OPTIMAL CONTROL

Optimal control theory is concerned with the development of techniques that allow one to control physical phenomena described by dynamical systems in such a manner that a prescribed performance criterion is minimized. The principal components of an optimal control problem are the mathematical model in the form of a differential equation, a description of how the control enters into this system, and a criterion describing the cost.

The start of optimal control theory, as a mathematical discipline, dates back to the mid 1940s. The increasing interest in and use of methods provided by optimal control theory is linked to the rise of the importance of mathematical models in many diverse areas of science—including chemistry, medicine, biology, management, and finance—and to ever increasing computing power, which allows the realization of optimal control strategies for practical systems of increasing difficulty and complexity. While optimal control theory has its roots in the classical calculus of variations, its specific nature has necessitated the development of new techniques. In contrast with general optimization problems, whose constraints are typically described by algebraic equations, the constraints in optimal control problems are given by dynamical systems. The dynamic programming principle, the Pontryagin maximum principle, the Hamilton–Jacobi–Bellman equation, the Riccati equation arising in the linear quadratic regulator problem, and (more recently) the theory of viscosity solutions are some of the milestones in the analysis of optimal control theory.

Analyzing an optimal control problem for a concrete system requires knowledge of the systems-theoretic properties of the control problem and its linearization (controllability, stabilizability, etc.). Its solution, in turn, may give significant additional insight. In some cases, a suboptimal solution that stabilizes the physical system under consideration may be the main purpose of formulating an optimal control problem, while an exact solution is of secondary importance.

In the first section we explain some of the concepts in optimal control theory by means of a classical example. The following sections describe some of the most relevant techniques in the mathematical theory of optimal control.

Many monographs, emphasizing either theoretical or control engineering aspects, are devoted to optimal control theory. Some of these texts are listed in the bibliography and reading list.

DESCRIPTIVE EXAMPLE AND BASIC CONCEPTS

Control Problem

We consider the controlled motion of a pendulum described by

$$m\frac{d^2}{dt^2}y(t) + mg\sin y(t) = u(t), \qquad t > 0 \tag{1}$$

with initial conditions

$$y(0) = y_0 \quad \text{and} \quad \frac{d}{dt}y(0) = v_0$$

Here $y(t)$ is the angular displacement, m is the mass, and g is the gravitational acceleration. Further, $u(t)$ represents the applied force, which will be chosen from a specified class of functions in such a way that the system described by Eq. (1) behaves in a desired way. We refer to y and u as the *state* and *control* variables. Due to the appearance of the sine function, Eq. (1) constitutes a nonlinear control system. It will be convenient to express Eq. (1) as a first-order system. For this purpose, we define $x(t) = \text{col}(x_1(t), x_2(t))$, where $x_1(t) = y(t)$ and $x_2(t) = (d/dt)y(t)$. Then we obtain the first-order form of Eq. (1), which is of dimension $n = 2$:

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -g\sin x_1(t) + u(t) \end{pmatrix} \tag{2}$$

with initial condition $x(0) = x_0 = (y_0, v_0) \in R^2$, where we assume $m = 1$. In general, a control system is written in the form

$$\frac{d}{dt}x(t) = f(t, x(t), u(t)), \qquad x(0) = x_0 \tag{3}$$

with state vector $x(t) \in R^n$, control input $u(t) \in R^m$, and $f: R^1 \times R^n \times R^m \to R^n$. If f is independent of time t [$f = f(x, u)$], then the system is said to be autonomous.

Next we formulate a sample control system associated to Eq. (1). For that purpose, note that the stationary solutions to the uncontrolled system, which are characterized by $f(x, \overline{u}) = 0$ for $\overline{u} = 0$, are given by (0, 0) and (π, 0). Our objective is to regulate the state $x(t) \in R^2$ to the stationary state (π, 0). Thus a control u must be determined that steers the system described by Eq. (1) from the initial state x_0 to the vertical position ($y = \pi$) or into its neighborhood (inverted-pendulum problem). This objective can be formulated as an optimal control problem: minimize the cost functional

$$J(x, u) = \int_0^{t_f} [|x_1(t) - \pi|^2 + |x_2(t)|^2 + \beta|u(t)|^2]\,dt + \alpha[|x_1(t_f) - \pi|^2 + |x_2(t_f)|^2] \tag{4}$$

subject to Eq. (2), over $u \in L^2(0, t_f; R^1)$, the space of square-integrable functions on $(0, t_f)$. The nonnegative constants β and α are the weights for the control cost and target constraint at the terminal time $t_f > 0$, respectively. The integrand $|x_1(t) - \pi|^2 + |x_2(t)|^2$ describes the desired performance of the trajectory [the square of distance of the current state $(x_1(t), x_2(t))$ to the target (π, 0)]. The choice of the cost functional J contains a certain freedom. Practical considerations frequently suggest the use of quadratic functionals.

A general form of optimal control problems is given by

$$\min \quad \int_0^{t_f} f^0(t, x(t), u(t))\,dt + g(t_f, x(t_f)) \tag{5}$$

subject to Eq. (3), over $u \in L^2(0, t_f; R^m)$ with $u(t) \in U$ a.e. in $(0, t_f)$, where U is a closed convex set in R^m describing con-

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straints that must be observed by the class of admissible controls. In the terminology of the calculus of variations, Eq. (5) is called a Bolza problem. The special cases with $f^0 = 0$ and with $g = 0$ are referred to as the Lagrange and the Mayer problem, respectively. If $t_f > 0$ is finite, then Eq. (5) is called a finite-time-horizon problem. In case $g = 0$ and $t_f = \infty$, we refer to Eq. (5) as an infinite-time-horizon problem. The significance of the latter is related to the stabilization of Eq. (3).

If Eq. (5) admits a solution u^*, we refer to it as the optimal control, and the associated state $x^* = x(u^*)$ is the optimal trajectory. Under certain conditions, the optimal control can be expressed as a function of x^*, that is, $u^*(t) = K(t, x^*(t))$ for an appropriate choice of K. In this case u^* is said to be given in feedback or closed-loop form.

If the final time t_f itself is a free variable and $f^0 = 1$, then Eq. (5) becomes the time optimal control problem.

For certain analytical and numerical considerations, the treatment of the fully nonlinear problem (5) can be infeasible or lengthy. In these cases, a linearization of the nonlinear dynamics around nominal solutions will be utilized.

Linearization

We discuss the linearization of the control system (3) around stationary solutions. Henceforth $\bar{x}$ stands for a stationary solution of $f(x, \bar{u}) = 0$, where $\bar{u}$ is a nominal constant control. Let $A \in R^{n\times n}$ and $B \in R^{n\times m}$ denote the Jacobians of f at $(\bar{x}, \bar{u})$; that is,

$$A = f_x(\bar{x}, \bar{u}) \quad \text{and} \quad B = f_u(\bar{x}, \bar{u}) \tag{6}$$

Defining $z(t) = x(t) - \bar{x}$ and $v(t) = u(t) - \bar{u}$ and assuming that f is twice continuously differentiable, Eq. (3) can be expressed as

$$\frac{d}{dt}z(t) = Az(t) + Bv(t) + r(z(t), v(t))$$

where the residual dynamics r satisfy

$$|r(z(t), v(t))|_{R^n} \le \text{const. } [|z(t)|^2 + |v(t)|^2]$$

This implies that the residual dynamics r are dominated by the linear part $Az(t) + Bv(t)$ if $|(z(t), v(t))|$ is sufficiently small. We obtain the linearization of the control system (3) around $(\bar{x}, \bar{u})$:

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 - \bar{x} \tag{7}$$

where now $x(t)$ and $u(t)$ represent the translated coordinates $x(t) - \bar{x}$ and $u(t) - \bar{u}$, respectively. We refer to Eq. (7) as a *linear control system.*

For the example of the pendulum we find

$$A = \begin{pmatrix} 0 & 1 \\ -g & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6a}$$

at $\bar{x} = (0, 0)$ and

$$A = \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6b}$$

at $\bar{x} = (\pi, 0)$. The linearized control system for the inverted pendulum is given by

$$\frac{d}{dt}x(t) = \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

where $x_1(t)$ is the relative angle from π.

Stability

In this and the following subsection we restrict our attention to linear control systems of the form (7). One of the main objectives of optimal control is to find controls in state feedback form $u(t) = -Kx(t)$ with $K \in R^{m\times n}$ such that the closed-loop system

$$\frac{d}{dt}x(t) = Ax(t) - BKx(t) = (A - BK)x(t) \tag{8}$$

is asymptotically stable, in the sense that $|x(t)|_{R^n} \to 0$ as $t \to \infty$ for all $x_0 \in R^n$. Recall that a system of the form

$$\frac{d}{dt}x(t) = Ax(t)$$

is asymptotically stable if and only if all eigenvalues λ of the matrix A satisfy Re $\lambda < 0$. For example, for the matrices in Eqs. (6a) and (6b) we have

$$\det(\lambda I - A) = \lambda^2 + g = 0 \implies \lambda = \{\pm\sqrt{g}i\} \quad \text{(marginal stability)}$$

and

$$\det(\lambda I - A) = \lambda^2 - g = 0 \implies \lambda = \{\pm\sqrt{g}\} \quad \text{(instability)}$$

respectively. In particular, this implies that the uncontrolled inverted-pendulum problem is unstable in the sense of Liapunov stability theory. For the closed-loop feedback system (8) associated with the inverted pendulum with feedback matrix chosen in the form

$$K = (0 \quad \gamma) \in R^{1\times 2}$$

we find that the eigenvalues of $A - BK$ are given by $-\frac{1}{2}(-\gamma \pm \sqrt{\gamma^2 - 4g})$ and hence the closed-loop system is asymptotically stable for appropriately chosen $\gamma > 0$. Moreover, if we apply the feedback $u(t) = -Kz(t)$ with $z(t) = x(t) - (\pi, 0)$ to the original system (2), then the closed system is locally asymptotically stable by Liapunov stability theory.

Linear Quadratic Regulator Problem

In order to construct the optimal stabilizing feedback law $u(t) = -Kx(t)$ for the linear system (7), we consider the infinite-time-horizon linear quadratic regulator (LQR) problem

$$\min \int_0^\infty [x^t(t)Qx(t) + u^t(t)Ru(t)]\,dt \tag{9}$$

subject to Eq. (7), where $Q, G \in R^{n\times n}$ are symmetric nonnegative definite matrices and $R \in R^{m\times m}$ is symmetric and positive definite. The optimal solution $u^*(\cdot)$ to Eq. (9) is given in feedback form by

$$u^*(t) = -R^{-1}B^tPx^*(t) \tag{10}$$

where the optimal trajectory $x^*(\cdot)$ satisfies

$$\frac{d}{dt}x^*(t) = (A - BR^{-1}B^tP)x^*(t), \qquad x^*(0) = x_0$$

and the symmetric nonnegative definite matrix $P \in R^{n\times n}$ satisfies the matrix Riccati equation

$$A^tP + PA - PBR^{-1}B^tP + Q = 0 \tag{11}$$

In the section titled "Linear Quadratic Regulator Problem" we shall return to a detailed discussion of this equation and its significance in optimal control theory.

EXISTENCE AND NECESSARY OPTIMALITY

In this section we consider the optimal control problems of Lagrange type

$$\min \quad J(x,u) = \int_0^{t_f} f^0(t, x(t), u(t))\,dt \tag{12}$$

subject to the dynamical system

$$\frac{d}{dt}x(t) = f(t, x(t), u(t)) \tag{13}$$

control constraints

$$u(t) \in U \qquad \text{(a closed set in } R^m) \tag{14}$$

and initial and target constraints

$$x(0) = x_0 \quad \text{and} \quad \varphi(t_f, x(t_f)) = 0 \tag{15}$$

over $u \in L^2(0, t_f; R^m)$, where $f: R^+ \times R^n \times R^m \to R^n$, $f^0: R^+ \times R^n \times R^m \to R$, and $\varphi: R \times R^n \to R^p$ are C^1 functions. In contrast with Eq. (5), we generalize the control problem in that we restrict the trajectory to reach a target described by the manifold $\varphi(t_f, x(t_f)) = 0$. If, for example, $t_f > 0$ is free, $f^0(t, x, u) = 1$, and $\varphi(t, x) = x$, then the objective is to bring the system to rest in minimum time. Typical forms of the control constraint set U are given by $U = \{u \in R^m : |u| \le \gamma\}$ and $U = \{u \in R^m : u_i \le 0, 1 \le i \le m\}$.

In order to obtain a first insight into the problem of (12)–(15), one needs to address the questions of (a) the existence of admissible candidates (x, u) satisfying Eqs. (13)–(15), (b) the existence and uniqueness of solutions to the optimal control problem (12)–(15), and (c) necessary optimality conditions.

In the remainder of this section we shall present some of the ideas that were developed to answer these questions. For detailed information we refer to the bibliography and reading list and to additional references given in the listed works.

In spite of their importance, in practice we shall not consider problems with constraints on the trajectories except at the initial and terminal times. Also we shall not systematically discuss the bang–bang principle, which states that, for certain control systems with controls constrained to lie in a convex compact set, the optimal controls are achieved in the extremal points of the admissible control set.

Existence of Optimal Controls

The problem of the existence of admissible control–trajectory pairs (x, u) and of solutions to the optimal control problem (12)–(15) has stimulated a significant amount of research. Here we can only give the flavor of some of the relevant aspects required to guarantee existence of optimal controls.

Let us assume in this subsection that t_f is fixed. Then the optimal control problem can be stated in the form of a nonlinear mathematical programming problem for $(x, u) \in H^1(0, t_f, R^n) \times L^2(0, t_f, R^m)$:

$$\min \quad J(x,u) \tag{16}$$

subject to the equality constraints

$$E(x,u) = \begin{Bmatrix} \dfrac{d}{dt}x(t) - f(t,x,u) \\ \varphi(t_f, x(t_f)) \end{Bmatrix} = 0 \tag{17}$$

and

$$u \in K = \{u(t) \in U \text{ a.e. in } (0, t_f)\} \tag{18}$$

with K a closed convex subset of $L^2(0, t_f; R^m)$ and E considered as a mapping from $H^1(0, t_f; R^n) \times L^2(0, t_f; R^n) \to L^2(0, t_f; R^n) \times R^p$. Question (a) above is equivalent to the existence of feasible points satisfying the equality and control constraints (17) and (18). The existence of optimal controls can be argued as follows: Under an appropriate assumption the state function $x = x(\cdot, u) \in L^2(0, t_f; R^n)$ can be defined as the unique solution to Eq. (13) with initial condition $x(0) = x_0$, so that the control problem (12)–(15) can be written as

$$\min \ J(x(u), u) \quad \text{over } u \in K \qquad \text{with} \quad \varphi(t_f, x(u)(t_f)) = 0 \tag{19}$$

Suppose that the admissible control set K is compact, that is, every bounded sequence in K has a strongly convergent subsequence in K, and the solution map $u \in L^2(0, t_f; R^m) \to (x(u), x(u)(t_f)) \in L^2(0, t_f, R^n) \times R^n$ is strongly continuous. Moreover, assume that the functional J is lower semicontinuous, that is,

$$J(\lim x_n, \lim u_n) \le \lim\inf J(x_n, u_n)$$

for all strongly convergent sequences $\{(x_n, u_n)\}$ in $L^2(0, t_f, R^n) \times L^2(0, t_f, R^m)$. Then the control problem (12)–(15) has a solution. In fact, let $\eta = \inf J(x(u), u)$ over $u \in K$ with $\varphi(t_f, x(u)(t_f)) = 0$, and let $\{u_n\}$ be a minimizing sequence, that is, $J(x(u_1), u_1) \le J(x(u_2), u_2) \le \cdots$, with $\lim_{n\to\infty} J(x(u_n), u_n) = \eta$ and the constraints in Eq. (19) are satisfied. Due to the compactness assumption for K, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u^*$ for some $u^* \in K$. The continuity assumption for the control to solution mapping implies that

$\varphi(t_f, x(u^*)(t_f)) = 0$, and from the semicontinuity of J it follows that

$$J(x(u^*), u^*) = J(\lim x(u_{n_k}), \lim u_{n_k}) \\ \le \liminf J(x(u_{n_k}), u_{n_k}) = \eta$$

This implies that $J(x(u^*), u^*) = \eta$ and u^* is a solution to Eqs. (12)–(15).

Alternatively to the compactness assumption for K, we may assume that either $\lim_{|u|\to\infty} J(x(u), u) = \infty$, or that K is bounded. We then also require that the solution map $u \in L^2(0, t_f; R^m) \to (x(u), x(u)(t_f)) \in L^2(0, t_f; R^n) \times R^n$ be continuous when $L^2(0, t_f; R^m)$ is endowed with the weak topology and that the functional J is weakly sequentially lower semicontinuous. Then, using arguments similar to the ones above, the existence of a solution to Eqs. (12)–(15) again follows.

Pontryagin Maximum Principle

An important step toward practical realization of optimal control problems is the derivation of systems of equations that must be satisfied by the optimal controls and optimal trajectories. The maximum principle provides such a set of equations. It gives a set of necessary optimality conditions for the optimal control problem (12)–(15).

We shall require the Hamiltonian associated with Eqs. (12)–(15) given by

$$H(t, x, u, \hat{\lambda}) = \lambda_0 f^0(t, x, u) + \lambda f(t, x, u) \tag{20}$$

where $\hat{\lambda} = (\lambda^0, \lambda) \in R \times R^n$.

Theorem 1. Assume that f^0, f, φ are sufficiently smooth, and suppose that (x^*, u^*) minimizes the cost functional in Eq. (12) subject to Eqs. (13)–(15). Then there exists $\hat{\lambda}(t) = (\lambda_0, \lambda(t)) \in R^{n\times 1}$ with $\lambda_0 \le 0$ such that $\hat{\lambda}(t)$ never vanishes on $[0, t_f]$, and

1. Maximum condition:

$$H(t, x^*(t), u^*(t), \hat{\lambda}(t)) \ge H(t, x^*(t), u, \hat{\lambda}(t)) \quad \text{for all} \quad u \in U$$

2. Adjoint equation:

$$\frac{d}{dt}\lambda(t) = -H_x(t, x^*(t), u^*(t), \hat{\lambda}(t))$$

3. Transversality:

$$(H(t_f, x^*(t_f), u^*(t_f), \hat{\lambda}(t_f)), -\lambda(t_f)) \perp T_f$$

where T_{t_f} is the tangent space to the manifold described by $\varphi(t, x) = 0$ at $(t_f, x^*(t_f))$.

An admissible triple $(x^*, u^*, \hat{\lambda}^*)$ that satisfies the conclusions of Theorem 1 is called an *extremal element.* The function x^* is called the *extremal trajectory,* and u^* is called the *extremal control.*

Remarks

1. The maximum principle provides a necessary condition for optimality. It is simple to find examples illustrating the fact that it is in general not a sufficient optimality condition, that is, $(x^*, u^*, \hat{\lambda}^*)$ can be an extremal element without (x^*, u^*) being a solution to the control problem in (12)–(15).
2. We refer to the literature for the proof of the maximum principle. A proof is sketched in the next subsection. We also mention that the following fact plays an essential role. Let $s \in [0, t_f]$, and consider the problem (12)–(15) with initial time 0 replaced by s and initial condition $x(s) = x^*(s)$. Then the optimal state–control pair restricted to $[s, t_f]$ is optimal for the control problem starting at s with $x(s) = x^*(s)$.
3. Suppose t_f is fixed, that is, $\varphi_0(t) = t - t_f$ and that the target constraint is described by p additional conditions $\varphi_i(x) = 0$, $i = 1, \ldots, p$. Then the transversality condition can be expressed as

$$(H(t_f, x^*(t_f), u^*(t_f), \hat{\lambda}(t_f)), -\lambda(t_f)) \\ = \mu_0(1, 0, \ldots, 0) + (0, \mu\varphi_x(x^*(t_f))$$

for some $(\mu_0, \mu) \in R \times R^p$. Here we set $\mu\varphi_x(x^*(t_f)) = \sum_{i=1}^p \mu_i \operatorname{grad} \varphi_i(x^*(t_f))$.

4. If one can ascertain that $\lambda_0 \ne 0$ (normality), then without loss of generality we can set $\lambda_0 = -1$, and conditions 2–3 of Theorem 1 can be equivalently expressed as

$$\frac{d}{dt}\lambda(t) = -H_x(t, x^*, u^*, \hat{\lambda}) \qquad \lambda(t_f) = -\mu\varphi_x(x^*(t_f)) \tag{21}$$

If t_f is fixed and no other target constraints are given, then normality holds. In fact, from the adjoint equation and the transversality condition we have

$$\frac{d}{dt}\lambda(t) = -\lambda_0 f_x^0(t, x^*(t), u^*) - \lambda(t) f_x(t, x^*(t), u^*(t))$$

with $\lambda(t_f) = 0$. If λ_0 were 0, then $\lambda(t) = 0$ on $[0, t_f]$, which gives a contradiction.

5. The maximum principle is based on first-order information of the Hamilton H. Additional assumptions involving, for example, convexity conditions or second-order information are required to ascertain that a pair (x^*, u^*) satisfying conditions 1–3 of Theorem 1 is, in fact, a solution to the problems in Eqs. (16)–(18). Some sufficient optimality conditions are discussed in the next two subsections.

Example. We conclude this subsection with an example. Let us denote by x_1, x_2, x_3 the rates of production, reinvestment, and consumption of a production process. Dynamical constraints are given by

$$\frac{d}{dt}x_1(t) = x_2(t), \qquad x_1(0) = c > 0$$

and it is assumed that $x_1 = x_2 + x_3$ and $x_i \ge 0$ for $i = 1, 2, 3$. The control function is related to the state variables and is chosen as $u(t) = x_2(t)/x_1(t)$. The objective consists in maximizing the total amount of consumption Φ given by

$$\Phi = \int_0^T x_3(t)\, dt$$

on the fixed operating period [0, T] with $T \geq 1$. Setting $x = x_1$, this problem can be formulated as a Lagrange problem:

$$\min \quad J = \int_0^T [u(t) - 1]x(t)\,dt \tag{22}$$

subject to

$$\frac{d}{dt}x(t) = u(t)x(t), \qquad x(0) = c \text{ and } u(t) \in U = [0, 1]$$

To apply the maximum principle, note first that $x(t) > 0$ on [0, T] for all admissible control u. The Hamiltonian H is given by

$$H = \lambda_0(u - 1)x + \lambda\, ux$$

the adjoint equation is

$$\frac{d}{dt}\lambda = -H_x = -\lambda_0(u - 1) - \lambda u$$

and the transversality condition implies that $\lambda(T) = 0$. Since $(\lambda_0, \lambda(t)) \neq 0$ on [0, T], it follows that $\lambda_0 \neq 0$. Thus normality of the extremals holds, and we set $\lambda_0 = -1$. The maximum condition implies that

$$[1 - u^*(t)]x^*(t) + \lambda(t)x^*(t)u^*(t) \geq (1 - u)x^*(t) + \lambda(t)x^*(t)u \quad \text{for all} \quad u \in [0, 1]$$

Since necessarily $x^*(t) > 0$, the sign of $\lambda(t) - 1$ determines $u^*(t)$, that is,

$$u^*(t) = \begin{cases} 1 & \text{if} \quad \lambda(t) - 1 > 0 \\ 0 & \text{if} \quad \lambda(t) - 1 \leq 0 \end{cases}$$

The adjoint equation is therefore given by

$$\frac{d}{dt}\lambda = (1 - \lambda)u^* - 1, \qquad \lambda(T) = 0$$

We can now derive the explicit expression for the extremal elements. Since λ is continuous, there exists a $\delta > 0$ such that $\lambda(t) \leq 1$ on [δ, T]. On [δ, T] we have $u^*(t) = 0$. It follows that $\lambda(t) = T - t$ on [δ, T], and hence λ reaches 1 at $\delta = T - 1$. Since $(d^+/dt)\lambda(\delta) = -1$ and $(d^-/dt)\lambda(\delta) < 0$, there exists an $\eta < \delta$ such that $(d/dt)\lambda \leq 0$ on [η, δ]. This implies that $\lambda(t) > 1$ and thus $u^*(t) = 1$ on [η, δ], and consequently

$$\lambda(t) = e^{-(t-\delta)} \text{ and } x^*(t) = \xi e^{t-\delta} \qquad \text{on } [\eta, \delta]$$

for some $\xi > 0$. Now we can argue that η is necessarily 0 and that on [0, δ]

$$u^*(t) = 1, \qquad x^*(t) = ce^t, \quad \text{and} \quad \lambda(t) = e^{-(t-\delta)}$$

We have thus derived the form of the only extremal on [0, T]. Since one can easily argue the existence of a solution to the problem (22), it follows that the optimal control is given by

$$u^*(t) = \begin{cases} 1 & \text{on } [0, T - 1] \\ 0 & \text{on } (T - 1, T] \end{cases}$$

Lagrange Multiplier Rule

Here we present a necessary optimality condition based on the Lagrange multiplier rule and establish the relationship to the maximum principle. As in the section titled "Existence of Optimal Controls," it is assumed that t_f is fixed. We recall the definition of E in Eq. (19) and define the Lagrange functional $L : H^1(0, t_f, R^n) \times L^2(0, t_f, R^m) \times L^2(0, t_f, R^n) \times R^p \to R$ given by

$$L(x, u, \lambda, \mu) = J(x, u) + ((\lambda, \mu), E(x, u))_{L^2(0,t_f,R^n)\times R^p}$$

Further define $H^1_L(0, t_f, R^n)$ as the set of functions in $H^1(0, t_f, R^n)$ that vanish at $t = 0$.

We have the Lagrange multiplier rule:

Theorem 2. Assume that $(x(u^*), u^*)$ minimizes the cost functional in Eq. (16) subject to Eqs. (17) and (18) and that the regular point condition

$$0 \in \text{int}\{E'(x(u^*), u^*)(h, v - u^*) : h \in H^1_L(0, t_f, R^n) \text{ and } v \in K\} \tag{23}$$

holds. Then there exists a Lagrange multiplier $(\lambda, \mu) \in L^2(0, t_f, R^n) \times R^p$ such that

$$L_x(h) = J_x(x^*, u^*)h + ((\lambda, \mu), E_x(x^*, u^*))h = 0 \quad \text{for all} \quad h \in H^1_L(0, t_f, R^n) \tag{24}$$
$$(L_u, u - u^*) \geq 0 \qquad \text{for all} \quad u \in K$$

where the partial derivatives L_x and L_u are evaluated at (x^*, u^*, λ, μ).

Let us establish the relationship between the Lagrange multiplier (λ, μ) and the adjoint variable λ of the maximum principle. From the first line in Eq. (24) one deduces

$$\int_0^{t_f} \left([f^0_x(t, x^*, u^*) - \lambda f_x(t, x^*, u^*)]h + \lambda \frac{d}{dt}h \right) dt + \mu\varphi_x(x^*(t_f))h(t_f) = 0$$

for all $h \in H^1_L(0, t_f; R^n)$. An integration-by-parts argument implies that

$$\int_0^{t_f} \left(\int_{f_t}^{t} [f^0_x(s, x^*, u^*) - \lambda f_x(s, x^*, u^*)]\,ds + \mu\varphi_x(x^*(t_f)) + \lambda \right) \frac{d}{dt}h(t)\,dt = 0$$

and thus

$$\lambda(t) = \int_t^{t_f} [-f^0_x(s, x^*, u^*) + \lambda f_x(s, x^*, u^*)]\,ds - \mu\varphi_x(x^*(t_f))$$

a.e. in $(0, t_f)$. If f and f^0 are sufficiently regular, then $\lambda \in H^1(0, t_f, R^n)$ and (λ, μ) satisfy Eq. (21).

For certain applications, a Hilbert space framework may be too restrictive. For example, $f(u) = \sin u^2$ is well defined but not differentiable on $L^2(0, t_f; R)$. In such cases, it can be more appropriate to define the Lagrangian L on $W^{1,\infty}(0, t_f; R^n) \times L^\infty(0, t_f; R^n) \times L^\infty(0, t_f; R^n) \times R^p$.

Let us briefly turn to sufficient optimality conditions of second order. To simplify the presentation, we consider the case of minimizing $J(x, u)$ subject to the dynamical system (13) and initial and target constraints [Eq. (15)], but without constraints on the controls. If (x^*, u^*) satisfy the maximum principle and f^0, f are sufficiently regular, then

$$H_{uu}(t) \le 0 \qquad \text{for} \quad t \in [0, t_f]$$

where $H(t) = H(t, x^*(t), u^*(t), \lambda(t))$. A basic assumption for second-order sufficient optimality is given by the Legendre–Clebsch condition

$$H_{uu}(t) < 0 \qquad \text{for} \quad t \in [0, t_f] \tag{25}$$

This condition, however, is not sufficient for u^* to be a local minimizer for the control problem in Eqs. (12)–(15). Sufficient conditions involve positivity of the Hessian of the Lagrange functional L at (x^*, u^*, λ, μ) with (λ, μ) as in Theorem 2. This condition, in turn, is implied by the existence of a symmetric solution Q to the following matrix Riccati equation:

$$\begin{cases} \dot{Q} = -Qf_x(t) - f_x(t)^T Q + H_{xx}(t) \\ \qquad - [Qf_u(t) - H_{xu}(t)]H_{uu}(t)^{-1}[f_u^T(t)Q - H_{xu}(t)] \\ Q(t_f) = \mu\varphi_{xx}(x(t_f)) \qquad \text{on } \ker \varphi_x(x(t_f)) \end{cases} \tag{26}$$

where $f(t) = f(t, x^*(t), u^*(t))$. We have the following result:

Theorem 3. Let (x^*, u^*) denote a pair satisfying Eqs. (13) and (15), and assume the existence of a Lagrange multiplier (λ, μ) in the sense of Theorem 2 with $U = R^m$. If, further, f and f^0 are sufficiently regular, Eq. (25) holds, and Eq. (26) admits a symmetric C^1-solution, then u^* is a local solution of Eq. (12). Moreover, there exist $c > 0$ and $\bar{\epsilon} > 0$ such that

$$J(x, u) \ge J(x^*, u^*) + c|(x, u) - (x^*, u^*)|^2_{L^2(0,t_f;R^{n+m})}$$

for all (x, u) satisfying Eq. (13), Eq. (15), and $|(x, u) - (x^*, u^*)|_{L^\infty(0,t_f;R^{n+m})} < \bar{\epsilon}$.

The fact that perturbations (x, u) are only allowed in L^∞ so as to obtain an L^2 bound on variations of the cost functional is referred to as the *two-norm discrepancy.*

Bolza Problem

Here we discuss the maximum principle for a Bolza type problem, where the cost functional of Eq. (12) is replaced by

$$\min \quad J(x, u) = \int_0^{t_f} f^0(t, x(t), u(t))\, dt + g(x(t_f))$$

with $g: R^n \to R$. Augmenting the system (13), the Bolza problem can be expressed as a Lagrange problem. For this purpose we introduce additional scalar components for the dynamical system and for the target constraint by

$$\frac{d}{dt}x_{n+1} = 0 \quad \text{and} \quad \varphi_{p+1}(x) = g(x) - tx_{n+1}$$

where $x = (x_1, \ldots, x_n)$ as before, and the augmented cost functional is

$$\tilde{f}^0 = f^0(t, x, u) + x_{n+1}$$

We find

$$\int_0^{t_f} \tilde{f}^0(t, x(t), u(t))\, dt = \int_0^{t_f} f^0(t, x(t), u(t))\, dt + t_f x_{n+1} = \int_0^{t_f} f^0(t, x(t), u(t))\, dt + g(x(t_f))$$

For the augmented system, the initial conditions are $x(0) = x_0$, while $x_{n+1}(0)$ is free. The maximum principle can be generalized to allow for trajectories that are constrained to lie on an initial manifold $M_0 \subset R^{n+1}$. In this case a transversality condition at $t = 0$ must be added to part 3 of Theorem 1:

$$(H(0, x^*(0), u^*(0), \hat{\lambda}(0)), -\lambda(0)) \perp T_0$$

where T_0 is the tangent space to M_0 at $(0, x^*(0))$. For the Bolza problem the initial manifold is characterized by $t = 0$ and $x - x_0 = 0$, and thus the transversality condition at $t = 0$ implies $\lambda_{n+1}(0) = 0$. The adjoint condition 2 of Theorem 1 turns out to be

$$\frac{d}{dt}\lambda(t) = -H_x(t, x^*(t), u^*(t), \hat{\lambda}(t))$$
$$\frac{d}{dt}\lambda_{n+1}(t) = -\lambda_0, \qquad \lambda_{n+1}(0) = 0$$

where the Hamiltonian H is defined by Eq. (20), and the transversality condition 3 of Theorem 1 is given by

$$(H(t_f), -\lambda(t_f)) + \lambda_0(0, g_x(x(t_f))) \perp T_{t_f}$$

If we assume that t_f fixed and that no target constraints at t_f are present, then normality holds and conditions 2–3 of Theorem 1 can be expressed as

$$\frac{d}{dt}\lambda(t) = -H_x(x^* u^*, \hat{\lambda}) \qquad \lambda(t_f) = -g_x(x^*(t_f)) \tag{27}$$

For a restricted class of Bolza problems, the maximum principle provides a sufficient optimality condition. We have the following result:

Theorem 4. Consider the Bolza problem

$$\min \quad \int_0^{t_f} [f^0(t, x(t)) + h^0(t, u(t))]\, dt + g(x(t_f))$$

subject to

$$\frac{d}{dt}x(t) = Ax(t) + h(t, u(t)), \qquad x(t) = x_0$$

and the control constraint $u(t) \in U$, where t_f is fixed and g and f^0 are C^1 functions that are convex in x. If $(\lambda_0 = -1, \lambda(t), x^*(t), u^*(t))$ is extremal, then u^* is optimal.

Proof. From the maximum condition

$$-h^0(t,u^*(t)) + \lambda(t)h(t,u^*(t)) \geq -h^0(t,u) + \lambda(t)h(t,u) \quad \text{for all} \quad u \in U \tag{28}$$

where $\lambda(t)$ satisfies

$$\frac{d}{dt}\lambda(t) = f_x^0(t,x^*(t)) - \lambda(t)A, \qquad \lambda(t_f) = -g_x(x^*(t_f))$$

For all admissible pairs (x, u) satisfying

$$\frac{d}{dt}x(t) = Ax + h(t,u), \qquad x(0) = x_0$$

we have

$$\frac{d}{dt}(\lambda x) = \frac{d}{dt}\lambda x + \lambda\frac{d}{dt}x = (f_x^0 - \lambda A)x + \lambda(Ax + h)$$
$$= f_x^0(\cdot, x^*)x + \lambda h(\cdot, u)$$

Combined with Eq. (28), this implies

$$\frac{d}{dt}(\lambda x^*) - f_x^0(\cdot,x^*)x^* - h^0(\cdot,u^*) \geq \frac{d}{dt}(\lambda x) - f_x^0(\cdot,x^*)x - h^0(\cdot,u)$$

Integration of this inequality on $[0, t_f]$ yields

$$\lambda(t_f)x^*(t_f) - \int_0^{t_f}[f_x^0(t,x^*)x^* + h^0(t,u^*)]\,dt$$
$$\geq \lambda(t_f)x(t_f) - \int_0^{t_f}[f_x^0(t,x^*)x + h^0(t,u)]\,dt$$

By Eq. (27), the last inequality implies

$$g_x(x^*(t_f))[x(t_f) - x^*(t_f)] + \int_0^{t_f} f_x^0(t,x^*)(x - x^*)\,dt$$
$$\geq \int_0^{t_f}[h^0(t,u^*) - h^0(t,u)]\,dt$$

Note that $\Phi(x) \geq \Phi(x^*) + \Phi_x(x^*)(x - x^*)$ for all x, x^* for any convex and C^1 function Φ. Since g, f^0 are convex, we have

$$g(x(t_f)) + \int_0^{t_f}[f^0(t,x) + h^0(t,u)]\,dt$$
$$\geq g(x^*(t_f)) + \int_0^{t_f}[f^0(t,x^*) + h^0(t,u^*)]\,dt$$

which implies that u^* is optimal.

LINEAR QUADRATIC REGULATOR PROBLEM

We consider the special optimal control problem

$$\min \quad J(x_0, u) = \frac{1}{2}\left(\int_0^{t_f}[x^t(t)Qx(t) + u^t(t)Ru(t)]\,dt + x^t(t_f)Gx(t_f)\right) \tag{29}$$

subject to the linear control system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + f(t), \qquad x(0) = x_0 \tag{30}$$

where $f(t) \in L^2(0, t_f; R^m)$ represents a disturbance or an external force, Q, $G \in R^{n\times n}$ are symmetric and nonnegative matrices, and $R \in R^{m\times m}$ is symmetric and positive definite. This problem is referred to as the finite-time-horizon linear quadratic regulator problem. The Hamiltonian H is given by

$$H = -\tfrac{1}{2}(x^tQx + u^tRu) + \lambda(Ax + Bu + f(t))$$

where we have used the fact that $\lambda_0 = -1$ established in the last subsection. From Eq. (27) we obtain the form of the adjoint equation for $\lambda(t)$:

$$\frac{d}{dt}\lambda(t) = -\lambda(t)A + x^t(t)Q, \qquad \lambda(t_f) = -x^t(t_f)G$$

and the maximum condition implies that

$$u^*(t) = R^{-1}B^t\lambda^t(t)$$

Thus the maximum principle reduces to a two-point boundary value problem. If we define $p = -\lambda$, then the optimal triple (x, u^*, p) is characterized by $u^*(t) = -R^{-1}B^tp(t)$ and

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) - BR^{-1}B^tp(t) + f(t), \qquad x(0) = x_0\\ \frac{d}{dt}p(t) &= -A^tp(t) - Qx(t), \qquad p(t_f) = Gx(t_f)\end{aligned} \tag{31}$$

In the section titled "Linear Quadratic Regulator Theory and Riccati Equations" we discuss the solution to Eq. (31) in terms of matrix Riccati equations. There we shall also consider the infinite-time-horizon problem with $t_f = \infty$.

Time Optimal Control

A time optimal control problem consists of choosing a control in such a way that a dynamical system reaches a target manifold in minimal time. Without control constraints, such problems may not have a solution. In the presence of control constraints, the optimal control will typically be of bang–bang type. The following class of examples illustrates this behavior. We consider the time optimal control problem

$$\min \quad t_f = \int_0^{t_f} 1\,dt \tag{32}$$

subject to the linear control system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) \qquad x(0) = x_0 \tag{33}$$

and control as well as target constraints

$$u_i \in [-1, 1] \quad \text{for } 1 \leq i \leq m, \quad \text{and} \quad x(t_f) = 0$$

Assume that (A, B) is controllable, that is, for every $x_0 \in R^n$ and every target x_1 at t_f there exists a control $u \in L^2(0, t_f, R^m)$ that steers the system (33) from x_0 to x_1. We recall that, for the linear autonomous control system (33), controllability is equivalent to the requirement that the Kalman rank $(B, AB, \ldots, AB^{n-1}) = n$. A sufficient condition for the existence of an optimal control to Eq. (32) for arbitrary initial conditions x_0 in R^n is that (A, B) is controllable and that A is strongly stable (the real parts of all eigenvalues of A are strictly negative).

The Hamiltonian for Eq. (32) is

$$H = \lambda_0 1 + \lambda(Ax + Bu)$$

and the adjoint equation is given by

$$\frac{d}{dt}\lambda = -\lambda A$$

The transversality condition implies that $H(t_f) = 0$ and hence $\lambda(t) = \mu e^{A(t_f-t)}$ for some $\mu \in R^{1\times n}$. As a consequence of the maximum condition, we find

$$\lambda(t)Bu^*(t) \geq \lambda(t)Bu \qquad \text{for all} \quad u \in [-1, 1]^n$$

and hence

$$u_i^*(t) = \operatorname{sign} g_i(t), \ \text{for} \quad 1 \leq i \leq m$$

where $g(t) := \lambda(t)B = \mu e^{A(t_f-t)}B$. We claim that $g(t)$ is nontrivial. In fact, if $g(t) = 0$ for some $t \in [0, t_f]$, then, since (A, B) is controllable, $\mu = 0$ and $\lambda(t) = 0$. We have

$$H(t_f) = \lambda_0 + \lambda(t_f)[Ax(t_f) + Bu(t_f)] = \lambda_0 = 0$$

and thus $(\lambda_0, \lambda(t))$ is trivial if $g(t) = 0$. This gives a contradiction to Theorem 1.

In the remainder of this subsection we consider a special linear control system (rocket sled problem) and provide its solution. Let $y(t)$, the displacement of a sled with mass 1 on a friction-free surface be controlled by an applied force $u(t)$ with constraint $|u(t)| \leq 1$. By Newton's second law of motion, $(d^2/dt^2)y(t) = u(t)$. If we define $x_1(t) = y(t)$ and $x_2(t) = (d/dt)y(t)$, then the state $x(t) = \text{col}(x_1(t), x_2(t))$ satisfies Eq. (33) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We observe that the system (A, B) is a single-input controllable system that is marginally stable. This implies existence of an optimal control u^*. From the above discussion it follows that u^* must satisfy

$$u^*(t) = \operatorname{sign} \lambda_2(t)$$

The adjoint equation implies that $\lambda = (\lambda_1(t), \lambda_2(t))$ is given by

$$\lambda_1(t) = \mu_1 \quad \text{and} \quad \lambda_2(t) = \mu_1(T - t) + \mu_2$$

for some nonzero $\mu = (\mu_1, \mu_2)$. Hence the optimal control assumes at most the values -1 and 1 (bang–bang control) and it has at most one switch between these two values. Assume that $u^* = 1$. Then the equations $(d/dt)x_1(t) = x_2(t)$, $(d/dt)x_2(t) = 1$ have solutions of the form

$$x_2(t) = t + c_1 \quad \text{and} \quad x_1(t) = \frac{(t + c_1)^2}{2} + c_2$$

Thus, the orbit is on the manifold $x_1 = x_2^2/2 + c_2$ oriented upwards. Similarly, for $u^* = -1$ the orbit is on the manifold $x_1 = -x_2^2/2 + \hat{c}_2$ oriented downwards. Since the optimal controls have at most one switch and the orbits must terminate at (0, 0), it follows that the optimal control u^* is given in feedback form by

$$u^*(t) = U(x_1(t), x_2(t)) = \begin{cases} -1 & \text{if } (x_1, x_2) \text{ is above } S \\ \ \ 1 & \text{if } (x_1, x_2) \text{ is below } S \end{cases}$$

where S is the switching curve consisting of $x_1 = -x_2^2/2$ $(x_1 \leq 0)$ and $x_1 = x_2^2/2$ $(x_1 \geq 0)$. The feedback law is in general robust, since it possesses the self-correcting property.

DYNAMIC PROGRAMMING PRINCIPLE AND HAMILTON–JACOBI–BELLMAN EQUATION

In this section we discuss Bellman's dynamic programming principle (Refs. 1,2) for optimal control problems.

Derivation of the Hamilton–Jacobi–Bellman

Consider the Bolza problem

$$\min \quad J(s, y; u) = \int_s^{t_f} f^0(t, x(t), u(t))\, dt + g(x(t_f)) \qquad (34)$$

subject to

$$\frac{d}{dt}x(t) = f(t, x(t), u(t)), \qquad x(s) = y, \qquad u(t) \in U \qquad (35)$$

where U is a closed convex set in R^m. Under appropriate conditions on f, Eq. (35) has a unique solution $x = x(t; (s, y))$, and moreover $x \in C(s, t_f; R^n)$ depends continuously on $(s, y) \in [0, t_f] \times R^n$ and $u \in L^1(s, t_f; R^m)$. As discussed in the preceding section, sufficient conditions on f^0 and g, which guarantee the existence of an optimal pair (x^*, u^*) for each $(s, y) \in [0, t_f] \times R^n$, are well known. We define the *minimum-value function* $V(s, y)$ by

$$V(s, y) = \min_{u \in K} J(s, y; u)$$

Then V satisfies the optimality principle:

$$\min \left\{ \int_s^{\sigma} f^0(t, x(t), u(t))\, dt + V(\sigma, x(\sigma)) : u \in U \text{ on } [s, \sigma] \right\} = V(s, y) \qquad (36)$$

In fact, the cost functional J is additive in the first variable: for $\sigma \in [s, t_f]$,

$$J(s, y; u) = \int_s^{\sigma} f^0(t, x(t), u(t))\, dt + J(\sigma, x(\sigma); u) \qquad (37)$$

and thus

$$\int_s^\sigma f^0(t,x(t),u(t))\,dt + V(\sigma,x(\sigma)) \le J(s,y;u)$$

for all $u \in K$. Thus,

$$V(s,y) \le \min\left\{\int_s^\sigma f^0(t,x(t),u(t))\,dt + V(\sigma,x(\sigma)) : u \in U_{ad} \text{ on } [s,\sigma]\right\} \le V(s,y)$$

which implies Eq. (36).

Suppose that V is continuously differentiable. Then V satisfies the so-called Hamilton–Jacobi–Bellman (HJB) equation:

$$V_t(s,y) + \min_{u\in U}[f(s,y,u)V_x(s,y) + f^0(s,y,u)] = 0 \qquad (38)$$

We derive Eq. (38) using the optimality principle (36). Let $\hat{u} \in K$ be of the form

$$\hat{u} = \begin{cases} u & \text{on } (s,\sigma) \\ \tilde{u} & \text{on } (\sigma,t_f) \end{cases}$$

where $u(t) \in U$ on (s, σ) and $\tilde{u}$ minimizes $J(\sigma, x(\sigma); u)$ over the interval $[\sigma, t_f]$. From Eq. (37) we have

$$J(s,x;\hat{u}) = \int_s^\sigma f^0(t,x(t),u(t))\,dt + V(\sigma,x(\sigma))$$

If we set $u(t) = u^*(t)$ on $[s, \sigma]$, then, from Eq. (36) $\tilde{u}(t) = u^*(t)$ on $[\sigma, t_f]$ minimizes $J(\sigma, x^*(\sigma);\ \cdot\ )$ on $[\sigma, t_f]$ and

$$V(s,y) = \int_s^\sigma f^0(t,x^*(t),u^*(t))\,dt + V(\sigma,x^*(\sigma)) \qquad (39)$$

Since V is assumed to be C^1, we have

$$\frac{d}{dt}V(t,x(t)) = V_t + V_x\frac{d}{dt}x(t) = V_t + f(t,x(t),u(t))\cdot V_x$$

or equivalently,

$$V(\sigma,x(\sigma)) = V(s,y) + \int_s^\sigma (V_t + fV_x)\,dt$$

Now, since $V(s, y) \le J(s, x; \hat{u})$, the equation above Eq. (39) implies

$$\int_s^\sigma [V_t(t,x(t),u(t)) + f(t,x(t),u(t))V_x(t,x(t)) + f^0(t,x(t),u(t))]\,dt \ge 0$$

where, from Eq. (39), the equality holds if $u = u^*$ on $[s, \sigma]$. Thus,

$$\lim_{\sigma\to s^+}\frac{1}{\sigma - s}\int_s^\sigma [V_t + f(t,x(t),u(t))\cdot V_x + f^0(t,x(t),u(t))]\,dt = V_t(s,y) + f(t,y,u(s))V_x(s,y) + f^0(s,y,u(s)) \ge 0$$

for all $u(s) = u \in U$, and

$$V_t(s,y) + f(s,y,u^*(s))V_x(s,y) + f^0(s,y,u^*(s)) = 0$$

which is Eq. (38). Moreover, we have the following dynamical programming principle.

Theorem 5. (Verification Theorem). Let V be a solution of the HJB equation (38) such that $V \in C^1((0, t_f) \times R^n)$ and $V(t_f, x) = g(x)$. Then we have

1. $V(s, x) \le J(s, x; u)$ for any admissible control u.
2. If $u^* = \mu(t, x) \in U$ is the unique solution to

$$f(t,x,u^*)V_x(t,x) + f^0(t,x,u^*) = \min_{u\in U}[f(t,x,u)V_x(t,x) + f^0(t,x,u)] \qquad (40)$$

and the equation

$$\frac{d}{dt}x(t) = f(t,x(t),\mu(t,x(t))) \qquad x(s) = y$$

has a solution $x^*(t) \in C^1(s, t_f; R^n)$ for each $(s, y) \in [0, t_f] \times R^n$, then the feedback solution $u^*(t) = \mu(t, x^*(t))$ is optimal, that is, $V(s, y) = J(s, y; u^*)$.

Proof. Note that for $u \in K$

$$\frac{d}{dt}V(t,x(t)) = V_t(t,x(t)) + f(t,x(t),u(t))V_x(t,x(t))$$

and for any pair $(x, u) \in R^n \times U$,

$$V_t + f(t,x,u)V_x(t,x) + f^0(t,x,u) \ge 0$$

and thus

$$\frac{d}{dt}V(t,x(t)) \ge -f^0(t,x(t),u(t))$$

Hence we have

$$V(s,y) \le \int_s^{t_f} f^0(t,x(t),u(t))\,dt + g(x(t_f)) = J(s,y;u)$$

for all admissible controls u. Similarly, if $u^*(t) \in U$ attains the minimum in Eq. (40) with $x = x^*(t)$, then

$$\frac{d}{dt}V(t,x^*(t)) = V_t(t,x^*(t)) + f(t,x^*(t),u^*(t))V_x(t,x^*(t)) = -f^0(t,x^*(t),u^*(t))$$

and thus

$$V(s,y) = \int_s^{t_f} f^0(t,x^*(t),u^*(t))\,dt + g(x^*(t_f)) = J(s,x;u^*)$$

Relation to the Maximum Principle

In this section we discuss the relation between the dynamic programming and the maximum principle. Define the Hamiltonian $\hat{H}$ by

$$\hat{H}(t,x,u,p) = -f^0(t,x,u) - (f(t,x,u),p)_{R^n} \qquad (41)$$

We assume that $\hat{H}(t, x, u, p)$ attains the maximum over $u \in U$ at the unique point $\hat{u} = \mu(t, x, p)$ and that μ is locally Lipschitz. Let us define

$$H(t,x,p) = \max_{u\in U} \hat{H}(t,x,u,p)$$

Then Eq. (38) can be written as

$$V_t - H(t,x,V_x) = 0 \tag{42}$$

Assume that $V \in C^{1,2}((0, t_f) \times R^n)$. Then, defining $p(t) = V_x(t, x^*(t))$, we obtain

$$\frac{d}{dt}p(t) = \hat{H}_x(t,x^*(t),u^*(t),p(t)), \qquad p(t_f) = g_x(x^*(t_f)) \tag{43}$$

and $u^*(t) = \mu(t, x^*(t), p(t))$ is an optimal control. In fact, by Eq. (42)

$$\begin{aligned}\frac{d}{dt}V_x(t,x^*(t)) &= \frac{\partial V_x}{\partial t}(t,x^*(t)) + V_{xx}(t,x^*(t))\frac{d}{dt}x^*(t)\\ &= H_x(t,x,V_x(t,x^*(t)))\\ &\quad + V_{xx}(t,x^*(t))H_p(t,x,V_x(t,x^*(t)))\\ &\quad + V_{xx}(t,x^*(t))f(t,x^*(t),u^*(t))\\ &= \hat{H}_x(t,x^*(t),u^*(t),V_x(t,x^*(t)))\end{aligned}$$

where $V_{xx} = \{\partial^2 V/\partial x_i\, \partial x_j\} \in R^{n\times n}$. Here we have used the fact that

$$H_p(t,x,p) = -f(t,x,\hat{u})$$

and

$$H_x(t,x,p) = -f_x(t,x,\hat{u})^t p - f_x^0(t,x,\hat{u})^t$$

where $\hat{u} = \mu(t, x, p) \in U$ maximizes $\hat{H}(t, x, u, p)$ over U. We observe that Eq. (43) represents the adjoint equation of the maximum principle with adjoint variable λ given by $-V_x(\cdot, x^*)$.

Next let us set $U = V_x$. Then U satisfies

$$\begin{aligned}&U_t(t,x) + f(t,x,\mu(t,x,U(t,x)))\\ &\quad \cdot U_x(t,x) - H_x(t,x,U(t,x)) = 0\end{aligned} \tag{44}$$

Hence, setting $u(t) = \mu(t, x(t), p(t))$, the necessary optimality conditions

$$\begin{aligned}\frac{d}{dt}x(t) &= f(t,x,u(t))\\ \frac{d}{dt}p(t) &= H_x(t,x,p(t))\\ \frac{d}{dt}V(t) &= -f^0(t,x,u(t))\end{aligned} \tag{45}$$

are the characteristic equations of the first order partial differential equations (PDEs) (42) and (44).

Viscosity Solution Method

In this section we discuss the viscosity solution method for the HJB equation. For motivation, we first consider the exit time problem

$$\min \quad J(y,u) = \int_0^\tau f^0(x(t),u(t))\,dt + g(x(\tau)) \tag{46}$$

subject to Eq. (35), where $\tau = \inf\{x(t) \notin \Omega\}$ is the exit time from an open set Ω in R^n. It can be shown that, if we define the value function $V(y) = \inf_{u\in U} J(y, u)$, then it satisfies the HJB equation

$$\min_{u\in U}[f^0(x,u) + f(x,u)\cdot V_x] = 0, \qquad V(x) = g(x) \quad \text{on } x \in \partial\Omega \tag{47}$$

For the specific case $f(x, u) = u$, $f^0 = 1$, $g = 0$, $U = [-1, 1]$, and $\Omega = (-1, 1)$, with $x, u \in R$, the HJB equation (47) becomes

$$-|V_x(x)| + 1 = 0 \qquad \text{with} \quad V(\pm 1) = 0 \tag{48}$$

It can be proved that Eq. (48) has no C^1 solution, but there are infinitely many Lipschitz continuous solutions that satisfy it a.e. in $(-1, 1)$. The viscosity method is developed as a mathematical concept that admits non-C^1 solutions and selects the solution corresponding to the optimal control problem to the HJB equation.

We return now to the general problem presented in the first subsection of this section.

Definition 1. A function $v \in C((0, t_f] \times R^n)$ is called a *viscosity solution* to the HJB equation $v_t - H(t, x, v_x) = 0$ provided that for all $\psi \in C^1(\Omega)$, if $v - \psi$ attains a (local) maximum at (t_0, x_0), then

$$\psi_t - H(t,x,\psi_x) \ge 0 \qquad \text{at } (t_0,x_0)$$

and if $v - \psi$ attains a (local) minimum at (t_0, x_0), then

$$\psi_t - H(t,x,\psi_x) \le 0 \qquad \text{at } (t_0,x_0)$$

It is clear that a C^1 solution to Eq. (42) is a viscosity solution, and if v is a viscosity solution of Eq. (42) and Lipschitz continuous, then $v_t - H(t, x, v_x) = 0$ a.e. in $(0, t_f) \times R^n$. The viscosity solution concept is derived from the vanishing viscosity method illustrated by the following theorem.

Theorem 6. Let $V^\epsilon(t, x) \in C^{1,2}((0, t_f) \times R^n)$ be a solution to the viscous equation

$$V_t^\epsilon - H(t,x,V^\epsilon) + \epsilon\,\Delta V^\epsilon = 0, \qquad V^\epsilon(t_f,x) = g(x) \tag{49}$$

If $V^\epsilon(t, x) \to V(t, x)$ uniformly on compact sets as $\epsilon \to 0^+$, then $V(t, x)$ is a viscosity solution to Eq. (42).

Proof. We need to show that

$$\psi_t - H(t,x,\psi_x) \ge 0 \qquad \text{at } (t_0,x_0)$$

for all $\psi \in C^1((0, t_f) \times R^n)$ such that $V - \psi$ attains a local maximum at (t_0, x_0).

Choose a function $\zeta \in C^1((0, t_f) \times R^n)$ such that $0 \le \zeta < 1$ for $(t, x) \neq (t_0, x_0)$ and $\zeta(t_0, x_0) = 1$. Then (t_0, x_0) is a strict local maximum of $V + \zeta - \psi$. Define $\Psi^\epsilon = V^\epsilon + \zeta - \psi$, and note that, since $V^\epsilon \to V$ uniformly on compact sets, there exists a sequence (t_ϵ, x_ϵ) such that $(t_\epsilon, x_\epsilon) \to (t_0, x_0)$ as $\epsilon \to 0^+$ and Ψ^ϵ attains a local maximum at (t_ϵ, x_ϵ). The necessary optimality condition yields

$$\Psi_t^\epsilon = 0, \quad \Psi_x^\epsilon = 0, \quad \Psi_{xx}^\epsilon \le 0 \qquad \text{at } (t,\epsilon,x_\epsilon)$$

It follows that

$$\psi_t - H(t, x, \psi_x) + \epsilon\ \Delta\psi \geq \zeta_t - H(t, x, \zeta_x) + \epsilon\ \Delta\zeta \qquad \text{at } (t_\epsilon, x_\epsilon)$$

Since ζ is an arbitrary function with the specified properties, $\psi_t - H(t, x, \psi_x) \geq 0$ at (t_0, x_0).

For example, let $V^\epsilon(x)$ be the solution to

$$-|V_x(x)| + 1 + \epsilon V_{xx} = 0, \qquad V(-1) = V(1) = 0$$

Then the solution V^ϵ is given by

$$V^\epsilon(x) = 1 - |x| + \epsilon(e^{-1/\epsilon} - e^{-|x|/\epsilon})$$

and we have $\lim V^\epsilon(x) = 1 - |x|$ as $\epsilon \to 0^+$. Moreover, $V(x) = 1 - |x|$ is a viscosity solution to $-|V_x(x)| + 1 = 0$. We can also check that any other Lipschitz continuous solution is not a viscosity solution. It can be proved, in a general context, that *the viscosity solution is unique* (Refs. 3,4).

As we saw for the exit time problem, the value function V is not necessarily differentiable. But it always superdifferentiable. Here we call a function φ *superdifferentiable* at y_0 if there exists $p \in R^n$ such that

$$\limsup_{y \to y_0} \frac{\varphi(y) - \varphi(y_0) - (p, y - y_0)}{|y - y_0|} \geq 0$$

and we denote the set of p such that the above inequality holds by $D^+\varphi(y_0)$. Based on the notion of viscosity solution, one can express the dynamic programming and the maximum principle without assuming that V is C^1 as follows (see, e.g., Ref. 2).

Theorem 7. The value function $V(s, y)$ is continuous on $(0, t_f) \times R^n$, and locally Lipschitz continuous in y for every $s \in [0, t_f]$. Moreover, V is a viscosity solution of the Hamilton–Jacobi–Bellman equation, and every optimal control u^* to the problem Eqs. (34)–(35) is given by the feedback law

$$u^*(t) = \mu(t, x^*(t), \eta(t)) \qquad \text{for some} \quad \eta(t) \in D_x^+ V(t, x^*(t))$$

for every $t \in [0, T]$, where $x^*(\cdot)$ is the optimal trajectory of Eq. (35) corresponding to u^*.

Applications

Here we consider the case

$$f(t, x(t), u(t)) = a(x) + b(x)u \quad \text{and} \quad f^0(t, x, u) = l(x) + h(u)$$

where it is assumed that $h : U \to R$ is convex, is lower semicontinuous, and satisfies

$$h(u) \geq \omega |u|^2 \qquad \text{for some} \quad \omega > 0$$

We find

$$\begin{aligned} &\min_{u \in U}\{f^0(t, x, u) + p \cdot f(t, x, u)\} \\ &\quad = a(x) \cdot p + l(x) - \max_{u \in U}\{-u \cdot b(x)^t p - h(u)\} \\ &\quad = a(x) \cdot p - h^*(-b(x)^t p) \end{aligned}$$

Here h^* denotes the conjugate function of h, which is defined by

$$h^*(v) = \sup_{u \in U}\{vu - h(u)\} \tag{50}$$

We assume that h^* is Gateaux-differentiable with locally Lipschitz Gateaux derivative h_p^*. Then we have $\hat{u} = h_p^*(v)$, where $\hat{u} \in U$ attains the maximum of $(v, u) - h(u)$ over $u \in U$. In this case, the HJB equation is written as

$$V_t + a(x) \cdot V_x - h^*(-b(x)^t V_x) + l(x) = 0 \tag{51}$$

with $V(t_f, x) = g(x)$. As a specific case, we may consider the linear quadratic control problem where

$$\begin{aligned} f(t, x, u) &= A(t)x + B(t)u + f(t), \\ f^0(t, x, u) &= \tfrac{1}{2}[x^t Q(t) x_u^t R(t) u] \end{aligned}$$

and $U = R^m$. Then we have

$$V_t + [A(t)x + f(t)]V_x - \tfrac{1}{2}|-R^{-1}(t)B^t(t)V_x|_R^2 + \tfrac{1}{2}x^t Q(t)x = 0 \tag{52}$$

Suppose that $g(x) = \frac{1}{2}x^t Gx$. Then $V(t, x) = \frac{1}{2}x^t P(t)x + xv(t)$ is a solution to Eq. (52), where $P(t) \in R^{n\times n}$ satisfies the differential Riccati equation

$$\begin{aligned} &\frac{dP}{dt}(t) + A^t(t)P(t) + P(t)A(t) \\ &\qquad - P(t)B(t)R^{-1}(t)B^t(t)P(t) + Q(t) = 0 \end{aligned} \tag{53}$$

with $P(t_f) = G$, and the feedforward $v(t)$ satisfies

$$\frac{d}{dt}v(t) = -[A - BR^{-1}B^t P(t)]^t v(t) = P(t)f(t), \qquad v(t_f) = 0 \tag{54}$$

In the control-constrained case with $U = \{u \in R^m : |u| \leq 1\}$ and $h(u) = \frac{1}{2}|u|^2$, we find

$$h^*(p) = \begin{cases} \frac{1}{2}|p|^2 & \text{if} \quad |p| < 1 \\ |p| - \frac{1}{2} & \text{if} \quad |p| \geq 1 \end{cases}$$

and

$$h_p^*(p) = \begin{cases} p & \text{if} \quad |p| < 1 \\ p/|p| & \text{if} \quad |p| \geq 1 \end{cases}$$

so that $h^* \in C^1(R^m) \in W^{2,\infty}(R^m)$.

LINEAR QUADRATIC REGULATOR THEORY AND RICCATI EQUATIONS

In this section we first revisit the finite-horizon LQR problem in Eqs. (29)–(30) and show that the optimal control $u^*(t)$ can be expressed in the feedback form

$$u^*(t) = -R^{-1}B^t[P(t)x(t) + v(t)] \tag{55}$$

where the symmetric matrix $P(t)$ and the feedforward $v(t)$ satisfy Eqs. (53) and (54). The matrix $K(t) = R^{-1}B^tP(t)$ describing the control action as a function of the state is referred to as the feedback gain matrix. The solution in the form of Eq. (55) can be derived from the dynamical programming principle in the preceding subsection, but here we prefer to give an independent derivation based on the two-point boundary value problem (31). Since this equation is affine, we can assume that

$$p(t) = P(t)x(t) + v(t) \tag{56}$$

More precisely, let $(x(t), p(t))$ denote a solution to Eq. (31). Then for each $t \in [0, t_f]$, the mapping from $x(t) \in R^n$ to $p(t) \in R^n$ defined by forward integration of the first equation in Eq. (31) with initial condition $x(t)$ on $[t, t_f]$ and subsequent backward integration of the second equation of Eq. (31) with terminal condition $p(t_f) = Gx(t_f)$ on $[t, t_f]$ is affine. Substituting Eq. (56) into the second equation of Eq. (31), we obtain

$$\frac{d}{dt}P(t)x(t) + P(t)\frac{d}{dt}x(t) + \frac{d}{dt}v(t) = -Qx(t) - A^t[P(t)x(t) + v(t)]$$

and from the first equation in Eq. (31) we derive

$$\left(\frac{d}{dt}P(t)^t + AP(t) + P(t)A - P(t)BR^{-1}B^tP(t) + Q\right)x(t) + \frac{d}{dt}v(t) + [A - BR^{-1}B^tP(t)]^t v(t) + P(t)f(t) = 0$$

This equation holds if Eqs. (53) and (54) are satisfied. By standard results from the theory of ordinary differential equations, there exists a unique symmetric and nonnegative solution $P_{t_f}(t) \in C^1(0, t_f; R^{n\times n})$ with $P_{t_f}(t) = G$ to the Riccati equation (53). If $x^*(t)$ is a solution to

$$\frac{d}{dt}x^*(t) = [A - BR^{-1}B^tP_{t_f}(t)]x^*(t) - BR^{-1}B^tv(t) + f(t) \tag{57}$$

where $v(t)$ is a solution to Eq. (54), and if we set $p(t) = P(t)x^*(t) + v(t)$, then the pair $(x^*(t), p(t))$ is a solution to Eq. (31). Thus, the triple $(x^*(t), u^*(t), -p^t(t))$ satisfies the maximum principle. From Theorem 3 or from Eq. (58) below, it follows that the feedback solution (55) is optimal.

The formula (56) is called the *Riccati transformation.* It transforms the TPBV problem (31) into a system of initial value problems backwards in time. Moreover, the feedback solution given by Eq. (55) is unique. This follows from the fact that for arbitrary $u \in L^2(0, t_f; R^m)$, multiplying Eq. (57) from the left by $[P_{t_f}(t)x^*(t)]^t$ using Eqs. (53) and (54), and integrating the resulting equation on $[0, t_f]$, we have

$$J(x_0, u) = J(x_0, u^*) + \frac{1}{2}\int_0^{t_f} |u(t) + R^{-1}B^t[P_{t_f}x(t) + v(t)]|_R^2\, dt \tag{58}$$

where $|y|_R^2 = y^tRy$ and

$$J(x_0, u^*) = \frac{1}{2}\left(x_0^tP_{t_f}(0)x_0 + 2v(0)^tx_0 + \int_0^{t_f}[-v^t(t)BR^{-1}B^tv(t) + 2v^t(t)f(t)]\, dt\right) \tag{59}$$

We turn to the infinite-time-horizon problem:

$$\min \quad J(x_0, u) = \frac{1}{2}\int_0^{\infty}[x^t(t)Qx(t) + u(t)Ru(t)]\, dt \tag{60}$$

subject to the linear control system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$$

This problem need not admit a solution. For example, consider the system

$$\frac{d}{dt}x(t) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}x(t) + \begin{pmatrix}1\\ 0\end{pmatrix}u(t)$$

and let $Q = I$ and R be arbitrary. Then there exists no admissible control $u \in L^2(0, \infty; R^m)$ such that $J(x_0, u)$ is finite, unless $x_2(0) = 0$.

Under the assumption that for each $x_0 \in R^n$ there exists at least one admissible control such that $J(x_0, u)$ is finite (finite-cost condition), it can be shown that the optimal control is given in feedback form by

$$u^*(t) = -R^{-1}B^tP_\infty x^*(t)$$

where the nonnegative symmetric matrix P_∞ is defined in the following theorem. A sufficient condition for the finite-cost condition is that the pair (A, B) is stabilizable, that is, there exists a matrix $K \in R^{m\times n}$ such that $A - BK$ is asymptotically stable. In this case, the closed-loop system with feedback control $u(t) = -Kx(t)$ is exponentially stable, and we have $J(x_0, -Kx(t)) \le M|x_0|^2$ for some $M > 0$ independent of $x_0 \in R^n$.

The following result is referred to as *LQR theory.* It relies on the notions of detectability and observability. The pair $(A, Q^{1/2})$ is called *detectable* if there exists a matrix G such that $A - GQ^{1/2}$ is asymptotically stable. Further $(A, Q^{1/2})$ is called *observable* if, for some $\tau > 0$, the kernel of the mapping $x \to Q^{1/2}e^{A\tau}x$ is trivial. Observability of $(A, Q^{1/2})$ is equivalent to controllability of $(A^t, Q^{1/2})$.

Theorem 8 (LQR)

1. Assume that for each $x_0 \in R^n$ there exists at least one admissible control such that $J(x_0, u)$ is finite. For any $t_f > 0$, let $P_{t_f}(\cdot)$ denote the solution to the Riccati equation Eq. (53) with $G = 0$. Then $P_{t_f}(0)$ converges monotonically to a nonnegative symmetric matrix P_∞ as $t_f \to \infty$, and P_∞ satisfies the algebraic Riccati equation

$$A^tP_\infty + P_\infty A - P_\infty BR^{-1}B^tP_\infty + Q = 0 \tag{61}$$

The control

$$u^*(t) = -R^{-1}B^tP_\infty x^*(t)$$

is the unique solution to the LQR problem (60), and

$$J(x_0, u^*) = \tfrac{1}{2}x_0^tP_\infty x_0 = \min_{u\in L^2(0,\infty;R^m)} J(x_0, u)$$

Conversely, if there exists a nonnegative symmetric solution P to Eq. (61), then for all $x_0 \in R^n$ there exists an admissible control u such that $J(x_0, u)$ is finite and $P_\infty \le P$.

2. Suppose that $(A, Q^{1/2})$ is detectable and that Eq. (61) admits a solution. Then the closed-loop matrix $A - BR^{-1}B^tP_\infty$ is asymptotically stable, and P_∞ is the unique nonnegative symmetric solution to Eq. (61). If, moreover, $(A, Q^{1/2})$ is observable, then P_∞ is positive definite.

Proof. For part 1 note that, due to Eq. (59), we have for $t_f \le \hat{t}_f$

$$x_0^t P_{t_f}(0)x_0 \le x_0^t P_{\hat{t}_f}(0)x_0$$

Thus, $P_{t_f}(0) \le P_{\hat{t}_f}(0)$ for $t_f \le \hat{t}_f$. The assumption on the existence of admissible controls shows that $e_i^t P_{t_f}(0)e_i$ and $(e_i + e_j)^t P_{t_f}(0)(e_i + e_j)$ are monotonically nondecreasing and bounded with respect to t_f. Here e_i denotes the ith unit vector in R^n. Defining $P_\infty = \lim_{t_f\to\infty} P_{t_f}(0)$, it follows that P_∞ is symmetric, is nonnegative, and moreover satisfies the steady-state equation Eq. (61). It can then be argued that the feedback control $u^*(t) = -R^{-1}B^tP_\infty x^*(t)$ is the unique optimal solution to the LQR problem (60). To prove the last assertion of part 1, suppose that P is a nonnegative symmetric solution to Eq. (61). Let $x(t)$ be the solution to $(d/dt)x(t) = (A - BR^{-1}B^tP)x(t)$ with $x(0) = x_0$, and let $u(t) = -R^{-1}B^tPx(t)$. Then

$$\begin{aligned}\frac{d}{dt}[x^t(t)Px(t)] &= 2x^t(t)P(A - BR^{-1}B^tP)x(t)\\ &= -x^t(t)(Q + PBR^{-1}B^tP)x(t)\end{aligned}$$

Integration of this equation on $[0, t_f]$ implies

$$\int_0^{t_f}(x^tQx + u^tRu)\,dt + x(t_f)^tPx(t_f) = x_0^tPx_0$$

and thus $J(x_0, u) \le \frac{1}{2}x_0^tPx_0$ and $x_0^tP_\infty x_0 \le x_0^tPx_0$ for every $x_0 \in R^n$.

To verify part 2, note that

$$\begin{aligned}(A - BR^{-1}B^tP_\infty)^tP_\infty + P_\infty(A - BR^{-1}B^tP_\infty)&\\ + Q + P_\infty BR^{-1}B^tP_\infty &= 0\end{aligned}$$

It can be shown that $(A - BR^{-1}B^tP_\infty, (Q + P_\infty BR^{-1}B^tP_\infty)^{1/2})$ is detectable (observable) if $(A, Q^{1/2})$ is detectable (observable). Hence, it follows from the Liapunov criterion (5) that $A - BR^{-1}B^tP_\infty$ is asymptotically stable and moreover that P_∞ is positive definite if $(A, Q^{1/2})$ is observable. For the proof of uniqueness we refer to the literature (see, e.g., Ref. 5).

In the following theorem we consider the LQR theory with external forcing.

Theorem 9. Consider the infinite-time-horizon problem

$$\min \quad \frac{1}{2}\int_0^\infty [x^t(t)Qx(t) + u^t(t)Ru(t)]\,dt$$

subject to

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + f(t), \qquad x(0) = x_0$$

where $f \in L^2(0, \infty; R^n)$. Assume that (A, B) is stabilizable and that $(A, Q^{1/2})$ is detectable. Then there exists a unique optimal solution u^* given by

$$u^*(t) = -R^{-1}B^t[P_\infty x^*(t) + v(t)]$$

where P_∞ is the unique solution to Eq. (61) and $v \in L^2(0, \infty; R^n)$ satisfies

$$\frac{d}{dt}v(t) + (A - BR^{-1}B^tP_\infty)v(t) + P_\infty f(t) = 0, \qquad v(\infty) = 0$$

Proof. From Theorem 8 it follows that $A - BR^{-1}B^tP_\infty$ is asymptotically stable and thus $v \in L^2(0, \infty; R^n)$ and $u^*(t) \in L^2(0, \infty; R^m)$. Since $(A, Q^{1/2})$ is detectable, there exists a matrix $G \in R^{n\times n}$ such that $A - GQ$ is asymptotically stable. For arbitrary admissible controls $u \in L^2(0, \infty; R^n)$

$$x(t) = e^{(A-GQ)t}x_0 + \int_0^t e^{(A-GQ)(t-s)}[GQx(t) + Bu(t) + f(t)]\,dt$$

From the Fubini inequality

$$\int_0^\infty |x(t)|^2\,dt \le M(|x_0|^2 + |u|^2_{L^2(0,\infty;R^m)} + |f|^2_{L^2(0,\infty;R^n)})$$

for some $M > 0$. Thus $\lim_{t\to\infty} x(t)$ exists and is zero. Taking the limit $t_f \to \infty$ in Eq. (58), we obtain

$$J(x_0, u) = J(x_0, u^*) + \frac{1}{2}\int_0^\infty |u(t) + R^{-1}B^t[P_\infty x(t) + v(t)]|_R^2\,dt$$

which proves the theorem.

Assume that (A, B) is stabilizable. Then there exists a solution P_∞ to Eq. (61), for which, however, $A - BR^{-1}B^tP_\infty$ is not necessarily asymptotically stable. The following theorem shows that there exists a maximal solution P_+ to the Riccati equation (61) and gives a sufficient condition such that $A - BR^{-1}B^tP_+$ is asymptotically stable.

Theorem 10. Assume that (A, B) is stabilizable. For $\epsilon > 0$ let P_ϵ be the nonnegative symmetric solution P_ϵ to the Riccati equation

$$A^tP + PA - PBR^{-1}B^tP + Q + \epsilon I = 0$$

Then P_ϵ converges monotonically to a nonnegative symmetric matrix P_+ as $\epsilon \to 0^+$. The matrix P_+ is a solution to Eq. (61), and $P \le P_+$ for all nonnegative symmetric solutions P to Eq. (61). Moreover, if we assume that the Hamiltonian matrix

$$H = \begin{pmatrix} A & -BR^{-1}B^t \\ -Q & -A^t \end{pmatrix} \qquad (62)$$

has no eigenvalues on the imaginary axis, then $A - BR^{-1}BP_+$ is asymptotically stable. For the stability-constrained LQR problem of minimizing Eq. (60) subject to

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + f(t),$$
$$x(0) = x_0 \text{ and } \int_0^\infty |x(t)|^2\,dt < \infty$$

the unique optimal control u^* is given by

$$u^*(t) = -R^{-1}[B^tP + x(t) + v(t)]$$

where $v(t) \in L^2(0, \infty, R^n)$ satisfies

$$\frac{d}{dt}v(t) + (A - BR^{-1}B^tP_+)v(t) + P_+f(t) = 0, \qquad v(\infty) = 0$$

Due to the importance of finding the stabilizing feedback gain, solving Eq. (61) is of considerable practical importance. We therefore close this section by describing the Potter–Laub method. We also refer to Refs. 6, 7 for iterative methods based on the Newton–Kleimann and Chandrasekhar algorithms. The Potter–Laub method uses the Schur decomposition of the Hamiltonian matrix (62) and is stated in the following theorem.

Theorem 11

1. Let Q, W be symmetric $n \times n$ matrices. Solutions P to the algebraic Riccati equation $A^tP + PA - PWP + Q = 0$ coincide with the set of matrices of the form $P = VU^{-1}$, where the $n \times n$ matrices $U = [u_1, \ldots, u_n]$, $V = [v_1, \ldots, v_n]$ are composed of upper and lower halves of n real Schur vectors of the matrix

$$H = \begin{pmatrix} A & -W \\ -Q & -A^t \end{pmatrix}$$

   and U is nonsingular.
2. There exist at most n eigenvalues of H that have negative real part.
3. Suppose $[u_1, \ldots, u_n]$ are real Schur vectors of H corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$, and $\lambda_i \neq -\overline{\lambda}_j$ for $1 \leq i, j \leq n$. Then the corresponding matrix $P = UV^{-1}$ is symmetric.
4. Assume that Q, W are nonnegative definite and $(A, Q^{1/2})$ is detectable. Then the solution P is symmetric and nonnegative definite if and only if Re $\lambda_k < 0$, $1 \leq k \leq n$.

Proof. We prove part 1. Let S be a real Schur form of H, that is, $HU = US$ with $U^tU = I$ and

$$S = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}$$

Thus

$$H\begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} S_{11}$$

where

$$\begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix}$$

is made up of n Schur vectors of H corresponding to the S_{11} block. We observe that

$$AU_{11} - WU_{21} = U_{11}S_{11} \quad \text{and} \quad -QU_{11} - A^tU_{21} = U_{21}S_{11}$$

Assume that U_{11} is nonsingular, and define $P = U_{21}U_{11}^{-1}$. Since $PU_{11} = U_{21}$, we have

$$(A - WP)U_{11} = U_{11}S_{11} \quad \text{and} \quad (-Q - A^tP)U_{11} = PU_{11}S_{11}$$

Thus $P(A - WP)U_{11} = (-Q - A^tP)U_{11}$, and moreover $A - WP = U_{11}S_{11}U_{11}^{-1}$.

Conversely, if P satisfies $A^tP + PA - PWP + Q = 0$. Then

$$H\begin{pmatrix} I & 0 \\ P & I \end{pmatrix} = \begin{pmatrix} A - WP & -W \\ -Q - A^tP & -A^t \end{pmatrix}$$

and

$$\begin{pmatrix} I & 0 \\ P & I \end{pmatrix}\begin{pmatrix} A - WP & -W \\ 0 & -A^t + PW \end{pmatrix} = \begin{pmatrix} A - WP & -W \\ P(A - WP) & -A^t \end{pmatrix}$$

Thus

$$H\begin{pmatrix} I & 0 \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}\begin{pmatrix} A - WP & -W \\ 0 & -A^t + PW \end{pmatrix}$$

The proof of assertions 2–4 can be found in Ref. 6.

In summary, the stabilizing solution P corresponds to the stable eigen subspace of H, and the eigenvalues of the resulting closed-loop system coincide with those of S_{11}.

NUMERICAL METHODS

In this section we discuss numerical methods for the nonlinear regulator problem

$$\min \quad J(x_0, u) = \int_0^T [l(x(t)) + h(u(t))]\,dt + g(x(T)) \tag{63}$$

subject to

$$\frac{d}{dt}x(t) = f(x(t), u(t)), \qquad x(0) = x_0, \quad u(t) \in U \tag{64}$$

We assume that

$$(f(x, u) - f(y, u), x - y) \leq \omega|x - y|^2 \qquad \text{for all} \quad u \in U \tag{65}$$

and moreover either that $U \subset R^m$ is bounded or that $h(u) \geq c_1|u|^2$ and

$$(f(x, u), x) \leq \omega|x|^2 + c_2|u|^2 \tag{66}$$

for constants ω, c_1, $c_2 > 0$, independent of $x, y \in R^n$ and $u \in U$. Also we assume that for each $(x, p) \in R^n \times R^n$ the mapping

$$u \to h(u) + (p, f(x, u))$$

admits a unique minimizer over U, denoted by $\Psi(x, p)$. Finally, we assume that l, h, g, and f are sufficiently smooth with l and g bounded from below.

Discrete-Time Approximation

We consider the discretized problem

$$\min \quad J^N(u^N) = \sum_{k=1}^{N} [l(x^k) + h(u^k)]\, \Delta t + g(x^N) \qquad (67)$$

subject to

$$\frac{x^k - x^{k-1}}{\Delta t} = f(x^k, u^k) \text{ and } u^k \in U \qquad 1 \le k \le N \qquad (68)$$

where $N\,\Delta t = T$, which realizes the implicit Euler scheme for time integration of Eq. (64) and first-order integration of the cost functional (63). Note that if $\omega\,\Delta t < 1$, then the mapping $\Phi(x) = x - \Delta t\, f(x, u)$ is dissipative, that is, $(F(x_1, u) - F(x_2, u), x_1 - x_2) \ge (1 - \Delta t\,\omega)|x_1 - x_2|^2$. Thus for $u = \{u^k\}_{k=1}^N$ in U there exists a unique $x = \{x^k\}_{k=1}^N$, satisfying the constraint Eq. (68) and depending continuously on u. Moreover, if $\omega\,\Delta t < 1$, then there exists an optimal pair (x^k, u^k) to the problem Eqs. (67), (68). The necessary optimality condition for that problem is given by

$$\begin{aligned} \frac{x^k - x^{k-1}}{\Delta t} &= f(x^k, u^k) \\ -\frac{p^{k+1} - p^k}{\Delta t} &= f_x(x^k, u^k)^t p^k + l_x(x^k) \\ u^k &= \Psi(x^k, p^k) \in U \end{aligned} \qquad (69)$$

for $1 \le k \le N$, with $x^0 = x_0$ and $p^{N+1} = g_x(x^N)$. It is noted that Eq. (69) is a sparse system of nonlinear equations for $\mathrm{col}(\mathrm{col}(x^1, \ldots, x^N), \mathrm{col}(p^1, \ldots, p^N)) \in R^{nN} \times R^{nN}$. We have the following result:

Theorem 12. Assume that Ψ is Lipschitz continuous, that $\omega\,\Delta t < 1$, and that $\{u^N\}_{N=1}^\infty$ is a sequence of solutions to Eqs. (67), (68) with associated primal and adjoint states $\{(x^N, p^N)\}_{N=1}^\infty$ such that Eq. (69) holds. Let $\tilde{u}^N$ denote the step function defined by $\tilde{u}^N(t) = u^k$ on (t_{k-1}, t_k), $1 \le k \le N$, and let $\tilde{x}^N$ and $\tilde{p}^N$ be the piecewise linear functions defined by

$$\tilde{x}^N(t) = x^{k-1} + \frac{x^k - x^{k-1}}{\Delta t}(t - t_{k-1})$$

and

$$\tilde{p}^N(t) = p^k + \frac{p^{k+1} - p^k}{\Delta t}(t - t_{k-1})$$

Then the sequence $(\tilde{x}^N, \tilde{u}^N, \tilde{p}^N)$ in $H^1(0, T; R^n) \times L^2(0, T; R^m) \times H^1(0, T; R^n)$ has a convergent subsequence as $\Delta t \to 0$, and for every cluster point (x, u, p), $u \in K$ is an optimal control of Eqs. (63), (64), and (x, u, p) satisfies the necessary optimality condition

$$\begin{aligned} \frac{d}{dt}x(t) &= f(x(t), u(t)), \qquad x(0) = x_0 \\ -\frac{d}{dt}p(t) &= f_x(x(t), u(t))^t p(t) + l_x(x(t)), \qquad p(T) = g_x(x(T)) \\ u(t) &= \Psi(x(t), p(t)) \in U \end{aligned} \qquad (70)$$

Proof. First, we show that $\tilde{x}^N$ and $\tilde{p}^N$ are uniformly Lipschitzian in N. It will be convenient to drop the tilde in the notation for these sequences. In the case that U is bounded, we proceed by taking the inner product of Eq. (68) with x^k and employing Eq. (65):

$$\begin{aligned} \tfrac{1}{2}(|x^k|^2 - |x^{k-1}|^2) &\le \Delta t\,[\omega|x^k|^2 + |f(0, u^k)|\,|x^k|] \\ &\le \Delta t\,[(\omega + \tfrac{1}{2})|x^k|^2 + \tfrac{1}{2}|f(0, u^k)|^2] \end{aligned}$$

In case U is unbounded, Eq. (66) implies that

$$\tfrac{1}{2}(|x^k|^2 - |x^{k-1}|^2) \le \Delta t\,(\omega|x^k|^2 + c_2|u^k|^2)$$

and, by assumption on h, $\sum_{k=1}^N |u^k|^2\,\Delta t$ is bounded uniformly in N. In either case, by the discrete-time Gronwall's inequality we obtain that $|x^k| \le M_1$ for some $M_1 > 0$ uniformly in k and N. The condition (65) implies that

$$(f_x(x, u)p, p) \le \omega|p|^2$$

and taking the inner product of Eq. (69) with p^k, we obtain

$$\begin{aligned} \tfrac{1}{2}(|p^k|^2 - |p^{k+1}|^2) &\le \Delta t\,[\omega|p^k|^2 + |l_x(x^k)|\,|p^k|] \\ &\le \Delta t\,[(\omega + \tfrac{1}{2})|p^k|^2 + \tfrac{1}{2}|l_x(x^k)|^2] \end{aligned}$$

Thus $|p^k| \le M_2$ for some M_2 uniformly in k and N. Due to the Lipschitz continuity of Ψ, we find that $|u^k|$ bounded uniformly in k and N, and from Eq. (69),

$$\left|\frac{x^k - x^{k-1}}{\Delta t}\right|, \left|\frac{p^k - p^{k+1}}{\Delta t}\right| \text{ are bounded uniformly}$$

Using Lipschitz continuity of Ψ a second time, we find that $(u^k - u^{k-1})/\Delta t$ is uniformly bounded as well. By the compactness of Lipschitz continuous sequences in $L^2(0, T)$, there exists a subsequence $\hat{N}$ such that $(x^{\hat{N}}, u^{\hat{N}}, p^{\hat{N}})$ converges to (x^*, u^*, p^*) in $L^2(0, T; R^n \times R^m \times R^n)$ and pointwise a.e. in $(0, T)$. From Eq. (69),

$$x^N(t) = x_0 + \int_0^t f(\hat{x}^N(t), u^N(t))\,dt$$

where $\hat{x}^N$ is the piecewise constant sequence defined by x^k, $1 \le k \le N$. By Lebesgue's dominated convergence theorem, we find that x^* coincides with the solution $x(t; u^*)$ to Eq. (64) associated with u^*. For $v \in L^2(0, T; R^m)$ let v^N be the piecewise constant approximation of v, defined by $v^k = (1/N) \int_{t_k-1}^{t_k} v(t)\,dt$, $1 \le k \le N$. Then

$$J^N(u^N) \le J^N(v^N) \qquad \text{for all } v$$

and $x(t, v^N) \to x(t, v)$ as $N \to \infty$, and thus, by the Lebesgue dominated convergence theorem, $J(x_0, u^*) \le J(v; x_0)$ for all admissible control, that is, (x^*, u^*) is an optimal pair.

It is not difficult to argue that the triple (x^*, u^*, p^*) satisfies the necessary optimality [Eq. (70)].

Construction of Feedback Synthesis

The numerical realization of feedback synthesis for problems that are not of the LQR type has not received much research attention up to now. In this subsection we propose a method for the construction of the feedback synthesis K to the problem (63)–(64), which is still under investigation.

As we discussed in the Section titled "Dynamic Programming Principle and Hamilton–Jacoby–Bellman Equation," the optimal feedback law is given by $K(t, x(t)) = K_T(t, x(t)) = \Psi(x(t), V_x(t, x(t)))$, where $V(t, x)$ is the solution to HJB Eq. (38) and we stress the dependence of K on T. Let us assume that $f(x, u) = f(x) + Bu$, and $h = \beta/2|u|^2$. Then

$$K_T(t, x(t)) = -\frac{1}{\beta} B^t V_x(t, x(t)) \tag{71}$$

Since the problem under consideration is autonomous, we can write $K_T(t, x(t)) = K_{T-t}(0, x(t))$. Using the relationship between the HJB equation and the Pontryagin maximum principle, as described in the section titled "Dynamic Programming Principle and Hamilton–Jacoby–Bellman Equation," we construct a suboptimal feedback law. It is based on the fact that if we set $p(t) = V_x(t, x(t))$, then the pair $(x(t), p(t))$ satisfies the TPBV problem (70). Thus, if we define the function $x_0 \in R^n \to p_{x_0}(0) \in R^n$, where $(x, p)_{x_0}$ is the solution to Eq. (70) with $x(0) = x_0$, then

$$K_T(0, x_0) = \Psi(x_0, p_{x_0}(0)), \qquad x_0 \in R^n \tag{72}$$

The dependence of the feedback gain K_T on T is impractical, and we replace it by a stationary feedback law $v(t) = K(x(t))$, which is reasonable if T is sufficiently large.

Based on these observations, a suboptimal feedback law can be constructed by carrying out the following steps:

1. Choose $T > 0$ sufficiently large, as well as a grid $\Sigma \subset R^n$, and calculate the solutions $(x, p)_{x_0}$ to the TPBV problem for all initial conditions determined by $x_0 \in \Sigma$. Thus we obtain the values of K at $x_0 \in \Sigma$.
2. Use an interpolation method based on K at the grid points of Σ to construct a suboptimal feedback synthesis $\overline{K}$.

The interpolation in step 2 above can be based on appropriate Green's functions, for example,

$$\tilde{K}(x) = \sum_{j=1}^{M} G(x, x_j)\eta_j \quad \text{with} \quad G(x_i, x_j)\eta_j = K(x_i), \quad 1 \le i \le M \tag{73}$$

The Green's function interpolation (73) has the following variational principle: Consider

$$\min \int_{R^n} |\Delta K(x)|^2 \, dx \qquad \text{subject to} \quad K(x_i) = \zeta_i, \quad 1 \le i \le M$$

Then the optimal solution K is given by Eq. (73), where Green's function G satisfies the biharmonic equation $\Delta^2 G(x) = \delta(x)$. For instance, $G(x, y) = |x - y|$ (biharmonic Green's function) in R^3. In our numerical testings we found that $G(x, y) = |x - y|^\alpha$, $1.5 \le \alpha \le 4$, works very well. Alternatively, we may employ the following optimization method. We select a class $W_{ad} \subset C^1(R^n)$ of parametrized solutions to the HJB equation and collocation points $\{x_k\}$ in R^n. Then we determine the best interpolation $W \in W_{ad}$ based on the stationary equation $f(x) \cdot V_x - h^*(-B^t V_x) + l(x) = 0$ by

$$\min \quad \sum_k |[f(x_k) \cdot W_x(x_k) - h^*(-B^t W_x(x_k)) + l(x_k)]^+|^2 + \sum_{x_0 \in \Sigma} |W_x(x_0) - p_{x_0}(0)|^2 \tag{74}$$

over W_{ad}, where $x^+ = \max(0, x)$. Then we set $\overline{K}(x) = \Psi(x, W_x)$.

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BIBLIOGRAPHY

1. R. Bellman, *Dynamic Programming,* Princeton, NJ: Princeton Univ. Press, 1957.
2. W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Control,* New York: Springer-Verlag, 1975.
3. M. G. Crandall and P. L. Lions, Viscosity solutions of Hamilton–Jacobi equations, *Trans. Amer. Math. Soc.,* **277**: 1–42, 1983.
4. W. Fleming and M. Soner, *Controlled Markov Processes and Viscosity Solutions,* Berlin: Springer-Verlag, 1993.
5. W. M. Wonham, *Linear Multivariable Control: A Geometric Approach,* Berlin: Springer-Verlag, 1974.
6. D. L. Russell, *Mathematics of Finite-Dimensional Control Systems, Theory and Design,* New York: Marcel Dekker, 1979.
7. H. T. Banks and K. Ito, A numerical algorithm for optimal feedback gains in high dimensional linear quadratic regulator problems, *SIAM J. Control Optim.,* **29**: 499–515, 1991.

Reading List

V. M. Alekseev, V. M. Tikhomirov, and S. V. Fomin, *Optimal Control,* New York: Plenum, 1987.

M. Athans and P. Falb, *Optimal Control,* New York: McGraw-Hill, 1963.

L. D. Berkovitz, *Optimal Control Theory,* New York: Springer-Verlag, 1974.

A. E. Bryson and Y. Ho, *Applied Optimal Control,* New York: Wiley, 1975.

M. R. Hestenes, *Calculus of Variations and Optimal Control,* New York: Wiley, 1963.

H. Hermes and J. L. LaSalle, *Functional Analysis and Time Optimal Control,* New York: Wiley, 1969.

H. Knobloch, *Higher Order Necessary Conditions in Optimal Control Theory,* Berlin: Springer-Verlag, 1981.

H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems,* New York: Wiley-Interscience, 1972.

B. Lee and L. Marcus, *Foundations of Optimal Control Theory,* New York: Wiley, 1967.

L. W. Neustadt, *Optimization: A Theory of Necessary Conditions,* Princeton, NJ: Princeton Univ. Press, 1976.

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