Solution for Chapter 26

(compiled by Xinkai Wu)

A. Arm waving [Xinkai Wu/02]

(a). Take the frequency to be $f \sim 2Hz$, then the wavelength is $\lambda \sim c/f \sim 1 \times 10^8 m$. The major contribution to the gravitational wave comes from the mass quadrupole moment and is given by eq. (26.112): $h_+ \sim h_\times \sim \frac{G}{c^4}\frac{Mv^2}{r}$, and we take $M \sim 10kg$, $v \sim Lf \sim 1m \times 2Hz \sim 2m/s$(where L is the size of the arm), and $r \sim \lambda \sim 10^8 m$, this gives $h_+ \sim h_\times \sim 10^{-51}$.

(b). The total power $\frac{dE}{dt}$ is given by eq. (26.113). Restoring G, c, we get $\frac{dE}{dt} \sim \frac{G}{c^5}\frac{M^2v^6}{L^2} \sim 10^{-49} J/s$, and the number of gravitons emitted per second is $\frac{1}{\hbar 2\pi f}\frac{dE}{dt} \sim 10^{-16}$, which means that in your entire lifetime ($\sim 3 \times 10^9$ sec) you have a probability of less than one part in a million to emit a single graviton.

B. Exercise 26.4 Behavior of h_+ and $h_\times$ under rotations and boosts [Kip Thorne and Xinkai Wu/02]

(a) Quantities with a tilde denote those in the new basis, and those without tilde in the old basis. In terms of real numbers, the change of basis $\tilde{\mathbf{e}}_x + i\tilde{\mathbf{e}}_y = (\mathbf{e}_x + i\mathbf{e}_y)e^{i\psi}$ is

$$\tilde{\mathbf{e}}_x = \mathbf{e}_x cos\psi - \mathbf{e}_y sin\psi, \quad \tilde{\mathbf{e}}_y = \mathbf{e}_y cos\psi + \mathbf{e}_x sin\psi$$

Plugging the above transformation matrix into eq.(26.41) we find the components of Riemann in the new basis

$$\begin{aligned} R_{\tilde{x}0\tilde{x}0} &= cos^2\psi R_{x0x0} + sin^2\psi R_{y0y0} - 2cos\psi sin\psi R_{x0y0} \\ &= cos2\psi\left(-\frac{1}{2}\ddot{h}_+\right) - sin2\psi\left(-\frac{1}{2}\ddot{h}_\times\right) \end{aligned}$$

on the other hand $R_{\tilde{x}0\tilde{x}0} = -\frac{1}{2}\ddot{\tilde{h}}_+$, thuse we get

$$\tilde{h}_+ = (cos2\psi)h_+ - (sin2\psi)h_\times$$

Similarly, by looking at $R_{\tilde{x}0\tilde{y}0}$, we find

$$\tilde{h}_\times = (cos2\psi)h_\times + (sin2\psi)h_+$$

Translated into complex numbers, this is just

$$\tilde{h}_+ + i\tilde{h}_\times = (h_+ + ih_\times)e^{2i\psi}$$

(b) The desired boost is a boost along the z direction, which gives

$$\vec{\tilde{e}}_0 = \vec{e}_0 cosh\beta + \vec{e}_z sinh\beta, \quad \vec{\tilde{e}}_z = \vec{e}_0 sinh\beta + \vec{e}_z cosh\beta$$

with $\vec{e}_x,\ \vec{e}_y$ unchanged. And the corresponding transformation for the coordinates is

$$\tilde{t} = t cosh\beta - z sinh\beta, \quad \tilde{z} = -t sinh\beta + z cosh\beta$$
$$\text{which gives } \tilde{t} - \tilde{z} = (cosh\beta + sinh\beta)(t - z)$$

with x, y unchanged.

Look at components of Riemann in the new basis using the above transformation matrix, we find

$$\begin{aligned} R_{x\tilde{0}x\tilde{0}} &= (cosh\beta - sinh\beta)^2 R_{x0x0} = e^{-2\beta} R_{x0x0} \\ &= \left(\frac{-1}{2}\right) e^{-2\beta} \frac{\partial^2}{\partial t^2} h_+ \end{aligned} \quad (1)$$

From the coordinate transformation we find $\frac{\partial h_+}{\partial t} = \left(cosh\beta \frac{\partial}{\partial \tilde{t}} - sinh\beta \frac{\partial}{\partial \tilde{z}}\right) h_+$ an since h_+ is a function of $t - z$ and thence of $\tilde{t} - \tilde{z}$, this gives $\frac{\partial h_+}{\partial t} = (cosh\beta + sinh\beta) \frac{\partial}{\partial \tilde{t}} h_+ = e^{\beta} \frac{\partial h_+}{\partial \tilde{t}}$. similarly $\frac{\partial^2 h_+}{\partial t^2} = e^{2\beta} \frac{\partial^2 h_+}{\partial \tilde{t}^2}$. Combining with eqn. (1) we get $R_{x\tilde{0}x\tilde{0}} = \frac{-1}{2} \frac{\partial^2 h_+}{\partial \tilde{t}^2}$. But in the tilded coordinate $R_{x\tilde{0}x\tilde{0}} = \frac{-1}{2} \frac{\partial^2 \tilde{h}_+}{\partial \tilde{t}^2}$. Therefore $h_+ = \tilde{h}_+$.

By looking at $R_{x\tilde{0}y\tilde{0}}$ one can show the invariance of $h_\times$ in a similar manner.

C. Exercise 26.5 Energy-momentum conservation in geometric optics limit [Alexander Putilin/00 and Kip Thorne/02]

$$T^{GW}_{\alpha\beta} = \frac{1}{16\pi} < h_{+,\alpha} h_{+,\beta} + h_{\times,\alpha} h_{\times,\beta} >$$

In geometric optics limit:

$$h_+ = \frac{Q_+(\tau_r; \theta, \phi)}{r}, \quad h_\times = \frac{Q_\times(\tau_r; \theta, \phi)}{r}$$

The wave vector $\vec{k} = -\vec{\nabla}\tau_r$ is null, and we have $\nabla_{\vec{k}} \vec{k} = 0,\ \nabla_{\vec{k}} r = \frac{1}{2}(\vec{\nabla} \cdot \vec{k}) r$.

Consider the contribution of h_+

$$16\pi T^{GW}_{\alpha\beta} =< \frac{\dot{Q}_+^2}{r^2} > \tau_{r,\alpha} \tau_{r,\beta} =< \frac{\dot{Q}_+^2}{r^2} > k_\alpha k_\beta$$

where the dot means $\partial/\partial\tau_r$. Thus

$$16\pi T^{\alpha\beta}_{GW|\beta} = \left(< \dot{Q}_+^2 > \frac{k^\beta}{r^2} \right)_{|\beta} k^\beta + \frac{< \dot{Q}_+^2 >}{r^2} k^\alpha{}_{|\beta} k^\beta$$

The second term vanishes by the geodesics equation for the rays. In the first term, $< \dot{Q}_+^2 >$ is constant along a ray, i.e. $< \dot{Q}_+^2 >_{,\beta} k^\beta = 0$, so it can be removed from the derivative. Thus

$$16\pi T^{\alpha\beta}_{GW|\beta} =< \dot{Q}_+^2 > \left(\frac{k^\beta}{r^2} \right)_{|\beta}$$

But the transport law $\nabla_{\vec{k}} r = \frac{1}{2}(\vec{\nabla} \cdot \vec{k})r$ implies that $(k^\beta/r^2)_{|\beta} = 0$. Therefore $16\pi T^{\alpha\beta}_{GW|\beta} = 0$.

The proof for the $\times$ polarization is identical.

D. Exercise 26.6 Transformation to TT gauge [Alexander Putilin/00]

(a) Consider gauge transformation generated by ξ_α: $h_{\alpha\beta} \to h'_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$, or, $\bar{h}_{\alpha\beta} \to \bar{h}'_{\alpha\beta} = \bar{h}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\xi_\mu{}^{,\mu}$. Then

$$\begin{aligned}
\bar{h}'{}_{\alpha\beta}{}^{,\beta} &= \bar{h}_{\alpha\beta}{}^{,\beta} - \xi_{\alpha,\beta}{}^{\beta} - \xi_{\beta,\alpha}{}^{\beta} + \xi_{\mu,\alpha}{}^{\mu} \\
&= \bar{h}_{\alpha\beta}{}^{,\beta} - \xi_{\alpha,\beta}{}^{\beta} \\
&= -\xi_{\alpha,\beta}{}^{\beta}
\end{aligned}$$

where to get the last expression we've used the fact that $\bar{h}_{\alpha\beta}{}^{,\beta} = 0$, since $\bar{h}_{\alpha\beta}$ is in Lorentz gauge.

If we want $\bar{h}'{}_{\alpha\beta}$ to remain in Lorentz gauge, we see that the generators ξ_α should satisfy wave equation: $\xi_{\alpha,\beta}{}^{\beta} = 0$

The general solution of this equation can be written as a sum of plane waves:

$$\xi_\alpha(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[A_\alpha(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + B_\alpha(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}+\omega t)}\right]$$

The first term describes the wave propagating in $\mathbf{k}$ direction and second one in $-\mathbf{k}$ direction. In our cases we need only the first term (since we consider a gravitational wave propagating in some particular direction). So

$$\xi_\alpha(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} A_\alpha(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

At time $t = 0$: $\xi_\alpha(0,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} A_\alpha(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$, or $A_\alpha(\mathbf{k}) = \int d^3\mathbf{x}\,\xi_\alpha(0,\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}$. We see that $\xi_\alpha(x)$ are completely determined by four functions of three spatial coordinates: $\xi_\alpha(0,\mathbf{x})$. These functions give initial conditions for wave equation at $t = 0$.

(b) Consider a plane gravitational wave propagating in z-direction.

$$\bar{h}_{\alpha\beta} = \bar{h}_{\alpha\beta}(t-z) = \bar{h}_{\alpha\beta}(\tau), \quad \tau \equiv t - z$$

$\bar{h}_{\alpha\beta}$ is in Lorentz gauge, i.e.

$$\begin{aligned}
\bar{h}_{\alpha\beta}{}^{,\beta} &= \bar{h}_{\alpha t}{}^{,t} + \bar{h}_{\alpha z}{}^{,z} = -\bar{h}_{\alpha t,t} + \bar{h}_{\alpha z,z} = -\bar{h}'_{\alpha t} - \bar{h}'_{\alpha z} \\
&= 0 \quad \text{(prime denotes derivatives w.r.t. } \tau)
\end{aligned}$$

Integrating: $\bar{h}_{\alpha z} = -\bar{h}_{\alpha t} + const$. Constant is irrelevant and we can set it to zero, thus $\bar{h}_{\alpha z} = -\bar{h}_{\alpha t}$.

These four gauge conditions reduce the number of independent components of $\bar{h}_{\alpha\beta}$ from 10 to 6: $\bar{h}_{tt}, \bar{h}_{tx}, \bar{h}_{ty}, \bar{h}_{xx}, \bar{h}_{xy}, \bar{h}_{yy}$.

Now make additional gauge transformation with

$$\xi_\alpha = \xi_\alpha(\tau) = \xi_\alpha(t-z), \qquad {\xi_{\alpha,\beta}}^{\beta} = 0$$

$$\bar{h}_{\alpha\beta} \to \bar{h}^{new}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}{\xi_\mu}^{,\mu}$$

${\xi_\mu}^{,\mu} = -\xi_{t,t} + \xi_{z,z} = -\xi'_t - \xi'_z$.

We want to choose ξ_α so that $\bar{h}^{new}_{\alpha\beta}$ satisfy additional constraints: $\bar{h}^{new}_{tt} = \bar{h}^{new}_{tx} = \bar{h}^{new}_{ty} = 0$, $\bar{h}^{new}_{xx} + \bar{h}^{new}_{yy} = 0$.

$$\begin{aligned}
\bar{h}^{new}_{tt} &= \bar{h}_{tt} - 2\xi_{t,t} + (\xi'_t + \xi'_z) = \bar{h}_{tt} + \xi'_z - \xi'_t \\
\bar{h}^{new}_{tx} &= \bar{h}_{tx} - \xi_{t,x} - \xi_{x,t} = \bar{h}_{tx} - \xi'_x \\
\bar{h}^{new}_{ty} &= \bar{h}_{ty} - \xi_{t,y} - \xi_{y,t} = \bar{h}_{ty} - \xi'_y \\
\bar{h}^{new}_{xx} &= \bar{h}_{xx} - 2\xi_{x,x} - \xi'_t - \xi'_z = \bar{h}_{xx} - \xi'_t - \xi'_z \\
\bar{h}^{new}_{yy} &= \bar{h}_{yy} - \xi'_t - \xi'_z \\
\bar{h}^{new}_{xy} &= \bar{h}_{xy}
\end{aligned}$$

This gives the system of equations:

$$\begin{aligned}
\xi'_x &= \bar{h}_{tx} \\
\xi'_y &= \bar{h}_{ty} \\
\xi'_t &= \frac{\bar{h}_{tt} + \frac{1}{2}(\bar{h}_{xx} + \bar{h}_{yy})}{2} \\
\xi'_z &= \frac{-\bar{h}_{tt} + \frac{1}{2}(\bar{h}_{xx} + \bar{h}_{yy})}{2}
\end{aligned}$$

These equations have unique solutions (up to an additive constant) given by simple integrations.

E. Exercise 26.7 Quadrupolar wave generation in linearized theory [Xinkai Wu/02]

(a) For a slow-motion source, the size of the source is much smaller than the wavelength. Only keeping the leading order when expanding $|\mathbf{x} - \mathbf{x}'|$, which is r, eq. (26.128) becomes eq. (26.129).

(b) Taking the divergence of both sides of the linearized Einstein equation (24.106) we get

$$-{\bar{h}_{\mu\nu,\alpha}}^{\alpha\nu} = 16\pi {T_{\mu\nu,}}^{\nu}$$

the r.h.s. vanishes by virtue of the Lorentz gauge condition ${\bar{h}_{\mu\nu,}}^{\nu} = 0$, thus we conclude ${T_{\mu\nu,}}^{\nu} = 0$, i.e. ${T^{\mu\nu}}_{,\nu} = 0$. (Another way to see this is, Bianchi identity combined with Einstein equation implies the covariant divergence of the stress-energy tensor always vanishes; in the linearized theory we can ignore the connection coefficients, which means the coordinate divergence, just like the covariant divergence, of the stress-energy tensor vanishes.)

(c) Let's first evaluate the l.h.s.: $[T^{00}x^jx^k]_{,00} = T^{00}{}_{,00}x^jx^k$.

The terms on the r.h.s. are: straightforward differentiation gives $[T^{lm}x^jx^k]_{ml} = T^{lm}{}_{,ml}x^jx^k + 2T^{jm}{}_{,m}x^k + 2T^{km}{}_{,m}x^j + 2T^{jk}$, and $-2[T^{lj}x^k + T^{lk}x^j]_{,l} = -2(T^{lj}{}_{,l}x^k + T^{lk}{}_{,l}x^j + 2T^{jk})$. When the three terms on the r.h.s. are added up, there're some cancellations and we find the r.h.s. to be $T^{lm}{}_{,ml}x^jx^k$.

Now using the $T^{\mu\nu}{}_{,\nu} = 0$ one finds $T^{00}{}_{,00} = T^{lm}{}_{,ml}$, because $T^{00}{}_{,00} = -T^{0l}{}_{,l0} = -T^{l0}{}_{,0l} = -[-T^{lm}{}_{,m}]_l = T^{lm}{}_{,ml}$

Thus we conclude $l.h.s. = r.h.s.$

(d)

$$\begin{aligned}
\frac{2}{r}\,\ddot{I}_{jk}(t-r) &= \frac{2}{r}\int \ddot{\rho}(t-r,\mathbf{x}')x'^jx'^k dV_{x'} \\
&= \frac{2}{r}\int T^{00}{}_{,00}(t-r,\mathbf{x}')x'^jx'^k dV_{x'} \\
&= \frac{2}{r}\int [T^{00}x'^jx'^k]_{,00}(t-r,\mathbf{x}')dV_{x'} \\
&= \frac{4}{r}\int T^{jk}(t-r,\mathbf{x}')dV_{x'}
\end{aligned}$$

where to reach the last expression we've used the result of part (c) and the fact that the integrals of total derivatives vanish. Comparing the expression above with (26.129), we conclude

$$\bar{h}_{jk}(t,\mathbf{x}) = \frac{2}{r}\,\frac{d^2 I_{jk}(t-r)}{dt^2}$$

(e) Since the expression for the trace-reversed metric perturbation obtained in the previous part has the "speed-of-light-propagation" form, namely, it's a function of $(t-r)$ [with the $1/r$ essentially consant on lengthscale of order a wavelength], and since $\bar{h}_{jk}$ and h_{jk} only differ in their trace which doesn't matter in the TT projection, we get by eq.(26.96)

$$h^{TT}_{jk} = (\bar{h}_{jk})^{TT} = 2\left[\frac{\ddot{I}_{jk}(t-r)}{r}\right]^{TT}$$

which is the desired eq. (26.111).

F. Exercise 26.10 Propagation of waves through an expanding universe [Alexander Putilin/00 and Kip Thorne/02]

$ds^2 = b^2[-d\eta^2 + d\chi^2 + \chi^2(d\theta^2 + sin^2\theta d\phi^2)]$, where $b = b_0\eta^2$.

(a) We can prove that curves of constant $\theta, \phi, \eta - \chi$ satisfy geodesic equation by explicit calculation of connection coefficients. But the easier way is to use symmetry. Spherical symmetry implies that a radial curve $\eta = \eta(\zeta)$, $\chi = \chi(\zeta)$, $\theta, \phi = const$ must be a geodesic for some parameter ζ. Since geodesic is null we have $-\left(\frac{d\eta}{d\zeta}\right)^2 + \left(\frac{d\chi}{d\zeta}\right)^2 = 0$, $\frac{d\eta}{d\zeta} = \frac{d\chi}{d\zeta}$, $\Rightarrow \eta - \chi = const$ along geodesic.

(b) Symmetry also helps here. Spherical symmetry guarantees that $\nabla_{\vec{k}}\vec{e}_{\hat{\theta}}$ cannot point in χ or ϕ direction. So $\nabla_{\vec{k}}\vec{e}_{\hat{\theta}} = a\vec{e}_{\hat{\theta}} + b\vec{k}$. $\vec{k} = k^{\eta}\vec{e}_{\hat{\eta}} + k^{\chi}\vec{e}_{\hat{\chi}} = k^{\eta}(\vec{e}_{\hat{\eta}}+\vec{e}_{\hat{\chi}})$, $k^{\eta} = k^{\chi}$ since $\vec{k}^2 = 0$. But $\vec{e}_{\hat{\theta}}\cdot\nabla_{\vec{k}}\vec{e}_{\hat{\theta}} = a = \frac{1}{2}\nabla_{\vec{k}}(\vec{e}_{\hat{\theta}}\cdot\vec{e}_{\hat{\theta}}) = \frac{1}{2}\nabla_{\vec{k}}(1) = 0$ gives $a = 0$. and $\nabla_{\vec{k}}\vec{e}_{\hat{\theta}} = b\vec{k} = bk^{\eta}(\vec{e}_{\hat{\eta}} + \vec{e}_{\hat{\chi}})$.

$$k^{\hat{\alpha}}\vec{e}_{\hat{\theta};\hat{\alpha}} = k^{\hat{\alpha}}\Gamma^{\hat{\mu}}{}_{\hat{\theta}\hat{\alpha}}\vec{e}_{\hat{\mu}} = bk^{\hat{\eta}}(\vec{e}_{\hat{\eta}} + \vec{e}_{\hat{\chi}})$$

Take a dot product of this eqn with $\vec{e}_{\hat{\chi}}$:

$bk^{\hat{\eta}} = k^{\hat{\alpha}}\Gamma^{\hat{\mu}}{}_{\hat{\theta}\hat{\alpha}}\eta_{\hat{\chi}\hat{\mu}} = k^{\hat{\alpha}}\Gamma_{\hat{\chi}\hat{\theta}\hat{\alpha}} = k^{\hat{\eta}}(\Gamma_{\hat{\chi}\hat{\theta}\hat{\eta}} + \Gamma_{\hat{\chi}\hat{\theta}\hat{\chi}})$, so $b = \Gamma_{\hat{\chi}\hat{\theta}\hat{\eta}} + \Gamma_{\hat{\chi}\hat{\theta}\hat{\chi}}$.

Now we need only to calculate two connection coefficients to verify that $\Gamma_{\hat{\chi}\hat{\theta}\hat{\eta}} = \Gamma_{\hat{\chi}\hat{\theta}\hat{\chi}} = 0$, so that $b = 0 \;\Rightarrow \nabla_{\vec{k}}\vec{e}_{\hat{\theta}} = 0$. The proof that $\nabla_{\vec{k}}\vec{e}_{\hat{\phi}} = 0$ is very similar.

(c) The general solutions are

$$h_+ = \frac{Q_+(\tau_r, \theta, \phi)}{r}, \quad h_\times = \frac{Q_\times(\tau_r, \theta, \phi)}{r}$$

where $\vec{k} = -\vec{\nabla}\tau_r$ and $\nabla_{\vec{k}}r = \frac{1}{2}(\vec{\nabla}\cdot\vec{k})r$.

To fix τ_r recall that it is the proper time of the ray's emission. If η_e is the coordinate time of emission, then $\tau_r = \int_0^{\eta_e} bd\eta = \int_0^{\eta_e} b_0\eta^2 d\eta = \frac{1}{3}b_0\eta_e^3$. But along the ray $\eta - \chi = \eta_e$, so

$$\tau_r = \frac{1}{3}b_0(\eta - \chi)^3$$

To determine r proceed as follows

$$\vec{k} = -\vec{\nabla}\tau_r \Rightarrow k^{\eta} = k^{\chi} = \frac{(\eta-\chi)^2}{b_0\eta^4}$$

$$\begin{aligned}(\vec{\nabla}\cdot\vec{k}) &= \frac{1}{\sqrt{-g}}(\sqrt{-g}\;k^{\alpha})_{,\alpha} = \frac{1}{\sqrt{-g}}\left[(\sqrt{-g}\;k^{\eta})_{,\eta} + (\sqrt{-g}\;k^{\chi})_{,\chi}\right] \\ &= \frac{2(\eta-\chi)^2(\eta+2\chi)}{b_0\eta^5\chi} \quad \text{(after some calculations)}\end{aligned}$$

Then

$$\nabla_{\vec{k}}r = \frac{1}{2}(\vec{\nabla}\cdot\vec{k})r = k^{\eta}(r_{,\eta} + r_{,\chi})$$

reduces to

$$\left(\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\chi}\right)r = \left(\frac{1}{\chi} + \frac{2}{\eta}\right)r$$

changing variables: $a = \eta - \chi, b = \eta + \chi$, we get:

$$\begin{aligned}\frac{\partial}{\partial b}\,r &= \left(\frac{1}{b-a} + \frac{2}{b+a}\right)r \\ r(a,b) &= C(a)e^{\int db\left(\frac{1}{b-a}+\frac{2}{b+a}\right)} = C(a)(b-a)(b+a)^2 \\ &\Rightarrow\; r(\chi,\eta) = C(\eta-\chi)\chi\eta^2\end{aligned}$$

where $C(\eta-\chi)$ is an arbitrary function.

Consider the region $\eta=\eta_0, \chi << \eta_0$. In this region we should have:

$$r(\chi,\eta)=r$$

where $dr^2 = ds^2 = b^2 d\chi^2 = b_0^2\eta_0^4 dx^2,\ r = b_0\eta_0^2\chi,\ \Rightarrow\ C(\eta_0)\chi\eta_0^2 = b_0\eta_0^2\chi,\ \Rightarrow$ $C(\eta_0)=b_0$. So finally we get

$$r=b_0\eta^2\chi$$

Notice that a bundle of rays which subtends solid angle $\Delta\Omega = sin\theta\Delta\theta\Delta\phi$ has cross section area $b^2\chi^2\Delta\Omega = (b_0\eta^2\chi)^2\Delta\Omega = r^2\Delta\Omega$; i.e. its cross section area is $\propto r^2$. This is true quite generally and is an easy way to compute r.

To determine $Q_+, Q_\times$, compare them to the solution of gravitational wave eqn. in the near zone: $\eta\approx\eta_0, \chi << \eta_0$ $(\tau_r = t-r)$

$$\begin{aligned} h_+ &= \frac{2}{r}\left[\ddot{I}_{\hat\theta\hat\theta}(t-r)\right]^{TT} = \frac{2}{r}\left[\ddot{I}_{\hat\theta\hat\theta}(\tau_r)\right]^{TT} \\ h_\times &= \frac{2}{r}\left[\ddot{I}_{\hat\theta\hat\phi}(t-r)\right]^{TT} = \frac{2}{r}\left[\ddot{I}_{\hat\theta\hat\phi}(\tau_r)\right]^{TT} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} Q_+(\tau_r,\theta,\phi) &= 2\left[\ddot{I}_{\hat\theta\hat\theta}(\tau_r)\right]^{TT} \\ Q_\times(\tau_r,\theta,\phi) &= 2\left[\ddot{I}_{\hat\theta\hat\phi}(\tau_r)\right]^{TT} \end{aligned}$$

G. Exercise 26.11 Gravitational waves emitted by a linear oscillator [Alexander Putilin/00]

Since the mass is moving along z-direction the second moment of mass distribution has only zz-component.

$$I_{zz}(t) = mz^2(t) = ma^2cos^2\Omega t$$

or

$$I(t) = ma^2cos^2\Omega t\ \vec{e}_z\otimes\vec{e}_z$$

and we have

$$h_{jk}^{TT} = 2\left[\frac{\ddot{I}_{jk}(t-r)}{r}\right]^{TT}$$

which gives

$$\begin{aligned} h^{TT} &= \frac{2}{r}ma^2\frac{-4\Omega^2cos2\Omega(t-r)}{2}\left[\vec{e}_z\otimes\vec{e}_z\right]^{TT} \\ &= -\frac{4m\Omega^2a^2}{r}cos\left(2\Omega(t-r)\right)\left[\vec{e}_z\otimes\vec{e}_z\right]^{TT} \end{aligned}$$

To perform TT-projection notice that $\vec{e}_z = cos\theta \vec{e}_{\hat{r}} - sin\theta \vec{e}_{\hat{\theta}}$, and thus

$$\vec{e}_z \otimes \vec{e}_z = cos^2\theta \vec{e}_{\hat{r}} \otimes \vec{e}_{\hat{r}} - cos\theta sin\theta(\vec{e}_{\hat{r}} \otimes \vec{e}_{\hat{\theta}} + \vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{r}}) + sin^2\theta \vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\theta}}$$

TT-projection on $(\vec{e}_{\hat{\theta}}, \vec{e}_{\hat{\phi}})$ plane gives:

$$[\vec{e}_z \otimes \vec{e}_z]^{TT} = \frac{1}{2} sin^2\theta \left(\vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\theta}} - \vec{e}_{\hat{\phi}} \otimes \vec{e}_{\hat{\phi}} \right)$$

so

$$h^{TT} = -\frac{2m\Omega^2 a^2}{r} sin^2\theta cos(2\Omega(t-r)) \left(\vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\theta}} - \vec{e}_{\hat{\phi}} \otimes \vec{e}_{\hat{\phi}} \right) = -\frac{2m\Omega^2 a^2}{r} sin^2\theta cos(2\Omega(t-r)) \mathbf{e}^{+}$$

It follows immediately from the result above that:

$$\begin{aligned} h_{+}(t,r,\theta,\phi) &= -\frac{2m\Omega^2 a^2}{r} sin^2\theta cos(2\Omega(t-r)) \\ h_{\times}(t,r,\theta,\phi) &= 0 \end{aligned}$$

In conventional units

$$h_{+}(t,r,\theta,\phi) = -\frac{2Gm\Omega^2 a^2}{rc^4} sin^2\theta cos(2\Omega(t-r))$$

H. Exercise 26.13 Light in an interferometric gravitational wave detector in TT gauge [Xinkai Wu/02]

(a) This expression for ϕ gives

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= -\omega_0 \left[1 + \frac{1}{2} h(t-x) - \frac{1}{2} h(t) \right] \\ \frac{\partial \phi}{\partial x} &= -\omega_0 \left[-1 - \frac{1}{2} h(t-x) \right] \end{aligned}$$

Ignoring quadratic (and higher) order terms in h, we find

$$\begin{aligned} \vec{k}^2 &= -\left(\frac{\partial \phi}{\partial t} \right)^2 + [1-h(t)] \left(\frac{\partial \phi}{\partial x} \right)^2 \\ &= -\omega_0^2[1 + h(t-x) - h(t)] + [1-h(t)]\omega_0^2[1 + h(t-x)] = 0 \end{aligned}$$

(b) Setting $x = 0$, we get $\frac{\partial \phi}{\partial t} = -\omega_0$.

(c)

$$(\nabla_{\vec{k}} \vec{k})_\nu = k^\mu \nabla_\mu k_\nu = k^\mu \nabla_\mu \nabla_\nu \phi = k^\mu \nabla_\nu \nabla_\mu \phi = k^\mu \nabla_\nu k_\mu = \frac{1}{2} \nabla_\nu (k^\mu k_\mu) = 0$$

(d) The null geodesic of the photon is given by $0 = ds^2 = -dt^2 + [1+h(t)]dx^2$, which gives $\frac{dx}{dt} = 1 - \frac{1}{2}h(t)$. Now $p_x = -\partial\phi/\partial x = -\omega_0[1 + \frac{1}{2}h(t-x)]$, and along the null geodesic we have

$$\frac{dp_x}{dt} = \left(\frac{\partial}{\partial t} + \frac{dx}{dt} \frac{\partial}{\partial x} \right) p_x = -\omega_0 \frac{1}{2} \dot{h}(t-x) \left(1 - \frac{dx}{dt} \right) = -\omega_0 \frac{1}{2} \dot{h}(t-x) \left(\frac{1}{2} h(t) \right) = 0$$

up to linear order in h. Thus we see p_x is indeed conserved along the geodesic.

(e) The observer at rest has 4-velocity $\vec{u} = (1, 0, 0, 0)$, thus the photon's energy measured by him is $\omega = -\vec{k} \cdot \vec{u} = -k_\alpha u^\alpha = -k_t = -\partial\phi/\partial t = \omega_0[1 + \frac{1}{2}h(t-x) - \frac{1}{2}h(t)]$. $\frac{d\omega}{dt} = \frac{\partial\omega}{\partial t} + \frac{\partial\omega}{\partial x}\frac{dx}{dt}$. Since $\frac{\partial\omega}{\partial x}$ is already of order h, we can approximate $\frac{dx}{dt}$ as unity, and thus getting $\frac{d\omega}{dt} \approx (\frac{\partial}{\partial t} + \frac{\partial}{\partial x})\omega$. And this is

$$\begin{aligned}
\frac{d\omega}{dt} &\approx (\frac{\partial}{\partial t} + \frac{\partial}{\partial x})\omega \\
&= \omega_0[\frac{1}{2}\dot{h}(t-x) - \frac{1}{2}\dot{h}(t) - \frac{1}{2}\dot{h}(t-x)] \\
&= -\frac{1}{2}\omega_0\dot{h}(t)
\end{aligned}$$

as desired.