MODEL REFERENCE ADAPTIVE CONTROL

The aim of control is to keep the relevant outputs of a given dynamic process within prescribed limits. Denoting the process to be controlled as a *plant* and denoting its input and output as u and y, respectively, the aim of control is to keep the error ($e_1 = y - y_d$) between the plant output and a desired output y_d within prescribed values. If y_d is a constant, the control problem is referred to as *regulation* and if y_d is a function of time, the problem is referred to as *tracking*. In the former case, the value of y_d around which the system is to be regulated is also referred to as a *set point* or *operating point*. The goal of control, in both cases, is to ensure that the output error e_1 is as small as possible, in the presence of disturbances and modeling errors, for all time, and that the controlled system is stable. Feedback control is one procedure by which regulation and tracking can be accomplished in a number of dynamic processes. When the differential equations describing the behavior of the plant are linear and known a priori, powerful analytical techniques in both time domain and frequency domain have been developed. When the characteristics of the plant are unknown, both regulation and tracking can be viewed as adaptive control problems.

The field of adaptive control in general, and model reference adaptive control in particular, has focused on problems where the uncertainties in the system are parametric. Such parametric uncertainties occur due to a variety of reasons on practical applications. Typically system dynamics, which are invariably nonlinear, are linearized to derive the requisite linear controller. The resulting linear model and its parameters are therefore dependent on and vary with the operating condition. Parameters also may vary due to aging, disturbances, or changes in the loading conditions. Parameters may be unknown due to approximations made in the modeling process. In all these cases, a controller that provides a uniformly satisfactory performance in the presence of the parametric uncertainties and variations is called for. The adaptive approach to this problem is to design a controller with varying parameters, which are adjusted in such a way that they adapt to and accommodate the uncertainties and variations in the plant to be controlled. By providing such a time-varying solution, the exact nature of which is determined by the nature and magnitude of the parametric uncertainty, the closed-loop adaptive system seeks to enable a better performance. The results that have accrued in the field of adaptive control over the past three decades have provided a framework within which such time-varying, adaptive controllers can be designed so as to yield stability and robustness in various control tasks.

Model reference adaptive control refers to a particular class of adaptive systems. In this class, adaptive controllers are designed by using a reference model to describe the desired characteristics of the plant to be controlled. The use of such reference models facilitates the analysis of the adaptive system and provides a stability framework. Two philosophically different approaches, indirect control and direct control, exist for synthesizing model reference adaptive controllers. In the indirect approach, the unknown plant parameters are estimated using a model of the plant before a control input is chosen. In the direct approach, an appropriate controller structure is selected and its parameters are directly adjusted so that the output error is minimized. For the sake of mathematical tractability, the desired output y_d needs to be characterized in a suitable form, which is generally accomplished by the use of a reference model. Thus in a model reference problem formulation, the indirect approach employs both an identification model and a reference model while the direct approach uses a reference model only. We describe these models in further detail below.

IDENTIFICATION MODEL

Mathematical modeling is an indispensable part of all sciences, whether physical, biological, or social. One often seeks to characterize the cause-and-effect relations in an observed phenomenon using a model and tune the model parameters so that the behavior of the model approximates the observed behavior for all cases of interest. One form of quantitative models is mathematical and in the cases of dynamic systems, these take the form of differential or difference equations.

Alternatively, a general mathematical model which represents the input–output behavior of a given process can also be used for identification. The model obtained in the latter case is often referred to as an *identification model,* since those from the first approach are either not available or too complex for control purposes. Often, especially for linear problems, frequency-domain methods are used to identify the system parameters. When measurement noise is present, the identification methods include statistical criteria so as to determine the model that best fits the observed data. Systems identification, which is based on such approaches, is a well-developed area of systems theory (1).

REFERENCE MODEL

The use of a reference model for controls can be traced to aircraft systems. Often, the situation therein is such that the controls designer is sufficiently familiar with the plant to be controlled and its desired properties; thus by choosing the structure and parameters of a reference model suitably, its outputs can be used as the desired plant response. While in principle such a model can be linear or nonlinear, considerations of analytical tractability have made linear reference models more common in practice.

Explicit and Implicit Model-Following

Two methods that have been studied extensively in this context include explicit and implicit model-following methods (2), both of which include the use of a reference model described by the homogeneous differential equation

$$\dot{y}_m = A_m y_m \tag{1}$$

where the constant matrix $A_m \in \mathbb{R}^{m\times m}$ is chosen so that the desired dynamics in terms of transient behavior, decoupling of modes, bandwidth, and handling qualities is captured. Suppose that the plant to be controlled is described adequately by an nth-order differential equation with $m(m \ll n)$ outputs as

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u \\ y_p &= C_p x_p \end{aligned} \tag{2}$$

The reference model in Eq. (1) is chosen so that the output y_p follows y_m as closely as possible. The explicit and implicit model-following methods are based on different performance indices of the model-following error $y_p - y_m$. In explicit model-following, the performance index is of the form

$$I_e = \int_0^\infty [(y_p - y_m)^T Q_e (y_p - y_m) + u^T Ru]\, dt$$

while in the latter the performance index implicitly includes the reference model as

$$I_i = \int_0^\infty \left[(\dot{y}_p - A_m y_p)^T Q_i (\dot{y}_p - A_m y_p) + u^T Ru\right] dt, \qquad Q_i > 0$$

In both cases, it can be shown that quadratic optimization theory can be used to determine the control input. In the former case, the optimal input has the form

$$u(t) = K_m y_m(t) + K_p x_p(t)$$

and in the latter case we have

$$u(t) = K_p x_p(t)$$

The structure of the controller can be used in an adaptive situation when the parameters of the plant are unknown, though the control parameters have to be estimated to compensate for parametric uncertainties.

Reference Model with Inputs

In Eq. (1), the output of the reference model was specified as the output of a homogeneous differential equation. A more general formulation of a reference model includes external inputs and is of the form

$$\dot{x}_m = A_m x_m + B_m r, \qquad y_m = C_m x_m \tag{3}$$

where A_m is a stable $n \times n$ matrix with constant elements, B_m and C_m are constant matrices with appropriate dimensions, and r is an arbitrary continuous uniformly bounded input. The goal of the control input u into the plant in Eq. (2) so that the output $y_p(t)$ tracks the output $y_m(t)$ as closely as possible. In this case, the reference input r along with the model in Eq. (3) with the parameters $\{A_m, B_m, C_m\}$ determines the output of the reference model. The introduction of the reference inputs significantly increases the class of desired trajectories that can be represented by a reference model. For a perfect model following to occur, the differential equations governing y_p and ym as well as the initial conditions $y_p(t)$ and $y_m(t)$ have to be identical. This imposes restrictive conditions on the matrices A_p, B_p, A_m, and B_m, in terms of their canonical forms. It has been shown by Berger that the requisite control input in this case is of the form

$$u(t) = K_p x_p(t) + K_m x_m(t) + K_r r(t)$$

In an adaptive situation, it is more reasonable to have the objective of asymptotic model-following where $y_p(t)$ is desired to follow $y_m(t)$ as $t \to \infty$. The problem in this case is to determine the conditions under which this can be achieved amidst parametric uncertainties.

MODEL REFERENCE ADAPTIVE CONTROL

The model reference adaptive control (MRAC) problem can be qualitatively stated as the following: Given a plant P with an input–output pair $\{u(\cdot), y_p(\cdot)\}$, along with a stable reference model M whose input–output pair is given by $\{r(\cdot), y_m(\cdot)\}$, where r is a bounded piecewise continuous function, determine the control input $u(t)$ for all $t \geq t_0$ so that

$$\lim_{t\to\infty} |y_p(t) - y_m(t)| = 0$$

Much of what is well known in MRAC concerns the case when the plant and model are linear and time-invariant though there have been a number of advances in the early 1990s in adaptive control of nonlinear systems (3,4) by making use of results in feedback linearization (5).

It becomes evident from the statement of the problem that considerable prior information regarding the plant P is needed to have a well-posed problem. Such information is critical while determining the structure of the reference model and the controller. For instance, the controller must be such that it makes use of all measurable signals in the system, is differentiator-free, and results in a bounded control input. For the plant output to follow the model output, the class of models M has to be constrained in some sense. Obviously, M depends on the prior information regarding the class P of plants. For example, if the reference input r is a pulse train and the model M has a unity transfer function, it is clear that the output of the plant cannot follow y_m asymptotically with a bounded input u and a differentiator-free controller. To determine M for linear time-invariant plants, results related to model-following in linear systems theory (6,7), LQG methods (8), and pole-placement can be utilized. Once the classes of plants P and M are determined, the structure of the controller that generates u can be found. When the parameters of the plant are known, the requisite controller has a linear structure. However, in order to compensate for the parametric uncertainty in P, the model reference adaptive controller has a nonlinear structure where the nonlinearity arises due to the fact that the controller parameters are adjusted on-line as a function of the system variables that are measured.

To better illustrate the nature of the nonlinearity in MRAC, we define the two parts of an MRAC, the algebraic and the analytic. In what follows, we focus our attention only on the case when the plant P and the model M are linear and time-invariant.

Algebraic Part and Analytic Part

We parameterize the controller $\mathscr{C}$ by a vector $\theta: \mathbb{R} \rightarrow \mathbb{R}^m$, where $\mathscr{C}$ is linear and time-invariant if θ is a constant. By using model reference approaches, one can determine the controller structure and a parameter θ^* such that if θ is equal to θ^* in $\mathscr{C}$, the closed-loop system determined by the plant together with the model has an output which asymptotically follows y_m. Such a design process marks the first step of an MRAC design and is referred to as the *algebraic part.*

The aim of adaptation is to generate the control input u such that $\lim_{t\to\infty}|y_p(t)| = 0$ when the plant parameters are unknown. Since $u(t)$ is determined by the manner in which the parameter $\theta(t)$ is adjusted in the controller, the problem can be equivalently stated in terms of $\theta(t)$. The second part of a MRAC design, referred to as the *analytic part,* consists of determining the rule by which $\theta(t)$ is to be adjusted at each instant of time so that the closed-loop remains stable and the output error $e(t)$, defined as $e(t) = y_p(t) - y_m(t)$, tends to zero as $t \rightarrow \infty$. The adjustment rule for $\theta(t)$ is referred to as the *adaptive law.*

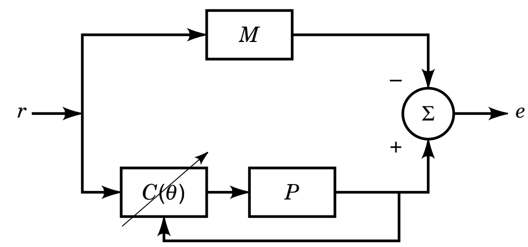

Figure 1. The MRAC problem.

The MRAC Problem

With the above definitions, the MRAC problem can be stated below (Fig. 1). Suppose the input–output pair of a linear time-invariant plant P with unknown parameters is $\{u(\cdot), y_p(\cdot)\}$.

1. Determine the class M of stable LTI reference models such that if the input–output pair of the model is given by $\{r(\cdot), y_m(\cdot)\}$, a uniformly bounded input u to the plant P, generated by a differentiator-free controller, exists which assures that
$$\lim_{t\to\infty} |y_p(t) - y_m(t)| = 0 \tag{4}$$
2. Determine a differentiator-free controller $C(\theta)$ parameterized by a vector $\theta(t) \in \mathbb{R}^m$, which generates u, such that for a constant value $\theta \equiv \theta^*$, the transfer function of the plant together with the controller is equal to the transfer function of M.
3. Determine a rule for adjusting $\theta(t)$ such that the closed-loop system is stable and Eq. (4) is satisfied.

When a disturbance is present, such an asymptotic output tracking may not be possible if very little information is available about the disturbance. In such a case, the goal of MRAC is to minimize the error between y_p and y_m as much as possible.

Since stability is vital to the satisfactory operation of any dynamic system and since in general adaptive systems are nonlinear, one of the major difficulties in designing adaptive systems is ensuring their stability properties. This often serves as a guideline while solving the MRAC problem stated above.

Error Model Approach

The solution of the MRAC problem is often significantly facilitated by an error model approach. This approach consists of studying the relationship between two kinds of errors commonly present in any adaptive system: (1) the tracking error e between the plant output and the model output and (2) the parameter error $\tilde{\theta}$ between the estimated adaptive parameter and its desired value. If the evolution of the error e is determined by the differential equation

$$\dot{e}(t) = f_1(e(t), \tilde{\theta}(t), t) \tag{5}$$

then the MRAC problem can be formulated as the determination of the adaptive law

$$\dot{\theta}(t) = f_2(e(t), t) \tag{6}$$

in such a way as to ensure closed-loop stability and asymptotic tracking. Focusing attention directly on the error rather than on the actual response of the plant or the reference model enables the designer to concentrate on the essential features of the problem and determine the adaptive law by inspection. Such an approach has facilitated the design of many adaptive systems both in the disturbance-free case as well as when disturbances and modeling errors are present.

Solution to the MRAC Problem

By the year 1980, several solutions to the MRAC problem when the plant to be controlled is linear and time-invariant were proposed. One such solution is summarized below.

Suppose the plant P to be controlled is described by the transfer function

$$W_p(s) = k_p \frac{Z_p(s)}{R_p(s)}$$

where k_p as well as the coefficients of the monic polynomials $Z_p(s)$ and $R_p(s)$ are unknown. The degree of $R_p(s)$, n, and the degree of $Z_p(s)$, $m \leq n - 1$, are assumed to be known. $Z_p(s)$ is assumed to be a Hurwitz polynomial. The sign of k_p is assumed to be known. (This assumption was relaxed in (9) by replacing sign(k_p) in the adaptive law by a nonlinear gain. In this article, we assume that sign(k_p) is known for ease of exposition). The reference model M is chosen to have a transfer function

$$W_m(s) = k_m \frac{Z_m(s)}{R_m(s)}$$

where $R_m(s)$ and $Z_m(s)$ are monic Hurwitz polynomials with degree n and $n - m$. The structure of the controller and the adaptive laws for adjusting the control parameters are given separately for the cases when the relative degree $n^* = n - m$ is unity and when it is greater than or equal to two. In both cases, the objective is to solve problems 1–3 stated above and accomplish the tracking stated in Eq. (4).

Case 1: n* = 1. In this case, the model transfer function is chosen to be strictly positive real (SPR) (10). This can be accomplished since the relative degree is one and $W_m(s)$ has asymptotically stable poles and zeros, by interlacing the zeros with the poles. The control input u is chosen as

$$\begin{aligned} u &= \theta^T(t)\omega(t) \\ \dot{\omega}_1 &= \Lambda\omega_1 + \ell u \\ \dot{\omega}_2 &= \Lambda\omega_2 + \ell y_p \\ \omega &= [r, \omega_1^T, y_p, \omega_2^T]^T \\ \theta &= [k, \theta_1^T, \theta_0, \theta_2^T]^T \end{aligned}$$

where $\Lambda \in \mathbb{R}^{(n-1)\times(n-1)}$ is asymptotically stable, with $\det(sI - \Lambda) = Z_m(s)$, (Λ, ℓ) is controllable, $\theta \in \mathbb{R}^n$ is the control parameter to be adjusted appropriately so that (4) is achieved, and ω is a sensitivity function which essentially estimates the state of the system on-line. The requisite adaptive law, assuming that $k_m > 0$, is given by

$$\dot{\theta}_\ell = -\text{sign}(k_p)\Gamma_\ell e_1 \omega, \qquad \Gamma_\ell > 0 \tag{7}$$

The structure of the model transfer function guarantees that a constant vector θ^* exists such that when $\theta(t) = \theta^*$, the plant together with the controller has the same transfer function as that of the model. The structure of the controller guarantees that the underlying error model is of the form

$$e_1 = \frac{k_p}{k_m} W_m(s)[\tilde{\theta}^T \omega] \tag{8}$$

where $\tilde{\theta} = \theta - \theta^*$. The structure of the adaptive law in Eq. (7) and the fact that $W_m(s)$ is SPR enables one to select an appropriate Lyapunov function of all of the states of the adaptive system. As a result, the closed-loop system remains bounded and $e_1(t)$ tends to zero as $t \to \infty$.

Case 2: n* ≥ 2. When the relative degree is greater than unity, even though the same error model structure as in Eq. (8) can be derived, it is not possible to choose an SPR model transfer function. This requires additional processing of the error signal and the sensitivity function in order to construct the necessary adaptive law. In particular, an augmented error ϵ_1 is constructed as

$$\begin{aligned} \epsilon_1 &= e_1 + e_2 \\ e_2 &= \theta^T W_m(s)\omega - W_m(s)\theta^T\omega \end{aligned}$$

Defining the filtered sensitivity function as ζ where

$$\zeta = W_m(s)\omega$$

one can show that the underlying error model, when k_p is known, is simplified from Eq. (8) to

$$\epsilon_1 = \tilde{\theta}^T \zeta. \tag{9}$$

As a result, an adaptive law of the form

$$\dot{\tilde{\theta}} = -\frac{\epsilon\zeta}{1 + \zeta^T\zeta} \tag{10}$$

can be chosen. Such an adaptive law guarantees that the closed-loop system remains bounded and that $\epsilon_1(t) \to 0$ asymptotically. The normalization in Eq. (10) is needed to establish the global boundedness of signals. Recently, other adaptive control structures and adaptive laws have been proposed (11) that do not employ such normalization, which has the potential to lead to better transient performance. The most important distinction between the approach in (10) and that in (9) is that the former prescribes an explicit Lyapunov function for the adaptive system and hence provides bounds on the tracking errors and parameter errors that can be estimated *a priori*.

Results in model reference adaptive control have been extended in several different directions, including robustness properties in the presence of disturbances and unmodeled dynamics, time-varying parameters and, most notably, adaptive control of nonlinear dynamic systems (3,4,10,12,13). Extensions to multivariable adaptive control and stable adaptive control in the presence of very few assumptions on the plant have also been proposed (10). Improvement of the transient response of the adaptive system by using multiple models and switching and tuning has also been proposed (14). Adaptive control techniques including self-tuning regulators, and automatic tuning, as well as practical aspects of control implementation and applications, can be found in Ref. (15).

PARAMETER IDENTIFICATION

The discussions earlier pertain to the global stability of the adaptive system and conditions under which the output error ϵ_1 will converge to zero. However, if one desires to match the closed-loop transfer function with that of the model, then the parameter estimate $\theta(t)$ must approach θ^* asymptotically. In other words, parameter identification has to be carried out. This also will ensure that the adaptive system will have an improved transient response as initial conditions and reference input change and better robustness with respect to different kinds of perturbations.

In the context of Case 1 above, the problem of parameter identification can be stated as follows: The output error ϵ_1 and the parameter error $\tilde{\theta}$ satisfy the differential equations

$$\begin{aligned} \dot{e} &= Ae + b\omega^T\tilde{\theta}, \qquad e_1 = h^T e \\ \dot{\tilde{\theta}} &= -e_1\omega \end{aligned} \tag{11}$$

where $h^T(sI - A)^{-1}b$ is SPR. Find the conditions on ω under which e_1 and $\tilde{\theta}$ will converge to zero. On the other hand, in Case 2 the parameter error evolves as

$$\dot{\tilde{\theta}} = \overline{u}\,\overline{u}^T\tilde{\theta} \tag{12}$$

where

$$\overline{u} = \frac{\zeta}{1 + \zeta^T\zeta}$$

The problem once again is the determination of conditions on $\overline{u}$ under which $\tilde{\theta}(t) \to 0$ as $t \to \infty$. These conditions are labeled as *persistent excitation* of the corresponding signal and are discussed further below.

PERSISTENT EXCITATION

A function $u: \mathbb{R}^+ \to \mathbb{R}^m$ is said to be persistently exciting in $\mathbb{R}^m$ if it satisfies the inequality

$$\int_t^{t+T_0} u(\tau)u^T(\tau)\,d\tau \geq \alpha I \qquad \forall t \geq t_0 \tag{13}$$

for some constants t_0, T_0, and α. Several statements equivalent to (13) can be given, one of which is that, for every unit vector w in $\mathbb{R}^m$, we obtain

$$\frac{1}{T_0}\int_t^{t+T_0} |u^T(\tau)w|\,d\tau \geq \epsilon_0 \qquad \forall t \geq t_0 \tag{14}$$

It can be shown that for $m = 2n$, if ζ satisfies Eq. (13), then $\lim_{t\to\infty}\tilde{\theta}(t) = 0$ in Eq. (12), and if ω satisfies Eq. (13), the errors in Eq. (11) converge to zero asymptotically, which ensures that parameter identification will take place.

Typically, a vector signal generated using n distinct frequencies can be shown to be persistently exciting in $\mathbb{R}^n$. The state of a $2n$th-order asymptotically stable system can be shown to be persistently exciting in $\mathbb{R}^{2n}$ if the input has n distinct frequencies and the system is controllable.

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MODELS, FUZZY. See FUZZY MODEL FUNDAMENTALS.
MODELS OF ELECTRICAL MACHINES. See ELECTRIC MACHINE ANALYSIS AND SIMULATION.
MODULAR INSTRUMENTATION. See CAMAC.
MODULARIZATION. See SUBROUTINES.
MODULATION. See DIGITAL AMPLITUDE MODULATION; INFORMATION THEORY OF MODULATION CODES AND WAVEFORMS.