Solution for Chapter 21

(compiled by Xinkai Wu)
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A. Exercise 21.1 Two-fluid Equation of Motion [by Xinkai Wu/02]

We'll first write things in components and in the end convert back to vector/tensor notation.

Vlasov eq.

$$\frac{\partial f_s}{\partial t} + v^j \frac{\partial f_s}{\partial x^j} + a^j \frac{\partial f_s}{\partial v^j} = 0$$

Also we'll make use of the following definitions

$$\begin{aligned} n_s &= \int f_s dV_v \\ n_s u_s^i &= \int f_s v^i dV_v \\ P_s^{ik} &= m_s \int f_s (v^i - u_s^i)(v^k - u_s^k) dV_v \\ &= m_s \int f_s v^i v^k dV_v - m_s n_s u_s^i u_s^k \end{aligned}$$

Multiplying the Vlasov eq. by v^k, integrating over velocity space, using integration by parts at various places and the fact that $\frac{\partial a^j}{\partial v^j} = 0$, also using the explicit expression eq. (21.3) for the acceleration due to external EM field, we get

$$\frac{\partial (n_s u_s^k)}{\partial t} + \frac{\partial}{\partial x^j}\left(\frac{P_s^{jk}}{m} + n_s u_s^j u_s^k\right) - \frac{n_s q_s}{m_s}(E^k + (\mathbf{u}_s \times \mathbf{B})^k) = 0 \tag{1}$$

Now using the continuity equation,

$$\frac{\partial n_s}{\partial t} = -\frac{\partial}{\partial x^j}(n_s u_s^j)$$

one immediately sees that eq. (1) reduces to eq. (21.11) after converting to vector notation.

B.

Exercise 21.3 Dielectric Tensor for Longitudinal, Electrostatic Waves [by Alexei Dvoretskii/99]

With all the hints given, it's very straightforward.

$$\boldsymbol{\nabla} \cdot \mathbf{P} = ikP_z = -\rho_e = \left(\sum_s \frac{iq_s^2}{m_s} \int_{-\infty}^{+\infty} \frac{F'_{s0}}{\omega - kv} dv\right) E_z$$

Therefore

$$\chi_{zz} = \frac{P_z}{\epsilon_0 E_z} = \left(\sum_s \frac{q_s^2}{m_s \epsilon_0 k} \int_{-\infty}^{+\infty} \frac{F'_{s0}}{\omega - kv} dv \right)$$

$$\epsilon_{zz} = 1 + \chi_{zz} = 1 + \left(\sum_s \frac{q_s^2}{m_s \epsilon_0 k} \int_{-\infty}^{+\infty} \frac{F'_{s0}}{\omega - kv} dv \right)$$

Using the standard procedure in Chapter 20 we then find L_{ij} to be diagonal, with matrix elements on the diagonal being $\left(-k^2 + \frac{\omega^2}{c^2}\epsilon_{xx}, -k^2 + \frac{\omega^2}{c^2}\epsilon_{yy}, \frac{\omega^2}{c^2}\epsilon_{zz}\right)$. And then $detL = 0$ immediately gives $\epsilon_{zz} = 0$, which is the desired eq. (21.20).

Exercise 21.10 Correlations in a Tokamak Plasma [by Xinkai Wu/00]

Plugging the relevant numbers for the Tokamak plasma as given in table 19.1 into the expression (21.74) for the two-point correlation function, we find:

(a) $\xi_{12}(\lambda_D) \sim 10^{-9}$

(b) $\xi_{12}(n^{-1/3}) \sim 10^{-6}$

C.Exercise 21.4 Landau Contour Deduced Using Laplace Transforms [by Xinkai Wu/00]

(a) This part is nothing but a definition of Laplace transformation.

(b) The z-dependence is e^{ikz}, giving $\partial/\partial z \to ik$. Also integration by parts gives

$$\int_0^{\infty} dte^{-pt} \partial F_{s1}/\partial t = -F_{s1}(v,0) + p \int_0^{\infty} dte^{-pt} F_{s1}(v,t)$$

Noticing the above facts, we get by Laplace transforming the Vlasov equation:

$$0 = -F_{s1}(v,0) + p\tilde{F}_{s1}(v,p) + vik\tilde{F}_{s1}(v,p) + (q_s/m_s)F'_{s0}\tilde{E}(p)$$

where $s = p, e$

Laplace transforming $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ gives us a second equation:

$$ik\tilde{E}(p) = \sum_s (q_s/\epsilon_0) \int_{-\infty}^{\infty} dv[F_{s0}(v)/p + \tilde{F}_{s1}(v,p)] = \sum_s (q_s/\epsilon_0) \int_{-\infty}^{\infty} dv\tilde{F}_{s1}(v,p)$$

Where to get the last equality we've used the fact that the unperturbed charge density is zero, i.e. the contribution from $F_{s0}(v)$ vanishes.

Combining these two equations we easily get (21.41).

(c) Setting $ip = \omega$, and plugging (21.41) into (21.42), we immediately get (21.26) without that overall minus sign(I guess that sign is a typo).

D.

Exercise 21.5 Ion Acoustic Dispersion Relation [by Xinkai Wu/02]

Recall the definitions of Debye length and plasma frequency for species s:

$$\omega_{ps} = \left(\frac{ne^2}{\epsilon_0 m_s}\right)^{1/2}$$
$$\lambda_{Ds} = \left(\frac{\epsilon_0 k_B T_s}{ne^2}\right)^{1/2}$$

and the Maxwellian distribution is

$$F_s(v) = n\left(\frac{m_s}{2\pi k_B T_s}\right)^{1/2} exp\left[-\frac{m_s v^2}{2k_B T_s}\right]$$

Now consider the integral

$$I_s(\frac{\omega_r}{k}) \equiv \int_P \frac{F_s'(v)}{v-\omega_r/k}dv$$

For $\frac{\omega_r}{k} >> \sqrt{\frac{k_B T_s}{m_s}}$, $I_s \approx \frac{nk^2}{\omega_r^2}$(see eq. (21.37)), and this is the formula we are going to use for I_p. For $\frac{\omega_r}{k} << \sqrt{\frac{k_B T_s}{m_s}}$, $I_s \approx -\frac{nm_s}{k_B T_s}$(as can be easily seen by ignoring the ω_r/k in the denominator in the integral), and this is the formula we'll use for I_e.

In our problem, the total $F(v)$ is given by $F = F_e + \frac{m_e}{m_p}F_p$, and thus the total $I(\omega_r/k)$ is given by $I = I_e + \frac{m_e}{m_p}I_p$.

Now that we have the explicit (approximate) expressions for F and I, it's then just a matter of straightforward substitution to find ω_r and ω_i using eq.'s (21.34) and (21.35) (which are the real and imaginary parts of eq. (21.33)) so we don't bother to write down the details here. The expression for ω_r/k in eq. (21.39) comes from eq. (21.34), and that for ω_i/ω_r comes from (21.35) (after approximating $F_e(\omega_r/k)$ as $n\left(\frac{m_e}{2\pi k_B T_e}\right)^{1/2}$ in the numerator of (21.35), which is justified by our assumption $\frac{\omega_r}{k} << \sqrt{\frac{k_B T_e}{m_e}}$). Two more things to mention: the λ_D's in eq. (21.39) are λ_{De}, the electron Debye length; and the r.h.s. of eq. (21.35) should be multiplied by an overall minus sign.

Exercise 21.6 Dispersion Relations for a Non-Maxwellian Distribution Function [by Xinkai Wu/00]

This problem is quite instructive in that it clarifies what approximations we are making when considering Langmuir waves and ion acoustic waves, respectively.

Now the distribution function is given by $F(v) = F_e(v) + (m_e/m_p)F_p(v)$, where $F_e(v) = nv_{0e}/[\pi(v_{0e}^2 + v^2)]$, and $F_p(v)$ is obtained by replacing v_{0e} with v_{0p}. Note that these distribution functions are normalized so that n is the electron(and also proton) density.

We then plug this $F(v)$ into the general expression (21.30) for $\epsilon(\omega, k)$. Let's first assume that $\text{Im}(\omega/k) > 0$. Thus we take the Landau contour to be the real axis in the v-plane. Closing the contour in the lower half plane and evaluating the residues carefully we get:

$$\begin{aligned} & \epsilon(\omega, k) \\ = \quad & 1 + e^2 n\{\epsilon_0 k^2 m_e v_{0e}^2 [1 - i\omega/(kv_{0e})]^2\}^{-1} + e^2 n\{\epsilon_0 k^2 m_p v_{0p}^2 [1 - i\omega/(kv_{0p})]^2\}^{-1} \end{aligned}$$

Obviously the above expression for $\epsilon(\omega, k)$ can be analytically continued to the whole ω-plane. Note that so far no approximation has been made.

(a) For Langmuir waves, $T_e \sim T_{p,}(i.e. m_e v_{0e}^2 \sim m_p v_{0p}^2)$, and $\omega/k >> v_{0e} >> v_{0p}$, thus we can ignore the third term in $\epsilon(\omega, k)$, namely the contribution from the protons. Then by letting $\epsilon(\omega, k) = 0$, we easily get $\omega = \omega_{pe} - ikv_{0e}$

(b) For ion acoustic waves, $T_e >> T_{p,}(i.e. m_e v_{0e}^2 >> m_p v_{0p}^2)$, and $v_{0e} >> \omega/k >> v_{0p}$. Thus we have to include both electron and proton contribution. We expand the electron contribution to the first power of $\omega/(kv_{0e})$, and the proton contribution to the leading order of $v_{0p}k/\omega$. When solving $\epsilon(\omega, k) = 0$ for $\omega(k)$, we will assume $\omega_i << \omega_r$, which we expect for the case of weak damping.

$$\mathrm{Im}(\epsilon(\omega, k)) = 0 \Longrightarrow \omega_i = -(m_p/m_e)[\omega_r^4/(kv_{0e})^3]$$

Plugging the above relation into the equation $\mathrm{Re}(\epsilon(\omega, k)) = 0$, we turn the equation $\mathrm{Re}(\epsilon(\omega, k)) = 0$ into:

$$0 = 1 + [\omega_{pe}/(kv_{0e})]^2 \{1 + (m_p/m_e)[2\omega_r/(kv_{0e})]^4 - (m_e/m_p)(kv_{oe}/\omega_r)^2\}$$

The ratio of the second to the third term in $\{....\}$ is $\sim (m_p/m_e)^2 [\omega_r/(kv_{0e})]^6$, which will be very small when T_e is very high(i.e. v_{0e} very large compared with ω_r/k). Thus we can neglect the second term. After making this approximation, we readily get ω_r, and in turn ω_i. The ω we get is precisely the same as given in (21.45), excpet that we use v_{0e} instead of v_0(which is just a difference in notation).