BILINEAR SYSTEMS

An electrical circuit or other engineering system often communicates with its external environment by input signals that control its behavior and output signals; it is then called a control system. If the components of a control system all obey Ohm's Law or one of its analogs, such as Hooke's Law, the system is called linear. In linear control systems, the effect of the controls is additive and the output measurement is linear. They are discussed in the article in this encyclopedia on MULTIVARIABLE SYSTEMS.

What distinguishes a *bilinear system* (BLS) is that although it is linear in its state variables, some control signal *u(t)* exerts its effect multiplicatively. BLS may be given as mathematical models of circuits or plants or may be chosen by a designer to obtain better system response than is possible with a linear system. Their study is a first step toward nonlinear control theory. Industrial process control, economics, and biology provide examples of BLS with multiplicative controls such as valve settings, interest rates, and neural signals, respectively. This topic of research began in the early 1960s with independent work in the USSR and in the USA; see the surveys of Bruni et al. (1) and Mohler (2, 3) for historical development and reviews of the early literature.

Notation in This Article. The symbol $\mathbf{R}$ means the real numbers and $\mathbf{R}^n$ the n-dimensional real linear space; $\mathbf{C}$ means the complex plane, with $\Re(s)$ the real part of $s \in \mathbf{C}$. The bold symbols $\mathbf{a}$–$\mathbf{h}$, $\mathbf{x}$, $\mathbf{z}$ will represent elements (column vectors) of $\mathbf{R}^n$; $\dot{\mathbf{x}}'$ ($\mathbf{x}$ transposed) is a row vector; $\mathbf{x}^*$. is the complex conjugate transpose of $\mathbf{x}$. Given a vector function $\mathbf{x}(t) = \mathrm{col}[x_1(t), \ldots, x_n(t)]$, its time derivative is $\dot{\mathbf{x}}(t) \overset{\mathrm{def}}{=} d\mathbf{x}(t)/dt$. Capital letters A, B, F, X are square matrices and I the identity matrix diag $(1, 1, \ldots, 1)$. The trace of matrix A is the sum of its diagonal elements, written $\mathrm{tr}(A)$; $\det(A)$ is its determinant. i–n are integers; r, s, t are real scalars, as are lowercase Greek letter quantities. German type $\mathfrak{g}$, $\mathfrak{sl}$, ... will be used for Lie algebras.

Control Systems: Facts and Terminology

The following discussion is a brief reminder of state-space methods; see Sontag (4). The state variables in an electrical circuit are currents through inductors and voltages across capacitors; in mechanics, they are generalized positions and momenta; and in chemistry, they are concentrations of molecules. The state variables for a given plant constitute a vector function depending on time $\mathbf{x}(t)$. Knowledge of an initial state $\mathbf{x}(0)$, of future external inputs, and the first-order vector differential equation that describes the plant determine the trajectory $\{\mathbf{x}(t), t \geq 0\}$. For the moment, we will suppose that the plant elements, such as capacitors, inductors, and resistors, are linear (Ohm's Law) and constant in value. The circuit equations can usually be combined into a single first-order vector differential equation, with interaction matrix F and a linear output function. Here is a single-input single-output example, in which the coefficient vector g describes the control transducer, the output transducer is described by the row vector h', and $v = v(t)$ is a control signal:

$$\dot{\mathbf{x}} = F\mathbf{x} + vg, \quad y = \mathbf{h}'\mathbf{x} \tag{1}$$

or written out in full,

$$\frac{dx_i}{dt} = \sum_{j=1}^{n}(F_{ij}x_j) + vg_i, \quad i = 1, \ldots, n; \quad y = \sum_{i=1}^{n} h_i x_i \tag{1'}$$

(It is customary to suppress in the notation the time-dependence of $\mathbf{x}, y$ and often the control v.)

As written, equation 1 has constant coefficients, and such a control system is called *time-invariant*, which means that its behavior does not depend on where we choose the origin of the independent variable t; the system can be initialized and control v exerted at any time. When the coefficients are not constant, that is, made explicit in the notation, e.g.,

$$\dot{\mathbf{x}} = F(t)\mathbf{x} + v\mathbf{g}(t), \quad y = \mathbf{h}'(t)\mathbf{x}$$

which is called a *time-variant* linear system.

For both linear and bilinear systems, we will need the solution of $\dot{\mathbf{x}} = F\mathbf{x}$, which for a given initial condition $\mathbf{x}(0)$ is $\mathbf{x}(t) = e^{Ft}\mathbf{x}(0)$. The matrix exponential function is defined by

$$e^{Ft} = I + Ft + \frac{F^2t^2}{2} + \ldots + \frac{F^kt^k}{k!} + \ldots \tag{2}$$

it is the inverse Laplace transform of the matrix $(sI - F)^{-1}$ and can be computed by numerical methods described by Golub and Van Loan (5). Its most familiar use in electrical engineering is to solve equation 1

$$\mathbf{x}(t) = e^{Ft}\mathbf{x}(0) + \int_0^t e^{(t-r)F}v(r)g\,dr$$

The polynomial

$$\mathcal{P}_F(s) \overset{\mathrm{def}}{=} \det(sI - F) = (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n)$$

is called the *characteristic polynomial* of F, and its roots are called the *eigenvalues* of F. The entries in the matrix e^{Ft} are linear combinations of terms like

$$t^{m_i}e^{\lambda_i t}, \quad i = 1, \ldots, n$$

If the eigenvalues are distinct, the integers m_i vanish, but if λ_i is a multiple eigenvalue, m_i may be positive. For a given F, the matrices $\{\exp(Ft), t \in \mathbf{R}\}$ are a group under matrix multiplication: $\exp((r+t)F) = \exp(rF)\exp(tF)$, $(\exp(tF))^{-1} = \exp(-tF)$.

For different applications, different restrictions may be placed on the class $\mathcal{U}$ of *admissible control signals*. In this article, $\mathcal{U}$ will usually be the class of *piecewise constant* (PWC) signals. The value of a control at an instant of transition between pieces need not be defined; it makes no difference. If there is a single number $\mu > 0$ that is an upper bound for all admissible controls, call the class $\mathcal{U}_\mu$ as a reminder.

A control system, linear or not, is said to be *controllable* on its state space if for any two states $\bar{\mathbf{x}}$, $\hat{\mathbf{x}}$, a time $T > 0$ exists and a control in $\mathcal{U}$ for which the trajectory starting at $\mathbf{x}(0) = |\bar{\mathbf{x}}$ ends at $\mathbf{x}(T) = \hat{\mathbf{x}}$. For time-invariant linear systems, there is a simple and useful condition necessary for controllability. Stated for equation 1, this *Kalman rank condition* is

$$\text{rank}(g, Fg, \ldots, F^{n-1}g) = n$$

If it is satisfied, the matrix-vector pair $\{F, g\}$ is called a *controllable pair*. If the controls are not bounded, this condition is sufficient for controllability, but if they are in some $\mathcal{U}_\mu$, the control may be small compared with $F\mathbf{x}$ for large $\mathbf{x}$ and have an insufficient effect.

BILINEAR SYSTEMS: WHAT, WHY, WHERE?

Precisely what are BLS and why should one use bilinear systems in control engineering? This section will give some answers to those questions, starting with a formal definition, and give some examples.

Definition. A bilinear system is a first-order vector differential equation $\dot{\mathbf{x}} = p(\mathbf{x}, u)$ in which $\mathbf{x}$ is a state vector; u is a control signal (scalar or vector); and the components of $p(\mathbf{x}, u)$ are polynomials in $(\mathbf{x}, u)$ that are linear in $\mathbf{x}$, u separately, but jointly quadratic, with constant real coefficients. Restating that, for any real numbers α, β

$$p(\alpha\mathbf{x}, \beta u) \equiv \alpha\beta\, p(\mathbf{x}, u) + \alpha\, p(\mathbf{x}, 0) + \beta\, p(0, u)$$

The (optional) output is linear, $y = \mathbf{h}'\mathbf{x}$.

To begin with, BLS are simpler and better understood than most other nonlinear systems. Their study involves a constant interplay between two profitable viewpoints: looking at BLS as time-invariant nonlinear control systems and as time-variant linear systems.

Another answer is that BLS are useful in designing control systems that use a very small control signal to modulate a large current of electricity of fluid, apply brakes, or change rates of growth.

A third answer is that the usual *linearization* of a nonlinear control system near an equilibrium point can be improved by using a BLS approximation; thus,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{a}(\mathbf{x}) + u\mathbf{g}(\mathbf{x}), \text{ with } \mathbf{a}(\mathbf{x}_e) = 0; \text{ let} \\ A &= \frac{\partial \mathbf{a}}{\partial \mathbf{x}}\,|_{\mathbf{x}=\mathbf{x}_e,}\ \mathbf{b} = q(\mathbf{x}_e),\ B = \frac{\partial \mathbf{q}}{\partial \mathbf{x}}\,|_{\mathbf{x}=\mathbf{x}_e} \end{aligned}$$

Translating the origin so that $\mathbf{x}_e = \mathbf{0}$, to first order in $\mathbf{x}$ and u separately:

$$\dot{\mathbf{x}} = A\mathbf{x} + u(B\mathbf{x} + \mathbf{b}),\ \ y = \mathbf{h}'\mathbf{x} \tag{3}$$

Although some results will be stated for equation 3, usually we will suppose that $\mathbf{b} = \mathbf{0}$; such BLS are called *homogeneous bilinear systems*, and for later reference, they are given here in both their single-input and k-input versions

$$\dot{\mathbf{x}} = A\mathbf{x} + uB\mathbf{x} \tag{4}$$

$$\dot{\mathbf{x}} = A\mathbf{x} + \sum_{j=1}^{k} u_j B_j \mathbf{x} \tag{5}$$

(We will see later how to recover facts about equation 3 from the homogeneous case.) If $A = 0$ in equations 4 or 5, the BLS is called *symmetric*:

$$\dot{x} = \sum_{j=1}^{k} B_j x \tag{5'}$$

As a control system, equation 4 is time-invariant, and we need to use controls that can start and stop when we wish. The use of PWC controls not only is appropriate for that reason but also allows us to consider switched linear systems as BLS, and in that case, only a discrete set of control values such as $\{-1, 1\}$ or $\{0, 1\}$ is used.

Later we will be concerned with state-dependent feedback controls, $u = u(\mathbf{x})$, which may have to satisfy conditions that guarantee that differential equations like $\dot{\mathbf{x}} = A\mathbf{x} + u(\mathbf{x})B\mathbf{x}$ have well-behaved solutions.

Discrete time bilinear systems (DBLS) are described by difference equations, rather than ordinary differential equations. DBLS applications have come from the discrete-time dynamics common in economic and financial modeling, in which the control is often an interest rate. DBLS also are used for digital computer simulation of continuous time systems like equation 3: Using Euler's point-slope method with a time-step τ, for times $k = 0, 1, 2, \ldots$, the discrete-time system is

$$\mathbf{x}(k+1) = (I + \tau A)\mathbf{x}(k) + \tau u(k)(B\mathbf{x}(k) + \mathbf{b}) \tag{6}$$

with output $y(k) = \mathbf{h}'\mathbf{x}(k)$. DBLS will be discussed briefly at appropriate places below. Their solutions are obtained by recursion from their initial conditions using their difference equations.

Some Application Areas

Reference (2) lists early applications of BLS to nuclear reactors, immunological systems, population growth, and compartmental models in physiology. For a recently analyzed BLS from a controlled compartmental model, see the work on cancer chemotherapy by Ladzewicz and Schättler (6) and its references.

Using linear feedback $u = K\mathbf{x}$ in a BLS results in a quadratic autonomous system. Recently some scientifically interesting quadratic systems, exhibiting chaotic behavior, have been studied by decomposing them into BLS of this type. Čelikovský and Vaněček (7) have studied the third-order Lorenz system as a BLS with output feedback:

$$\dot{\mathbf{x}} = A\mathbf{x} + uB\mathbf{x},\ \ u(\mathbf{x}) = x_1,\ \text{ with } \sigma > 0,\ \rho > 0,\ \beta > 0,\ \text{ and}$$

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix},\quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

For small ρ all eigenvalues of A are negative, but for $\rho > 1$ one becomes positive and B generates a rotation. This description seems to be characteristic of several such examples of strange attractors and bounded chaos, including the Rössler attractor.

In electrical engineering, BLS viewpoints can be used to discuss switched and clocked circuits, in which the control takes on only a discrete set of values like $\{0, 1\}$ and the plant is linear in each switched condition. The solutions are then easy to compute numerically. Sometimes the duty cycle of a switch is under control, as in motors and DC-to-DC power conversion systems. A simple example is a conventional ignition sparking system for automobiles, in which the primary circuit can be modeled by assigning x_1 to voltage across capacitor of value C, x_2 to current in the primary coil of inductance L. The control is a distributor or electronic switch, either open (infinite resistance) or closed (small resistance R) with duty cycle specified by the crankshaft rotation and timing controls. Then with a battery of emf V,

$$\dot{x}_1 = -\frac{1}{C}x_2 - \frac{u}{C}(x_1 - V), \quad \dot{x}_2 = -\frac{1}{L}x_1, \quad u = \begin{cases} 1/R, & \textit{closed} \\ 0, & \textit{open} \end{cases}$$

Other automotive BLS include mechanical brakes and controlled suspension systems, among the many applications discussed in Reference 3.

One advantage of piecewise constant control in BLS is that the solutions, being piecewise linear, are readily computed. For that reason, in aerospace and process engineering, control designs with gain scheduling (see GAIN SCHEDULING) are an area where BLS methods should be useful; such control schemes change the F matrix and equilibrium point to permit locally linear control. Recent work on *hybrid systems* (finite-state machines interacting with continuous plants) also falls into the category of switched linear systems.

STABILIZATION I: CONSTANT CONTROLS

This section will introduce an important engineering design goal, stability, and the beginning of a running discussion of stabilization. Stabilization is an active area of research, in an effort to find good design principles.

A matrix F is called a *Hurwitz* matrix if all n of the eigenvalues of F lie in the left half of the complex plane; i.e., $\Re(\lambda_i) < -\epsilon < 0$. Then $\dot{\mathbf{x}} = F\mathbf{x}$ is said to be *exponentially stable* (ES); as time increases, all solutions are bounded and $||\mathbf{x}(t)|| < ||\mathbf{x}(0)||e^{-t\epsilon}$. If even one eigenvalue lies in the right half plane, almost all solutions will grow unboundedly and the system is called *unstable*. Multiple imaginary eigenvalues $\lambda = j\omega$ can give $t^m\cos(\omega t)$ (resonance) terms that also are unstable. Warning: Even if the time-varying eigenvalues of a time-variant linear differential equation all lie in the left half plane, that does not guarantee stability!

If A is a Hurwitz matrix, equation 4 is ES when $u = 0$. Suppose that A is not Hurwitz or that ϵ is too small; finding a feedback control u such that equation 4 is ES with a desirable ϵ is called *stabilization*.

The problem of stabilization of BLS and other nonlinear control systems is still an active area of engineering research. In this section, we consider only the use of constant controls $u = \mu$ in equation 4; the result of applying this feedback is a linear dynamical system $\dot{\mathbf{x}} = (A + \mu B)\mathbf{x}$. To find a range of values for μ that will stabilize, the BLS is somewhat difficult, but for small n one can find $\mathcal{P}_{A+\mu B}(\lambda)$ and test possible values of μ by the Routh–Hurwitz stability test for polynomials (see STABILITY THEORY, ASYMPTOTIC). For $n = 2$

$$\mathcal{P}_{A+\mu B}(\lambda) = \lambda^2 - (\mathrm{tr}(A) + \mu\,\mathrm{tr}(B))\lambda + \det(A + \mu B), \text{ so} \quad \mathrm{tr}(A) + \mu\,\mathrm{tr}(B) < 0 \text{ and } det(A + \mu B) > 0 \tag{7}$$

guarantee stability. Graphing these two expressions against μ is an appropriate method for finding good values of μ. A complete set of conditions A and B that are necessary and sufficient for stabilizability of second-order BLS with constant feedback were given by Chabour et al. (8).

Other criteria for stabilization by constant control have been found, such as this one from Luesink and Nijmeijer (9). Suppose the eigenvalues of A are λ_i, $i = 1,\ldots, n$, and the eigenvalues of B are $\hat{\lambda}_i$. If there is some nonsingular matrix P, real or complex, for which $P^{-1}AP$ and $P^{-1}BP$ are simultaneously upper triangular, then the eigenvalues of $A + \mu B$ are $\lambda_i + \mu\hat{\lambda}_i$, $i = 1, \ldots, n$. If some real μ satisfies the n linear inequalities $\Re(\lambda_i + \mu\hat{\lambda}_i) < 0$, it will be the desired constant control. For more about such triangularizable BLS, see the section below on "The Lie Algebra of a BLS."

SOLUTIONS OF BILINEAR SYSTEMS

From one viewpoint a BLS with a specific nonconstant control history $\{u(t), t \geq 0\}$ should be thought of as a time-variant linear differential equation. We will use the single-input case of equation 4 as an example, with $A + u(t)B$ as time-variant matrix. The solution depends on the initial time t_0 at which the state is $\mathbf{x}(t_0)$, and is of the form $\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0)$, where $\Phi(t, t_0)$ is called a *transition matrix*. For more about the general theory of these matrices, see MULTIVARIABLE SYSTEMS, Chap. 9 of Kailath 10, or vol. 1, Chap. 4 of Reference 3. Once having written that expression for $\mathbf{x}(t)$, it can be observed that Φ must satisfy the matrix differential equation

$$\dot{\Phi} = (A + u(t)B)\Phi, \quad \Phi(t_0, t_0) = I$$

It has the composition property, also called the semigroup property $\Phi(t, t_1)\Phi(t_1, t_0) = \Phi(t, t_0)$.

However, as a *control system*, equation 4 is time-invariant, by definition. Then the most convenient families of admissible controls for BLS are the PWC and other piecewise-defined controls; such a control can be specified by its values on an interval of definition of duration τ, for instance $\{u(t), t \in (t_0, t_0 + \tau)\}$. From the time-invariance of the system, a basic interval of definition, $(0,\tau)$, can be used without any loss. Given a particular control signal u on $(0,\tau)$, its *time shift* by σ can be denoted $u^{\sigma}(t) = u(t - \sigma)$, on $(\sigma, \tau + \sigma)$, as is usual in system theory.

The *concatenation* of two controls u and v with respective durations τ_1 and τ_2 is written $u \; o \; v$ and is another admissible control with duration $\tau_1 + \tau_2$:

$$(u \circ v)(t) = \begin{cases} u(t), & t \in [0, \tau_1) \\ v^{\tau_1}(t), & t \in [\tau_1 < t \leq \tau_2] \end{cases}$$

For the general *multi-input* BLS of equation 5, the control is a k-component vector $\mathbf{u} = [u_1, \ldots, u_k]$, and the transition matrix satisfies $\dot{\Phi} = (A + \sum_{j=1}^{k} u_j B_j)\Phi$. Concatenation is defined in the same way as for scalar controls: $\mathbf{u} \circ \mathbf{v}$ is $\mathbf{u}$ followed by the translate of $\mathbf{v}$.

The time-invariance of the BLS leads to useful properties of the transition matrices. The transition matrix depends on the control $\mathbf{u}$ and its starting time, so the matrix Φ should be labeled accordingly as $\Phi(\mathbf{u}; t, t_0)$, and the state trajectory corresponding to $\mathbf{u}$ is

$$\mathbf{x}(t) = \Phi(\mathbf{u}; t, t_0)\mathbf{x}(0)$$

Given two controls and their basic intervals $\{\mathbf{u}, 0 < t < \sigma\}$ and $\{\mathbf{v}, 0 < t < \tau\}$, the composition property for BLS transition matrices can be written in a nice form that illustrates concatenation ($\mathbf{u}$ followed by the translate of $\mathbf{v}$)

$$\Phi(\mathbf{v}^{\sigma}; \tau, \sigma)\Phi(\mathbf{u}; \sigma, 0) = \Phi(\mathbf{u} \circ \mathbf{v}; \tau, 0) \tag{8a}$$

A transition matrix always has an inverse, but it is not always a transition matrix for the BLS. However, if the BLS is symmetric (5') and the admissible controls $\mathcal{U}$ are sign-symmetric (i.e., if $\mathbf{u} \in \mathcal{U}$, then $-\mathbf{u} \in \mathcal{U}$), the transition matrix $\Phi(\mathbf{u}; \tau, 0)$ resulting from control history $\{\mathbf{u}(t), 0 \le t \le \tau\}$ has an inverse that is again a transition matrix, obtained by using the control that reverses what has been done before, $\mathbf{u}^*_{\tau}(t) = -\mathbf{u}(\tau - t),\ \tau \le t \le 2\tau$;

$$\Phi(\mathbf{u}^*_{\tau}; 2\tau, \tau)\Phi(\mathbf{u}; \tau, 0) = I \tag{8b}$$

An asymmetric BLS, $\dot{\mathbf{x}} = (A + u_1 B + \cdots + u_k B_k)\mathbf{x}$, is like a symmetric one in which one of the matrices $B_0 = A$ has a constant "control" $u_0 \equiv 1$, whose sign cannot be changed. The controllability problem for asymmetric BLS involves finding ways around this obstacle by getting to I some other way.

From a mathematical viewpoint, the set of transition matrices for equation 4 is a *matrix semigroup* with identity element I. See the last section of this article.

MORE ABOUT TRANSITION MATRICES

Transition matrices for BLS have some additional properties worth mentioning. For instance, in the rare situation that $A, B_1, \ldots, B_k$ all commute, the transition matrix has a comforting formula

$$\Phi(\mathbf{u}; \tau, 0) = e^{At + \int_0^t \sum_{i=1}^k u_i(s) B_i ds}$$

Warning: If the matrices do not commute, this formula is invalid!

The solution of a single-input inhomogeneous BLS like equation 3, $\dot{\mathbf{x}} = A\mathbf{x} + u(B\mathbf{x} + \mathbf{b})$, is much like the solution of a linear system. If $\Phi(u; t, 0)$ is the solution of the homogeneous matrix system

$$\dot{\Phi} = A\Phi + uB\Phi, \quad \Phi(0) = I$$

then for equation 3 with initial condition $\mathbf{x}(0)$,

$$\mathbf{x}(t) = \Phi(u; t, 0)\mathbf{x}(0) + \int_0^t \Phi(u; t, s)u(s)b ds$$

One advantage of using piecewise constant controls is that they not only approximate other signals, but suggest a construction of the transition matrix. For a PWC control u given by m constant pieces $\{u(t) = u(\tau_{k-1}), \tau_{k-1} \le t < \tau_k\}$ on intervals that partition $\{0 \le t < \tau_m = T\}$, the transition matrix for $\dot{X} = (A + uB)X$ is clearly

$$\Phi(u; T, 0) = \prod_{k=1}^{m} e^{(A + u(\tau_{k-1})B)\tau_k} \tag{9}$$

This idea can be carried much further with more analysis: More general (measurable) inputs can be approximated by PWC controls, and in analogy to the definition of an integral as a limit of sums, the solution to equation 4 for measurable inputs can be written as (in an appropriate sense) the limit of products like equation 9, called a product-integral.

The representation given by equation 9 generalizes to the multi-input BLS equation 5 in the obvious way. With equation 9, one can also easily verify the composition and (for $A = 0$) inverse properties. To emphasize that exponential formulas for noncommuting matrices have surprising behavior, here is a standard example in which you should notice that $A^2 = 0$ and $B^2 = 0$.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};\ AB - BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

$$e^{At}e^{Bt} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1 + t^2 & 1 \\ t & 1 \end{pmatrix}, \text{ but}$$

$$e^{(A+B)t} = \exp\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

OBSERVABILITY AND OBSERVERS

This section is concerned with BLS that have an output measurement y with $m < n$ components, so that $x(t)$ is not directly available. For many purposes in control (stabilization, prediction of future outputs, and optimization), it is necessary 1) to ensure that different states can be distinguished and 2) to obtain estimates of $\mathbf{x}(t)$ from available information. An important question is whether an *input–output history*

$$\mathcal{H}_T = \{u(t), y(t) | 0 \le t \le T\}$$

will *uniquely* determine the initial or final state.

Let C be an $m \times n$ matrix (think of it as m row vectors); its null-space $\{\mathbf{x} | C\mathbf{x} = 0\}$ is denoted by $C^{\perp}$. The BLS is the m-output system given by

$$\dot{\mathbf{x}} = (A + uB)\mathbf{x},\ y = C\mathbf{x}. \tag{10}$$

The initial state is not known, only the history $\mathcal{H}_T$. Suppose that $u(t),\ t \ge 0$, is given. Call two states $\bar{\mathbf{x}}, \hat{\mathbf{x}} \in \mathbf{R}^n$ *u-indistinguishable* on the interval $(0, T)$ if the two corresponding outputs are equal on that interval, i.e., if $C\Phi(u; t, 0)\bar{\mathbf{x}} = C\Phi(u; t, 0)\hat{\mathbf{x}},\ 0 \le t \le T$. This relation, written $\bar{\mathbf{x}} \sim_u \hat{\mathbf{x}}$, is transitive, reflexive, and symmetric (an equivalence relation) so it partitions the state space into disjoint sets (see Chap. 5 of Reference 4); it is also linear in the state. Therefore we need only be concerned with the set of states u-indistinguishable from the origin, namely,

$$\mathcal{I}_u \stackrel{\text{def}}{=} \{\mathbf{x} | \mathbf{x} \sim_u 0\} = \{\mathbf{x} | C\Phi(u; t, 0)\mathbf{x} = 0,\ 0 \le t \le T\}$$

which is a linear subspace called the u-unobservable subspace of the BLS; the u-observable subspace is the quotient space $\mathcal{O}_u = \mathbf{R}^n/\mathcal{I}_u$. That can be rephrased as $\mathbf{R}^n = \mathcal{I}_u \oplus \mathcal{O}_u$. If $\mathcal{I}_u = 0$, we say that the given system is u-observable. In Grasselli and Isidori (11) u-observability is given the name observability under single experiment.

If two states $\mathbf{x}$, $\mathbf{z}$ are u-indistinguishable for *all* admissible u, we say they are *indistinguishable* and write $\mathbf{x} \sim \mathbf{z}$. The set of unobservable states is the linear subspace

$$\mathcal{I} = \{\mathbf{x} | C\Phi(u; t, 0)\mathbf{x} = 0 \text{ for all } u \in U\}$$

Its quotient subspace (complement) $\mathcal{O} = \mathbf{R}^n/\mathcal{I}$ is called the observable subspace, and the system is called *observable* if $\mathcal{I} = 0$. The unobservable subspace is invariant for the BLS; trajectories that begin in $\mathcal{I}$ remain there. If the largest invariant linear subspace of $C^\perp$ is 0, the BLS is *observable*; this is also called observability under multiple experiments, because to test it one would have to have duplicate systems, each initialized at $\mathbf{x}(0)$ but using its own control u.

Theorem 1 of Reference 11 states that for PWC controls and piecewise continuous controls, the BLS is observable if and only if a u exists for which it is u-observable. The proof constructs a universal input $\tilde{u}$ that distinguishes all states from 0 by concatenating at most $n + 1$ inputs: $\tilde{u} = u^0 o \cdots o\, u^n$. At the kth stage in the construction, the set of states indistinguishable from 0 is reduced in dimension by a well-chosen u^k. The test for observability that comes out of this analysis is that the rank of the matrix $\Gamma(C; A, B)$ is n, where

$$\Gamma(C; A, B) = \text{col}[C, CA, CB, CA^2, CAB, CBA, CB^2, \ldots]$$

That is, $\Gamma(C; A, B)$ contains C and all matrices obtained by repeated multiplications on the right by A and B. This is the first theorem on the existence of universal inputs, and the idea has been extended to other nonlinear systems by Sontag and Sussmann.

The simplest situation in which to look for observability criteria is for a system with input zero, an autonomous time-invariant linear system $\dot{\mathbf{x}} = A\mathbf{x}$, $y = C\mathbf{x}$. It is no surprise that the Kalman rank criterion for observability is appropriate for such systems. The (time-invariant) observability Gramian is $W = \text{col}[C, CA, \ldots, C(A)^{n-1}]$; we say $\{C, A\}$ is an *observable pair* if rank$(W) = n$, and that is both necessary and sufficient for linear system observability.

How can we extend this to the case where the input is unknown? To derive the answer, from williamson (12), choose our admissible controls to be polynomials in t of degree n on any fixed time interval. Assume $\mathbf{x}(0) \neq 0$. It is still necessary that rank$(W) = n$, to preserve observability when $u = 0$. Repeatedly differentiate $y = C\mathbf{x}$ at $t = 0$ (not that one would do this in practice) to obtain $Y \stackrel{\text{def}}{=} \text{col}\{y_0, \dot{y}_0, \ldots, y_0^{(n-1)}\}$. If $\dot{y}_0 = CA\mathbf{x}(0) + u(0)CB\mathbf{x}(0) = 0$ for some $u(0)$, the information from $\dot{y}_0$ would be lost; this gives a necessary condition that $CB = 0$; continuing this way, necessarily $CAB = 0$, and so on. All the necessary conditions for observability can be summarized as

$$\text{rank}(W) = n \text{ and } c'A^kB = 0,\ 0 \leq k \leq n-2 \tag{11}$$

To show the sufficiency of these conditions for observability, just note that no matter what control u is used, it and its derivatives do not appear in any of the output derivatives, so $Y = W\mathbf{x}(0)$ and $\mathbf{x}(0) = W^{-1}Y$ from the rank condition.

State Observers

Given a u-observable system with A, B known, it is possible to estimate the initial state (or current state) from the history $\mathcal{H}_T$. The theory of time-variant linear systems (see Reference 10 or vol. 1 of Reference 3) shows that $\mathcal{O}_u$ is the range of the time-variant version of the observability Gramian,

$$W_T \stackrel{\text{def}}{=} \int_0^T \Phi'(u; t, 0)C'C\Phi(u; t, 0)$$

If rank$(W_T) = n$, the initial state can be recovered; in our notation

$$\mathbf{x}(0) = W_T^{-1}(T) \int_0^T \Phi'(u; t, 0)C'y(t)dt$$

The current state can be obtained from $\Phi(u; t, 0)\mathbf{x}(0)$ or by more efficient means. Even though observability may fail for any constant control, it still may be possible, using some piecewise constant control u, to achieve u-observability. Frelek and Elliott (13) pointed out that the Gramian could be optimized in various ways (e.g., by minimizing its condition number) with PWC controls, using a finite sequence of u values, to permit accurate recovery of the entire state or certain preferred state variables. The larger the linear span of the trajectory, the more information is acquired about $\mathbf{x}(0)$.

One recursive estimator of the current state is an asymptotic state observer. There are many variations on this idea; see KALMAN FILTERS AND OBSERVERS. A state observer can be regarded as a simplification of the Kalman filter in which no assumptions about noise statistics are made, nor is a Riccati equation used, and so can be extended to nonlinear systems for which no Kalman filter can be found.

The asymptotic state observer to be described is sufficiently general for BLS. For a given control system, it is a model of the plant to be observed, with state vector denoted by z and an input proportional to the output error. (Grasselli and Isidori (14) showed that there is nothing to be gained by more general ways of introducing an error term.) To show how this works, we generalize slightly to allow an inhomoge-neous BLS. Here are the plant, observer, and error equations; K is an $n \times m$ gain matrix at our disposal.

$$\dot{\mathbf{x}} = A\mathbf{x} + u(B\mathbf{x} + \mathbf{b}),\ \ y = C\mathbf{x} \tag{12a}$$

$$\dot{z} = Az + u(Bz + b) + uK(y - Cz);\ \text{let } e = \mathbf{z} - \mathbf{x} \tag{12b}$$

$$\dot{\mathbf{e}} = (A + uB - uKC)\mathbf{e} \tag{12c}$$

Observer design for linear systems is concerned with finding K for which the observer is *convergent*, meaning that $\|\mathbf{e}(t)\| \to 0$, under some assumptions about u. (Observability is more than what is needed for the error to die out; a weaker concept, detectability, will do. Roughly put, a system is detectable if what cannot be observed is asymptotically stable.) At least three design problems can be posed for equation 12.

1. Design an observer that will converge for all choices of (unbounded) u in $\mathcal{U}$. This requires only the conditions of equation 11, replacing B with $B - KC$, but then the convergence depends only on the eigenvalues of A; using the input has gained us no information.
2. Assume that u is known and fixed; in which case, the methods of finding K for observers of time-variant linear systems are employed, such as Riccati-type equations. There is no advantage to the BLS form in this problem.
3. Design a control and simultaneously choose K to get best convergence. Currently, this is a difficult nonlinear programming problem, although Sen (15) shows that a random choice of the values of PWC u should suffice; from the invariance of the problem under dilation (zooming in toward the origin), it seems likely that a periodic control would be a good choice. This observer design problem is much like an identification problem for a linear system, but identification algorithms are typically not bilinear.

Using digital computer control, one is likely to have not a continuous history $\mathcal{H}$ but the history of a PWC input u_d and a sampled output

$$\mathcal{H}_d = \{(u_d(t_1), y(t_1)), (u_d(t_2), y(t_2)), \ldots\}$$

and the BLS can be dealt with as a discrete-time BLS. Using only a, finite sequence of $N > n$ values, the initial state z [or current state $\mathbf{x}(t_N)$] can be estimated by the least-squares method, which at best projects z onto the closest $\hat{z}$ in the observable subspace:

$$\hat{z} = \arg\min_z \sum_{k=1}^{N} \| y(t_k) - C\Phi(u_d; t_k, 0)z\|^2$$

CONSEQUENCES OF NONCOMMUTIVITY

The noncommutativity of the coefficient matrices A, B of a BLS is crucial to its controllability, raising questions that suggest for their answers some interesting mathematical tools: Lie algebras and Lie groups, named for Norwegian mathematician Sophus Lie, pronounced "lee." See the Reading List at the end of this article for books about them.

Lie Brackets

If A and B do *not* commute, then solving equation 4 is more interesting and difficult. In addition to A, B, we will need $AB - BA$, which is written $[A, B]$ and is called the *Lie bracket* of A and B. To see how this matrix might come into the picture, and obtain a geometric interpretation of the Lie bracket, consider the two-input BLS with piecewise constant controls

$$\dot{\mathbf{x}} = uA\mathbf{x} + vB\mathbf{x};\ u, v \in \{-1, 0, 1\} \tag{13a}$$

Starting at any $x(0)$, use a control with four control segments (u, v), each of small duration $\tau > 0$:

$$\{1, 0\},\ \{0, 1\},\ \{-1, 0\},\ \{0, -1\} \text{ on intervals} \tag{13b}$$

$$\{0, \tau\},\ \{\tau, 2\tau\},\ \{2\tau, 3\tau\},\ \{3\tau, 4\tau\},\ \text{respectively} \tag{13c}$$

The final state is (using the Taylor series of Eq. (7) for the exponential and keeping only terms up to degree 2)

$$\begin{aligned}\mathbf{x}(4\tau) &= 7e^{-B\tau}e^{-A\tau}e^{B\tau}e^{A\tau}\mathbf{x}(0)\\ &= (I + \tau^2[A, B] + \tau^3(\text{higher order terms}) + \cdots)\mathbf{x}(0)\end{aligned} \tag{13d}$$

The Lie Algebra of a BLS

In much the same way that controllability of the linear system equation 1 is related to the linear span of $\{g, Fg, \cdots, F^{n-1}g\}$, controllability of BLS is related to a linear space of matrices generated from A and B by repeated bracketing, called the *Lie algebra* of the BLS. A survey of Lie algebra facts is given in Belinfante and Kolman (16).

The primary concern of this subsection is homogeneous BLS of the types given in equations 4 and 5. To make clear what is being discussed, we need a definition.

Definition. A linear space g over a field $\mathbf{K}$ (usually the real or complex numbers) with a multiplication $g \times g \to g$: $\{X, Y\} \to [X, Y] \in g$ will be called a *Lie algebra* if it satisfies the properties

1. $[X, \alpha Y] = \alpha[X, Y] = [\alpha X, Y], \alpha \in K$.
2. $[X, Y] + [X, X] = 0$.
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity).

In mathematical writing, Lie algebras are usually given abstractly, by relations among their elements, and only then does one represent the elements by matrices acting on some vector space. In contrast, the Lie algebras of control theory have specified generators. (In this article, the Lie algebras are all matrix Lie algebras; for nonlinear vector field Lie algebras, see CONTROLLABILITY AND OBSERVABILITY.)

The Lie algebra generated by a BLS is constructed as follows. Let $\mathcal{M}^n$ denote the linear space of all $n \times n$ real matrices; its dimension is r^2. A real linear subspace $\mathfrak{g} \subseteq \mathcal{M}^n$ that is closed under the bracket $[X, Y] = XY - YX$ is called a (real) *matrix Lie algebra*. The space $\mathcal{M}^n$ itself can be identified with a Lie algebra of dimension n^2 over $\mathbf{R}$ called the general linear algebra $\mathfrak{gl}(n, \mathbf{R})$.

Two matrix Lie algebras $\mathfrak{g}$, $\hat{\mathfrak{g}}$ are said to be *equivalent* if their elements are related by a common similarity transformation $\hat{\mathbf{X}} = P^{-1}XP$. This definition is justified by the identity

$$P^{-1}[A, B]P = [P^{-1}AP, P^{-1}BP]$$

For homogeneous BLS with system matrices A and B as in equation 4, we generate a Lie algebra in the following way, based on the theory of free Lie algebras. $\{A, B\}_{LA}$ is the subspace of $\mathcal{M}^n$ containing A and B and closed under the Lie bracket and linear span operations. It is not hard to compute $\{A, B\}_{LA}$, using the properties 1–3 of Lie brackets. Start with two generators A and B, we write down a tree of brackets as a data structure $\mathcal{T}(A, B)$:

$$\begin{array}{cccc} & B\ A & & \\ & [A, B] & & \\ [A, [A, B]] & & [B, [B, A]] & \\ [A, [A, [A, B]]] & [B, [A, [A, B]]] & [A, [B, [B, A]]] & [B, [B, [B, A]]] \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

The tree's indicated structure depends only on the definition of a Lie algebra; $\mathcal{T}(A, B)$ is built level by level, bracketing each terminal leaf by A and by B; as shown it has already been pruned of obviously linearly dependent leaves by the identities $[X, Y] = -[Y, X]$ and $[X, X] = 0$. By using the Jacobi identity, more leaves can be pruned; e.g., note that $[B, [A, [A, B]]] = -[A, [B, [B, A]]]$ at level 4 before building level 5, and so forth. It is a known feature of Lie algebras that all higher order Lie brackets generated by A and B can be obtained from those in this tree using the Jacobi identity. The linear span of the members of $\mathcal{T}(A, B)$ is the desired Lie algebra $g = \{A, B\}_{LA}$. As our matrices are in $\mathcal{M}^n$ there can be no more than n^2 of them that are linearly independent; working with the specific matrices A, B more dependencies can be found (the Cayley-Hamilton Theorem provides them) so the construction of the tree stops when the matrices at some level are linearly dependent on their ancestors.

In this process, we find enough matrices to obtain the dimension l and a basis $\mathcal{B}$ for the linear space g; we shall write it as an array $\mathcal{B} = \{C_1, \ldots, C_l\}$. As we assume A and B are not linearly dependent, it is convenient to take $C_1 = A$, $C_2 = B$, $C_3 = [A, B]$. If the entries of A and B are rational numbers, symbolic algebra computer programs can generate $\mathcal{T}(A, B)$ and produce $\mathcal{B}$.

It is known that generic pairs A, B will generate a tree of matrices whose span is $\mathcal{M}^n$. In other words, if you fill out the entries with randomly chosen real numbers, then for almost all sample pairs, $\{A, B\}_{LA} = \mathfrak{gl}(n, \mathbf{R})$. However, control systems have structural relationships among their components that may lead to smaller Lie algebras.

For example, here are some properties of generators that are preserved under bracketing and linear combination, and the name of the Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbf{R})$ with the given property and largest dimension d.

- Commutativity: g is called *abelian* and has $d = 2$.
- Zero trace: the *special linear* algebra $\mathfrak{sl}(n, \mathbf{R})$ has $d = n^2 - 1$.
- Skew-symmetry: the *orthogonal* algebra $\mathfrak{so}(n, \mathbf{R})$ has $d = n(n-1)/2$.
- Simultaneously triangularizable over $\mathbf{R}$ or $\mathbf{C}$: *solvable* Lie algebras have $d \leq n(n+1)/2$.

In the stabilizability criterion of Reference 9 discussed above, solvable Lie algebras were mentioned; they are characterized by the property that the sequence of Lie algebras

$$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \mathfrak{f}\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}_1], \ldots, \mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j], \ldots$$

terminates at the trivial Lie algebra $\{0\}$.

For multi-input BLS like equation 5 or symmetric systems (5'), a similar (if harder to diagram) tree construction can be carried out. If the dimension n is low and there are many independent matrices $A, B_1, \ldots, B_k$, only a few brackets may be needed to obtain the basis $\mathcal{B}$ of the Lie algebra.

One of the more useful ideas in computing with Lie brackets is to notice that $[A, X]$ is a linear operation on the matrix X; define $\mathrm{ad}_A(X) \stackrel{\mathrm{def}}{=} [A, X]$ and powers of ad_A recursively: $\mathrm{ad}_A^0(X) = X$, $\mathrm{ad}_A^k(X) = [A, \mathrm{ad}_A^{k-1}(X)]$, $k > 0$. This simplifies the discussion of an important part of the tree $\mathcal{T}(A, B)$ because its leftmost leaves $\mathrm{ad}_A^k(B)$ are used in an important rank condition (the ad-condition, discussed in the "Controllabity Conditions" section below). There are many useful formulas involving ad_A, such as

$$e^{t\,\mathrm{ad}_A}(B) = e^{tA} B e^{-tA}$$

Accessibility and the Lie Rank Condition. What "state space" is most appropriate for a bilinear system? For inhomogeneous systems, $\mathbf{R}^n$ is appropriate. However, for homogeneous BLS, a trajectory starting at 0 can never leave it, and 0 can never be reached in finite time; the state space may as well be punctured at 0. This punctured n-space is denoted by $\mathbf{R}^n \backslash 0$ or $\mathbf{R}_0^n$, and it is of interest in understanding controllability. For $n = 1$, it is the union of two open half-lines; in the scalar BLS $\xi = u\xi$, the state can never change sign. For $n = 2$, the punctured plane is not simply connected. For $n \geq 3$ the puncture has negligible effect.

In some applications, other unusual state spaces may be appropriate. There is an n-dimensional generalization of our scalar example, the diagonalizable BLS, which are (up to similarity) given by $\dot{x}_i = u_i x_i$, $i = 1, \ldots, n$. If the initial state is on a coordinate half-axis, a quadrant of a coordinate plane, ..., or on one of the 2^n orthants, the state must stay there forever. Other BLS that live this way on positive orthants occur rather often, and their controllability properties will be discussed in the "Positive Systems" section below. In economics, chemistry, ecology, and probability applications, the state variables are usually positive and the dynamical model must respect that; bilinear systems and quadratic systems are the simplest models needed in such applications.

Sometimes all we need to establish or can establish will be a property weaker than controllability but still very useful: if the set of states $\{\Phi(u; t, 0)\mathbf{x}, t > 0\}$ has an open interior, the BLS is said to satisfy the *accessibility condition at* $\mathbf{x}$. If that condition is satisfied for all initial states, we say the BLS has the *accessibility property* on its state space. Controllable systems have this property, always, but it is not enough to ensure controllability, as is shown in our next example, which will also motivate the concept of *strong accessibility*, which means that $\{\Phi(u; t, 0)\mathbf{x}\}$ has an open interior at each t.

Example I. On $\mathbf{R}_0^2$ consider a BLS

$$\dot{\mathbf{x}} = (I + uJ)\mathbf{x}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In polar coordinates, this system becomes $\dot{r} = r$, $\dot{\theta} = u$ and its trajectories are, for constant control values, expanding spirals, clockwise or counterclockwise. Starting at r_0, θ_0

at the set of states that can be reached at τ, any fixed positive time, it is a circle with radius $r(\tau) = e^{\tau} r_0$, which is not an open set, so the BLS does not have the strong accessibility property. No state with $r(\tau) < r_0$ can be reached, which establishes that the system is not controllable. The set $\{r(t), \theta(t) | 0 < t < \tau\}$ is the open annulus $\{(r, \theta) | r_0 < r < r_0 e^{\tau}\}$, so the control system does have the accessibility property.

Reverse the roles of the two generators. The BLS becomes $\dot{\mathbf{x}} = (J + uI)\mathbf{x}$; in polar coordinates $\dot{r} = ur$, $\dot{\theta} = 1$. At later time T, the radius is arbitrary but $\theta(T) = 2\pi T + \theta_0$. This system has the accessibility property (but not strong accessibility) and *is controllable*. If the trajectory misses a target state, it takes 2π seconds before it has a second chance. This peculiar state of affairs reflects the fact that our state space is punctured at the origin, and as we remarked before, it is topologically a cylinder.

Sometimes it is easy to see from the structure or known symmetries of generators that a given system cannot have the accessibility property; the most obvious of these, for equation 5, are 1) the dimension of $\{A, B\}_{LA}$ is less than $n^2 - 1$ or 2) A and B are simultaneously triangularizable.

How can we guarantee accessibility on $\mathbf{R}_0^n$? For a given homogeneous BLS, e.g., equation 4 whose Lie algebra is $\mathfrak{g} = \{A, B\}_{LA}$, construct the $n \times l$ matrix $\mathcal{B}\mathbf{x} \stackrel{\text{def}}{=} [C_1\mathbf{x}, C_2\mathbf{x}, \ldots, C_l\mathbf{x}]$, where l is the dimension of $\mathfrak{g}$. Define the Lie rank $\rho(\mathbf{x})$ as the dimension of the linear span of $\{X\mathbf{x} | X \in \mathfrak{g}\}$, or more constructively, $\rho(\mathbf{x}) = \text{rank}(\mathcal{B}\mathbf{x})$.

For homogeneous BLS, a necessary and sufficient condition for the accessibility property is the *Lie rank condition* introduced by J. Kučera (17):

$$\rho(\mathbf{x}) = n, \text{ for all } \mathbf{x} \neq 0 \tag{14}$$

If $\rho(\mathbf{x}(0)) = k$, then $\rho(\mathbf{x}(t)) = k, t \in \mathbf{R}$. If at some point $\mathbf{x}$ the BLS can move in $\rho(\mathbf{x})$ directions, this must remain true at all points that the trajectories can reach from $\mathbf{x}$. (Consider the "diagonal" systems mentioned above, for example.) Due to the radial scaling properties of BLS, the Lie rank actually needs to be checked only on the unit sphere and is the same at antipodal points. If $\mathfrak{g}$ satisfies the condition in equation 14, it is called *transitive* on $\mathbf{R}_0^n$; see the "Matrix Groups" section to see why the word transitive is used. To check a BLS for transitivity, find the $n \times n$ minors of $\mathcal{B}x$; these are nth degree polynomials. If 0 is their only common zero, the Lie rank is n. This algebraic task is performed by symbolic algebra; see the book by Elliott in the Reading List.

For symmetric systems (Eq. 5'), transitivity of the Lie algebra generated by $\{B_1, \ldots, B_k\}$ is necessary and sufficient for controllability on $\mathbf{R}_0^n$; the "Matrix Groups" section will explain why. For asymmetric systems (Eq. 4) and (Eq. 5) transitivity is necessary but far from sufficient; that can be seen from Example I. Its Lie algebra is the span of I and J, $\det(\mathbf{x}, J\mathbf{x}) = x_1^2 + x_2^2$ so $\rho(\mathbf{x}) = 2$; but all paths have $\|x(t)\| = e^t \|x(0)\|$.

At each state $\mathbf{x} \in \mathbf{R}_0^n$, the set $\mathbf{h}_{\mathbf{x}} \stackrel{\text{def}}{=} \{X \in \mathfrak{g} | X\mathbf{x} = 0\}$ is a linear subspace of $\mathfrak{g}$ and contains the Lie bracket of any two of its elements, so it is called the *isotropy Lie algebra* at $\mathbf{x}$; it is the same, up to similarity equivalence, at all points reachable from $\mathbf{x}$ by trajectories of the BLS. Transitivity of the Lie algebra $\mathfrak{g}$ on state space $\mathbf{R}_0^n$ also means that for *every* state $\mathbf{x}$ the quotient space of its isotropy subalgebra $\mathbf{h}_{\mathbf{x}}$ in $\mathfrak{g}$ satisfies $\mathfrak{g}/\mathbf{h}_{\mathbf{x}} \simeq \mathbf{R}_0^n$.

Returning once more to Example I, $\mathfrak{g} = \{\alpha I + \beta J | \alpha, \beta \in \mathbf{R}\}$, so

$$\mathcal{B}\mathbf{x} = \{I\mathbf{x},\ J x\} = \begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix}, \ \det(\mathcal{B}\mathbf{x}) = x_1^2 + x_2^2$$

As, $x_1^2 + x_2^2 \neq 0$, the equation $\alpha I\mathbf{x} + \beta J\mathbf{x} = 0$ has the unique solution $\alpha = 0$, $\beta = 0$; therefore, the isotropy algebra is $\{0\}$; on $\mathbf{R}_0^2$ the Lie rank is $\rho(\mathbf{x}) = 2$.

Those interested in computability will notice that the Lie rank criterion (Eq. 14) is computationally of exponential complexity as n increases, but as a negative criterion, it can be easily checked by choosing a random state $\mathbf{x} \in \mathbf{R}_0^n$; if $\rho(\mathbf{x}) < n$, the system cannot have the accessibility property nor controllability. However, if $\rho(\mathbf{x}) = n$, there still may be states, as in our diagonal-system example, which cannot be reached from $\mathbf{x}$.

Proving accessibility or establishing useful tests was made easier through the work of Boothby and Wilson (18), which lists, for each state-space dimension n, the (*transitive Lie algebras*) and provides a rational algorithm to determine whether a homogeneous BLS has a Lie algebra on the list. This list was completed recently by the independent work of Kramer (19) with an additional Lie algebra known as $\mathfrak{spin}$ (9, 1).

Example II. Another Lie algebra helps relate linear and bilinear systems; its system theoretic interpretation was pointed out by Brockett (20).

$$\text{aff}(n, \mathbf{R}) = \left\{ \mathbf{X} \in M^{n+1} | \mathbf{X} = \begin{pmatrix} X & \mathbf{x} \\ 0' & 0 \end{pmatrix}, X \in \mathcal{M}^n, \ \mathbf{x} \in \mathbf{R}^n \right\}$$

The appropriate state space will be an n-dimensional hyperplane

$$P = \{\mathbf{z} \in \mathbf{R}^{n+1} | z_{n+1} = 1\}$$

To see what is happening, consider first a linear system on $\mathbf{R}^n$, $\dot{x} = Ax + u(t)b$. Now on $\mathbf{R}^n$ let

$$\mathbf{z} = \begin{pmatrix} x_1 \\ \vdots \\ z_{n+1} \end{pmatrix}, \mathbf{A} = \begin{pmatrix} A & 0 \\ 0' & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & \text{b} \\ 0' & 0 \end{pmatrix}$$

with $\dot{z}_{n+1} = 0$ and $z_{n+1}(0) = 1$. On $\mathbf{R}^{n+1}$, the bilinear system $\dot{\mathbf{z}} = \mathbf{A}\mathbf{x} + u(t)\mathbf{B}\mathbf{x}$ has aff$(n, \mathbf{R})$ for its Lie algebra but on P and is equivalent to the linear system we started with. Note that

$$\text{ad}_{\mathbf{A}}(B) = \begin{pmatrix} 0 & A\mathbf{b} \\ 0' & 0 \end{pmatrix}, \ \text{ad}^2_{\mathbf{A}}(\mathbf{B}) = \begin{pmatrix} 0 & A^2\mathbf{b} \\ 0' & 0 \end{pmatrix}, \ \text{etc.}$$

Brackets containing the factor $\mathbf{B}$ twice will vanish. The hyperplane $z_{n+1} = 1$ is invariant, and the BLS is controllable on that hyperplane under the usual Kalman rank condition that rank$\{\mathbf{b}, A\mathbf{b}, \ldots\} = n$.

The controllability properties of inhomogeneous BLS like equation 3 $\dot{\mathbf{x}} = A\mathbf{x} + u(B\mathbf{x} + \mathbf{b})$ can be studied using the idea and notation of Example II. This BLS system can, using $n + 1$ coordinates, be given as a homogeneous BLS with an invariant hyperplane,

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + u(t)\mathbf{B}\mathbf{z}, \ \text{where}\, \mathbf{A} = \begin{pmatrix} A & 0 \\ 0' & 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} B & \mathbf{b} \\ 0' & 0 \end{pmatrix}$$

A Left Inverse System. A problem closely related to observability can be posed at this point: left invertibility of a single-input single-output BLS $\dot{\mathbf{x}} = A\mathbf{x} + uB\mathbf{x},\ y = \mathbf{c}'\mathbf{x}$. In the terminology of Hirschorn (21), this system is called *left invertible* if the output history on some interval $[0, T)$ uniquely determines the input history on $[0, T)$.

The *relative order* α of the BLS is the least positive integer k such that

$$\mathbf{c}'\mathrm{ad}_A^k(B) \neq 0 \tag{15}$$

or $\alpha = \infty$ if all $\mathbf{c}'\mathrm{ad}_A^{k-1}B = 0,\ k > 0$. The BLS is invertible if $\alpha < \infty$ and $\mathbf{c}'\mathrm{ad}_A^{k-1}B\mathbf{x}(0) \neq 0$. Invertibility fails if and only if every control results in the same output.

The conditions of Hirschorn (Eq. 15) and Williamson (Eq. 11) are interestingly related when one takes into account that in the system-inverse problem $\mathbf{x}(0)$ is known. Assume that the Williamson condition holds, i.e., the rank of W is n and $\mathbf{c}'A^kB = 0,\ 0 \le k \le n-2$. They do not force $B = 0$, because we can have $c'A^{n-1}B \neq 0$. Now evaluating the Hirschorn conditions, in turn

$$\mathbf{c}'B = 0;\ \mathbf{c}'[A, B] = \mathbf{c}'AB = 0;\ \mathbf{c}'[A, [A, B]] = \mathbf{c}'A^2B = 0; \ldots$$

but $\mathbf{c}'\mathrm{ad}_A^{n-1}B = \mathbf{c}'A^{n-1}B \neq 0$; so the relative order is $\alpha = n$. Reference 21 points out, in this case the inverse system becomes an observer, tracking $\mathbf{x}(t)$ when $y^{(\alpha)}$ is supplied as its input.

CONTROLLABILITY PROBLEMS

Criteria for Controllability. The necessary conditions for controllability of homogeneous BLS begin with the Lie rank condition (Eq. 14). Using it is made easier by the list of transitive Lie algebras in References 18–19. A necessary condition for controllability given by Elliott (22) is that controllable homogeneous BLS on $\mathbf{R}_0^n,\ n > 2$ have the strong accessibility property. (The peculiarity of Example I occurs because the punctured plane is not simply connected.)

For inhomogeneous BLS like equation 3, at $\mathbf{x} = \mathbf{0}$, the Kalman rank condition on $\{b, Ab, \ldots\}$ is a sufficient condition for *local* controllability with unbounded controls; as that rank condition is an open one in the space of coefficients, for sufficiently small $\mathbf{x}$, the $B\mathbf{x}$ term does no harm. There is usually a family of possible equilibria corresponding to solutions of $(A + \mu B)\mathbf{x} = -\mu\mathbf{b}$ and at each of these one can make such a test. In the first paper on BLS, Rink and Mohler (23) assumed such local controllability conditions to show controllability with bounded control u of

$$\dot{\mathbf{x}} = A\mathbf{x} + \sum_{k=1}^{m}(B_k\mathbf{x} + b_k)u_k$$

provided that the equilibrium set, for admissible u, is connected and all eigenvalues of $A + \sum_{k=1}^{m} u_kB_k$ can be made strictly stable and strictly unstable using admissible constant values of u. This use of bilinear terms to make up for the deficiencies of linear control methods, when controls are bounded, is emphasized by Mohler.

For equation 4, in Cheng et al. (24), the ad-condition

$$\mathrm{rank}(A\mathbf{x}, B\mathbf{x}, \mathrm{ad}_A(B)\mathbf{x}, \ldots, \mathrm{ad}_A^{n^2-1}(B)\mathbf{x}) = n \text{ on } \mathbf{R}_0^n \tag{16}$$

plus the hypothesis that A is similar to a skew-symmetric matrix are shown sufficient for controllability with bounded control. A partial converse is that controllability of equation 4 for arbitrarily small bounds on the controls implies that all eigenvalues of A are imaginary.

Many papers on controllability and stabilization cite Jurdjevic and Quinn (25). For BLS their condition specializes as follows, for piecewise continuous signals. If A has eigenvaiues that are purely imaginary and distinct and the Lie rank condition is satisfied, then $\dot{\mathbf{x}} = A\mathbf{x} + uB\mathbf{x}$ is controllable. This extends to multiple-input BLS immediately. A stabilization result using the ad-condition is also obtained.

For symmetric systems, the Lie rank condition is a sufficient condition for controllability, which is connected to the fact that in that case the transition matrices constitute a Lie group (discussed below).

For asymmetric BLS, the set of transition matrices is only a semigroup, meaning that it may not contain the inverses of some transition matrices. A broad principle of controllability theory is that you must be able to get back to an open neighborhood of the state where you started, somehow.

Several conditions sufficient for controllability of BLS were given by Jurdjevic and Kupka (26), for unbounded controls. Here is a sample. Assume that the eigenvalues of B are simple and real, in the order $q_1 > q_2 \ldots > q_n$; choose coordinates so that $B = \mathrm{diag}(q_1, \ldots, q_n)$. Then if the numbers $q_i - q_j$ are all distinct, if the elements of A satisfy $A_{ij} \neq 0$ for all i, j such that $|i - j| = 1$, and if $A_{1n}A_{n1} < 0$, then $\dot{\mathbf{x}} = (A + uB)\mathbf{x}$ is controllable on $\mathbf{R}^n \backslash 0$. Jurdjevic and Sallet (27) and others have extended the approach of Reference 24 to nonhomogeneous BLS like equation 3, in the m-input case.

Positive Systems. There are several ways in which BLS models arise that involve positive variables. As a simple example, diagonal systems

$$\dot{x}_i = u_ix_i,\ i = 1, \ldots, n,\ x_i > 0 \text{ on the orthant}$$
$$\mathbf{R}_+^n = \{\mathbf{x} \in \mathbf{R}^n | x_i > 0,\ i = 1, \ldots, n\}$$

can obviously be transformed by the substitution $x_i = \exp(z_i)$ to the system of coordinate-translations $\dot{z}_i = u_i$.

Controllability for systems of a more interesting nature that have positive orthants as their natural state spaces was studied by Boothby (28). Here the BLS is

$$\dot{\mathbf{x}} = A\mathbf{x} + uB\mathbf{x} \text{ on } \mathbf{R}_+^n \tag{17}$$

under the hypothesis that the $n \times n$ matrix A is *essentially positive*: $a_{ij} > 0,\ i \neq j$, written $A > 0$. It is well known and easy to show that, for $A > 0$, if $\mathbf{x}(0) \in \mathbf{R}_+^n$, then $\mathbf{x}(t) \in \mathbf{R}_+^n,\ t > 0$.

The conditions on B used in Reference 28 are that B is nonsingular and diagonal; $B = \mathrm{diag}[\beta_1, \ldots, \beta_n]$; and for all $i \neq j,\ \beta_i - \beta_j \neq 0$, which is an invertibility condition for ad_B. If none of the eigenvalue differences is repeated, $\{A, B\}_{LA}$ is $\mathfrak{sl}(n, \mathbf{R})$ if the trace of A is zero, $\mathfrak{gl}(n, \mathbf{R})$ otherwise, so the

Lie rank condition is satisfied. Controllability and noncontrollability results are established for several families of A, B pairs, especially for $n = 2$.

Bacciotti (29) completed the study for $n = 2$. He assumes the same conditions: $A > 0$; B is diagonal and nonsingular, with no repeated eigenvalues; and $\beta_2 > 0$ (if not, reverse the sign of the control). Then for the BLS $\dot{\mathbf{x}} = A\mathbf{x} + uB\mathbf{x}$:

1. If $\beta_1 > 0$, the BLS is completely controllable on $\mathbf{R}^n_+$.
2. If $\beta_1 < 0$ but $\delta = (\beta_2 a_{11} - \beta_1 a_{22})^2 + 4\beta_1\beta_2 a_{12}a_{21} > 0$ and $\beta_1 a_{22} - \beta_2 a_{11} > 0$, then the BLS is completely controllable on $\mathbf{R}^n_+$. In any other case, controllability fails.

Sachkov (30) gives answers for m-input problems (Eq. 5) with $m = n - 1$ or $m = n - 2$, using an idea that has been successful in other controllability problems: if the symmetric system $\dot{\mathbf{x}} = \sum_{i=1}^{m} u_i B_i \mathbf{x}$ is controllable on hypersurfaces $V(\mathbf{x}) = \nu$ that fill up the (simply connected) state space $\mathbf{R}^n_+$, and if tne zero-control trajectories of equation 5 can cross all the hypersurfaces in both directions, then equation 5 is globally controllable on $\mathbf{R}^n_+$.

Stabilization II. At this point it is appropriate to look at stabilization by state feedback controls that are not constant. For BLS the usual way of attacking this has been by quadratic Lyapunov functions. Given a vector differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f}(0) = 0$, the basic idea behind Lyapunov's Direct Method is to find a family of nested smooth hypersurfaces around 0, such as concentric ellipsoids, which the trajectories enter and never leave. For example, let us start with a test for stability of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. Choose a symmetric matrix Q that is *positive definite*, which means that all its eigenvalues are positive and is easy to check by the criterion that each of the n leading minor determinants of Q is positive:

$$Q_{11} > 0, \quad \begin{vmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{vmatrix} > 0, \ldots, \det(Q) > 0$$

Let $V(\mathbf{x}) = \mathbf{x}'Q\mathbf{x}$, whose level surfaces are ellipsoids; then along trajectories of the differential equation, $\dot{V} = \mathbf{x}'(QA + A'Q)\mathbf{x}$. If $(QA + A'Q)$ is negative definite (all eigenvalues negative), then $\dot{V} < 0$, which is what we wanted. There are various recipes for choosing Q, such as solving a linear Lyapunov equation $QA + A'Q = -I$ for Q and testing for positive definiteness afterward.

One of the applications of BLS is to controlled *switched linear systems*

$$\dot{x} = A_u x, \quad A_u \in \mathcal{A}$$

where $\mathcal{A}$ is (in the simplest version) a finite family of $n \times n$ matrices indexed by an integer valued control $u(t)$ that can be assigned in any way that enforces some delay between different values. A basic question is to find conditions are needed to ensure that the switched system will be ES for arbitrary control sequences. Necessarily all the A_u are Hurwitz; otherwise a fixed u would not provide ES. Agrachev and Liberzon (31) showed that it is sufficient to also impose the condition that the Lie algebra generated by the $A_u \in \mathcal{A}$ is solvable, as in Reference 9. In the coordinates (real or complex) in which the matrices are triangular, their common Lyapunov function is $\mathbf{x}^*\mathbf{x}$.

Returning to equation 4, suppose A has all its eigenvalues on the imaginary axis. Then there is a positive definite Q for which $QA + A'Q = 0$; by a change of variables, $Q = I$. That is the assumption in Reference 25, which therefore uses the Lyapunov function $V(\mathbf{x}) = \mathbf{x}'\mathbf{x}$; the the ad-condition on A and B (Eq. 16) is assumed to hold. The proposed feedback control is $u(\mathbf{x}) = -\mathbf{x}'B\mathbf{x}$; so along the trajectories of the closed-loop system $\dot{\mathbf{x}} = A\mathbf{x} - (\mathbf{x}'B\mathbf{x})B\mathbf{x}$, we have $\dot{V} = -(\mathbf{x}'B\mathbf{x})^2 \leq 0 \,\forall\, \mathbf{x} \neq 0$. From the ad-condition, trajectories cannot remain in the set $\{\mathbf{x}'B\mathbf{x} = 0 | \mathbf{x} \neq 0\}$, and $V(\mathbf{x}(t)) \to 0$ as $t \to \infty$. However, looking at the one-dimensional case, one sees that for the differential equation $\dot{\xi} = -\xi^3$, the approach to the origin is of order $t^{-1/2}$, and that this is also true for the n-dimensional problem using quadratic feedback. Using the same hypotheses and nearly the same short proof as in Reference 25, the reader can verify that the bounded feedback $u = -\mathbf{x}'B\mathbf{x}/\mathbf{x}'\mathbf{x}$ provides exponential stability.

In linear system work, the choice of feedback control is linear, $u = \mathbf{c}'\mathbf{x}$. Applied to BLS $\dot{x} = A\mathbf{x} + uB\mathbf{x}$, this results in a quadratic system, such as the Lorenz system, and is the topic of References 6 and 7. Reference 7 is concerned with linear feedback for a class of BLS in which B generates a rotation and, for some constant μ, $A + \mu B$ has real eigenvalues of mixed sign. For $n = 2$, after a similarity transformation

$$A + \mu B = \mathrm{diag}(\lambda_1, \lambda_2); \; \lambda_1 > 0, \; \lambda_2 < 0, \; B = \beta J$$

Using a constant control $\gamma > \mu$, the BLS can be globally "practically stabilized"; i.e., the trajectory eventually enters a ball of radius of order $1/\gamma$. For $n = 3$, the same type of system [now with $\lambda_3 < 0$, $B \in \mathbf{so}(3)$] can be globally asymptotically stabilized given certain polynomial inequalities in the λ_i and B. Given some simpler inequalities, such as $\mathrm{tr}(A) < 0$ but allowing $\lambda_1 > 0$, the system is practically stabilized by a family of linear feedbacks with parameter γ.

Stabilization of homogeneous BLS in the plane has been fully analyzed; it is not necessary that A have imaginary eigenvalues. Bacciotti and Boieri (32) using constant, linear, and quadratic feedbacks, and Reference 8 using feedbacks differentiable except perhaps at $\mathbf{0}$, have given complete classifications of the possibilities for stabilizability of equation 4 on $\mathbf{R}^2_0$. The methods of analysis in these papers include Lyapunov functions, center manifolds, and properties of plane curves, depending on the various cases. These cases are specified by the determinants and traces of A and B, diagonalizability, and a few other structural features. In Reference 8, a feedback control, useful even when A is not stable, is $u = \mathbf{x}'R\mathbf{x}/\mathbf{x}'T\mathbf{x}$ where matrices R, T are found on a case by case basis. These controls are homogeneous of degree zero, and if not constant are discontinuous at 0 but differentiate for $\mathbf{x} \neq 0$. The stabilized trajectories typically approach the origin in a spiraling fashion, often with large excursions; over each revolution the distance to $\mathbf{0}$ decreases in a constant ratio. In that sense they generalize the constant controls; the signum controls are essentially homogeneous of degree zero, and there may be other possibilities.

Recent work on more general problems of nonlinear system stabilization suggests that a time-periodic feedback will be needed for three or more dimensions, in order to bring into play the higher order Lie brackets of A and B.

A Note on Optimal Control. At the time that work began on BLS one of the motivations for such studies (2, 17) was their application to optimal control; see OPTIMAL CONTROL. Reference 9 treats optimal control of a BLS arising in cancer chemotherapy. Case studies for vehicle control and nuclear reactor control were summarized in Reference 3, vol. 2; its bibliography lists some studies of biological and ecological systems with BLS models, in which there is some evidence that certain biosystems switch behaviors optimally; see the paper of Oster (33) on bees.

The most studied cost was the time-to-target; that is, given an initial state and a target (a state or closed set), use controls in some admissible class of bounded controls $\mathcal{U}_\mu$ to find the trajectory connecting the initial state and the target in least elapsed time [See Jurdjevic's book (34)]. In linear system versions of this problem, it was known that the set accessible from a given state was the same for $\mathcal{U}_\mu$ as for the set of PWC controls with values in $\{-\mu, \mu\}$, called bang-bang controls. One formulation of the bang-bang principle is "the set attainable for bounded controls can be obtained by using only the extreme values of the controls." It holds true for BLS in which all matrices commute. The computation of time-optimal controls, as well as the first attempts at studying controllability (17), assumed that one could simplify their solution to the problem of finding optimal switching times (open-loop) or finding hypersurfaces on which the control would switch values. However, early on it was discovered by Sussmann (35) that the bang-bang principle did not apply to bilinear systems in general. A value between the extremes may be required. There are examples of simply formulated optimal control problems for which the control may have to switch infinitely often in a finite time interval.

MATRIX GROUPS

The set of all $n \times n$ nonsingular matrices equipped with the usual matrix multiplication, identity I, and inverse constitutes a group, called $\mathbf{GL}(n,\mathbf{R})$ where GL stands for "general linear." A subgroup of $\mathbf{GL}(n,\mathbf{R})$ is called a *matrix group*; the matrix groups we need for BLS are *matrix Lie groups*.

The derivation of the results of References 17 and 23 other nonlocal results for BLS depends not only on Lie algebras, but also the corresponding matrix Lie groups. Begin by considering the k-input symmetric homogeneous BLS of equation 5, where it is assumed that the matrices B_i are linearly independent. Again write $\mathbf{u} = (u_1, \ldots, u_k)$. The Lie algebra of this BLS is $\{B_1, \ldots, B_k\}_{LA}$. We let $\{\Phi\}$ be the set of transition matrices for this BLS, i.e., the solutions of the matrix system

$$\dot{\Phi} = \left(\sum_{1}^{k} u_i B_i)\Phi,\ \Phi(0, 0 \right) = I \tag{18}$$

As $\{\Phi\}$ contains I and is closed under composition [eq. 8a] and inverse [eq. 8b], we see that it constitutes a group, in fact a subgroup of $\mathbf{GL}(n,\mathbf{R})$. As all matrices in $\{\Phi\}$ have positive determinants, $\{\Phi\}$ actually lies in $\mathbf{GL}^+(n, \mathbf{R})$.

On the other hand, corresponding to any matrix Lie algebra $\mathfrak{g}$, a *Lie group* we shall call $G(\mathfrak{g})$ can be constructed, consisting of all products of exponentials of matrices in g; see Reference 16 or Rossmann's book in the Reading List. If a basis of $\mathfrak{g}$ is $\{C_1, \ldots, C_l\}$, in a neighborhood of the identity $G(\mathfrak{g})$ has coordinates $\{\exp(C_1t_1), \ldots, \exp(C_1t_l)\} \to (t_1, \ldots, t_l)$. (Thus, $\mathfrak{g}$ is the tangent space to $\mathbf{G}$ at I.) Using elements of G to translate this coordinate patch anywhere on the group, it can be observed that $G(\mathfrak{g})$ has an atlas of coordinate charts that are related by differentiable transformations where they overlap, like a geographic atlas, and on which the group multiplication and inverse are differentiable functions. Lie groups occur in many applications of mathematics to classical mechanics, quantum mechanics, chemistry, and the control of robots and aerospace vehicles.

At this point we can note that the mathematics of controllability for symmetric BLS is rather simple. If a matrix Lie group $\mathbf{G}$ has the property that given any two states $\mathbf{x}$ and $\mathbf{z}$ in $\mathbf{R}_0^n$, $X \in \mathbf{G}$ exists such that $X\mathbf{x} = \mathbf{z}$, then $\mathbf{G}$ is called *transitive* on $\mathbf{R}_0^n$ and $\mathfrak{g}$ is a transitive Lie algebra.

In Reference 20, it was shown that $G(\{B_1, \ldots, B_k\}_{LA}) = \{\Phi\}$; that is, all matrices in the Lie group can be obtained as transition matrices. This is a simple version of the Chow Theorem of nonlinear control. Its meanings for BLS are that, once the Lie algebra has been identified, the structure of the group of transition matrices is completely known; and that any matrix $M \in G(\{B_1, \ldots, B_k\}_{LA})$ can be written as a product

$$M = e^{t_1B_1}e^{s_1B_2}\ldots e^{t_kB_1}e^{s_kB_2}$$

for some finite sequence of reals $\{t_1, s_1, \ldots, t_k, s_k\}$. Thus, a few generators are as good as l of them.

Some examples of Lie groups are useful; in some of those listed below, the identity $\det(e^A) = e^{\mathrm{tr}(A)}$ is relevant:

Lie algebra $\mathfrak{g}$	$\mathbf{G} \subset \mathbf{GL}(n, \mathbf{R})$
$\mathfrak{gl}(n, \mathbf{R}) = \mathcal{M}^n$	$\mathbf{GL}^+(n, \mathbf{R}) = \{Q \mid \det(Q) > 0\}$
$\mathfrak{sl}(n, \mathbf{R}) = \{X \mid \mathrm{tr}(X) = 0\}$	$\mathbf{SL}(n, \mathbf{R}) = \{Q \mid \det(Q) = 1\}$
$\mathbf{so}(n) = \{X \mid X' = -X\}$	$\mathbf{SO}(n) = \{Q \mid Q'Q = I\}$

and if $J^2 = -I$ and θ is irrational,

$$\{X \mid X = \mathrm{diag}\{J, \theta J\} \in \mathcal{M}^4\} \quad \tilde{R} \subset GL(4, \mathbf{R})$$

where $\tilde{R}$ is a densely wound curve that fills up a 2-torus, so it is not a closed (Lie) subgroup of $GL(4, \mathbf{R})$.

Any element $Z \in \mathbf{G}$ is a product of exponentials of elements of the corresponding Lie algebra g but not necessarily an exponential of any such element; the standard counterexample is $Z = \mathrm{diag}(-e, -e^{-1}) \in \mathbf{SL}(2, \mathbf{R})$. Here $-I = \exp(J)$, $\mathrm{diag}(e, e^{-1}) = \exp(\mathrm{diag}(1, -1))$, but their product Z is not the exponential of any real matrix.

For asymmetric systems $\dot{\mathbf{x}} = (A + uB)\mathrm{x}$, in $\mathbf{GL}(n, \mathbf{R})$ the set of transition matrices $S = \{\Phi(u; t, 0) \mid u \in \mathcal{U}, t \geq 0\}$ contains I and is closed under multiplication but not under matrix inversion; it is a matrix *semigroup*. Example I showed an uncontrollable system for which S is a semigroup but with the accessibility property. Here is another two-dimensional example with bounded controls, showing that controllability and stabilizability are possible for semigroups acting on $\mathbf{R}_0^2$.

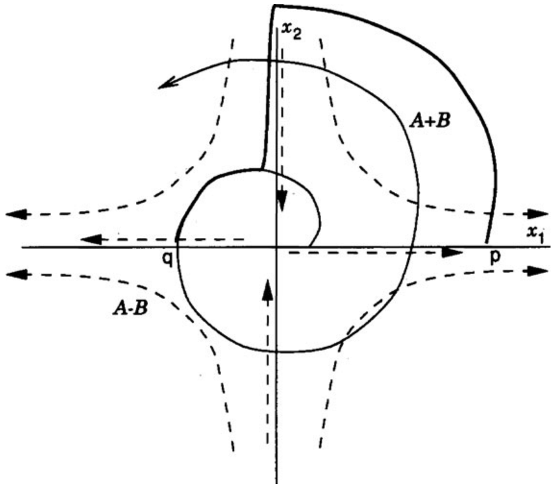

Figure 1. Phase portrait and path from p to q for Example III. From p use $u = 1$ to reach the x_2 axis; then use $u = -1$; switch to $u = 1$ at the arc of the spiral that leads to q.

Example III. Consider this BLS with $u = \pm 1$ and $0.5 > \alpha > 0$:

$$\dot{\mathbf{x}} = A\mathbf{x} + uB\mathbf{x};\ A = \begin{pmatrix} 1+\alpha & -1 \\ 1 & \alpha - 1 \end{pmatrix},\quad B = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix};$$
$$A - B = \begin{pmatrix} 2+\alpha & 0 \\ 0 & \alpha - 2 \end{pmatrix},\qquad A + B = \begin{pmatrix} \alpha & -2 \\ 2 & \alpha \end{pmatrix} \tag{19}$$

As $\mathrm{tr}(A \pm B) = 2\alpha$, for either choice of control this is not a stable system. Paths with $u = 1$ are spirals that provide rotations. With $u = -1$ the paths are hyperbolas, which near the coordinate axes permit movement toward the origin x_2-axis) or away from it (x_1-axis). System l9 is controllable, because dilations and rotations of arbitrary extent are possible. Furthermore, system 19 satisfies the conditions $\mathrm{tr}\, B = 0$, $\mathrm{tr}\, A > 0$, and $\mathrm{tr}\, AB \neq 0$ of Th. 2.2.3 of Reference 33, so it can be stabilized with a feedback control $u(\mathbf{x}) = \mathbf{x}'R\mathbf{x}/\mathbf{x}'\mathbf{x}$.

Other Aspects of Bilinear System Theory.

Discrete-Time Systems. There is a large literature on DBLS on most of the topics we have covered, sometimes employing very different mathematical tools and sometimes highly parallel to continuous-time systems. To emphasize this we use a “successor” (time-shift) notation; instead of $\mathbf{x}(t+1) = (A + uB)\mathbf{x}(t)$, for DBLS, we use $\mathbf{x}^s = (A + uB)\mathbf{x}$.

The observability theory of discrete-time bilinear systems is very much like that of the continuous version, with the same criteria, e.g., equation 11 to assure observability for all u. Inverting an observaole discrete-time system requires no time-derivatives of the output $y = \mathbf{c}'\mathbf{x}$.

When the DBLS is obtained by sample-and-hold operations with interval δ on a continuous-time system like equation 4, the sampled system is

$$\mathbf{x}^s = e^{\delta(A+uB)} \tag{20}$$

If the BLS is observable to begin with, observability is preserved when the sampling interval δ is sufficiently small; the condition (the same as for linear systems) is that $\delta(\lambda - \mu)$ is not of the form $2k\pi i$ for any pair of eigenvalues λ, μ of A and integer k.

However, Sontag (36) has shown that for a controllable BLS (eq. 4), the sampled system (eq. 20) will be controllabile if this condition is imposed: $\delta(\lambda + \lambda' - \mu - \mu') \neq 2k\pi$, $k \neq 0$ for any four eigenvalues of A.

The discretization of an uncontrollable BLS can be artifactually controllable,. depending on the BLS and the numerical method used. For the Euler discretization shown in equation 6, here is a two-dimensional example. The BLS is $\dot{\mathbf{x}} = u(2J + I)\mathbf{x}$, which has $A = 0$, $B = 2J + I$, and is not controllable. The system can move back and forth along one spiral trajectory through $\mathbf{x}(0)$. The discrete-time approximation is $\mathbf{x}^s = \mathbf{x} + \tau uB\mathbf{x}$. This DBLS is controllable on R_0^2; the trajectories move on the tangent lines to the spiral.

Control and optimal control problems for DBLS $\mathbf{x}^s = A\mathbf{x} + B\mathbf{x}u$ are investigated in Swamy and Tarn (37); in the optimal control area, the case that B has rank one is notable, because it can be reduced to a linear system problem by the factorization $B = bc'$. Perform the optimization, with whatever cost, for the linear system $\mathbf{x}^s = A\mathbf{x} + vb$ and obtain the optimal control v^* then in the DBLS, let $u = v^*/(\mathbf{c}'\mathbf{x})$,

so

$$\mathbf{x}^s = A\mathbf{x} + \mathrm{bc}'\mathbf{x}\frac{v^*}{\mathbf{c}'\mathbf{x}}$$

when $\mathbf{c}'\mathbf{x} \neq 0$, and perturb the computation slightly to avoid its vanishing. This “division controller” reflects the fact that the discrete motion can avoid the hyperplane $\mathbf{c}'\mathbf{x} = 0$.

Controllability with bounded controls is associated (as it is in continuous-time systems) with zero-control motion on an ellipsoid and an accessibility condition. For the DBLS $\mathbf{x}^s = A\mathbf{x} + uB\mathbf{x}$, Reference 24 has a DBLS version of the ad-condition obtained from the Jacobian of the map from the input history to output history

$$\mathrm{rank}\{A\mathbf{x}, A^m B\mathbf{x}, A^{m-1}BA\mathbf{x}, \ldots, ABA^{m-1}\mathbf{x}, BA^m\mathbf{x}\} = n$$

on R_0^n, where m need be no larger than $n^2 - 1$. When the rank condition holds, the condition for controllability with controls bounded by any positive constant is that A is similar to an orthogonal matrix. As a partial converse, it was shown that requiring controllability for inputs with arbitrarily small bounds implies the spectrum of A lies on the unit circle, and that the DBLS ad-condition is sufficient for controllability when some power of A is orthogonal.

The solution of a DBLS $\mathbf{x}^s = (A + uB)\mathbf{x}$ is given by the right-to-left ordered product

$$\Phi(u; t, 0)\mathbf{x}(0) = (\Pi_{k=t}^{0}(A + u(k)B))\mathbf{x}(0)$$

The set of states attainable from $\mathbf{x}(0)$ may be pathological. For example take

$$A = \begin{pmatrix} \cos(\alpha\pi) & -\sin(\alpha\pi) \\ \sin(\alpha\pi) & \cos(\alpha\pi) \end{pmatrix},\quad B = I$$

For α rational, S is a finite set of rays and for α irrational, a dense set of rays.

For linear systems, the realization of an input–output map as a system on a state space is essentially the same in continuous and discrete time, using the concept of Hankel matrix

$$\begin{pmatrix} c'b & c'Ab & c'A^2b & \cdots \\ c'Ab & c'A^2b & c'A^3b & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

For discrete-time bilinear systems, realization theory involves a different looking but highly analogous Hankel matrix; Isidori (38), Fliess (39), and Tarn and Nonoyama (40) introduced variations on this idea. The situation differs from linear systems in that the input–putput map is $u \to c'\Phi(u; k, 0)$, with the single output a row vector, or it may be multiplied on the right by the initial state. Discrete-time bilinear systems were then, like linear systems, used as examples of adjoint systems in category theory, with the Hankel matrix being the fundamental description. A category, loosely speaking, is a collection of objects and maps between them; for instance, the category **Lin** has linear spaces and linear maps. A machine (adjoint system) in a category has input, state, and output objects; dynamics (iterated maps on the state space) and input and output maps; and Hankel matrices. The category of discrete sets underlies automata theory; **Lin** leads to discrete-time linear system theory, etc., and discrete-time bilinear systems "occur naturally"; see Arbib and Manes (41). Continuous-time dynamics does not fit into category theory.

Transforming Nonlinear Systems to Bilinear Form. As an application of BLS ideas to nonlinear control theory, consider the nonlinear system

$$\begin{aligned} \dot{z}_1 &= -z_1 + (z_2 + z_1^2)u \\ \dot{z}_2 &= z_2 + 3z_1^2 + (z_1 - 2z_1z_2 - 2z_1^3)u \end{aligned}$$

This was cooked by using the one-to-one coordinate transformation $x_1 = z_1, x_2 = z_2 + z_1^2$, from the BLS $\dot{\mathbf{x}} = A\mathbf{x} + uB\mathbf{x}$ where

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Lie algebra of the BLS is $\mathfrak{sl}(2, \mathbf{R})$. The matrices A,B are the Jacobians at 0 of the nonlinear vector functions in the first system, so the BLS is also the classic "linearization" of the nonlinear system, but what we have here is not an approximation but a system equivalence. The Lie algebra of nonlinear vector fields (see CONTROLLABILITY AND OBSERVABILITY, Section "Nonlinear Finite-Dimensional Systems") generated by our example is also $\mathfrak{sl}(2, \mathbf{R})$.

It has been shown by Sedwick and Elliott (42) that given a family $\mathcal{F}$ of real-analytic nonlinear vector fields on $\mathbf{R}^n$ that vanish at 0, if $\mathcal{F}$ and the family $\mathcal{F}_1$ of their linear terms isomorphically generate a transitive Lie algebra, then a real-analytic coordinate transformation exists that transforms $\mathcal{F}$ system to its equivalent BLS $\mathcal{F}_1$, and that can be found by solving linear partial differential equations. Of Lie algebras not on that list, the compact and the semisimple ones also permit this linearization, but all others either fail to have differentiable transformations or do not have any, as far as is known; and nonlinear systems axe prone to have infinite-dimensional Lie algebras.

A Note on Volterra Series. The connection between BLS and the Volterra series representation of input–output maps is surveyed in Reference 1. Volterra series are used extensively in the identification of nonlinear systems, especially in biological work, because algorithms exist to evaluate the first few terms purely from input–output data when no state-space model is known. The series is

$$\begin{aligned} y(t) \;=\;& W_0(t) + \textstyle\sum_{n=1}^{\infty} \int_0^t \cdots \int_0^{\sigma_{n-1}} \\ & W_n(t, \sigma_1, \ldots, \sigma_n)u(\sigma_1)\cdots u(\sigma_n)d\sigma_1 \cdots d\sigma_n \end{aligned}$$

Treat our usual system $\dot{\mathbf{x}} = A\mathbf{x} + u(t)B\mathbf{x}$, as if it were inhomogeneous to get the integral equation

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t u(t_1)e^{A(t-t_1)}B\mathbf{x}(t_1)dt_1$$

then the Volterra kernels are easily observed, by iteration, to be

$$\begin{aligned} W_0(t) &= c'e^{At}\mathbf{x}(0) \\ W_1(t, \sigma_1) &= c'e^{A(t-\sigma_1)}Be^{A\sigma_1}\mathbf{x}(0) \\ W_2(t, \sigma_1, \sigma_2) &= c'e^{A(t-\sigma_1)}Be^{A(\sigma_1-\sigma_2)}Be^{A\sigma_2}\mathbf{x}(0) \end{aligned}$$

For bounded u, the series converges on any time interval and represents the solution. The Volterra series has a finite number of terms precisely when the system's Lie algebra $\mathbf{g}$ is nilpotent, which means that all brackets with a sufficiently high number of factors must vanish. Several approximation theorems for analytic nonlinear systems have been based on this approach.

For an account of this theory in its general form for analytic systems, see Isidori (43); the book is also a source for many other topics omitted here, such as Fliess functional expansions and continuous-time realization theory. Also see NONLINEAR CONTROL SYSTEMS: ANALYTICAL METHODS.

Systems on Lie Groups; Quantum Systems. The transition matrices for Eq. (14) satisfy

$$\dot{\Phi} = A\Phi + \sum_{i=1}^{k} B_i\Phi \qquad (20)$$

with the initial condition $\Phi(0) = I$. It is also worthwhile to look at equation 20 as a control system on $\mathbf{GL}(n,\mathbf{R})$ or on one of its Lie subgroups such as (if A and the B_i are skew-symmetric) $\mathbf{SO}(3)$, the group of rigid-body rotations.

Such control systems are also called bilinear and have inspired much work such as References 20 and 26 and especially Reference 34. The controllability problem for systems on Lie groups is closely related to the study of matrix semigroups; see Lawson (44) for a survey of this area.

Typical applications have been to the angular attitude control of spacecraft (satellites) and undersea vehicles. A new application in which electronic engineers have been active is in quantum control; see QUANTUM SYSTEMS. D'Alessandro and Dahleh (45) and much subsequent work by D'Alessandro on quantum bits has made the study of BLS on complex Lie groups an attractive subject of study.

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Further Reading

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Lipschutz, S., *Schaum's Outline of Linear Algebra*, 2nd ed.; McGraw-Hill: New York, 1991. (Fine self-study text for students and as a refresher; the accounts of the Cayley–Hamilton theorem, the Jordan canonical form, and quadratic forms will be helpful in studying BLS.)

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