LARGE-SCALE AND DECENTRALIZED SYSTEMS

The size of a system is a subjective notion, and so is the notion of large-scale systems. In the following we will take a pragmatic view, and consider a system large whenever it is conceptually or computationally attractive to decompose it into interconnected subsystems. Typically the subsystems are of small size and can be solved efficiently. The individual solutions can then be combined in some way to obtain a solution for the overall system.

If the system is decomposed along the boundaries of the physical subsystems, the subsequent analysis may produce important information about the interplay between subsystems behavior and the nature of interconnections. It may, however, be computationally inefficient. To reduce the computational effort, one should develop universal decomposition

techniques, which need not be constrained by physical boundaries of subsystems, but may lead to efficient solutions of large problems utilizing modern computer architectures. With that in mind, our first objective will be to describe several decomposition schemes that can either be used as preconditioners for decentralized control design, or can serve to speed up computations involving the control of large-scale systems (usually through the use of parallel processing).

To efficiently decompose a large-scale system, it is generally convenient to represent it in the form of a *graph.* Depending on whether or not the graph is weighted, we can roughly distinguish between two broad classes of decomposition algorithms. In problems where weights are not assigned to edges of the graph, decomposition schemes typically exploit topological properties such as structure and sparsity to obtain an appropriate partitioning (1–8). Whenever possible, it is also useful to incorporate any existing information regarding the physical attributes of the system (such as hierarchical structures or repetitive blocks that are built into the design). Decomposition algorithms of this type are commonly applied for solving large sparse systems of linear equations, and a number of them have been successfully utilized in parallelizing control-related computational problems.

A conceptually different class of algorithms arises when the system is viewed as a *weighted* graph. In this case the partitioning strategy changes significantly, since we can now utilize edge weights to identify weakly coupled subsystems and establish hierarchical relationships between them. Decomposition schemes based on this approach have found numerous applications not only in parallel computing, but also in decentralized control design. In the following sections particular attention will be devoted to the *epsilon decomposition* algorithm (9–11), due to its efficiency and simplicity; in this context, we will also examine the concept of *overlapping,* and its application to the control of large-scale systems.

Our second objective in this paper will be to provide the motivation and describe the basic ideas and techniques for *decentralized control* of dynamic systems. The accumulated experience in controlling large complex systems suggests three basic reasons for using decentralized control structures: dimensionality, information structure constraints, and uncertainty (6). By decomposing a system of large dimension into subsystems, a designer can devise decentralized strategies for solving control problems that would be either impractical or impossible to solve using a single centralized controller. Furthermore, in a large complex system where databases are developed around the plant with distributed sources of data, a need for fast control actions in response to local inputs and perturbations dictates use of distributed (that is, decentralized) measurement and control structures.

A restriction on what and where the information is delivered in a large system is a common *structural constraint* in building controllers and estimators. A good example is the standard automatic generation control in power systems, where the decentralized schemes are used to reduce the cost of communication that would be demanded by a centralized control strategy spread over distant geographical areas.

In modeling and control of large systems, it has been long recognized that models of subsystems can be obtained with increasing levels of accuracy and versatility. The essential *uncertainty* resides in the interconnections between the subsystems, since these interconnections are often poorly known in both deterministic and stochastic terms. One of the reasons for using decentralized control strategies is their inherent robustness to a wide variety of structured and unstructured perturbations in the subsystems and their interconnections. Furthermore, the strategies can be made *reliable* with respect to both interconnection and controller failures involving individual subsystems.

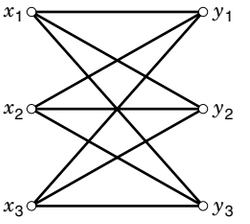

Figure 1. Bipartite graph **B**.

Epsilon Decompositions

A natural way to introduce epsilon decompositions is to consider the system of linear algebraic equations

$$Ax = b \tag{1}$$

and its solution by the Jacobi iterative method (12). In this context, the epsilon decomposition algorithm will be used to permute matrix A into a form that ensures rapid convergence to the solution.

The algorithm itself is remarkably simple—given a matrix $A = (a_{ij})$ and a value of parameter $\epsilon > 0$, all elements satisfying $|a_{ij}| \leq \epsilon$ are set to zero. The resulting sparsified matrix is then permuted into a block-diagonal form, and all the variables in the same block are considered to be strongly coupled. After such a permutation, the matrix A can be represented as

$$A = A_D + \epsilon A_C \tag{2}$$

where A_D is block-diagonal and all elements of A_C are less than or equal to one in magnitude. The following example illustrates how such a permutation can be identified, and subsequently utilized to iteratively solve Eq. (1).

Example 1. Consider the matrix

$$A = \begin{bmatrix} 1 & 0.05 & 2 \\ 0.01 & 1 & 0.1 \\ 0.1 & 0.1 & 2 \end{bmatrix} \tag{3}$$

and the corresponding bipartite graph **B** in Fig. 1, in which vertices y_i and x_j are connected if and only if $a_{ij} \neq 0$. If we remove all edges that correspond to elements $|a_{ij}| \leq 0.1$, we obtain the subgraph $\mathbf{B}^\epsilon$ shown in Fig. 2. It is easily seen that

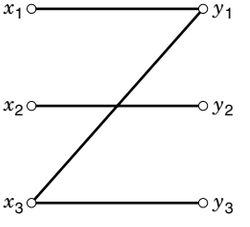

Figure 2. Subgraph $\mathbf{B}^\epsilon$.

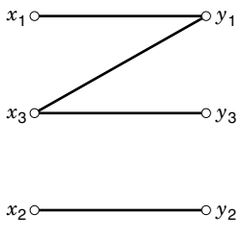

Figure 3. Components of $\mathbf{B}^\epsilon$.

the vertices and edges of $\mathbf{B}^\epsilon$ can now be regrouped into two disconnected components, as indicated in Fig. 3. The permutation defined by the vector $p = (1\ 3\ 2)$ now produces a matrix which satisfies Eq. (2), with $\epsilon = 0.1$ and

$$A_D = \left[\begin{array}{cc:c} 1 & 2 & 0 \\ 0.1 & 2 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right], \qquad A_C = \left[\begin{array}{cc:c} 0 & 0 & 0.5 \\ 0 & 0 & 1 \\ \hdashline 0.1 & 1 & 0 \end{array}\right] \tag{4}$$

The impact of weak coupling is easily seen when Eq. (1) is solved using the Jacobi iterative method

$$x_{k+1} = x_k - A_D^{-1}(Ax_k - b) \qquad (k = 0, 1, \ldots) \tag{5}$$

Namely, if the original matrix A of Eq. (3) is partitioned so that

$$A_D = \left[\begin{array}{cc:c} 1 & 0.05 & 0 \\ 0.01 & 1 & 0 \\ \hdashline 0 & 0 & 2 \end{array}\right] \tag{6}$$

we obtain $\|I - A_D^{-1}A\|_2 > 1$, and the process diverges; on the other hand, the partitioning in Eq. (4) obtained by epsilon decomposition results in rapid convergence.

Overlapping Epsilon Decompositions. Given a matrix A and a particular choice of parameter ϵ, there is no guarantee that A can be permuted into the form shown in Eq. (2). The obvious remedy in such cases is to repeat the decomposition with a larger value of ϵ; alternatively, we can use the concept of *overlapping*. The following example illustrates the basic ideas behind overlapping epsilon decomposition.

Example 2. Consider the following matrix

$$A = \begin{bmatrix} * & * & \epsilon \\ \epsilon & * & \epsilon \\ \epsilon & * & * \end{bmatrix} \tag{7}$$

where all entries larger than ϵ in magnitude are denoted by $*$. The corresponding bipartite graph $\mathbf{B}^\epsilon$ is given in Fig. 4, and

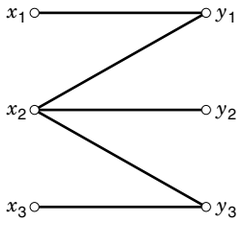

Figure 4. Subgraph $\mathbf{B}^\epsilon$.

it is easily verified that no permutation can produce weakly coupled diagonal blocks. On the other hand, if *repeated* vertices are allowed, we obtain the *expanded* bigraph $\tilde{\mathbf{B}}^\epsilon$ in Fig. 5, which now has two disconnected components. This transformation corresponds to a *rectangular* permutation matrix V, which is uniquely defined by the ordering of x and y vertices in the graph. Specifically,

$$VA = \tilde{A}V \tag{8}$$

where

$$\tilde{A} = \begin{bmatrix} * & * & \epsilon & 0 \\ \epsilon & * & \epsilon & 0 \\ \epsilon & 0 & * & * \\ \epsilon & 0 & \epsilon & * \end{bmatrix} \tag{9}$$

and

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{10}$$

It is easily verified that the expanded matrix $\tilde{A}$ now has an epsilon decomposition in the sense of Eq. (2).

Scaling. Another issue that arises in the practical application of epsilon decompositions is that elements of the matrix A can widely vary in size. In such cases it may not be possible to find a meaningful value for ϵ, and row scaling needs to be utilized to obtain a more uniform distribution of element values. This process is demonstrated by the following example.

Example 3. Let matrix A be defined as

$$A = \begin{bmatrix} 10 & 5 & 0.3 \\ 0.1 & 0.2 & 0.02 \\ 4.5 & 1 & 100 \end{bmatrix} \tag{11}$$

In this case $\epsilon = 0.3$ is obviously not a feasible choice, since the entire second row would be eliminated. However, if each row is scaled by the element with the maximal absolute value, we obtain

$$\overline{A} = \overline{A}_D + \epsilon\overline{A}_C \tag{12}$$

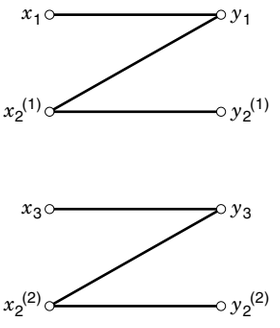

Figure 5. Expanded subgraph $\tilde{\mathbf{B}}^\epsilon$.

where $\epsilon = 0.1$ and

$$\overline{A}_D = \left[\begin{array}{cc:c} 1 & 0.5 & 0 \\ 0.5 & 1 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right], \qquad \overline{A}_C = \left[\begin{array}{cc:c} 0 & 0 & 0.3 \\ 0 & 0 & 1 \\ \hdashline 0.45 & 0.1 & 0 \end{array}\right] \tag{13}$$

Note that in this case A can be represented only as $A = A_D + A_C$ (without ϵ), but scaling does identify block-diagonal dominance, since

$$\|A_D^{-1}A_C\| = \epsilon\|\overline{A}_D^{-1}\overline{A}_C\| \ll 1 \tag{14}$$

This feature is frequently used to enhance the convergence of the iterative process in Eq. (5).

Structural Decompositions

By their very nature, epsilon decompositions are ideally suited for iterative solutions of Eq. (1). In contrast, there is an entire class of decompositions that are aimed at solving Eq. (1) *directly;* in this approach, no matrix entries are discarded, and the decompositions are designed to achieve certain desirable structures. A structure that is of particular interest in the analysis and simulation of large-scale systems is the *bordered block-diagonal* (BBD) form shown in Fig. 6. The appeal of this structure lies in its inherent potential for parallel computation, a feature that has been widely exploited in different areas of engineering.

Numerous algorithms have been developed for permuting a matrix into the BBD form, based on diverse concepts ranging from node clustering (2) and diakoptics (1,13) to various forms of graph dissection (3,4,7). Despite their obvious differences, all these methods have a common goal in identifying a *minimal* border that induces a block-diagonal structure in the rest of the matrix. The border size is an important issue in the solution of the system in Eq. (1), since the computation time depends heavily on the number of nonzero elements in the border.

In this section, we will briefly describe three methods for obtaining BBD structures. We begin our analysis with the classic nested dissection algorithm of George and Liu (3).

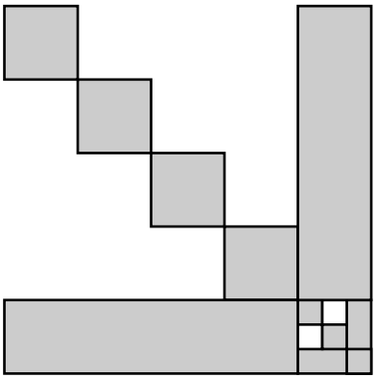

Figure 6. The BBD matrix structure.

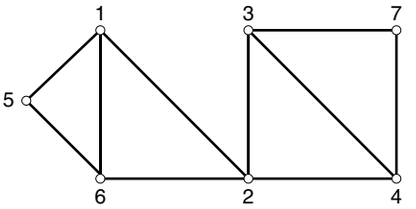

Figure 7. The undirected graph.

Nested Dissection

Nested dissection is a typical representative of BBD decomposition methods. It is relatively simple algorithm, in which the matrix is assumed to be structurally symmetric and can therefore be represented by an *undirected* graph. The following example illustrates the decomposition procedure.

Example 4. Let us consider the structurally symmetric matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} * & * & 0 & 0 & * & * & 0 \\ * & * & * & * & 0 & * & 0 \\ 0 & * & * & * & 0 & 0 & * \\ 0 & * & * & * & 0 & 0 & * \\ * & 0 & 0 & 0 & * & * & 0 \\ * & * & 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & 0 & 0 & * \end{pmatrix} \end{array} \tag{15}$$

and the corresponding undirected graph in Fig. 7. The basic idea of nested dissection is to arrange the vertices of this graph into a *rooted level structure,* such as the one in Fig. 8. To achieve optimal results, this structure should have as many levels as possible, which is assured by choosing an appropriate vertex as the root (14,15).

Once a rooted level structure is formed, the nested dissection algorithm identifies the "middle" level (in this example, vertex {2}) and removes it from the graph, so that two disconnected components are left. The middle level is then placed in the border, as illustrated in Eq. (16). In general, this proce-

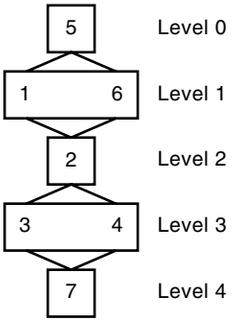

Figure 8. A rooted level structure.

dure can be recursively repeated on each of the remaining blocks:

$$\begin{array}{c} \\ 5 \\ 1 \\ 6 \\ 3 \\ 4 \\ 7 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccccc} 5 & 1 & 6 & 3 & 4 & 7 & 2 \end{array} \\ \left(\begin{array}{ccc:ccc:c} * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & * \\ * & * & * & 0 & 0 & 0 & * \\ \hdashline 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * & 0 \\ \hdashline 0 & * & * & * & * & 0 & * \end{array}\right) \end{array} \tag{16}$$

The nested dissection algorithm was found to be very successful for matrices with a regular structure (such as those arising in the numerical solution of partial differential equations using finite elements). However, this decomposition is much less effective for systems such as electric circuits, where the matrix structure is typically highly irregular (4). Several different solutions where proposed to alleviate this problem, dating back to the work of Sangiovanni-Vincentelli et al. (2). We now describe two recent decomposition algorithms, which have been successful over a wide range of matrix structures.

Decompositions Using Eigenvectors of Graphs. The idea of eigenvector decompositions was introduced by Pothen et al. (5), and is based on the *Laplacian* matrix of a graph. This matrix is defined as

$$Q \equiv D - A \tag{17}$$

where A is the adjacency matrix of the graph, and D is a diagonal matrix whose entries represent vertex degrees. It can be shown that matrix Q is always positive semidefinite, with at least one zero eigenvalue; the smallest positive eigenvalue of Q is denoted λ_2, and the corresponding eigenvector is denoted by X_2. The decomposition procedure can now be summarized as follows:

1. Compute eigenvector X_2, and determine its *median* component x_l.
2. Partition the vertices of the graph in the following way: for any vertex i, if $x_i < x_l$, set $i \in A$; otherwise, $i \in B$. In this way, the vertices will be partitioned into two approximately equal sets, A and B.
3. All the edges connecting sets A and B constitute an *edge separator, H*. The objective now is to find a minimal *vertex cover* for H (that is, the minimal number of vertices that need to be removed so that all edges in set H are removed). This vertex cover constitutes the *separator,* which appears in the border of the BBD structure.
4. Repeat steps 1–3 on the remaining components after the separator is removed.

Decompositions based on eigenvectors of graphs were found to be effective and applicable to a wide range of matrix structures. However, for large matrices computing the second eigenvector can be difficult, if not impossible. This consideration has motivated the development of the *balanced* BBD decomposition (11,16), which is described next.

The Balanced Bordered Block-Diagonal Decompositions. The balanced BBD decomposition algorithm is recursive and consists of two basic steps.

Step 1. Select a *maximal allowable block size* $N_{\max}$. Given this choice, move as many vertices as necessary to the border so that each block has size $\leq N_{\max}$. A typical situation after this step is shown in Fig. 9.

Step 2. The border is obviously too large after the first step; consequently, in step 2 we reconnect border vertices one by one. In this process, the next vertex to be reconnected is always the one that results in the *smallest* increase in block sizes (such an algorithm is called *greedy*). The process continues as long as there are at least two blocks left (in other words, it terminates when we establish that the next reconnection will result in a single block).

Once two diagonal blocks and an initial border have been obtained, steps 1 and 2 are repeated on *each* block (which makes the algorithm nested); the local borders are then added to the initial border. This procedure continues until we obtain a desired number of diagonal blocks of approximately equal size (hence the term "balanced").

Advantages of this algorithm are its execution speed and numerical simplicity, which result from the fact that only the *sizes* of blocks are considered, not their contents. In addition, since all diagonal blocks are balanced in size, the workload can be evenly distributed across different processors; this feature is critical for an efficient parallel solution of Eq. (1).

To illustrate the effectiveness of this decomposition, in Figs. 10 and 11 we show how it is applied to a highly irregular matrix that arises in the modeling of the US electric power network.

DECENTRALIZED CONTROL: AN EXAMPLE

The underlying idea of decentralized control is decomposition. A dynamic system is considered as an interconnection of subsystems, which have independent inputs and outputs. A satisfactory performance of the overall system is achieved by controlling each individual subsystem using local feedback, whereby local inputs are connected to local outputs (or states).

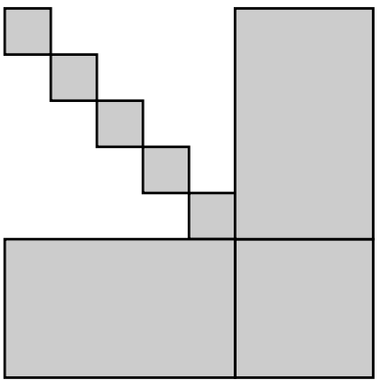

Figure 9. Situation after step 1.

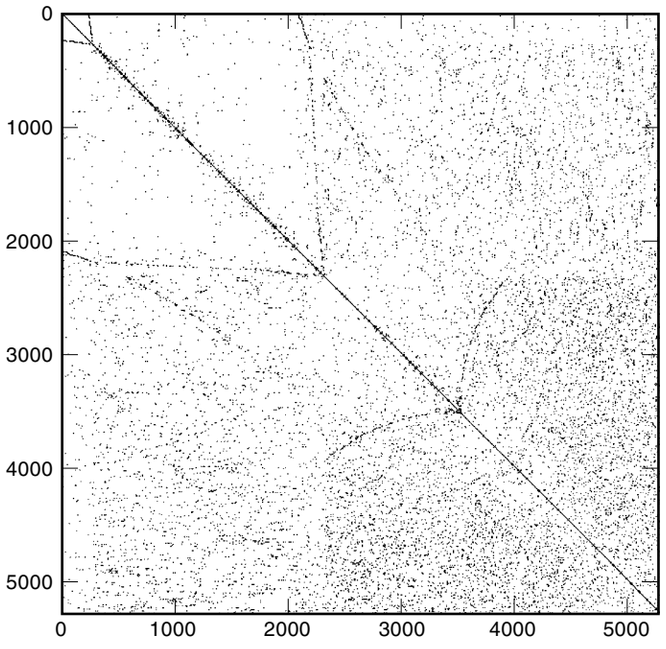

Figure 10. A model for the US electric power network (5300 × 5300).

To introduce the decentralized control problem, let us consider the two inverted pendulums interconnected by a spring as shown in Fig. 12. The control objective is to keep the pendulums upright by using the inputs u_1 and u_2. After linearization around the equilibrium state $\theta_1 = \theta_2 = 0$, the equations of motion are

$$\begin{aligned} ml^2\ddot{\theta}_1 &= mgl\theta_1 - ka^2(\theta_1 - \theta_2) + u_1 \\ ml^2\ddot{\theta}_2 &= mgl\theta_2 - ka^2(\theta_2 - \theta_1) + u_2 \end{aligned} \tag{18}$$

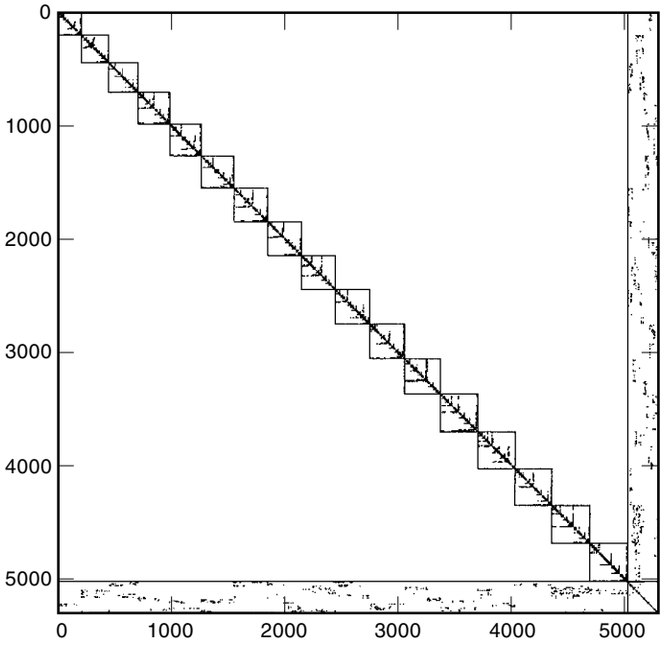

Figure 11. The matrix after a balanced BBD decomposition.

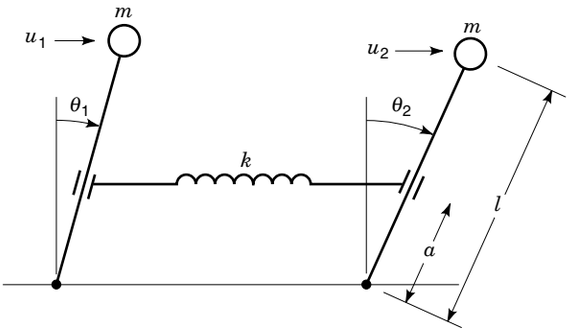

Figure 12. Inverted pendulums.

When the state and input vectors are chosen as $x = (\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)^T$ and $u = (u_1, u_2)^T$, the interconnected system is represented by the state equations

$$\mathbf{S}: \quad \dot{x} = \left[\begin{array}{c|c|c|c} 0 & 1 & 0 & 0 \\ \hline \dfrac{g}{l} - \dfrac{ka^2}{ml^2} & 0 & \dfrac{ka^2}{ml^2} & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline \dfrac{ka^2}{ml^2} & 0 & \dfrac{g}{l} - \dfrac{ka^2}{ml^2} & 0 \end{array}\right] x + \left[\begin{array}{c|c} 0 & 0 \\ \hline \dfrac{1}{ml^2} & 0 \\ \hline 0 & 0 \\ \hline 0 & \dfrac{1}{ml^2} \end{array}\right] u \tag{19}$$

In choosing the feedback control laws to achieve the objective, the *information structure constraint* is essential: Each input u_1 and u_2 can depend only on local states $x_1 = (\theta_1, \dot{\theta}_1)^T$ and $x_2 = (\theta_2, \dot{\theta}_2)^T$ of each individual pendulum. In other words,

$$u_1 = u_1(x_1), \qquad u_2 = u_2(x_2) \tag{20}$$

Since the system **S** is linear, a reasonable choice is a linear control law

$$u_1 = -k_1^T x_1, \qquad u_2 = -k_2^T x_2 \tag{21}$$

where feedback gain vectors

$$k_1 = (k_{11}, k_{12})^T, \qquad k_2 = (k_{21}, k_{22})^T \tag{22}$$

need to be selected to stabilize **S**, that is, to keep the pendulums in the upright position. When the two pendulums are considered as two subsystems

$$\begin{aligned} \mathbf{S}_1: \quad \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u_1 \\ \mathbf{S}_2: \quad \dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u_2 \end{aligned} \tag{23}$$

the overall system breaks up into two interconnected subsystems as

$$\begin{aligned} \mathbf{S}: \quad \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u_1 + e \begin{bmatrix} 0 & 0 \\ -\gamma & 0 \end{bmatrix} x_1 + e \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} x_2 \\ \dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u_2 + e \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} x_1 + e \begin{bmatrix} 0 & 0 \\ -\gamma & 0 \end{bmatrix} x_2 \end{aligned} \tag{24}$$

where $\alpha = g/l$, $\beta = 1/ml^2$, $\gamma = \bar{a}^2 k/ml^2$, and $e = (a/\bar{a})^2$.

By choosing the decentralized control of Eq. (21), we effectively intend to stabilize the interconnected system $\mathbf{S}$ by stabilizing the two subsystems in Eq. (23). This turns out to be a *robust control strategy,* since it can produce an overall closed-loop system that can remain stable despite the essential uncertainty about the height a of the spring. This uncertainty is represented by the normalized interconnection parameter e, which can take any value between 0 and 1. When an interconnected system is stabilized in this way, it is said to be *connectively stable* (6).

By using the control law of Eq. (21) in Eq. (23) we obtain the closed-loop subsystems as

$$\begin{aligned} \hat{\mathbf{S}}_1: \qquad \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ \alpha - \beta k_{11} & -\beta k_{12} \end{bmatrix} x_1 \\ \hat{\mathbf{S}}_2: \qquad \dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ \alpha - \beta k_{21} & -\beta k_{22} \end{bmatrix} x_2 \end{aligned} \tag{25}$$

and the overall closed-loop system becomes

$$\begin{aligned} \hat{\mathbf{S}}: \quad \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ \alpha - \beta k_{11} & -\beta k_{12} \end{bmatrix} x_1 + e \begin{bmatrix} 0 & 0 \\ -\gamma & 0 \end{bmatrix} x_1 + e \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} x_2 \\ \dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ \alpha - \beta k_{21} & -\beta k_{22} \end{bmatrix} x_2 + e \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} x_1 + e \begin{bmatrix} 0 & 0 \\ -\gamma & 0 \end{bmatrix} x_2 \end{aligned} \tag{26}$$

It is interesting that the system in this example belongs to the class of interconnected systems that can always be stabilized by decentralized feedback. A decentralized control law can be chosen to connectively stabilize the closed-loop system $\hat{\mathbf{S}}$ even if the spring is shifting up and down the length of the pendulums in an unpredictable way. The class of decentrally stabilizable systems is described next.

INTERCONNECTED PLANTS AND CONTROLLERS

To describe representations of plants, which are required in the design of decentralized controllers, let us consider a linear system

$$\begin{aligned} \mathbf{S}: \qquad \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{27}$$

as an interconnection

$$\begin{aligned} \mathbf{S}: \qquad \dot{x}_i &= A_i x_i + B_i u_i + \sum_{j \in \mathcal{N}} (A_{ij} x_j + B_{ij} u_j) \\ y_i &= C_i x_i + \sum_{j \in \mathcal{N}} C_{ij} x_j, \qquad i \in \mathcal{N} \end{aligned} \tag{28}$$

of N subsystems

$$\begin{aligned} \mathbf{S}_i: \qquad \dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= C_i x_i, \qquad i \in \mathcal{N} \end{aligned} \tag{29}$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, $y_i(t) \in \mathbb{R}^{l_i}$ are the state, input, and output of the subsystem $\mathbf{S}_i$ at a fixed time $t \in \mathbb{R}$, all matrices have proper dimensions, and $\mathcal{N} = \{1, 2, \ldots, N\}$. At present we are interested in *disjoint* decompositions, that is,

$$\begin{aligned} x &= (x_1^T, x_2^T, \ldots, x_N^T)^T \\ u &= (u_1^T, u_2^T, \ldots, u_N^T)^T \\ y &= (y_1^T, y_2^T, \ldots, y_N^T)^T \end{aligned} \tag{30}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$ are the state, input, and output of the overall system $\mathbf{S}$, such that

$$\begin{aligned} \mathbb{R}^n &= \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_N} \\ \mathbb{R}^m &= \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_N} \\ \mathbb{R}^l &= \mathbb{R}^{l_1} \times \mathbb{R}^{l_2} \times \cdots \times \mathbb{R}^{l_N} \end{aligned} \tag{31}$$

A compact description of the interconnected system $\mathbf{S}$ is

$$\begin{aligned} \mathbf{S}: \qquad \dot{x} &= A_D x + B_D u + A_C x + B_C u \\ y &= C_D x + C_C x \end{aligned} \tag{32}$$

where

$$\begin{aligned} A_D &= \operatorname{diag}\{A_1, A_2, \ldots, A_N\} \\ B_D &= \operatorname{diag}\{B_1, B_2, \ldots, B_N\} \\ C_D &= \operatorname{diag}\{C_1, C_2, \ldots, C_N\} \end{aligned} \tag{33}$$

and the coupling block matrices are

$$A_C = (A_{ij}), \qquad B_C = (B_{ij}), \qquad C_C = (C_{ij}) \tag{34}$$

The collection of N decoupled subsystems is described by

$$\begin{aligned} \mathbf{S}_D: \qquad \dot{x} &= A_D x + B_D u \\ y &= C_D x \end{aligned} \tag{35}$$

which is obtained from Eq. (32) by setting the coupling matrices to zero.

Important special classes of interconnected systems are *input* ($B_C = 0$) and *output* ($C_C = 0$) *decentralized systems,* where inputs and outputs are not shared among the subsystems. Input–output decentralized systems are described as

$$\begin{aligned} \mathbf{S}: \qquad \dot{x} &= A_D x + B_D u + A_C x \\ y &= C_D x \end{aligned} \tag{36}$$

where both B_C and C_C are zero. This structural feature helps to a great extent when decentralized controllers and estimators are designed for large plants.

A *static decentralized state feedback*

$$u = -K_D x \tag{37}$$

is characterized by a block-diagonal gain matrix

$$K_D = \operatorname{diag}\{K_1, K_2, \ldots, K_N\} \tag{38}$$

which implies that each subsystem $\mathbf{S}_i$ has its individual control law

$$u_i = -K_i x_i, \qquad i \in \mathcal{N} \tag{39}$$

with a constant gain matrix K_i. The control law u of Eq. (37), which is equivalent to the totality of subsystem control laws of Eq. (39), obeys the decentralized information structure constraint requiring that each subsystem $\mathbf{S}_i$ is controlled using its locally available state x_i. The resulting closed-loop system is described as

$$\hat{\mathbf{S}}: \qquad \dot{x} = (A_D - B_D K_D)x + A_C x \tag{40}$$

When *dynamic output feedback* is used under decentralized constraints, then controllers of the following type are considered:

$$\begin{aligned} \mathbf{C}_i: \qquad & \dot{z}_i = F_i z_i + G_i y_i \\ & u_i = -H_i z_i - K_i y_i, \qquad i \in \mathcal{N} \end{aligned} \tag{41}$$

which can be rewritten in a compact form as a single decentralized controller defined as

$$\begin{aligned} \mathbf{C}_D: \qquad & \dot{z} = F_D z + G_D y \\ & u = -H_D z - K_D y \end{aligned} \tag{42}$$

where

$$\begin{gathered} z = (z_1^T, z_2^T, \ldots, z_N^T)^T, \qquad y = (y_1^T, y_2^T, \ldots, y_N^T)^T \\ u = (u_1^T, u_2^T, \ldots, u_N^T)^T \end{gathered} \tag{43}$$

are the state $z \in \mathbb{R}^r$, input $y \in \mathbb{R}^l$, and output $u \in \mathbb{R}^m$ of the controller $\mathbf{C}_D$. By combining the system $\mathbf{S}$ and the decentralized dynamic controller $\mathbf{C}_D$, we get the composite closed-loop system as

$$\mathbf{S}\&\mathbf{C}_D: \qquad \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_D - B_D K_D C_D + A_C & -B_D H_D \\ G_D C_D & F_D \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{44}$$

DECENTRALIZED FEEDBACK STRUCTURES

Before a search for stabilizing decentralized feedback begins, it is necessary to determine whether such a feedback exists for a given plant. It is well known that if there are no restrictions on the information structure of the linear system $\mathbf{S}$ of Eq. (27), it can be stabilized if and only if the uncontrollable or unobservable modes of the system are stable. However, this is not the case when information constraints are present and the plant has unstable decentralized fixed modes.

Fixed Modes

Let us consider the system $\mathbf{S}$ of Eq. (27) in the form

$$\begin{aligned} \mathbf{S}: \qquad & \dot{x} = Ax + \sum_{i \in \mathcal{N}} \tilde{B}_i u_i \\ & y_i = \tilde{C}_i x, \quad i \in \mathcal{N} \end{aligned} \tag{45}$$

where only the inputs and outputs are partitioned as in Eq. (28), but the state (and thus the matrix A) is considered as a whole. Either the subsystems are ignored for technical reasons, or there are no natural or useful decompositions of the system into interconnected subsystems. In this case, the controllers $\mathbf{C}_i$ described in Eq. (41) still use local measurements y_i and inputs u_i, but are collectively responsible for the stabilization of $\mathbf{S}$. The closed-loop system is

$$\mathbf{S}\&\mathbf{C}_D: \qquad \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A - BK_D C & -BH_D \\ G_D C & F_D \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{46}$$

The basic result of Ref. 17 states that the closed-loop system $\mathbf{S}\&\mathbf{C}_D$ is stabilizable by decentralized controllers $\mathbf{C}_i$ if and only if the set of *decentralized fixed modes*

$$\Lambda_D = \bigcap_{K_D} \sigma(A - BK_D C) = \bigcap_{K_1, \ldots, K_N} \sigma\left(A - \sum_{i \in \mathcal{N}} \tilde{B}_i K_i \tilde{C}_i\right) \tag{47}$$

lies in the open left half plane. This result includes the interconnected system of Eq. (44) in an obvious way.

A simple characterization of decentralized fixed modes was provided in Ref. 18. For any subset $\mathcal{I} = \{i_1, \ldots, i_P\}$ of the index set $\mathcal{N}$, let $\mathcal{I}^C = \{j_1, \ldots, j_{N-P}\}$ denote the complement of $\mathcal{I}$ in $\mathcal{N}$, and define

$$\tilde{B}_{\mathcal{I}} = [\tilde{B}_{i_1}, \tilde{B}_{i_2}, \ldots, \tilde{B}_{i_P}], \qquad \tilde{C}_{\mathcal{I}^c} = \begin{bmatrix} \tilde{C}_{j_1} \\ \tilde{C}_{j_2} \\ \vdots \\ \tilde{C} j_{N-P} \end{bmatrix} \tag{48}$$

Then a complex number $\lambda \in \mathbb{C}$ is a decentralized fixed mode of $\mathbf{S}$ if and only if

$$\operatorname{rank} \begin{bmatrix} A - \lambda I & \tilde{B}_{\mathcal{I}} \\ \tilde{C}_{\mathcal{I}^c} & 0 \end{bmatrix} < n \tag{49}$$

for some $\mathcal{I} \subset \mathcal{N}$. Thus, the appearance of a fixed mode can be attributed to a special pole–zero cancellation, which cannot be removed by constant decentralized feedback. However, under relatively mild conditions, such fixed modes can be eliminated by time-varying decentralized feedback.

Structurally Fixed Modes

Graph theory serves as a suitable environment for both conceptual and numerical analysis of large-scale systems (6), because it allows the designer to take advantage of the special structural features of a large system before attempting a costly quantitative analysis. In particular, the theory is suitable for handling the lack of exact knowledge of system parameters by considering the existence rather than the true value of a connection between any two variables in the system (19).

A graph-theoretic characterization of controllability was introduced by Lin (20). He established that in an uncontrollable pair (A, B), loss of controllability is either due to an insufficient number of nonzero parameters (indicating a lack of sufficient linkage among system variables), or due to a perfect matching of system parameters. In the former case, the pair (A, B) is *structurally uncontrollable* in the sense that all pairs having the same structure as (A, B) are uncontrollable. By describing the structure of (A, B) using a directed graph, structural controllability can be checked via efficient graph-theoretic algorithms (6). In this context, a pair (A, B) is *structurally controllable* if and only if:

1. The system graph is *input-reachable,* that is, each state variable can be reached along a directed path by at least one input, and
2. The system graph has no *dilation,* that is, there exists no subset of state variables whose number exceeds the total number of all state and input variables directly affecting these variables.

The concept of *structurally fixed modes* represents a generalization of the idea of structural controllability (21). Let $\mathbf{D} = (\mathscr{V}, \mathscr{E})$ be a directed graph associated with the system $\mathbf{S}$ of Eq. (27), where $\mathscr{V} = \mathscr{U} \cup \mathscr{X} \cup \mathscr{Y}$ is a set of vertices corresponding to inputs, states, and outputs of $\mathbf{S}$, and $\mathscr{E}$ is a set of directed edges corresponding to nonzero elements of the system matrices A, B, and C. To every nonzero a_{ij} there corresponds an edge from vertex x_j to vertex x_i, to every nonzero b_{ij} an edge from u_j to x_i, and to every nonzero c_{ij} one from x_j to y_i. Given a feedback pattern $\overline{K}$, which adds a feedback edge from y_j to u_i for every $\overline{k}_{ij} = 1$, one obtains a digraph $\mathbf{D}_{\overline{K}} = (\mathscr{V}, \mathscr{E} \cup \mathscr{E}_{\overline{K}})$ that completely describes the structure of both the system $\mathbf{S}$ and the feedback constraint specified by $\overline{K}$, a special case of which is the decentralized constraint. In this case, permissible controllers have the structure

$$\mathbf{C}_{\overline{K}}: \qquad \begin{aligned} \dot{z}_i &= F_i z_i + \sum_{j \in \mathscr{J}_i} g_{ij} y_j \\ u_i &= -h_i^T z_i - \sum_{j \in \mathscr{J}_i} k_{ij} y_i \end{aligned} \tag{50}$$

where $\mathscr{J}_i = \{j : \overline{k}_{ij} = 1\}$.

If we choose a gain matrix K to conform with the feedback structure $\overline{K}$, then the set

$$\Lambda_{\overline{K}} = \bigcap_K \sigma(A - BKC) \tag{51}$$

can be conveniently specified as the set of fixed modes with respect to the decentralized feedback structure constraint defined by $\overline{K}$. It is fairly easy to show (6) that the system $\mathbf{S}$ can be stabilized by the constrained controller $\mathbf{C}_{\overline{K}}$ if and only if $\Lambda_{\overline{K}}$ is contained in the open left half plane. To characterize $\Lambda_{\overline{K}}$ as in Eq. (49), let us consider the index sets $\mathscr{I} \subset \mathscr{M} = \{1, 2, \ldots, M\}$ and replace $\mathscr{I}^C$ by $\mathscr{J} = \bigcup_{i \in \mathscr{I}^C} \mathscr{J}_i$, where $\mathscr{I}^C$ now refers to the complement of $\mathscr{I}$ in $\mathscr{M}$.

In order to formulate graph-theoretic conditions for the existence of structurally fixed modes, let us recall that two systems are said to be *structurally equivalent* if they have the same graph. Then, a system $\mathbf{S}$ is said to have structurally fixed modes with respect to a given feedback structure $\overline{K}$ if every system structurally equivalent to $\mathbf{S}$ has fixed modes with respect to $\overline{K}$. "Structurally fixed" is a generic concept: Having structurally fixed modes is a property of a class of systems sharing the same graph. Most importantly, if a system has no structurally fixed modes, then it either has no fixed modes, or the fixed modes can be removed by arbitrarily small perturbations of system parameters. This means that if a system has no structurally fixed modes for a given $\overline{K}$, then (generically) it can be stabilized by a controller with a gain K and structure $\overline{K}$, decentralized structure being a special case.

It was shown in Ref. 21 that a system $\mathbf{S}$ has no structurally fixed modes with respect to a feedback pattern $\overline{K}$ if and only if:

1. All state vertices of $\mathbf{D}_{\overline{K}}$ are covered by vertex disjoint cycles, and
2. No strong component of $\mathbf{D}_{\overline{K}}$ contains only state vertices, where a strong component is a maximal subgraph whose vertices are reachable from each other.

How this graph-theoretic criterion can be used to choose a minimum number of feedback links that avoid structurally fixed modes is explained in Refs. 6 and 22.

STABILIZATION

It has been common practice to use decentralized control in the stabilization of large interconnected systems. Each subsystem is stabilized independently using local feedback, and stability of the overall system is established using the Matrosov–Bellman concept of vector Liapunov functions. While a vector Liapunov function may offer more flexibility and computational simplicity than a scalar one, it remains inherently conservative. The use of vector functions, however, has been justified by the presence of uncertainty in the interconnections, which can cause a breakup of the system during operation along the subsystem boundaries. The method of vector Liapunov functions is a natural tool for making decentrally controlled systems robustly stable to interconnection failures, that is, *connectively stable* (6).

Vector Liapunov Functions

Let us assume that the plant is governed by linear time-invariant equations

$$\mathbf{S}: \qquad \dot{x}_i = A_i x_i + B_i u_i + \sum_{j=1}^{N} e_{ij} A_{ij} x_j, \quad i \in \mathscr{N} \tag{52}$$

which are an obvious derivative of Eq. (28) save for the insertion of the interconnection parameters $e_{ij} \in [0, 1]$. The parameters are coefficients of the $N \times N$ interconnection matrix $E = (e_{ij})$, which are used to model the uncertain strength of interconnections.

To stabilize $\mathbf{S}$ we use the decentralized control laws of Eq. (39) to stabilize each individual closed-loop system

$$\hat{\mathbf{S}}_i: \qquad \dot{x}_i = (A_i - B_i K_i) x_i, \quad i \in \mathscr{N} \tag{53}$$

This is presumably a relatively easy task, because the subsystems have low dimensions. Stability of each decoupled closed-loop system follows from a subsystem Liapunov function $v_i : \mathbb{R}^{n-i} \to \mathbb{R}_+$. These individual functions are then stacked up to form a vector Liapunov function $v(x) = [v_1(x_1), v_2(x_2), \ldots, v_N(x_N)]^T$. Finally, the vector Liapunov function is used to form a scalar Liapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ of the form

$$V(x) = d^T v \tag{54}$$

where $d = (d_1, d_2, \ldots, d_N)^T$ is a positive vector ($d_i > 0$, $i \in \mathscr{N}$). The Liapunov function $V(x)$ is utilized to establish stabil-

ity of the overall closed-loop system

$$\hat{\mathbf{S}}: \qquad \dot{x}_i = \hat{A}_i x_i + \sum_{j=1}^{N} e_{ij} A_{ij} x_j, \qquad i \in \mathcal{N} \tag{55}$$

where $\hat{A}_i = A_i - B_i K_i$.

Taking the total time derivative of $V(x)$ with respect to $\hat{\mathbf{S}}$, after lengthy but straightforward computations (6), we get

$$\dot{V}(x)_{\hat{S}} \leq -d^T \overline{W} z \tag{56}$$

where $z = (\|x_1\|, \|x_2\|, \ldots, \|x_N\|)^T$, and $\overline{W} = (\overline{w}_{ij})$ is the *aggregate matrix* defined as

$$\overline{w}_{ij} = \begin{cases} \alpha_i - \overline{e}_{ii}\beta_{ii}, & i = j \\ -\overline{e}_{ij}\beta_{ij}, & i \neq j \end{cases} \tag{57}$$

The positive numbers α_i depend on the choice of v_i's, and the nonnegative numbers β_{ij} are bounds on interconnection matrices A_{ij}. The binary numbers $\overline{e}_{ij}$ are elements of the $N \times N$ fundamental interconnection matrix $\overline{E} = (\overline{e}_{ij})$, which define the nominal structure of $\mathbf{S}$,

$$\overline{e}_{ij} = \begin{cases} 1 & \text{if } \mathbf{S}_j \text{ acts on } \mathbf{S}_i \\ 0 & \text{if } \mathbf{S}_j \text{ does not act on } \mathbf{S}_i \end{cases} \tag{58}$$

It is a well-known result (6,19) that:

The interconnected system $\hat{\mathbf{S}}$ is connectively stabilized by decentralized control $u_i = -K_i x_i$, $i \in \mathcal{N}$, if the aggregate matrix $\overline{W}$ is an M matrix.

A matrix $\overline{W} = (\overline{w}_{ij})$ with nonpositive off-diagonal elements is an M matrix if and only if there exists a positive vector d such that the vector

$$c = \overline{W}^T d \tag{59}$$

is a positive vector as well, and stability of $\hat{\mathbf{S}}$ follows by the standard Liapunov theorem involving $V(x) > 0$ and $\dot{V}(x)_{\hat{\mathbf{S}}} < 0$. The connective property of stability, which requires stability to hold for all $E \leq \overline{E}$, is concluded from the fact that $W(E) \geq \overline{W}(\overline{E})$ element by element, and W is an M matrix whenever $\overline{W}$ is.

Epsilon Decompositions

Epsilon decompositions are ideal preconditioners for the stabilization of large-scale systems using decentralized control and vector Liapunov functions (6,9). To see this, assume that $C = 0$ and form the augmented matrix

$$M = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \tag{60}$$

for the system $\mathbf{S}$ of Eq. (27). By applying the epsilon decomposition to M and regrouping the variables, we get $\mathbf{S}$ as

$$\mathbf{S}: \qquad \dot{x}_i = A_i x_i + B_i u_i + \epsilon \left(\sum_{\substack{j=1 \\ j \neq i}}^{N} A_{ij} x_j + B_{ij} u_j \right), \quad i \in \mathcal{N} \tag{61}$$

By applying the decentralized feedback of Eq. (39), we obtain the closed-loop system as

$$\hat{\mathbf{S}}: \qquad \dot{x}_i = \hat{A}_i x_i + \epsilon \sum_{\substack{j=1 \\ j \neq i}}^{N} \hat{A}_{ij} x_j, \quad i \in \mathcal{N} \tag{62}$$

where $\hat{A}_{ij} = A_{ij} - B_{ij} K_j$. In this case, the aggregate matrix $W = (w_{ij})$ is defined as

$$\overline{w}_{ij} = \begin{cases} \alpha_i, & i = j \\ -\epsilon\beta_{ij}, & i \neq j \end{cases} \tag{63}$$

We recall (6) that if the threshold ϵ is sufficiently small, then W is an M matrix: The smaller the absolute values of w_{ij}'s, the easier it is for W to satisfy the M-matrix conditions. We note, however, that the smaller ϵ is, the smaller the number of subsystems will be, implying a smaller reduction in the dimensionality of the stability problem via decomposition.

Decentrally Stabilizable Systems

Vector Liapunov functions provide only sufficient conditions for stability of interconnected systems, and one may search in vain for stabilizing control. For this reason, there are a number of results aimed at identifying classes of interconnected systems which can always be stabilized by decentralized feedback.

The most popular (but also the most restrictive) conditions for decentral stabilizability are the *matching conditions* (23),

$$\operatorname{Im} B_i \supset \operatorname{Im} A_{ij} \tag{64}$$

These conditions simply imply that the interconnections may be made to enter the subsystem through the input matrices B_i, that is, we have

$$\mathbf{S}: \qquad \dot{x}_i = A_i x_i + B_i \left(u_i + \sum_{j=1}^{N} e_{ij} D_{ij} x_j \right), \quad i \in \mathcal{N} \tag{65}$$

which means that $A_{ij} = B_i D_{ij}$ for some matrices D_{ij}.

Nonmatching conditions for decentral stabilizability of single-input subsystems have been considered in Ref. 24. The system $\mathbf{S}$ is described as

$$\mathbf{S}: \qquad \dot{x}_i = A_i x_i + b_i u_i + \sum_{j \in \mathcal{N}} e_{ij} A_{ij} x_j, \quad i \in \mathcal{N} \tag{66}$$

where, without loss of generality, the subsystem pairs (A_i, b_i) are assumed to be in the controllable canonical form. For each

interconnection matrix A_{ij}, define an integer

$$m_{ij} = \begin{cases} \max\{q - p : a_{pq}^{ij} \neq 0\}, & A_{ij} \neq 0 \\ n, & A_{ij} = 0 \end{cases} \tag{67}$$

Thus, m_{ij} is the distance between the main diagonal and a line parallel to the main diagonal that borders all nonzero elements of A_{ij}. Given an index set $\mathscr{I} \subset \mathscr{N}$, let $\mathscr{J}$ denote any permutation of $\mathscr{I}$. Then, the following holds:

The system $\mathbf{S}$ of Eq. (66) is stabilizable by decentralized state feedback $u_i = -k_i^T x_i$, $i \in \mathscr{N}$, if

$$\sum_{\substack{i \in \mathscr{I} \\ j \in \mathscr{J}}} (m_{ij} - 1) < 0 \tag{68}$$

for all $\mathscr{I}$ and all permutations $\mathscr{J}$.

In the special case of matching conditions, $m_{ij} = n_j - n_i$. The condition in Eq. (68) was obtained via vector Liapunov functions, and therefore guarantees decentral stabilizability even when the elements of the interconnection matrices A_{ij} are bounded nonlinear time-varying functions of the state x.

There are a large number of results that characterize wider classes of decentrally stabilizable systems (e.g., Refs. 24–27), most of them surveyed in Ref. 6. Some of the results broadened the scope of the given framework to include multivariable systems, time delays in the interactions, time-varying models, and stochastic perturbations. Especially interesting are schemes for decentralized output control, which utilize dynamic output controllers or *decentralized estimators* for reconstruction of the state of the overall system from the subsystem state estimates.

Robustness. There are many useful and interesting properties of systems stabilized by decentralized feedback. First, stability of decentrally controlled systems can tolerate nonlinearities in the interconnections; the nonlinear interconnections need not be known, since only their size must be limited. Once the closed-loop system $\hat{\mathbf{S}}$ is shown to be stable, it automatically follows (6) that any nonlinear time-varying version

$$\hat{\mathbf{S}}_N : \qquad \dot{x}_i = \hat{A}_i x_i + h_i(t, x), \quad i \in \mathscr{N} \tag{69}$$

of $\hat{\mathbf{S}}$ is connectively stable, provided the conical constraints

$$\|h_i(t, x)\| \leq \sum_{j=1}^{N} \bar{e}_{ij} \beta_{ij} \|x_j\|, \qquad i \in \mathscr{N} \tag{70}$$

on the interconnection functions $h_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n_i}$ hold for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. This robustness result is useful in practice because, typically, either interconnections are poorly known, or they change during the operation of the controlled system. Obviously, the result includes the case when the interconnection parameters e_{ij} are considered as nonlinear time-varying functions $e_{ij} : \mathbb{R} \times \mathbb{R}^n \to [0, 1]$.

Connective decentralized stabilization was considered by Tan and Ikeda (28) and, when applied to robotic manipulators, by Stokić and Vukobratović (29), Mills and Goldenberg (30), and Mills (31). New results in robust design of decentralized control have been obtained by Chen et al. (32,33), Zhang et al. (34), and Hassan et al. (35).

Since the introduction of multiple control system concepts for *reliable stabilization* (36), there have been numerous papers dealing with controller or sensor/actuator failures. Solutions to this problem in the algebraic setting were given by Vidyasagar and Viswanadham (37), Özgüler (38), Tan et al. (39), and Gündes and Kabuli (40); relying on adaptation, by Cho et al. (41); using $\mathscr{H}_\infty$, by Veillette et al. (42), Medanić (43), and Park and Bien (44); and in the context of linear quadratic control by Veillette (45). An application of the multiple control system concept to reliable control of steam generators was offered by Wu and Lin (46).

There are a number of interesting recent results concerning fixed-gain designs of decentralized control for interconnected systems. In Ref. 47 a stabilization scheme was proposed in the parameter space of linear systems, which was formulated in the standard framework of convex programming. This opened up the possibility of using a variety of concepts and algorithms available in the linear matrix inequalities approach (48) for robust decentralized designs. A promising result is the decentralized quadratic stabilization of interconnected systems (49), which can be reduced to an $\mathscr{H}_\infty$ control problem (see also Refs. 50 and 51).

There is a large body of literature on decentralized control design using frequency-domain methods, which is based on the work of Rosenbrock (52). The initial results were obtained in Ref. 53 and further developed in Refs. 54–56 using the M-matrix theory and block-diagonal dominance. Parametrization of decentralized stabilizing controllers by the fractional representation approach was considered in Refs. 57–60. Finally, we should mention a number of books (61–65) where a wide variety of useful methods and techniques can be found for the design of stabilizing decentralized control.

ADAPTIVE CONTROL

Decentrally stabilizable systems with suitable interconnection structures can (always) be stabilized by using local states or outputs, often employing high-gain feedback. How high the gains should be depends on how strong the interconnections are. More often than not, it is difficult, or impossible, to predict and limit the size of the coupling among the subsystems, implying that fixed-gain controllers may be incapable in stabilizing the system. In such cases, one has to use *adaptive controllers,* which can adjust the gains to values needed for overall stability (66).

Let us consider a single-input, single-output version of $\mathbf{S}$,

$$\begin{aligned} \mathbf{S} : \qquad & \dot{x}_i = A_i x_i + b_i u_i + P_i v_i \\ & y_i = c_i^T x_i \\ & w_i = Q_i x_i, \qquad i \in \mathscr{N} \end{aligned} \tag{71}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}$, and $y_i \in \mathbb{R}$ are the state, input, and output of the subsystem $\mathbf{S}_i$, and $v_i \in \mathbb{R}^{m_i}$ and $w_i \in \mathbb{R}^{l_i}$ are the interconnection inputs and outputs of $\mathbf{S}_i$ from and to other subsystems $\mathbf{S}_j$, $j \in \mathscr{N}$, which are related as

$$v_i = h_i(t, w), \qquad i \in \mathscr{N} \tag{72}$$

The crucial restriction on the functions $h_i : \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^{m_i}$ is that they are bounded:

$$\|h_i(t, w)\| \leq \sum_{j=1}^{N} \beta_{ij} \|w_j\| \tag{73}$$

where β_{ij} are nonnegative, but unknown, numbers. The matrices A_i and vectors b_i and c_i defining the subsystem $\mathbf{S}_i$ are not specified, except for the fact that pairs (A_i, b_i) are controllable and that pairs (A_i, c_i) are observable. The uncertainty about the triples (A_i, b_i, c_i) compounds the essential uncertainty about the overall system $\mathbf{S}$ caused by our ignorance of the interconnections.

The control objective is to force each state $x_i(t)$ of $\mathbf{S}_i$ to track the state $x_{mi}(t)$ of the corresponding reference model

$$\begin{aligned} \mathbf{M}_i : \qquad & \dot{x}_{mi} = A_{mi} x_{mi} + b_{mi} r_i \\ & y_{mi} = c_{mi}^T x_{mi}, \qquad i \in \mathcal{N} \end{aligned} \tag{74}$$

where $r_i \in \mathbb{R}$ is the external (reference) input. To achieve this objective, we assume that there exist subsets $\mathcal{I} \subset \mathcal{N}$ and $\mathcal{J} = \mathcal{N} - \mathcal{I}$ such that

$$\begin{aligned} P_i &= b_i p_i^T, \qquad i \in \mathcal{I} \\ Q_i &= q_i c_i^T, \qquad i \in \mathcal{J} \end{aligned} \tag{75}$$

for some constant vectors $p_i \in \mathbb{R}^{m_i}$ and $q_i \in \mathbb{R}^{l_i}$, which are matching conditions requiring that either incoming disturbances enter through the control channel or outgoing disturbances pass through the measurement channel. These conditions, as shown in Ref. 67, make the system stabilizable by high-gain decentralized feedback, and are crucial for the adaptation scheme to work.

The basic requirement of the adaptive regulator is to drive the state of the overall system $\mathbf{S}$ to zero. For this purpose, the local control laws are chosen as

$$u_i = \theta_i^T x_i, \qquad i \in \mathcal{N} \tag{76}$$

where $\theta_i \in \mathbb{R}^{n_i}$ is the time-varying adaptation gain vector. To arrive at suitable adaptation laws $\theta_i(t)$, we choose A_{mi}'s as (satisfactorily) stable constant matrices and set

$$\dot{\theta}_i = -R_i (k_i^T x_i) x_i, \qquad i \in \mathcal{N} \tag{77}$$

where R_i and k_i are appropriate constant matrices and $\theta_i(t_0)$ is finite. Finally, the closed-loop system is

$$\begin{aligned} \hat{\mathbf{S}} : \qquad & \dot{x}_i = (A_{mi} + b_i \phi_i^T) x_i + b_i p_i^T h_i \\ & \dot{\phi}_i = -R_i (k_i^T x_i) x_i, \qquad i \in \mathcal{N} \end{aligned} \tag{78}$$

where $\phi_i \in \mathbb{R}^{n_i}$ is the ith parameter adaptation error defined as $\phi_i = \theta_i - \theta_i^*$, and θ_i^* is a model-matching constant parameter vector. We denote the solutions of the closed-loop system $\mathbf{S}$ by $(x, \phi)(t; t_0, x_0, \phi_0)$, where $x = (x_1^T, x_2^T, \ldots, x_N^T)$ and $\phi = (\phi_1^T, \phi_2^T, \ldots, \phi_N^T)^T$. Relying on the decentral stabilizability condition of Eq. (75), Gavel and Šiljak (66) established the following basic result:

The solutions $(x, \phi)(t; t_0, x_0, \phi_0)$ of $\hat{\mathbf{S}}$ are globally bounded, and $x(t; t_0, x_0, \phi_0) \to 0$ as $t \to \infty$.

The adaptive stabilizability conditions, which are based on structural restriction in Eq. (75), allow the controller gains to rise to whatever level is necessary to ensure that stability of the subsystem overrides the perturbations caused by interconnection fluctuations, so long as they are finite. The boundedness part of the above result ensures the boundedness of the adaptation gains and therefore the realizability of the decentralized adaptive control scheme.

The above basic result has many extensions. In the state-tracking problem, the state $x(t)$ of the plant $\mathbf{S}$ follows the state $x_m(t)$ of the reference model $\mathbf{M}$ despite the change in the size and shape of interconnections. Furthermore, the steady-state tracking error $e(t) = x(t) - x_m(t)$ can be made as small as desired. When the subsystem models are known, which is often the case in practice, the adaptive decentralized scheme extends to multiinput, multioutput subsystems with added simplicity in implementation; the scheme requires only one adaptation parameter per subsystem. Under relatively mild conditions the scheme can accommodate output feedback controllers as well.

Since the initial work on adaptive decentralized control by Hmamed and Radouane (68), a wide variety of schemes have been developed. Unmodeled dynamics with fast and slow modes in the subsystems was considered by Ioannou and Kokotović (69). The use of input–output models was initiated by Wiemer and Unbehauen (70) for a discrete-time problem of decentralized adaptive control. A state-space approach to the same problem was proposed by Reed and Ioannou (71) as well as Kamoun et al. (72) and Yang and Papavassilopoulos (73). Motivated by models of mechanical systems, Shi and Singh (74,75) considered higher-order interconnections with polynomial bounds. Poor transient behavior of the standard adaptive schemes, which is caused by insufficient knowledge of subsystem parameters, can be improved by exchanging the output signals between individual subsystems and assuming weak coupling (76).

A major drawback of the early adaptive decentralized schemes was the relative-degree restriction on the subsystems. This restriction was first removed by Ortega and Herrera (77) by using the concept of higher-order tuning. The same result was later achieved by Wen (78) by applying the integrator backstepping procedure. The original scheme of Gavel and Šiljak (66) was extended by Lyon (79) to the case when the relative order of each subsystem does not exceed two. Due to the high-gain nature of the adaptive scheme proposed in Refs. 66, 80, a variable-structure decentralized control (81,82) became a logical candidate for improving the design. Sasaki et al. (83) developed a variable-structure version of their decentralized adaptive scheme for control of distributed systems.

Input–output models with relay-type controllers were used by Brusin and Ugrinovskaya (84); see also Refs. 85 and 86. In Refs. 87 and 88, coordinate transformations have been utilized to broaden the class of systems that can be stabilized by decentralized adaptive control. Indirect adaptive schemes have been proposed by Wen (89) and Spooner and Passino (90). Finally, a partially decentralized adaptive control has been developed in Ref. 91.

OVERLAPPING DECENTRALIZED CONTROL

In a wide variety of natural and industrial systems, subsystems share common parts (6). In these cases, for either con-

ceptual or computational reasons it is advantageous to use overlapping decentralized control. We will use the overlapping epsilon decomposition to provide a justification for such a practice.

Let us consider a linear constant system

$$\mathbf{S}: \qquad \dot{x} = Ax + Bu \tag{79}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and the block matrices

$$A = \left[\begin{array}{ccc:c} A_{11} & A_{12} & \epsilon A_{13} & \\ \epsilon A_{21} & A_{22} & \epsilon A_{23} & \\ \hdashline \epsilon A_{31} & A_{32} & A_{33} & \end{array}\right], \qquad B = \left[\begin{array}{c:c} B_{11} & 0 \\ \hdashline 0 & 0 \\ \hdashline 0 & B_{32} \end{array}\right] \tag{80}$$

are decomposed along the dashed lines defining the two overlapping subsystems. By using a linear transformation

$$\tilde{x} = Vx \tag{81}$$

where V is the $\tilde{n} \times n$ matrix

$$V = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{82}$$

and the identity matrices are compatible with the blocks of A, we get the *expansion* of $\mathbf{S}$ as

$$\tilde{\mathbf{S}}: \qquad \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \tag{83}$$

where $x \in \mathbb{R}^{\tilde{n}}$, $\tilde{n} = n_1 + 2n_2 + n_3$, and n_1, n_2, n_3 are dimensions of the square matrices A_{11}, A_{22} and A_{33}. The system matrices are

$$\tilde{A} = VAU + M, \qquad \tilde{B} = VB + N \tag{84}$$

and

$$\begin{aligned} U &= \begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}I_2 & \frac{1}{2}I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix} \\ M &= \begin{bmatrix} 0 & \frac{1}{2}A_{12} & -\frac{1}{2}A_{12} & 0 \\ 0 & \frac{1}{2}A_{22} & -\frac{1}{2}A_{22} & 0 \\ 0 & -\frac{1}{2}A_{22} & \frac{1}{2}A_{22} & 0 \\ 0 & -\frac{1}{2}A_{32} & \frac{1}{2}A_{32} & 0 \end{bmatrix} \\ N &= 0 \end{aligned} \tag{85}$$

Then, the expansion $\tilde{\mathbf{S}}$ of Eq. (83) has the form

$$\tilde{\mathbf{S}}: \qquad \dot{\tilde{x}} = \left[\begin{array}{cc:cc} A_{11} & A_{12} & 0 & \epsilon A_{13} \\ \epsilon A_{21} & A_{22} & 0 & \epsilon A_{23} \\ \hdashline \epsilon A_{21} & 0 & A_{22} & \epsilon A_{23} \\ \epsilon A_{31} & 0 & A_{32} & A_{33} \end{array}\right] \tilde{x} + \left[\begin{array}{c:c} B_{11} & 0 \\ 0 & 0 \\ \hdashline 0 & 0 \\ 0 & B_{32} \end{array}\right] u \tag{86}$$

where the dashed lines delineate two *disjoint* subsystems, so that we can write the expansion $\tilde{\mathbf{S}}$ as

$$\tilde{\mathbf{S}}: \qquad \dot{\tilde{x}} = \left[\begin{array}{c:c} \tilde{A}_{11} & \epsilon\tilde{A}_{12} \\ \hdashline \epsilon\tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] \tilde{x} + \left[\begin{array}{c:c} \tilde{B}_{11} & 0 \\ \hdashline 0 & \tilde{B}_{22} \end{array}\right] u \tag{87}$$

It is crucial to note from Eq. (87) that the expansion $\tilde{\mathbf{S}}$ of Eq. (86) has an epsilon decomposition, while the original system $\mathbf{S}$ of Eq. (80) does not. Decentralized stabilization schemes can now take advantage of the weak coupling among the two subsystems, and use local feedback to stabilize the expansion $\tilde{\mathbf{S}}$ by independent stabilization of the two subsystems. It is easy to show that stability of the expanded system $\tilde{\mathbf{S}}$ implies stability of the original system $\mathbf{S}$.

The circle of ideas and methods surrounding the overlapping decompositions and control structures have been organized into a general mathematical framework known as the *inclusion principle* (6). In a wide variety of problems arising in the theory and practice of large-scale systems, the principle produced interesting conceptual insights and useful solution procedures (92–98).

OPTIMIZATION

Despite considerable efforts and a large number of new results in the theory of large complex systems, the fundamental problem of *optimal decentralized control* has remained unsolved for over two decades. The simple reason has been the fact that the decentralized information structure constraints have not been successfully incorporated into any of the standard optimization frameworks. Neither Pontryagin's maximum principle nor Bellman's dynamic programming can handle the lack of complete state observation, a difficulty recognized by Fleming (99) as far back as the late sixties. For this reason, there have been a large number of results relying on pragmatic *suboptimality* concepts, which have capitalized on effective solutions of robustness issues in the suboptimality framework (6).

The standard practice has been to optimize each decoupled subsystem using linear quadratic (LQ) control laws. Then, suboptimality of the interconnected closed-loop system, which is driven by the union of the locally optimal LQ control laws, is determined with respect to the sum of the quadratic costs chosen for the subsystems. Under relatively mild conditions, suboptimality implies stability. Furthermore, the degree of suboptimality, which is computed with respect to the globally optimal union of decoupled subsystems, can serve as a measure of robustness with respect to a wide spectrum of uncertainties residing in both the subsystems and their interactions. It has been shown (100) how the classical measures of gain and phase margins, as well as the gain reduction tolerance, can be incorporated in the decentralized LQ control of large interconnected systems.

Recently, a solution to the *optimal decentralized stochastic control problem* has been offered in the classical optimization framework of Lagrange (101) relying on the constrained optimization approach proposed in Ref. 102. The principal idea is to redefine the information structure constrains as differential equations, and attach Lagrange multipliers to each constraint to obtain sufficient as well as necessary conditions for optimality of decentralized control laws. The multipliers are functions of time and state, and, because of the role they play in the optimization process, they are termed the Lagrange–Liapunov multipliers. The sufficient conditions are formulated in terms of the Hamilton–Jacobi-like equations and are proved by using the method of global optimization (103). In contrast, the necessary conditions of optimality, which are de-

rived very much in the spirit of the Fleming's result, require the solution of a two-point boundary value problem with a minimum condition of the Pontryagin type.

Let us present the main idea of this new development with a minimum of technical elements. A system **S**, which is composed of N interconnected subsystems, is described by a stochastic differential equation of Ito's type,

$$\mathbf{S}: \qquad dx_i = \left(A_{ii}(t)x_i + B_i(t)u_i(t,x) + \sum_{j\neq i}^{n} A_{ij}(t)x_j \right) dt + C_i(t,x)\, dw_i, \qquad i \in \mathcal{N} \tag{88}$$

With the system **S** we associate a quadratic cost

$$J = \frac{1}{2}\, \mathcal{E} \sum_{i=1}^{N} \int_{t_0}^{t_1} [x_i^T Q_i(t) x_i + u_i^T R_i(t) u_i]\, dt \tag{89}$$

The design objective is to determine a control $u(t, x)$ that drives the system **S** optimally with respect to cost J and, at the same time, satisfies the essential information restriction that the control law u_i at each individual subsystem can utilize only the locally available state x_i. More precisely, we have the following problem:

Determine a control $u(t, x)$ that minimizes the functional J with respect to the system **S** under the decentralized information structure constraints

$$u_i = u_i(t, x_i), \qquad i \in \mathcal{N} \tag{90}$$

By reformulating the constraints in Eq. (90) as differential equations

$$\frac{\partial u_i(t,x)}{\partial x^i} = 0, \qquad i \in \mathcal{N} \tag{91}$$

where the vector x^i is the state vector x without the state x_i of the ith subsystem, we can append Eq. (91) to the equations of motion in Eq. (88) and use the standard Lagrange optimization framework in the manner suggested by Pontryagin in his maximum principle and by Bellman in his version of the Hamilton–Jacobi equation. When the additional assumption of Gaussian state evolution is added, the optimality conditions obtained in the new approach provide a feedback structure for decentralized control, which involves Riccati-type equations in the same way as in the classical regulator theory of Kalman (104). This fact is expected to play a major role in applications of optimal decentralized control to complex interconnected systems.

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LARGE SCALE INTEGRATION. See VLSI CIRCUIT LAYOUT.

LARGE-SCALE SYSTEMS. See LARGE-SCALE AND DECENTRALIZED SYSTEMS.

LARGE SUPERCONDUCTING DETECTOR SOLENOIDS. See HIGH-ENERGY PHYSICS PARTICLE DETECTOR MAGNETS.

LASER ABLATION. See LASER APPLICATIONS IN MEDICINE.