ABSOLUTE STABILITY

Analysis of dynamical systems and circuits is mostly done under the assumption that the system is linear and time-invariant. Powerful mathematical techniques are then available for analysis of stability and performance of the system, for example, superposition and frequency domain analysis. In fact, even if the system is nonlinear and time-varying, such assumptions can often be used to get a first estimate of the system properties.

The purpose of absolute stability theory is to carry the analysis one step further and get a bound on the possible influence of the nonlinear or time-varying components. The approach was suggested by the Russian mathematician Lur'e in the 1940s and has, since then, developed into a cornerstone of nonlinear systems theory.

The basic setup is illustrated in Fig. 1, where the linear time-invariant part is represented by the transfer function $G(s)$ and the nonlinear parts of the system are represented by a feedback loop $w = \phi(v, t)$. The analysis of the system is based on conic bounds on the nonlinearity (Fig. 2):

$$\alpha \le \phi(v,t)/v \le \beta \qquad \text{for all } v \ne 0 \tag{1}$$

This problem was studied in Russia during the 1950s. In particular, a conjecture by Aiserman was discussed, hoping that the system would be stable for all continuous functions ϕ in the cone [Eq. (1)], if and only if it was stable for all linear functions in the cone. This conjecture was finally proved to be false, and it was not until the early 1960s that a major breakthrough was achieved by Popov (1).

THE CIRCLE CRITERION

Popov and his colleagues made their problem statements in terms of differential equations. The linear part then has the form

$$\begin{cases} \dot{x} = Ax - Bw \\ v = Cx \end{cases} \tag{2}$$

and the corresponding transfer function is

$$G(s) = C(sI - A)^{-1}B$$

Theorem 1 (Circle criterion). Suppose that the system $\dot{x} = Ax$ is exponentially stable and that $\phi : \mathbf{R}^2 \to \mathbf{R}$ is Lipschitz continuous and satisfies Eq. (1). If

$$0 < \mathrm{Re}\, \frac{\beta G(i\omega) + 1}{\alpha G(i\omega) + 1} \qquad \omega \in \mathbf{R} \tag{3}$$

then also the feedback system

$$\dot{x}(t) = Ax(t) - B\phi[Cx(t), t] \tag{4}$$

is exponentially stable.

The name of this result comes from its graphical interpretation. The *Nyquist plot,* that is, the plot of $G(i\omega)$ in the complex plane as $\omega \in \mathbf{R}$, must not cross or circumscribe the circle centered on the real axis and passing through $-1/\alpha$ and $-1/\beta$ (Fig. 3).

An important aspect of the circle criterion is that it demonstrates how frequency domain properties can be used in a nonlinear setting. It is instructive to compare with the Nyquist criterion, which states that the closed-loop system with linear feedback $w(t) = kv(t)$ is stable for all $k \in [\alpha, \beta]$, provided that $G(i\omega)$ does not intersect the real axis outside the interval $[-1/\beta, -1/\alpha]$. The circle criterion replaces the interval condition with a circular disk. As a consequence, the stability assertion is extended from constant feedback to nonlinear and time-varying feedback.

The proof of the circle criterion is based on a quadratic *Lyapunov function* of the form

$$V(x) = x'Px$$

where the matrix P is positive definite. It can be verified that $V(x)$ is decreasing along all possible trajectories of the system, provided that the frequency condition [Eq. (3)] holds. As a consequence, the state must approach zero, regardless of the initial conditions.

THE POPOV CRITERION

In the case that ϕ has no explicit time dependence, the circle criterion can be improved. For simplicity, let $\alpha = 0$ and hence

$$0 \le \phi(v)/v \le \beta \qquad \text{for all } v \ne 0 \tag{5}$$

The Popov criterion can then be stated as follows.

Theorem 2 (Popov criterion). Suppose that $\phi : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and satisfies Eq. (5). Suppose the system $\dot{x} = Ax$ is exponentially stable and let $G(i\omega) = C(i\omega I - A)^{-1}B$. If there exists $\eta \in \mathbf{R}$ such that

$$\mathrm{Re}[(1 + i\omega\eta)G(i\omega)] > -\frac{1}{\beta} \qquad \omega \in \mathbf{R} \tag{6}$$

then the system

$$\dot{x}(t) = Ax(t) - B\phi[Cx(t)] \tag{7}$$

is exponentially stable.

Note that the circle criterion is recovered with $\eta = 0$. Also the Popov criterion can be illustrated graphically. Introduce the *Popov plot,* where $\omega \mathrm{Im} G(i\omega)$ is plotted versus $\mathrm{Re} G(i\omega)$. Then

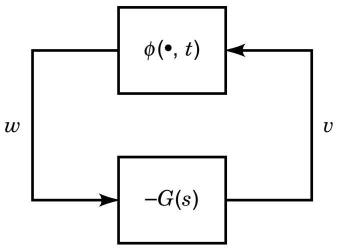

Figure 1. Absolute stability deals with linear time-invariant systems in interconnection with nonlinear functions.

stability can be concluded from the Popov criterion if and only if there exists a straight line separating the plot from the point $-1/\beta$. The slope of the line corresponds to the parameter η. See the following example.

Example. To apply the Popov criterion to the system

$$\dot{x}_1 = -5x_1 - 4x_2 + \phi(x_1)$$
$$\dot{x}_2 = -x_1 - 2x_2 + 21\phi(x_1)$$

let

$$G(s) = C(i\omega - A)^{-1}B = \frac{-s + 82}{s^2 + 7s + 6}$$

The plot in Fig. 4 then shows that the Popov criterion is satisfied for $\beta = 1$.

The theory of absolute stability had a strong development in the 1960s, and various improvements to the circle and Popov criteria were generated, for example, by Yakubovich. Many types of nonlinearities were considered and stronger criteria were obtained in several special cases (2–4). Important aspects of the theory were summarized by Willems (5), using the notions of dissipativity and storage function.

GAIN AND PASSIVITY

In the results described so far, the results were stated in terms of differential equations. A parallel theory was developed by Zames (6) avoiding the state-space structure and studying stability purely in terms of input–output relations.

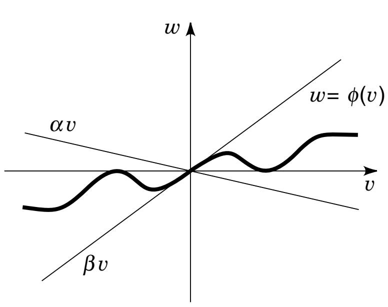

Figure 2. The nonlinearity is bounded by linear functions.

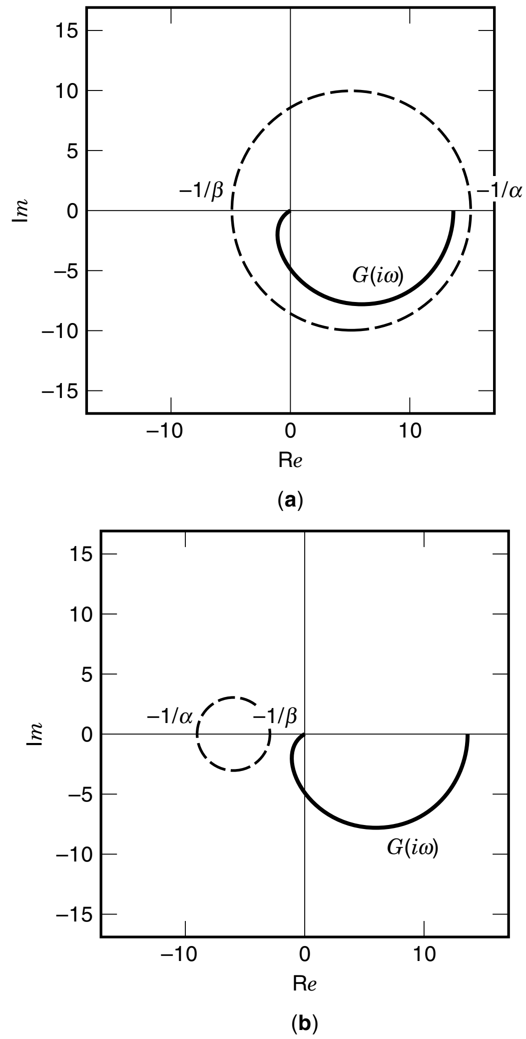

Figure 3. The circle criterion proves stability as long as the Nyquist plot does not cross or circumscribe the circle corresponding to the conic bounds on the nonlinearity. (a) $\alpha < 0 < \beta$. (b) $0 < \alpha < \beta$.

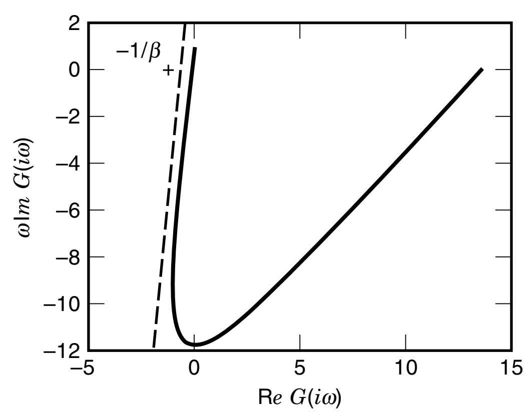

Figure 4. The Popov criterion can be applied when there exists a straight line separating the Popov plot from the point $-1/\beta$.

For this purpose, a dynamical system is viewed as a map F from the input u to the output Fu. The map F is said to be bounded if there exists $C > 0$ such that

$$\int_0^T |Fu|^2\,dt \le C^2 \int_0^T |u|^2\,dt$$

for all $T > 0$. The gain of F is denoted $\|F\|$ and defined as the minimal such constant C. The map is said to be *causal* if two inputs that are identical until time T will generate outputs that are also identical until time T.

This makes it possible to state the following well-known result:

Theorem 3 (Small gain theorem). Suppose that the input–output maps F and G are bounded and casual. If

$$\|F\| \cdot \|G\| < 1 \tag{8}$$

then the feedback equations

$$v = Gw + f$$
$$w = Fv + e$$

define a bounded causal map from the inputs (e, f) to the outputs (v, w).

It is worthwhile to make a comparison with the circle criterion. Consider the case when $\alpha = -\beta$. Then the condition [Eq. (3)] becomes

$$\beta|G(i\omega)| < 1 \qquad \omega \in [0, \infty]$$

This is the same as Eq. (8), since β is the gain of ϕ and $\max_\omega|G(i\omega)|$ is the gain of the linear part [Eq. (2)].

Another important notion closely related to gain is passivity. The input–output map F is said to be *passive* if

$$0 \le \int_0^T u(t)y(t)\,dt \qquad \text{for all } T \text{ and all } y = Fu$$

For example, if the input is a voltage and the output is a current, then passivity property means that the system only can consume electrical power, not produce it.

Stability criteria can also be stated in terms of passivity. For example, the circle criterion can be interpreted this way, if $\alpha = 0$ and β is large.

MULTIPLIERS AND INTEGRAL QUADRATIC CONSTRAINTS

Less conservative stability criteria can often be obtained by exploiting more information about the nonlinearity. One way to do this is to introduce so-called multipliers. Consider, for example, a system with a saturation nonlinearity:

$$\dot{x} = Ax - B\text{sat}(Cx) = \begin{cases} Ax + B & \text{if } Cx \le 1 \\ Ax + BCx & \text{if } |Cx| < 1 \\ Ax - B & \text{if } Cx \ge 1 \end{cases}$$

The only property of saturation that would be exploited by the Popov criterion is that $0 \le \text{sat}(v)/v \le 1$ and, consequently,

$$0 \le \int_0^T w(t)[v(t) - w(t)]\,dt$$

for all $w = \text{sat}(v)$, $T > 0$. However, the inequality will remain valid even if some perturbation of amplitude smaller than one is added to the factor w in the product $w(v - w)$. One way to do this is to introduce a function $h(t)$ with the property $\int_{-\infty}^{\infty} |h(t)|dt \le 1$ and replace the previous expression by $(w + h * w)(v - w)$, where $h * w$ is a convolution. The integral inequality then becomes

$$0 \le \int_0^T (w + h * w)(v - w)\,dt \tag{9}$$

Using this inequality, the Popov criterion [Eq. (6)] can be replaced by the condition

$$\text{Re}[(1 + i\omega\eta + H(i\omega))(G(i\omega) + 1)]] > 0 \qquad w \in \mathbf{R} \tag{10}$$

where $H(i\omega) = \int_{-\infty}^{\infty} e^{-i\omega} h(t)dt$. The factor $1 + i\omega\eta + H(i\omega)$ is called *multiplier*.

The theory and applications of absolute stability have recently had a revival since new computer algorithms make it possible to optimize multipliers numerically and to address applications of much higher complexity than previously. The inequality [Eq. (9)] is a special case of what is called an *integral quadratic constraint,* IQC. Such constraints have been verified for a large number of different model components such as relays, various forms of hysteresis, time delays, time variations, and rate limiters. In principle, all such constraints can be used computationally to improve the accuracy in stability and performance analysis. A unifying theory for this purpose has been developed by Megretski and Rantzer (8), while several other authors have contributed with new IQCs that are ready to be included in a computer library for system analysis.

BIBLIOGRAPHY

1. V. M. Popov, Absolute stability of nonlinear systems of automatic control, *Autom. Remote Control,* **22**: 857–875, 1962. (Original in Russian, August, 1961.)
2. V. A. Yakubovich, Absolute stability of nonlinear control systems in critical cases, parts 1–3, *Avtomaika i Telemechanika,* **24** (3): 293–302; **24** (6): 717–731, 1963; **25** (25): 601–612, 1964. (English translation in *Autom. Remote Control.*)
3. V. A. Yakubovich, On an abstract theory of absolute stability of nonlinear systems, *Vestnik Leningrad Univ. Math.,* **10**: 341–361, 1982. (Original in Russian, 1977.)
4. S. Lefschetz, *Stability of Nonlinear Control Systems,* New York: Academic Press, 1963.
5. J. C. Willems, Dissipative dynamical systems, part 1, General theory; part 2, Linear systems with quadratic supply rates, *Arch. Rational Mech. Anal.,* **45** (5): 321–393, 1972.
6. G. Zames, On the input–output stability of nonlinear time-varying feedback systems—part 1, Conditions derived using concepts of loop gain; part 2, Conditions involving circles in the frequency plane and sector nonlinearities, *IEEE Trans. Autom. Control,* **11**: 228–238, 1966.

7. G. Zames and P. L. Falb, Stability conditions for systems with monotone and slope-restricted nonlinearities, *SIAM J. Control,* **6** (1): 89–108, 1968.

8. A. Megretski and A. Rantzer, System analysis via Integral Quadratic Constraints, *IEEE Trans. Autom. Control,* **47**: 819–830, 1997.

ANDERS RANTZER
Lund Institute of Technology

ABSORBER. See ELECTROMAGNETIC FERRITE TILE ABSORBER.

ABSORPTION MODULATION. See ELECTROABSORPTION.