Solution for Chapter 23

(compiled by Xinkai Wu)

A. Ex 23.2 Causality [A. Dvoretskii/April 2000]

Consider two different reference frames - primed and unprimed. Assume without loss of generality that event P_1 occurs at a point $(0, 0)$ in spacetime in both frames and event P_2 at a point $(t, 0)$ in the unprimed frame (i.e. at the same spacial point) and at a point $(t', \mathbf{x}')$ in the primed frame.

Now using the invariance of the interval

$$\Delta s^2 = \Delta s'^2 = -t^2 = -t'^2 + x'^2$$

so

$$t' = \pm\sqrt{t^2 + x'^2}$$

The transformations from one frame to another are continuous and in the limit of very small transformations $t' \approx t$, so we must use the + sign.

Therefore,

$$t' = \sqrt{t^2 + x'^2} > 0$$

and so the temporal order of events is the same in all inertial frames. Of course, were this not true, causality would be violated.

It's also obvious that $t' \geq t$ and that apart from that there are no limits on the values of the spacial and temporal separation of the two events: t' and x' can be made arbitrarily large.

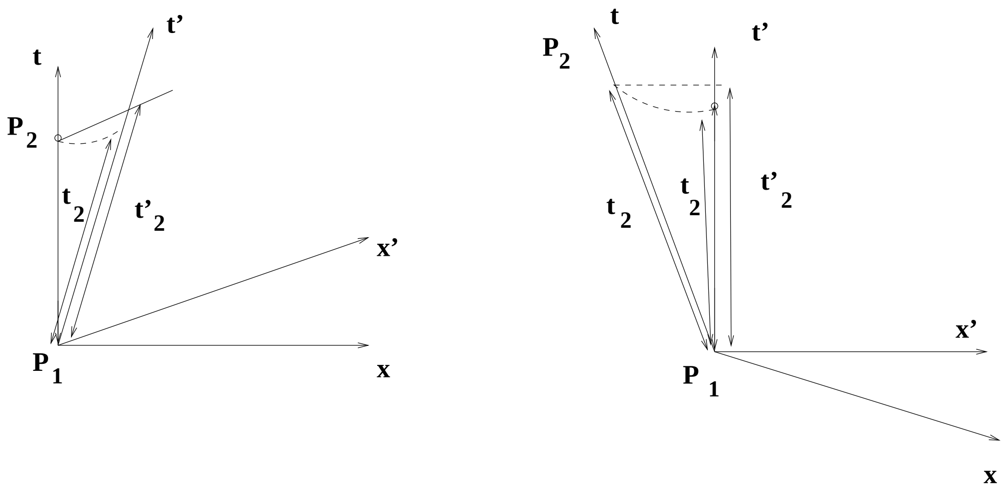

Figure 1: Causality

See Fig. 1 for the spacetime diagrams. As the velocity of the primed frame increases, the time of event P_2, t'_2 moves up the t' axis.

Clearly $t_2' > t_2$ (c.f. dashed hyperbola of events all at same interval from P_1). Similiar diagram can be drawn for velocity in the opposite direction. The diagram shows that for t_2', $t_2 \leq t_2'$ and t_2' can be made arbitrarily large.

B. Ex. 23.4 Index manipulation rules from duality [A. Dvoretskii/April 2000]

(a) Let's expand a given dual basis vector in the original basis

$$\vec{e}^{\mu} = f^{\mu\beta}\vec{e}_{\beta}$$

Now to find the coefficients $f^{\mu\beta}$ multiply both sides of the equation by $\vec{e}^{\alpha}$ and use the duality relation to obtain

$$\vec{e}^{\mu} \cdot \vec{e}^{\alpha} = f^{\mu\beta}\delta^{\alpha}_{\beta} = f^{\mu\alpha}$$

But

$$\vec{e}^{\mu} \cdot \vec{e}^{\alpha} = \mathbf{g}(\vec{e}^{\mu}, \vec{e}^{\alpha}) = g^{\mu\alpha} = f^{\mu\alpha}$$

This proves the first relation.

$$\vec{e}^{\mu} = g^{\mu\beta}\vec{e}_{\beta}$$

The proof of the second relation is similar.

(b) Using the result above

$$F^{\mu\nu} = \mathbf{F}(\vec{e}^{\mu}, \vec{e}^{\nu}) = \mathbf{F}(g^{\mu\alpha}\vec{e}_{\alpha}, \vec{e}^{\nu})$$

Now use the linearity of the tensor to write

$$F^{\mu\nu} = g^{\mu\alpha}\mathbf{F}(\vec{e}_{\alpha}, \vec{e}^{\nu}) = g^{\mu\alpha}F_{\alpha}{}^{\nu}$$

The proof of the second relation is similar.

(c) Consider for example the first identity

$$\mathbf{F} = F^{\mu\nu}\vec{e}_{\mu} \otimes \vec{e}_{\nu}$$

Now consider

$$\mathbf{F}(\vec{e}^{\alpha}, \vec{e}^{\beta}) = F^{\mu\nu}(\vec{e}_{\mu} \cdot \vec{e}^{\alpha})(\vec{e}_{\nu} \cdot \vec{e}^{\beta}) = F^{\mu\nu}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} = F^{\alpha\beta}$$

So all the components of the tensors on the left-hand side and the right hand side are equal which proves the identity. Similar proof can be given in the other cases.

C.

Ex. 23.6 Part (a) and (b) Connection coefficients for circular coordinates [A. Dvoretskii/April 2000]

First let's consider the Coordinate basis

$$\vec{e}_\varpi = \partial_\varpi, \vec{e}_\phi = \partial_\phi$$

$$[\vec{e}_\varpi, \vec{e}_\varpi] = [\vec{e}_\varpi, \vec{e}_\phi] = [\vec{e}_\phi, \vec{e}_\phi] = 0$$

Hence

$$c_{\alpha\beta\gamma} = 0$$

The metric tensor is given by

$$\mathbf{g} = (\begin{array}{cc} 1 & 0 \\ 0 & \varpi^2 \end{array})$$

The only non-zero Christoffel symbols are then

$$\Gamma_{\varpi\phi\phi} = (-1/2)g_{\phi\phi,\varpi} = -\varpi$$
$$\Gamma_{\phi\phi\varpi} = \Gamma_{\phi\varpi\phi} = \varpi$$

and the connection coefficients are

$$\Gamma^{\varpi}{}_{\phi\phi} = -\varpi, \quad \Gamma^{\phi}{}_{\varpi\phi} = \Gamma^{\phi}{}_{\phi\varpi} = 1/\varpi$$

(a) Orthornomal basis

$$\vec{e}_{\hat{\varpi}} = \partial_\varpi, \vec{e}_{\hat{\phi}} = 1/\varpi\partial_\phi$$

In this basis the metric tensor is of course just

$$\mathbf{g} = (\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array})$$

The commutation coefficients are readily computed from

$$[\vec{e}_{\hat{\varpi}}, \vec{e}_{\hat{\phi}}] = [\partial_\varpi, (1/\varpi)\partial_\phi] = -(1/\varpi^2)\partial_\phi = -(1/\varpi)\vec{e}_{\hat{\phi}}$$

Hence, the only non-zero commutation coefficients are

$$c_{\hat{\varpi}\hat{\phi}}{}^{\hat{\phi}} = -1/\varpi, \quad , c_{\hat{\phi}\hat{\varpi}}{}^{\hat{\phi}} = 1/\varpi$$

And the Christoffel symbols and the connection coefficients are

$$\Gamma_{\hat{\varpi}\hat{\phi}\hat{\phi}} = \Gamma^{\hat{\varpi}}{}_{\hat{\phi}\hat{\phi}} = -1/\varpi$$

$$\Gamma_{\hat{\phi}\hat{\varpi}\hat{\phi}} = \Gamma^{\hat{\phi}}{}_{\hat{\varpi}\hat{\phi}} = 1/\varpi$$

Ex.23.5 Part (b) Transformation matrices for circular polar bases [Xinkai Wu/02]

By chain rule, we have

$$\begin{aligned}
\frac{\partial f}{\partial \varpi} &= \frac{\partial x}{\partial \varpi}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \varpi}\frac{\partial f}{\partial y} = cos\phi\frac{\partial f}{\partial x} + sin\phi\frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial \phi} &= -\varpi sin\phi\frac{\partial f}{\partial x} + \varpi cos\phi\frac{\partial f}{\partial y}
\end{aligned}$$

for any function f. Combining this with

$$\vec{e}_{\hat{\varpi}} = \frac{\partial}{\partial \varpi}, \quad \vec{e}_{\hat{\phi}} = \frac{1}{\varpi}\frac{\partial}{\partial \phi}$$
$$\vec{e}_x = \frac{\partial}{\partial x}, \quad \vec{e}_y = \frac{\partial}{\partial y}$$

we get the transformation matrix

$$L^x{}_{\hat{\varpi}} = cos\phi,\ L^y{}_{\hat{\varpi}} = sin\phi,\ L^x{}_{\hat{\phi}} = -sin\phi,\ L^y{}_{\hat{\phi}} = cos\phi$$

inverting which gives

$$L^{\hat{\varpi}}{}_x = cos\phi,\ L^{\hat{\phi}}{}_x = -sin\phi,\ L^{\hat{\varpi}}{}_y = sin\phi,\ L^{\hat{\phi}}{}_y = cos\phi$$

D. Ex. 23.9 Index gymnastics [Kip and Xinkai Wu/02]

(a) First notice that $P_{\alpha\beta}u^\beta = u_\alpha + u_\alpha u_\beta u^\beta = 0$, using the fact that $\vec{u}^2 = -1$. Thus

$$P_{\alpha\beta}P^\beta_{\ \gamma} = P_{\alpha\beta}(g^\beta{}_\gamma + u^\beta u_\gamma) = P_{\alpha\gamma}$$

(b) $P_{\alpha\beta}A^\beta u^\alpha = 0$ because $P_{\alpha\beta}u^\alpha = 0$, thus $P_{\alpha\beta}A^\beta$ is orthogonal ot $\vec{u}$. $P_{\alpha\beta}A^\beta = A_\alpha + u_\alpha u_\beta A^\beta = A_\alpha$ if $u_\beta A^\beta = 0$.

(c) In the fluid's local rest frame, $g_{\alpha\beta} = \eta_{\alpha\beta}$, and $u_\alpha = -\delta^{\alpha 0}$. Thus $P_{\alpha\beta}$ is diagonal in this frame, with $P_{00} = \eta_{00} - u_0 u_0 = 1 - 1 = 0$, $P_{11} = P_{22} = P_{33} = 1$.

(d)

$$(\nabla_{\vec{u}}\vec{u})_\alpha = u^\beta u_{\alpha;\beta} = -a_\alpha \vec{u}^2 = a_\alpha$$

where we've used eq. (23.53) and the fact that $P_{\alpha\beta}$, $\sigma_{\alpha\beta}$, and $\omega_{\alpha\beta}$ are all orthogonal to $\vec{u}$. Thus we see $\nabla_{\vec{u}}\vec{u} = \vec{a}$. Also

$$\vec{a}\cdot\vec{u} = a_\alpha u^\alpha = u^\beta u_{\alpha;\beta}u^\alpha = \frac{1}{2}(u_\alpha u^\alpha)_{;\beta}u^\beta = 0$$

because $u_\alpha u^\alpha = -1$ is a constant.

(e) Contracting eq. (23.53) with $g^{\alpha\beta}$, using $\vec{a}\cdot\vec{u} = 0$, tracelessness of $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$, and $g^{\alpha\beta}P_{\alpha\beta} = g^{\alpha\beta}g_{\alpha\beta} + \vec{u}^2 = 4 - 1 = 3$, we get $\boldsymbol{\nabla}\cdot\vec{u} = \theta$.

(f) Notice that the term $a_\alpha u_\beta$ in (23.53) is not orthogonal to $\vec{u}$ on its second slot. We can get rid of it by projecting with P^β_γ. Thus $u_{\alpha;\beta}P^\beta_\gamma = \frac{1}{3}\theta P_{\alpha\gamma} + \sigma_{\alpha\gamma} +$

$\omega_{\alpha\gamma}$ where in the second term we have used $P_{\alpha\beta}P^{\beta}_{\gamma} = P_{\alpha\gamma}$. Now $\sigma_{\alpha\beta}$ is the symmetric, traceless part of this tensor: $\sigma_{\alpha\gamma} = \frac{1}{2}(u_{\alpha;\beta}P^{\beta}_{\gamma} + u_{\gamma;\beta}P^{\beta}_{\alpha}) - \frac{1}{3}u^{\rho}{}_{;\rho}P_{\alpha\gamma}$. Similarly $\omega_{\alpha\gamma}$ is the antisymmetric part: $\omega_{\alpha\gamma} = \frac{1}{2}(u_{\alpha;\beta}P^{\beta}_{\gamma} - u_{\gamma;\beta}P^{\beta}_{\alpha})$.

(g) (i) The four velocity is given by $\vec{u} = (\gamma, \gamma v^j)$, where $\gamma = 1/\sqrt{1 - v^j v^j}$. So to first order in v^j, we have $u^0 = 1$, and $u^j = v^j$. (ii) $\theta = u^{\alpha}{}_{;\alpha} = u^{\alpha}{}_{,\alpha}$, to first order in v^j, this becomes $\theta = u^j{}_{,j} = v^j{}_{,j}$ (since $u^0 = 1$ to first order of v^j, $u^0{}_{,0} = 0$). (iii) Using the expression of $\sigma_{\alpha\gamma}$ given in the previous part and noticing that ,to first order in v^j, $P_{jk} = g_{jk}$, and in the brackets we can take $P^{\beta}{}_{k} = \delta^{\beta}{}_{k}$(since the $u_{j;\beta}$ terms is already linear in v^j) ,we get $\sigma_{jk} = \frac{1}{2}(v_{j,k} + v_{k,j}) - \frac{1}{3}\theta g_{jk}$,which is the fluid's nonrelativistic shear. (iv)Similar to part (iii), we get $\omega_{jk} = \frac{1}{2}(v_{j,k} - v_{k,j})$, which is the nonrelativstic rotation.

E. Ex. 23.10 Integration–Gauss's law [Xinkai Wu/02]

$\mathbf{E} \cdot d\mathbf{\Sigma} = E^j d\Sigma_j$, where $d\Sigma_j = \epsilon(\vec{e}_j, d\theta\partial/\partial\theta, d\phi\partial/\partial\phi)$. By the antisymmetry of ϵ, only $d\Sigma_r$ doesn't vanish, and is given by $\epsilon_{r\theta\phi}d\theta d\phi = \sqrt{det||g_{jk}||}d\theta d\phi = R^2 sin\theta d\theta d\phi$. On the r.h.s. of eq. (23.55), $d\Sigma$ was already worked out in the text: $d\Sigma = r^2 sin\theta dr d\theta d\phi$. Thus eq. (23.55) becomes

$$\int_{r=R} E^r R^2 sin\theta d\theta d\phi = \int_{r<R} \frac{\rho_e}{\epsilon_0} r^2 sin\theta dr d\theta d\phi$$

F. Ex. 23.11 Stress energy tensor for a perfect fluid [A. Dvoretskii/00 and Kip/02]

(a) If the components of two tensors are equal in a given frame, then they will be equal in any frame, so we just need to verify that

$$\mathbf{T} = (\rho + P)\vec{u} \otimes \vec{u} + P\mathbf{g}$$

reduces to

$$T^{00} = \rho, \quad T^{ij} = P\delta^{jk}$$

in the rest frame. It's a trivial exercise given the simple form of $\vec{u}$ in the rest frame

$$\vec{u} = (1, 0, 0, 0)$$

(b) If the observer is moving with a speed v much smaller than the speed of light with respect to the rest frame of the fluid, then the momentum density in this frame can be written as

$$T^{0j} = \rho^{ij}_{\text{inertial}} v_i$$

which is the definition of the tensorial inertial mass density. In the limit of small v the momentum density can be written as

$$T^{0j} = (\rho + P)u^0 u^j = (\rho + P)v^j$$

to first order in v, and so

$$\rho^{ij}_{\text{inertial}} = (\rho + P)\delta^{ij}$$

(c) First notice that

$$\rho = \rho_0(1+u) = \rho_N(1-\frac{v^2}{2})(1+u) = \rho_N(1+u-\frac{v^2}{2})$$

which we'll use frequently in the following derivation.

$$\begin{aligned} T^{00} &= (\rho + P)u^0 u^0 - P = (\rho + P)(1+\frac{1}{2}v^2)^2 - P \\ &= \rho(1+v^2) = \rho_N(1+u-\frac{v^2}{2})(1+v^2) \\ &= \rho_N + \frac{1}{2}\rho_N v^2 + \rho_N u \end{aligned}$$

as desired.

$$\begin{aligned} T^{j0} &= (\rho + P)u^j u^0 = \left[\rho_N(1+u-\frac{v^2}{2}) + P\right](1+\frac{v^2}{2})v^j(1+\frac{v^2}{2}) \\ &= \rho_N(1+u-\frac{v^2}{2})(1+v^2)v^j + Pv^j = \rho_N(1+u+\frac{v^2}{2})v^j + Pv^j \\ &= \rho_N v^j + \left(u + \frac{1}{2}v^2 + \frac{P}{\rho_N}\right)\rho_N v^j \end{aligned}$$

since $T^{0j} = T^{j0}$, this gives the desired expression for T^{0j}. And keeping only the term linear in v, this gives

$$T^{j0} = \rho_N v^j$$

as desired.

$$\begin{aligned} T^{jk} &= (\rho + P)u^j u^k + Pg^{jk} = \left[\rho_N(1+u-\frac{v^2}{2}) + P\right](1+v^2)v^j v^k + Pg^{jk} \\ &= \rho_N v^j v^k + Pg^{jk} \end{aligned}$$

as desired.

(d) as in (b)

$$T^{0j} = \rho^{ij}_{\text{inertial}} v_i = (\rho + P)v_j$$

G. Ex. 23.14 Stress-energy tensor for a point particle [A. Dvoretskii/00]

We want to prove that

$$p^\alpha(\zeta_0) = \int_\mathcal{S} p^\alpha(\zeta) p^\beta(\zeta) \delta(\mathcal{Q}, \mathcal{P}(\zeta)) d\Sigma_\beta d\zeta$$

The δ function vanishes everywhere except at the point $\mathcal{Q}$ in the 3-surface $\mathcal{S}$ at which the particle's worldline pierces $\mathcal{S}$. The value of ζ at that point is ζ_0, so the only nonzero contribution to the ζ integral comes from ζ_0. The right hand side reduces to

$$RHS = p^\alpha(\zeta_0) p^\beta(\zeta_0) \int \delta(\mathcal{Q}, \mathcal{P}(\zeta)) d\Sigma_\beta d\zeta$$

Let x^μ be the coordinates of $\mathcal{Q}$ (not necessarily Lorentzian) and $y^\mu(\zeta)$ - coordinates of $\mathcal{P}$ in the same coordinate system. Since the expression is frame invariant we can choose the coordinate system any way we like. To simplify the calculation we make it satisfy the following requirements:

- The surface Σ is given by the eqn. $x^0 = 0$.
- The world line of the particle intersects Σ at $x^j = 0$.
- The coordinate system is a local Lorentz one at the point of intersection $x^\mu = 0$.

Then the surface element $d\Sigma_\beta$ has only its zero component non-vanishing:

$$d\Sigma_0 = d^3\mathbf{x} = dx^1 dx^2 dx^3, \qquad d\Sigma_j = 0,\ j = 1,2,3$$

and the δ-function can be written as a product

$$\delta(\mathcal{Q}, \mathcal{P}(\zeta)) = \delta(x_0 - y_0(\zeta))\delta^3(\mathbf{x} - \mathbf{y}(\zeta)) = \delta(y_0(\zeta))\delta^3(\mathbf{x} - \mathbf{y}(\zeta))$$

The resulting integral can be easily calculated.

$$\begin{aligned} RHS &= p^\alpha(\zeta_0) p^0(\zeta_0) \int \delta(y^0(\zeta))\delta^3(\mathbf{x} - \mathbf{y}(\zeta)) d^3\mathbf{x} d\zeta = \\ &= p^\alpha(\zeta_0) p^0(\zeta_0) \int \delta(y^0(\zeta)) d\zeta = p^\alpha(\zeta_0) p^0(\zeta_0) \frac{1}{\frac{dy^0}{d\zeta}|_{\zeta_0}} \end{aligned}$$

But $\frac{dy^0}{d\zeta}|_{\zeta_0} = p^0(\zeta_0)$ by definition of momentum, so

$$RHS = p^\alpha(\zeta_0)$$

H. Ex. 23.15 Proper Reference Frame [A. Dvoretskii/April 2000]

(a) It's fairly straightforward to obtain the transformation law in differential form. The hats on the right-hand side are dropped to simplify notation, and to avoid confusion we use primes for the inertial coordinates(left hand side)

$$d\mathbf{x}' = d\mathbf{x} + \mathbf{a}x^0 dx^0 + (\Omega \times d\mathbf{x})x^0 + (\Omega \times \mathbf{x})dx^0$$
$$dx^{0\prime} = dx^0(1 + \mathbf{a}\cdot\mathbf{x}) + x^0\mathbf{a}\cdot d\mathbf{x}$$

Squaring and only keeping terms linear in $\mathbf{x}$ get

$$d\mathbf{x}'^2 = d\mathbf{x}^2 + 2\mathbf{a}\cdot d\mathbf{x}x^0 dx^0 + 2(\Omega \times \mathbf{x})\cdot d\mathbf{x}dx^0$$
$$(dx^{0\prime})^2 = (dx^0)^2(1 + 2\mathbf{a}\cdot\mathbf{x}) + 2\mathbf{a}\cdot d\mathbf{x}x^0 dx^0$$

Given this it's easy to see that the new metric is indeed

$$ds^2 = -(dx^{0\prime})^2 + (d\mathbf{x}')^2 = -(1 + 2\mathbf{a}\cdot\mathbf{x})(dx^0)^2 + 2(\Omega\times\mathbf{x})\cdot d\mathbf{x}dx^0 + d\mathbf{x}^2$$

(b) Recall that the components of the metric in the proper frame are (again dropping the hats for simplicity of notation)

$$g_{00} = -(1 + 2a_j x^j),\ g_{0i} = \epsilon_{ijk}\Omega^j x^k,\ g_{jk} = \delta_{jk}$$

And to linear order in x^j, the inverse metric $g^{\mu\nu}$ is given by taking $x^j \to -x^j$ in $g_{\mu\nu}$. And to computate the connection coefficients along the world line, we only need the inverse metric at $x^j = 0$, which is just $\eta^{\mu\nu}$. We have

$$\Gamma^{\mu}{}_{\alpha 0} = \eta^{\mu\nu}\frac{1}{2}(g_{\nu 0,\alpha} - g_{\alpha 0,\nu})$$

which gives

$$\Gamma^0{}_{00} = 0,\ \Gamma^0{}_{j0} = a_j;\ \ \Gamma^j{}_{00} = a_j,\ \Gamma^i{}_{j0} = \epsilon_{ikj}\Omega^k$$

Also it's not hard to see that $\Gamma^{\mu}{}_{ij}$ all vanish. The above results can be verified by, say, GRtensor, which is straightforward and we omit here.

(c) Using the connection coefficients we obtained in the previous part, we find

$$\nabla_{\vec{U}}\vec{e}_0 = \Gamma^{\mu}{}_{00}\vec{e}_\mu = \Gamma^i{}_{00}\vec{e}_i = a^i\vec{e}_i = \vec{a}$$

and

$$\begin{aligned}\nabla_{\vec{U}}\vec{e}_j &= \Gamma^{\mu}{}_{j0}\vec{e}_\mu = \Gamma^0{}_{j0}\vec{e}_0 + \Gamma^i{}_{j0}\vec{e}_i \\ &= a_j\vec{e}_0 + \epsilon_{ikj}\Omega^k\vec{e}_i = (\vec{a}\cdot\vec{e}_j)\vec{U} + \boldsymbol{\epsilon}(\vec{U},\vec{\Omega},\vec{e}_j,...)\end{aligned}$$

(d) Now we are away from the world line, $x^j \neq 0$. However, we see that at our order of approximation , $\Gamma^i{}_{00}$ and $\Gamma^i{}_{j0}$ are still given by the expressions worked out in part (b). Plugging them into eq. (23.95), we readily get

$$\frac{d^2\mathbf{x}}{(dx^0)^2} = -\mathbf{a} - 2\boldsymbol{\Omega}\times\mathbf{v}$$