$u(\tau) = K_o(\tau)\hat{x}(\tau)$$

Thus, the control scheme of the controller takes the form of a cascade of two blocks, as can be seen in Fig. 3. Periodic optimal filtering and control problems for periodic systems have been intensively investigated (see Refs. 84–93). For numerical issues see, for example, Ref. 94.

Receding Horizon Periodic Control

The infinite horizon optimal control law can be implemented provided that the periodic solution of the matrix Riccati equation is available. Finding such a solution may be computationally demanding so that the development of simpler control design tools has been considered. Among them, an interesting approach is provided by the so-called *receding horizon* control strategy, which has its roots in optimal control theory and remarkable connections with the field of adaptive and predictive control (see Refs. 95 and 96). Among the many research streams considered in such a context, the periodic stabilization of time-invariant systems is dealt with in Ref. 97 under the heading of "intervalwise receding horizon control"; see also Ref. 96. The approach was then extended to periodic systems in Ref. 98. The problem can be stated as follows. Consider the optimal control problem with $S(\cdot) = 0$ and $R(\cdot) = I$, and write the performance index over the interval $(t_f - T, t_f)$ as

$$J = x(t_f)'P_{t_f}x(t_f) + \int_{t_f-T}^{t_f} \{x(\tau)'Q(\tau)x(\tau) + u(\tau)'u(\tau)\}\,d\tau$$

Assume that the solution $\Pi(\cdot, t_f)$ of the Riccati equation with terminal condition $\Pi(t_f, t_f) = P_{t_f}$ is such that

$$\Delta_{t_f} = P_{t_f} - \Pi(t_f - T, t_f) \geq 0$$

This condition is usually referred to as cyclomonotonicity condition. Now, consider the periodic extension $P_e(\cdot)$ of $\Pi(\cdot, t_f)$

$$P_e(t + kT) = \Pi(t, t_f), \quad t \in (t_f - T, t_f], \quad \forall \text{integer } k$$

Then, under mild assumptions on $[A(\cdot), B(\cdot), Q(\cdot)]$, it turns out that the receding horizon control law

$$u(\tau) = -B(\tau)'P_e(\tau)x(\tau)$$

is stabilizing. Although such a control law is suboptimal, it has the advantage of requiring the integration of the Riccati equation over a finite interval (precisely over an interval of length T, which must be selected by the designer in the case of time-invariant plants and coincides with the system period—or a multiple—in the periodic case). However, for feedback stability, it is fundamental to check if the cyclomonotonicity condition is met. If not, we are led to consider a different selection of matrix P_{t_f}. Some general guidelines for the choice of P_{t_f} can be found in the literature. The simplest way is to choose P_{t_f} "indefinitely large," ideally such that $P_{t_f}^{-1} = 0$. Indeed, as is well known in optimal control theory, such condition guarantees that the solution of the differential Riccati equation enjoys the required monotonicity property.

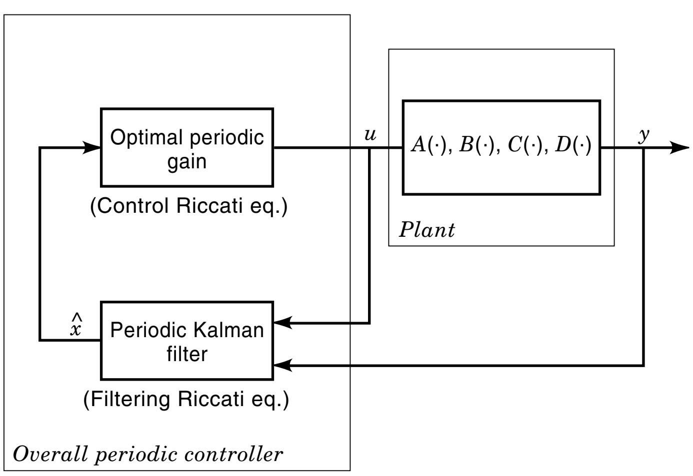

Figure 3. Periodic optimal control based on the measurement of the output signal. The controller is constituted of two blocks. The first one (Kalman filter) elaborates the external signals of the plant (u and y) to provide an estimate $\hat{x}$ of the unmeasureable state. The second block consists in an algebraic gain providing the command u from the estimated state $\hat{x}$.

CONCLUSION

Optimal control ideas have been used in a variety of contexts, and have been adequately shaped for the needs of the specific problem dealt with. In particular, ad hoc control design techniques have been developed for the rejection or attenuation of periodic disturbances, a problem of major importance in the emerging field of active control of vibrations and noise; see Refs. 99 and 100.

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S. Bittanti
P. Colaneri
Politecnico di Milano