Solution for Chapter 19 & 20

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A.

Exercise 19.1 Boundary of Degeneracy [by Alexander Putilin/00]

We'll ignore factors of order unity in what follows.

(a) $l = n_e^{-1/3} >> \lambda_{dB} = \hbar/(momentum) \simeq \hbar/\sqrt{m_e kT}$, which immediately gives $n_e << (m_e kT)^{3/2}/h^3$.

(b) Using the uncertainty principle, $\Delta x \simeq h/\Delta p \simeq h/\sqrt{m_e kT} << n_e^{-1/3}$, which again reduces to $n_e << (m_e kT)^{3/2}/h^3$.

(c) The quantum mechanical zero-point energy is given by $(\Delta p)^2/m_e \simeq h^2/(l^2 m_e) \simeq h^2/(n_e^{-2/3} m_e) << kT$, which reduces to $n_e << (m_e kT)^{3/2}/h^3$.

Exercise 19.4 Ion-ion Equilibration Rate for a He^3 Plasma [by Alexander Putilin/00]

The ion equilibration rate for a pure He^3 plasma is derived by the same method as proton-proton equilibration rate. We start with electron-electron equilibration rate (B.T. eq. (19.27))

$$\begin{aligned} \nu_{ee} &= \frac{n_e \sigma_T c \ln\Lambda}{2\sqrt{\pi}} \left(\frac{kT}{m_e c^2}\right)^{-3/2} \\ &= \frac{n_e c \ln\Lambda}{2\sqrt{\pi}} \, \frac{8\pi}{3} \left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right)^2 \left(\frac{kT}{m_e c^2}\right)^{-3/2} \end{aligned}$$

Replace electron charge, density, and mass with corresponding values for He^3 ions: $e \longrightarrow 2e$, $n_e \longrightarrow n_{He} = \frac{1}{2}n_e$, $m_e \longrightarrow m_{He} = 3m_p$. We get

$$\begin{aligned} \nu_{He\ He} &= \frac{n_{He} c \ln\Lambda}{2\sqrt{\pi}} \, \frac{8\pi}{3} \left(\frac{4e^2}{4\pi\epsilon_0 m_{He} c^2}\right)^2 \left(\frac{kT}{m_{He} c^2}\right)^{-3/2} \\ &= \frac{16}{\sqrt{3}}\sqrt{\frac{m_e}{m_p}} \frac{n_{He}\sigma_T c \ln\Lambda}{2\sqrt{\pi}} \left(\frac{kT}{m_e c^2}\right)^{-3/2} \\ &= \frac{16}{\sqrt{3}} \cdot 5.8 \times 10^{-7} s^{-1} \left(\frac{n_{He}}{1m^{-3}}\right) \left(\frac{T}{1K}\right)^{-3/2} \left(\frac{\ln\Lambda}{10}\right) \end{aligned}$$

and we have $n_{He} = \frac{1}{2}n_e = 0.5 \times 10^{20} m^{-3}$, $T = 10^8 K$.

Now estimate $\ln\Lambda$. $\Lambda = \frac{\lambda_D}{b_{min}}$, with $\lambda_D = \left(\frac{\epsilon_0 kT}{n_{He}(2e)^2}\right)^{1/2} = 4.9 \times 10^{-5} m$, and $b_{min} = MAX[b_0 = \frac{2(2e)^2}{m_{He} v^2}, \frac{\hbar}{m_{He} v}]$, where $v \simeq \sqrt{\frac{3kT}{m_{He}}}$. We find $b_0 = 4.4 \times 10^{-13} m$ and $\frac{\hbar}{m_{He} v} = 2 \times 10^{-14} m$. Thus we take $b_{min} = 4.4 \times 10^{-13} m$, which gives $\ln\Lambda = \ln\frac{4.9\times10^{-5}}{4.4\times10^{-13}} \simeq 18$.

Plugging it into the formula for $\nu_{He\ He}$ we get

$$\nu_{He\ He} \simeq 500 s^{-1}$$

B. Exercise Parameters for Various Plasmas [by X.M. Henry Huang/98]

Text eq. (19.10) $\lambda_D = \left(\frac{\epsilon_0 kT}{ne^2}\right)^{1/2} = 69\left(\frac{T/1K}{n/1m^{-3}}\right)^{1/2} m.$

Text eq. (19.11) $N_D = n\frac{4\pi}{3}\lambda_D^3 = 1.4\times10^6\frac{(T/1K)^{3/2}}{(n/1m^{-3})^{1/2}}.$

Text eq. (19.13) $f_p = \frac{\omega_p}{2\pi} = \frac{1}{2\pi}\left(\frac{ne^2}{\epsilon_0 m_e}\right)^{1/2} = \frac{56.4}{2\pi}\left(n/1m^{-3}\right)^{1/2} Hz.$

Text eq. (19.21) $t_D^{ee} = \frac{1}{\nu_D^{ee}} = \frac{1}{2.5\times10^{-5}}\left(n/1m^{-3}\right)^{-1}(T/1K)^{3/2}(ln\Lambda/10)^{-1} s.$ Using the fact that $\Lambda = \frac{9}{2}N_D$(see Exercise 19.2 part (a)), we get $t_D^{ee} = 4\times10^4\left(n/1m^{-3}\right)^{-1}(T/1K)^{3/2}\left(ln(\frac{9N_D}{2})/10\right)^{-1} s.$

And we only need to know T and n to get numerical values of these.

(a) Atomic bomb.

Text eq. (16.61) gives $T\sim4\times10^4\,(t/1ms)^{-1.2}\,K$; at $t=1ms$, $T\sim4\times10^4K$. The discussion above eq. (16.61) gives $\rho\sim5kg/m^3$, which means $n\sim\frac{\rho}{\mu m_p}\sim\frac{5kg/m^3}{29\times1.66\times10^{-27}kg}\sim10^{26}m^{-3}$.

(b) Space shuttle

Box 16.1 gives $T\sim9000K$. And since shuttle moves at $\sim7000m/s >>$ sound speed $280m/s$ (all given in Box 16.1), we can use eq. (16.50) which says $\frac{\rho_1}{\rho_2}\simeq\frac{\gamma-1}{\gamma+1}$ and gives $\rho_2\sim5\rho_1$ taking $\gamma\sim1.5$. The density at the altitude of $70km$ is $\rho_1\sim\rho_{ground}exp(-70km/8km)\sim10^{-4}kg/m^3$, which gives $n\sim\frac{\rho_2}{\mu m_p}\sim\frac{5\rho_1}{\mu m_p}\sim10^{22}m^{-3}$.

(c) Expanding universe

Text Fig. 16.1 gives: at recombination threshold, $logT\sim3.5\Rightarrow T\sim10^{3.5}K\sim3\times10^3K$. Also from chapter 26, $\rho\propto T^3\Rightarrow\rho_{then}=\rho_{now}\left(\frac{T_{then}}{T_{now}}\right)^3\sim10^{-29}g/cm^3\left(\frac{3\times10^3K}{3K}\right)^3\sim10^{-20}g/cm^3$. So we get $n\sim\frac{\rho_{then}}{m_p}\sim10^{10}m^{-3}$.

Plugging the above values of T and n into the equations on the top of this page, we get

	$\lambda_D(m)$	N_D	$f_p(Hz)$	$t_D(s)$
A-bomb	1×10^{-9}	1	9×10^{13}	2×10^{-14}
Shuttle	7×10^{-8}	10	9×10^{11}	9×10^{-12}
Universe	4×10^{-2}	2×10^6	9×10^5	4×10^{-1}

C. Exercise 19.7 Equilibration Time for a Globular Cluster [by unknown author :)]

(a) For single deflections when $b\le b_0$, $\sigma=\pi b_0^2$. While for cumulative deflections, in which each deflection has $b>>b_0$, then $\Delta E=-(b_0/b)^2E$ for each deflection. Since we are interested in the case where the test star has high kinetic energy compared to the field stars, then we add up ΔE linearly.

$$\frac{\Delta E}{E} = -\int_{b_{min}}^{b_{max}}\left(\frac{b_0}{b}\right)^2 nvt2\pi bdb = -2\pi vtnb_0^2ln\left(\frac{b_{max}}{b_{min}}\right)$$

where b_{min} is b_0 and b_{max} is $R=$ the radius of the star cluster.

So the energy change timescale is dominated by cumulative deflections:

$$t_E = \frac{1}{2\pi b_0^2 n v ln\Lambda}$$

In the gravitational case, $b_0 = 2Gm/v^2$ and we get

$$t_E = \frac{v^3}{8nG^2m^2ln\Lambda}$$

To estimate this, use the Virial theorem(e.g. Goldstein) which says that the cluster's kinetic energy is half the potential energy, so $\frac{1}{2}Nmv^2 \sim \frac{1}{2}\frac{G(Nm)^2}{R}$, $\Rightarrow v \sim \left(\frac{GNm}{R}\right)^{1/2}$.

$$ln\Lambda = ln\frac{R}{b_0} = ln\frac{R}{2Gm/v^2} = lnN = ln10^6 = 14$$

and

$$t_E = \frac{N^{3/2}}{14n(8GmR^3)^{1/2}}$$

Also we can put in $n = N/(\frac{4\pi}{3}R^3)$ to get

$$t_E = \frac{\frac{4\pi}{3}N^{1/2}R^{3/2}}{14(8Gm)^{1/2}} = 4 \times 10^{17} seconds = 1.3 \times 10^{10} years$$

which is about the age of the universe.

(b) The cluster will try to develop a distribution function that is a function of the constant of motion (so it satisfies the collisionless Boltzmann equation, i.e. Liouville's theorem of chapter 2). The velocity distribution will try to become isotropic, so

$$\mathcal{N} = \frac{dN}{d^3xd^3p} = f(E) = f(m\Phi + \frac{1}{2}mv^2)$$

where Φ is the gravitational potential which is less than zero. In true equilibrium, this $f(E)$ should become an exponential, so $\mathcal{N} = C\ exp(-E/kT)$. However, only stars with $E < 0$ are gravitationally bound in the cluster; those with $E > 0$ escape and fly away. This means that $\mathcal{N} = 0$ for $E > 0$ and $\mathcal{N} \simeq C\ exp(-E/kT)$ for $E < 0$.

Stellar encounters then keep kicking stars, occasionally, to energies $E > 0$, and those stars evaporate from the cluster. Since the evaporated stars have larger energy than average, the rest of the cluster keeps shrinking and becoming more and more tightly bound.

D.

Exercise 20.2 Effect of Collisional Damping [by Alexei Dvoretskii/99]

Write the equation of motion for the electrons with a collision term added (since electron is the only relevant species here, we drop all the subscript e):

$$-i\omega mn\mathbf{u} = qn\mathbf{E} - \nu nm\mathbf{u}$$

Hence

$$\mathbf{u} = \frac{q\mathbf{E}}{m}\frac{\nu + i\omega}{\nu^2 + \omega^2}$$

For $\omega >> \nu$ and using $\omega_p^2 = \frac{nq^2}{m\epsilon_0}$ the current is then

$$\mathbf{j} = qn\mathbf{u} = \epsilon_0\omega_p^2\frac{\nu + i\omega}{\omega^2}\mathbf{E}$$

So the conductivity tensor is

$$\kappa_{ij} = \delta_{ij}\epsilon_0\omega_p^2\frac{\nu + i\omega}{\omega^2}$$

and the dielectric tensor is

$$\epsilon_{ij} = \delta_{ij}\left(1 - \frac{\omega_p^2(\omega - i\nu)}{\omega^3}\right)$$

Hence the dispersion relation is

$$\omega^2 - \omega_p^2 + i\nu\frac{\omega_p^2}{\omega} = c^2k^2$$

or

$$k = \frac{\sqrt{\omega^2 - \omega_p^2}}{c}\left(1 + \frac{i\nu\omega_p^2}{2\omega(\omega^2 - \omega_p^2)}\right)$$

The imaginary part describes the dissipation and therefore

$$S = c\epsilon_0 E^2 exp\left(-\frac{x}{c}\frac{\nu\omega_p^2}{\omega\sqrt{\omega^2 - \omega_p^2}}\right)$$

which can be obtained by using the facts that $\mathbf{S} = < \mathbf{E} \times \mathbf{B} >$ and $\mathbf{E}$ and $\mathbf{B}$ are of the form $\propto exp[ikx]$.

Again assuming $\omega >> \omega_p$, it's now trivial to compute the rate of energy loss per unit volume and see that it's equal to the Ohmic dissipation

$$-\frac{dS}{dx} = \epsilon_0\frac{\nu\omega_p^2}{\omega^2}E^2(x) = Re(j(x)E(x))$$

Exercise 20.4 Ion Acoustic Solitons [by Xinkai Wu/02]

(a) The derivation of these equations is trivial, so we omit it here.

(b) Write the proton density as $n = n_0 + \delta n$ and substitute it into the equations of part (a), keeping only terms linear in δn, u, and Φ. We find

$$\frac{\partial \delta n}{\partial t} + n_0 \frac{\partial u}{\partial z} = 0$$
$$\frac{\partial u}{\partial t} = -\frac{e}{m_p}\frac{\partial \Phi}{\partial z}$$
$$\frac{\partial^2 \Phi}{\partial z^2} = -\frac{e}{\epsilon_0}\left(\delta n - \frac{n_0 e}{k_B T_e}\Phi\right)$$

Plugging the plane-wave solution where δn, u, and Φ are all of the form $\propto exp[i(kz - \omega t)]$ into the above linearized equations, we find

$$-i\omega \delta n + n_0 iku = 0$$
$$-i\omega u = -\frac{e}{m_p} ik\Phi$$
$$-k^2\Phi = -\frac{e}{\epsilon_0}\left(\delta n - \frac{n_0 e}{k_B T_e}\Phi\right)$$

For the above algebraic equation to possess a solution, the determinant of its coefficient matrix must vanish, which gives

$$\omega^2\left(-k^2 - \frac{n_0 e^2}{\epsilon_0 k_B T_e}\right) + \frac{n_0 e^2}{\epsilon_0 m_p}k^2 = 0$$

solving which gives the dispersion relation

$$\omega = \omega_{pp}(1 + 1/k^2\lambda_D^2)^{-1/2}$$

with $\lambda_D = \left(\frac{\epsilon_0 k_B T_e}{n_0 e^2}\right)^{1/2}$ and $\omega_{pp} = \left(\frac{n_0 e^2}{\epsilon_0 m_p}\right)^{1/2}$.

For long-wavelength, $k\lambda_D << 1$, one finds

$$\omega \approx \omega_{pp} k \lambda_D = k\left(\frac{k_B T_e}{m_p}\right)^{1/2}$$

which agrees with eq. (20.35).

(c) Upon the substitution, the continuity equation gives, at leading order in ϵ

$$\frac{\partial n_1}{\partial \xi} = \frac{\partial u_1}{\partial \xi} \qquad (1A)$$

and at subleading order in ϵ

$$-\frac{1}{\lambda_D}\sqrt{\frac{k_B T_e}{m_p}}\frac{\partial n_2}{\partial \xi} + \omega_{pp}\frac{\partial n_1}{\partial \tau} + \sqrt{\frac{k_B T_e}{m_p}}\frac{1}{\lambda_D}\frac{\partial(u_2 + n_1 u_1)}{\partial \xi} = 0 \qquad (1B)$$

The equation of motion for the proton gives, at leading order

$$\frac{\partial u_1}{\partial \xi} = \frac{\partial \Phi_1}{\partial \xi} \qquad (2A)$$

and at subleading order

$$-\frac{k_B T_e}{\lambda_D m_p}\frac{\partial u_2}{\partial \xi} + \sqrt{\frac{k_B T_e}{m_p}}\omega_{pp}\frac{\partial u_1}{\partial \tau} + \frac{k_B T_e}{2\lambda_D m_p}\frac{\partial u_1^2}{\partial \xi} = -\frac{k_B T_e}{\lambda_D m_p}\frac{\partial \Phi_2}{\partial \xi} \qquad (2B)$$

The Poisson equation gives, at leading order

$$n_1 = \Phi_1 \qquad (3A)$$

and at subleading order

$$\frac{k_B T_e}{e\lambda_D^2}\frac{\partial^2 \Phi_1}{\partial \xi^2} = -\frac{e n_0}{\epsilon_0}\left(n_2 - \Phi_2 - \frac{1}{2}\Phi_1^2\right) \qquad (3B)$$

Combining (1A), (2A), and (3A) we conclude

$$n_1 = u_1 = \Phi_1$$

Now

$$(1B)\cdot\sqrt{\frac{k_B T_e}{m_p}} + (2B)$$

gives us an expression for $\left(\frac{\partial n_2}{\partial \xi} - \frac{\partial \Phi_2}{\partial \xi}\right)$ in terms of derivatives of n_1, which we then plug into $\frac{\partial (3B)}{\partial \xi}$. After using the explicit expressions for λ_D and ω_{pp}, it reduces to

$$\frac{\partial n_1}{\partial \tau} + n_1\frac{\partial n_1}{\partial \xi} + \frac{1}{2}\frac{\partial^3 n_1}{\partial \xi^3} = 0$$

and u_1, Φ_1, being equal to n_1, satisfy the same equation.

(d) Actually to make this the standard KdV equation (15.34) we need to rescale $\tilde{\xi} \equiv \sqrt{2}\xi$ and $\tilde{\tau} \equiv \sqrt{2}\tau$. Then eq. (15.35) gives us the single-soliton solution

$$y = y_0 sech^2\left[\left(\frac{y_0}{12}\right)^{1/2}\left(\tilde{\xi} - \frac{1}{3}y_0\tilde{\tau}\right)\right]$$

the speed of which is

$$\frac{dz}{dt} = \sqrt{\frac{k_B T_e}{m_p}}\left(1 + \frac{1}{3}y_0\epsilon\right)$$

For this to be equal to $\sqrt{\frac{k_B T_e}{m_p}}(1+\epsilon)$ we must set $y_0 = 3$. The final form of the solution is thus

$$y = 3sech^2\left[\frac{1}{\sqrt{2}}(\xi - \tau)\right]$$