Solution for Chapter 22

(compiled by Xinkai Wu)

A.

Exercise 22.1 Non-resonant Particle Energy in Wave [Chris Hirata/00]

The rate of change of the non-resonant electron kinetic energy is given by

$$\begin{aligned}
\frac{dU_e}{dt} &= \frac{d}{dt}\int \frac{1}{2}m_e v^2 F_0 dv \\
&= \int \frac{1}{2}m_e v^2 \frac{\partial F_0}{\partial t} dv \\
& \quad \text{using eq. (22.20)} \\
&= \int \frac{1}{2}m_e v^2 \left[-\frac{\partial}{\partial v}\left(D\frac{\partial F_0}{\partial v}\right) - v\frac{\partial F_0}{\partial z}\right] dv \\
& \quad \text{the 2nd term in the square brackets gives zero b/c it's odd in } v; \\
& \quad \text{also, integrate by part gives} \\
&= \int D m_e v \frac{\partial F_0}{\partial v} dv \qquad (1)
\end{aligned}$$

Now the D appearing the above expression should be the difference between eq. (22.21) and eq. (22.22). It's not hard to see this corresponds to, for non-resonant electrons with $v << \frac{\omega_r}{k}$, setting v to zero in the denominator of eq. (22.21) (and also ignoring the ω_i^2 in the denominator). Thus we get

$$\begin{aligned}
D &= \frac{e^2}{\epsilon_0 m_e^2}\int dk \mathcal{E}_k \frac{\omega_i}{\omega_r^2} \\
& \quad \text{using } \omega_r^2 \approx \omega_p^2 = \frac{ne^2}{\epsilon_0 m_e}, \text{ see eq. (22.11)} \\
&= \frac{\omega_i}{n m_e}\int dk \mathcal{E}_k
\end{aligned}$$

Plugging the above expression for D(which is actually v-independent) into eq. (1) and integrating by part, we get

$$\frac{dU_e}{dt} = D m_e \int F_0 dv = D m_e n = 2\omega_i \times \frac{1}{2}\int_0^{+\infty} dk \mathcal{E}_k$$

which is the desired eq. (22.23).

Exercise 22.2 Energy Conservation [Alexander Putilin/99]

The electron kinetic energy density and momentum density are given by

$$\begin{aligned}
U^e &= \int \frac{1}{2}m_e v^2 F_0 dv \\
S_z^e &= \int \frac{1}{2}m_e v^3 F_0 dv
\end{aligned}$$

and we have

$$\begin{aligned}\frac{\partial U^e}{\partial t}+\frac{\partial S_z^e}{\partial z} &= \int \frac{1}{2}m_e v^2\left(\frac{\partial F_0}{\partial t}+v\frac{\partial F_0}{\partial z}\right)dv \\ &= \int \frac{1}{2}m_e v^2\frac{\partial}{\partial v}\left(D\frac{\partial F_0}{\partial v}\right)dv \\ &= -\int D\frac{\partial F_0}{\partial v}m_e v dv\end{aligned}$$

Now using eq. (22.22) for the resonant electrons

$$D=\frac{e^2\pi}{\epsilon_0 m_e^2}\int dk\mathcal{E}_k\delta(\omega_r-kv)$$

we find

$$\frac{\partial U^e}{\partial t}+\frac{\partial S_z^e}{\partial z}=\frac{-e^2\pi}{\epsilon_0 m_e}\int dk\frac{\mathcal{E}_k\omega_r}{k^2}F_0'\left(\frac{\omega_r}{k}\right)$$

which becomes, upon using eq. (22.12) $F_0'\left(\frac{\omega_r}{k}\right)=\frac{2\epsilon_0 m_e}{\pi e^2}\frac{k^2}{\omega_r}\omega_i$,

$$\begin{aligned}&-\int dk\ 2\omega_i\mathcal{E}_k \\ &=-\frac{\partial}{\partial t}\int dk\mathcal{E}_k-\frac{\partial}{\partial z}\int dk\mathcal{E}_k\frac{\omega_r}{k}\quad \text{using eq. (22.19)} \\ &=-\frac{\partial U^w}{\partial t}-\frac{\partial S_z^w}{\partial z}\end{aligned}$$

where we've used the facts that $\int dk\mathcal{E}_k$ is the wave's energy density U^w, and $\int dk\mathcal{E}_k\frac{\partial\omega_r}{\partial k}$ is the wave's energy flux S_z^w.

Thus we finally have

$$\frac{\partial U^e}{\partial t}+\frac{\partial S_z^e}{\partial z}=-\frac{\partial U^w}{\partial t}-\frac{\partial S_z^w}{\partial z}$$

which is energy conservation law.

B. Exercise 22.3 Cerenkov Power in Electrostatic Waves [Alexander Putilin/99]

The emission rate of plasmons is given by (22.43)

$$W=\frac{\pi e^2\omega_r}{\epsilon_0 k^2\hbar}\delta(\omega_r-\mathbf{k}\cdot\mathbf{v})$$

Each plasmon has energy $\hbar\omega_r$, so the radiated power per unit time is

$$P=\frac{1}{(2\pi)^3}\int d^3\mathbf{k}W\hbar\omega_r=\frac{e^2}{8\pi^2\epsilon_0}\int d^3\mathbf{k}\frac{\omega_r^2}{k^2}\delta(\omega_r-\mathbf{k}\cdot\mathbf{v})$$

The integration is over the region $k<k_{max}$(outside this region waves are strongly Landau damped). A good estimate of k_{max} is inverse Debye length:

$k_{max} \sim 1/\lambda_D$(see the discussion at the end of Chapter 21.3.5). Since $k\lambda_D < 1$, we can approximate $\omega_r(k)$ by a constant ω_p.

Choosing $\mathbf{v}$ to point along z-axis, we have

$$\begin{aligned}
P &= \frac{e^2\omega_p^2}{8\pi^2\epsilon_0}\int_{k<k_{max}} d^3\mathbf{k}\frac{1}{k^2}\delta(\omega_r - \mathbf{k}\cdot\mathbf{v}) \\
&= \frac{e^2\omega_p^2}{8\pi^2\epsilon_0}\int_{k<k_{max}} d^2k_\perp dk_z\frac{1}{k_\perp^2 + k_z^2}\delta(\omega_r - \mathbf{k}\cdot\mathbf{v}) \\
&= \frac{e^2\omega_p^2}{8\pi^2\epsilon_0 v}\int_{k_\perp^2<k_{max}^2-(\omega_p^2/v^2)} d^2k_\perp\frac{1}{k_\perp^2 + (\omega_p^2/v^2)} \\
&= \frac{e^2\omega_p^2}{8\pi^2\epsilon_0 v}\int_0^{k_{max}^2-(\omega_p^2/v^2)} 2\pi k_\perp dk_\perp\frac{1}{k_\perp^2 + (\omega_p^2/v^2)} \\
&= \frac{e^2\omega_p^2}{4\pi\epsilon_0 v}\ln\left(\frac{k_{max}v}{\omega_p}\right)
\end{aligned}$$

Note that P depends on k_{max} logarithmically. So if v is sufficiently large, it doesn't make much difference what particular definition we use for k_{max}.

C. Exercise 22.5 Three-Wave Processes Parts (a),(c),(d),(e) [Alexander Putilin/99 and Xinkai Wu/02]

(a) Diagram (b) in Fig. 22.4 gives the rate of creation of B,

$$\left(\frac{d\eta}{dt}\right)_{creation} = \int W_{C\to AB}\eta_C(1+\eta_A)(1+\eta_B)\frac{dV_{k_A}}{(2\pi)^3}\frac{dV_{k_C}}{(2\pi)^3}$$

where in the above expression, the unity in $(1+\eta_A)$ corresponds to spontaneous emission and η_A corresponds to induced emission. Simlilar for $(1+\eta_B)$.

Diagram (a) in Fig. 22.4 gives the rate of destruction of B,

$$\left(\frac{d\eta}{dt}\right)_{destruction} = \int W_{AB\to C}(\eta_C + 1)\eta_A\eta_B\frac{dV_{k_A}}{(2\pi)^3}\frac{dV_{k_C}}{(2\pi)^3}$$

The net rate of change is given by the diffrence of these two, and by principle of detailed balance $W_{AB\to C} = W_{C\to AB}$. So we get

$$\frac{d\eta}{dt} = \int W_{AB\leftrightarrow C}\left[(1+\eta_A+\eta_B)\eta_C - \eta_A\eta_B\right]\frac{dV_{k_A}}{(2\pi)^3}\frac{dV_{k_C}}{(2\pi)^3}$$

(c) Under the approximation stated in the problem, we have

$$\eta_C(1+\eta_A+\eta_B) - \eta_A\eta_B \approx \eta_B(\eta_C - \eta_A)$$

Also let's change notation: $\mathbf{k}_B \equiv \mathbf{k}$, $\mathbf{k}_C \equiv \mathbf{k}'$, $\eta_B(\mathbf{k}_B) \equiv \eta_{ia}(\mathbf{k})$, $\eta_A(\mathbf{k}_A) \equiv \eta_L(\mathbf{k}_A)$, $\eta_C(\mathbf{k}_C) \equiv \eta_L(\mathbf{k}')$, also change ω_A and ω_C to ω_L. Then using the Taylor expansion

$$\begin{aligned}
\eta_L(\mathbf{k}') - \eta_L(\mathbf{k}'-\mathbf{k}) &\approx \mathbf{k}\cdot\boldsymbol{\nabla}_{\mathbf{k}'}\eta_L(\mathbf{k}') \\
\omega_L(\mathbf{k}'-\mathbf{k}) - \omega_L(\mathbf{k}') &\approx -\mathbf{k}\cdot\boldsymbol{\nabla}_{\mathbf{k}'}\omega_L(\mathbf{k}') = -\mathbf{k}\cdot\mathbf{V}_g(\mathbf{k}')
\end{aligned}$$

we easily get eq. (22.59).

(d) eq. (22.59) becomes almost identical to eq. (22.42) upon $\eta_{ia}(\mathbf{k}) \to \eta(\mathbf{k})$, and $\eta_L(\mathbf{k}') \to f(\mathbf{v})$. The underlying physical reason is of course in the former case the emitted particle is an ion acoustic plasmon and the emitting particle is a Langmuir plasmon; while in the latter case the emitted particle is a Langmuir plasmon and the emitting particle is an electron.

(e) Sorry folks, not here yet; will appear soon.

D.

Exercise 22.4 Electron Fokker-Planck Equation [Xinkai Wu/02]

It's very straightforward to carry out the Taylor expansion in $\hbar$ to second order, so we don't bother to write down all the formulae here. Rather we'll just point out a few things worth noticing in the expansion. Fristly you may find it's easier to use component notation rather than tensor notation. Secondly, you may want to group terms coming out of expanding the r.h.s. of eq. (22.47) into three categories: f terms, $\frac{\partial f}{\partial v^j}$ terms, and $\frac{\partial^2 f}{\partial v^j \partial v^n}$ terms. Thirdly, to get the final answer, notice that in the classical limit both η and W are large, i.e. of order $1/\hbar$. After all these, you can set $\hbar \to 0$ and recover eq. (22.48).

Exercise 22.5 Three-Wave Processes: Parts (b), (f), (g) [Xinkai Wu/02]

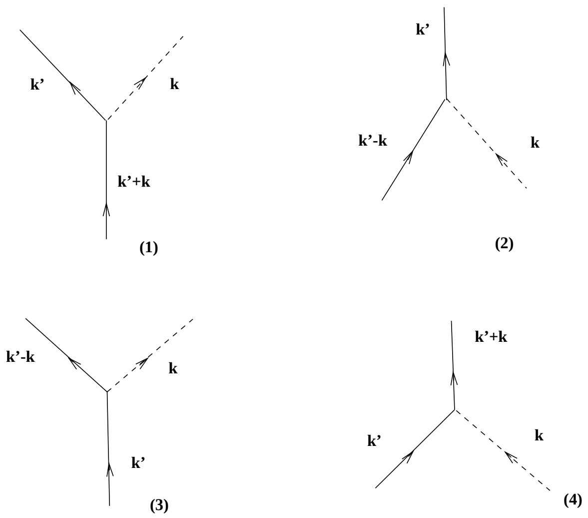

Figure 1: Diagrams for the change of Langmuir occupation number (we use solid lines to denote Langmuir plasmons, and dashed lines ion acoustic plasmons)

(b) See Fig. 1 for the four relevant diagrams. The desired rate of change for the Langmuir occupation number of is given by the sum/difference of these four diagrams:

$$\frac{d\eta_L(\mathbf{k}')}{dt} = \int \frac{d^3\mathbf{k}}{(2\pi)^6}[(1) + (2) - (3) - (4)]$$

where

$$\begin{aligned}
(1) &= \eta_L(\mathbf{k}'+\mathbf{k})[1+\eta_L(\mathbf{k}')][1+\eta_{ia}(\mathbf{k})]R(\mathbf{k}'+\mathbf{k},\mathbf{k},\mathbf{k}')\delta(\omega_L(\mathbf{k}'+\mathbf{k})-\omega_L(\mathbf{k}')-\omega_{ia}(\mathbf{k}))\\
(2) &= \eta_L(\mathbf{k}'-\mathbf{k})\eta_{ia}(\mathbf{k})[1+\eta_L(\mathbf{k}')]R(\mathbf{k}'-\mathbf{k},\mathbf{k},\mathbf{k}')\delta(\omega_L(\mathbf{k}')-\omega_L(\mathbf{k}'-\mathbf{k})-\omega_{ia}(\mathbf{k}))\\
(3) &= \eta_L(\mathbf{k}')[1+\eta_L(\mathbf{k}'-\mathbf{k})][1+\eta_{ia}(\mathbf{k})]R(\mathbf{k}'-\mathbf{k},\mathbf{k},\mathbf{k}')\delta(\omega_L(\mathbf{k}')-\omega_L(\mathbf{k}'-\mathbf{k})-\omega_{ia}(\mathbf{k}))\\
(4) &= \eta_L(\mathbf{k}')\eta_{ia}(\mathbf{k})[1+\eta_L(\mathbf{k}'+\mathbf{k})]R(\mathbf{k}'+\mathbf{k},\mathbf{k},\mathbf{k}')\delta(\omega_L(\mathbf{k}'+\mathbf{k})-\omega_L(\mathbf{k}')-\omega_{ia}(\mathbf{k}))
\end{aligned}$$

(f) Now under the approximation stated in part (c), all the δ-functions in the previous part reduce to the same expression $\delta(\omega_{ia}(\mathbf{k}) - \mathbf{k}\cdot\mathbf{V}_g(\mathbf{k}'))$. We also have(dropping this δ-function for the moment being)

$$\begin{aligned}
(1)-(4) &= \{\eta_L(\mathbf{k}'+\mathbf{k})[1+\eta_L(\mathbf{k}')+\eta_{ia}(\mathbf{k})] - \eta_L(\mathbf{k}')\eta_{ia}(\mathbf{k})\}\, R(\mathbf{k}'+\mathbf{k},\mathbf{k},\mathbf{k}')\\
&\approx [\eta_L(\mathbf{k}'+\mathbf{k}) - \eta_L(\mathbf{k}')]R(\mathbf{k}'+\mathbf{k},\mathbf{k},\mathbf{k}')\eta_{ia}(\mathbf{k})\\
&\approx \mathbf{k}\cdot\boldsymbol{\nabla}_{\mathbf{k}'}\eta_L(\mathbf{k}'+\mathbf{k})R(\mathbf{k}'+\mathbf{k},\mathbf{k},\mathbf{k}')\eta_{ia}(\mathbf{k})
\end{aligned}$$

and

$$\begin{aligned}
(2)-(3) &= \{\eta_L(\mathbf{k}'-\mathbf{k})\eta_{ia}(\mathbf{k}) - \boldsymbol{\eta}_{\mathbf{L}}(\mathbf{k}')[\mathbf{1}+\boldsymbol{\eta}_{\mathbf{L}}(\mathbf{k}'-\mathbf{k})+\boldsymbol{\eta}_{\mathbf{ia}}(\mathbf{k})]\}\, R(\mathbf{k}'-\mathbf{k},\mathbf{k},\mathbf{k}')\\
&\approx [\eta_L(\mathbf{k}'-\mathbf{k}) - \eta_L(\mathbf{k}')]R(\mathbf{k}'-\mathbf{k},\mathbf{k},\mathbf{k}')\eta_{ia}(\mathbf{k})\\
&\approx -\mathbf{k}\cdot\boldsymbol{\nabla}_{\mathbf{k}'}\eta_L(\mathbf{k}')R(\mathbf{k}',\mathbf{k},\mathbf{k}'-\mathbf{k})\eta_{ia}(\mathbf{k})
\end{aligned}$$

where to reach the last expression in the above equation, we've used the fact that R is symmetric w.r.t. the two Langmuir wave momenta.

It's not hard to see that $(1)-(4)$ and $(2)-(3)$ are the same expression of $\mathbf{K}$ evaluated at $\mathbf{K} = \mathbf{k}'+\mathbf{k}$ and $\mathbf{k}'$ respectively So integrating their sum over phase space we immediately get the desired answer

$$\frac{d\eta_L(\mathbf{k}')}{dt} = \nabla_{\mathbf{k}'}\cdot[\mathbf{D}(\mathbf{k}')\cdot\boldsymbol{\nabla}_{\mathbf{k}'}\eta_L(\mathbf{k}')]$$

with

$$\mathbf{D}(\mathbf{k}') \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^6}\eta_{ia}(\mathbf{k})\mathbf{k}\otimes\mathbf{k}\; R(\mathbf{k}',\mathbf{k},\mathbf{k}'-\mathbf{k})\delta(\omega_{ia}(\mathbf{k}) - \mathbf{k}\cdot\mathbf{V}_g(\mathbf{k}'))$$

(g) The diffusion equation we get in the previous part is very similar to eq. (22.30) and (22.31) because in both cases we are looking at the EMITTING particle(c.f. part (d), where we were looking at the EMITTED particles).