Solution for Chapter 24

(compiled by Xinkai Wu)

Exercise 24.4 Constant of geodesic motion in a spacetime with symmetry [Alexander Putilin/99]

(a) Geodesic equation $\nabla_{\vec{p}}\vec{p} = 0$, i.e.

$$p^\beta p_{\alpha;\beta} = 0$$

$$\begin{aligned}(p_{\alpha,\beta} - \Gamma^\mu{}_{\alpha\beta}p_\mu)p^\beta &= \frac{dx^\beta}{d\zeta}\frac{\partial p_\alpha}{\partial x^\beta} - \Gamma^\mu{}_{\alpha\beta}p_\mu p^\beta \\ &= \frac{dp_\alpha}{d\zeta} - \Gamma_{\mu\alpha\beta}p^\mu p^\beta = 0\end{aligned}$$

which gives

$$\frac{dp_\alpha}{d\zeta} = \frac{1}{2}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu})p^\mu p^\beta$$

where in the brackets the first and the third terms are antisymmetric over $(\beta\mu)$ so their contraction with the symmetric tensor $p^\beta p^\mu$ is zero. Thus

$$\frac{dp_\alpha}{d\zeta} = \frac{1}{2}g_{\mu\beta,\alpha}p^\mu p^\beta$$

Take α to be A and using $g_{\alpha\beta,A} = 0$, we find

$$\frac{dp_A}{d\zeta} = 0$$

namely p_A is a constant of motion.

(b) Let $x^j(t)$ be the trajectory of a particle. Its proper time is

$$\begin{aligned}d\tau^2 &= -ds^2 = dt^2\left[1 + 2\Phi - (\delta_{jk} + h_{jk})v^j v^k\right] \\ &= dt^2(1 + 2\Phi - \delta_{jk}v^j v^k + O(\frac{v^4}{c^4}))\end{aligned}$$

thus

$$d\tau = dt\sqrt{1 + 2\Phi - \mathbf{v}^2} = dt(1 + \Phi - \frac{1}{2}\mathbf{v}^2)$$

where we have omitted terms of order v^4/c^4(i.e. $|\Phi|^2$). The 4-velocity is given by

$$\begin{aligned}u^\alpha &= \frac{dx^\alpha}{d\tau} = \frac{dx^\alpha}{dt(1 + \Phi - \frac{1}{2}\mathbf{v}^2)} \\ &= \frac{dx^\alpha}{dt}(1 - \Phi + \frac{1}{2}\mathbf{v}^2)\end{aligned}$$

thus in particular $u^0 = 1 - \Phi + \frac{1}{2}\mathbf{v}^2$.

4-momentum: $p^\alpha = mu^\alpha$, and in particular $p^0 = mu^0 = m(1 - \Phi + \frac{1}{2}\mathbf{v}^2)$. And the conserved quantity is then given by

$$\begin{aligned} p_t &= g_{0\alpha}p^\alpha = g_{00}p^0 = -(1+2\Phi)m(1-\Phi+\frac{1}{2}\mathbf{v}^2) \\ &= -m - (m\Phi + \frac{1}{2}m\mathbf{v}^2) \end{aligned}$$

we see that p_t is indeed the non-relativistic energy of a particle aside from an additive constant $-m$ and an overall minus sign.

Exercise 24.5 Action Principle for Geodesic Motion [Xinkai Wu/00]

The action is given by:

$S[x^\alpha(\lambda)] = \int_0^1 (-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2} d\lambda$

$\delta S = \int_0^1 \delta(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2} d\lambda$

$= \int_0^1 \frac{1}{2}(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{-1/2}\delta(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})d\lambda$

$= -\int_0^1 \frac{1}{2}(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{-1/2}\{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\delta x^\rho \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}\}d\lambda$

(by renaming $\mu \longleftrightarrow \nu$, and noticing $g_{\mu\nu} = g_{\nu\mu}$, we get:)

$= -\int_0^1 \frac{1}{2}(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{-1/2}\{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\delta x^\rho \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\}d\lambda$

Integrating the 2nd term in {...}by parts, we find, after renaming some indices:

$\delta S = \int_0^1 (-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{-1/2}\{g_{\mu\nu}\frac{d^2x^\nu}{d\lambda^2} + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{1}{2}\frac{\partial g_{\rho\nu}}{\partial x^\mu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{d\ln(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}}{d\lambda}g_{\mu\nu}\frac{dx^\nu}{d\lambda}\}\delta x^\mu d\lambda$

Thus $\delta S = 0$ if and only if

$g_{\mu\nu}\frac{d^2x^\nu}{d\lambda^2} + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{1}{2}\frac{\partial g_{\rho\nu}}{\partial x^\mu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{d\ln(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}}{d\lambda}g_{\mu\nu}\frac{dx^\nu}{d\lambda} = 0$

Contracting both sides with $g^{\pi\mu}$, we get

$\frac{d^2x^\pi}{d\lambda^2} + \frac{1}{2}g^{\pi\mu}\{2\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{\partial g_{\rho\nu}}{\partial x^\mu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda}\} - \frac{d\ln(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}}{d\lambda}\frac{dx^\pi}{d\lambda} = 0$

By renaming $\rho \longleftrightarrow \nu$ for the first term in {..}, the above equation becomes

$\frac{d^2x^\pi}{d\lambda^2} + \frac{1}{2}g^{\pi\mu}\{\frac{\partial g_{\mu\nu}}{\partial x^\rho} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\rho\nu}}{\partial x^\mu}\}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{d\ln(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}}{d\lambda}\frac{dx^\pi}{d\lambda} = 0$

which is just, using the expression for the Christoffel symbols,

$\frac{d^2x^\pi}{d\lambda^2} + \Gamma^\pi_{\rho\nu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{d\ln(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}}{d\lambda}\frac{dx^\pi}{d\lambda} = 0$

Now let's reparametrize the world line, $\lambda \rightarrow s(\lambda)$, then the equation becomes,

$(\frac{d^2x^\pi}{ds^2} + \Gamma^\pi_{\rho\nu}\frac{dx^\rho}{ds}\frac{dx^\nu}{ds})(\frac{ds}{d\lambda})^2 + \frac{dx^\pi}{ds}[\frac{d^2s}{d\lambda^2} - \frac{d\ln(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}}{d\lambda}\frac{ds}{d\lambda}] = 0$

Integrating [...] twice we readily find that [..] vanishes for

$s = \int A(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}d\lambda + B$, where A and B are arbitrary constants.

After this reparametrization, we get the familiar geodesic equation:

$\frac{d^2x^\pi}{ds^2} + \Gamma^\pi_{\rho\nu}\frac{dx^\rho}{ds}\frac{dx^\nu}{ds} = 0$

Exercise 24.7 Orders of magnitude of the radius of curvature [Alexander Putilin/99]

Eq. (24.43) tells us that, if a system has characteristic mass M and characteristic length R, order of magnitude estimate gives,

$$\frac{1}{\mathcal{R}^2} \sim \frac{GM}{R^3}$$

where $\mathcal{R}$ is the radius of curvature

$$\mathcal{R} \sim \sqrt{\frac{R^3}{M}} \quad \text{in units } G = c = 1$$

1. near earth's surfae: $R \sim R_\oplus \sim 6.4 \times 10^6 m$ (earth's radius), $M \sim M_\oplus \sim 4.4mm$ (earth's mass), and $\mathcal{R} \sim 2.4 \times 10^{11} m \sim 1$ astronomical unit $\equiv 1AU$.

2. near sun's surface: $R \sim R_{sun} \sim 7 \times 10^8 m$, $M \sim M_{sun} \sim 1.5km$, and $\mathcal{R} \sim 5 \times 10^{11} m \sim 1AU$.

3. near the surface of a white-dwarf star: $R \sim 5000km$, $M \sim M_{sun} \sim 1.5km$, and $\mathcal{R} \sim 3 \times 10^8 m \sim \frac{1}{2}$(sun radius).

4. near the surface of a neutron star: $R \sim 10km$, $M \sim M_{sun} \sim 3km$, and $\mathcal{R} \sim 20km$.

5. near the surface of a one-solar-mass black hole: $M \sim M_{sun} \sim 1.5km$, $R \sim 2M \sim 3km$, and $\mathcal{R} \sim 4km$.

6. in intergalactic space: $R \sim 10\times$(galaxy diameter) $\sim 10^6$ light-year, $M \sim$(galaxy mass)~ 0.03 light-year (for Milky way), and $\mathcal{R} \sim 6 \times 10^9$ light-years $\sim$ Hubble Distance.

Exercise 24.8 Components of Riemann in an arbitrary basis [Xinkai Wu/02]

$$p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma} = -R^\alpha{}_{\beta\gamma\delta} p^\beta$$

we have

$$\begin{aligned} p^\alpha{}_{;\gamma\delta} &= (p^\alpha{}_{;\gamma})_{;\delta} = (p^\alpha{}_{,\gamma} + p^\mu \Gamma^\alpha{}_{\mu\gamma})_{;\delta} \\ &= (p^\alpha{}_{,\gamma} + p^\mu \Gamma^\alpha{}_{\mu\gamma})_{,\delta} + \Gamma^\alpha{}_{\mu\delta}(p^\mu{}_{,\gamma} + p^\nu \Gamma^\mu{}_{\nu\gamma}) - \Gamma^\mu{}_{\gamma\delta}(p^\alpha{}_{,\mu} + p^\nu \Gamma^\alpha{}_{\nu\mu}) \end{aligned}$$

interchaging γ and δ in the above expression and then taking the difference, we get

$$\begin{aligned} p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma} &= (\Gamma^\alpha{}_{\beta\gamma,\delta} - \Gamma^\alpha{}_{\beta\delta,\gamma} + \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} - \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta})p^\beta + \\ &\quad + (\Gamma^\mu{}_{\delta\gamma} - \Gamma^\mu{}_{\gamma\delta})\Gamma^\alpha{}_{\beta\mu} p^\beta + (p^\alpha{}_{,\gamma\delta} - p^\alpha{}_{,\delta\gamma}) + (\Gamma^\mu{}_{\delta\gamma} - \Gamma^\mu{}_{\gamma\delta}) p^\alpha{}_{,\mu} \\ &= (\Gamma^\alpha{}_{\beta\gamma,\delta} - \Gamma^\alpha{}_{\beta\delta,\gamma} + \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} - \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta})p^\beta + \\ &\quad + c_{\gamma\delta}{}^\mu \Gamma^\alpha{}_{\beta\mu} p^\beta + (p^\alpha{}_{,\gamma\delta} - p^\alpha{}_{,\delta\gamma}) + c_{\gamma\delta}{}^\mu p^\alpha{}_{,\mu} \end{aligned}$$

where in the last step we've used $c_{\gamma\delta}{}^\mu = \Gamma^\mu{}_{\delta\gamma} - \Gamma^\mu{}_{\gamma\delta}$ (eq. (23.44)). We can see that the last two terms cancel, because

$$\begin{aligned} p^\alpha{}_{,\gamma\delta} - p^\alpha{}_{,\delta\gamma} &= \nabla_{\vec{e}_\delta}\nabla_{\vec{e}_\gamma} p^\alpha - \nabla_{\vec{e}_\gamma}\nabla_{\vec{e}_\delta} p^\alpha \\ &= \nabla_{[\vec{e}_\delta, \vec{e}_\gamma]} p^\alpha = c_{\delta\gamma}{}^\mu \nabla_{\vec{e}_\mu} p^\alpha \\ &= c_{\delta\gamma}{}^\mu p^\alpha{}_{,\mu} = -c_{\gamma\delta}{}^\mu p^\alpha{}_{,\mu} \end{aligned}$$

where to get to the second line, we've used the fact that for any scalar f, $\nabla_{\vec{A}}\nabla_{\vec{B}}f - \nabla_{\vec{B}}\nabla_{\vec{A}}f = A^\alpha(B^\beta f_{;\beta})_{;\alpha} - B^\beta(A^\alpha f_{;\alpha})_{;\beta} = A^\alpha B^\beta f_{;\beta\alpha} + A^\alpha B^\beta{}_{;\alpha} f_{;\beta} - B^\beta A^\alpha f_{;\alpha\beta} - B^\beta A^\alpha{}_{;\beta} f_{;\alpha} = (A^\alpha B^\beta{}_{;\alpha} - B^\alpha A^\beta{}_{;\alpha}) f_{;\beta} = [\vec{A},\vec{B}]^\beta f_{;\beta} = \nabla_{[\vec{A},\vec{B}]} f$. (note $f_{;\alpha\beta} = f_{;\beta\alpha}$ by the "torsion free" condition).

Thus we finally conclude that

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} - \Gamma^\alpha{}_{\beta\mu}c_{\gamma\delta}{}^\mu$$

Exercise 24.9 Curvature of the surface of a sphere [Alexander Putilin/99]

Hard copies of computerized part of this problem will be distributed in class.

(a) We read off the metric components from the line element:

$$g_{\theta\theta} = a^2,\ g_{\phi\phi} = a^2 sin^2\theta,\ g_{\theta\phi} = 0$$
$$g^{\theta\theta} = \frac{1}{a^2},\ g^{\phi\phi} = \frac{1}{a^2 sin^2\theta},\ g^{\theta\phi} = 0$$

There are six independent connection coefficients

$$\begin{aligned}
\Gamma^\theta{}_{\theta\theta} &= g^{\theta\theta}\Gamma_{\theta\theta\theta} = g^{\theta\theta}\frac{1}{2}g_{\theta\theta,\theta} = 0 \\
\Gamma^\theta{}_{\theta\phi} &= \Gamma^\theta{}_{\phi\theta} = g^{\theta\theta}\Gamma_{\theta\theta\phi} = \frac{1}{a^2}\ \frac{1}{2}(g_{\theta\theta,\phi} + g_{\theta\phi,\theta} - g_{\theta\phi,\theta}) = 0 \\
\Gamma^\theta{}_{\phi\phi} &= g^{\theta\theta}\frac{1}{2}(2g_{\theta\phi,\phi} - g_{\phi\phi,\theta}) = -\ \frac{1}{2a^2}(a^2 sin^2\theta)_{,\theta} = -sin\theta cos\theta \\
\Gamma^\phi{}_{\theta\theta} &= g^{\phi\phi}\frac{1}{2}(2g_{\phi\theta,\theta} - g_{\theta\theta,\phi}) = 0 \\
\Gamma^\phi{}_{\theta\phi} &= \Gamma^\phi{}_{\phi\theta} = g^{\phi\phi}\frac{1}{2}(g_{\phi\phi,\theta} + g_{\phi\theta,\phi} - g_{\phi\theta,\phi}) = \frac{1}{2a^2 sin^2\theta}(a^2 sin^2\theta)_{,\theta} = cot\theta \\
\Gamma^\phi{}_{\phi\phi} &= g^{\phi\phi}\frac{1}{2}g_{\phi\phi,\phi} = 0
\end{aligned}$$

(b) We can think of the Riemann tensor as a symmetric matrix $R_{[ij][kl]}$ with indices $[ij]$ and $[kl]$. Since R_{ijkl} is antisymmetric in the first and the second pairs of indices, the only nontrivial component is $[ij] = [\theta\phi]$, $[kl] = [\theta\phi]$

$$R_{\theta\phi\theta\phi} = -R_{\phi\theta\theta\phi} = -R_{\theta\phi\phi\theta} = R_{\phi\theta\phi\theta}$$

(c) Using eq. (24.57) and the fact that in a coordinate basis the $c_{\gamma\delta}{}^\mu$'s all vanish, we get

$$\begin{aligned}
R^\theta{}_{\phi\theta\phi} &= \Gamma^\theta{}_{\phi\phi,\theta} - \Gamma^\theta{}_{\phi\theta,\phi} + \Gamma^\theta{}_{\mu\theta}\Gamma^\mu{}_{\phi\phi} - \Gamma^\theta{}_{\mu\phi}\Gamma^\mu{}_{\phi\theta} \\
&= -\ \frac{1}{2}(sin2\theta)_{,\theta} - \Gamma^\theta{}_{\phi\phi}\Gamma^\phi{}_{\phi\theta} \\
&= -cos2\theta - (-sin\theta cos\theta)cot\theta \\
&= sin^2\theta
\end{aligned}$$

and thus

$$R_{\theta\phi\theta\phi} = g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} = a^2 sin^2\theta$$

(d) The new basis is related to the old by $\vec{e}_{\hat{\theta}} = \frac{1}{a}\vec{e}_\theta$, $\vec{e}_{\hat{\phi}} = \frac{1}{asin\theta}\vec{e}_\phi$. Thus by the multilinearity of tensors in their slots, we have

$$g_{\hat{\theta}\hat{\theta}} = \frac{1}{a^2} g_{\theta\theta} = 1,\ g_{\hat{\phi}\hat{\phi}} = \frac{1}{a^2 sin^2\theta} g_{\phi\phi} = 1,\ g_{\hat{\theta}\hat{\phi}} = \frac{1}{a^2 sin\theta} g_{\theta\phi} = 0. \text{ i.e. } g_{\hat{j}\hat{k}} = \delta_{\hat{j}\hat{k}}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{a^4 sin^2\theta} R_{\theta\phi\theta\phi} = \frac{1}{a^2}$$

$$R_{\hat{j}\hat{k}} = g^{\hat{m}\hat{n}} R_{\hat{m}\hat{j}\hat{n}\hat{k}} = \delta^{\hat{m}\hat{n}} R_{\hat{m}\hat{j}\hat{n}\hat{k}}$$

thus

$$\begin{aligned} R_{\hat{\theta}\hat{\theta}} &= R_{\hat{\theta}\hat{\theta}\hat{\theta}\hat{\theta}} + R_{\hat{\phi}\hat{\theta}\hat{\phi}\hat{\theta}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{a^2} \\ R_{\hat{\phi}\hat{\phi}} &= R_{\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}} + R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{a^2} \\ R_{\hat{\theta}\hat{\phi}} &= R_{\hat{\theta}\hat{\theta}\hat{\theta}\hat{\phi}} + R_{\hat{\phi}\hat{\theta}\hat{\phi}\hat{\phi}} = 0 \end{aligned}$$

namely, $R_{\hat{k}\hat{k}} = \frac{1}{a^2} g_{\hat{j}\hat{k}}$.

$$R = R_{\hat{k}\hat{k}} g^{\hat{j}\hat{k}} = \frac{1}{a^2} g^{\hat{j}}_{\hat{j}} = \frac{2}{a^2}$$

Exercise 24.10 Geodesic deviation on a sphere [Alexander Putilin/99]

(a) $ds^2 = a^2(d\theta^2 + sin^2\theta d\phi^2)$. on the equator, $\theta = \frac{\pi}{2}$, $dl^2 = a^2 d\phi^2$, $l = a\phi$ is the proper distance.

(b) Geodesic deviation eqn: $\nabla_{\vec{p}}\nabla_{\vec{p}}\vec{\xi} = -\mathbf{R}(..., \vec{p}, \vec{\xi}, \vec{p})$, with

$$\vec{p} = \frac{d}{dl} = \frac{1}{a}\frac{\partial}{\partial\phi},\ p^\theta = 0,\ p^\phi = \frac{1}{a}$$

At $\theta = \frac{\pi}{2}$, connection coefficients vanish (see Ex. 24.9)

$$\nabla_{\vec{p}}\nabla_{\vec{p}}\xi^\alpha = \frac{1}{a^2}\left(\xi^\alpha{}_{;\phi}\right)_{;\phi} = \frac{1}{a^2}\left(\xi^\alpha{}_{;\phi}\right)_{,\phi}$$

$$\begin{aligned} \xi^\theta{}_{;\phi} &= \xi^\theta{}_{,\phi} + \Gamma^\theta{}_{\mu\phi}\xi^\mu = \xi^\theta{}_{,\phi} - sin\theta cos\theta \xi^\phi \\ \xi^\phi{}_{;\phi} &= \xi^\phi{}_{,\phi} + \Gamma^\phi{}_{\mu\phi}\xi^\mu = \xi^\phi{}_{,\phi} + cot\theta\xi^\theta \end{aligned}$$

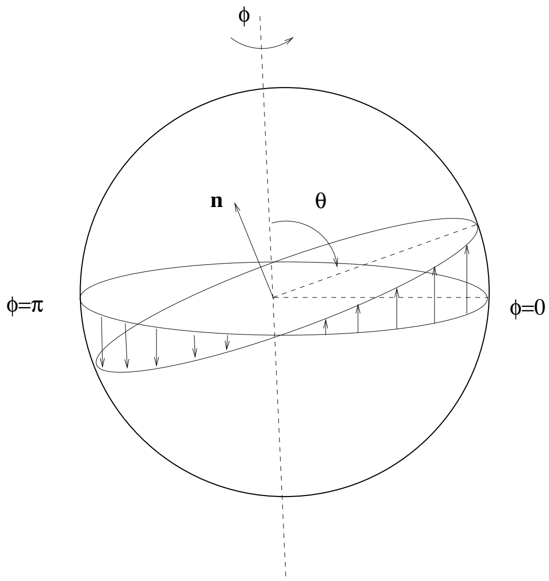

Figure 1: geodesic deviation on a sphere

thus

$$\begin{aligned}(\nabla_{\vec{p}}\nabla_{\vec{p}}\xi)^{\theta} &= \frac{1}{a^2}\left(\xi^{\theta}{}_{,\phi} - sin\theta cos\theta\xi^{\phi}\right)_{,\phi}|_{\theta=\frac{\pi}{2}} = \frac{1}{a^2}\xi^{\theta}{}_{,\phi\phi} \\ (\nabla_{\vec{p}}\nabla_{\vec{p}}\xi)^{\phi} &= \frac{1}{a^2}\left(\xi^{\phi}{}_{,\phi} + cot\theta\xi^{\theta}\right)_{,\phi}|_{\theta=\frac{\pi}{2}} = \frac{1}{a^2}\xi^{\phi}{}_{,\phi\phi}\end{aligned}$$

On the other hand

$$\begin{aligned}\nabla_{\vec{p}}\nabla_{\vec{p}}\xi^{\theta} &= -R^{\theta}{}_{\alpha\beta\gamma}p^{\alpha}\xi^{\beta}p^{\gamma} = -\frac{1}{a^2}R^{\theta}{}_{\phi\beta\phi}\xi^{\beta} = -\frac{1}{a^2}R^{\theta}{}_{\phi\theta\phi}\xi^{\theta} \\ &= -\frac{sin^2\theta}{a^2}\xi^{\theta}|_{\theta=\frac{\pi}{2}} = -\frac{1}{a^2}\xi^{\theta}\end{aligned}$$

thus

$$\frac{1}{a^2}\xi^{\theta}{}_{,\phi\phi} = -\frac{1}{a^2}\xi^{\theta} \Rightarrow \frac{d^2\xi^{\theta}}{d\phi^2} = -\xi^{\theta}$$

$$\nabla_{\vec{p}}\nabla_{\vec{p}}\xi^{\phi} = -\frac{1}{a^2}R^{\phi}{}_{\phi\mu\phi}\xi^{\mu} = 0 \Rightarrow \frac{d^2\xi^{\phi}}{d\phi^2} = 0$$

(c) Initial conditions (note that the geodesics are parallel at $\phi = 0$):

$$\xi^{\theta}(0) = b,\ \dot{\xi}^{\theta}(0) = 0;\ \xi^{\phi}(0) = 0,\ \dot{\xi}^{\phi}(0) = 0$$

This gives $\xi^{\phi} = A\phi + B = 0$. And

$$\xi^{\theta}(\phi) = A'cos\phi + B'sin\phi = bcos\phi$$

Let $\theta = \theta(\phi)$ be the eqn. for a "tilted" great circle. It's given by $\mathbf{n}\cdot\mathbf{x} = 0$, where $\mathbf{n} = (-\sin\Delta\theta, 0, cos\Delta\theta) \approx (-\Delta\theta, 0, 1)$ is the orthogonal vector and $\Delta\theta = \frac{b}{a}$, while $\mathbf{x} = (asin\theta cos\phi, asin\theta sin\phi, acos\theta)$. $\mathbf{n}\cdot\mathbf{x} = a(-sin\theta cos\phi\cdot\Delta\theta + cos\theta) = 0$ then gives: $cot\theta = \Delta\theta cos\phi = tan(\frac{\pi}{2} - \theta) \approx \frac{\pi}{2} - \theta$, i.e. $\theta = \frac{\pi}{2} - \Delta\theta cos\phi$.

From Fig. 1 we see that the separation vectors points along θ-direction (i.e. $\xi^\phi = 0$), and its magnitude is $\xi^\theta = a(\frac{\pi}{2} - \theta) = a\Delta\theta cos\phi = bcos\phi$, which is precisely what we got before.

Exercise 24.12 Newtonian limit of general relativity [Alexander Putilin/99]

(a) $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, $|h_{\alpha\beta}| << 1$. Proper time: $d\tau^2 = -g_{\alpha\beta}dx^\alpha dx^\beta \approx -\eta_{\alpha\beta}dx^\alpha dx^\beta \approx dt^2 - d\mathbf{x}^2 \approx dt^2$. (in non-relativistic limit,$|dx|/|dt| \sim |v/c| << 1$). Thus $d\tau \approx dt$, and $u^\alpha = \frac{dx^\alpha}{d\tau} \approx \frac{dx^\alpha}{dt}$: $u^0 = \frac{dt}{d\tau} \approx 1$, $u^j = \frac{dx^j}{d\tau} \approx \frac{dx^j}{dt} = v^j$.

(b) Geodesic eqn: $\frac{du^\alpha}{d\tau} = -\Gamma^\alpha{}_{\beta\gamma}u^\beta u^\gamma$.

$$\begin{aligned}\frac{du^j}{d\tau} &\approx \frac{dv^j}{dt} \approx -\Gamma^j{}_{00} = -\Gamma_{j00} = -\frac{1}{2}(2g_{j0,0} - g_{00,j}) \\ &= -h_{j0,0} + \frac{1}{2}h_{00,j} \approx \frac{1}{2}h_{00,j}\end{aligned}$$

where in the last step we've used $|h_{\alpha\beta,t}| << |h_{\alpha\beta,j}|$.

$$\frac{dv^j}{dt} = u^\alpha v^j{}_{,\alpha} \approx \frac{\partial v^j}{\partial t} + v^k\frac{\partial v^j}{\partial x^k} \text{ i.e. } \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}$$

$\frac{dv^j}{dt} = -\Phi_{,j} \Rightarrow h_{00} = -2\Phi$.

(c) $\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}) = \frac{1}{2}\eta^{\alpha\mu}(h_{\mu\beta,\gamma} + h_{\mu\gamma,\beta} - h_{\beta\gamma,\mu}) + O(h^2)$. And the Riemann tensor is:

$$\begin{aligned}R^\alpha{}_{\beta\gamma\delta} &= \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + O(\Gamma^2) \\ &= \frac{1}{2}\eta^{\alpha\mu}(h_{\mu\beta,\delta} + h_{\mu\delta,\beta} - h_{\beta\delta,\mu})_{,\gamma} - \frac{1}{2}\eta^{\alpha\mu}(h_{\mu\beta,\gamma} + h_{\mu\gamma,\beta} - h_{\beta\gamma,\mu})_{,\delta} + O(h^2) \\ &= \frac{1}{2}(h^\alpha{}_{\beta,\gamma\delta} + h^\alpha{}_{\delta,\beta\gamma} - h_{\beta\delta}{}^{\alpha}{}_{,\gamma} - h^\alpha{}_{\beta,\delta\gamma} - h^\alpha{}_{\gamma,\beta\delta} + h_{\beta\gamma}{}^{\alpha}{}_{,\delta}) + O(h^2)\end{aligned}$$

Notice that in the last line the first and fourth terms cancel. Thus we get

$$R_{\alpha\beta\gamma\delta} \approx \frac{1}{2}(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma})$$

(d) $R_{j0k0} = \frac{1}{2}(h_{j0,k0} + h_{k0,j0} - h_{jk,00} - h_{00,jk})$. Recall that in non-relativistic limit, time derivatives are small compared to spatial ones, thus the last term in the brackets dominates. And we get

$$R_{j0k0} \approx -\frac{1}{2}h_{00,jk} = \Phi_{,jk}$$

Exercise 24.13 Gauge transformation in linearized theory [Alexander Putilin/99]

(a) $x^\alpha_{new} = x^\alpha_{old} + \xi^\alpha$,

$$g^{new}_{\alpha\beta}(x_{new}) = \frac{\partial x^\mu_{old}}{\partial x^\alpha_{new}} \frac{\partial x^\nu_{old}}{\partial x^\beta_{new}} g_{\mu\nu}(x_{old})$$

Evaluate l.h.s. and r.h.s. up to linear order in ξ^α and $h_{\alpha\beta}$:

$$l.h.s. = \eta_{\alpha\beta} + h^{new}_{\alpha\beta}(x_{old} + \xi) \approx \eta_{\alpha\beta} + h^{new}_{\alpha\beta}(x_{old})$$

$$\begin{aligned}
r.h.s. \quad &= \quad (\delta^\mu{}_\alpha - \xi^\mu{}_{,\alpha})(\delta^\nu{}_\beta - \xi^\nu{}_{,\beta}) g_{\mu\nu}(x_{old}) \\
&= \quad g_{\alpha\beta}(x_{old}) - g_{\mu\beta}(x_{old})\xi^\mu{}_{,\alpha} - g_{\alpha\nu}(x_{old})\xi^\nu{}_{,\beta} \\
&\approx \quad \eta_{\alpha\beta} + h^{old}_{\alpha\beta} - \eta_{\mu\beta}\xi^\mu{}_{,\alpha} - \eta_{\alpha\nu}\xi^\nu{}_{,\beta} \\
&\approx \quad \eta_{\alpha\beta} + h^{old}_{\alpha\beta}(x_{old}) - \xi_{\alpha,\beta}(x_{old}) - \xi_{\beta,\alpha}(x_{old}) \\
&\Rightarrow \quad h^{new}_{\alpha\beta} = h^{old}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}
\end{aligned}$$

(b)

$$\bar{h}^{new}_{\mu\nu} = h^{new}_{\mu\nu} - \frac{1}{2} h^{new} \eta_{\mu\nu} = \bar{h}^{old}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^\alpha{}_{,\alpha}$$

Lorentz gauge: $\bar{h}^{new,\nu}_{\mu\nu} = 0$.

$$\bar{h}^{new,\nu}_{\mu\nu} = \bar{h}^{old,\nu}_{\mu\nu} - \xi_{\mu,\nu}{}^\nu - \xi_{\nu,\mu}{}^\nu + \xi_{\alpha,\mu}{}^\alpha = 0$$

thus we need

$$\Box \xi_\mu \equiv \xi_{\mu,\nu}{}^\nu = \bar{h}^{old,\nu}_{\mu\nu}$$

(c) In Lorentz gauge, all terms on the l.h.s. of eq. (24.102) vanish except the first one, thus it reduces to

$$-\bar{h}_{\mu\nu,\alpha}{}^\alpha = 16\pi T_{\mu\nu}$$