KALMAN FILTERS AND OBSERVERS

FILTERING THEORY

GAUSSIAN PROCESSES

NONLINEAR FILTERS

Theoretically, a Kalman filter is an estimator for what is called the linear quadratic Gaussian (*LQG*) problem, which is the problem of estimating the instantaneous "state" of a linear dynamic system perturbed by Gaussian white noise, by using measurements linearly related to the state, but corrupted by Gaussian white noise. The resulting estimator is statistically optimal with respect to any quadratic function of estimation error. R. E. Kalman introduced the "filter" in 1960 (1).

Practically, the Kalman filter is certainly one of the greater discoveries in the history of statistical estimation theory, and one of the greatest discoveries in the twentieth century. It has enabled humankind to do many things that could not have been done without it, and it has become as indispensable as silicon in the makeup of many electronic systems. Its most immediate applications have been for the control of complex dynamic systems such as continuous manufacturing processes, aircraft, ships, or spacecraft.

In order to control a dynamic system, one must first know what the system is doing. For these applications, it is not always possible or desirable to measure every variable that one wants to control. The Kalman filter provides a means for inferring the missing information from indirect (and noisy) measurements. In such situations, the Kalman filter is used to estimate the complete state vector from partial state measurements and is called an observer. The Kalman filter is also used for predicting the outcome of dynamic systems that people are not likely to control, such as the flow of rivers during flood conditions, the trajectories of celestial bodies, or the prices of traded commodities.

From a practical standpoint, this article will present the following perspectives:

1. Kalman filtering is an algorithm made from mathematical models. The Kalman filter makes it easier to solve a problem, but it does not solve the problem all by itself. As with any algorithm, it is important to understand its use and function before it can be applied effectively. The purpose of this article is to ensure sufficient familiarity with the use of the Kalman filter that it can be applied correctly and efficiently.
2. The Kalman filter is a recursive algorithm. It has been called "ideally suited to digital computer implementation," in part because it uses a finite representation of the estimation problem—by a finite number of variables (2). It does, however, assume that these variables are real numbers with infinite precision. Some of the problems encountered in its use arise from the distinction between finite dimension and finite information, and the distinction between finite and manageable problem sizes. These are all issues on the practical side of Kalman filtering that must be considered along with the theory.
3. It is a complete statistical characterization of an estimation problem. It is much more than an estimator, because it propagates the entire probability distribution of the variables it is tasked to estimate. This is a complete characterization of the current state of knowledge of the dynamic system, including the influence of all past measurements. These probability distributions are also useful for statistical analysis and predictive design of sensor systems.
4. In a limited context, the Kalman filter is a learning process. It uses a model of the estimation problem that distinguishes between phenomena (what we are able to observe), noumena (what is really going on), and the state of knowledge about the noumena that we can deduce from the phenomena. That state of knowledge is represented by probability distributions. To the extent that those probability distributions represent knowledge of the real world, and the cumulative processing of knowledge is learning, this is a learning process. It is a fairly simple one, but quite effective in many applications.

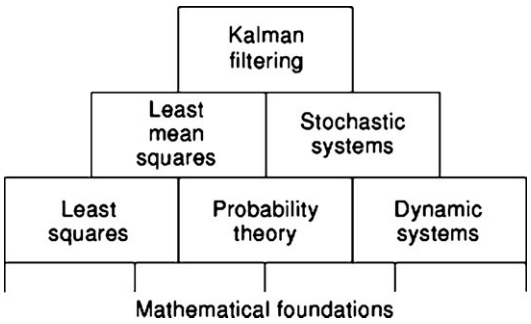

Figure 1. Foundational concepts in Kalman filtering.

Figure 1 depicts the essential subjects forming the foundations for Kalman filtering theory. Although this shows Kalman filtering as the apex of a pyramid, it is but part of the foundations of another discipline—modern control theory—and a proper subset of statistical decision theory (3).

Applications of Kalman filtering encompass many fields. As a tool, the algorithm is used almost exclusively for estimation and performance analysis of estimators and as observers for control of a dynamical system. Except for a few fundamental physical constants, there is hardly anything in the universe that is truly constant. The orbital parameters of the asteroid Ceres are not constant, and even the "fixed" stars and continents are moving. Nearly all physical systems are dynamic to some degree. If we want very precise estimates of their characteristics over time, then we must take their dynamics into consideration. Table 1 gives examples of common estimation problems.

We do not always know the dynamics very precisely. Given this state of partial ignorance, the best we can do is express ignorance more precisely—using probabilities.

The Kalman filter allows us to estimate the state of such systems with certain types of random behavior by using such statistical information. A few examples of such systems are listed in Table 1.

The third column of Table 1 lists some sensor types that we might use to estimate the state of the corresponding dynamic systems. The objective of design analysis is to determine how best to use these sensor types for a given set of design criteria. These criteria are typically related to estimation accuracy and system cost.

Because the Kalman filter uses a complete description of the probability distribution of its estimation errors in determining the optimal filtering gains, this probability distribution may be used in assessing its performance as a function of the design parameters of an estimation system, such as:

- The types of sensors to be used
- The locations and orientations of the various sensor types with respect to the system to be estimated
- The allowable noise characteristics of the sensors
- The prefiltering methods for smoothing sensor noise
- The data sampling rates for the various sensor types
- The level of model simplification to reduce implementation requirements

This analytical capability of the Kalman filter enables system designers to assign "error budgets" to subsystems of an estimation system and to trade off the budget allocations to optimize cost or other measures of performance while achieving a required level of estimation accuracy. Furthermore, it acts like an observer by which all the states not measured by the sensors can be constructed for use in the control system applications.

WHITE NOISE

It is common engineering practice to model uncertainty in terms of Gaussian probability distributions and dynamic uncertainty in terms of linear dynamic systems disturbed by uncorrelated (white noise) processes—even though empirical analysis may indicate that the probability distributions are not truly Gaussian, the random processes are not truly white, or the relationships are not truly linear. Although this approach may discard useful information, we continue the practice for the following reasons:

1. Approximation Probability distributions may not be precisely Gaussian, but it is close enough. Nonlinear systems are often smooth enough that local linearization is adequate. Even though the "flicker" noise observed in electronic systems cannot be modeled precisely using only white noise, it can often be done closely enough for practical purposes.
2. Simplicity These models have few parameters to be estimated. Gaussian distributions are characterized by their means and variances, and white noise processes are characterized by their variances.
3. Consistency Linearity preserves Gaussianity. That is, Gaussian probability distributions remain Gaussian under linear transformations of the variates.
4. Tractability These models allow us to derive estimators minimizing expected squared errors.
5. Good Performance The resulting estimators have performed well for many important applications, despite apparent discrepancies between models and reality.
6. Adaptability These estimators can often be extended to estimate parameters of the model or to track slow random variations in parameters.
7. Extendability The variances used for calculating feedback gains can also be used for comparing performance to modeled performance, detecting anomalous behavior, and rejecting anomalous sensor data.

Vector-valued random processes $x(t)$ and $y(t)$ are called uncorrelated if their cross-covariance matrix is identically zero for all times t_1 and t_2:

$$\mathbf{E}\left\langle [x(t_1) - \mathbf{E}\langle x(t_1)\rangle][y(t_2) - \mathbf{E}\langle y(t_2)\rangle]^T \right\rangle = 0 \tag{1}$$

where E is the expected value operator and T is the transpose of the vector.

The random process $x(t)$ is called uncorrelated if

$$\mathbf{E}\left\langle [x(t_1) - \mathbf{E}\langle x(t_1)\rangle][x(t_2) - \mathbf{E}\langle x(t_2)\rangle]^T \right\rangle = Q_1(t_1)\delta(t_1 - t_2) \tag{2}$$

where $\delta(t)$ is the Dirac delta "function" (actually, a generalized function), defined by

$$\int_a^b \delta(t)\,dt = \begin{cases} 1 & \text{if } a \le 0 \le b \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Similarly, a random sequence x_k in discrete time is called uncorrelated if

$$\mathbf{E}\left\langle [x_k - \mathbf{E}x_k][x_j - \mathbf{E}x_j]^T \right\rangle = Q_2(k)\Delta(k - j) \tag{4}$$

where $\Delta(\cdot)$ is the Kronecker delta function, defined by

$$\Delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$Q_1(t)$ and $Q_2(k)$ are the intensity matrices of the white noise process and sequence. If $Q_1(t)$ and $Q_2(t)$ are constant, the processes and sequences are stationary. If the probability distribution of a white noise process at each instant of time is Gaussian, then it is completely defined by its first two moments, mean and variance. If $\mathrm{E}x(t) = 0$, the process is called zero mean.

A white noise process or sequence is an example of an uncorrelated process or sequence. Generally, a white noise process has no time structure. In other words, knowledge of the white process value at one instant of time provides no knowledge of what its value will be (or was) at any other time point.

LINEAR ESTIMATION

Linear estimation addresses the problem of estimating the state of a linear stochastic system by using measurements or sensor outputs that are linear functions of the state. We suppose that the stochastic systems can be represented by the types of plant and measurement models (for continuous and discrete time) shown as equations in Table 2, with dimensions of the vector and matrix quantities. The measurement and plant noise v_k and w_k, respectively, are assumed to be zero-mean Gaussian processes, and the initial value x_o is a Gaussian random variable with known mean x_0 and known covariance matrix P_o. Although the noise sequences w_k and v_k are assumed to be uncorrelated, this restriction can be removed, modifying the estimator equations accordingly.

The objective of statistical optimization is to find an estimate of the n state vector x_k represented by $\hat{x}_k$, a linear function of the measurements $z_i, \ldots, z_k$, which minimizes the weighted mean-squared error

$$\mathbf{E}[x_k - \hat{x}_k]^T M[x_k - \hat{x}_k] \tag{6}$$

where M is any symmetric nonnegative definite weighting matrix.

We will now derive the mathematical form of an optimal linear estimator for the states of linear stochastic systems given in Table 2. This is called the linear quadratic Gaussian estimation problem. The dynamic systems are linear, the performance cost functions are quadratic, and the random processes are Gaussian.

Let us consider similar types of estimators for the LQG problem:

- *Filters* use observations up to the time that the state of the dynamic system is to be estimated:

$$t_{obs} \leq t_{est}$$

- *Predictors* estimate the state of the dynamic system beyond the time of the observations:

$$t_{obs} < t_{est}$$

This is a relatively minor distinction, and the differences between the respective estimators are correspondingly slight.

A straightforward and simple approach using orthogonality principles is used in the derivation of estimators. These estimators will have minimum variance and the unbiased and consistent. Interested readers may refer to Refs. 2–12.

The Kalman filter can be characterized as an algorithm for computing the conditional mean and covariance of the probability distribution of the state of a linear stochastic system with uncorrelated Gaussian process and measurement noise. The conditional mean is the unique unbiased estimate. It is propagated in feedback form by a system of linear differential equations or by the corresponding discrete-time equations. The conditional covariance is propagated by a nonlinear differential equation or its discrete-time equivalent. This implementation automatically minimizes the expected risk associated with any quadratic loss function of the estimation error.

The statistical performance of the estimator can be predicted a priori (i.e., before it is actually used) by solving the nonlinear differential (or difference) equations used in computing the optimal feedback gains of the estimator. These are called Riccati equations, named in 1763 by Jean le Rond D'Alembert (1717–1783) for Count Jacopo Francesco Riccati (1676–1754), who had studied a second-order scalar differential equation, although not the form that we have here (13, 14). Kalman gives credit to Richard S. Bucy for the discovery that the Riccati differential equation serves the same role as the Wiener-Hopf integral equation in defining optimal gains. The Riccati equation also arises naturally in the problem of separation of variables in ordinary differential equations and in the transformation of two-point boundary value problems to initial value problems (15). The behavior of their solutions can be shown analytically in trivial cases. These equations also provide a means for verifying the proper performance of the actual estimator when it is running.

THE LINEAR OPTIMAL ESTIMATOR IN DISCRETE TIME (KALMAN FILTER)

Suppose that a measurement has been made at time t_k and that the information that it provides is to be applied in updating the estimate of the state x of a stochastic system at time t_k. It is assumed that the measurement is linearly related to the state by an equation of the form $z_k = Hx_k + v_k$, where H is the measurement sensitivity matrix and v_k is the measurement noise.

The optimal linear estimate is equivalent to the general (nonlinear) optimal estimator if the random variables x and z are jointly Gaussian. Therefore, it suffices to seek an updated estimate $\hat{x}_k(+)$ (observation z_k is included in the estimate) that is a *linear* function of the a priori estimate and the measurement z:

$$\hat{x}_k(+) = K^1_k \hat{x}_k(-) + \overline{K}_k z_k \tag{7}$$

where $\hat{x}_k(-)$ is the a priori estimate (observation z_k is not included in the estimate) of x_k and $\hat{x}_k(+)$ is the a posteriori value of the estimate.

The weighting matrices $K^1{}_k$ and $\bar{K}_k$ are as yet unknown. We seek those values of $K^1{}_k$ and $\bar{K}_k$ such that the new estimate $\hat{x}_k(+)$ will satisfy the orthogonality principle. This orthogonality condition can be written in the form

$$\mathbf{E}\left\{[x_k - \hat{x}_k(+)]z_i^T\right\} = 0, i = 1, 2, \ldots, k-1 \tag{8}$$

$$\mathbf{E}\left\{[x_k - \hat{x}_k(+)]z_k^T\right\} = 0 \tag{9}$$

Equations Table 2 and for $\hat{x}_k(+)$ from Eq. (9) into Eq. (10), then we will observe from Eqs. 1 and 2 that the data $z_1, \ldots, z_k$ do not involve the noise term w_k. Therefore, because the random sequences w_k and v_k are uncorrelated, it follows that $Ew_k z^T{}_i = 0$ for $1 \leq i \leq k$.

Using this result, we can obtain the following relation:

$$\mathrm{E}\langle \Phi_{k-1}x_{k-1} + w_{k-1} - K_k^1\hat{x}_k(-) - \overline{K}_k z_k\rangle z_i^T = 0, i = 1, \ldots, k-1 \quad (10)$$

It can be shown that $\mathrm{E}(w_{k-1}z^{T}_{i}) = 0$ because w_k is a white noise sequence. But because $z_k = H_k x_k + v_k$, Eq. (12) can be rewritten as

$$\mathrm{E}[\Phi_{k-1}x_{k-1} - K_k^1\hat{x}_k(-) - \overline{K}_k H_k x_k - \overline{K}_k v_k] z_i^T = 0, i = 1, \ldots, k-1 \quad (11)$$

We also know that Eq. (11) holds at the previous step, that is

$$\mathrm{E}\left\langle [x_{k-1} - \hat{x}_{k-1}(+)] z_i^T \right\rangle = 0, i = 1, \ldots, k-1$$

and

$$\mathrm{E}\langle v_k z_i^T\rangle = 0$$

because v_k is a white noise sequence uncorrelated with w_k white noise $i = 1, \ldots k-1$. Then Eq. (13) can be reduced to the form

$$\Phi_{k-1}\mathrm{E}x_{k-1}z_i^T - K_k^1\mathrm{E}\hat{x}_k(-)z_i^T - \overline{K}_k H_k \Phi_{k-1}\mathrm{E}x_{k-1}z_i^T - \overline{K}_k \mathrm{E}v_k z_i^T = 0$$

$$\Phi_{k-1}\mathrm{E}x_{k-1}z_i^T - K_k^1\mathrm{E}\hat{x}_k(-)z_i^T - \overline{K}_k H_k \Phi_{k-1}\mathrm{E}x_{k-1}z_i^T = 0$$

$$\mathrm{E}\left\langle [x_k - \overline{K}_k H_k x_k - K_k^1 x_k] - K_k^1(\hat{x}_k(-) - x_k)\right\rangle z_i^T = 0$$

$$[I - K_k^1 - \overline{K}_k H_k]\mathrm{E}x_k z_i^T = 0 \quad (12)$$

Equation (16) can be satisfied for any given x_k if

$$K_k^1 = I - \overline{K}_k H_k \quad (13)$$

Clearly, this choice of K^1_k causes Eq. (9) to satisfy a portion of the condition given by Eq. (10) where $\bar{K}_k$ is chosen such that Eq. (11) is satisfied.

Let

$$\tilde{x}_k(+) \triangleq \hat{x}_k(+) - x_k \quad (14)$$

$$\tilde{x}_k(-) \triangleq \hat{x}_k(-) - x_k \quad (15)$$

$$\tilde{z}_k \overset{\text{def}}{=} \hat{z}_k(-) - z_k = H_k\hat{x}_k(-) - z_k \quad (16)$$

Vectors $x^{\neg}_{k}(+)$ and $x^{\neg}_{k}(-)$ are the estimation errors after and before updates, respectively.

From Eq. (11)

$$\mathrm{E}[x_k - \hat{x}_k(+)]\hat{z}_k^T = 0 \quad (17)$$

and also [subtract Eq. (11) from Eq. (21)]

$$\mathrm{E}[x_k - \hat{x}_k(+)]\tilde{z}_k^T = 0 \quad (18)$$

Substitute for x_k, $\hat{x}_k(+)$ and $z^{\neg}_{k}$ from Eqs. <xref target="W1020-mdis-0001 W1020-mdis-0009"/>, and <xref target="W1020-mdis-0020"/>, respectively. Then

$$\mathrm{E}[\Phi_{k-1}x_{k-1} + w_{k-1} - K_k^1\hat{x}_k(-) - \overline{K}_k z_k][H_k\hat{x}_k(-) - z_k]^T = 0$$

However, by the system structure

$$\mathrm{E}w_k z_k^T = \mathrm{E}w_k\hat{x}_k^T(+) = 0$$

$$\mathrm{E}[\Phi_{k-1}x_{k-1} - K_k^1\hat{x}_k(-) - \overline{K}_k z_k][H_k\hat{x}_k(-) - z_k]^T = 0$$

Substituting for K^1_k, z_k, and $x^{\neg}_{k}(-)$,

$$\mathrm{E}[\Phi_{k-1}x_{k-1} - \hat{x}_k(-) + \overline{K}_k H_k\hat{x}_k(-) - \overline{K}_k H_k x_k - \overline{K}_k v_k][H_k\hat{x}_k(-) - H_k x_k - v_k]^T = 0$$

$$\mathrm{E}[(x_k - \hat{x}_k(-)) - \overline{K}_k H_k(x_k - \hat{x}_k(-)) - \overline{K}_k v_k][H_k\tilde{x}_k(-) - v_k]^T = 0$$

$$\mathrm{E}[(-\tilde{x}_k(-) + \overline{K}_k H_k\tilde{x}_k(-) - \overline{K}_k v_k][H_k\tilde{x}_k(-) - v_k]^T = 0$$

(Note that $\mathrm{E}\tilde{x}_k(-)v_k^T = 0$.)

Let

$$P_k(-) = \mathrm{E}\tilde{x}_k(-)\tilde{x}_k^T(-)$$

be the error covariance matrix before update. Then

$$[I - \overline{K}_k H_k]P_k(-)H_k^T - \overline{K}_k R_k = 0$$

and therefore

$$\overline{K}_k = P_k(-)H_k^T[H_k P_k(-)H_k^T + R_k]^{-1} \quad (19)$$

Let

$$P_k(+) = \mathrm{E}[\tilde{x}_k(+)\tilde{x}_k^T(+)] \quad (20)$$

be the error covariance matrix after update.

Substituting Eq. (17) into Eq. (9), we obtain

$$\hat{x}_k(+) = (I - \overline{K}_k H_k)\hat{x}_k(-) + \overline{K}_k z_k$$

$$\hat{x}_k(+) = \hat{x}_k(-) + \overline{K}_k[z_k - H_k\hat{x}_k(-)] \quad (21)$$

Subtract x_k from both sides to obtain

$$\hat{x}_k(+) - x_k = \hat{x}_k(-) + \overline{K}_k H_k x_k + \overline{K}_k v_k - \overline{K}_k H_k\hat{x}_k(-) - x_k$$

$$\tilde{x}_k(+) = \tilde{x}_k(-) - \overline{K}_k H_k\tilde{x}_k(-) + \overline{K}_k v_k$$

$$\tilde{x}_k(+) = (I - \overline{K}_k H_k)\tilde{x}_k(-) + \overline{K}_k v_k \quad (22)$$

Substituting Eq. (31) into Eq. (29),

$$P_k(+) = \mathrm{E}\left\{[I - \overline{K}_k H_k]\tilde{x}_k(-)\tilde{x}_k^T(-)[I - \overline{K}_k H_k]^T + \overline{K}_k v_k v_k^T\overline{K}_k^T\right\}$$

(Note that $\mathrm{E}\tilde{x}_k(-)v_k^T = 0$.)

$$P_k(+) = (I - \overline{K}_k H_k)P_k(-)(I - \overline{K}_k H_k)^T + \overline{K}_k R_k\overline{K}_k^T \quad (23)$$

Substituting for $\bar{K}_k$ from Eq. (28),

$$\begin{aligned} P_k(+) &= P_k(-) - \overline{K}_k H_k P_k(-) \\ &\quad - P_k(-) H_k^T \overline{K}_k^T + \overline{K}_k H_k P_k(-) H_k^T \overline{K}_k^T + \overline{K}_k R_k \overline{K}_k^T \\ &= (I - \overline{K}_k H_k) P_k(-) - P_k(-) H_k^T \overline{K}_k^T \\ &\quad + \underbrace{\overline{K}_k (H_k P_k(-) H_k^T + R_k)}_{P_k(-) H_k^T} \overline{K}_k^T \end{aligned}$$

Therefore,

$$P_k(+) = (I - \overline{K}_k H_k) P_k(-) \tag{24}$$

Let

$$P_k(-) = \mathbf{E}[\tilde{x}_k(-) \tilde{x}_k^T(-)]$$

$$\text{and} \quad \tilde{x}_k(-) = \Phi_{k-1} \tilde{x}_{k-1}(+) \tag{25}$$

Subtract x_k from both sides

$$\begin{aligned} \hat{x}_k(-) - x_k &= \Phi_{k-1} \hat{x}_{k-1}(+) - x_k \\ \tilde{x}_k(-) &= \Phi_{k-1} [\hat{x}_{k-1}(+) - x_{k-1}] - w_{k-1} \\ \tilde{x}_k(-) &= \Phi_{k-1} \tilde{x}_{k-1}(+) - w_{k-1} \end{aligned}$$

Postmultiply by $x\neg^{\mathrm{T}}{}_k(-)$ both sides and take the expected value:

$$\begin{aligned} \mathbf{E}[\tilde{x}_k(-) \tilde{x}_k^T(-)] &= \Phi_{k-1} P_{k-1}^{(+)} \Phi_{k-1}^T + \mathbf{E}[w_{k-1} w_{k-1}^T] \\ P_k(-) &= \Phi_{k-1} P_{k-1}^{(+)} \Phi_{k-1}^T + Q_{k-1} \qquad (26) \\ &\text{(Note that } E \tilde{x}_{k-1} w_{k-1}^T = 0.) \end{aligned}$$

Summary of Equations for the Discrete-Time Kalman Estimator

The equations derived in the previous section are summarized in Table 3. In this formulation of the filter equations, G has been combined with the plant covariance by multiplying G_{k-1} and $G^{\mathrm{T}}{}_{k-1}$, for example,

$$\begin{aligned} Q_{k-1} &= G_{k-1} \mathbf{E}(w_{k-1} w_{k-1}^T) G_{k-1}^T \\ &= G_{k-1} \overline{Q}_{k-1} G_{k-1}^T \end{aligned}$$

The relation of the filter to the system is illustrated in the block diagram of Fig. 2. The computational procedure for the discrete-time Kalman estimator follows:

1. Compute $P_k(-)$ with $P_{k-1}(+)$, Φ_{k-1}, Q_{k-1} given to initialize the procedure.
2. Compute $\bar{K}_k$ with $P_k(-)$ computed from step 1 and H_k, R_k given to initialize the procedure.
3. Compute $P_k(+)$ with $\bar{K}_k$ computed from step 2 and $P_k(-)$ from step 1.
4. Compute $\hat{x}_k(+)$ with computed values of $\bar{K}_k$ from step 3 and with given initial estimate x_0 and data set z_k.

Figure 3 shows a typical time sequence of values assumed by the ith component of the estimated state vector (plotted with solid circles) and its corresponding variance of estimation uncertainty (plotted with open circles). The arrows show the successive values assumed by the variables, with the annotation (in parentheses) on the arrows indicating which input variables define the indicated transitions.

Note that each variable assumes two distinct values at each discrete time: its a priori value corresponding to the value before the information in the measurement is used and the a posteriori value corresponding to the value after the information is used.

At the beginning of the design phase of a measurement and estimation system, when neither real nor simulated data are available, just the covariance calculations can be used to obtain preliminary indications of filter performance. Covariance calculations consist of solving the estimator equations with steps 1–3, repeatedly. It is important to notice that the covariance calculations are independent of data z_k. Covariance calculations will involve the plant noise matrix Q, measurement noise matrix R, state transition matrix Φ, measurement matrix H, and initial covariance matrix P_o.

Step 4 of the Kalman filter implementation [computation of $\hat{x}_k(+)$] can be implemented only for state vector propagation where simulator or real data sets are available.

In the design tradeoffs, the covariance matrix update (steps 1 and 3) should be checked for symmetry and positive definiteness. Failure to attain either condition is a sign that something is wrong. One possibility is that it is an ill-conditioned problem. In order to overcome ill-conditioning, another equivalent expression for $P_k(+)$ is called the Joseph form, as shown Eq. (32)

$$P_k(+) = [I - \overline{K}_k H_k] P_k(-) [I - \overline{K}_k H_k]^T + \overline{K}_k R_k \overline{K}_k^T$$

Note that the right-hand side of this equation is the summation of two symmetric matrices. The first of these is positive definite and the second is nonnegative definite, thereby making $P_k(+)$ a positive definite matrix. Other techniques are described in the implementation methods to alleviate the ill-conditioning.

Other forms for $\bar{K}_k$ and $P_k(+)$ are not that useful (8–17). It can be shown that state vector update, Kalman gain, and error covariance equations represent an asymptotically stable system; therefore, the estimate of state $\hat{x}_k$ becomes independent of the initial estimate $\hat{x}_o$, P_o as k is increased. It is also obvious that the Kalman gain and error covariance equations are independent of the observation. These equations are used for covariance analysis purposes.

THE CONTINUOUS-TIME OPTIMAL ESTIMATOR (KALMAN-BUCY FILTER)

Analogous to the discrete-time case, the continuous-time random process $x(t)$ and the observation $z(t)$ are given by

$$\dot{x}(t) = F(t) x(t) + G(t) w(t) \tag{27}$$

$$z(t) = H(t) x(t) + v(t) \tag{28}$$

$$\begin{aligned} \mathbf{E} w(t) &= \mathbf{E} v(t) = 0 \\ \mathbf{E} w(t_1) w^T(t_2) &= Q(t) \delta(t_2 - t_1) \end{aligned} \tag{29}$$

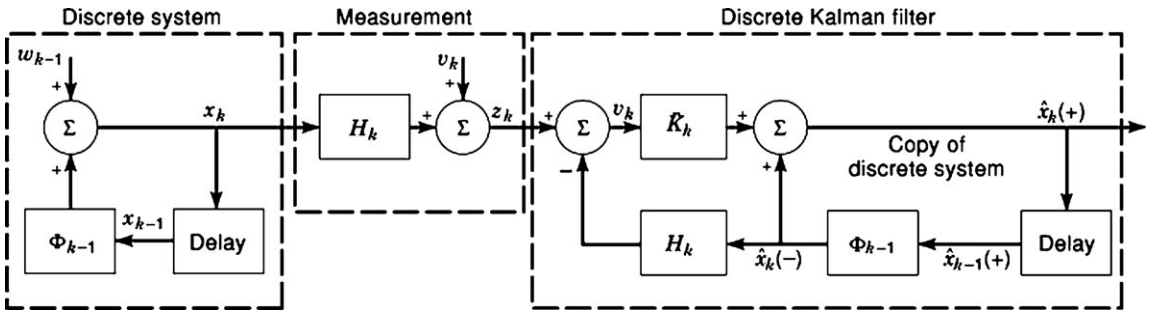

Figure 2. Block diagram of discrete-time Kalman filter.

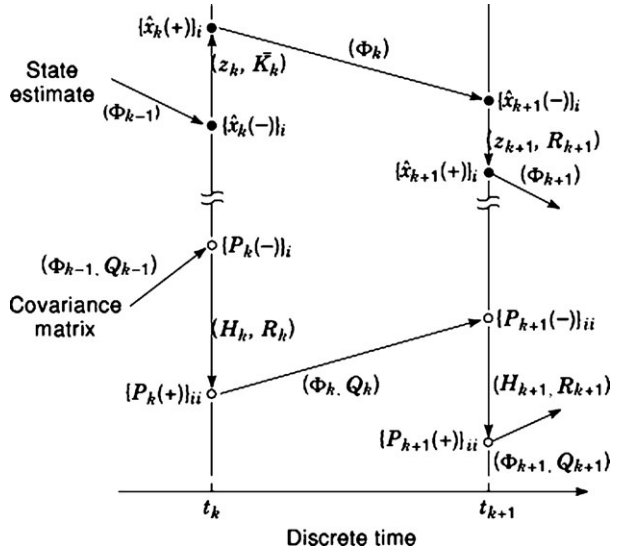

Figure 3. Sequence of values of filter variable in discrete time.

$$\mathbf{E}v(t_1)v^T(t_2) = R(t)\delta(t_2 - t_1) \tag{30}$$

$$\mathbf{E}w(t)v^T(\eta) = 0 \tag{31}$$

where $F(t)$, $G(t)$, $H(t)$, $Q(t)$, and $R(t)$ are $n \times n$, $n \times n$, $l \times n$, $n \times n$, and $l \times l$ matrices, respectively. The covariance matrices Q and R are positive definite.

It is desired to find the estimate of n state vector $x(t)$ represented by $\hat{x}(t)$, which is a linear function of the measurements $z(t) o \le t \le T$ which minimizes the scalar equation

$$\mathbf{E}[x(t) - \hat{x}(t)]^T M[x(t) - \hat{x}(t)] \tag{32}$$

where M is a symmetric positive definite matrix. The initial estimate and covariance matrix are $\hat{x}_0$ and P_0.

This section provides a formal derivation of the continuous-time Kalman estimator. A rigorous derivation can be achieved by using the orthogonality principle as in the discrete-time case. In view of the main objective (to obtain efficient and practical estimators), less emphasis is placed on continuous-time estimators, except in academia.

Let Δt be the time interval $[t_k - t_{k-1}]$. The following relationships are used:

$$\Phi(t_k, t_{k-1}) = \Phi_k = I + F(t_{k-1})\Delta t + 0(\Delta t^2)$$

where $0(\Delta t^2)$ consists of terms with powers of Δt greater than and equal to two.

Discrete measurement noise covariance in terms of continuous covariance is given by

$$R_k = \frac{R(t_k)}{\Delta t}$$

Discrete process noise covariance in terms of continuous covariance is given by

$$Q_k = G(t_k)Q(t_k)G^T(t_k)\Delta t$$

Equations <xref target="W1020-mdis-0034"/> and <xref target="W1020-mdis-0037"/> can be combined. By substituting these relations, we can get the result

$$\begin{aligned} P_k(-) = {} & [I + F(t)\Delta t][I - \overline{K}_{k-1}H_{k-1}]P_{k-1}(-) \\ & [I + F(t)\Delta t]^T + G(t)Q(t)G^T(t)\Delta t \end{aligned} \tag{33}$$

$$\begin{aligned} \frac{P_k(-) - P_{k-1}(-)}{\Delta t} = {} & F(t)P_{k-1}(-) + P_{k-1}(-)F^T(t) \\ & + G(t)Q(t)G^T(t) - \frac{\overline{K}_{k-1}H_{k-1}P_{k-1}(-)}{\Delta t} \\ & - F(t)\overline{K}_{k-1}H_{k-1}P_{k-1}(-)F^T(t)\Delta t \\ & + \text{higher order terms} \end{aligned} \tag{34}$$

The Kalman gain of Eq. (28) becomes, in the limit,

$$\begin{aligned} \lim_{\Delta t \to 0}\left[\frac{\overline{K}_{k-1}}{\Delta t}\right] = {} & \lim_{\Delta t \to 0} \\ & \left\{P_{k-1}(-)H_{k-1}^T[H_{k-1}P_{k-1}(-)H_{k-1}^T\Delta t + R(t)]^{-1}\right\} \\ & \overline{K}(t) = PH^TR^{-1} \end{aligned} \tag{35}$$

Substituting Eq. (50) into Eq. (51) and taking the limit as $\Delta t \to 0$, we obtain the desired result

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) \quad (36)$$

with $P(t_0)$ as the initial condition. This is called the matrix Riccati differential equation. Methods for solving it will be discussed in the next section. The differential equation can be rewritten by using the identity

$$P(t)H^T(t)R^{-1}(t)R(t)R^{-1}(t)H(t)P(t) = \overline{K}(t)R(t)\overline{K}^T(t)$$

to transform Eq. (52) to the form

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t) - \overline{K}(t)R(t)\overline{K}^T(t) \quad (37)$$

In similar fashion, the state vector update equation can be derived from Eqs. 30 and 35 by taking the limit as $\Delta t \to 0$ to obtain the differential equation for the estimate:

$$\dot{\hat{x}}(t) = F(t)\hat{x}(t) + \overline{K}(t)[z(t) - H(t)\hat{x}(t)] \quad (38)$$

with initial condition $\hat{x}(0)$. Equations <xref target="W1020-mdis-0051 W1020-mdis-0054"/>, and <xref target="W1020-mdis-0055"/> define the continuous-time Kalman estimator, which is also called the Kalman-Bucy filter (1–19).

The Wiener filter is defined for stationary systems in continuous time, and the Kalman filter is defined for either stationary or nonstationary systems in either discrete time or continuous time, but finite state dimension. To demonstrate the connections on problems satisfying both sets of constraints, like the continuous-time Kalman Bucy estimator Eqs. <xref target="W1020-mdis-0051"/>, <xref target="W1020-mdis-0054"/>, and <xref target="W1020-mdis-0055"/>, letting F, G, H be constants; the noises be stationary (Q and R constants), and the filter reach steady state (P constant). That is, as $t \to \infty$, then $P^2(xt) \to 0$. The Riccati differential Eq. (54) becomes the algebraic Riccati equation for continuous time systems. Taking the Laplace of Eq. (55) leads in a transfer function with constant gain $\bar{K}$ and represents the steady state Kalman-Bucy filter, which is identical to the Wiener filter (3).

NONLINEAR ESTIMATION

Linear estimators for discrete and continuous systems have been derived. The combination of functional linearity, quadratic performance criteria, and Gaussian statistics is essential to this development. The resulting optimal estimators are simple in form and powerful in effect.

Many dynamic systems and sensors are not absolutely linear, but they are not far from it. Following the considerable success enjoyed by linear estimation methods on liner problems, extensions of these methods were applied to such nonlinear problems. This section investigates the model extensions and approximation methods used for applying the methodology of Kalman filtering to these "slightly nonlinear" problems. More formal derivations of these nonlinear filters and predictors can be found in References 2, 4, and 7–9.

Suppose that a continuous or discrete stochastic system can be represented by nonliner plant and measurement models as shown in Table 4. Although affine (i.e., linear and additive) transformations of Gaussian random variables have Gaussian distributions, the same is not always true in the nonlinear case. Consequently, it is not necessary that w and v be gaussian. They may be included as arguments of the nonlinear functions f and h, respectively. However, the initial value x_0 may be assumed to be a gaussian random variate with known mean and known $n \times n$ covariance matrix P_0. The objective is to estimate x_k or $x(t)$ to satisfy a specified performance criterion (as given previously). Applying linearization techniques (comparison with Taylor series expansion and discarding the 2nd and higher order terms) to get simple approximate solutions to nonlinear estimation problems requires that f and h are twice continuously differentiable (7, 20).

Linearization About a Nominal Trajectory

A trajectory is a particular solution of a stochastic system, with a particular instantiation of the random variates involved. The trajectory is a vector-valued sequence $\{x_k | k = 0, 1, 2, 3, \dots\}$ for discrete-time systems, and a vector-valued function $x(t)$, $0 \le t$, for continuous-time systems.

The term *nominal* in this case refers to that trajectory obtained when the random variates assume their expected values. For example, the sequence $\{x^N{}_k\}$ obtained as a solution of the equation

$$x_k^N = f(x_{k-1}^N, k-1) \quad (39)$$

with zero process noise and with the mean $x^N{}_0$ as the initial condition would be a nominal trajectory for a discrete-time system.

The word *perturbation* has been used by astronomers to describe a minor change in the trajectory of a planet (or any free-falling body) due to secondary forces—such as those produced by other gravitational bodies. Astronomers had learned long ago that the actual trajectory can be accurately modeled as the sum of the solution of the two-body problem (which is available in closed form) and a linear dynamic model for the perturbations due to the secondary forces. This technique also works well for many other nonlinear problems, including the problem at hand. In this case, the perturbations are due to the presence of random process noise and errors in the assumed initial conditions.

If the function f in the previous example is continuous, then the state vector x_k at any instant on the trajectory will vary smoothly with small perturbations of the state vector x_{k-1} at the previous instant. These perturbations are the result of "off-nominal" (i.e., off-mean) values of the random variates involved. These random variates include the initial value of the state vector x_0, the process noise w_k, and (in the case of the estimated trajectory) the measurement noise v_k.

If f is continuously differentiable, then the influence of the perturbations on the trajectory can be represented by a Taylor series expansion about the nominal trajectory. The likely magnitudes of the perturbations are determined by

the variances of the variates involved. If these perturbations are sufficiently small relative to the higher-order coefficients of the expansion, then we can obtain a good approximation by ignoring terms beyond some order. (However, we must usually evaluate the magnitudes of the higher-order coefficients before making such an assumption.)

Let the symbol δ denote perturbations from the nominal

$$\delta x_k = x_k - x_k^{\mathrm{N}}$$
$$\delta z_k = z_k - h(x_k, k)$$

so that the Taylor series expansion of $f(x, k-1)$ with respect to x at $x = x^{\mathrm{N}}{}_{k-1}$ is

$$x_k = f(x_{k-1}, k-1) \tag{40}$$

$$= f(x_{k-1}^{\mathrm{N}}, k-1) + \left.\frac{\partial f(x, k-1)}{\partial x}\right|_{x=x_{k-1}^{\mathrm{N}}} \delta_{x_{k-1}} \tag{41}$$

$$= x_k^{\mathrm{N}} + \left.\frac{\partial f(x, k-1)}{\partial x}\right|_{x=x_{k-1}^{\mathrm{N}}} \delta_{x_{k-1}} \tag{42}$$

or

$$\delta x_k = x_k - x_k^{\mathrm{N}} \tag{43}$$

$$= \left.\frac{\partial f(x, k-1)}{\partial x}\right|_{x=x_{k-1}^{\mathrm{N}}} \delta_{x_{k-1}} + \text{higher-order terms} \tag{44}$$

If the higher-order terms in δx can be neglected, then

$$\delta x_k \approx \Phi_k^{[1]} \delta x_{k-1} + w_{k-1} \tag{45}$$

where the first-order approximation coefficients are given by

$$\Phi_k^{[1]} = \left.\frac{\partial f(x, k-1)}{\partial x}\right|_{x=x_{k-1}^{\mathrm{N}}} \tag{46}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \cdots & \frac{\partial f_3}{\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_3} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x=x_{k-1}^{\mathrm{N}}} \tag{47}$$

a $n \times n$ constant matrix.

Let h be sufficiently differentiable, then the measurement can be represented by a Taylor series:

$$h(x_k, k) = h(x_k^{\mathrm{N}}, k) + \left.\frac{\partial h(x, k)}{\partial x}\right|_{x=x_k^{\mathrm{N}}} \delta x_k + \text{higher-order terms} \tag{48}$$

or

$$\delta z_k = \left.\frac{\partial h(x, k)}{\partial x}\right|_{x=x_k^{\mathrm{N}}} \delta x_k + \text{higher-order terms} \tag{49}$$

If the higher-order terms in this expansion can be ignored, then we can represent the perturbation in z_{k} as

$$\delta z_k = H_k^{[1]} \delta x_k \tag{50}$$

where the first-order variational term is

$$H_k^{[1]} = \left.\frac{\partial h(x, k)}{\partial x}\right|_{x=x_k^{\mathrm{N}}} \tag{51}$$

$$= \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} & \frac{\partial h_2}{\partial x_n} \\ \frac{\partial h_3}{\partial x_1} & \frac{\partial h_3}{\partial x_2} & \frac{\partial h_3}{\partial x_3} & \frac{\partial h_3}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_l}{\partial x_1} & \frac{\partial h_l}{\partial x_2} & \frac{\partial h_l}{\partial x_3} & \frac{\partial h_l}{\partial x_n} \end{bmatrix}_{x=x_k^{\mathrm{N}}} \tag{52}$$

which is an $l \times n$ constant matrix.

In the continuous case, the corresponding nonlinear differential equations for plant and observation are

$$\dot{x}(t) = f(x(t), t) + G(t) w(t) \tag{54}$$

$$z(t) = h(x(t), t) + v(t) \tag{55}$$

with the dimensions of the vector quantities the same as in the discrete case.

Similar to the case of the discrete system, the linearized differential equations can be derived as

$$\delta\dot{x}(t) = \left\{ \left.\frac{\partial f(x(t), t)}{\partial x(t)}\right|_{x(t)=x^{\mathrm{N}}} \right\} \delta x(t) + G(t) w(t) \tag{56}$$

$$= F^{[1]} \delta x(t) + G(t) w(t) \tag{57}$$

$$\delta z(t) = \left\{ \left.\frac{\partial h(x(t), t)}{\partial x(t)}\right|_{x(t)=x^{\mathrm{N}}} \right\} \delta x(t) + v(t) \tag{58}$$

$$= H^{[1]} \delta x(t) + v(t) \tag{59}$$

Equations <xref target="W1020-mdis-0075"/> and <xref target="W1020-mdis-0077"/> represent linearized continuous model equations. The variables $\delta_{\mathrm{x}}(t)$ and $\delta_{\mathrm{z}}(t)$ are the

perturbations about the nominal values as in the discrete case (3).

Linearization About the Estimated Trajectory

The problem with linearization about the nominal trajectory is that the deviation of the actual trajectory from the nominal trajectory tends to increase with time. As the deviation increases, the significance of the higher-order terms in the Taylor series expansion of the trajectory also increases.

A simple but effective remedy for the deviation problem is to replace the nominal trajectory with the estimated trajectory—that is, to evaluate the Taylor series expansion about the estimated trajectory. If the problem is observable (as evidenced by the covariance of estimation uncertainty getting smaller until it reaches a steady state), then the deviations between the estimated trajectory (along which the expansion is made) and the actual trajectory will remain sufficiently small that the linearization assumption is valid (7, 20).

The principal drawback to this approach is that it tends to increase the real-time computational burden. Even though Φ, H, and $\bar{K}$ for linearization about a nominal trajectory may have been precomputed offline, they must be computed in real time as functions of the estimate for linearization about the estimated trajectory.

The only modification required is to replace $x^{\mathrm{N}}{}_{\mathrm{k}-1}$ by $\hat{x}_{\mathrm{k}-1}$ and $x^{\mathrm{N}}{}_{\mathrm{k}}$ by $\hat{x}_{\mathrm{k}}$ in the evaluations of partial derivatives. Now the matrices of partial derivatives become

$$\Phi^{[1]}(\hat{x}, k) = \left.\frac{\partial f(x, k)}{\partial x}\right|_{x=\hat{x}_k(-)} \tag{60}$$

and

$$H^{[1]}(\hat{x}, k) = \left.\frac{\partial h(x, k)}{\partial x}\right|_{x=\hat{x}_k(-)} \tag{61}$$

The matrices have the same general form as for linearization about a nominal trajectory, except for the evaluations of the partial derivatives:

$$F^{[1]}(t) = \left.\frac{\partial f(x(t), t)}{\partial x(t)}\right|_{x=\hat{x}_k(-)} \tag{62}$$

and

$$H^{[1]}(t) = \left.\frac{\partial h(x(t), t)}{\partial x(t)}\right|_{x=\hat{x}_k(-)} \tag{63}$$

Linearized and Extended Kalman Filters. The block diagram of Fig. 4 shows the data flow of the estimator linearized about a nominal trajectory of the state dynamics. Note that the operations within the dashed box have no inputs. These are the computations for the nominal trajectory. Because they have no inputs from the rest of the estimator, they can be precomputed offline. The models and implementation equations for the linearized discrete Kalman filter are summarized in Table 5. Note that the last three equations in this table are identical to those of the "standard" Kalman filter.

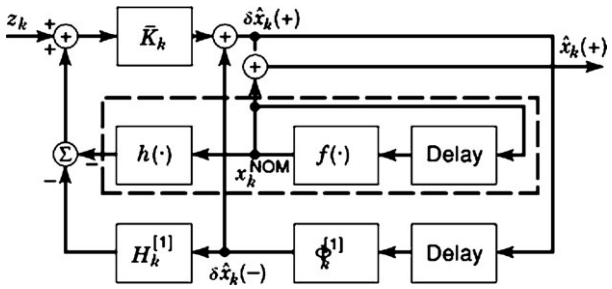

Figure 4. Estimator linearized about a "nominal" state.

The models and implementation equations of the extended Kalman filter are summarized in Table 6. The last three equations in this table are the same as those for the "standard" Kalman filter, but the other equations are noticeably different from those of the linearized Kalman filter in Table 5.

It has been said that modeling is the "hardest" part of Kalman filtering. This is especially true when there are nonlinearities in the physical equation that must be linearized. Developing a good Kalman filter model is partly "art" and partly "science." As a general rule, we look for models that are simple enough to be implemented but, at the same time, still represent the physical situation with a reasonable degree of accuracy (3).

THE MATRIX RICCATI DIFFERENTIAL EQUATION

In order to implement a Kalman filter, the Riccati equation must be solved. This section presents a brief discussion of solution methods for the Riccati differential equation for the Kalman-Bucy filter. A more thorough treatment of the Riccati equation can be found in Ref. 21.

Transformation to a Linear Equation

The Riccati differential equation was first studied in the eighteenth century as a nonlinear scalar differential equation, and a method was derived for transforming it to a linear matrix differential equation. That same method works when the dependent variable of the original Riccati differential equation is a matrix. That solution method is derived here for the matrix Riccati differential equation of the Kalman-Bucy filter. An analogous solution method for the discrete-time matrix Riccati equation of the Kalman filter is derived in the next section.

A matrix product of the sort AB^{-1} is called a *matrix fraction*, and a representation of a matrix N in the form

$$N = AB^{-1}$$

will be called a *fraction decomposition* of N. The matrix A is the numerator of the fraction, and the matrix B is its denominator. It is necessary that the matrix denominator be nonsingular.

The Riccati differential equation is nonlinear. However, a fraction decomposition of the covariance matrix results in a linear differential equation for the numerator and denominator matrices. The numerator and denominator matrices will be functions of time, such that the product

$A(t)B^{-1}(t)$ satisfies the matrix Riccati differential equation and its boundary conditions.

By taking the derivative of the matrix fraction $A(t)B^{-1}(t)$ with respect to t and using the fact that

$$\frac{d}{dt}B^{-1}(t) = -B^{-1}(t)\dot{B}(t)B^{-1}(t)$$

we can arrive at the following decomposition of the matrix Riccati differential equation:

$$\dot{A}(t)B^{-1}(t) - A(t)B^{-1}(t)\dot{B}(t)B^{-1}(t) = \frac{d}{dt}\{A(t)B^{-1}(t)\} \quad (64)$$

$$= \frac{d}{dt}P(t) \quad (65)$$

$$= F(t)P(t) + P(t)F^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) + Q(t) \quad (66)$$

$$= F(t)A(t)B^{-1}(t) + A(t)B^{-1}(t)F^T(t) - A(t)B^{-1}(t)H^T(t)R^{-1}(t)H(t)A(t)B^{-1}(t) + Q(t) \quad (67)$$

$$\dot{A}(t) - A(t)B^{-1}(t)\dot{B}(t) = F(t)A(t) + A(t)B^{-1}(t)F^T(t)B(t) - A(t)B^{-1}(t)H^T(t)R^{-1}(t)H(t)A(t) + Q(t)B(t) \quad (68)$$

$$\dot{A}(t) - A(t)B^{-1}(t)\{\dot{B}(t)\} = F(t)A(t) + Q(t)B(t) - A(t)B^{-1}(t)\{H^T(t)R^{-1}(t)H(t)A(t) - F^T(t)B(t)\} \quad (69)$$

$$\dot{A}(t) = F(t)A(t) + Q(t)B(t) \quad (70)$$

$$\dot{B}(t) = H^T(t)R^{-1}(t)H(t)A(t) - F^T(t)B(t) \quad (71)$$

Combining Eqs. 88 and 89 gives

$$\frac{d}{dt}\begin{bmatrix}A(t)\\B(t)\end{bmatrix} = \begin{bmatrix}F(t) & Q(t)\\H^T(t)R^{-1}(t)H(t) & -F^T(t)\end{bmatrix}\begin{bmatrix}A(t)\\B(t)\end{bmatrix} \quad (72)$$

The last equation is a linear first-order matrix differential equation. The dependent variable is a $2n \times n$ matrix, where n is the dimension of the underlying state variable.

The "Hamiltonian matrix" is given by

$$\Psi(t) = \begin{bmatrix}F(t) & Q(t)\\H^T(t)R^{-1}(t)H(t) & -F^T(t)\end{bmatrix} \quad (73)$$

the initial values of $A(t)$ and $B(t)$ must also be constrained by the initial value of $P(t)$. This is easily satisfied by taking $A(t_0) = P(t_0)$ and $B(t_0) = I$, the identity matrix.

In the time-invariant case, the Hamiltonian matrix Ψ is also time-invariant. As a consequence, the solution for the numerator A and denominator B of the matrix fraction can be represented in matrix form as the product

$$\begin{bmatrix}A(t)\\B(t)\end{bmatrix} = e^{\Psi t}\begin{bmatrix}P(0)\\I\end{bmatrix}$$

where $e^{\Psi t}$ is a $2n \times 2n$ matrix.

Solution of the Algebraic Riccati Equation

We have seen in the previous subsections the difficulty of obtaining a solution of the general Riccati differential equation in "closed form" (i.e., as a formula in the parameters of the mode), even for the simplest (scalar) problem. There is no general formula for solving higher-order polynomial equations (i.e., beyond quartic).This is at the limit of complexity for finding closed-form solutions to algebraic Riccati equations by purely algebraic means. Beyond this relatively low level of complexity, it is necessary to employ numerical solution methods. Numbers do not always provide us as much insight into the characteristics of the solution as formulas do, but they are all we can get for most problems of practical significance.

The MacFarlane-Potter-Fath Eigenstructure Method

MacFarlane, Potter, and Fath discovered (independently) that the solution $P(\infty)$ of the continuous-time form of the steady state matrix Riccati differential equation can be expressed in the form (22)

$$P(\infty) = AB^{-1}$$

$$\begin{bmatrix}A\\B\end{bmatrix} = [e_{i_1} \quad e_{i_2} \quad e_{i_3} \quad \cdots \quad e_{i_n}]$$

where the matrices A and B are $n \times n$ and the $2n$-vectors e_{i_k} are characteristic vectors of the continuous-time system Hamiltonian matrix

$$\Psi_c = \begin{bmatrix}F & Q\\H^T R^{-1} H & -F^T\end{bmatrix}$$

This can be formalized in somewhat greater generality as a lemma.

Lemma 1 If A and B are $n \times n$ matrices such that B is nonsingular and

$$\Psi_c\begin{bmatrix}A\\B\end{bmatrix} = \begin{bmatrix}A\\B\end{bmatrix}D \quad (74)$$

for a $n \times n$ matrix D, then $P = AB^{-1}$ satisfies the steady state matrix Riccati differential equation

$$0 = FP + PF^T - PH^T R^{-1} HP + Q$$

Proof Equation (95) can be written as two equations,

$$AD = FA + QB$$

$$BD = H^T R^{-1} HA - F^T B$$

If we multiply both of these on the right by B^{-1} and the last of these on the left by AB^{-1}, we obtain the equivalent equations

$$ADB^{-1} = FAB^{-1} + Q$$

$$ADB^{-1} = AB^{-1}H^T R^{-1} HAB^{-1} - AB^{-1}F^T$$

or taking the differences of the left-hand sides and substituting P for AB^{-1}

$$0 = FP + PF^T - PH^T R^{-1} HP + Q$$

which was to be proved.

In the case that A and B are formed in this way from n characteristic vectors of Ψ_c, the matrix D will be a diagonal matrix of the corresponding characteristic values. Therefore, to obtain the steady state solution of the matrix Riccati differential equation by this method, it suffices to find n characteristic vectors of Ψ_c such that the corresponding B-matrix is nonsingular. As will be shown in the next section, the same trick works for the discrete-time matrix Riccati equation.

The Matrix Riccati Equation in Discrete Time

The representation of the covariance matrix as a matrix fraction is also sufficient to transform the nonlinear discrete-time Riccati equation for the estimation uncertainty into a linear form. The discrete-time problem differs from the continuous-time problem in two important aspects:

1. The numerator and denominator matrices will be propagated by a $2n \times 2n$ transition matrix, and not by differential equations. The approach is otherwise similar to that for the continuous-time Riccati equation, but the resulting $2n \times 2n$ state transition matrix for the recursive updates of the numerator and denominator matrices is a bit more complicated than the coefficient matrix for the linear form of the continuous-time matrix Riccati equation.
2. There are two distinct values of the discrete-time covariance matrix at any discrete time-step—the a priori value and the a posteriori value. The a priori value is of interest in computing Kalman gains, and the a posteriori value is of interest in the analysis of estimation uncertainty.

The linear equations for matrix fraction propagation of the a priori covariance matrix are derived later. The method is then applied to obtain a closed-form solution for the scalar time-invariant Riccati equation in discrete time and to obtain a method for exponential speedup of convergence to the asymptotic solution.

Lemma 2 If the state transition matrices Φ_k are nonsingular and

$$P_k(-) = A_k B_k^{-1} \tag{75}$$

is a nonsingular matrix solution of the discrete-time Riccati equation at time t_k, then

$$P_{k+1}(-) = A_{k+1} B_{k+1}^{-1} \tag{76}$$

is a solution at time t_{k+1}, where

$$\begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix} = \begin{bmatrix} Q_k & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \Phi_k^{-T} & 0 \\ 0 & \Phi_k \end{bmatrix} \begin{bmatrix} H_k^T R_k^{-1} H_k & I \\ I & 0 \end{bmatrix} \begin{bmatrix} A_k \\ B_k \end{bmatrix} \tag{77}$$

$$= \begin{bmatrix} (\Phi_k + Q_k \Phi_k^{-T} H_k^T R_k^{-1} H_k) & Q_k \Phi_k^{-T} \\ \Phi_k^{-T} H_k^T R_k^{-1} H_k & \Phi_k^{-T} \end{bmatrix} \begin{bmatrix} A_k \\ B_k \end{bmatrix} \tag{78}$$

Proof The following annotated sequence of equalities starts with the product $A_{k+1} B^{-1}{}_{k+1}$ as defined and proves that it equals P_{k+1}:

$$\begin{aligned}
A_{k+1} B_{k+1}^{-1} &= \{[\Phi_k + Q_k \Phi_k^{-T} H_k^T R_k^{-1} H_k] A_k + Q_k \Phi_k^{-T} B_k\} \\
&\quad \times \{\Phi_k^{-T} [H_k^T R_k^{-1} H_k A_k B_k^{-1} + I] B_k\}^{-1} && \text{(definition)} \\
&= \{[\Phi_k + Q_k \Phi_k^{-T} H_k^T R_k^{-1} H_k] A_k + Q_k \Phi_k^{-T} B_k\} \\
&\quad \times B_k^{-1} \{H_k^T R_k^{-1} H_k A_k B_k^{-1} + I\}^{-1} \Phi_k^T && \text{(factor } B_k\text{)} \\
&= \{[\Phi_k + Q_k \Phi_k^{-T} H_k^T R_k^{-1} H_k] A_k B_k^{-1} + Q_k \Phi_k^{-T}\} \\
&\quad \times \{H_k^T R_k^{-1} H_k A_k B_k^{-1} + I\}^{-1} \Phi_k^T && \text{(distribute} B_k\text{)} \\
&= \{[\Phi_k + Q_k \Phi_k^{-T} H_k^T R_k^{-1} H_k] P_k(-) + Q_k \Phi_k^{-T}\} \\
&\quad \times \{H_k^T R_k^{-1} H_k P_k(-) + I\}^{-1} \Phi_k^T && \text{(definition)} \\
&= \{\Phi_k P_k(-) + Q_k \Phi_k^{-T} [H_k^T R_k^{-1} H_k P_k(-) + I]\} \\
&\quad \times \{H_k^T R_k^{-1} H_k P_k(-) + I\}^{-1} \Phi_k^T && \text{(regroup)} \\
&= \Phi_k P_k(-) \{H_k^T R_k^{-1} H_k P_k(-) + I\}^{-1} \Phi_k^T \\
&\quad + Q_k \Phi_k^{-T} \Phi_k^T && \text{(distribute)} \\
&= \Phi_k \{H_k^T R_k^{-1} H_k + P_k^{-1}(-)\}^{-1} \Phi_k^T + Q_k \\
&= \Phi_k \{P_k(-) - P_k(-) H_k^T [H_k P_k(-) H_k^T + R_k]^{-1} \\
&\quad \times H_k P_k(-)\} \Phi_k^T + Q_k && \text{(Hemes)} \\
&= P_{k+1}(-) && \text{(Riccati)}
\end{aligned}$$

where the "Hemes inversion formula" is given in Ref. 3. This completes the proof.

This lemma is used later to derive a closed-form solution for the steady state Riccati equation in the scalar time-invariant case.

The MacFarlane-Potter-Fath Eigenstructure Method

The method presented for the steady state solution of the time-invariant matrix Riccati *differential* equation (i.e., in continuous time) also applies to the Riccati equation in discrete time (22). As before, it is formalized as a lemma.

Lemma 3 If A and B are $n \times n$ matrices such that B is nonsingular and

$$\Psi_d \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} D \tag{79}$$

for a $n \times n$ nonsingular matrix D, then $P_\infty = AB^{-1}$ satisfies the steady state discrete-time matrix Riccati equation

$$P_\infty = \Phi\{P_\infty - P_\infty H^T[HP_\infty H^T + R]^{-1}HP_\infty\}\Phi^T + Q$$

Proof If $P_k = AB^{-1}$, then it was shown in Lemma 2 that $P_{k+1} = \acute{A}\acute{B}'^{-1}$ where

$$\begin{bmatrix}\acute{A}\\ \acute{B}\end{bmatrix} = \begin{bmatrix}(\Phi_k + Q_k\Phi_k^{-T}H_k^TR_k^{-1}H_k) & Q_k\Phi_k^{-T}\\ \Phi_k^{-T}H_k^TR_k^{-1}H_k & \Phi_k^{-T}\end{bmatrix}\begin{bmatrix}A\\ B\end{bmatrix}$$

$$= \Phi_d\begin{bmatrix}A\\ B\end{bmatrix}$$

$$= \begin{bmatrix}A\\ B\end{bmatrix}D$$

$$= \begin{bmatrix}AD\\ BD\end{bmatrix}$$

Consequently,

$$\begin{aligned}P_{k+1} &= \acute{A}\acute{B}^{-1}\\ &= (AD)(BD)^{-1}\\ &= ADD^{-1}B^{-1}\\ &= AB^{-1}\\ &= P_k\end{aligned}$$

That is, AB^{-1} is a steady state solution, which was to be proved.

In practice, A and B are formed from n characteristic vectors of Ψ_d. The matrix D will be a diagonal matrix of the corresponding nonzero characteristic values.

The algebraic Riccati equation can be solved by using *MATLAB*. The algebraic matrix Riccati equation for the continuous case uses Schur's method which is slower, but more robust when the system is not able to be diagonalized (3, 23).

The matrix Riccati equation for the continuous case can be converted to vector nonlinear coupled differential equations. Fourth-order Runge-Kutta (self-starting), fourth-order Adams method (non-self-starting), and Adams-Moulton's predictor corrector can be used to solve these equations (24).

CONTROLLERS, OBSERVERS, AND THE SEPARATION PRINCIPLE

Optimal control theory was developed under the influence of such great researchers as Pontryagin, Bellman, Kalman, and Bucy (25–30). Kalman introduced a number of state variable concepts. Among these were controllability, observability, optimal linear quadratic regulator (LQR), state feedback, and optimal state estimation (Kalman filtering).

In LQR problems, the dynamics of the system to be controlled are represented by the state-space model, a set of linear first-order differential equations. We deal with only continuous and linear differential equation models. Nonlinear models can be linearized as shown previously:

$$\dot{x}(t) = F(t)x(t) + C(t)u(t) \tag{80}$$

where $x(t)$ represents the n-dimensional state vector and u is r-dimensional deterministic control input. $F(t)$ and $G(t)$ are known time-varying matrices. The objective of control is to keep $x(t)$ close to zero without excessive control effort. This objective is to be achieved by minimizing the quadratic cost function.

$$J = x^T(T)M_0x(T) + \int_{t_0}^{T}[x^T(t)Q_0(t)x(t) + u^T(t)R_0(t)u(t)]\,dt \tag{81}$$

where

$$\begin{aligned}M_0 &= M_0^T \geq 0 && (n \times n \text{ matrix})\\ Q_0(t) &= Q_0^T \geq 0 && \text{for all } t \in [t_0, T], n \times n \text{ matrix}\\ R_0(t) &= R_0^T(t) \geq 0 && \text{for all } t \in [t_0, T], r \times n \text{ matrix}\end{aligned}$$

The solution is provided by the optimal state feedback control

$$u(t) = -\overline{K}_0(t)x(t) \tag{82}$$

where $\bar{K}_0(t)$ is an $r \times n$ time-varying control gain matrix. The value of $\bar{K}_0(t)$ is given by

$$\overline{K}_0(t) = R_0^{-1}(t)C^T(t)P_0(t)$$

where the $n \times n$ matrix $P_0(t)$ is the solution of the Riccati matrix differential equation

$$\begin{aligned}\frac{d}{dt}P_0(t) =& P_0(t)F(t) - F^T(t)P_0(t) - Q_0(t)\\ &+ P_0(t)C(t)R_0^{-1}(t)C_0^T(t)P_0(t)\end{aligned} \tag{83}$$

subject to boundary conditions at the terminal time T, $P_0(T) = M_0$.

The same solving techniques can be applied as developed previously. This Riccati equation has boundary conditions given as compared to initial condition in the case of estimation problems. For time-invariant cases, $x(t) \to 0$ as $t \to \infty$. For this reason, a terminal weighing matrix M_0 need not be included in Eq. (110) and $F(t) = F$, $C(t) = C$, $Q_0(t) = Q_0$, $R_0(t) = R_0$. The optimal control correction is

$$u(t) = -\overline{K}_0x(t) \tag{84}$$

$$\overline{K}_0 = R_0^{-1}C^TP_0 \tag{85}$$

and P_0 is a constant symmetric position definite $n \times n$ matrix, which is the solution of the algebraic matrix Riccati equation

$$0 = -P_0F - F^TP_0 - Q_0 + P_0\,C\,R_0^{-1}C^T\,P_0 \tag{86}$$

The same solving technique can be applied as developed previously.

In implementation, the state variables, which are generally unavailable for direct measurement, would be substituted by their estimates (see Fig. 5) by an observer or

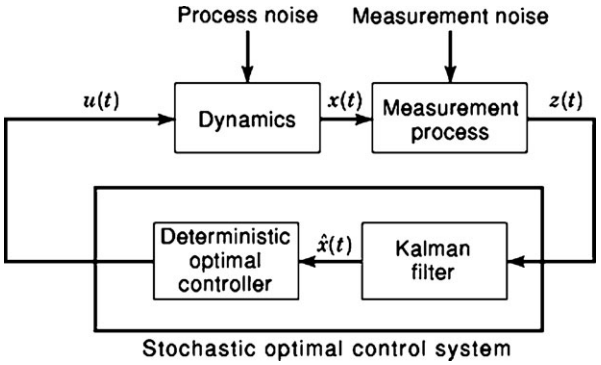

Figure 5. Block diagram of separation principle.

Kalman filter. The remarkable property of the LQG control problem is that the optimal control $u(t)$ is generated from the estimated state $\hat{x}(t)$, generated by the Kalman filter by means of the relationship

$$u(t) = -\overline{K}_0(t)\hat{x}(t)$$

where the gain matrix $\bar{K}_0$ is precisely the one determined in the solution of the deterministic LQR [see Eq. (116)] problem. That is, $u(t) = -\bar{K}_0(t)\ x(t)$ under the assumption that the complete state vector is measured exactly. The LQG problem solution separates into the solution of a linear quadratic deterministic problem and the solution of a linear Gaussian estimation problem. The key theorem that shows this property is often called the separation theorem.

The importance of the separation principle is that the LQG regulator design procedures can be accomplished in two separate stages: (1) the Kalman filter design and (2) the control feedback design. This means that all results derived separately for the deterministic optimal control problem and the optimal estimation problems are still valid (20, 26).

IMPLEMENTATION METHODS

We have discussed what Kalman filters are and how they are supposed to behave. Their theoretical performance has been shown to be characterized by the covariance matrix of estimation uncertainty, which is computed as the solution of a matrix Riccati differential and difference equation. A relationship between optimal deterministic control and optimal estimation problems has been described via the separation principle.

Soon after the Kalman filter was first implemented on computers, it was discovered that the observed mean-square estimation errors were often much larger than the values predicted by the covariance matrix, even with simulated data. The variances of the filter estimation errors were observed to diverge from their theoretical values, and the solutions obtained for the Riccati equations were observed to have negative variances. Riccati equations should have positive or zero variances.

Current work on the Kalman filter primarily focuses on development of robust and numerically stable implementation methods. Numerical stability refers to robustness against roundoff errors. Numerically stable implementation methods are called square root filtering because they use factors of the covariance matrix of estimation uncertainty or its inverse, called the information matrix.

Numerical solution of the Riccati equation tends to be more robust against roundoff errors if Cholesky factors of a symmetrical nonnegative definite matrix P is a matrix C such that $CC^{\mathrm{T}} = P$. Cholesky decomposition algorithms solve for C that is either upper triangular or lower triangular. Another method is modified Cholesky decomposition. Algorithms solve for diagonal factors and either a lower triangular factor L or an upper triangular factor U such that $P = UD_{\mathrm{u}}U^{\mathrm{T}} = LD_{\mathrm{L}}L^{\mathrm{T}}$ where D_{L} and D_{u} are diagonal factors with nonnegative diagonal elements. Another implementation method uses square root information filters that use a symmetric product factorization of the information matrix P^{-1}.

Alternative Kalman filter implementations use these factors of the covariance matrix (or its inverse) in three types of filter operations: (1) temporal updates, (2) observation updates, and (3) combined updates (temporal and observation). The basic algorithm methods used in these alternative Kalman filter implementations fall into four general categories. The first three of these categories are concerned with decomposing matrices into triangular factors and maintaining the triangular form of the factors through all the Kalman filtering operation. The fourth category includes standard matrix operations (multiplication, inversion, etc.) that have been specialized for triangular matrices. These implementation methods have succeeded where the conventional Kalman filter implementations have failed (3, 31).

Even though uses are being explored in virtually every discipline, research is particularly intense on successful implementation of Kalman filtering to global positioning systems (*GPS*), inertial navigation systems (*INS*), and guidance and navigation. GPS is a satellite-based system that has demonstrated unprecedented levels of positioning accuracy, leading to its extensive use in both military and civil arenas. The central problem for GPS receivers is the precise estimation of position, velocity, and time based on noisy observations of satellite signals. This provides an ideal setting for the use of Kalman filtering. GPS technology is used in automobile, aircraft, missiles, ships, agriculture, and surveying.

In 1995 the United States began development of the Wide Area Augmentation System (WAAS) under the auspices of the Federal Aviation Administration (FAA) and the Department of Transportation (DOT), to provide precision approach capability for aircraft. Without WAAS, ionospheric disturbances, satellite clock drift, and satellite orbit errors cause too much error in the GPS signal for aircraft to perform a precision landing approach. Additionally, signal integrity information as broadcast by the satellites is insufficient for the demanding needs of public safety in aviation. WAAS provides additional integrity messages to aircraft to meet these needs.

WAAS includes a core of approximately twenty-five wide area ground reference stations (WRS) positioned throughout the United States which have precisely surveyed coordinates. These stations compare the GPS signal measure-

ments with the measurements that should be obtained at the known coordinates. The WRS send their findings to a WAAS master station (WMS) using a land-based communications network, and theWMS calculates correction algorithms and assesses the integrity of the system. The WMS then sends correction messages via a ground uplink system (GUC) to geostationary (GEO) WAAS satellites covering the United States. The satellites in turn broadcast the corrections on a per-GPS satellite basis at the same L, 1575.42 MHz frequency as GPS. WAAS-enabled GPS receivers receive the corrections and use them to derive corrected GPS signals which enable highly accurate positioning.

On July 10, 2003, the WAAS system was activated for general aviation, covering 95% of the United States and portions of Alaska. In September 2003, improvements enabled WAAS-enabled aircraft to approach runways to within 250 feet altitude before requiring visual control. Currently, there are two Inmarsat III GEO satellites serving the WAAS area, the Pacific Ocean Region (POR) satellite, and the East Atlantic Ocean Region (AOR-W) satellite.

In March 2005, two additional WAAS GEO satellites were launched (PanAmSat Galaxy XV and Telesat (Anik F1R), and are planned to be operational in 2006. These satellites plus the two existing satellites will improve coverage of North America and all but the northwest part of Alaska. The four GEO satellites will be positioned at 54°, 107°, and 133° West longitude, and at 178° East longitude.

In 2006, WAAS is projected to be available over 99% of the time, and its coverage will include the full continental United States and most of Alaska. Although primarily intended for aviation applications, WAAS will be useful for improving the accuracy of any WAAS-enabled GPS receiver. Such receivers are already available in low-cost handheld versions for consumer use.

Positioning accuracy using WAAS is currently quoted at less than 2 meters of lateral error and less than 3 meters of vertical error, which meets the aviation Category I precision approach requirement of 16 meters lateral error and 4 meters vertical error. Kalman filters are an integral part of the WAAS system (32).

Kalman filters are used in bioengineering, traffic systems, photogrammetry, and myriad process controls. The Kalman filter is observer, parameter identifier in modeling, predictor, filter, and smoother in a wide variety of applications. It has become integral to twenty-first century technology.

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Table 1. Examples of Estimation Problems

Application	Dynamic System	Sensor Types
Process control	Chemical plant	Pressure Temperature Flow rate Gas Analyzer
Flood prediction	River system	Water level Rain gauge Weather radar
Tracking	Spacecraft	Radar Imaging system
Navigation	Ships Aircraft, missiles Smart bombs Automobiles Golf carts	Sextant Log Gyroscope Accelerometer GPS receiver

Table 2. Linear Plant and Measurement Models

Model	Continuous Time		Discrete Time	
Plant	$\dot{x}(t) = F(t)x(t) + w(t)$		$x_k = \Phi_{k-1}x_{k-1} + w_{k-1}$	
Measurement	$z(t) = H(t)x(t) + v(t)$		$z_k = H_k x_k + v_k$	
Plant noise	$E\langle w(t)\rangle = 0$		$E\langle w_k\rangle = 0$	
	$E\langle w(t)w^T(s)\rangle = (t-s)Q(t)$		$E\langle w_k w_i^T\rangle = \Delta(k-i)Q_k$	
Obs. noise	$E\langle v(t)\rangle = 0$		$E\langle v_k\rangle = 0$	
	$E\langle v(t)v^T(s)\rangle = (t-s)R(t)$		$E\langle v_k v_i^T\rangle = \Delta(k-i)R_k$	
(Linear Model)	Symbol	Dimension	Symbol	Dimensions
Dimensions of vectors and matrices	x, w	$n \times 1$	Φ, Q	$n \times n$
	z, v	$l \times 1$	H	$l \times n$
	R	$l \times l$	Δ,	Scalar

Table 3. Discrete-Time Kalman Filter Equations

System Dynamic Model

$$x_k = \Phi_{k-1}x_{k-1} + w_{k-1}$$

$$w_k \sim N(0, Q_k)$$

Measurement Model

$$z_k = H_k x_k + v_k$$

$$v_k \sim N(0, R_k)$$

Initial Conditions

$$E\langle x_0\rangle = \hat{x}_0$$

$$E\langle \tilde{x}_0 \tilde{x}_0^T\rangle = P_0$$

Independence Assumption

$E\langle w_k v_j^T\rangle = 0$ for all k and j

State Estimate Extrapolation (Eq. (25))

$$\hat{x}_k(-) = \Phi_{k-1}\hat{x}_{k-1}(-)$$

Error Covariance Extrapolation (Eq. (26))

$$P_k(-) = \Phi_{k-1}P_{k-1}(+)\Phi_{k-1}^T + Q_{k-1}$$

State Estimate Observational Update (Eq. (21))

$$\hat{x}_k(+) = \hat{x}_k(-) + \bar{K}_k[z_k - H_k\hat{x}_k(-)]$$

Error Covariance Update (Eq. (24))

$$P_k(+) = [I - \bar{K}_k H_k]P_k(-)$$

Kalman Gain Matrix (Eq. (19))

$$\bar{K}_k = P_k(-)H_k^T[H_k P_k(-)H_k^T + R_k]^{-1}$$

Table 4. Nonlinear Plant and Measurement Models

Model	Continuous Time		Discrete Time	
Plant	$\dot{x}(t) = f(x,t) + w(t)$		$x_k = f(x_{k-1},k-1) + w_{k-1}$	
Measurement	$z(t) = h(x(t),t) + v(t)$		$z_k = h(x_k,k) + v_k$	
Plant noise	$E\langle w(t)\rangle = 0$		$E\langle w_k\rangle = 0$	
	$E\langle w(t)w^T(s)\rangle = (t-s)Q(t)$		$E\langle w_k w_i^T\rangle = \Delta(k-i)Q_k$	
Meas. noise	$E\langle v(t)\rangle = 0$		$E\langle v_k\rangle = 0$	
	$E\langle v(t)v^T(s)\rangle = (t-s)R(t)$		$E\langle v_k v_i^T\rangle = \Delta(k-i)R_k$	
(Nonlinear Model)	Symbol	Dimension	Symbol	Dimensions
Dimensions of vectors and matrices	x, f, w	$n \times 1$	z, h, v	$l \times 1$
	Q	$n \times n$	R	$l \times l$
	Δ,	Scalars		

Table 5. Discrete Linearized Kalman Filter Equations

Nonlinear Nominal Trajectory Model

$$x_k^N = f_{k-1}(x_{k-1}^N)$$

Linearized Perturbed Trajectory Model

$$x \overset{\text{def}}{=} x - x^N$$

$$x_k = \left.\frac{\partial f_{k-1}}{\partial x}\right|_{x\, x_{k-1}^N} x_{k-1} + w_{k-1}$$

$$w_k \sim N(0, Q_k)$$

Nonlinear Measurement Model

$$z_k = h_k(x_k) + v_k$$

$$v_k \sim N(0, R_k)$$

Linearized Approximation Equations

Linear perturbation prediction

$$\hat{x}_k(-) = \Phi_{k-1}^{[1]} \hat{x}_{k-1}(-)$$

$$\Phi_{k-1}^{[1]} \approx \left.\frac{\partial f_{k-1}}{\partial x}\right|_{x\, x_{k-1}^N}$$

Conditioning the predicted perturbation on the measurement

$$\hat{x}_k(+) = \hat{x}_k(-) + K_k[z_k - h_k(x_k^N) - H_k^{[1]} \hat{x}_k(-)]$$

$$H_k^{[1]} \approx \left.\frac{\partial h_k}{\partial x}\right|_{x\, x_k^N}$$

Computing the a priori covariance matrix

$$P_k(-) = \Phi_{k-1}^{[1]} P_{k-1}(+) \Phi_{k-1}^{[1]T} + Q_{k-1}$$

Computing the Kalman gain

$$\bar{K}_k = P_k(-) H_k^{[1]T} [H_k^{[1]} P_k(-) H_k^{[1]T} - R_k]^{-1}$$

Computing the a posteriori covariance matrix

$$P_k(\cdot) = \{I \quad \bar{K}_k H_k^{[1]}\} P_k(\quad)$$

Table 6. Discrete Extended Kalman Filter Equations

Nonlinear Dynamic Model

$$x_k = f_{k-1}(x_{k-1}) + w_{k-1}$$

$$w_k \sim N(0, Q_k)$$

Nonlinear Measurement Model

$$z_k = h_k(x_k) + v_k$$

$$v_k \sim N(0, R_k)$$

Nonlinear Implementation Equations

Computing the predicted state estimate

$$\hat{x}_k(-) = f_{k-1}(\hat{x}_{k-1}(+))$$

Computing the predicted measurement

$$\hat{z}_k = h_k(\hat{x}_k(\quad))$$

Linear Approximation Equations

$$\Phi_{k-1}^{[1]} \approx \left.\frac{\partial f_k}{\partial x}\right|_{x_k \hat{x}_{k-1}(\quad)}$$

Conditioning the predicted estimate on the measurement

$$\hat{x}_k(\quad) = \hat{x}_k(\quad) + \bar{K}_k(z_k \quad \hat{z}_k)$$

$$H_k^{[1]} \approx \left.\frac{\partial h_k}{\partial x}\right|_{x\, \hat{x}_k(\quad)}$$

Computing the a priori covariance matrix

$$P_k(\quad) = \Phi_{k-1}^{[1]} P_{k-1}(\cdot) \Phi_{k-1}^{[1]T} + Q_{k-1}$$

Computing the Kalman gain

$$\bar{K}_k = P_k(\quad) H_k^{[1]T} [H_k^{[1]} P_k(\quad) H_k^{[1]T} + R_k]^{-1}$$

Computing the a posteriori covariance matrix

$$P_k(+) = \{I - K_k H_k^{[1]}\} P_k(-)$$