DELAY SYSTEMS

In most applications of mathematics to engineering it is tacitly assumed that the systems under consideration are causal. That is, the future state of the system depends only on its present state. In reality most electrical systems, particularly control systems, are subject to transportation and/or processing delays. Usually these delays are ignored, either because they are considered "small" or because they complicate the mathematical model. Thus a dilemma arises.

When does the realistic modeling of a physical system require the introduction of time delays into the mathematical model? One purpose of this article is to introduce the reader to the fundamental properties of time delay differential equations and to compare these to the corresponding ones for ordinary differential equations. The other is to sum up the structure and fundamental properties of delay systems of the type most frequently encountered in electrical engineering.

Ordinary differential equations (ODE) in which a part of the past history affects the present state are called delay differential equations or functional differential equations (FDE). Some examples are

$$\dot{x}(t) = ax(t) + bx(t-1) + \int_{-1}^{0} x(t+\sigma)\, d\sigma \qquad (1)$$

$$\frac{d}{dt}(x(t) - dx(t-1)) = ax(t) + bx(t-1) + c\int_{-1}^{0} x(t+\sigma)\, d\sigma \qquad (2)$$

Although special examples of delay differential equations were investigated as early as the eighteenth century by Euler and Lagrange, their systematic development did not occur until this century. The initial impetus was the study of certain mathematical models in mechanics and the physical sciences which incorporated time delays in their dynamics. One of the most interesting control models was constructed by N. Minorsky in 1942 (1) in which he incorporated the fact that the automatic steering mechanism of a ship was subject to a time delay between a course deviation and the turning angle of the rudder. Perhaps the two most important contributors to the initial mathematical development of delay systems were A. D. Mishkis (2) and N. N. Krasovskii (3). Myshkis gave the first systematic treatment of the existence and uniqueness problems for delay systems, and Krasovskii not only extended the second method of Lyapunov for stability to delay systems but also showed that the correct mathematical setting for linear time invariant delay differential equations was an infinite-dimensional space and not the finite-dimensional space where the system was defined. This is a crucial observation because a delay differential equation may be treated both as a process in a finite-dimensional space and one in an infinite-dimensional space. Some properties of delay systems do not depend on their infinite-dimensional character. For other properties, this is prerequisite, and yet other properties, whose computational nature is finite-dimensional, can only be established by considering the infinite-dimensional system. An example of the first situation is given by the representation of solutions of linear time invariant equations. These are obtained, as in the case of ODE, by using a combination of linear algebra and complex analysis. An infinite-dimensional property is the notion of a solution to the initial value problem. A property which is infinite-dimensional in nature, but sometimes computationally finite-dimensional, is the stability behavior of the homogeneous time independent systems considered in this article. The stability of these systems is determined by the zeros of an entire analytic function, as in the ODE case. However, the justification for this is based on the infinite-dimensional nature of these systems. This finite-infinite-dimensional duality is of critical importance in studying delay systems. The monograph by R. Bellman and K. Cooke (4) develops many fundamental stability properties of delay systems using only the finite-dimensional approach. The monograph by J. K. Hale and S. M. Verduyn-Lunel (5) develops both the coarse and fine structure of delay systems using the powerful tools of infinite-dimensional analysis together with the more mundane methods of linear algebra and complex analysis.

There are two main categories of FDE considered in the engineering literature, retarded functional differential equations (RFDE) and neutral functional differential equations (NFDE). A delay system, written as a first order system, is a RFDE if the derivative contains no delay terms, Eq. (1) is a RFDE. If the derivative contains delay terms in a first-order system, the equation is called a neutral functional differential equation. Eq. (2) for $d \neq 0$ is a NFDE. In engineering practice, only a certain class of NFDE is considered; namely, D-stable NFDE. A D-stable NFDE is one in which the difference equation associated with the derivative is uniformly exponentially stable (u.e.s.) Eq. (2) is a D-stable NFDE if $|d| < 1$ since the respective difference equation,

$$y(t) - dy(t-1) = 0 \qquad (3)$$

is u.e.s. If $|d| > 1$, there is an unbounded solution of Eq. (3), and so the equation is not stable. If $|d| = 1$, then Eq. (3) is stable, but not u.e.s. The definition of D-stability is due to M. A. Cruz and J. K. Hale (6). They showed that the stability properties of linear time invariant (LTI) RFDE and LTI D-stable NFDE are determined by the exponential solutions, just as in the case of LTI ODE. D. Henry (7) proved that a LTI NFDE cannot be uniformly exponentially stable unless it is D-stable. Until the papers of Cruz, Hale and Henry, engineers routinely gave conditions for the stability of neutral systems without realizing that D-stability was an essential requirement.

PROPERTIES OF DELAY SYSTEMS

The study of delay systems is a natural extension of the theory of ODE, and we shall describe the basic properties of these systems by comparing them with the analogous properties of ODE. Let R^n be Euclidean n-space.

Initial Conditions and Solutions

The initial condition for the solution of an ODE is given in the finite dimensional phase space R^n. The initial condition for the solution of an FDE or delay equation is an infinite-dimension space, called a Banach space. The reason for this is that the initial value of an FDE is a vector-valued function on R^n defined over an interval $[-h, 0]$, $h > 0$ representing the delay. The point values of the solution evolve in R^n, which also is called the phase space of the system. This is the dimension duality property of an FDE. The space of initial conditions is infinite-dimensional, and the phase space is finite-dimensional.

The notion of the solution of an FDE is weaker than for an ODE. This is because not all initial functions result in differentiable solutions. However for LTI FDE the Laplace transform provides a convenient alternative definition. The formal Laplace transform of an LTI FDE does not explicitly contain the derivative of the FDE, only its initial value. This is also true for ODE. Thus, if the formal Laplace transform of an

FDE is actually the Laplace transform of a vector-valued function in R^n, we call this function a solution to the equation. It is shown in Ref. 7 that this is indeed the case for LTI FDE. In the case where the FDE is nonlinear or linear with time dependent coefficients, the definition of a solution is more complex. In the applications to control problems in electrical engineering, LTI FDE occur most frequently.

Solutions of nonhomogeneous LTI FDE have a variation of parameters representation. This is given by the convolution of an $n \times n$ matrix-valued function, $S(t)$, and the forcing function. The matrix function, $S(t)$, is the inverse Laplace transform of the matrix version of the homogeneous system where the initial value is the zero matrix when $t < 0$ and the identity matrix at $t = 0$. This matrix-valued function is the analogue of the fundamental matrix of an LTI ODE and as a consequence is called the fundamental matrix of the system. However, it is not a matrix exponential function as it would be in the ODE case. Time varying nonhomogeneous linear FDE also have their solutions represented by fundamental matrices. However their computation is more difficult. One numerical procedure used to compute these matrices is known as the method of steps (4). This method works for the systems given by equations (1) and (2) if $c = 0$ because the time delays are discrete. The method consists of finding the fundamental matrix of the system over the interval [0, h], then using this information to compute the fundamental matrix over the interval $[h, 2h]$, etc. However, in many control problems, the indeterminate time behavior of the system is desired, and the method of steps is unsuitable for this purpose.

Stability

In the applications of delay systems to electrical engineering, which is practically synonymous with control theory, linear systems are used almost exclusively, and of these LTI systems predominate. For LTI systems, the Lyapunov stability theory is superficially the same as in the ODE case. However, in place of symmetric matrices, one uses symmetric functionals called Liapunov-Krasovskii functionals. The mathematical structure of these functionals was described by Yu. M. Repin (8) and N. N. Krasovskii (3). Their practical application to stability is very limited for two reasons. They are difficult to construct, even for very simple systems, and once a functional has been constructed, it is not easy to determine positive or negative definiteness. Their use in stability theory is usually on an ad hoc basis. That is, one "guesses" the form of the functional, applies it to the system, and hopes it will have the required positivity or negativity when applied to differentiable trajectories of the system. There is also an offshoot of the Lyapunov theory, called the method of Razumikhin (9). This method uses Lyapunov functions in place of functionals to determine stability. Its application requires that the FDE satisfy certain side conditions which are not always met in practice. However, when applicable, the method of Razumikhin is preferable to the standard Lyapunov method. On the other hand, unlike the Lyapunov method, the Razumikhin method does not have converse theorems on stability.

The application of Lyapunov functionals or Razumikhin functions to LTI FDE is a means to the end of locating the eigenvalues of the system. The eigenvalues of an LTI FDE are the zeros of an associated entire analytic function and determine, as in the ODE case, the qualitative behavior of the system. For LTI RFDE or LTI D-stable NFDE, if these zeros lie in the open left half of the complex plane, the system is u.e.s. Determining this condition or its absence is much more difficult than for LTI ODE, since no simple criterion is available, such as the Routh-Hurwitz criterion.

There is a bounded input-bounded output (BIBO) criteria for linear homogeneous FDE which applies to both time invariant and time dependent systems, provided the coefficients of the time dependent system are uniformly bounded on the real line. This is known as the Perron condition (10). This condition states that if the nonhomogeneous version of such a system with zero initial conditions has bounded solutions for all bounded forcing terms, the system is u.e.s.

Control and Stabilization

There are several versions of the Pontryagin Maximum Principle for delay systems, and the most popular method of solving optimal control problems for delay systems is Bellman's Method of Dynamic Programming (11,12). However, even for ODE systems, these theories are rarely used in applications to electrical engineering. Their main drawback is that they are nearly impossible to apply to practical, nonlinear delay systems. Moreover, most electrical engineering control problems are design problems, not optimal control problems. The two major exceptions to this statement are Linear Quadratic Regulator (LQR) problems and H^∞-optimization problems. The solutions of these problems result in linear feedback controls which stabilize a system. However, neither method requires the Pontryagin Maximum Principle or Dynamic Programming.

The main emphasis in engineering control is stabilization, particularly for linear systems. One popular method of stabilization for single input-single output systems with one time delay is the use of a Smith Predictor. This is a stabilization procedure which uses a proportional–integral–derivative (PID) feedback. This is a standard method and is discussed in ADAPTIVE CONTROL and CLASSICAL DESIGN METHODS FOR CONTINUOUS TIME SYSTEMS. In this article, we concentrate on the other two major methods of feedback stabilization for delay system, LQR- and H^∞-optimization.

The LQR method attempts to minimize an integral called the cost over the positive real axis. The integrand is quadratic and positive semidefinite in the space variable and quadratic and positive definite in the control variable. If this optimization is possible for all initial values of the systems, the optimal control is a feedback control which stabilizes the system. H^∞-optimization optimizes an LTI FDE control problem with an unknown disturbance. Here, the cost is positive and quadratic in the space and control variables as in the LQR problem but quadratic and negative definite in the disturbance variable. For a fixed admissible control function u, one attempts to maximize the cost in terms of the disturbance, then to minimize the resulting functional with respect to u. This is called a min-max problem. The optimal solution, if it exists, leads to a feedback control which is u.e.s. in the so-called worst case disturbance.

For ODE, the LQR problem is solved using routine numerical packages. The numerical treatment of H^∞-optimization for ODE is much more difficult than LQR-optimization, but more robust with respect to uncertainties in the system dynamics.

Theoretically, both methods may be used for LTI FDE and the properties of the resulting feedback controls are known. However, the practical implementation of either method is a project for future investigation (13).

ANALYTIC PROPERTIES OF DELAY SYSTEMS

We introduce some additional notation which will be used in the remainder of this article. Let $R = (-\infty, \infty)$, $R^+ = [0, \infty)$, Z be the complex plane, Z^n be complex n-space, I be the n-dimensional identity matrix, c^T be the transpose of an n-column vector c, $\bar{c}$ be the complex conjugate of an n-column vector, B^T be the transpose of an $n \times m$ matrix B, $|c| = \sqrt{\overline{cTc}}$ be the length of an n-vector c, and $\det(A)$ be the determinant of a square matrix A.

The set of all continuous n-vector functions from a closed interval $[-h, 0]$ into Z^n is denoted by $C(h)$. If ϕ is in $C(h)$, then $|\phi| = \sup\{|\phi(t)|: -h \le t < 0\}$.

If $x(t)$ is an n-vector function defined on $[-h, \infty)$, then, for $t \ge 0$, $x_t = \{x(t + \sigma): -h \le \sigma \le 0\}$, and $x_t(\sigma) = x(t + \sigma)$, $-h \le \sigma \le 0$.

If $x(t)$ is a vector or matrix function on Z^n, then

$$\mathscr{L}(x(t))(\lambda) = \hat{x}(\lambda) = \int_0^\infty x(t)e^{-\lambda t}\,dt$$

is the Laplace transform of $x(t)$. The inverse Laplace transform of any vector or matrix valued function, provided it exists, is denoted by $\mathscr{L}^{-1}(\hat{S}(\lambda))(t)$.

The delay systems most commonly encountered in electrical engineering are LTI systems with discrete time delays. The fundamental properties of these systems serve as a paradigm for most other systems one encounters. These systems which include RFDE and NFDE often have their dynamics described by

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^{r} D_j x(t - h_j)\right] = A_0 x(t) + \sum_{j=1}^{r} A_j x(t - h_j) + Bu(t), \quad t \ge 0 \tag{4}$$

and initial values

$$x(t) = \phi(t), -h \le t \le 0, \quad \phi \in C(h) \tag{5}$$

In Eq. (4), the matrices $\{D_j\}$, $\{A_j\}$, and A_0 are $n \times n$-matrices with real entries; the matrix B is an $n \times m$-matrix with real entries and $0 \le h_j \le h$, $1 \le j \le r$. The m-vector $u(t)$ is called the control.

A solution $x(t, \phi, u)$ of Eq. (4), Eq. (5) is a function which satisfies Eq. (5), is continuously differentiable for $t > 0$, and has a right hand derivative at $t = 0$ which satisfies Eq. (4). We can use the Laplace transform to obtain the existence of and a specific representation for a solution. Let

$$\hat{S}(\lambda) = \left[\lambda\left(I - \sum_{j=1}^{r} D_j e^{-\lambda h_j}\right) - A_0 - \sum_{j=1}^{r} A_j e^{-\lambda h_j}\right]^{-1} \tag{6}$$

In terms of the operator-valued complex matrix $\hat{S}(\lambda)$, the formal Laplace transform of Eqs. (4) and (5) is

$$\hat{x}(\lambda, \phi) = \hat{S}(\lambda)\left[\phi(0) - \sum_{j=1}^{r} D_j \phi(-h_j)\right] + \hat{S}(\lambda)\int_{-h_j}^{0} \sum_{j=1}^{r} (A_j + \lambda D_j) e^{-\lambda(\sigma + h_j)} \phi(\sigma)\,d\sigma + \hat{S}(\lambda)B\hat{u}(\lambda) \tag{7}$$

For any ϕ in $C(h)$, the inverse Laplace transform exists and is called a solution to the initial-value problem in Eqs. (4) and (5) (14). If $\phi \in C(h)$ has a derivative, then the inverse Laplace transform of Eq. (7) is

$$x(t, \phi, u) = S(t)\left[\phi(0) - \sum_{j=1}^{r} D_j \phi(-h_j)\right] + \sum_{j=1}^{r}\int_{-h_j}^{0} S(t - \sigma - h_j)(A_j \phi(\sigma) + D_j \dot{\phi}(\sigma))\,d\sigma + \int_0^t S(t - \sigma)Bu(\sigma)\,dv \tag{8}$$

where

$$S(t) = \mathscr{L}^{-1}(\hat{S}(\lambda))(t) \tag{9}$$

The function $S(t)$ formally satisfies Eq. (4) with the initial matrix $S(0) = I$, $S(\sigma) = 0$ for $\sigma < 0$ and is referred to as the fundamental matrix solution of Eq. (4).

If the matrices in Eq. (4) are time varying, there is a representation of the solution similar to Eq. (8), but it is not obtained by using the Laplace transform. The matrix $S(t)$ is replaced by a matrix $S(t, \tau)$, where $S(\tau, \tau) = I$, and $S(t, \tau) = 0$ if $\tau < t$. The matrix function $S(t, \tau)$ is formally a matrix solution of Eq. (4) for $t \ge \tau$ (5).

Stability

The difference equation associated with the NFDE in Eq. (8) is

$$x(t) - \sum_{j=1}^{r} D_j x(t - h_j) = 0 \tag{10}$$

The condition for Eq. (4) to be D-stable is that the equation

$$\det\left[I - \sum_{j=1}^{r} D_j e^{-\lambda h_j}\right] = 0 \tag{11}$$

has all of its solutions in $Re\lambda \le -\delta$ for some $\delta > 0$. If we seek solutions of Eq. (10) of the form $e^{\lambda t}\xi$ for some nonzero complex n-vector ξ, then λ must be an eigenvalue of the matrix in Eq. (11) and ξ must be a corresponding eigenvector. For this reason, we say that λ satisfying Eq. (11) are eigenvalues of Eq. (10). We remark that if Eq. (10) is D-stable at one collection of the delays h_j, $1 \le j \le r$, then it is D-stable for all other values of the delays (5).

The stability behavior of the homogeneous version of (4)-(5), (i.e., when $u(t) = 0$) is completely determined by the eigenvalues of the system. For the same reason as indicated for Eq. (10), these are the zeros of the entire analytic function

$$\det\left[\lambda\left(I-\sum_{j=1}^{r}D_je^{-\lambda h_j}\right)-A_0-\sum_{j=1}^{r}A_je^{-\lambda h_j}\right]=0 \qquad (12)$$

If the system is D-stable, then it is u.e.s. if and only if all solutions of Eq. (12) satisfy $Re\lambda < 0$ (5).

As mentioned above, before the papers of Cruz and Hale (6) and Henry (7), engineers often assumed a system was u.e.s. if solutions of Eq. (12) satisfied $Re\lambda < 0$. If the D-stability condition is not satisfied, showing that the solutions of Eq. (12) satisfy $Re\lambda < 0$ may be insufficient to determine stability or instability as the following two examples show (15). Consider the scalar systems

$$\frac{d}{dt}[x(t)-x(t-1)]=-x(t) \qquad (13)$$

$$\frac{d^2}{dt^2}[x(t)-2x(t-1)+x(t-2)]+2\frac{d}{dt}[x(t)-x(t-1)]+x(t)=0 \qquad (14)$$

The eigenvalues of both systems are solutions of the equation $\lambda(1 - e^{-\lambda}) + 1 = 0$ which has all its solutions in $Re\lambda < 0$. Equation (13) has all its solutions tending to zero as t tends to infinity, but it is not u.e.s. Equation (14) has solutions which tend to infinity as t tends to infinity.

The stability or instability of LTI FDE can be determined by Lyapunov–Krasovskii functions and sometimes by Razumikhin functions. However as was mentioned above, these are difficult to find for all but the simplest systems and even then are usually selected on an ad hoc basis. To illustrate this, consider the scalar system

$$\dot{x}(t)=-ax(t)+bx(t-r), \quad a>0, a>|b|>0, r>0 \qquad (15)$$

For a constant c to be determined, choose the Lyapunov–Krasovskii functional

$$V(\phi)=\frac{1}{2}(\phi(0))^2+c\int_{-r}^{0}(\phi(\sigma))^2\,d\sigma \qquad (16)$$

Along differentiable trajectories of Eq. (15)

$$\frac{dV}{dt}(x_t)=(-a+c)(x(t)^2+bx(t-r)x(t)-c(x(t-r))^2$$

This functional will be negative on $C(r)$ if we choose $c = a/2$. Therefore, by Theorem 2.1, Chapter 5, in Ref. 5, the region of u.e.s. contains the set of coefficients a, b with $|a| > |b|$. Notice that this simple choice for the Lyapunov–Krasovskii functional yielded the stability region which is completely independent of the size of the delay.

Now, consider the Razumikhin function for Eq. (15) given by $V(x(t)) = x^2/2$. Along differentiable trajectories of Eq. (15),

$$\frac{dV}{dt}(x(t))=x(t)(ax(t)+bx(t-r))$$

If we consider only those solutions $x(t)$ that satisfy the relation $|x(t)| \geq |x(t - r)|$, then

$$\frac{dV}{dt}(x(t))\leq-(a-|b|)(x(t))\leq 0$$

Thus, by Theorem 4.2, Chapter 5, in Ref. 5, the system is u.e.s. for all $r > 0$. In this case, the Razumikhin method yielded in a much more straightforward way the same result as above using the Lyapunov–Krasovskii functional.

There is yet another way to determine u.e.s. of LTI FDE. This is to treat the delay terms as parameters in a family of LTI FDE, which reduce to an ODE when all the delays are zero. If the ODE is u.e.s., one tries to estimate the size of the delays that the family can tolerate and yet remain u.e.s. This is possible since for LTI RFDE or LTI D-stable NFDE, the maximal exponential rate of expansion or contraction of a system depends continuously on the delay parameters (16). This condition is an easy consequence of the Hurwitz Theorem in Complex Analysis (17) since the maximal rate is determined by an eigenvalue for which the real part can be chosen to be continuous in the delay parameters. To illustrate this method, consider Eq. (15). When $r = 0$, the system is u.e.s. We try to find the smallest positive value r for which the system has a nontrivial periodic solution of period $2\pi/w$. This value, if it exists, satisfies the equation

$$iw+a+b(\cos\alpha-i\sin\alpha)=0, \quad \alpha=wr, w>0$$

Since $a > |b|$, for fixed w, this equation has no solution for real α. Thus, we conclude that the system is u.e.s. for all values of r.

There is an important class of nonlinear FDE whose stability is determined by frequency domain methods. A typical example of such a system is one whose dynamics are described by the equations

$$\dot{x}(t)=A_0x(t)+A_1x(t-h)+bf(\sigma), \quad \sigma=c^Tx \qquad (17)$$

where b is an n-vector, and f is a scalar function satisfying the sector condition

$$a_1\sigma^2\leq\sigma f(\sigma)\leq a_2\sigma^2, \quad 0<a_1<a_2<a \qquad (18)$$

The following theorem holds (18)

Theorem. Assume that the system

$$\dot{y}(t)=A_0y(t)+A_1x(t-h) \qquad (19)$$

is u.e.s. and let

$$K(iw)=c^T(iwI-A_0-A_1e^{iwh})^{-1}b \qquad (20)$$

If there exists a $q > 0$ such that for all w in R,

$$Re(1+iwq)K(iw)-\frac{1}{a}\leq 0 \qquad (21)$$

then each solution of Eq. (17) tends to zero as t tends to infinity.

The above theorem is also true for D-stable systems of the types in Eqs. (4) and (5) where $bf(\sigma)$ replaces Bu. This theorem is one of a class of theorems which are known collectively as Popov-type theorems. An interesting corollary to the above theorem is that condition [Eq. (21)] guarantees the existence of a Lyapunov functional for the system [Eq. (17)] which has a particular structure.

Corollary. If Eq. (20) is satisfied, then there exists a Lyapunov functional on $C(h)$ of the type

$$V(\phi, \psi) = Q(\phi, \psi) + \beta \int_0^{\sigma} f(s)\,ds, \quad \beta > 0 \tag{22}$$

such that Q is bilinear on $C(h)$, $Q(\phi, \phi) > 0$ if $|\phi| \neq 0$ and, along differentiable trajectories of Eq. (17), $dV(x_t, x_t)/dt \leq 0$.

The proof the above corollary is a simple extension of the same result for ODE given on p. 169 of Ref. 10. The converse is also true; that is, if the corollary is satisfied, then so is the theorem. The Lyapunov approach is in general not feasible, whereas the frequency domain or Popov approach is easily checked, especially by modern computing packages.

As was mentioned above, there is a necessary and sufficient condition for determining the u.e.s. of time varying FDE. This is the Perron condition or bounded-input, bounded-output criterion. We give an analytic form of this condition for the system whose dynamic is described by the equation

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h) + f(t) \tag{23}$$

but remark that the basic condition holds for any linear FDE whose coefficient matrices are uniformly bounded.

We assume that the square matrices $A_0(t)$ and $A_1(t)$ have all their entries uniformly bounded on R^+. It is known (5) that the solutions of Eq. (23) with initial conditions zero in $C(h)$ may be represented in the form

$$x(t, t_0, f) = \int_{t_0}^{t} S(t, \sigma) f(\sigma)\,d\sigma \tag{24}$$

where $S(t, \sigma) = 0$ if $\sigma > t$, $S(\sigma, \sigma) = I$ and, for $t > \sigma$,

$$\frac{d}{dt}(S(t, \sigma)) = A_0 S(t, \sigma) + A_1 S(t-h, \sigma) \tag{25}$$

Theorem. A necessary and sufficient condition for the homogeneous version of Eq. (23) to be u.e.s. is that for all f which are uniformly bounded on R^+, the vector function in Eq. (24) satisfies an inequality of the form $|x(t, t_0, f)| \leq M_f$, where M_f is finite and depends only on f (10).

Although the Perron condition or BIBO condition may be theoretically difficult to verify, a modified form often is used in control engineering. The linear system is subjected to periodic forcing functions at a variety of frequencies and with uniformly bounded gains. If the outputs are uniformly bounded over long time periods, the system is considered u.e.s.

Control and Stabilization

Controllability for an FDE system is function space control; that is, one seeks to control a given initial point in $C(h)$ to a given terminal point in a finite time. There is also the notion of ϵ-controllability, that is, control from a given point to an ϵ-ball of another given point. This latter form of control is more realistic for FDE systems, but in practice, neither form is much used in engineering design. A simple example may indicate the reason. Consider the scalar system

$$\dot{x} = ax(t) + bx(t-h) + u(t) \tag{26}$$

where $h > 0$ and $b \neq 0$. For $\phi \in C(h)$ and $|\phi| \neq 0$, suppose that one desires to find a $u(t)$ with $|u(t)| \leq 1$ which drives the solution of Eq. (26) with initial value ϕ to the zero function in some finite time T. The Laplace transform of the resulting motion is given by

$$\hat{x}(\lambda, \phi, u) = \frac{1}{\lambda - a - be^{-\lambda h}} \left(\phi(0) + \int_{-h}^{0} be^{-\lambda(\sigma+h)} \phi(\sigma)\,d\sigma + \hat{u}(\lambda) \right) \tag{27}$$

Since both $x_t(\phi, u)$ and $u(t)$ are identically zero after time T, the functions $\hat{x}(\lambda, \phi, u)$ and $\hat{u}(\lambda)$ in Eq. (27) must be entire analytic functions (19). This means that $\hat{u}(\lambda)$ must be chosen so that the numerator in Eq. (27) is zero when the denominator is zero. But the zeros of $\lambda - a - be^{-\lambda h}$ are infinite in number and can at best be approximated.

There are several versions of the Pontryagin Maximum Principle for FDE control problems, and the theoretical method used to solve control problems is the Method of Dynamic Programming. From an engineering point of view, these are only of academic interest. Comprehensive reference sources for this area of control are Refs. 12 and 20.

For multiple-input-multiple-output LTI ODE, there are three basic methods of feedback stabilization. These are pole placement, linear quadratic regulator (LQR)-optimization, and H^∞-optimization pole placement. The latter has the simplest numerical structure but has less eclat than the other two methods. Pole placement methods are possible for LTI FDE. In practice the best one can hope for are constant gain feedbacks which guarantee a given decay rate. However, LQR-optimization and H^∞-optimization have in theory been completely extended from LTI ODE to LTI FDE. There are at least two ways to look at these extensions. One way relates to the specific delay structure and minimizes the use of Banach space theory. The other embeds LTI delay systems into a general class of LTI infinite-dimensional systems known as Pritchard-Salamon systems (P-S systems) and makes extensive use of the theory of Sobelev spaces (21). Here, we confine ourselves to the first approach.

A typical LQR-optimization problem is the following. For a given positive definite matrix W and a given $\phi \in C(h)$, choose the control function $u(t)$ to minimize the functional

$$\mathscr{F}(u) = \int_0^{\infty} [x^T(t)Wx(t) + u^T(t)u(t)]\,dt \tag{28}$$

subject to the constraint that $x(t)$ is the solution with initial value ϕ of the equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t) \tag{29}$$

If the minimum for all ϕ in $C(h)$ is finite, then there is a bounded linear mapping K from $C(h)$ into Z^n such the optimal

u is given by a feedback $u(t) = B^T K x_t$. Moreover, if $q(t) = Kx_t$, then

$$\dot{q}(t) = -Wx(t) - A_0^T q(t) - A_1^T q(t+h) \tag{30}$$

The eigenvalues of the feedback system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) - BB^T K x_t \tag{31}$$

are the solutions with Re $\lambda < 0$ of the equation

$$\det \begin{bmatrix} \lambda I - A_0 - A_1 e^{-\lambda h} & BB^T \\ W & \lambda I + A_0^T + A_1^T e^{\lambda h} \end{bmatrix} = 0 \tag{32}$$

There are variants of the optimization problem in Eq. (28). One is to optimize Eq. (28) where W is only positive semidefinite. This condition requires some additional assumptions on the system [Eq. (29)], which are technical but do not alter the basic structure of the solution. The fundamental assumption needed to solve the LQR-problem is that system [Eq. (29)] can be stabilizable. This appears to be putting the cart before the horse, and in some sense, it is. For example, an LQR problem for the ODE system $\dot{x} = Ax + Bu$, where A is an $n \times n$ matrix, is solvable if rank $[B, AB, \cdots, A^{n-1}B] = n$. There is no such simple condition for LTI FDE. For instance, the n-dimensional system [Eq. (29)] for $B = b$ an n-vector is stabilizable if, for $Re\lambda \geq 0$,

$$\text{rank}[\lambda I - A_0 - A_1 e^{-\lambda h}, b] = n \tag{33}$$

An example of H^∞-optimization is to find the min-max of the following system. Consider the functional

$$\mathscr{F}(u,d) = \int_0^\infty \left[x^T(t) W x(t) + u^T(t) u(t) - \frac{1}{\gamma^2} d^T(t) d(t) \right] dt \tag{34}$$

subject to the constraint

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Ld(t) + Bu(t) \tag{35}$$

where A_0, A_1, B, and W are the matrices in Eqs. (28) and (29), and L in an $n \times r$-matrix. It is assumed that the systems

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t) \tag{36}$$

and

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Ld(t) \tag{37}$$

are stabilizable. The object is to find, if possible, the

$$\min_u \left[\max_d \mathscr{F}(u,d) \right] \tag{38}$$

and to show that it is nonegative for all initial values ϕ in $C(h)$ of the solution of Eq. (35). In this problem, the constant γ plays a critical role. There exist γ for which there is either no optimum or for which the optimum is negative for at least one ϕ in $C(h)$. However, there is a smallest $\gamma_0 > 0$ for which Eq. (38) has a nonegative solution for all ϕ in $C(h)$ and all γ in (γ_0, ∞) (21). The optimal solution is a linear feedback control, K, which maps $C(h)$ into Z^n. The optimal u and d satisfy

$$u(t) = -B^T K x_t, \; d(t) = \frac{1}{\gamma^2} L^T K x_t \tag{39}$$

If $q(t) = Kx_t$, then

$$\dot{q}(t) = -Wx(t) - A_0^T q(t) - A_1^T q(t+h) \tag{40}$$

If $\int_0^\infty (d^T(\sigma) d(\sigma)) d\sigma < \infty$, then the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + BB^T K x_t + Ld(t) \tag{41}$$

has all solution converging to the zero vector as t tends to infinity. The system [Eq. (35)] with feedbacks [Eq. (39)] is u.e.s.

There are many variants of the above problem. For example, one could attempt to optimize the functional

$$C(u,d) = \int_0^\infty \left[x^T(t-h) x(t-h) + u^T(t) u(t) - \frac{1}{\gamma^2} d^T(t) d(t) \right] dt$$

The basic structure of the optimizable problem is the same. The critical problem is to find efficient numerical methods. Since this problem has not been adequately solved for LQR-optimization, it will be much more difficult for H^∞-optimization.

Fine Structure

The solutions of the homogeneous version of Eqs. (4) and (5) generate a C_0-semigroup on $C(h)$. The domain of the infinitesimal generator of this semigroup are those points ϕ in $C(h)$ which are continuously differential and satisfy the condition

$$\dot{\phi}(0) - \sum_{j=1}^{r} D_j \dot{\phi}(-h_j) - A_0 \phi(0) - \sum_{j=1}^{r} A_j \phi(-h_j) = 0 \tag{42}$$

If all $D_j = 0$, Eq. (42) is an RFDE. If the solutions of Eq. (11) lie in $Re\lambda < -\delta < 0$ for some $\delta > 0$, then the system is a D-stable NFDE. For RFDE, the spectra of the semigroup at $t = 1$ is the origin plus eigenvalues e^λ where λ is an eigenvalue which is a solution of the characteristic equation [Eq. (12)] with all $D_j = 0$ (5). For D-stable NFDE, the spectra of the semigroup at $t = 1$ has the essential spectrum with moduli $<$ 1 plus eigenvalues e^λ, where λ is an eigenvalue which is a solution of the characteristic equation [Eq. (11)] (5). An LTI RFDE has only a finite number of its eigenvalues in any right half plane, whereas a D-stable LTI NFDE has a vertical strip in $Re\lambda < 0$ which contains an infinite number of its eigenvalues. The solutions of LTI RFDE become more differentiable as time increases. They pick up one derivative for each interval of length h. For example, even if ϕ in $C(h)$ is not continuously differentiable over $[-h, 0]$, the solution $x_t(\phi)$ will be k-times differentiable for $t \geq kh$. LTI NFDE retain the smoothness of their initial conditions. For these reasons, LTI RFDE have been compared to LTI parabolic partial differential equations (PDE), and LTI D-stable NFDE have been compared to LTI hyperbolic PDE. However, the fine structures of

LTI RFDE and LTI parabolic PDE are dissimilar. For example, parabolic PDE generate analytic semigroups, and RFDE generate C_0-semigroups which are compact for $t \geq h$ but are not analytic. One basic reason for this difference is that the eigenvalues of LTI RFDE do not belong to a sector in the complex plane. On the other hand, some LTI hyperbolic PDE and LTI D-stable NFDE have identical mathematical structures. In particular, the equations for the dynamics of transmission lines described by the telegraph equation can be transformed to D-stable NFDE (5). An example which illustrates the similarity between some hyperbolic PDE and D-stable NFDE, are the PDE system

$$w_{tt} = w_{xx} - 2aw_t - a^2w = 0, \quad 0 < x < 1, t > 0 \tag{43}$$

$$w(0,t) = 0, \quad w_x(1,t) = -Kw_t(1,t) \tag{44}$$

where $a > 0$ and $K > 0$ are constant, and the D-stable NFDE

$$\frac{d}{dt}\left[x(t) - \frac{1-K}{1+K}e^{-2a}x(t-2)\right] = \frac{-a}{1+k}[x(t) + e^{-2a}x(t-2)] \tag{45}$$

These systems have the same spectrum (22).

Small Solutions

Linear homogeneous FDE may have small solutions; that is, nontrivial solutions which decay faster than any exponential function. A characterization of the set of small solutions for D-stable LTI NFDE and LTI RFDE is contained in Ref. 5. A remarkable property of any D-stable LTI NFDE and LTI RFDE is that there is a $\tau > 0$ such that any small solution is identically zero in $C(h)$ after time τ. An example of a system with small solutions is

$$\dot{x} = y(t-1), \quad \dot{y} = x(t) \tag{46}$$

All solutions of Eq. (46) whose initial functions satisfy $x(0) = 0$ and $y(t) \equiv 0$ for $-1 \leq t \leq 0$ are small solutions which vanish in $C(1)$ for $t \geq 1$ [(5), p. 74].

Linear periodic systems also have small solutions, but these are not necessarily zero after a finite time (5). An example is the system

$$\dot{x}(t) = \left(\frac{1}{2} + \sin 2\pi t\right)x(t-1)$$

[see e.g. (5) p. 250].

Stability

As mentioned above, N. N. Krasovskii extended the Second Method of Lyapunov to FDE. An example of this extension is the following theorem (5).

Theorem. Let f be a continuously differentiable mapping from $C(h)$ into R^n with $f(0) = 0$ and consider the RFDE

$$\dot{x}(t) = f(x_t) \tag{47}$$

Let V be a continuous mapping from $C(h)$ into R^+ which satisfies the following two conditions:

(i) There is a nonnegative continuous function $a(r)$ with $a(r) \to \infty$ as $r \to \infty$ such that, for all ϕ in $C(h)$,

$$a(|\phi(0)|) \leq V(\phi) \tag{48}$$

(ii) There is a nonnegative continuous function $b(r)$ such that

$$\lim\sup_{h\to 0^+} \frac{1}{h}[V(x_h)(\phi) - V(\phi)] \leq -b(|\phi(0)|) \tag{49}$$

Then the solution $x_t = 0$ is stable and every solution of Eq. (47) is bounded. If $b(r)$ is positive definite every solution of Eq. (47) tends to zero as t tends to infinity.

Similar results exist concerning stability and instability for autonomous and nonautonomous FDE. If the function V is continuously differentiable, then the computation of the left side of relation [Eq. (49)] may be performed on solutions with smooth initial functions. The corresponding differential inequality gives estimates on these smooth solutions. Since the initial data of these smooth solutions are dense in the space $C(h)$, one obtains estimates on all solutions. In this sense, there is no essential difference in the method than for ODE. A complete description for the Lyapunov method is given in Ref. 5, Chapter 5.

The stability of linear homogeneous periodic RFDE and linear homogeneous periodic D-stable NFDE can be determined by examining their solutions after any integer multiple of their period which is larger than the delay. This results in a bounded linear mapping, U, from $C(h)$ into itself. The eigenvalues of U, called the characteristic multipliers of the system, determine the stability behavior of the system. If all of the multipliers lie inside the unit circle in the complex plane, then the system is u.e.s. If some are outside the unit circle, the system is unstable. If the multipliers lie inside or on the unit circle, the geometric multiplicity of those on the unit circle determines stability or instability of the system.

Feedback Stabilization

LTI FDE of the type given in Eqs. (4) and (5) are particular examples of Pritchard–Salamon control systems. Their acronym is P–S system (21). They are the largest class of infinite-dimensional control systems to which the theory of finite-dimensional LTI control theory can be most easily extended. The most diverse class of P–S systems are those described by LTI FDE whose spaces of initial conditions are Hilbert spaces and whose solutions evolve in their space of initial conditions.

LQR- and H^∞-stabilization are in theory completely developed for P–S systems, but in a very abstract setting (21). In the case of FDE, this setting requires the use of the infinitesimal generator of the associated semigroup, which is an unbounded linear operator. This operator is not as easy to manipulate as the Laplace transform of the solution of the system and is the main reason why LQR- and H^∞-stabilization is theoretically possible but computationally difficult for these systems. To illustrate the difficulty, consider the following LQR problem for the finite-dimensional system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{50}$$

where the function to be optimized is given by Eq. (28). It is known that the optimal solutions have controls $u(t) = -B^TKx(t)$, where K is a unique positive definite $n \times n$ matrix with the property that, for any n-vector, x_0 in Z^n, the analytic vector-valued function

$$\begin{pmatrix} \lambda I - A & BB^T \\ W & \lambda I + A^T \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ Kx_0 \end{pmatrix} \tag{51}$$

in Z^{2n} has no poles in the right half complex plane. Since the poles of the matrix function in Eq. (51) are symmetric with respect to the imaginary axis, the matrix K is uniquely determined once the solutions of

$$\det \begin{pmatrix} \lambda I - A & BB^T \\ W & \lambda I + A^T \end{pmatrix} = 0 \tag{52}$$

are known. This method of finding the feedback is known as spectral factorization. If the matrices A, B, and W in Eqs. (52) and (50) are replaced by linear operators α, β, and $\mathscr{W}$, where α is unbounded, a P–S LQR problem will symbolically be represented by an expression of the form in Eq. (51), and a spectral factorization exists for such a system (21). However, how does one in practice carry it out? This is the crux of the computational difficulties for LQR- and H^∞-optimization in P–S systems. On the other hand, the LQR-optimization described by Eqs. (28) and (29) has the eigenvalues of the feedback system given by the solutions in $Re\lambda < 0$ of Eq. (32). The methods used to obtain this result were system specific (23); that is, they depended on the explicit structure of the delay system and not its abstract representation, and in this instance, yielded more information. The same is true of the frequency domain criterion [Eq. (21)] used in the Popov problem described by Eq. (17). This problem has a P–S setting (24). However, in this setting, one has to unravel the simple Popov criterion. Another instance of this is the Perron condition. This condition exists for the evolution of the solutions of the system in Eq. (23) in a Banach space setting, but in practice, one examines the output of a system in R^n when the forcing function is an n-vector not the output of an infinite-dimensional vector.

EXTENSIONS TO INFINITE DIMENSIONAL PHASE SPACES

Delays may appear in PDE as well as ODE. For example,

$$u_t - du_{xx} = -\left(\frac{\pi}{2} + \mu\right) u(x, t-1)(1_u(x, t)) \tag{53}$$

(25), which is a nonlinear diffusion equation with a time delay. Extensions of time delay systems to PDE and abstract Banach spaces may be found in Refs. 26–29.

Time independent versions of these systems are often of the form

$$\dot{x}(t) = Ax(t) + f(x_t) \tag{54}$$

where A generates a C_0-semigroup in a Banach space, X, and f is a continuous mapping from the Banach $C = \{\phi:[-h, 0] \to X \text{ is continuous}\}$. The proofs of existence, uniqueness, continuation of solutions, continuous dependence on data, and parameters, etc. for these systems are similar to the corresponding ones for delay systems in R^n. The major exception to this statement occurs for properties which depend on the compactness of closed bounded sets in R^n or Z^n. These systems are often encountered in models in population ecology, genetic repression, control theory, climatology, coupled oscillators, age dependent populations, etc. (29,30).

TIME DELAYS IN CONTROL SYSTEMS

Time delays are sometimes desired in the design of control systems. For example in self-tuning control, one encounters systems of the form

$$\begin{aligned} y(t) + a_1 y(t-1) a_2 y(t-2) + \cdots a_{na} y(t-na) \\ = b_1 w(t-1) + b_2 u(t-2) + \cdots + b_{ub} u(t-nb) \end{aligned}$$

where the u-terms are the controls. These are known as DARMA (deterministic autoregressive and moving average) systems (31). They have their own methodology which is described in ADAPTIVE CONTROL in this encyclopedia. Our interest here is in systems where unpredicted delays appear in the controls, particularly feedback stabilized controls.

If the control system is finite-dimensional of the type

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{55}$$

and is stabilized by a feedback of the form

$$u(t) = Rx(t) \tag{56}$$

then a "small" time delay in the control will not destroy the stabilization. Of course, the word small depends on the particular system. However, if system [Eq. (55)] is infinite-dimensional, it may be unable to tolerate any delays, particularly if it is an abstract representation of a boundary stabilized hyperbolic PDE. The simplest example of such a system is given by

$$w_{tt} = w_{xx}, \quad 0 < x < 1, t > 0 \tag{57}$$

$$w(0,t) = 0, \quad w_x(1,t) = u(t) \tag{58}$$

If, in Eq. (58), the control is feedback and given by

$$u(t) = -w_t(1,t) \tag{59}$$

then all of the solutions of the resulting feedback system are identically zero after time $t = 2$. However, if

$$u(t) = -w_t(1, t-h), \quad h > 0 \tag{60}$$

then the system is unstable—so much so that the following result holds (32).

Theorem. Given any $\beta > 0$ there exists $h_n \to 0^+$ as $n \to \infty$ and λ_n in Z, $Re\lambda_n > \beta$ such that the system [Eqs. (57) and (58)] with the feedback

$$u(t) = -w_t(1, t-h_n) \tag{61}$$

has solutions

$$w(x,t) = e^{\lambda_n t} \sinh \lambda_n t \tag{62}$$

Systems of the type [Eqs. (57) and (58)] are often approximated by finite-dimensional oscillatory systems of the form

$$\ddot{x} + Ax = bu \tag{63}$$

where A is a positive definite $n \times n$ matrix, with eigenvalues $0 < \sigma_1^2 < \cdots \sigma_n^2$, and b is an n-vector, which is not an eigenvector of A. Suppose the system [Eq. (63)] is stabilized by a feedback

$$u(t) = c_1^T x(t) + c_2^T \dot{x}(t) \tag{64}$$

This could be accomplished by pole placement, LQR-optimization, or H^∞-optimization.

Theorem. The maximum time delay which the system

$$\ddot{x}(t) + Ax(t) = (c_1^T x(t-h) + c_2^T \dot{x}(t-h))\, b \tag{65}$$

can tolerate and remain u.e.s. is in the interval

$$0 < h < \frac{2\pi}{\sigma_n} \tag{66}$$

This result implies that large-dimensional Galerkin approximations to systems of the type in Eqs. (57) and (58) become unstable for small time delays. Thus, there is a tradeoff between the dimension of the approximation and the tolerance for delays.

The above example illustrates one of the important differences between LTI evolutionary systems whose initial space is infinite dimensional and those for which this space is finite dimensional. It is instructive to make some general remarks about why such a situation might occur. Suppose that τ is a parameter varying in a subset S of a Banach space, and $T_\tau(t)$, $t \geq 0$ is a C^0-semigroup of linear transformations on a Banach space X which is continuous in τ; that is, $T_\tau(t)x$ is continuous in (τ, t, x). Let r_τ be the radius of the spectrum of $T_\tau(1)$. The asymptotic behavior of the semigroup is determined by the spectrum $\sigma(T_\tau(1))$ of $T_\tau(1)$. If $\sigma(T_\tau(1))$ is inside the unit circle in the complex plane, then each orbit $\{T_\tau(t)\phi, t \geq 0\}$ will approach zero exponentially and uniformly. If there is a point in $\sigma(T_\tau(1))$ outside the unit circle, then there is an unbounded orbit, and we have instability. A fundamental role in the study of stability and the preservation of stability under perturbations in the parameter τ is the behavior of the essential spectrum $\sigma_e(T_\tau(1))$ of $T_\tau(1)$. Let $r_{e\tau} \equiv r(\sigma_e(T_\tau(1)))$ denote the radius of the essential spectrum of $T_\tau(1)$. If it is known that $r_{e\tau} < 1$, then the stability or instability of 0 is determined by eigenvalues of $\sigma(T_\tau(1))$. Furthermore, if 0 is unstable, then it is due to only a finite number of eigenvalues; that is, the instability occurs in a finite dimensional subspace of X. If the latter situation occurs, then it is natural to expect that the stabilization of the system could be accomplished by using a finite dimensional control. However, if it is required that the stabilization be insensitive to small changes in the parameter, then this may not be the case. If $r_{e\tau} > 1$, then the instability of the system is of such a nature that it cannot be controlled by a finite dimensional control.

In general, eigenvalues of $T_\tau(1)$ can be chosen to be continuous functions of τ. On the other hand, the function $r_{e\tau}$ may not be continuous in τ. The function $r_{e\tau}$ will be continuous in τ at a point τ_0 if it is known that $T_\tau(1) - T_{\tau_0}(1)$ is compact. This condition is not necessary, but it is sufficient. If this difference is only bounded, then there is the possibility of a large shift in $r_{e\tau}$ if we vary τ. For example, if $r_{e\tau_0} < 1$, and the perturbation is only bounded, it is possible to have $r_{e\tau} > 1$ for a sequence of $\tau_j \to \tau_0$, and the semigroup $T_{\tau_j}(t)$ will be unstable for each j.

Let us interpret these remarks in terms of the examples that we have been discussing above. For finite dimensional problems, the semigroup is compact and thus the asymptotic behavior of orbits is determined by the eigenvalues. Since the semigroup for a LTI RFDE is compact for $t \geq h$, the continuous spectrum always is the point zero, and so the asymptotic behavior is again determined by dominant eigenvalues (finite in number), and these are generally continuous in parameters.

For D-stable NFDE, the essential spectrum lies in the unit circle for all values of the delay. Therefore, the introduction of small delays in the control function does not disturb the stabilization property of the feedback control.

It can happen that the solutions of the difference equation associated with the difference operator D for an NFDE has all solutions approaching zero exponentially and uniformly for a particular value of the delays, and a small change in the delays leads to exponential instability. If this is the case, then the asymptotic behavior of solutions of NFDE subjected to small variations in the delays is not determined by the eigenvalues of the semigroup, but by the essential spectrum, which in turn is determined by the eigenvalues of the difference equations associated to the difference operator D.

In the example [Eqs. (57) and (59)], the natural space of initial data is $H_B^1(0, 1) \times L^2(0, 1)$, where B represents the homogeneous boundary conditions $w = 0$ at $x = 0$, $w_x = 0$ at $x = 1$. In this case, the boundary control $-w_t$ is bounded but not compact. If this control is implemented with a delay, then the radius of the essential spectrum is increased considerably and, in fact, leads to instability.

It is possible to consider a more physical version of Eqs. (57–59), for which the boundary control problem is insensitive to small delays in the time at which it is implemented. Consider the equation

$$w_{tt} - w_{xx} - cw_{xxt} = 0, \quad 0 < x < 1, t > 0 \tag{67}$$

with the boundary conditions

$$w(0,t) = 0, \quad w_x(1,t) + cw_{xt}(1,t) = -kw_t(1, t-h) \tag{68}$$

where $h \geq 0$, $c > 0$, $k > 0$ are constants. In any space for which one can define a C^0-semigroup for Eqs. (67) and (68), the control function is compact. Furthermore, the radius of the essential spectrum is determined by the same problem with $k = 0$ and is given by $e^{-(1/c)} < 1$. Therefore, stability is preserved with small perturbations in the delay.

For further discussion of this topic, see Refs. 32–37.

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DELTA-SIGMA MODULATORS.

See Sigma-delta modulation.