GAIN SCHEDULING

MOTIVATION

A classical tradeoff in control design is model accuracy versus model simplicity. While sophisticated models better represent a physical system's behavior, the resulting analysis and control design are more involved.

One manifestation of this tradeoff is the use of nonlinear versus linear models. Most physical systems exhibit nonlinear behavior. Some common examples are saturations, rate limiters, hysteresis, and backlash. Predominantly nonlinear behavior may be found in robotic manipulator dynamics, aircraft or missile flight dynamics, undersea vehicle dynamics, jet engine combustion dynamics, and satellite attitude dynamics. Although analysis and control design for nonlinear systems remains an active topic of research, analysis of linear systems is significantly less complicated, and there is an abundance of control design methodologies, ranging from classical control to multivariable robust control.

One compromise is to *linearize* the system behavior, that is, approximate the behavior of a nonlinear system near a particular operating condition by a linear system. This simplification allows one to draw upon analysis and design methods for linear systems. However, this simplification comes at the cost of certain limitations:

- The nonlinear system must be confined to operating near the specified operating condition of the linearization.
- The linearization analysis may give misleading or inconclusive results.
- The linearization may ignore important nonlinear phenomena which dominate the system behavior.

Despite these limitations, linearization remains a widely used method for control system analysis and design.

In many cases, confining a nonlinear system to operating near a specified operating condition is too restrictive. One example is flight control. An aircraft typically experiences several flight conditions, including take-off, cruising at various altitudes, specialized maneuvers, and landing. No single linearization can adequately describe the aircraft dynamics at all of these conditions. Another example is boiler-turbine control in power generation. Typical operating conditions include power ramp up, steady power delivery at various levels, and power ramp down. Again, no linearization can adequately describe the dynamics at all operating conditions.

WHAT IS GAIN SCHEDULING?

Gain scheduling is an approach to overcome the local limitations associated with linearizations. The idea is simple and intuitively appealing. Given a nonlinear plant with a wide range of operating conditions, one can select *several* representative operating conditions within the operating regime, perform several linearizations of the nonlinear dynamics, and design several linear controllers, one for each operating condition.

Each individual controller is expected to achieve good performance whenever the nonlinear plant is near the controller's associated operating condition. As the plant varies from one operating condition to another, the gains of the individual controllers are interpolated, or scheduled, to match the changes in operating conditions. The final result is a nonlinear controller which is constructed out of several local linear controllers.

The implementation of a gain scheduled controller is depicted in Fig. 1. An auxiliary variable, usually called the scheduling variable, is used to update the gains of the linear controller. The scheduling variable should be a good indication of the current operating condition of the plant, and hence should be correlated with the plant nonlinearities. The scheduling variable can be a combination of endogenous signals, such as a plant measurements, or exogenous parameters which reflect environmental conditions.

One example is missile autopilot design. Useful scheduling variables are the angle-of-attack and dynamic pressure, both of which characterize the aerodynamic flight coefficients of the missile. The angle-of-attack is the angle between the missile body and velocity vector and can be considered a state variable, and hence endogenous to the missile dynamics. The dynamic pressure, which is a function of missle velocity and atmospheric pressure, is indicative of the environmental conditions. Atmospheric pressure is clearly an exogenous signal. Since the dynamic pressure is also a function of missile velocity, it can be considered an endogenous variable. However, the velocity variations in a simplified model are decoupled from the attitude dynamics, and hence, dynamic pressure may be modeled as an exogenous signal. This sort of ambiguity, namely that an "exogenous" signal is really an endogenous signal in a more sophisticated model, is common.

Gain scheduling has seen widespread industrial application. It is perhaps the most prevalent nonlinear method for aircraft flight control and missle autopilot design. Other applications include power systems, process control, and automotive control. Despite its widespread usage, traditional gain scheduling has been an ad hoc design approach accompanied by heuristic guidelines.

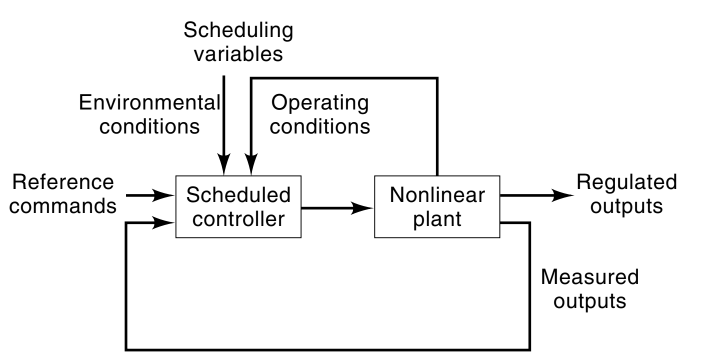

Figure 1. Gain scheduled control implementation.

Some immediate drawbacks of gain scheduling are the following:

- The design of several controllers at several linearization points can be a tedious task.
- Gain scheduled designs typically assume a fixed operating condition, even though the operating condition is varying.
- Although the resulting controller is nonlinear, it is still based on linearizations of plant dynamics, and hence may neglect important nonlinear phenomena.

Two heuristic rules-of-thumb which guide successful gain scheduled designs are the following:

- The scheduling variable should vary slowly.
- The scheduling variable should capture plant nonlinearities.

It is easy to interpret these guidelines in terms of the aforementioned drawbacks. Since gain scheduled designs assume a constant operating condition, slow variations among operating conditions should be tolerable. Similarly, since gain scheduling relies on a family of linearizations, the changes within this family should be indicative of plant nonlinearities.

This article provides an overview of the gain scheduling design procedure, discusses the theoretical foundations behind gain scheduling, as well as limitations of traditional gain scheduling, and presents emerging techniques for gain scheduling which address these limitations.

LINEARIZATION

Linearization of Functions

We begin by recalling some concepts from multivariable calculus. Let $f: \mathcal{R}^n \to \mathcal{R}^p$ denote a multivariable function which maps vectors in $\mathcal{R}^n$ to vectors in $\mathcal{R}^p$. In terms of the individual components,

$$f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_p(x_1, \ldots, x_n) \end{bmatrix} \tag{1}$$

In case f is differentiable, $Df(x)$ denotes the $p \times n$ Jacobian matrix of partial derivatives; i.e.,

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(x) & \cdots & \frac{\partial f_p}{\partial x_n}(x) \end{bmatrix} \tag{2}$$

For f continuously differentiable, i.e., if $Df(x)$ has continuous elements, we may approximate $f(x)$ by the truncated Taylor's series

$$f(x) \approx f(x_0) + Df(x_0)(x - x_0) \tag{3}$$

Let $r(x)$ denote the residual error of approximation; that is,

$$r(x) = f(x) - \big(f(x_0) + Df(x_0)(x - x_0)\big) \tag{4}$$

Then,

$$\lim_{x \to x_o} \frac{r(x)}{|(x - x_0)|} = 0 \tag{5}$$

where $|v|$ denotes the Euclidean norm of $v \in \mathcal{R}^n$,

$$|v| = (v^T v)^{1/2} \tag{6}$$

Example 1. Let $f: \mathcal{R}^2 \to \mathcal{R}^2$ be defined as

$$f(x) = \begin{pmatrix} x_1|x_1| + x_2 \\ x_1^2 x_2 \end{pmatrix} \tag{7}$$

Then

$$Df(x) = \begin{pmatrix} 2|x_1| & 1 \\ 2x_1x_2 & x_1^2 \end{pmatrix} \tag{8}$$

Approximating $f(x)$ near

$$x_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

leads to

$$\begin{aligned} f(x) &\approx f(x_0) + Df(x_0)(x - x_0) \\ &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 - 1 \\ x_2 - 2 \end{pmatrix} \end{aligned} \tag{9}$$

Now, let $f: \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^p$ denote a multivariable function which maps vectors in $\mathcal{R}^n$ and $\mathcal{R}^m$ together to vectors in $\mathcal{R}^p$. In terms of the individual components,

$$f(x,u) = \begin{bmatrix} f_1(x_1, \ldots, x_n, u_1, \ldots, u_m) \\ \vdots \\ f_p(x_1, \ldots, x_n, u_1, \ldots, u_m) \end{bmatrix} \tag{10}$$

In case f is differentiable, $D_1f(x, u)$ denotes the $p \times n$ Jacobian matrix of partial derivatives with respect to the first variable,

$$D_1 f(x,u) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x,u) & \cdots & \frac{\partial f_1}{\partial x_n}(x,u) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(x,u) & \cdots & \frac{\partial f_p}{\partial x_n}(x,u) \end{bmatrix} \tag{11}$$

and $D_2f(x, u)$ denotes the $p \times m$ Jacobian matrix of partial derivatives with respect to the second variable,

$$D_2 f(x,u) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(x,u) & \cdots & \frac{\partial f_1}{\partial u_m}(x,u) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial u_1}(x,u) & \cdots & \frac{\partial f_p}{\partial u_m}(x,u) \end{bmatrix} \tag{12}$$

As before, if f is continuously differentiable, we may approximate $f(x, u)$ by

$$f(x,u) \approx f(x_0,u_0) + D_1 f(x_0,u_0)(x - x_0) + D_2(x_0,u_0)(u - u_0) \tag{13}$$

Let

$$r(x,u) = f(x,u) - \big(f(x_0,u_0) + D_1 f(x_0,u_0)(x - x_0) + D_2(x_0,u_0)(u - u_0)\big) \tag{14}$$

denote the approximation residual. Then as before,

$$\lim_{\substack{x \to x_0 \\ u \to u_0}} \frac{r(x,u)}{\sqrt{|x - x_0|^2 + |u - u_0|^2}} = 0 \tag{15}$$

Equations (5) and (15) indicate that the approximations are accurate up to the first order.

Linearization of Autonomous Systems

In the last section, we saw how to approximate the static behavior of a nonlinear function by using a truncated Taylor's series. We now show how similar tools can be used to approximate the dynamic behavior of a nonlinear system.

Consider an autonomous nonlinear system

$$\dot{x} = f(x) \tag{16}$$

Definition 2. The vector x_0 is an equilibrium of (16) if

$$f(x_0) = 0 \tag{17}$$

The reasoning for this terminology is that the initial condition $x(0) = x_0$ leads to the solution $x(t) = x_0$ for all time. So if the solution starts at x_0, it remains at x_0, hence the term equilibrium.

In case f is continuously differentiable, we may rewrite (16) as

$$\dot{x} = f(x_0) + Df(x_0)(x - x_0) + r(x) \tag{18}$$

where $r(x)$ denotes the residual error in the approximation

$$f(x) \approx f(x_0) + Df(x_0)(x - x_0) \tag{19}$$

Since x_0 is both fixed and an equilibrium,

$$\frac{d}{dt}(x - x_0) = \frac{dx}{dt} \tag{20}$$

and

$$f(x_0) = 0 \tag{21}$$

Substituting these formulas into (18) and neglecting the residual term, $r(x)$, leads to the approximate dynamics

$$\dot{\tilde{x}} = A\tilde{x} \tag{22}$$

where

$$A = Df(x_0) \tag{23}$$

Equation (22) is called the *linearization* of Eq. (16) about the equilibrium x_0. Intuitively, whenever $x - x_0$ is small, then $\tilde{x}$ in the linearization should be a good approximation of $x - x_0$, and hence the linear dynamics of Eq. (22) should be a good approximation of the nonlinear dynamics of Eq. (16).

It is often possible to make more definite statements about nonlinear dynamics based on an analysis of the linearization, in particular regarding the stability of the nonlinear system. First, recall the following stability definitions.

Definition 3. Let x_0 be an equilibrium of Eq. (16).

- x_0 is stable if for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x(0) - x_0| < \delta \Longrightarrow |x(t) - x_0| \le \epsilon \tag{24}$$

  Otherwise, x_0 is unstable.
- x_0 is asymptotically stable if in addition to being stable,

$$|x(0) - x_0| < \delta \Longrightarrow \lim_{t \to \infty} x(t) = x_0 \tag{25}$$

In words, stability implies that the solution to Eq. (16) stays near x_0 whenever it starts sufficiently close to x_0, whereas asymptotic stability implies that the solution also asymptotically approaches x_0.

Theorem 4. Let f in Eq. (16) be continuously differentiable. The equilibrium x_0 is asymptotically stable if all of the eigenvalues of $Df(x_0)$ have negative real parts. It is unstable if $Df(x_0)$ has at least one eigenvalue with a positive real part.

Since the eigenvalues of $Df(x_0)$ determine the stability of the linearization (22), Theorem 4 states that one can assess the stability of a nonlinear system based on its linearization.

Example 5. The equations of motion for a pendulum of length ℓ are

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} - \frac{g}{\ell}\sin(\theta) = 0 \tag{26}$$

where θ is the pendulum angle measure positive clockwise with $\theta = 0$ being the upright position, c is a friction coefficient, and g is gravitational acceleration. In state space form, these equations becomes

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{g}{\ell}\sin(x_1) - cx_2 \end{pmatrix} \tag{27}$$

where $x_1 = \theta$, and $x_2 = \dot{\theta}$.

Linearizing about the upright equilibrium $x_0 = 0$ leads to

$$\dot{\tilde{x}} = \begin{pmatrix} 0 & 1 \\ g/\ell & -c \end{pmatrix} \tilde{x} \tag{28}$$

The linearization has an eigenvalue with positive real part. Hence, the upright equilibrium of the pendulum is unstable, as expected.

Linearizing about the hanging equilibrium

$$x_0 = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

leads to

$$\dot{\tilde{x}} = \begin{pmatrix} 0 & 1 \\ -g/\ell & -c \end{pmatrix} \tilde{x} \tag{29}$$

which is asymptotically stable. Therefore, the hanging equilibrium of the nonlinear pendulum is asymptotically stable, as expected.

The above example demonstrates that different equilibrium points of the same nonlinear system can have different stability conditions.

In some cases, using the linearization to assess stability may be inconclusive.

Example 6. Consider the scalar nonlinear systems

$$\dot{x} = -x^3 \tag{30}$$

and

$$\dot{x} = x^3 \tag{31}$$

It is easy to see that the equilibrium $x_0 = 0$ is asymptotically stable for the former system, while the same equilibrium is unstable for the latter system.

Both systems have the same linearization at the equilibrium $x_0 = 0$,

$$\dot{\tilde{x}} = 0 \tag{32}$$

(note that $\tilde{x}$ represents different quantities in the two linearizations). In this case, stability analysis of linearization is inconclusive; it does not indicate either the stability or instability of the equilibrium $x_0 = 0$.

Linearization of Systems with Controls

It is also possible to use linearization methods to *synthesize* controllers for a nonlinear system.

Consider the controlled system

$$\dot{x} = f(x, u) \tag{33}$$

Definition 7. The pair (x_0, u_0) is an equilibrium of Eq. (33) if

$$f(x_0, u_0) = 0 \tag{34}$$

The reasoning behind the term equilibrium is similar to the autonomous case. The initial condition $x(0) = x_0$ along with the *constant* input $u(t) = u_0$ leads to the constant solution $x(t) = x_0$.

Proceeding as in the autonomous case, whenever f is continuously differentiable, we may rewrite (33) as

$$\begin{aligned} \dot{x} = f(x_0, u_0) + D_1 f(x_0, u_0)(x - x_0) \\ + D_2 f(x_0, u_0)(u - u_0) + r(x, u)] \end{aligned} \tag{35}$$

where $r(x, u)$ denotes the residual error in the approximation

$$\begin{aligned} f(x, u) \approx f(x_0, u_0) + D_1 f(x_0, u_0)(x - x_0) \\ + D_2 f(x_0, u_0)(u - u_0) \end{aligned} \tag{36}$$

Dropping this residual term and using that $f(x_0, u_0) = 0$ leads to the approximate linear dynamics

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u} \tag{37}$$

where

$$A = D_1 f(x_0, u_0), \quad B = D_2 f(x_0, u_0) \tag{38}$$

As before, Eq. (37) is called the *linearization* of the nonlinear Eq. (33) about the equilibrium (x_0, u_0). The quantity $\tilde{x}$ approximates $x - x_0$, whereas the quantity $\tilde{u}$ equals $u - u_0$, exactly.

Definition 8. The equilibrium (x_0, u_0) of Eq. (33) is stabilized by the state feedback $u = G(x)$ if

- $u_0 = G(x_0)$
- x_0 is a stable equilibrium of the closed loop dynamics

$$\dot{x} = f(x, G(x)) \tag{39}$$

The following is a direct consequence of Theorem 4.

Theorem 9. Let f in Eq. (33) be continuously differentiable, and let Eq. (37) be the linearization of Eq. (33) about the equilibrium (x_0, u_0). Suppose the static linear feedback,

$$\tilde{u} = -K\tilde{x} \tag{40}$$

stabilizes the linearization of Eq. (37). Then the equilibrium (x_0, u_0) of Eq. (33) is stabilized by the feedback

$$u = u_0 - K(x - x_0) \tag{41}$$

Theorem 9 states that we can construct stabilizing feedback for a nonlinear system by designing stabilizing feedback for its linearization.

Example 10. Recall the simple pendulum example, but now with a control torque input,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{g}{\ell}\sin(x_1) - cx_2 + u \end{pmatrix} \tag{42}$$

Linearizing about the upright equilibrium leads to

$$\dot{\tilde{x}} = \begin{pmatrix} 0 & 1 \\ g/\ell & -c \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tilde{u} \tag{43}$$

The feedback

$$\tilde{u} = -(k_1 \quad k_2)\tilde{x} \tag{44}$$

stabilizes the linearization for any $k_1 > g/\ell$ and $k_2 > -c$. From Theorem 9, the feedback

$$\begin{aligned} u &= u_0 - (k_1 \quad k_2)(x - x_0) \\ &= -(k_1 \quad k_2)x \end{aligned} \tag{45}$$

stabilizes the upright equilibrium, where we used that $x_0 = 0$ and $u_0 = 0$.

In some cases, analysis of the linearization does not aid in the construction of stabilizing feedback.

Example 11. Consider the scalar nonlinear system

$$\dot{x} = x + xu \tag{46}$$

Linearizing about the equilibrium $(x_0, u_0) = (0, 0)$ leads to

$$\dot{\tilde{x}} = \tilde{x} \tag{47}$$

which is not stabilizable. However, the constant "feedback" $u = -2$ leads to the closed loop equations

$$\dot{x} = -x \tag{48}$$

which is stable.

Now suppose that the state is not available for feedback. Rather, the control is restricted to measurements

$$y = g(x) \tag{49}$$

By using a similar analysis, we can construct stabilizing output feedback for the nonlinear system based on stabilizing output feedback for the linearization.

Definition 12. The equilibrium (x_0, u_0) of Eq. (33) is stabilized by the dynamic output feedback

$$\begin{aligned} \dot{z} &= F(z, y) \\ u &= G(z) \end{aligned} \tag{50}$$

if for some z_0,

- $(z_0, g(x_0))$ is an equilibrium of Eq. (50)
- $u_0 = G(z_0)$
- (x_0, z_0) is an asymptotically stable equilibrium of the closed loop dynamics

$$\begin{aligned} \dot{x} &= f(x, G(z)) \\ \dot{z} &= F(z, g(x)) \end{aligned} \tag{51}$$

Let (x_0, u_0) be an equilibrium of Eq. (33), and define $y_0 = g(x_0)$. In case g is continuously differentiable, we can approximate Eq. (49) as

$$y \approx y_0 + Dg(x_0)(x - x_0) \tag{52}$$

Then, the linearization about the equilibrium (x_0, u_0),

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\tilde{x} \\ \tilde{y} &= C\tilde{x} \end{aligned} \tag{53}$$

where

$$A = D_1 f(x_0, u_0), \quad B = D_2 f(x_0, u_0), \quad C = Dg(x_0) \tag{54}$$

approximates the input-output behavior of Eq. (33) with measurement of Eq. (49). Here, $\tilde{x}$ approximates $x - x_0$, $\tilde{y}$ approximates $y - y_0$, and $\tilde{u}$ exactly represents $u - u_0$.

Theorem 13. Let f in Eq. (33) and g in Eq. (49) be continuously differentiable, and let Eq. (53) be the linearization of Eq. (33) about the equilibrium (x_0, u_0). Suppose the linear feedback,

$$\begin{aligned} \dot{z} &= \overline{A}z + \overline{B}\tilde{y} \\ \tilde{u} &= \overline{C}z \end{aligned} \tag{55}$$

stabilizes the linearization of Eq. (53). Then the equilibrium (x_0, u_0) of Eq. (33) is stabilized by the output feedback

$$\begin{aligned} \dot{z} &= \overline{A}z + \overline{B}(y - g(x_0)) \\ u &= u_0 + \overline{C}z \end{aligned} \tag{56}$$

Example 14. Suppose we wish to control the simple pendulum under the output feedback

$$y = (1 \quad 0)x \tag{57}$$

Linearizing about the upright equilibrium leads to

$$\begin{aligned} \dot{\tilde{x}} &= \begin{pmatrix} 0 & 1 \\ g/\ell & -c \end{pmatrix}\tilde{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\tilde{u} \stackrel{\text{def}}{=} A\tilde{x} + B\tilde{u} \\ \tilde{y} &= (1 \quad 0)\tilde{x} \stackrel{\text{def}}{=} C\tilde{x} \end{aligned} \tag{58}$$

The observer based controller

$$\begin{aligned} \dot{z} &= (A - BK - HC)z + H\tilde{y} \\ \tilde{u} &= -Kz \end{aligned} \tag{59}$$

stabilizes the linearization for appropriate gain matrices

$$K = (k_1 \quad k_2), \quad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \tag{60}$$

Since x_0, u_0, $y_0 = 0$, the same controller (with input y and output u) stabilizes the nonlinear pendulum.

BASIC GAIN SCHEDULING

Gain Scheduled Command Following

We now outline the basic procedure of gain scheduling in the context of command following. The nonlinear plant of interest is Eq. (33). The objective is to make the measured output Eq. (49) approximately follow reference commands, r.

The primary motivation of gain scheduling is to address local limitations associated with a control design based on a single linearization. The main problem is that the performance and even stability of the closed loop system can deteriorate significantly when the system is not operating in the vicinity of the equilibrium.

Example 15. Consider the scalar system

$$\begin{aligned} \dot{x} &= x|x| + u \\ y &= x \end{aligned} \tag{61}$$

By linearizing about the equilibrium $(x_0, u_0) = (0, 0)$, we obtain the linear control $u = -x + r$. For $r = 0$, this control law stabilizes the equilibrium (0, 0). The resulting closed loop system is

$$\dot{x} = x|x| - x \tag{62}$$

For $|x(0)| < 1$, the solution asymptotically approaches 0. However, for $|x(0)| > 1$, the solution diverges to infinity.

Step 1: Construction of Linearization Family. Gain scheduling attempts to overcome local limitations by considering a *family* of linearizations, rather than a single linearization.

Definition 16. The functions $(x_{eq}(\cdot), u_{eq}(\cdot))$ define an equilibrium family for the nonlinear system Eq. (33) over the set S if

$$f(x_{eq}(s), u_{eq}(s)) = 0 \tag{63}$$

for all $s \in S$.

Associated with an equilibrium family are the output equilibrium values

$$y_{eq}(s) \stackrel{\text{def}}{=} g(x_{eq}(s)) \tag{64}$$

The equilibrium family induces the following linearization family for Eq. (33) with measurement Eq. (49),

$$\begin{aligned} \dot{\tilde{x}} &= A(s)\tilde{x} + B(s)\tilde{u} \\ \tilde{y} &= C(s)\tilde{x} \end{aligned} \tag{65}$$

where

$$\begin{aligned} A(s) &= D_1 f(x_{eq}(s), u_{eq}(s)), \\ B(s) &= D_2 f(x_{eq}(s), u_{eq}(s)), \\ C(s) &= Dg(x_{eq}(s)) \end{aligned} \tag{66}$$

The variable, s, which we will call the *scheduling variable,* parameterizes a family of equilibrium points and plant linearizations. Typically, s can be a combination of both endogenous and exogenous signals (recall discussion of missile autopilot earlier). Any fixed s will be called an *operating condition,* and the set S defines the *operation envelope,* or range of operating conditions.

Example 17. Recall the controlled pendulum

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \dfrac{g}{\ell}\sin(x_1) - cx_2 + u \end{pmatrix} \tag{67}$$

An equilibrium family over the set $S = [-\pi, \pi]$ is

$$x_{eq}(s) = \begin{pmatrix} s \\ 0 \end{pmatrix}, \quad u_{eq}(s) = -\frac{g}{\ell}\sin s \tag{68}$$

The associated linearization family is

$$\dot{\tilde{x}} = \begin{pmatrix} 0 & 1 \\ \dfrac{g}{\ell}\cos(s) & -c \end{pmatrix}\tilde{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\tilde{u} \tag{69}$$

Step 2: Fixed Operating Condition Designs. Let us select several operating conditions,

$$\{s_1, s_2, \ldots, s_N\} \subset S \tag{70}$$

which characterize the variations within the operating envelope.

At the i^{th} equilibrium, s_i, we can linearize the plant dynamics about the equilibrium $(x_{eq}(s_i), u_{eq}(s_i))$ and design a stabilizing linear controller to achieve approximate command following using any suitable linear design methodology. The result is an indexed collection of controllers,

$$\begin{aligned} \dot{z} &= \overline{A}_i z + \overline{B}_i \tilde{y} + \overline{L}_i \tilde{r} \\ \tilde{u} &= \overline{C}_i z \end{aligned} \tag{71}$$

where $\tilde{r}$ denotes the reference command in local coordinates,

$$\tilde{r} = r - y_{eq}(s_i) \tag{72}$$

This step constitutes the core of gain scheduling, and accordingly, accounts for the bulk of the effort in a gain scheduled control design. Designing fixed operating point controllers is especially tedious in the case of several design operating conditions.

Step 3: Scheduling. The remaining step is to piece together a global controller from the individual local controllers. As the scheduling variable varies in time, the control gains are updated to reflect the current operating condition of the plant. The resulting overall controller is

$$\begin{aligned} \dot{z} &= \overline{A}(s)z + \overline{B}(s)(y - \overline{y}_{eq}(s)) + \overline{L}(s)(r - \overline{y}_{eq}(s)), \\ u &= \overline{u}_{eq}(s) + \overline{C}(s)z \end{aligned} \tag{73}$$

The matrices $\overline{A}(s)$, $\overline{B}(s)$, $\overline{C}(s)$, and $\overline{L}(s)$ are functions of the scheduling variable, as are the vectors $\overline{y}_{eq}(s)$ and $\overline{u}_{eq}(s)$. It is important to note that the scheduling variable, s, which was held constant during the design phase, is now time varying. These matrix and vector functions are used to update the control parameters according to the variations in the scheduling variable.

There are different options in how to schedule the controller parameters in Eq. (73).

- *Switched Scheduling.* The operating envelope is divided into disjoint regions, R_i, so that

$$S \subset R_1 \cup \ldots \cup R_N \tag{74}$$

and the controller matrices in (73) are scheduled according to

$$\overline{A}(s) = \begin{cases} \overline{A}_1, & s \in R_1; \\ \vdots & \\ \overline{A}_N, & s \in R_N. \end{cases}, \ldots, \overline{u}_{\text{eq}}(s) = \begin{cases} u_{\text{eq}}(s_1), & s \in R_1; \\ \vdots & \\ u_{\text{eq}}(s_N), & s \in R_N \end{cases} \tag{75}$$

- *Continuous Scheduling.* Any interpolation algorithm is used to construct continuous matrices which interpolate the design conditions so that

$$\overline{A}(s_i) = \overline{A}_i, \ldots, \overline{u}_{\text{eq}}(s_i) = u_{\text{eq}}(s_i) \tag{76}$$

Some important points to consider are the following:

- Gain scheduling is still based on linearizations, and hence can ignore important nonlinear phenomena.
- The fixed operating point designs assume a constant scheduling variable which is actually time varying.
- Implementing the gain scheduled controller introduces feedback loops which are not present in the fixed operating point designs.

Example 18. Recall the system of Example 15. The equilibrium family

$$x_{\text{eq}}(s) = s, \quad u_{\text{eq}}(s) = -s|s| \tag{77}$$

leads to the linearization family

$$\dot{\tilde{x}} = 2|s|\tilde{x} + \tilde{u} \tag{78}$$

Because of the simplicity of this system, we are able to design controls for all s, rather than selected s. A suitable linear design for command following is

$$\tilde{u} = -3|s|\tilde{x} + (\tilde{r} - \tilde{x}) \tag{79}$$

Implementing this design using smooth scheduling leads to the gain scheduled control

$$\begin{aligned} u &= u_{\text{eq}}(s) + \tilde{u} \\ &= u_{\text{eq}}(s) - 3|s|(x - x_{\text{eq}}(s)) + ((r - x_{\text{eq}}(s)) - (x - x_{\text{eq}}s)) \end{aligned} \tag{80}$$

For the scheduling variable $s = x$, the control becomes

$$u = -x|x| + (r - x) \tag{81}$$

This feedback leads to the closed loop dynamics

$$\dot{x} = -x + r \tag{82}$$

which are globally stable, as opposed to the local stability of Example 15.

Notice in this example that the linear control term $(x - x_{\text{eq}}(s))$ has no effect on the closed loop equations. This is because of the smooth scheduling implementation with $s = x$ as the scheduling variable. In this case, a desirable feedback loop was eliminated in the scheduling implementation. It is also possible to *introduce* undesirable feedback during implementation.

Theoretical Foundations

The gain scheduled controller is designed so that stability and performance are achieved whenever the plant is in the vicinity one of the design operating conditions. Since the plant actually varies throughout the entire operating regime, an important question is to what degree the *local* properties of the individual operating point designs carry over to the *global* system.

The overall closed loop equations for a gain scheduled system are

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{z} &= \overline{A}(s)z + \overline{B}(s)\tilde{y} + \overline{L}(s)\tilde{r} \\ u &= \overline{C}(s)z + \overline{u}_{\text{eq}}(s) \\ \tilde{y} &= g(x) - \overline{y}_{\text{eq}}(s) \\ \tilde{r} &= r - \overline{y}_{\text{eq}}(s) \end{aligned} \tag{83}$$

In general, the scheduling variable, s, can be written as

$$s = \gamma(x, r) \tag{84}$$

for an appropriate function, γ. Clearly, the overall system is nonlinear and hence, requires nonlinear methods for analysis.

An analysis of these equations (see Bibliography for sources) leads to the conclusion that the overall gain scheduled system will exhibit similar stability and performance as the local designs whenever (1) the scheduling variable, s, changes "sufficiently slowly," and (2) the plant dynamics are predominantly nonlinear in the scheduling variable.

The following sections provide some insight into these restrictions.

LPV Systems. It is convenient to consider slow variation restriction in the context of linear parameter varying (LPV) systems. LPV systems are defined to be linear systems whose dynamics depend on exogenous time varying parameters which are unknown a priori, but can be measured upon operation of the control system.

An LPV system can be represented in state space form as

$$\begin{aligned} \dot{x} &= A(\theta)x + B(\theta)u \\ y &= C(\theta)x \end{aligned} \tag{85}$$

where θ is a time varying parameter. Typical assumptions on θ are magnitude bounds; for example,

$$|\theta| \le \theta_{\max} \tag{86}$$

and rate bounds; for example,

$$|\dot{\theta}| \le \dot{\theta}_{\max} \tag{87}$$

LPV systems form the underlying basis of gain scheduling. It is convenient to associate the "parameter" with the scheduling variable and the LPV structure with the linearization family, although this is not always the case as will be seen.

The following is a classical result from differential equations stated in an LPV context.

Theorem 2. If the equilibrium $x_o = 0$ of Eq. (85) is asymptotically stable for all constant θ, then it is asymptotically stable for all time varying θ provided that $\dot{\theta}_{\max}$ is sufficiently small.

The relevance of Theorem 19 to gain scheduling is as follows. A closed loop LPV system is such that good stability and performance is expected for fixed values of the parameter/scheduling variable. However, performance and even stability can deteriorate in the presence of parameter time variations. Theorem 19 provides a sufficient condition for the fixed parameter properties to carry over to the varying parameter setting.

Example 20. A classical example of instability from fixed parameter stability is the time-varying oscillator,

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -(1+\theta(t)/2) & -0.2 \end{pmatrix} x(t) \tag{88}$$

These equations can be viewed as a mass-spring-damper system with time-varying spring stiffness. For fixed parameter values, $\theta(t) = \theta_o$, the equilibrium $x_o = 0$ is asymptotically stable. However, for the parameter trajectory $\theta(t) = \cos(2t)$, it becomes unstable. An intuitive explanation is that the stiffness variations are timed to pump energy into the oscillations.

Quasi-LPV Representation. It is also convenient to consider the relationship between the scheduling variable and plant nonlinearities in an LPV setting.

The relationship between LPV systems and gain scheduling is not limited to linearization families. Consider the following special nonlinear plant in which the scheduling variable is a subset of the state,

$$\frac{d}{dt}\begin{pmatrix} s \\ v \end{pmatrix} = \phi(s) + M(s)\begin{pmatrix} s \\ v \end{pmatrix} + Bu \tag{89}$$

These equations represent the extreme case where the nonlinearities are *entirely* captured in the scheduling variable, s. Let $(x_{\text{eq}}(s), u_{\text{eq}}(s))$ be an equilibrium family, with

$$x_{\text{eq}}(s) = \begin{pmatrix} s \\ v_{\text{eq}}(s) \end{pmatrix} \tag{90}$$

so that

$$0 = \phi(s) + M(s)\begin{pmatrix} s \\ v_{\text{eq}}(s) \end{pmatrix} + Bu_{\text{eq}}(s) \tag{91}$$

By subtracting equation (91) from equation (89), we obtain

$$\frac{d}{dt}\begin{pmatrix} s \\ v \end{pmatrix} = M(s)\begin{pmatrix} 0 \\ v - v_{\text{eq}}(s) \end{pmatrix} + B(u - u_{\text{eq}}(s)) \tag{92}$$

If $v_{\text{eq}}(s)$ is differentiable,

$$\begin{aligned} \frac{d}{dt}v_{\text{eq}}(s) &= Dv_{\text{eq}}(s)\dot{s} \\ &= Dv_{\text{eq}}(s)M_{12}(s)(v - v_{\text{eq}}(s)) + Dv_{\text{eq}}(s)B_1(u - u_{\text{eq}}(s)) \end{aligned} \tag{93}$$

where the matrices $M_{12}(s)$ and B_1 are appropriate sub-matrices of $M(s)$ and B.

Combining these equations leads to the alternate form of Eq. (89),

$$\frac{d}{dt}\begin{pmatrix} s \\ v - v_{\text{eq}}(s) \end{pmatrix} \tag{94}$$

$$\begin{aligned} &= \begin{pmatrix} 0 & M_{12}(s) \\ 0 & M_{22}(s) - Dv_{\text{eq}}(s)M_{12}(s) \end{pmatrix}\begin{pmatrix} s \\ v - v_{\text{eq}}(s) \end{pmatrix} \\ &\quad + \begin{pmatrix} B_1 \\ B_2 - Dv_{\text{eq}}(s)B_1 \end{pmatrix}(u - u_{\text{eq}}(s)) \end{aligned} \tag{95}$$

which can be written as

$$\frac{d}{dt}\begin{pmatrix} s \\ \tilde{v} \end{pmatrix} = A_{\text{new}}(s)\begin{pmatrix} s \\ \tilde{v} \end{pmatrix} + B_{\text{new}}(s)\tilde{u} \tag{96}$$

where

$$\tilde{v}(t) = v(t) - v_{\text{eq}}(s(t)), \quad \tilde{u}(t) = u(t) - u_{\text{eq}}(s(t)) \tag{97}$$

The original equations now take a quasi-LPV form, where the "parameter" is actually an endogenous variable. Note that no linearization approximations were made to bring Eq. (89) to the form Eq. (96).

This transformation shows that an underlying LPV structure exists, even without linearizations, in the extreme case that the plant dynamics are nonlinear only in the scheduling variable. Any additional nonlinearities not captured by the scheduling variable enter as high order perturbations in Eq. (96). This transformation then reveals the importance of the scheduling variable to capture the plant nonlinearities.

Example 21. Consider the nonlinear system

$$\dot{x} = \begin{pmatrix} x_1|x_1| + x_2 \\ x_1^2 x_2 + u \end{pmatrix} \tag{98}$$

and let $s = x_1$ be the scheduling variable. These equations take the form of Eq. (89). The resulting equilibrium family is

$$x_{\text{eq}}(s) = \begin{pmatrix} s \\ -s|s| \end{pmatrix}, \quad u_{\text{eq}}(s) = s^3|s| \tag{99}$$

Performing the transformations described above leads to the quasi-LPV form

$$\frac{d}{dt}\begin{pmatrix} s \\ x_2 - (-s|s|) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & s^2 - 2|s| \end{pmatrix}\begin{pmatrix} s \\ x_2 - (-s|s|) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(u - (-s^3|s|) \tag{100}$$

ADVANCED METHODS FOR GAIN SCHEDULING

Convex Optimization for LPV Systems

Because of the availability of numerically efficient methods for large scale problems, convex optimization is an emerging technique for gain scheduling design of for LPV and quasi-LPV systems. As seen in the previous section, the development of methods for LPV systems is directly pertinent to gain scheduling, since LPV systems form the underlying structure of a gain scheduled design.

The main idea in convex optimization methods for LPV systems is to combine stability and performance parameters with controller parameters in a single convex optimization objective.

We will demonstrate these methods in the following simple context. Consider the open loop LPV system

$$\dot{x} = (\theta A_1 + (1-\theta)A_2)x + Bu \tag{101}$$

where the parameter is constrained by

$$0 \le \theta \le 1 \tag{102}$$

We are interested in constructing stabilizing gain scheduled state feedback. Let us impose the feedback structure

$$u = -(\theta K_1 + (1-\theta)K_2)x \tag{103}$$

which mimics the LPV variations of the system. The closed loop dynamics are then

$$\dot{x} = (\theta(A_1 - BK_1) + (1-\theta)(A_2 - BK_2))x \tag{104}$$

A sufficient condition which guarantees the stability of (104) is the following.

Theorem 22. The equilibrium $x_o = 0$ of Eq. (104) is asymptotically stable if there exists a positive definite matrix, $P = P^T > 0$, such that for both $i = 1$ and $i = 2$,

$$P(A_i - BK_i) + (A_i - BK_i)^T P < 0 \tag{105}$$

It is important to note that Theorem 22 only provides a sufficient condition for stability. The main idea is that the matrix P defines the Lyapunov function, $V(x) = x^T Px$, for the closed loop system Eq. (104).

Our objective is to find matrices K_1, K_2, and P which satisfy Eq. (105). It can be shown that the set of matrices which satisfy Eq. (105) is *not* convex. This lack of convexity significantly complicates any direct search process.

In order to convexify the problem, consider the change in variables

$$Q = P^{-1}, \quad Y_i = K_i P^{-1} \tag{106}$$

Given $Q > 0$ and Y_i, one can solve for the original variables P and K_i. With these variables, condition Eq. (105) is equivalent to

$$A_i Q - BY_i + QA_i^T - Y_i^T B^T < 0 \tag{107}$$

Now the set of $Q > 0$, Y_1, and Y_2 which satisfy Eq. (107) *is* convex. This allows one to employ efficient convex feasibility algorithms which either produce a feasible set of matrices or determine definitely that no solution exists.

Some important points to consider are the following:

- The scheduling process is built into the construction of the state feedback; it is *not* necessary to perform several fixed parameter designs.
- Stability for arbitrarily fast parameter variations is assured.
- Theorem 22 is only a sufficient condition for stability, and hence may be conservative.

The method extends to more general control objectives, other than stabilizing state feedback, including

- disturbance rejection and command following
- output feedback
- rate constrained parameter variations

Extended/Pseudo-Linearization

The objective in extended and pseudo-linearization is to impose that the closed loop system has a linearization family which is invariant in some desired sense.

Let us consider the special case of a tracking problem with full state feedback for the nonlinear system Eq. (33). The objective is for the first state, x_1, to approximately track reference commands, r. Let the equilibrium family $(x_{eq}(s), u_{eq}(s))$ be an such that

$$(1 \quad 0 \quad \ldots \quad 0)x_{eq}(s) = s \tag{108}$$

Now consider the nonlinear feedback

$$u = G(x, r) \tag{109}$$

where

$$u_{eq}(s) = G(x_{eq}(s), s) \tag{110}$$

Then, a linearization family for the closed loop system

$$\dot{x} = f(x, G(x, r)) \tag{111}$$

is

$$\begin{aligned}\dot{\tilde{x}} = (&D_1 f(x_{eq}(s), G(x_{eq}(s), s)) \\ &+ D_2 f(x_{eq}(s), G(x_{eq}(s), s))D_1 G(x_{eq}(s), s))\tilde{x} \\ &+ D_2 f((x_{eq}(s), G(x_{eq}(s), s))D_2 G(x_{eq}(s), s)\tilde{r}\end{aligned} \tag{112}$$

One invariance objective is for the closed loop linearization family to be constant; for example,

$$\dot{\tilde{x}} = A_{\text{des}}\tilde{x} + B_{\text{des}}\tilde{r} \tag{113}$$

One can state appropriate conditions involving partial differential constraint equations under which there exists any G which achieves this objective.

The intention is that improved closed loop performance is possible if the closed loop linearizations have some sort of invariance property. Such improved performance, if any, is difficult to quantify, but simulation studies indicate the potential benefits.

Example 23. Consider the nonlinear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1x_2 + x_1^2 + u \end{pmatrix} \tag{114}$$

The equilibrium family suggested by x_1 is

$$x_{\text{eq}}(s) = \begin{pmatrix} s \\ 0 \end{pmatrix}, \quad u_{\text{eq}}(s) = -s^2 \tag{115}$$

Suppose we want the closed loop linearization family to have dynamics matrices

$$A_{\text{des}} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \qquad B_{\text{des}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{116}$$

An obvious selection for $G(x, r)$ is

$$G(x, r) = -x_1x_2 - x_1^2 - x_1 - 2x_2 + r \tag{117}$$

Then setting $u = G(x, r)$ leads to the linear closed loop dynamics

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r \tag{118}$$

The above choice for G achieves what is called *feedback linearization,* since the resulting closed loop dynamics are linear.

It is not necessary that feedback linearization is achieved. For example, consider the modified dynamics

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 + x_2^2 u \\ x_1x_2 + x_1^2 + u \end{pmatrix} \tag{119}$$

The resulting equilibrium families and linearization families are the same. Furthermore, the same choice of G achieves the desired invariance objective.

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