J. Webster (ed.), *Wiley Encyclopedia of Electrical and Electronics Engineering*

ADAPTIVE CONTROL

According to Webster's dictionary, to adapt means "to change (oneself) so that one's behavior will conform to new or changed circumstances." The words "adaptive systems" and "adaptive control" were used as early as 1950 (1). The design of autopilots for high-performance aircraft was one of the primary motivations for active research on adaptive control in the early 1950s. Aircraft operate over a wide range of speeds and altitudes, and their dynamics are nonlinear and conceptually time-varying. For a given operating point, specified by the aircraft speed (Mach number) and altitude, the complex aircraft dynamics can be approximated by a linear model of the form (2)

$$\begin{aligned} \dot{x} &= A_i x + B_i u, \qquad x(0) = x_0 \\ y &= C_i^{\mathrm{T}} x + D_i u \end{aligned} \tag{1}$$

where A_i, B_i, C_i, and D_i are functions of the operating point i. As the aircraft goes through different flight conditions, the operating point changes, leading to different values for A_i, B_i, C_i, and D_i. Because the output response $y(t)$ carries information about the state x as well as the parameters, one may argue that in principle, a sophisticated feedback controller should be able to learn about parameter changes by processing $y(t)$ and use the appropriate gains to accommodate them. This argument led to a feedback control structure on which adaptive control is based. The controller structure consists of a feedback loop and a controller with adjustable gains as shown in Fig. 1. The way the controller parameters are adjusted on line leads to different adaptive control schemes.

Gain Scheduling

The approach of *gain scheduling* is illustrated in Fig. 2. The gain scheduler consists of a lookup table and the appropriate logic for detecting the operating point and choosing the corresponding value of the control parameter vector θ. For example, let us consider the aircraft model in Eq. (1) where for each operating point i, $i = 1, 2, \ldots, N$, the parameters A_i, B_i, C_i, and D_i are known. For each operating point i, a feedback controller with constant gains, say θ_i, is designed to meet the performance requirements for the corresponding linear model. This leads to a controller, say $C(\theta)$, with a set of gains $\{\theta_1, \theta_2, \ldots, \theta_i, \ldots, \theta_N\}$ covering N operating points. Once the operating point, say i, is detected, the controller gains can be changed to the appropriate value of θ_i obtained from the precomputed gain set. Transitions between different operating points that lead to significant parameter changes may be handled by interpolation or by increasing the number of operating points. The two elements that are essential in implementing this approach are a lookup table to store the values of θ_i and the plant auxiliary measurements that correlate well with changes in the operating points.

Direct and Indirect Adaptive Control

A wide class of adaptive controllers is formed by combining an on-line parameter estimator, which provides estimates of unknown parameters at each time instant, with a control law that is motivated from the known-parameter case. The way the parameter estimator, also referred to as the *adaptive law,* is combined with the control law gives rise to two different approaches. In the first approach, referred to as *indirect adaptive control*

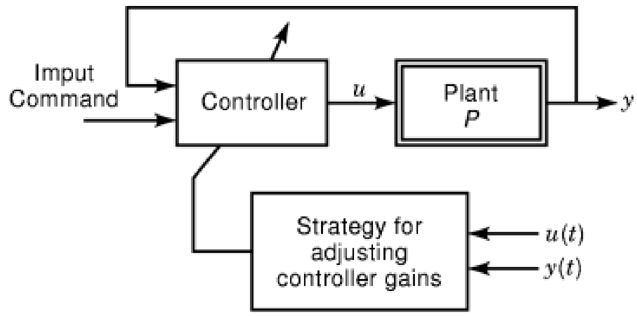

Fig. 1. Controller structure with adjustable controller gains.

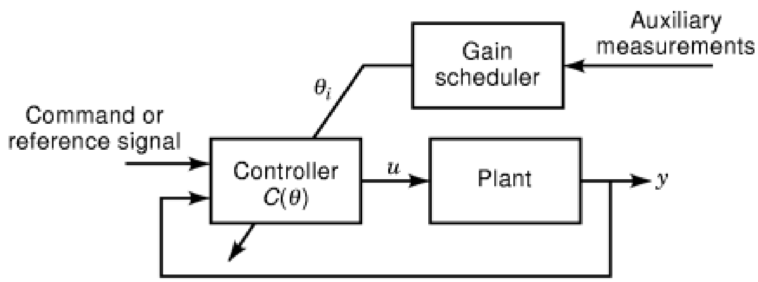

Fig. 2. Gain scheduling.

(shown in Fig. 3), the plant parameters are estimated online and used to calculate the controller parameters. This approach has also been referred to as *explicit adaptive control,* because the design is based on an explicit plant model.

In the second approach, referred to as *direct adaptive control* (shown in Fig. 4), the plant model is parametrized in terms of the controller parameters, which are estimated directly without intermediate calculations involving plant parameter estimates. This approach has also been referred to as *implicit adaptive control,* because the design is based on the estimation of an implicit plant model.

The principle behind the design of direct and indirect adaptive control shown in Figs. 3 and 4 is conceptually simple. The design of $C(\theta_c)$ treats the estimates $\theta_c(t)$ (in the case of direct adaptive control) or the estimates $\theta(t)$ (in the case of indirect adaptive control) as if they were the true parameters. This design approach is called *certainty equivalence* and can be used to generate a wide class of adaptive control schemes by combining different on-line parameter estimators with different control laws.

The idea behind the certainty equivalence approach is that as the parameter estimates $\theta_c(t)$ and $\theta(t)$ converge to the true ones θ^*_c and $\theta*$, respectively, the performance of the adaptive controller $C(\theta_c)$ tends to that achieved by $C(\theta^*_c)$ in the case of known parameters.

Model Reference Adaptive Control

Model reference adaptive control (*MRAC*) is derived from the model-following problem or model reference control (*MRC*) problem. The structure of an MRC scheme for a *LTI*, single-input single-output (*SISO*) plant is shown in Fig. 5. The transfer function $W_m(s)$ of the reference model is designed so that for a given reference input signal $r(t)$ the output $y_m(t)$ of the reference model represents the desired response the plant output $y(t)$ should follow. The feedback controller, denoted by $C(\theta^*_c)$, is designed so that all signals are bounded and the closed-loop plant transfer function from r to y is equal to $W_m(s)$. This transfer function matching guarantees that for any given reference input $r(t)$, the tracking error $e_1 \triangleq y - y_m$, which represents the deviation of the

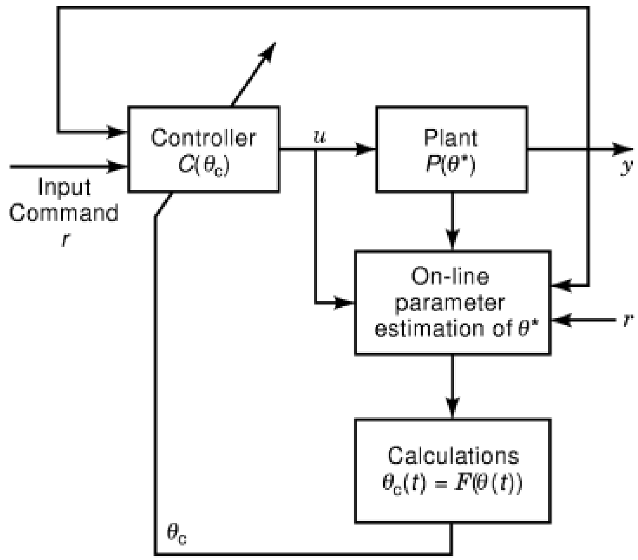

Fig. 3. Indirect adaptive control.

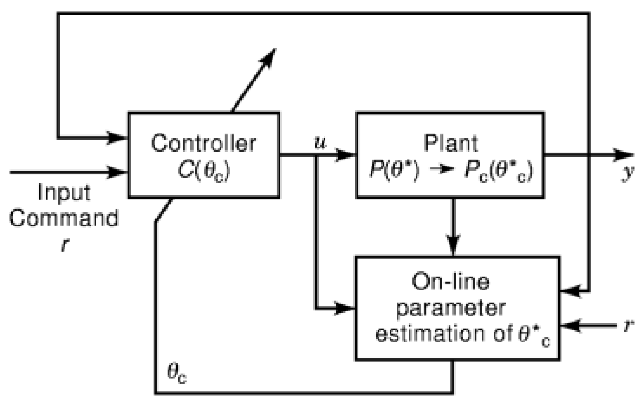

Fig. 4. Direct adaptive control.

plant output from the desired trajectory y_m, converges to zero with time. The transfer function matching is achieved by canceling the zeros of the plant transfer function $G(s)$ and replacing them with those of $W_m(s)$ through the use of the feedback controller $C(\theta^*_c)$. The cancellation of the plant zeros puts a restriction on the plant to be minimum-phase, that is, have stable zeros. If any plant zero is unstable, its cancellation may easily lead to unbounded signals.

The design of $C(\theta^*_c)$ requires the knowledge of the coefficients of the plant transfer $G(s)$. When $\theta*$ is unknown we use the certainty equivalence approach to replace the unknown θ^*_c in the control law with its estimate $\theta_c(t)$ obtained using the direct or the indirect approach. The resulting control schemes are known as MRAC and can be classified as *indirect MRAC* of the structure shown in Fig. 3 and *direct MRAC* of the structure shown in Fig. 4.

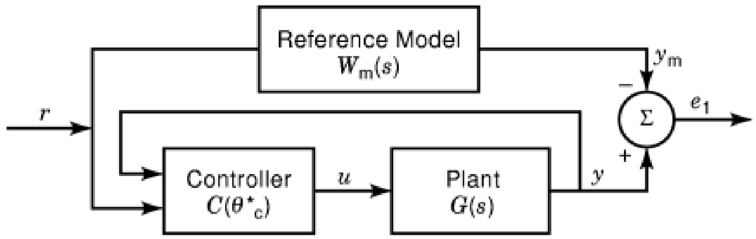

Fig. 5. Model reference control.

Adaptive Pole Placement Control

Adaptive pole placement control (*APPC*) is derived from the pole placement control (*PPC*) and regulation problems used in the case of LTI plants with known parameters. In PPC, the performance requirements are translated into desired locations of the poles of the closed-loop plant. A feedback control law is then developed that places the poles of the closed-loop plant at the desired locations. The structure of the controller $C(\theta^*_c)$ and the parameter vector θ^*_c are chosen so that the poles of the closed-loop plant transfer function from r to y are equal to the desired ones. The vector θ^*_c is usually calculated using an algebraic equation of the form

$$\theta^*_c = F(\theta^*) \tag{2}$$

where $\theta*$ is a vector with the coefficients of the plant transfer function $G(s)$.

As in the case of MRC, we can deal with the unknown-parameter case by using the certainty equivalence approach to replace the unknown vector θ^*_c with its estimate $\theta_c(t)$. The resulting scheme is referred to as *adaptive pole placement control* (*APPC*). If $\theta_c(t)$ is updated directly using an on-line parameter estimator, the scheme is referred to as *direct APPC.* If $\theta_c(t)$ is calculated using the equation

$$\theta_c(t) = F(\theta(t)) \tag{3}$$

where $\theta(t)$ is the estimate of $\theta*$ generated by an on-line estimator, the scheme is referred to as *indirect APPC.* The structure of direct and indirect APPC is the same as that shown in Figs. 3 and 4, respectively, for the general case. The design of APPC schemes is very flexible with respect to the choice of the form of the controller $C(\theta_c)$ and of the on-line parameter estimator.

Design of On-Line Parameter Estimators

As we mentioned in the previous sections, an adaptive controller may be considered as a combination of an on-line parameter estimator with a control law that is derived from the known-parameter case. The way this combination occurs and the type of estimator and control law used give rise to a wide class of different adaptive controllers with different properties. In the literature of adaptive control the on-line parameter estimator has often been referred to as the *adaptive law, update law,* or *adjustment mechanism.*

Some of the basic methods used to design adaptive laws are

(1) Sensitivity methods
(2) Positivity and Lyapunov design
(3) Gradient method and least-squares methods based on estimation error cost criteria

The sensitivity method is one of the oldest methods used in the design of adaptive laws and is briefly explained below.

Sensitivity Method. This method became very popular in the 1960s (3), and it is still used in many industrial applications for controlling plants with uncertainties. In adaptive control, the sensitivity method is used to design the adaptive law so that the estimated parameters are adjusted in a direction that minimizes a certain performance function. The adaptive law is driven by the partial derivative of the performance function with respect to the estimated parameters multiplied by an error signal that characterizes the mismatch between the actual and desired behavior. This derivative is called the *sensitivity function,* and if it can be generated online, then the adaptive law is implementable. In most earlier formulations of adaptive control, the sensitivity function cannot be generated online, and this constitutes one of the main drawbacks of the method. The use of approximate sensitivity functions that are implementable leads to adaptive control schemes whose stability properties are either weak or cannot be established.

Positivity and Lyapunov Design. This method of developing adaptive laws is based on the direct method of Lyapunov and its relationship to *positive real* functions. In this approach, the problem of designing an adaptive law is formulated as a stability problem where the differential equation of the adaptive law is chosen so that certain stability conditions based on Lyapunov theory are satisfied. The adaptive law developed is very similar to that based on the sensitivity method. The only difference is that the sensitivity functions in the approach are replaced with ones that can be generated online. In addition, the Lyapunov-based adaptive control schemes have none of the drawbacks of the MIT rule-based schemes.

The design of adaptive laws using Lyapunov's direct method was suggested by Grayson (4) and Parks (5) in the early 1960s. The method was subsequently advanced and generalized to a wider class of plants by Phillipson (6), Monopoli (7), and others (8,9,10,11).

Gradient and Least-Squares Methods Based on Estimation Error Cost Criteria. The main drawback of the sensitivity methods used in the 1960s is that the minimization of the performance cost function led to sensitivity functions that are not implementable. One way to avoid this drawback is to choose a cost function criterion that leads to sensitivity functions that are available for measurement. A class of such cost criteria is based on an error, referred to as the *estimation error* (12), that provides a measure of the discrepancy between the estimated and actual parameters. The relationship of the estimation error with the estimated parameters is chosen so that the cost function is convex, and its gradient with respect to the estimated parameters is implementable. Several different cost criteria may be used, and methods such as the gradient and least-squares may be adopted to generate the appropriate sensitivity functions.

On-Line Parameter Estimation

The purpose of this section is to present the design and analysis of a wide class of schemes that can be used for on-line parameter estimation. The essential idea behind on-line estimation is the comparison of the observed system response $y(t)$ with the output of a parametrized model $\hat{y}(\theta,t)$ whose structure is the same as that of the plant model. The parameter vector $\theta(t)$ is adjusted continuously so that $\hat{y}(\theta,t)$ approaches $y(t)$ as t increases. The on-line estimation procedure involves three steps: In the first step, an appropriate parametrization of the plant model is selected. The second step involves the selection of the adjustment law, referred to as the *adaptive law,* for generating or updating $\theta(t)$. The third and final step is the design of the plant input so that the properties of the adaptive law imply that $\theta(t)$ approaches the unknown plant parameter vector $\theta *$ as $t \to \infty$.

We start by considering the SISO plant

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x(0) = x_0 \\ y &= C^{\mathrm{T}} x \end{aligned} \tag{4}$$

where $x \in \Re^n$ and only y, u are available for measurement. Note that the plant equation can also be written as an nth-order differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_{n-1}u^{(n-1)} + b_{n-2}u^{(n-2)} + \cdots + b_0 u$$

The constants a_i, b_i are the plant parameters. We can express y^n as

$$y^n = \theta^{*\mathrm{T}} Y$$

where

$$\theta^* = [b_{n-1}, b_{n-1}, \ldots, b_0, a_{n-1}, a_{n-2}, \ldots a_0]^{\mathrm{T}}$$
$$Y = [u^{(n-1)}, u^{(n-2)}, \ldots, u, -y^{(n-1)}, -y^{(n-2)}, \ldots, -y]^{\mathrm{T}}$$

We can avoid the use of differentiators by filtering with our nth-order stable filter $1/\Lambda(s)$ to obtain

$$z = \theta^{*\mathrm{T}} \phi \qquad (5)$$

where

$$z = \frac{1}{\Lambda(s)} y^n = \frac{s^n}{\Lambda(s)} y$$
$$\phi = \left[\frac{a_{ni1}^{\mathrm{T}}(s)}{\Lambda(s)} u, -\frac{a_{ni1}^{\mathrm{T}}(s)}{\Lambda(s)} y \right]^{\mathrm{T}}$$
$$\Lambda(s) = s^n + \lambda_{n-1} s^{n-1} + \cdots + \lambda_0$$

In a similar way we can express the plant dynamics as follows:

$$z = W(s)\theta_{\lambda}^{*\mathrm{T}} \phi_{\lambda} \qquad (6)$$

where $W(s)$ is an appropriate proper stable transfer function and $\theta^*{}_{\lambda}$, ϕ_{λ} are defined similarly to $\theta*$, ϕ.

In Eqs. (5), (6) the unknown vectors $\theta*$, $\theta^*{}_{\lambda}$ appear linearly in equations where all the other terms are known a priori or can be measured online. We use these parametric models to estimate the unknown vectors $\theta*$ or $\theta^*{}_{\lambda}$ by using the following approaches.

SPR–Lyapunov Design Approach. We start by rewriting Eq. (6) as follows (for simplicity we drop the subscript λ):

$$z = W(s)L(s)\theta^{*\mathrm{T}} \psi \qquad (7)$$

where $\phi = L^{-1}(s)\psi$ and $L(s)$ is chosen so that $L^{-1}(s)$ is a proper transfer function and $W(s)L(s)$ is a proper *strictly positive real* (*SPR*) transfer function. Let $\theta(t)$ denote the estimate of $\theta*$ at time t. Then the estimated value of z

based on $\theta(t)$ is given by

$$\hat{z} = W(s)L(s)\theta^{\mathrm{T}}\psi$$

and the estimation error is given by

$$\epsilon_1 = z - \hat{z}$$

Let

$$\epsilon = z - \hat{z} - W(s)L(s)\epsilon n_s^2$$

denote the *normalized estimation error,* where $n^2{}_{\mathrm{s}}$ is the normalizing signal, which we design to satisfy

$$\frac{\phi}{m} \in \mathscr{L}_\infty, \qquad m^2 = 1 + n_s^2$$

Typical choices for n_{s} that satisfy this condition are $n^2{}_{\mathrm{s}} = \psi^{\mathrm{T}}P\psi$ for any $P = P^{\mathrm{T}} > 0$, and the like. When $\psi \in \mathscr{L}_\infty$, the condition is satisfied with $m = 1$, that is, $n_{\mathrm{s}} = 0$, in which case $\varepsilon = \varepsilon_1$. We express ε in terms of the parameter error $\tilde{\theta} = \theta - \theta^*$:

$$\epsilon = WL(-\tilde{\theta}^{\mathrm{T}}\phi - \epsilon n_s^2) \tag{8}$$

For simplicity, let us assume that $L(s)$ is chosen so that WL is strictly proper and consider the following state space representation of Eq. (8):

$$\begin{aligned} \dot{e} &= A_c e + B_c(-\tilde{\theta}^{\mathrm{T}}\phi - \epsilon n_s^2) \\ \epsilon &= C_c^{\mathrm{T}} e \end{aligned} \tag{9}$$

where A_{c}, B_{c}, and C_{c} are the matrices associated with a state-space representation that has a transfer function $W(s)L(s) = C^{\mathrm{T}}{}_{\mathrm{c}}(sI - A_{\mathrm{c}})^{-1}B_{\mathrm{c}}$.

Let us now consider the following Lyapunov-like function for the differential equation (9):

$$V(\tilde{\theta}, e) = \frac{e^{\mathrm{T}}P_c e}{2} + \frac{\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}}{2} \tag{10}$$

where $\Gamma = \Gamma^{\mathrm{T}} > 0$ is a constant matrix and $P_{\mathrm{c}} = P^{\mathrm{T}}{}_{\mathrm{c}} > 0$ satisfies the algebraic equations

$$\begin{aligned} P_c A_c + A_c^{\mathrm{T}} P_c &= -qq^{\mathrm{T}} - \nu L_c \\ P_c B_c &= C_c \end{aligned} \tag{11}$$

for some vector q, matrix $L_{\mathrm{c}} = L^{\mathrm{T}}{}_{\mathrm{c}} > 0$, and small constant $\nu > 0$. The existence of $P_{\mathrm{c}} = P^{\mathrm{T}}{}_{\mathrm{c}} > 0$ satisfying the above equations is guaranteed by the SPR property (12) of $W(s)L(s) = C^{\mathrm{T}}{}_{\mathrm{c}}(sI - A_{\mathrm{c}})^{-1}B_{\mathrm{c}}$. The time derivative of

V is given by

$$\dot{V}(\tilde{\theta}, e) = -\tfrac{1}{2}e^{\mathrm{T}}qq^{\mathrm{T}}e - \frac{\nu}{2}e^{\mathrm{T}}L_{c}e + e^{\mathrm{T}}P_{c}B_{c}(-\tilde{\theta}^{\mathrm{T}}\phi - \epsilon n_{s}^{2}) + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\tilde{\theta}} \tag{12}$$

We now need to choose $\tilde{\theta} = \dot{\theta}$ as a function of signals that can be measured so that the indefinite terms in $\dot{V}$ are canceled out. Because e is not available for measurement, $\dot{\theta}$ cannot depend on e explicitly.

We know that $P_cB_c = C_c$, which implies that $e^{\mathrm{T}}P_cB_c = e^{\mathrm{T}}C_c = \varepsilon$. Therefore

$$\dot{V}(\tilde{\theta}, e) = -\tfrac{1}{2}e^{\mathrm{T}}qq^{\mathrm{T}}e - \frac{\nu}{2}e^{\mathrm{T}}L_{c}e - \epsilon\tilde{\theta}^{\mathrm{T}}\phi - \epsilon^{2}n_{s}^{2} + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\tilde{\theta}} \tag{13}$$

The choice for $\dot{\theta} = \dot{\theta}$ to make $\dot{V} \le 0$ is now obvious, namely, for

$$\dot{\theta} = \dot{\tilde{\theta}} = \Gamma\epsilon\phi \tag{14}$$

we have

$$\dot{V}(\tilde{\theta}, e) = -\tfrac{1}{2}e^{\mathrm{T}}qq^{\mathrm{T}}e - \frac{\nu}{2}e^{\mathrm{T}}L_{c}e - \epsilon^{2}n_{s}^{2} \le 0 \tag{15}$$

Using the above inequality, we can prove the following theorem.

Theorem 1. The adaptive law in Eq. (4) guarantees that:

(i) $\theta, \varepsilon \in \mathscr{L}_\infty$.
(ii) $\varepsilon, \varepsilon n_s, \dot{\theta} \in \mathscr{L}_2$, independent of the boundedness of ϕ.
(iii) If $n_s, \phi, \dot{\phi} \in \mathscr{L}_\infty$, and ϕ is *perstistently exciting* (*PE*)—that is, there exist positive constant α_1, α_0, T_0 such that

$$\alpha_1 I \ge \frac{1}{T_0}\int_t^{t+T_0} \phi(\tau)\phi^{\mathrm{T}}(\tau)\,d\tau \ge \alpha_0 I \qquad \forall t \ge 0 \tag{16}$$

—then $\theta(t) \to \theta*$ exponentially fast.

The proof of the theorem can be found in Ref. 12.

Gradient Method. In this method, we consider the parametric model in Eq. (5). Similar to the previous subsection, we define $\theta(t)$ to be the on line estimate of $\theta*$ and the normalized estimation error as

$$\epsilon = \frac{z - \hat{z}}{m^2}$$

where $\hat{z} = \theta^{\mathrm{T}}(t)\phi$ and $m^2 = 1 + n^2{}_s$ and $n^2{}_s$ is chosen so that $\phi/m \in \mathscr{L}_\infty$. The adaptive law is designed to minimize the performance index $J(\cdot)$, i.e.,

$$\min_{\theta} J(\epsilon(\theta))$$

which gives

$$\dot{\theta} = -\Gamma \nabla J, \qquad \Gamma = \Gamma^{\mathrm{T}} > 0$$

Different choices for the performance index lead to different adaptive laws.

Let us consider the simple quadratic cost function (*instantaneous cost function*)

$$J(\theta) = \frac{\epsilon^2 m^2}{2} = \frac{(z - \theta^{\mathrm{T}}\phi)^2}{2m^2} \tag{17}$$

Applying the gradient method, the minimizing trajectory $\theta(t)$ is generated by the differential equation

$$\dot{\theta} = -\Gamma \nabla J(\theta)$$

where $\Gamma = \Gamma^{\mathrm{T}} > 0$ is a scaling matrix that we refer to as the adaptive gain. We have

$$\dot{\theta} = -\Gamma \nabla J(\theta) = \Gamma \frac{(z - \theta^{\mathrm{T}}\phi)\phi}{m^2} = \Gamma \epsilon \phi \tag{18}$$

The following theorem holds:

Theorem 2. The adaptive law in Eq. (18) guarantees that:

(i) ε, $\varepsilon n_{\mathrm{s}}\theta$, $\dot{\theta} \in \mathscr{L}_\infty$.
(ii) ε, $\varepsilon n_{\mathrm{s}}$, $\dot{\theta} \in \mathscr{L}_2$, independent of the boundedness of ϕ.
(iii) If n_{s}, $\phi \in \mathscr{L}_\infty$ and ϕ is PE, then $\theta(t) \to \theta*$ exponentially fast.

The proof of the theorem can be found in Ref. 12.

Least Squares. Let θ(t), ε, $\hat{z}$ be defined as above, and let $m^2 = 1 + n^2{}_{\mathrm{s}}$, $\theta(t)$ be the estimate of $\theta*$ at time t, and m satisfy $\phi/m \in \mathscr{L}_\infty$. We consider the following cost function:

$$\begin{aligned} J(\theta) = {} & \frac{1}{2}\int_0^t e^{-\beta(t-\tau)} \frac{[z(\tau) - \theta^{\mathrm{T}}(t)\phi(\tau)]^2}{m^2(\tau)}\, d\tau \\ & + \frac{1}{2} e^{-\beta t}(\theta - \theta_0)^{\mathrm{T}} Q_0 (\theta - \theta_0) \end{aligned} \tag{19}$$

where $Q_0 = Q^{\mathrm{T}}{}_0 > 0$, $\beta \geq 0$, $\theta_0 = \theta(0)$, which includes discounting of past data and a penalty on the initial estimate θ_0 of $\theta*$. Because z/m, $\phi/m \in \mathscr{L}_\infty$, we have that $J(\theta)$ is a convex function of θ over $\Re^n$ at each time t. Hence, any local minimum is also global and satisfies

$$\nabla J(\theta(t)) = 0 \qquad \forall t \geq 0$$

which yields the so-called *nonrecursive least-squares* algorithm

$$\theta(t) = P(t)\left(e^{-\beta t} Q_0 \theta_0 + \int_0^t e^{-\beta(t-\tau)} \frac{z(\tau)\phi(\tau)}{m^2(\tau)}\, d\tau\right) \tag{20}$$

where

$$P(t) = \left(e^{-\beta t}Q_0 + \int_0^t e^{-\beta(t-\tau)} \frac{\phi(\tau)\phi^{\mathrm{T}}(\tau)}{m^2(\tau)}\, d\tau\right)^{-1} \tag{21}$$

We can show that P, θ satisfy the differential equations

$$\dot{P} = \beta P - P\frac{\phi\phi^{\mathrm{T}}}{m^2}P, \qquad P(0) = P_0 = Q_0^{-1} \tag{22}$$

$$\dot{\theta} = P\epsilon\phi \tag{23}$$

We refer to Eqs. (22) and (23) as the *continuous-time recursive least-squares algorithm with forgetting factor.* The stability properties of the least-squares algorithm depend on the value of the forgetting factor β.

In the identification literature, Eqs. (22) and (23) with $\beta = 0$ are referred to as the *pure least-squares algorithm* and have a very similar form to the Kalman filter. For this reason, the matrix P is usually called the *covariance matrix.* The pure least-squares algorithm has the unique property of guaranteeing parameter convergence to constant values as described by the following theorem:

Theorem 3. The pure least-squares algorithm guarantees that:

(i) ε, $\varepsilon n_{\mathrm{s}}$, θ, $\dot{\theta}$, $P \in \mathscr{L}_\infty$.
(ii) ε, $\varepsilon n_{\mathrm{s}}$, $\dot{\theta} \in \mathscr{L}_2$.
(iii) $\lim_{t\to\infty} \theta(t) = \bar{\theta}$, where $\bar{\theta}$ is a constant vector.
(iv) If n_{s}, $\phi \in \mathscr{L}_\infty$ and ϕ is PE, then $\theta(t)$ converges to $\theta*$ as $t \to \infty$.

The proof of the theorem can be found in 12.

Bilinear Parametric Model. As will be shown in the next sections, a certain class of plants can be parametrized in terms of their desired controller parameters, which are related to the plant parameters via a Diophantine equation. Such parametrizations and their related estimation problem arise in direct adaptive control and, in particular, direct MRAC, which is discussed in the next section.

In these cases $\theta*$ appears in the form

$$z = W(s)[\rho^*(\theta^{*\mathrm{T}}\psi + z_0)] \tag{24}$$

where $\rho*$ is an unknown constant, z, ψ, z_0 are signals that can be measured, and $W(s)$ is a known proper transfer function with stable poles. Because the unknown parameters $\rho*$, $\theta*$ appear in a special bilinear form, we refer to Eq. (24) as the *bilinear parametric model.*

Known Sign of $\rho*$. The SPR–Lyapunov design approach and the gradient method with an instantaneous cost function discussed in the linear parametric case extend to the bilinear one in a rather straightforward manner.

Let us start with the SPR–Lyapunov design approach. We rewrite Eq. (24) in the form

$$z = W(s)L(s)\rho^*(\theta^{*\mathrm{T}}\phi + z_1) \tag{25}$$

where $z_1 = L^{-1}(s)z_0$, $\phi = L^{-1}(s)\psi$, and $L(s)$ is chosen so that $L^{-1}(s)$ is proper and stable and WL is proper and SPR. The estimate $\hat{z}$ of z and the normalized estimation error are generated as

$$\hat{z} = W(s)L(s)\rho(\theta^{\mathrm{T}}\phi + z_1) \tag{26}$$

$$\epsilon = z - \hat{z} - W(s)L(s)\epsilon n_s^2 \tag{27}$$

where n_s is designed to satisfy

$$\frac{\phi}{m}, \frac{z_1}{m} \in \mathcal{L}_\infty, \qquad m^2 = 1 + n_s^2 \tag{28}$$

and $\rho(t)$, $\theta(t)$ are the estimates of $\rho*$, $\theta*$ at time t, respectively. Letting $\dot{\rho} \triangleq \rho - \rho*$, $\tilde{\theta} \triangleq \theta - \theta*$, it follows from Eqs. (25) to (27) that

$$\epsilon = W(s)L(s)[\rho^*\theta^{*\mathrm{T}}\phi - \tilde{\rho}z_1 - \rho\theta^{\mathrm{T}}\phi - \epsilon n_s^2]$$

Now $\rho*\theta*^{\mathrm{T}}\phi - \rho\theta^{\mathrm{T}}\phi = \rho*\theta*^{\mathrm{T}}\phi - \rho*\theta^{\mathrm{T}}\phi + \rho*\theta^{\mathrm{T}}\phi - \rho\theta^{\mathrm{T}}\phi = -\rho*\tilde{\theta}^{\mathrm{T}}\phi - \dot{\rho}\theta^{\mathrm{T}}\phi$, and therefore,

$$\epsilon = W(s)L(s)[-\rho^*\tilde{\theta}^{\mathrm{T}}\phi - \tilde{\rho}\xi - \epsilon n_s^2], \qquad \xi = \theta^{\mathrm{T}}\phi + z_1 \tag{29}$$

By choosing

$$\begin{aligned} \dot{\tilde{\theta}} &= \dot{\theta} = \Gamma\epsilon\phi \operatorname{sgn}(\rho^*) \\ \dot{\tilde{\rho}} &= \dot{\rho} = \gamma\epsilon\xi \end{aligned} \tag{30}$$

we can see that the following theorem holds.

Theorem 4. The adaptive law in Eq. (30) guarantees that:

(i) $\varepsilon, \theta, \rho \in \mathcal{L}_\infty$.
(ii) $\varepsilon, \varepsilon n_s, \dot{\theta}, \dot{\rho} \in \mathcal{L}_2$.
(iii) If $\phi, \dot{\phi} \in \mathcal{L}_\infty$, ϕ is PE, and $\xi \in \mathcal{L}_2$, then $\theta(t)$ converges to $\theta*$ as $t \to \infty$.
(iv) If $\xi \in \mathcal{L}_2$, the estimate ρ converges to a constant $\bar{\rho}$ independent of the properties of ϕ.

The proof of the theorem can be found in Ref. 12. The case where the sign of $\rho*$ is unknown is also given in Ref. 12.

Model Reference Adaptive Control

Problem Statement. Consider the SISO LTI plant described by

$$y_p = G_p(s)u_p \tag{31}$$

where $G_p(s)$ is expressed in the form

$$G_p(s) = k_p \frac{Z_p(s)}{R_p(s)} \tag{32}$$

where Z_p, R_p are monic polynomials and k_p is a constant referred to as the *high-frequency gain*.

The reference model, selected by the designer to describe the desired characteristics of the plant, is described by

$$y_m = W_m(s)r$$

which is expressed in the same form as Eq. (32), that is,

$$W_m(s) = k_m \frac{Z_m(s)}{R_m(s)} \tag{33}$$

where $Z_m(s)$, $R_m(s)$ are monic polynomials and k_m is a constant.

The MRC objective is to determine the plant input u_p so that all signals are bounded and the plant output y_p tracks the reference model output y_m as close as possible for *any* given reference input $r(t)$ of the class defined above. We refer to the problem of finding the desired u_p to meet the control objective as the *MRC problem.* In order to meet the MRC objective with a control law that is implementable (i.e., a control law that is free of differentiators and uses only measurable signals), we assume that the plant and reference model satisfy the following assumptions:

Plant Assumptions.

P1. $Z_p(s)$ is a monic Hurwitz polynomial of degree m_p.
P2. An upper bound n on the degree n_p of $R_p(s)$ is known.
P3. The relative degree $n* = n_p - m_p$ of $G_p(s)$ is known.
P4. The sign of the high-frequency gain k_p is known.

Reference Model Assumptions.

M1. $Z_m(s)$, $R_m(s)$ are monic Hurwitz polynomials of degree q_m, p_m, respectively, where $p_m \leq n$.
M2. The relative degree $n^*{}_m = p_m - q_m$ of $W_m(s)$ is the same as that of $G_p(s)$, that is, $n^*{}_m = n*$.

MRC Schemes: Known Plant Parameters. In addition to assumptions P1 to P4 and M1, M2, let us also assume that the plant parameters, that is, the coefficients of $G_p(s)$, are known exactly. Because the plant is LTI and known, the design of the MRC scheme is achieved using linear system theory.

We consider the feedback control law

$$u_p = \theta_1^{*T} \frac{\alpha(s)}{\Lambda(s)} u_p + \theta_2^{*T} \frac{\alpha(s)}{\Lambda(s)} y_p + \theta_3^* y_p + c_0^* r \tag{34}$$

where

$$\alpha(s) \triangleq \begin{cases} \alpha_{n-2}(s) = [s^{n-2}, s^{n-3}, \ldots, s, 1]^{\mathrm{T}} & \text{for} \quad n \geq 2 \\ 0 & \text{for} \quad n = 1 \end{cases}$$

$c^*{}_0$, $\theta^*{}_3 \in \Re^1$, $\theta^*{}_1$, $\theta^*{}_2 \in \Re^{n-1}$ are constant parameters to be designed, and $\lambda(s)$ is an arbitrary monic Hurwitz polynomial of degree $n-1$ that contains $Z_{\mathrm{m}}(s)$ as a factor, i.e., $\Lambda(s) = \Lambda_0(s)Z_{\mathrm{m}}(s)$, which implies that $\lambda_0(s)$ is monic, Hurwitz, and of degree $n_0 = n - 1 - q_{\mathrm{m}}$. The controller parameter vector

$$\theta^* = [\theta_1^{*\mathrm{T}}, \theta_2^{*\mathrm{T}}, \theta_3^*, c_0^*]^{\mathrm{T}} \in \Re^{2n}$$

is to be chosen so that the transfer function from r to y_{p} is equal to $W_{\mathrm{m}}(s)$.

The input–output properties of the closed-loop plant are described by the transfer function equation

$$y_{\mathrm{p}} = G_{\mathrm{c}}(s)r \tag{35}$$

where

$$G_{\mathrm{c}}(s) = \frac{c_0^* k_{\mathrm{p}} Z_{\mathrm{p}} \Lambda^2}{\Lambda[\Lambda - \theta_1^{*\mathrm{T}}\alpha(s)]R_{\mathrm{p}} Z_{\mathrm{p}}[\theta_2^{*\mathrm{T}}\alpha(s) + \theta_3^*\Lambda]} \tag{36}$$

We can now meet the control objective if we select the controller parameters $\theta^*{}_1$, $\theta^*{}_2$, $\theta^*{}_3$, $c^*{}_0$ so that the closed-loop poles are stable and the closed-loop transfer function $G_{\mathrm{c}}(s) = W_{\mathrm{m}}(s)$ is satisfied for all $s \in C$. Choosing

$$c_0^* = \frac{k_{\mathrm{m}}}{k_{\mathrm{p}}} \tag{37}$$

and using $\Lambda(s) = \Lambda_0(s)Z_{\mathrm{m}}(s)$, the matching equation $G_{\mathrm{c}}(s) = W_{\mathrm{m}}(s)$ becomes

$$\begin{aligned} &\theta_1^{*\mathrm{T}}\alpha(s)R_{\mathrm{p}}(s) + k_{\mathrm{p}}[\theta_2^{*\mathrm{T}}\alpha(s) + \theta_3^*\Lambda(s)]Z_{\mathrm{p}}(s) \\ &\qquad = \Lambda(s)R_{\mathrm{p}}(s) - Z_{\mathrm{p}}(s)\Lambda_0(s)R_{\mathrm{m}}(s) \end{aligned} \tag{38}$$

Equating the coefficients of the powers of s on both sides of Eq. (38), we can express it in terms of the algebraic equation

$$S\bar{\theta}^* = p \tag{39}$$

where $\bar{\theta}* = [\theta^{*\mathrm{T}}{}_1, \theta^{*\mathrm{T}}{}_2, \theta^*{}_3]^{\mathrm{T}}$; S is an $(n + n_{\mathrm{p}} - 1) \times (2n - 1)$ matrix that depends on the coefficients of R_{p}, $k_{\mathrm{p}}Z_{\mathrm{p}}$, and Λ and p is an $n + n_{\mathrm{p}} - 1$ vector with the coefficients of $\Lambda R_{\mathrm{p}} - Z_{\mathrm{p}}\Lambda_0 R_{\mathrm{m}}$. The existence of $\bar{\theta}*$ to satisfy Eq. (39) and, therefore, Eq. (38) will very much depend on the properties of the matrix S. For example, if $n > n_{\mathrm{p}}$, more than one $\bar{\theta}*$ will satisfy Eq. (39), whereas if $n = n_{\mathrm{p}}$ and S is nonsingular, Eq. (39) will have only one solution.

Lemma 1. *Let the degrees of R_p, Z_p, Λ, Λ_0 and R_m be as specified in Eq. (34). Then*

(i) The solution $\bar{\theta}*$ of Eq. (38) or (39) always exists.
(ii) In addition, if R_{p}, Z_{p} are coprime and $n = n_{\mathrm{p}}$, then the solution $\bar{\theta}*$ is unique.

The proof of the lemma can be found in Ref. 12.

MRAC for SISO Plants. The design of MRAC schemes for the plant in Eq. (31) with unknown parameters is based on the certainty equivalence approach and is conceptually simple. With this approach, we develop a wide class of MRAC schemes by combining the MRC law, where $\theta*$ is replaced by its estimate $\theta(t)$, with different adaptive laws for generating $\theta(t)$ online. We design the adaptive laws by first developing appropriate parametric models for $\theta*$, which we then use to pick up the adaptive law of our choice from the preceding section.

Let us start with the control law

$$u_{\mathrm{p}} = \theta_1^{\mathrm{T}}(t)\frac{\alpha(s)}{\Lambda(s)}u_{\mathrm{p}} + \theta_2^{\mathrm{T}}(t)\frac{\alpha(s)}{\Lambda(s)}y_{\mathrm{p}} + \theta_3(t)y_{\mathrm{p}} + c_0(t)r \qquad (40)$$

whose state-space realization is given by

$$\begin{aligned} \dot{\omega}_1 &= F\omega_1 + gu_{\mathrm{p}}, \qquad \omega_1(0) = 0 \\ \dot{\omega}_2 &= F\omega_2 + gy_{\mathrm{p}}, \qquad \omega_2(0) = 0 \\ u_{\mathrm{p}} &= \theta^{\mathrm{T}}\omega \end{aligned} \qquad (41)$$

where $\theta = [\theta^{\mathrm{T}}{}_1, \theta^{\mathrm{T}}{}_2, \theta_3, c_0]^{\mathrm{T}}$ and $\omega = [\omega^{\mathrm{T}}{}_1, \omega^{\mathrm{T}}{}_2, y_{\mathrm{p}}, r]^{\mathrm{T}}$, and search for an adaptive law to generate $\theta(t)$, the estimate of the desired parameter vector $\theta*$.

It can be seen that under the above control law, the tracking error satisfies

$$e_1 = W_{\mathrm{m}}(s)\rho^*[u_{\mathrm{p}} - \theta^{*\mathrm{T}}\omega] \qquad (42)$$

where $\rho* = 1/c^*{}_0$, $\theta* = [\theta^{*\mathrm{T}}{}_1, \theta^{*\mathrm{T}}{}_2, \theta^*{}_3, c^*{}_0]^{\mathrm{T}}$. The above parametric model may be developed by using the matching Eq. (38) to substitute for the unknown plant polynomial $R_{\mathrm{p}}(s)$ in the plant equation and by canceling the Hurwitz polynomial $Z_{\mathrm{p}}(s)$. The parametric model in Eq. (42) holds for any relative degree of the plant transfer function.

A linear parametric model for $\theta*$ may also be developed from Eq. (42). Such a model takes the form

$$z = \theta^{*\mathrm{T}}\phi_{\mathrm{p}} \qquad (43)$$

where

$$\begin{aligned} z &= W_{\mathrm{m}}(s)u_{\mathrm{p}} \\ \phi_{\mathrm{p}} &= [W_{\mathrm{m}}(s)\omega_1^{\mathrm{T}}, W_m(s)\omega_2^{\mathrm{T}}, W_m(s)y_p, y_p]^{\mathrm{T}} \\ \theta^* &= [\theta_1^{*\mathrm{T}}, \theta_2^{*\mathrm{T}}, \theta_3^*, c_0^*]^{\mathrm{T}} \end{aligned}$$

The main equations of several MRAC schemes formed by combining Eq. (41) with an adaptive law based on Eq. (42) or (43). The following theorem gives the stability properties of the MRAC scheme.

Theorem 5. The closed-loop MRAC scheme in Eq. (41), with $\theta(t)$ adjusted with any adaptive law with normalization based on the model in Eq. (42) or (43) as described in the preceding section, has the following properties:

(i) All signals are uniformly bounded.
(ii) The tracking error $e_1 = y_{\mathrm{p}} - y_{\mathrm{m}}$ converges to zero as $t \to \infty$.

(iii) If the reference input signal r is sufficiently rich of order $2n$, $\dot{r} \in \mathscr{L}_\infty$, and R_p, Z_p are coprime, then the tracking error e_1 and parameter error $\tilde{\theta} = \theta - \theta*$ converge to zero for the adaptive law with known sgn(k_p).

The proof of the theorem can be found in Ref. 12.

Adaptive Pole Placement Control

In the preceding section we considered the design of a wide class of MRAC schemes for LTI plants with stable zeros. The assumption that the plant is minimum-phase, that is, it has stable zeros, is rather restrictive in many applications. A class of control schemes that is popular in the known-parameter case are those that change the poles of the plant and do not involve plant zero–pole cancellations. These schemes are referred to as *pole placement* schemes and are applicable to both minimum- and nonminimum-phase LTI plants. The combination of a pole placement control law with a parameter estimator or an adaptive law leads to an *adaptive pole placement control* (*APPC*) scheme that can be used to control a wide class of LTI plants with unknown parameters.

Problem Statement. Consider the SISO LTI plant

$$y_p = G_p(s)u_p, \qquad G_p(s) = \frac{Z_p(s)}{R_p(s)} \tag{44}$$

where $G_p(s)$ is proper and $R_p(s)$ is a monic polynomial. The control objective is to choose the plant input u_p so that the closed-loop poles are assigned to those of a given monic Hurwitz polynomial $A*(s)$. The polynomial $A*(s)$, referred to as the desired closed-loop characteristic polynomial, is chosen according to the closed-loop performance requirements. To meet the control objective, we make the following assumptions about the plant:

P1. $R_p(s)$ is a monic polynomial whose degreen n is known.
P2. $Z_p(s)$, $R_p(s)$ are coprime, and degree(Z_p) $< n$.

Assumptions P1 and P2 allow Z_p, R_p to be non-Hurwitz, in contrast to the MRC case, where Z_p is required to be Hurwitz. If, however, Z_p is Hurwitz, the MRC problem is a special case of the general pole placement problem defined above with $A*(s)$ restricted to have Z_p as a factor.

In general, by assigning the closed-loop poles to those of $A*(s)$, we can guarantee closed-loop stability and convergence of the plant output y_p to zero provided there is no external input. We can also extend the PPC objective to include tracking, where y_p is required to follow a certain class of reference signals y_m, by using the internal model principle as fol8lows: The reference signal $y_m \in \mathscr{L}_\infty$ is assumed to satisfy

$$Q_m(s)y_m = 0 \tag{45}$$

where $Q_m(s)$, the internal model of y_m, is a known monic polynomial of degree q with nonrepeated roots on the $j\omega$ axis and satisfies

P3. $Q_m(s)$, $Z_p(s)$ are coprime.

For example, if y_p is required to track the reference signal $y_m = 2 + \sin 2t$, then $Q_m(s) = s(s^2 + 4)$ and therefore, according to assumption P3, $Z_p(s)$ should not have s or $s^2 + 4$ as a factor.

In addition to assumptions P1 to P3, let us also assume that the coefficients of $Z_p(s)$, $R_p(s)$, i.e., the plant parameters are known exactly.

We consider the control law

$$Q_m(s)L(s)u_p = -P(s)y_p + M(s)y_m \qquad (46)$$

where $P(s)$, $L(s)$, $M(s)$ are polynomials [with $L(s)$ monic] of degree $q + n - 1$, $n - 1$, $q + n - 1$, respectively, to be found, and $Q_m(s)$ satisfies Eq. (45) and assumption P3.

Applying Eq. (46) to the plant in Eq. (44), we obtain the closed-loop plant equation

$$y_p = \frac{Z_p M}{LQ_m R_p + PZ_p} y_m \qquad (47)$$

whose characteristic equation

$$LQ_m R_p + PZ_p = 0 \qquad (48)$$

has order $2n + q - 1$. The objective now is to choose P, L such that

$$LQ_m R_p + PZ_p = A^* \qquad (49)$$

is satisfied for a given monic Hurwitz polynomial $A*(s)$ of degree $2n + q - 1$. It can be seen that assumptions P2 and P3 guarantee that L, P satisfying Eq. (49) exist and are unique. The solution for the coefficients of $L(s)$, $P(s)$ of Eq. (49) may be obtained by solving the algebraic equation

$$S_l \beta_l = \alpha_l^* \qquad (50)$$

where S_l is the Sylvester matrix of $Q_m R_p$, Z_p of dimension $2(n + q) \times 2(n + q)$,

$$\beta_l = [l_q^T, p^T]^T, \qquad \alpha_l^* = [\underbrace{0, \ldots, 0}_{q}, 1, \alpha^{*T}]^T$$

$$l_q = [\underbrace{0, \ldots, 0}_{q}, 1, l^T]^T \in \Re^{n+q}$$

and $l, p, a*$ are the vectors whose entries are the coefficients of $L(s), P(s)$ and $A*(s)$, respectively. The coprimeness of $Q_m R_p$, Z_p guarantees that S_l is nonsingular; therefore, the coefficients of $L(s)$, $P(s)$ may be computed from the equation

$$\beta_l = S_l^{-1} \alpha_l^*$$

The tracking error $e_1 = y_p - y_m$ is given by

$$e_1 = \frac{Z_p M - A^*}{A^*} y_m = \frac{Z_p}{A^*}(M - P) y_m - \frac{LR_p}{A^*} Q_m y_m \qquad (51)$$

For zero tracking error, Eq. (51) suggests the choice of $M(s) = P(s)$ to null the first term. The second term in Eq. (51) is nulled by using $Q_m y_m = 0$. Therefore, the pole placement and tracking objective are achieved by using the control law

$$Q_m L u_p = -P(y_p - y_m) \tag{52}$$

which is implemented using $n + q - 1$ integrators to realize $C(s) = P(s)/Q_m(s)L(s)$. Because $L(s)$ is not necessarily Hurwitz, the realization of Eq. (52) with $n + q - 1$ integrators may have a transfer function, namely $C(s)$, with poles outside C^-. An alternative realization of Eq. (52) is obtained by rewriting it as

$$u_p = \frac{\Lambda - LQ_m}{\Lambda} u_p - \frac{P}{\Lambda}(y_p - y_m) \tag{53}$$

where Λ is any monic Hurwitz polynomial of degree $n + q - 1$. The control law (53) is implemented using $2(n + q - 1)$ integrators to realize the proper stable transfer functions $(\Lambda - LQ_m)/\Lambda$, P/Λ.

APPC. The APPC scheme that meets the control objective for the unknown plant is formed by combining the control law in Eq. (53) with an adaptive law based on the parametric model in Eq. (5). The adaptive law generates on-line estimates θ_a, θ_b of the coefficient vectors θ^*_a of $R_p(s) = s^n + \theta^{*T}{}_a\alpha_{n-1}(s)$ and θ^*_b of $Z_p(s) = \theta^{*T}{}_b\alpha_{n-1}(s)$, respectively, to form the estimated plant polynomials $\hat{R}_p(s, t) = s^n + \theta^T{}_a\alpha_{n-1}(s)$, $\hat{Z}_p(s, t) = \theta^T{}_b\alpha_{n-1}(s)$. The estimated plant polynomials are used to compute the estimated controller polynomials $\hat{L}(s, t)$, $\hat{P}(s, t)$ by solving the Diophantine equation

$$\hat{L}Q_m \cdot \hat{R}_p + \hat{P} \cdot \hat{Z}_p = A^* \tag{54}$$

for $\hat{L}$, $\hat{P}$ pointwise in time, or the algebraic equation

$$\hat{S}_l \hat{\beta}_l = \alpha^*_l \tag{55}$$

for $\hat{\beta}_l$, where $\hat{S}_l$ is the Sylvester matrix of $\hat{R}_p Q_m$, $\hat{Z}_p$; $\hat{\beta}_l$ contains the coefficients of $\hat{L}$, $\hat{P}$; and $\alpha^*{}_l$ contains the coefficients of $A*(s)$. The control law in the unknown-parameter case is then formed as

$$u_p = (\Lambda - \hat{L}Q_m)\frac{1}{\Lambda}u_p - \hat{P}\frac{1}{\Lambda}(y_p - y_m) \tag{56}$$

Because different adaptive laws may be picked up from the section "On-Line Parameter Estimation" above, a wide class of APPC schemes may be developed.

The implementation of the APPC scheme requires that the solution of the polynomial Eq. (54) for $\hat{L}$, $\hat{P}$ or of the algebraic Eq. (55) for $\hat{\beta}_l$ exists at each time. The existence of this solution is guaranteed provided that $\hat{R}_p(s, t)Q_m(s)$, $\hat{Z}_p(s, t)$ are coprime at each time t, that is, the Sylvester matrix $\hat{S}_l(t)$ is nonsingular at each time t.

Theorem 6. Assume that the estimated plant polynomials $\hat{R}_p Q_m$, $\hat{Z}_p$ are strongly coprime at each time t. Then all the signals in the closed-loop APPC scheme are u.b., and the tracking error converges to zero asymptotically with time.

The proof of the theorem can be found in Ref. 12. The assumption that the estimated polynomials are strongly coprime at each time t is restrictive and cannot be guaranteed by the adaptive law. Methods that relax this assumption are given in Ref. 12.

Adaptive Control of Nonlinear Systems

In the previous sections we dealt with the problem of designing controllers for linear time-invariant systems. In this section, we show how the techniques of adaptive control of linear systems can be extended or modified for nonlinear systems. Although the techniques presented can be applied to a variety of nonlinear systems, we will concentrate our attention on adaptive state feedback control of SISO feedback-linearizable systems.

Feedback-Linearizable Systems in Canonical Form. We start with an nth-order SISO feedback-linearizable system in canonical form, whose dynamics are as follows:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ &\vdots\\ \dot{x}_{\mathrm{n}} &= f(x_1, x_2, \ldots, x_n) + g(x_1, x_2, \ldots, x_n)u\\ y &= x_1\end{aligned} \tag{57}$$

where y, $u \in \Re$ are the scalar system output and input, respectively, f, g are smooth vector fields, and $x \triangleq [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$ is the state vector of the system. In order for the system in Eq. (57) to be controllable and feedback-linearizable we assume that

A1. A lower bound $\varepsilon*$ for $g(x)$ [i.e., $|g(x)| > \varepsilon* > 0\ \forall x \in \Re^n$] and the sign of $g(x)$ are known.

The control objective is to find the control input u that guarantees signal boundedness and forces y to follow the output y_{m} of the reference model

$$\begin{aligned}\dot{x}_{\mathrm{m}} &= Ax_{\mathrm{m}} + br\\ y_{\mathrm{m}} &= c^{\mathrm{T}}x_{\mathrm{m}}\end{aligned} \tag{58}$$

where A is a Hurwitz $n \times n$ matrix, $r \in \mathscr{L}_\infty$, and therefore $x_{\mathrm{m}} \in \mathscr{L}_\infty$. In order to have a well-posed problem, it is assumed that the relative degree of the reference model is equal to n. If $e \triangleq y_{\mathrm{m}} - y$ is the tracking error, then its nth time derivative satisfies

$$e^{(n)} = c^{\mathrm{T}}A^n x_{\mathrm{m}} + c^{\mathrm{T}}A^{n-1}br - f(x) - g(x)u \tag{59}$$

Let $\bar{h}(s) = s^n + k_1 s^{n-1} + \cdots + k_n$ be a Hurwitz polynomial (here s denotes the d/dt operator). Also let $\varepsilon \triangleq [e, \dot{e}, \ldots, e^{(n-1)}]^{\mathrm{T}}$. Under assumption A1, the system Eq. (57) is a feedback-linearizable system. Therefore, if we know the vector fields f and g, we can apply the static feedback

$$u = \frac{-f(x) + c^{\mathrm{T}}A^n x_{\mathrm{m}} + c^{\mathrm{T}}A^{n-1}br + k^{\mathrm{T}}\epsilon}{g(x)} \tag{60}$$

where $k \triangleq [k_n, k_{n-1}, \ldots, k_1]^{\mathrm{T}}$. Then the error system in Eq. (59) becomes

$$\bar{h}(s)[e] = 0$$

which implies that e, $\varepsilon \in \mathscr{L}_\infty$ and therefore all closed-loop signals are bounded, and $\lim_{t\to\infty} e(t) = 0$.

In many cases, the vector fields f and g are not completely known and thus adaptive versions of the feedback law (60) have to be applied. For instance, using the usual assumption of linear parametrization, if the vector fields f and g are of the form

$$\begin{aligned} f(x) &= \theta_1^{\mathrm{T}}\phi_f(x) \\ g(x) &= \theta_2^{\mathrm{T}}\phi_g(x) \end{aligned} \tag{61}$$

where θ_i, $i = 1, 2$, are vectors with unknown constant parameters, one may replace the feedback law in Eq. (60) with the *certainty-equivalent* one [the *certainty-equivalent feedback-linearizing* (*CEFL*) controller]

$$u = \frac{-\hat{\theta}_1^{\mathrm{T}}\phi_f(x) + c^{\mathrm{T}}A^n x_{\mathrm{m}} + c^{\mathrm{T}}A^{n-1}br + k^{\mathrm{T}}\epsilon}{\hat{\theta}_2^{\mathrm{T}}\phi_g(x)} \tag{62}$$

where $\hat{\theta}_i$, $i = 1, 2$, are the estimates of the unknown parameter vectors θ_i, $i = 1, 2$. These estimates are generated by an on-line adaptive law. We next propose the following adaptive laws for updating $\hat{\theta}_i$:

$$\dot{\hat{\theta}}_i = -\Gamma_i \overline{\phi}_i^{\mathrm{T}} \epsilon \tag{63}$$

where Γ_i, $i = 1, 2$, are symmetric positive definite matrices and $\bar{\phi}_1 = b_c \phi^{\tau}{}_f$, $\bar{\phi}_2 = -u b_c \phi^{\tau}{}_g$, $b_c = [0, \ldots, 0, 1]^{\tau}$. The next theorem summarizes the properties of the proposed control law.

Theorem 7. Consider the system in Eq. (57) and the feedback control law in Eqs. (62) and (63). Let assumption A1 hold. Then, if $\hat{\theta}^{\mathrm{T}}{}_2(t)\phi_g(x(t)) \neq 0$ for all t, all the closed-loop signals are bounded and the tracking error converges to asymptotically to zero.

Parametric-Pure-Feedback Systems. Let us now try to extend the results of the previous section to nonlinear systems that take the form

$$\begin{aligned} \dot{z}_i &= f_{i0}(z_1, \ldots, z_{i+1}) + \theta^{\mathrm{T}} f_{i2}(z_1, \ldots, z_{i+1}), \qquad 1 \le i \le n-1 \\ \dot{z}_n &= f_{n1}(z_1, \ldots, z_n) + \theta^{\mathrm{T}} f_{n2}(z_1, \ldots, z_n) + [g_{n1}(z_1, \ldots, z_n) \\ &\quad + \theta^{\mathrm{T}} g_{n2}(z_1, \ldots, z_n)]u \\ y &= z_1 \end{aligned} \tag{64}$$

where $u, z_i \in \Re, f_{ij}, g_{nj}$ are smooth functions, and $\theta \in \Re^p$ is the vector of constant but unknown system parameters. Let us rewrite Eq. (64) as

$$\begin{aligned} \dot{z}_i &= z_{i+1} + f_{i1}(z_1, \ldots, z_{i+1})_\theta^{\mathrm{T}} f_{i2}(z_1, \ldots, z_{i+1}), \qquad 1 \le i \le n-1 \\ \dot{z}_n &= f_{n1}(z_1, \ldots, z_n) + \theta^{\mathrm{T}} f_{n2}(z_1, \ldots, z_n) + [g_{n1}(z_1, \ldots, z_n) \\ &\quad + \theta^{\mathrm{T}} g_{n2}(z_1, \ldots, z_n)]u \\ y &= z_1 \end{aligned} \tag{65}$$

where $f_{i1}(\cdot) = f_{i0}(\cdot) - z_{i+1}$. Systems of the form in Eq. (65) are called parametric-pure-feedback (*PPF*) systems (13,14). Note that the above class of systems includes as a special case the system in Eq. (57) of the previous section.

The control objective is to force the system output y to asymptotically track a reference signal y_m. We assume that the first $n-1$ time derivatives of y_m are known. Also it will be assumed that y_m and its first $n-1$ time derivatives are bounded and smooth signals.

Let us now assume that the parameter vector θ is known and construct a control law that meets the control objectives. Before we design the feedback law, we will transform the system in Eq. (64) into a suitable form. The procedure we will follow is based on the *backstepping integrator* principle (13).

Step 0. Let $\zeta_1 \triangleq z_1 - y_m$. Let also $c_1, \ldots, c_n$ be positive constants to be chosen later.

Step 1. Using the chain-of-integrators method, we see that, if z_2 were the control input in the z_1 part of Eq. (65) and θ were known, then the *control law*

$$z_2 = -f_{11}(z_1, z_2) - \theta^{\mathrm{T}} f_{12}(z_1, z_2) + \dot{y}_m - c_1\zeta_1 \qquad (66)$$

would result in a globally asymptotically stable tracking, since such a control law would transform the z_1 part of Eq. (65) as follows:

$$\dot{\zeta}_1 = -c_1\zeta_1$$

However, the state z_2 is not the control. We therefore define ζ_2 to be the difference between the actual z_2 and its desired expression in Eq. (66):

$$\zeta_2 \triangleq z_2 + f_{11}(z_1, z_2) + \theta^{\mathrm{T}} f_{12}(z_1, z_2) - \dot{y}_m + c_1\zeta_1 \qquad (67)$$

Step 2. Using the above definition of ζ_2, the definition of ζ_1, and the z_1 part of Eq. (65), we find that

$$\begin{aligned}\dot{\zeta}_1 &= z_2 + f_{11}(z_1, z_2) + \theta^{\mathrm{T}} f_{12}(z_1, z_2) - \dot{y}_m \\ &= -c_1\zeta_1 + \zeta_2\end{aligned} \qquad (68)$$

Step 2. Using the above definitions of ζ_1, ζ_2, we have that

$$\begin{aligned}\dot{\zeta}_2 = &\left(\frac{\partial f_{11}}{\partial z_1} + \theta^{\mathrm{T}}\frac{\partial f_{12}}{\partial z_1}\right)[z_2 + f_{11}(z_1, z_2) + \theta^{\mathrm{T}} f_{12}(z_1, z_2)] - \ddot{y}_m \\ &+ \left(1 + \frac{\partial f_{11}}{\partial z_2} + \theta^{\mathrm{T}}\frac{\partial f_{12}}{\partial z_2}\right) \\ &\times [z_3 + f_{21}(z_1, z_2, z_3) + \theta^{\mathrm{T}} f_{22}(z_1, z_2, z_3)] \\ &+ c_1[z_2 + f_{11}(z_1, z_2) + \theta^{\mathrm{T}} f_{12}(z_1, z_2) - \dot{y}_m] \\ \triangleq\, & z_3 + \varphi_2(z_1, z_2, z_3, y_m, \dot{y}_m, \ddot{y}_m) + \vartheta_{(2)}^{\mathrm{T}} w_2(z_1, z_2, z_3)\end{aligned} \qquad (69)$$

where $\vartheta_{(2)}$ is a $(p + p^2)$-dimensional vector that consists of all elements that are either of the form $\theta_{2,i}$ or of the form $\theta_{2,i}\theta_j$, where by $\theta_{2,i}$ we denote the ith entry of the vector θ. In the system (69) we will think of z_3 as our control input. Therefore, as in step 1, we define the new state ζ_3 as

$$\zeta_3 \triangleq z_3 + \varphi_2(z_1, z_2, z_3, y_m \dot{y}_m, \ddot{y}_m) + \vartheta_{(2)}^{\mathrm{T}} w_2(z_1, z_2, z_3) + c_2\zeta_2 \qquad (70)$$

Substituting Eq. (70) into Eq. (69) yields

$$\dot{\zeta}_2 = -c_2\zeta_2 + \zeta_3 \tag{71}$$

Step i ($2 < i \le n-1$). Using the definitions of $\zeta_1, \ldots, \zeta_i$ and working as in the previous steps, we may express the derivative of ζ_i as

$$\begin{aligned}\dot{\zeta}_i = {} & z_{i+1} + \varphi_i(z_1, \ldots, z_{i+1}, y_\mathrm{m}, \dot{y}_\mathrm{m}, \ddot{y}_\mathrm{m}, \ldots, y_\mathrm{m}^{(i)}) \\ & + \vartheta_{(i)}^\mathrm{T} w_i(z_1, \ldots, z_{i+1})\end{aligned} \tag{72}$$

where the vector $\vartheta_{(i)}$ contains all the terms of the form $\theta_{i1}\theta_{i2}\cdots\theta_i$ with $1 \le j \le i$. Defining now ζ_{i+1} as

$$\begin{aligned}\zeta_{i+1} \triangleq {} & z_{i+1} + \varphi_i(z_1, \ldots, z_{i+1}, y_\mathrm{m}, \dot{y}_\mathrm{m}, \ddot{y}_\mathrm{m}, \ldots, y_\mathrm{m}^{(i)}) \\ & + \vartheta_{(i)}^\mathrm{T} w_i(z_1, \ldots, z_{i+1}) + c_i\zeta_i\end{aligned} \tag{73}$$

we obtain that

$$\dot{\zeta}_i = -c_i\zeta_i + \zeta_{i+1} \tag{74}$$

Step n. Using the definitions of $\zeta_1, \ldots, \zeta_{n-1}$ and working as in the previous steps, we may express the derivative of ζ_n as follows:

$$\begin{aligned}\dot{\zeta}_n = {} & \varphi_n(z_1, \ldots, z_n, Y_\mathrm{m}) + \vartheta^\mathrm{T} w_n(z_1, \ldots, z_n) \\ & + [\gamma_0(z_1, \ldots, z_n) + \vartheta^\mathrm{T}\gamma_1(z_1, \ldots, z_n)]u\end{aligned} \tag{75}$$

where the vector ϑ contains all the terms of the form $\theta_{i1}\theta_{i2}\cdots\theta_{ij}$ with $1 \le j \le n$, $Y_\mathrm{m} \triangleq [y_\mathrm{m}, \dot{y}_\mathrm{m}, \ddot{y}_\mathrm{m}, \ldots, y^{(n-1)}{}_\mathrm{m}]^\mathrm{T}$, and $[\gamma_0(z_1, \ldots, z_n) + \vartheta^\mathrm{T}\gamma_1(z_1, \ldots, z_n)]$ is given by

$$\begin{aligned}[\gamma_0(z) + \varphi^\mathrm{T}\gamma_1(z)] \triangleq {} & \left(1 + \frac{\partial f_{11}}{\partial z_2} + \theta^\mathrm{T}\frac{\partial f_{12}}{\partial z_2}\right)\cdots \\ & \left(1 + \frac{\partial f_{(n-1)1}}{\partial z_n} + \theta^\mathrm{T}\frac{\partial f_{(n-1)2}}{\partial z_n}\right) \times [g_{n1}(z) + \theta^\mathrm{T} g_{n2}(z)]\end{aligned} \tag{76}$$

Using the definitions of $\zeta_1, \ldots, \zeta_n$, and rearranging terms, we may rewrite Eq. (75) as follows:

$$\begin{aligned}\dot{\zeta}_n = {} & -c_n\zeta_n + \varphi(z_1, \ldots, z_n, Y_\mathrm{m}) + \vartheta^\mathrm{T} w(z_1, \ldots, z_n) \\ & + [\gamma_0(z_1, \ldots, z_n) + \vartheta^\mathrm{T}\gamma_1(z_1, \ldots, z_n)]u\end{aligned} \tag{77}$$

Using the above methodology, we have therefore transformed the system in Eq. (65) into the following one:

$$\begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \\ \vdots \\ \dot{\zeta}_{n-1} \\ \dot{\zeta}_n \end{bmatrix} = \underbrace{\begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 \\ 0 & -c_2 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -c_{n-1} & 1 \\ 0 & \cdots & 0 & 0 & -c_n \end{bmatrix}}_{A_0} \underbrace{\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{n-1} \\ \zeta_n \end{bmatrix}}_{\zeta} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{b_0} (\varphi + \vartheta^{\mathrm{T}} w + [\gamma_0 + \vartheta^{\mathrm{T}} \gamma_1] u) \tag{78}$$

The above system is feedback-linearizable if the following assumption holds.

A1. $\gamma_0(z) + \vartheta^{\mathrm{T}}\gamma_1(z) \neq 0$ for all z.

Note now that in the case where θ (and thus ϑ) is known, a controller that meets the control objective is the controller of the form

$$u = -\frac{\varphi + \vartheta^{\mathrm{T}} w}{\gamma_0 + \vartheta^{\mathrm{T}} \gamma_1} \tag{79}$$

Under the above control law, the closed-loop dynamics become

$$\dot{\zeta} = A_0 \zeta$$

It can be shown that the matrix A_0 is a stability matrix, provided that $c_i > 2$.

Theorem 8. The control law in Eq. (79) guarantees that all the closed-loop signals are bounded and that the tracking error converges to zero exponentially fast, provided that the design constants c_i satisfy $c_i > 2$.

In the case where the vector θ is not known, the certainty-equivalence principle can be employed in order to design an adaptive controller for the system. However, the problem of designing parameter estimators for the unknown parameters is not as easy as it was in the linear case. This can be seen from the fact that the "states" ζ_i, $i > 1$, are not available for measurement, since they depend on the unknown parameters. To overcome this problem a recursive design approach similar to the approach above can be constructed. The difference between this approach [called *adaptive backstepping* (13)] and the approach presented above is the following: in the approach presented above the "states" ζ_i, $i > 1$, depend on the unknown vector of parameters θ; in the new approach they are appropriately redefined so they depend on the parameter estimate vector $\hat{\theta}$.

Then the derivatives of ζ_i, $i > 1$, depend on the derivatives of the parameter estimates $\hat{\theta}$. In order to overcome this problem, the adaptive backstepping approach makes use of the so-called *tuning functions* (13).

Next we present the adaptive controller that results from applying the adaptive backstepping procedure to the system in Eq. (65) for the case where

A2. $\theta^{\mathrm{T}} f_{i2}(\cdot)$ are independent of z_{i+1} and $\theta^{\mathrm{T}} g_{n2} = 0$.

Also for simplicity, and without loss of generality, we will assume that $f_{i1}(\cdot) = 0$. The case where assumption A2 is not valid will be treated in the next subsection. The adaptive controller that results from applying the adaptive backstepping procedure is recursively defined as follows:

- Control law:

$$u = \frac{\alpha_n + y_{\mathrm{m}}^{(n)}}{g_{n1}} \tag{80}$$

- Parameter update law:

$$\dot{\hat{\theta}} = -\Gamma \tau_n \tag{81}$$

- Tuning functions:

$$\begin{aligned}
\tau_i &= \tau_{i-1} + \omega_i \bar{\zeta}_i \\
\bar{\zeta}_i &= z_i - y_{\mathrm{m}}^{(i-1)} - \alpha_{i-1} \\
\alpha_{i-1} &= -\bar{\zeta}_{i-1} - c_i \bar{\zeta}_i - \omega_i^{\mathrm{T}} \hat{\theta} \\
&\quad + \sum_{k=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial z_k} z_{k+1} + \frac{\partial \alpha_{i-1}}{\partial y_{\mathrm{m}}^{(k-1)}} y_{\mathrm{m}}^{(k)} \right) \\
&\quad - \kappa_i |\omega_i|^2 \bar{\zeta}_i + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i + \sum_{k=2}^{i-1} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \omega_i \bar{\zeta}_k
\end{aligned}$$

- Regressor vectors:

$$\omega_i = f_{i2} - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial z_k} f_{k2}$$

Here $c_i > 2$, κ_i are positive design constants, and $\Gamma = \Gamma^{\mathrm{T}} > 0$ is a positive definite design matrix. The next theorem summarizes the properties of the above control law.

Theorem 9. Suppose that assumptions A1′, A2 hold. Then the above adaptive control law guarantees that all the closed-loop signals are bounded and that the tracking error converges asymptotically to zero.

The proof of the theorem can be found in Ref. 13.

A Control Law That Overcomes the Loss-Of-Controllability Problem. A significant problem that arises in adaptive control of linear-in-the-parameters feedback-linearizable systems is the computation of the feedback control law when the identification model becomes uncontrollable although the actual system is controllable; so far, there is no known solution to this problem. For instance, for the case of the system in Eq. (57) the parameter estimation techniques used in adaptive control cannot guarantee, in general, that $|\hat{\theta}_2(t)^{\mathrm{T}}\phi_{\mathrm{g}}(x(t))| > 0$ for each time t, that is, they cannot guarantee that the identification model is controllable. Also, for the case of PPF systems presented in the previous subsection, the adaptive backstepping techniques guarantee global stability only in the case where assumption A2 is valid. Such restrictions are made because the computation of the adaptive control law depends on the existence of the inverse of the matrix that consists of the estimated input vector fields (or the Lie derivatives of the output functions along those vector fields). Even in the case of known parameters where the inverse of the corresponding matrix exists (this is trivially satisfied for feedback-linearizable systems), the inverse of the estimate of this matrix might not exist at each time due to insufficiently rich regressor signals, large initial parameter estimation errors, and so on.

We next show how one can overcome the problem where the estimation model becomes uncontrollable, by appropriately using switching adaptive control. We will apply the switching adaptive control methodology to the PPF system of the previous subsection, by removing assumption A2.

Consider the Lyapunov function for the PPF system of the previous subsection.

$$V = \frac{1}{2}\sum_{i=1}^{n} \zeta_i^2$$

By differentiating V with respect to time, we obtain that

$$\dot{V} = -\sum_{i=1}^{n} c_i\zeta_i^2 + \sum_{i=1}^{n-1} \zeta_i\zeta_{i+1} + \zeta_n(\varphi + \vartheta^{\mathrm{T}}w + [\gamma_0 + \vartheta^{\mathrm{T}}\gamma_1]u) \quad (82)$$

Let us define

$$B(z) = \zeta_n([\gamma_0 + \vartheta^{\mathrm{T}}\gamma_1]u)$$

Note now that, using the definition of ζ_i, we can rewrite the ζ_i's as follows:

$$\zeta_i = \bar{\varphi}_i + \vartheta^{\mathrm{T}}\bar{w}_i$$

where $\bar{\varphi}_i$ and $\bar{w}_i$ are appropriately defined known functions. Therefore, we have that

$$B(z) = (\bar{\varphi}_n + \vartheta^{\mathrm{T}}\bar{w}_n)([\gamma_0 + \vartheta^{\mathrm{T}}\gamma_1]) \equiv \beta_0 + \bar{\vartheta}^{\mathrm{T}}\beta_1$$

where $\bar{\vartheta}$ is defined to be the vector whose entries are the elements $\vartheta_i\vartheta_j$, and β_0, β_1 are—appropriately defined—known functions.

We are now ready to present the proposed controller. The control input is chosen as follows:

$$u = \varrho u_1 + (1 - \varrho)u_2 \quad (83)$$

where:

- One has

$$u_1 = -sK^2 \frac{k(\cdot)}{1 + sK^2\hat{B}}, \quad u_2 = -\frac{k(\cdot)}{\hat{B}}, \qquad \hat{B} = \beta_0 + \hat{\bar{\vartheta}}^{\mathrm{T}}\beta_1 \tag{84}$$

  where $\hat{\bar{\vartheta}}$ denotes the estimate of $\bar{\vartheta}$.

- $k(\cdot)$ is a positive design function satisfying $k(\zeta_1, \cdot) = 0$ iff $\zeta_1 = 0$ and

$$k(\cdot) \geq \frac{1}{4}\left| -\sum_{i=1}^{n} c_i \zeta_i^2 + \sum_{i=1}^{n-1} \zeta_i \zeta_{i+1} + \zeta_n(\varphi + \vartheta^{\mathrm{T}} w)\right| \tag{85}$$

- ϱ is a continuous-switching signal that is used to switch from control u_1 to control u_2 and vice versa:

$$\varrho = \begin{cases} 1 & \text{if} \quad |\hat{B}| < 1/2K^4 \\ -2K^4|\hat{B}| + 2 & \text{if} \quad 1/2K^4 \leq |B| < 1/K^4 \\ 0 & \text{if} \quad |B| \geq 1/K^4 \end{cases} \tag{86}$$

- s is *hysteresis-switching* variable defined as follows:

$$s = \begin{cases} s^- & \text{if} \quad |\hat{B}| < 1/2K^4 \\ \operatorname{sgn}(\hat{B}) & \text{otherwise} \end{cases} \tag{87}$$

  where $s^-(t) \triangleq \lim_{\tau \to t^-} s(\tau)$, where $\tau \leq t$.

The parameter estimates $\hat{\bar{\vartheta}}$ are updated using the following smooth projection update law (15)

$$\dot{\hat{\bar{\vartheta}}} = \mathscr{P}_{\mathrm{C}}\{\hat{\bar{\vartheta}}, \Gamma\beta_1 u\} \tag{88}$$

where Γ is a symmetric positive definite design matrix and P_{C} is defined as follows (15):

$$\mathscr{P}_{\mathrm{C}}(\hat{\bar{\vartheta}}, y) = \begin{cases} y & \text{if } P(\hat{\bar{\vartheta}} < 0 \\ y & \text{if } P(\hat{\bar{\vartheta}} \geq 0 \text{ and } \\ & \left(\dfrac{\partial P}{\partial \hat{\bar{\vartheta}}}\right)^{\mathrm{T}}(\hat{\bar{\vartheta}}) \leq 0 \\ y - \dfrac{P(\hat{\bar{\vartheta}})\dfrac{\partial P}{\partial \hat{\bar{\vartheta}}}(\hat{\bar{\vartheta}})y}{\left|\dfrac{\partial P}{\partial \hat{\bar{\vartheta}}}(\hat{\bar{\vartheta}})\right|^2}\left(\dfrac{\partial P}{\partial \hat{\bar{\vartheta}}}\right)^{\mathrm{T}}(\hat{\bar{\vartheta}}) & \text{otherwise} \end{cases} \tag{89}$$

where

$$P(\hat{\hat{\vartheta}}) = \frac{2}{\delta}\left[\sum_{j=1}^{L}\left|\frac{\hat{\hat{\vartheta}}_j}{\rho_j}\right|^q - 1 + \delta\right]$$

where $0 < \delta < 1$, $q \geq 2$, and ρ_j are positive design constants.

The following theorem summarizes the properties of the control law in Eq. (83, 84, 85, 86, 87, 88, 89).

Theorem 10. Consider the system in Eq. (65) and the control law in Eqs. (83, 84, 85, 86, 87, 88, 89). Let assumption A1′ hold. Moreover assume that the following hold:

C1. $K > 1$; $k(\cdot)$ satisfies Eq. (85).

C2. ρ_j are sufficiently small. Moreover, $\hat{\hat{\vartheta}}(0) \in C$, where

$$\mathscr{C} \triangleq \left\{\hat{\hat{\vartheta}} : \sum_{j=1}^{L}\left|\frac{\hat{\hat{\vartheta}}_j}{\rho_j}\right|^q \leq 1 - \frac{\delta}{2}\right\}$$

Then for any compact $X_0 \subset \Re^n$ and for any positive constant $\bar{c}$ the following holds: there exist a positive constant $K*$ such that, for any initial state $x_0 \in X_0$, the control law in Eqs. (83, 84, 85, 86, 87, 88, 89) with $K > K*$ guarantees that all the closed-loop signals are bounded and, moreover, that the tracking error ζ_1 converges in finite time to the residual set

$$\Omega \triangleq \{\zeta_1 \in \Re : |\zeta_1| \leq \bar{c}\}$$

The idea of using switching adaptive controllers of the form presented above was first introduced in Ref. 16, where the proposed methodology was applied to systems of the form in Eq. (57). The controller of Ref. 16 was extended in Ref. 17 for PPF systems of the form in Eq. (65).

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