Solution for Chapter 27

(compiled by Xinkai Wu)

Exercise 27.2 The 3-sphere geometry of a closed universe [Xinkai Wu/02]
(a) We claim that

$$w = acos\chi,\ x = asin\chi sin\theta cos\phi,\ y = asin\chi sin\theta sin\phi,\ z = asin\chi cos\theta$$

As can be verified easily, the above expressions of (w, x, y, z) satisfy $w^2 + x^2 + y^2 + z^2 = a^2$, and when we plug them into the 4-d metric $ds^2 = dw^2 + dx^2 + dy^2 + dz^2$, the line element reduces to

$$ds^2 = a^2[d\chi^2 + sin^2\chi(d\theta^2 + sin^2\theta d\phi^2)]$$

(b) The volume of this 3-sphere is given by

$$\begin{aligned} V &= \int_0^\pi d\chi \int_0^\pi d\theta \int_0^{2\pi} d\phi\ \sqrt{|det\ g|} \\ &= \int_0^\pi d\chi \int_0^\pi d\theta \int_0^{2\pi} d\phi\ a^3 sin^2\chi sin\theta \\ &= 2\pi^2 a^3 \end{aligned}$$

Exercise 27.3 Energy conservation for a perfect fluid [Xinkai Wu/02]
(a) The conservation law of 4-momentum says ${T^{\alpha\beta}}_{;\beta} = 0$. Contracting this with the 4-velocity of the observer (the fluid), we get $u_\alpha {T^{\alpha\beta}}_{;\beta} = 0$.(It's tempting to put u_α inside the covariant derivative and say that we have ${J^\beta}_{;\beta} = 0$ with the 4-momentum flux measured by the observer being $J^\beta = u_\alpha T^{\alpha\beta}$. However this is in general wrong in curved spacetime unless $u_{\alpha;\beta} + u_{\beta;\alpha} = 0$, namely $\vec{u}$ is a Killing vector field, which is in general not the case.)
(b) Plugging the expression of the fluid's stress tensor into the conservation law

$$\begin{aligned} 0 &= u_\alpha {T^{\alpha\beta}}_{;\beta} \\ &= u_\alpha \left\{ (\rho + P)_{;\beta} u^\alpha u^\beta + (\rho + P)[u^\alpha {u^\beta}_{;\beta} + u^\beta {u^\alpha}_{;\beta}] + P_{;\beta} g^{\alpha\beta} \right\} \\ &\quad [\text{using } \vec{u}^2 = -1 \Rightarrow u_\alpha {u^\alpha}_{;\beta} = 0] \\ &= -u^\beta (\rho + P)_{;\beta} - (\rho + P) {u^\beta}_{;\beta} + u^\beta P_{;\beta} \\ &= -u^\beta \rho_{;\beta} - (\rho + P) {u^\beta}_{;\beta} \end{aligned}$$

which is

$$\frac{d\rho}{d\tau} = -(\rho + P)\vec{\nabla} \cdot \vec{u}$$

(c) In the (t, χ, θ, ϕ) coordinate system, $\vec{u} = (1, 0, 0, 0)$. Thus we have

$$\frac{d\rho}{d\tau} = u^\beta \rho_{;\beta} = u^t \frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial t} = \frac{d\rho}{dt}$$

noting that ρ(also a) only depends on t by the homogeneity and isotropy of our universe. Also we have

$$\vec{\nabla} \cdot \vec{u} = \frac{1}{\sqrt{-g}} (\sqrt{-g}\ u^\alpha)_{,\alpha} = \frac{1}{a^3 \Sigma^2 sin\theta} (a^3 \Sigma^2 sin\theta)_{,t} = \frac{1}{a^3} \frac{da^3}{dt}$$

Thus eq. (27.31) becomes

$$\frac{d\rho}{dt} = -(\rho + P) \frac{1}{a^3} \frac{da^3}{dt}$$

which can be rewritten as

$$\frac{d(\rho a^3)}{dt} = -P \frac{da^3}{dt}$$

Exercise 27.5 Einstein's static universe [Xinkai Wu/02]

The Einstein equation with a cosmological constant Λ is

$$G_{ab} = 8\pi T_{ab} - \Lambda g_{ab} = 8\pi (T_{ab} - \frac{\Lambda}{8\pi} g_{ab})$$

For our homogeneous and isotropic universe, the stress tensor is that of perfect fluid $T_{ab} = (\rho + P) u_a u_b + P g_{ab}$, and we have

$$\begin{aligned} G_{ab} &= 8\pi \left[(\rho + P) u_a u_b + P g_{ab} - \frac{\Lambda}{8\pi} g_{ab} \right] \\ &= 8\pi \left[\left(\rho + \frac{\Lambda}{8\pi} + P - \frac{\Lambda}{8\pi} \right) u_a u_b + \left(P - \frac{\Lambda}{8\pi} \right) g_{ab} \right] \end{aligned}$$

So we can use eq. (27.24) and (27.26) with $\rho \to \rho + \frac{\Lambda}{8\pi}$ and $P \to P - \frac{\Lambda}{8\pi}$.

Eq. (27.24) becomes

$$\frac{d\left[\left(\rho + \frac{\Lambda}{8\pi}\right) a^3\right]}{dt} = -\left(P - \frac{\Lambda}{8\pi} \right) \frac{da^3}{dt} \Rightarrow \frac{d(\rho a^3)}{dt} = -P \frac{da^3}{dt}$$

Eq. (27.26) becomes

$$\left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi}{3} \left(\rho + \frac{\Lambda}{8\pi} \right)$$

Now we assume the equation of state for the fluid $P = 0$. So eq. (27.24) gives $\rho = \rho_0 \left(\frac{a_0}{a}\right)^3$. Plugging this into eq. (27.26) we get

$$\frac{1}{2} \dot{a}^2 + V(a) = \frac{-k}{2}$$

with $V(a) = -\frac{4\pi}{3}a^2\left(\rho + \frac{\Lambda}{8\pi}\right) = -\frac{4\pi}{3}\left(\frac{\rho_0 a_0^3}{a} + \frac{\Lambda}{8\pi}a^2\right)$.

(a) The evolution equation for a is thus

$$\ddot{a} = -V'(a) = \frac{4\pi}{3}\left(-\frac{\rho_0 a_0^3}{a^2} + \frac{\Lambda}{4\pi}a\right)$$

Hence to get $V'(a_0) = 0$ we must have $\Lambda = 4\pi\rho_0$. And in that case, $\ddot{a}_0 = 0$. To get a static universe we also need $\dot{a}_0 = 0$, which requires

$$0 = \frac{k}{2} + V(a_0) = \frac{k}{2} - \frac{4\pi}{3}a_0^2\left(\rho_0 + \frac{\Lambda}{8\pi}\right) = \frac{k}{2} - 2\pi\rho_0 a_0^2$$

which implies $k = +1$ and $4\pi\rho_0 a_0^2 = 1$.

(b) Today $\rho_0 \sim 10^{-31} g/cm^3$, thus $\Lambda = 4\pi\rho_0 \sim 1.2 \times 10^{-30} g/cm^3$, and $\rho_\Lambda = \frac{\Lambda}{8\pi} = \frac{1}{2}\rho_0 \sim 0.5 \times 10^{-31} g/cm^3$.

(c) $V''(a_0) = -4\pi\rho_0 < 0$, thus this model is unstable against gravitational collapse. Also see Fig 1.

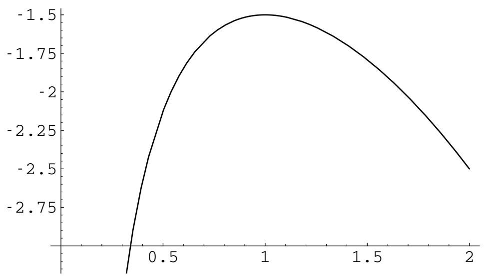

Figure 1: Einstein's static universe: the x-axis is a/a_0, the y-axis is $V(a)$ (not normalized)

Exercise 27.6 Cosmological redshift [Xinkai Wu/02]

(a) From the solution to Exercise 24.4(a), we have

$$\frac{dp_\alpha}{d\zeta} = \Gamma_{\mu\alpha\beta}p^\mu p^\beta = \frac{1}{2}g_{\mu\beta,\alpha}p^\mu p^\beta$$

In our case the only nonvanishing components of $\vec{p}$ are p^t and p^χ. So

$$\frac{dp_\chi}{d\zeta} = \frac{1}{2}g_{\mu\beta,\chi}p^\mu p^\beta = \frac{1}{2}(g_{tt,\chi}(p^t)^2 + g_{\chi\chi,\chi}(p^\chi)^2) = 0$$

since $g_{tt,\chi} = 0$ and $g_{\chi\chi,\chi} = 0$. So p_χ is conserved along the particle's worldline.

(b)

$$-m^2 = \vec{p}^2 = g^{tt}(p_t)^2 + g^{\chi\chi}(p_\chi)^2 = -(p_t)^2 + \left(\frac{p_\chi}{a}\right)^2$$

which gives

$$p_t = -\sqrt{m^2 + \left(\frac{p_\chi}{a}\right)^2}$$

also note that $E = -p_\alpha u^\alpha = -p_t = \sqrt{m^2 + \left(\frac{p_\chi}{a}\right)^2}$. Thus we find

$$p = \sqrt{E^2 - m^2} = \frac{p_\chi}{a}, \quad p_0 = \sqrt{E_0^2 - m^2} = \frac{p_\chi}{a_0}$$

and thus

$$\frac{p_0}{p} = \frac{a}{a_0} = \frac{1}{1+z}$$

(c) For a photon we simply set $m = 0$ and notice that $\lambda = \frac{\hbar}{p}$, then eq. (27.43) and (27.44) follow from eq. (27.54).

(d) The particle's velocity is given by $v = p/E$. For $v << 1$, we can approximate $E \approx m$, thus

$$\frac{v_0}{v} = \frac{p_0}{p} = \frac{a}{a_0}$$

Exercise 27.7 Cosmic microwave radiation in an anisotropic cosmological model [Xinkai Wu/02]

(a) Take $t = const$, $ds^2 = a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2$. This 3-d metric is positive definite, thus having Euclidean geometry. Since the 3-d metric components $a^2(t), b^2(t), c^2(t)$ are independent of x, y, z, the connection coefficients on the const-t slice all vanish, namely it's spatially flat. Also the x, y, z-independence of the 3-d metric components means that we have translation symmetry along the spatial directions, namely it's spatially homogeneous. Consider the observers whose 4-velocities are given by $u^\alpha = (1, 0, 0, 0)$ in the (t, x, y, z) coordinates (i.e. they have constant x, y, z coordinates). Since the 4-d metric is diagonal, $\vec{u}$ is orthogonal to the const-t slices and thus these slices are hypersurfaces of simultaneity for these observers. Also $u^t = dt/d\tau = 1$ gives $\tau = t$.

(b) In the local Lorentz frame of the homogeneous observer

$$\mathcal{N} = \frac{2}{h^3} \frac{1}{e^{E/kT_e} - 1}$$

where E is the energy measured by the observer: $E = -p_t$. Using $0 = \vec{p}^2 = g^{tt}(p_t)^2 + g^{xx}(p_x)^2 + g^{yy}(p_y)^2 + g^{zz}(p_z)^2$ we find p_t and thus E

$$E = \left[\left(\frac{p_x}{a_e}\right)^2 + \left(\frac{p_y}{b_e}\right)^2 + \left(\frac{p_z}{c_e}\right)^2\right]^{1/2}$$

(c) Since the metric has no dependence on either x, y, or z, by Exercise 24.4 (a), p_x, p_y, p_z are all conserved along the photon's worldline.

(d) Since we are assuming that there's no collision along the photon's trajectory, by Vlasov equation, $\mathcal{N}$ is conserved along the worldline of any photon. Since (27.57) is expressed solely in terms of the conserved p_x, p_y, p_z and T_e (which is constant by definitioin), $\mathcal{N}$ remains precisely the form (27.57) for all later times.

(e) The basis vetors of the local Lorentz frame of a homogeneous observer at time t_0 is given by

$$\vec{e}_{\hat{0}} = \vec{e}_0, \quad \vec{e}_{\hat{x}} = \frac{1}{a_0}\vec{e}_x, \quad \vec{e}_{\hat{y}} = \frac{1}{b_0}\vec{e}_y, \quad \vec{e}_{\hat{z}} = \frac{1}{c_0}\vec{e}_z$$

(f) The photon energy measured by the observer at t_0 is $E_0 = -p_{\hat{t}}$. By $0 = \vec{p}^2 = -(p_{\hat{t}})^2 + (p_{\hat{x}})^2 + (p_{\hat{y}})^2 + (p_{\hat{z}})^2$, we get

$$E_0 = \sqrt{(p_{\hat{x}})^2 + (p_{\hat{y}})^2 + (p_{\hat{z}})^2}$$

which means

$$p_{\hat{x}} = E_0 n_{\hat{x}}, \quad p_{\hat{y}} = E_0 n_{\hat{y}}, \quad p_{\hat{z}} = E_0 n_{\hat{z}}$$

Using the result of part (c) this gives

$$p_x = E_0 a_0 n_{\hat{x}}, \quad p_y = E_0 b_0 n_{\hat{y}}, \quad p_z = E_0 c_0 n_{\hat{z}}$$

Using the above result we see that E/kT_e in the expression for $\mathcal{N}$ can be written as

$$\frac{E}{kT_e} = \frac{E_0}{kT_0}$$

with

$$T_0 = T_e \left[\left(\frac{a_0}{a_e} n_{\hat{x}} \right)^2 + \left(\frac{b_0}{b_e} n_{\hat{y}} \right)^2 + \left(\frac{c_0}{c_e} n_{\hat{z}} \right)^2 \right]^{-1/2}$$

(g) When $a = b = c$, eq.(27.58) reduces to (noting that $\mathbf{n}^2 = 1$)

$$T_0 = T_e \frac{a_e}{a_0} \Rightarrow \frac{T_0}{T_e} = \frac{a_e}{a_0} = \frac{1}{1+z}$$

(note that in this isotropic case $E_0/E = a_e/a_0$ gives the photon's redshift.)

Exercise 27.8 Angular-diameter distance [Xinkai Wu/02]

(a) By definition the angular-diameter distance of an object is given by

$$r_{AD} = \frac{D}{\Theta}$$

with D being the object's physical size and Θ being its observed angular diameter. Put the object in the equatorial plane $\theta = \pi/2$, at coorinate distance $\Delta\chi$, then

$$D = \int ds = \int_0^{\Theta} a\Sigma(\Delta\chi)d\phi = \Theta a\Sigma(\Delta\chi)$$

Thus we get

$$r_{AD} = \frac{\Theta a\Sigma(\Delta\chi)}{\Theta} = a\Sigma(\Delta\chi) = \frac{a_0\Sigma(\Delta\chi)}{a_0/a} = \frac{R}{1+z}$$

with $R \equiv a_0\Sigma(\Delta\chi)$.

(b) The worldline of a radial photon is given by $0 = -dt^2 + a^2 d\chi^2$

$$\frac{d\chi}{dt} = -\frac{1}{a(t)}$$

integrating gives

$$\Delta\chi = \int_{t_e}^{t_0} \frac{1}{a}dt = \int_1^{1+z} \frac{a}{a_0}\frac{dt}{da}d\frac{a_0}{a}$$

Using equations (27.40) and (27.41) we have

$$\begin{aligned} \dot{a} &= \sqrt{-k - 2V(a)} \\ &= \sqrt{-k + \frac{8\pi}{3}\rho_{crit}a_0^2\left(\Omega_R u^2 + \Omega_M u + \Omega_\Lambda \frac{1}{u^2}\right)} \end{aligned}$$

where we have denoted $u \equiv \frac{a_0}{a}$. The definition of ρ_{crit} eq. (27.36) gives $\frac{8\pi}{3}\rho_{crit}a_0^2 = (\dot{a}_0)^2$. Eq. (27.60) tells us $\dot{a}_0 = \left(\frac{k}{\Omega-1}\right)^{1/2} = \left(\frac{1}{|\Omega-1|}\right)^{1/2}$(assuming $k = \pm 1$), where $\Omega \equiv \Omega_M + \Omega_R + \Omega_\Lambda$. Plugging all these into the expression for $\Delta\chi$, we get

$$\Delta\chi = |1-\Omega|^{1/2}\int_1^{1+z} \frac{du}{\sqrt{\Omega_M u^3 + \Omega_R u^4 + \Omega_\Lambda + (1-\Omega)u^2}}$$

Since Ω_R is very small, we can neglect it and $\Omega \approx \Omega_M + \Omega_\Lambda$, and we get

$$\Delta\chi = |1-\Omega_M-\Omega_\Lambda|^{1/2}\int_1^{1+z} \frac{du}{\sqrt{\Omega_M u^3 + (1-\Omega_M-\Omega_\Lambda)u^2 + \Omega_\Lambda}}$$

(c) See Figures 2 through 4. Note that for the flat universe case $k = 0, \Omega = 1$, the expression for $\Delta\chi$ is given by

$$\Delta\chi = \frac{1}{H_0 a_0}\int_1^{1+z} \frac{du}{\sqrt{\Omega_R u^4 + \Omega_M u^3 + \Omega_\Lambda}} \approx \frac{1}{H_0 a_0}\int_1^{1+z} \frac{du}{\sqrt{\Omega_M u^3 + \Omega_\Lambda}}$$

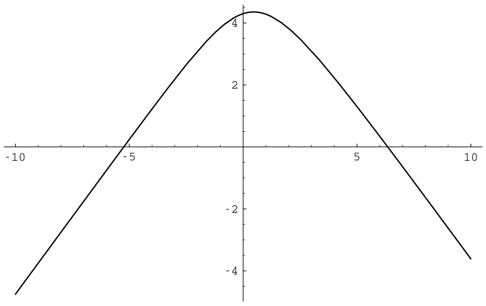

Figure 2: $r_{AD}(z)$: the x-axis is $Log(z)$, the y-axis is $Log(r_{AD})$; $\Omega_M = 0.3, \Omega_\Lambda = 0.73$, it's a closed universe.

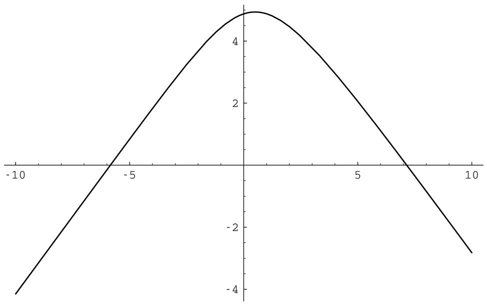

Figure 3: $r_{AD}(z)$: the x-axis is $Log(z)$, the y-axis is $Log(r_{AD})$; $\Omega_M = 0.3, \Omega_\Lambda = 0.6$, it's an open universe.

Exercise 27.9 Magnitude-redshift relation [Xinkai Wu/02]

Assume that the supernova emits monochromatic photons. In time interval dt(as measured in the supernova's local Lorentz frame) the supernova emits $dN = \frac{Ldt}{\hbar\omega}$ photons. These photons arrive at the earth in a time interval measured in earth's local Lorentz frame $dt_0 = dt(\omega/\omega_0)$(namely, because of time-dilation, the arrival rate is smaller than the emission rate), each photon having energy $\hbar\omega_0$ as measured by earth bound observer. Also note that the area of the two-sphere at the earth's distance is given by $S = 4\pi(a_0\Sigma(\Delta\chi))^2 = 4\pi R^2$. Thus the energy flux measured at earth today is

$$\mathcal{F} = \frac{dN\hbar\omega_0}{dt_0 S} = \frac{Ldt}{\hbar\omega}\,\frac{\hbar\omega_0}{dt(\omega/\omega_0)}\,\frac{1}{4\pi R^2} = \frac{L}{4\pi R^2}\left(\frac{\omega_0}{\omega}\right)^2 = \frac{L}{4\pi R^2(1+z)^2}$$

Exercise 27.11 Inflationary explanation of the isotropy of the CMB [Xinkai Wu/02]

A portion of the universe with size L_{PW} at the Planck-Wheeler time is in

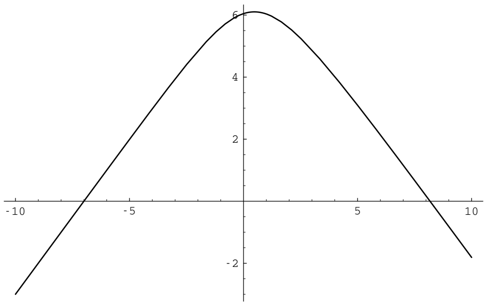

Figure 4: $r_{AD}(z)$: the x-axis is $Log(z)$, the y-axis is $Log(r_{AD})$; $\Omega_M = 0.3, \Omega_\Lambda = 0.7$, it's a flat universe.

causal contact with itself. This portion inflates, during the inflationary era, to size

$$a_I \equiv a(t = N\mu L_{PW}) = L_{PW} exp(N) \tag{1}$$

After that the expansion is dominated by radiation during which $\rho \propto 1/a^4$, hence

$$\frac{a_I}{a_0} = \left(\frac{\rho_0}{\rho_I}\right)^{1/4}$$

where the size of the universe today is $a_0 \sim H_0^{-1}$, and we can take ρ_I, the radiation energy density right after the inflation, to be of order $\rho_I \sim \Lambda$, with Λ being the the vacuum energy density driving inflation. Thus we have

$$a_I = H_0^{-1}\left(\frac{\rho_0}{\Lambda}\right)^{1/4} \tag{2}$$

Combining eq. (1) and eq. (2), we get

$$N = ln\left[\frac{H_0^{-1}}{L_{PW}}\left(\frac{\rho_0}{\Lambda}\right)^{1/4}\right] \approx 70$$

is the minimum number of e-foldings we need to solve the horizon problem.