Solution for Chapter 25

(compiled by Xinkai Wu)

Exercise 25.1 Connection coefficients and Riemann tensor in Schwarzschild metric [A. Dvoretskii/99]

Solution to this problem is mostly GRTensor output and will be distributed in class. Note that in part (b) of the solution, the last upper index of the connection coefficients corresponds to the first upper index of those in the text; aslo in part (d) of the solution, the orthonormal basis vectors $\hat{r}, \hat{\theta}, \hat{\phi}, \hat{0}$ are denoted as $1, 2, 3, 4$ respectively.

Exercise 25.2 Bertotti-Robinson solution of Einstein's field equation [A. Dvoretskii/99]

Let's drop the unimportant normalization constant Q and examine the form of line element

$$ds^2 = -dt^2 + sin^2 t dz^2 + d\theta^2 + sin^2\theta d\phi^2$$

Clearly, the metric coefficients have a well-defined sign and hence t is a timelike coordinate and z, θ, ϕ are spacelike everywhere. An observer at rest in the coordinate system (z, θ, ϕ fixed) experiences proper time $d\tau = \sqrt{-ds^2} = dt$, so t is that observer's proper time. Now consider the spatial part of the metric. Compare it with the metrics for the surface of a cylinder and a sphere.

$$\begin{aligned} ds^2 &= dz^2 + r^2(d\phi)^2 \\ ds^2 &= r^2(d\theta^2 + sin^2\theta d\phi^2) \end{aligned}$$

Evidently, the Bertotti-Robinson metric can be interpreted as a metric on the $3D$ surface of a cylindrical pipe of unit radius embedded in a $4D$ space, where the axial coordinate scale depends on time.

The spacetime defined by the Bertotti-Robinson metric possesses spherical symmetry as it is invariant under rotations in (θ, ϕ) and it's also axially symmetric as it is invariant under rotation in ϕ and translation in z.

In the (θ, ϕ) space orthogonal to z the curvature is constant and therefore such universe is not asymptotically flat.

For $t = 0$ the z dimension collapses and the universe becomes 2 dimensional living on the surface of a sphere with constant curvature everywhere. As t grows from 0 to π the universe expands in the z direction and then contracts back in the z direction.

Exercise 25.3 Schwarzschild Geometry in Isotropic Coordinates [Xinkai/00]

(a)We just want to change $r \to \bar{r}$ with t, θ, ϕ untouched. Then equating the time-time components of eq. (25.52) and (25.1) suggests the following coordinate change

$$r = \bar{r}\left(1 + \frac{M}{2\bar{r}}\right)^2$$

Plugging the above relation into eq. (25.1) and using $dr = \frac{dr}{d\bar{r}} d\bar{r}$ indeed brings it into the form of eq. (25.52).

(b) Expanding the metric components to leading order in $(\frac{1}{\bar{r}})$ is straightforward, and we find

$(\frac{1-\frac{M}{2\bar{r}}}{1+\frac{M}{2\bar{r}}})^2 \approx 1 - \frac{2M}{\bar{r}}$

$(1 + \frac{M}{2\bar{r}})^4 \approx 1 + \frac{2M}{\bar{r}}$

Thus we see (25.52) takes the form (24.113) but with vanishing spin angular momentum(b/c we don't have $dtd\phi$ term in our metric).

Exercise 25.5 Gravitational redshift [A. Dvoretskii/99]

(a) At the origin of the inertial reference frame of the atom the metric is flat:$\vec{e}_{\hat{\alpha}} \cdot \vec{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$. This is automatically satisfied if the basis vectors coincide with those of an orthonormal basis quoted in the problem. See Exercise 25.1 for a discussion.

(b) In the inertial frame the atom's 4-velocity is $\vec{u} = \vec{e}_{\hat{0}}$, so the energy of the photon as measured by the atom is

$$E = h\nu_{em} = -\vec{p} \cdot \vec{u} = -\vec{p} \cdot \vec{e}_{\hat{0}} = -p_{\hat{0}} = p^{\hat{0}}$$

(c) In the orthonormal basis

$$p_{\hat{0}} = \vec{p} \cdot \vec{e}_{\hat{0}} = -h\nu_{em}$$

and since $\vec{e}_t = \sqrt{1 - 2M/r}\vec{e}_{\hat{0}}$

$$p_t = \vec{p} \cdot \vec{e}_t = \sqrt{1 - 2M/R}\, \vec{p} \cdot \vec{e}_{\hat{0}} = -\sqrt{1 - 2M/R}\, h\nu_{em}$$

(d) The metric components do not depend on t and therefore, as was shown in part (a) of Exercise 24.4, p_t is constant.

(e) The observer who is measuring the frequency of the photon, projects the photon's 4-momentum on her timelike basis vector,

$$\nu_{rec} = -\vec{p} \cdot \vec{e}_{\hat{0}}/h$$

Since the observer is far away, the metric is flat and the coordinate and orthonomal bases are the same. Therefore

$$\nu_{rec} = -\vec{p} \cdot \vec{e}_0/h = -p_t/h$$

(f) From here it is trivial to calculate that the redshift is equal to

$$RS = \frac{\lambda_{rec} - \lambda_{em}}{\lambda_{em}} = \frac{1}{\sqrt{1 - 2M/R}} - 1 \approx \frac{M}{R}$$

(g)For the earth

$$RS = \frac{4.4 \times 10^{-3} m}{6.4 \times 10^{6} m} \approx 6.9 \times 10^{-10}$$

For the sun

$$RS = \frac{1.5 \times 10^3 m}{7.0 \times 10^8 m} \approx 2.1 \times 10^{-6}$$

For a neutron star

$$2M/R = 2\frac{1.4 \times 1.5 \times 10^3 m}{1.0 \times 10^4 m} \approx 0.4$$

Exercise 25.7 Implosion of the Surface of a Zero-Pressure Star [Xinkai Wu/00]
(a) There is no t-dependence in the metric components, thus
$\vec{u} \cdot \frac{\partial}{\partial t} = u_t$ is conserved along the particle's world line.
$u_t = g_{tt}u^t = -(1 - \frac{2M}{R})u^t = -(1 - \frac{2M}{R_0})u^t(R = R_0)$
Also we have the normalization of the 4-velocity:
$-1 = \vec{u}^2 = g_{tt}(u^t)^2 + g_{rr}(u^r)^2$
$u^r(R = R_0) = 0$, giving us $u^t(R = R_0) = \frac{1}{\sqrt{1-\frac{2M}{R_o}}}$
Thus we find

$$u_t = -\sqrt{1 - \frac{2M}{R_o}}$$

and as an aside $u^t = \frac{u_t}{g_{tt}} = \frac{\sqrt{1-\frac{2M}{R_o}}}{1-\frac{2M}{R}}$

(b) Plugging u^t obtained above into the equation of the 4-velocity normalization(as written out in part (a)), we easily get
$\frac{dR}{d\tau} = u^r = -[-\frac{2M}{R_0} + \frac{2M}{R}]^{1/2}$ (25.67)

In Newtonian gravity energy conservation for a freely falling particle (a particle on the star's surface) says

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{M}{R} = const = -\frac{M}{R_0}$$

so

$$\frac{dR}{dt} = -\left[-\frac{2M}{R_0} + \frac{2M}{R}\right]^{1/2}$$

which agrees with the GR result.

(c) Integrating eq. (25.67) and using the initial condition $\tau = 0, R = R_0$, one finds

$$\tau = \frac{R_0}{\sqrt{2M}}\left[R\sqrt{\frac{M}{R} - \frac{M}{R_0}} + R_0\sqrt{\frac{M}{R_0}}arctan\sqrt{\frac{R_0}{R} - 1}\right]$$

Setting $R = 2M$ and expanding the above expression to leading order in R_0 for $R_0 >> M$, we find

$$\tau \approx \frac{\pi}{2}\left(\frac{R_0^3}{2M}\right)^{1/2}$$

To find out the orbit period at leading order of large R_0 we can use Newtonian mechanics, which gives

$$\tau_{orbit} = \frac{2\pi R_0}{v} = \frac{2\pi R_0}{\sqrt{M/R_0}} = 2\pi \left(\frac{R_0^3}{M}\right)^{1/2}$$

and indeed we see $\tau/\tau_{orbit} = 1/4\sqrt{2}$.

(d) In Eddington-Finklestein coordinates, by the same argument as given in (a), we have the conserved quantity

$u_{\tilde{t}} = g_{\tilde{t}\tilde{t}} u^{\tilde{t}} + g_{\tilde{t}r} u^r$

$= -(1 - \frac{2M}{R}) u^{\tilde{t}} + \frac{2M}{R} u^r$

And the 4-velocity normalization now reads

$-1 = \vec{u}^2 = g_{\tilde{t}\tilde{t}} (u^{\tilde{t}})^2 + g_{rr}(u^r)^2 + 2 g_{\tilde{t}r} u^{\tilde{t}} u^r$

$= -(1 - \frac{2M}{R})(u^{\tilde{t}})^2 + (1 + \frac{2M}{R})(u^r)^2 + \frac{4M}{R} u^{\tilde{t}} u^r \qquad (*)$

Evaluating the above two equations at $R = R_0$ and noticing that $u^r(R = R_0) = 0$,

we obtain the value of the conserved quantity

$u_{\tilde{t}} = -\sqrt{1 - \frac{2M}{R_0}}$

and the first equation becomes

$-\sqrt{1 - \frac{2M}{R_0}} = -(1 - \frac{2M}{R}) u^{\tilde{t}} + \frac{2M}{R} u^r \qquad (**)$

Using (**) to express $u^{\tilde{t}}$ in terms of u^r and substituting the resulting expression into (*), we readily get

$\frac{dR}{d\tau} = u^r = -[-\frac{2M}{R_0} + \frac{2M}{R}]^{1/2}$

and $\frac{d\tilde{t}}{d\tau} = u^{\tilde{t}} = \frac{-\frac{2M}{R}\sqrt{\frac{2M}{R} - \frac{2M}{R_0}} + \sqrt{1 - \frac{2M}{R_0}}}{1 - \frac{2M}{R}}$

Since the expression for u^r is the same as in parts (b), $\tau(R)$ is given by the same solution obtained in part (c)[this is expected: because R is the same in both coordinates ($2\pi R =$ circumference around star) and τ is the same (proper time)]. In particular, the proper time τ for the surface to go from R_0 to $2M$ is the same as in (c). We could write out $d\tilde{t}/dR = u^{\tilde{t}}/u^r$ and then integrate it to find $\tilde{t}(R)$. The integral is doable but the answer is long and not so illuminating. So we use the following analysis:

The E-F coordinate time $\tilde{t}$ for the surface to go from R_0 to $2M$ is given by

$\tilde{t} = \int_{R=R_0}^{R=2M} \frac{d\tilde{t}}{d\tau} d\tau$

$\frac{d\tilde{t}}{d\tau}(R = R_0) = \frac{1}{\sqrt{1 - \frac{2M}{R_0}}} \approx 1$ when $R_0 >> 2M$

$\frac{d\tilde{t}}{d\tau}(R = 2M) = \sqrt{1 - \frac{2M}{R_0}} + \frac{1}{2\sqrt{1 - \frac{2M}{R_0}}} \approx \frac{3}{2}$ when $R_0 >> 2M$

Thus the E-F coordinate time $\tilde{t}$ for the surface to go from R_0 to $2M$ is finite and of the same order as the proper time given by (25.57).

Note that the expression (25.67) is valid also inside the gravitational radius because as seen above $\frac{d\tilde{t}}{d\tau}$ is finite at $R = 2M$.

By Taylor expansion it's easy to show that as $R \to 0$

$R \propto [2(\frac{2M}{R_o^3})^{1/2}\tau - \pi]^{2/3}$
and $\frac{d\widetilde{t}}{d\tau} \propto R^{-1/2} \propto [2(\frac{2M}{R_o^3})^{1/2}\tau - \pi]^{-1/3}$

Thus the integral $\int_{R=0} \frac{d\widetilde{t}}{d\tau} d\tau$ converges, which means the E-F coordinate time $\widetilde{t}$ to reach R=0 is finite and of the order $\frac{1}{2}(\frac{R_0^3}{2M})^{1/2}$.(The proper time is, of course, of the same order, see part (c))

(e) $\frac{dt}{dR} = \frac{u^t}{u^r}$, $\frac{d\widetilde{t}}{dR} = \frac{u^{\widetilde{t}}}{u^r}$. and we've worked out all these 4-velocity components in the previous parts. It is then trivial to check that:

$\frac{d\widetilde{t}}{dR}$ is always negative(in particular, $\frac{d\widetilde{t}}{dR}(R \to 0) = -1$)

$\frac{dt}{dR} < 0$ when $R > 2M$

$\longrightarrow \infty$ when $R \longrightarrow 2M$

> 0 when $R < 2M$

which verifies that the world lines in E-F coordinates and Schwarzschild coordinates are given by Fig 25.6 (a) and (b), respectively.

Exercise 25.8 Gore at the singularity [A. Dvoretskii/99]

(a) See Fig. 25.6 of the text. For $r < 2M$ let's continue to work in the Schwarzschild metric. Its advantage is that it is by now quite familiar and has a simple form. (The angular coodinates are assumed to be fixed.)

$$ds^2 = -\frac{dr^2}{2M/r - 1} + (2M/r - 1)dt^2$$

Thus the t and r coordinates have "switched places" and now have a somewhat counterintuitive meaning, r being timelike and t being spacelike. However, the light cones are still given by $ds^2 = 0$ and so

$$dt/dr = \frac{1}{2M/r - 1}$$

The geodesics of the matter molecules must lie inside them and as they approach the singularity the light cones become narrower and narrower so that at $r = 0$, $dt/dr = 0$.

(b) [Kip] The curve to which the worldline asymptotes $(t, \theta, \phi) = const$ is a timelike geodesic since it is (1) timelike (2) radial (3) has $P_t = 0 = const$.

(c) It's straightforward to compute $g_{\hat{\mu}\hat{\nu}}$ using (25.68) and the $g_{\mu\nu}$ in Schwarzschild coordinates. One finds $g_{\hat{0}\hat{0}} = -1$, $g_{\hat{1}\hat{1}} = g_{\hat{2}\hat{2}} = g_{\hat{3}\hat{3}} = 1$, thus (25.68) gives the basis vectors of the infalling observer's local Lorentz frame. The components of Riemann in this basis are related to those given in Box 25.1 by linear transformation and are

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}} = -R_{\hat{2}\hat{3}\hat{2}\hat{3}} = -\frac{2M}{r^3}$$

$$R_{\hat{2}\hat{1}\hat{2}\hat{1}} = R_{\hat{3}\hat{1}\hat{3}\hat{1}} = -R_{\hat{0}\hat{3}\hat{0}\hat{3}} = -R_{\hat{0}\hat{2}\hat{0}\hat{2}} = -\frac{M}{r^3}$$

The geodesic deviation equation is (25.58)

$$\frac{d^2\xi^{\hat{j}}}{d\tau^2} = -R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}}\xi^{\hat{k}}$$

in our case

$$
\begin{aligned}
\frac{d^2\xi^{\hat{1}}}{d\tau^2} &= -R^{\hat{1}}{}_{\hat{0}\hat{1}\hat{0}}\xi^{\hat{1}} = \frac{2M}{r^3}\xi^{\hat{1}} \\
\frac{d^2\xi^{\hat{2}}}{d\tau^2} &= -R^{\hat{2}}{}_{\hat{0}\hat{2}\hat{0}}\xi^{\hat{2}} = -\frac{M}{r^3}\xi^{\hat{2}} \\
\frac{d^2\xi^{\hat{3}}}{d\tau^2} &= -R^{\hat{3}}{}_{\hat{0}\hat{3}\hat{0}}\xi^{\hat{3}} = -\frac{M}{r^3}\xi^{\hat{3}}
\end{aligned}
$$

thus we see the radial direction gets stretched and the tangential ones get squeezed. When $r \to 0$ the r.h.s. of the above expressions diverge, giving an infinite stretching/squeezing force.

(e) The acceleration measured in earthly $g_{\oplus}$'s the observer is experiencing can be estimated as

$$a/g_{\oplus} \approx \frac{(M/M_{\oplus})}{(r/r_{\oplus})^3}\,\frac{h}{r_{\oplus}}$$

substituting

$$M = 5\times 10^9 M_{\odot} = (5\times 10^9)\cdot(3\times 10^5)M_{\oplus} = 1.5\times 10^{15}M_{\oplus}$$

$$r_{\oplus} = 0.01 r_{\odot} = 0.01\cdot 2.3\times 10^5(2M_{\odot}) = 0.01\cdot 2.3\times 10^5\frac{2M}{5\times 10^9} = 5\times 10^{-7}(2M)$$

$$h = 2meters = 3\times 10^{-7} r_{\oplus}$$

get

$$a/g_{\oplus} = 6\times 10^{-11}\left(\frac{2M}{r}\right)^3$$

A typical observer could only withstand a few $g_{\oplus}$'s so she would normally die at $r^{\dagger} \approx 2\times 10^{-4}(2M) = 10^6(2M_{\odot}) = 10s$, in other words deep inside the black hole.

Now, in this region the interval is

$$ds^2 = -d\tau^2 = -\frac{dr^2}{2M/r - 1} + (2M/r-1)dt^2 \approx -(r/2M)dr^2$$

Integrating from $r^{\dagger}$ to 0 gives

$$\tau^{\dagger} = \frac{2r^{\dagger}}{3}\sqrt{\frac{r^{\dagger}}{2M}} = 0.01 r^{\dagger} = 0.1s$$

(one can also use the expression $\tau(R)$ worked out in part (c) of exercise 25.7 to find out this $\tau^{\dagger}$, which is given by $\tau(R=0) - \tau(R=r^{\dagger})$, expanded to $O[(R_0)^0]$ for large R_0. This gives the same answer as obtained above.)

Thus the observer is torn apart by tidal force about 0.1 second before hitting the singularity.

Exercise 25.9 Wormholes [A. Dvoretskii/99]

For the line element in isotropic coordinates

$$ds^2 = \left(1 + \frac{M}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 d\phi^2) \qquad (*)$$

substitute $\rho = \left(\frac{M}{2}\right)^2 / \bar{r}$ to get

$$\left(1 + \frac{M}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\phi^2)$$

which has precisely the same form as (*) but with $\bar{r}$ replaced by ρ. Thus there are two asymptotically flat spaces, at $r \to \infty$ and $\rho \to \infty$ with the intermediate region connecting them together. Topologically, this wormhole could look like Figure 31.5 of MTW. (I'll distribute copies of this figure in class) [Remark by Kip] Both top and bottom halves have the form given in eq. (25.51).

Exercise 25.11 Penrose process, Hawking radiation, and black-hole thermodynamics [Kip Thorne and Xinkai Wu/02]

(a) Consider the event P inside the ergosphere at which the plunging particle is created. At P $\partial/\partial t$ is spacelike. Choose a local Lorentz frame in which it lies in the plane spanned by $\vec{e}_{\hat{0}}$ and $\vec{e}_{\hat{1}}$. Then by performing a boost in the $\vec{e}_{\hat{1}}$ direction we can make $\vec{e}_{\hat{1}}$ point in the same direction as $\partial/\partial t$, so $\partial/\partial t = K\vec{e}_{\hat{1}}$ for some $K > 0$. We are free to choose the direction of $\vec{p}^{plunge}$, so long as it is timelike. Any choice for which the plunging particle moves in the positive $\vec{e}_{\hat{1}}$ direction, so $p_{\hat{1}} = \vec{p} \cdot \vec{e}_{\hat{1}} > 0$ will lead to $p_t = \vec{p} \cdot \partial/\partial t = \vec{p} \cdot (K\vec{e}_{\hat{1}}) = Kp_{\hat{1}} > 0$ and hence $E = -p_t < 0$.

(b) In the Kerr coordinate basis, $\xi^t = 1, \xi^r = \xi^\theta = 0, \xi^\phi = \Omega_H$. And the horizon's generators are the world lines given in eq. (25.77), $r = r_H, \theta = const, \phi = \Omega_H t + const$ whose tangent vectors are $u^t(1, 0, 0, \Omega_H)$. So we see $\vec{\xi}$ is tangent to the horizon's generators.

On the horizon, $\Delta = 0$, which means $\alpha^2 = 0$, and the norm of any vector field $\vec{\chi}$ is then given by

$$\vec{\chi}^2 = \frac{\rho^2}{\Delta} (\chi^r)^2 + \rho^2 \left(\chi^\theta\right)^2 + \varpi^2 \left(\chi^\phi - \Omega_H \chi^t\right)^2$$

Each of the three terms on the r.h.s. of the above equation is ≥ 0, i.e. the norm of $\vec{\chi}$ becomes zero(null vector field) only when $\chi^r = 0, \chi^\theta = 0$ and $\chi^\phi = \Omega_H \chi^t$ which is the case of $\vec{\xi}$, for all other vectors this norm is positive, i.e. those vectors are all spacelike.

(c) Since the Kerr metric is independent of t and ϕ(time-translation symmetry and axisymmetry), we have the two conserved quantities $E \equiv -p_t$ and

$L \equiv p_\phi$ which are interpreted as energy and angular momentum of the particle, respectively. We have $\Delta M = E$ and $\Delta J_H = L$, hence

$$\Delta M - \Omega_H \Delta J_H = E - \Omega_H L = -p_t - \Omega_H p_\phi$$

On the other hand $-\vec{p} \cdot \vec{\xi}_H = -p_t - \Omega_H p_\phi$ using the expression for $\vec{\xi}$ given in part (b). Thus eq. (25.89) is true.

(d)Choose a Lorentz frame in which the timelike vector $\vec{A}$ points in the time direction so $\vec{A} = A^0 \vec{e}_0$ with $A^0 > 0$, and the null vector $\vec{K}$ points in the $\vec{e}_1$ direction so $\vec{K} = K^0(\vec{e}_1 + \vec{e}_0)$ with $K^0 > 0$, thus $\vec{A} \cdot \vec{K} = -A^0 K^0 < 0$. Thus $-\vec{p}^{plunge} \cdot \vec{\xi}_H$ is positive ($\vec{p}^{plunge}$ is timelike, future directed, and $\vec{\xi}_H$ is null.)

(e) Combining the results of part (c) and (d), we have $\Delta M - \Omega_H \Delta J_H > 0$, which implies $\Delta J_H < \Delta M / \Omega_H$. Thus $\Delta M < 0$ implies $\Delta J_H < 0$.

(f) Ω_H, J_H, g_H, A_H are all functions of (M, a) with the explicit expressions given in the text. Thus one can express the r.h.s. of eq. (25.90) in terms of ΔM and Δa in a staightforward manner. Δa terms cancel, and the ΔM terms sum to give ΔM, which is the l.h.s. of eq. (25.90).

(g) $T_H \Delta S_H = \frac{\hbar}{2\pi k_B} g_H \Delta S_H = \frac{g_H}{8\pi} \Delta A_H$ implies $S_H = \frac{k_B}{4} \frac{A_H}{\hbar}$. Restoring G and c, this becomes $S_H = \frac{k_B}{4} \frac{A_H}{l_p^2}$.

(i) Recall that $M_\odot \approx 1.5km \approx 0.5 \times 10^{-5}s$. Thus we find for a ten solar mass black hole $T_H \approx 6 \times 10^{-9}$degrees Kelvin, and its entropy $S_H \approx 1 \times 10^{79} k_B$.

Exercise 25.12 Slices of simultaneity in Schwarzschild spacetime [A. Dvoretskii/99]

The Schwarzschild spacetime can be sliced by surfaces $t = const$. Since the Schwarzschild coordinates are orthogonal to each other the world lines of observers with constant spacelike coordinates and varying t will be orthogonal to those surfaces and therefore for those observers the surfaces will be simultaneities.

If t is kept constant as the black hole horizon is crossed the form of the line element is changed and t becomes spacelike. Therefore slices of $t = const$ can no longer be viewed as simultaneities.

For Eddington-Finklestein coordinates no such problem exists and the simultaneities cover the interior as well as the exterior of the black hole. Copies of the figures will be distributed in class(they are basically Figure 25.4 of the text).