Solution for Chapter 12

(compiled by Xinkai Wu)

A.

Ex. 12.3 Earth's atmosphere [by Alexei Dvoretskii/99]

(i) Isothermal air

Let's use the ideal gas equation to express the density of air as a function of pressure in the equation of hydrostatic equilibrium

$$\begin{aligned} P &= \frac{\rho k T}{\mu m_p} \\ \nabla P &= \rho \mathbf{g}_e \end{aligned}$$

we get

$$\frac{dP}{dz} = -P\frac{\mu m_p g_e}{kT}$$

and hence

$$P \propto exp(-z/H)$$

where

$$H = \frac{kT}{\mu m_p g_e}$$

Taking the following numerical values

$$k = 1.38 \times 10^{-23} J s^{-1},\ g_e = 10 m s^{-2},\ m_p = 1.7 \times 10^{-27} kg,\ \mu = 29,\ T = 300K$$

we get

$$H \approx 8km$$

(ii) Isentropic air

We have the following three equations for the three unknowns P, ρ, T:

$$\begin{aligned} \frac{dP}{dz} &= -\rho g_e \\ P &= const \times \rho^\gamma \\ P &= \frac{\rho k T}{\mu m_p} \end{aligned}$$

which can be easily solved to give

$$\frac{dT}{dz} = -\frac{\gamma - 1}{\gamma}\frac{g_e \mu m_p}{k}$$

Using $\gamma \approx 1.4$ we find the lapse rate to be $\approx 10K/km$.

Ex. 12.5 Jupiter and Saturn [by Alexei Dvoretskii/99]

Plugging the relation $P = K\rho^2$ into equation (12.12) in the text, one finds

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\rho}{dr}\right) = -\frac{2\pi G}{K}\rho$$

which can be easily solved by letting $\rho(r) = f(r)/r$. And the two linearly independent solutions are

$$\frac{sin(\alpha r)}{r},\ \frac{cos(\alpha r)}{r},\ \text{with } \alpha \equiv \sqrt{\frac{2\pi G}{K}}$$

Taking the solution that is finite at $r = 0$, we get

$$\rho(r) = \frac{C}{r} sin(\alpha r)$$

The edge of the planet is where the density becomes zero, which gives

$$R = \frac{\pi}{\alpha} = \sqrt{\frac{\pi K}{2G}} \Rightarrow K = \frac{2GR^2}{\pi}$$

I.e. we see that such polytropic planets should all have roughly the same size (assuming K is the same for all such planets), $R_S = R_J = 7 \times 10^4 km$. And the constant C can be found by normalizing to the mass of the planet

$$M = \int_0^R 4\pi r^2 dr \frac{C}{r} sin(\alpha r) = \frac{2\pi CK}{G}$$

whence

$$C = \frac{M}{4R^2}$$

And the central pressure is

$$P(0) = K\rho^2(0) = \frac{\pi G M^2}{8R^4}$$

Using the numerical values given, we get

$$P_J(0) = 4.4 \times 10^{12} Pa,\ P_S(0) = 4 \times 10^{11} Pa$$

The gravitational binding energy is given by

$$U = -\frac{1}{2}\int d\mathbf{r}\rho(\mathbf{r})\Phi(\mathbf{r})$$

Integrating the following equation using the $\rho(r)$ we obtained above and the boundary conditions $\frac{d\Phi}{dr} = 0$ as $r \to 0$ and $\Phi(r = R) = -\frac{GM}{R}$(which comes from $\Phi(r = \infty) = 0$)

$$\boldsymbol{\nabla}^2\Phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho$$

we get

$$\Phi(r) = -\frac{GM}{R}\left[1 + \frac{sin\left(\frac{\pi}{R}r\right)}{\frac{\pi}{R}r}\right]$$

which when plugged into the integral for U gives

$$U = \frac{3GM^2}{4R}$$

The numerical values are

$$U_J = 2.9 \times 10^{36} J,\ U_S = 2.6 \times 10^{35} J$$

To find the moments of inertia again let's use spherical symmetry. Obviously,

$$I_{xx} = I_{yy} = I_{zz} = \frac{1}{3}\int 4\pi r^2 dr \rho(r) r^2 = \frac{MR^2}{3}\left(1 - \frac{6}{\pi^2}\right) = 0.13 \times MR^2$$

and

$$I = I_{xx} + I_{yy} = 0.26 \times MR^2$$

the numbers are then

$$I_J = 2.6 \times 10^{42} kg \cdot m^2,\ I_J = 7.7 \times 10^{41} kg \cdot m^2$$

B

Ex. 12.8 Rotating plantes, stars, and disks [by Xinkai Wu/00]

(i) For a stationary flow, Euler equation becomes

$$(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} + \frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{\nabla}\Phi = 0$$

Taking the curl of both sides and noticing that

$$\boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} P}{\rho}\right) = -\frac{1}{\rho^2}\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} P = -\frac{1}{\rho^2}\frac{dP}{d\rho}\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}\rho = 0$$

by virtue of the barotropic equation of state $P = P(\rho)$.

Thus we get

$$\boldsymbol{\nabla} \times [(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}] = 0$$

By axisymmetry the only nonvanishing component of $\mathbf{v}$ is $v_\phi(\varpi, z)$.

$$[(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}]_i = v_k v_{i;k} = v_\phi v_{i;\phi} = v_\phi(v_{i,\phi} + \Gamma_{ij\phi} v_j) = v_\phi \Gamma_{i\phi\phi} v_\phi = \delta_{i\varpi}\left(-\frac{v_\phi^2}{\varpi}\right)$$

namely $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ only has a nonvanishing ϖ-component. So we get

$$0 = \{\boldsymbol{\nabla}\times[(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}]\}_\phi = \frac{\partial}{\partial z}[(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}]_\varpi = -\frac{\partial}{\partial z}\left(\frac{v_\phi^2}{\varpi}\right)$$

which tells us that $v_\phi = v_\phi(\varpi)$, namely the "angular velocity" $\frac{v_\phi}{\varpi}$ only depends on ϖ.

(ii) Denote the angular momentum per unit mass as $l \equiv v(\varpi, z)\varpi$.

We already know that $\boldsymbol{\nabla}s\times\boldsymbol{\nabla}l = 0$ (i.e. the surfaces of constant entropy and constant angular momentum coincide), and we want to show that $\boldsymbol{\nabla}B\times\boldsymbol{\nabla}l = 0$, which means that the surfaces of constant B and constant angular momentum also coincide.

$$\begin{aligned}
\boldsymbol{\nabla}B &= \boldsymbol{\nabla}\left(\frac{1}{2}v^2 + h + \Phi\right) \\
\text{(using 1st law of thermodynamics} &\quad \boldsymbol{\nabla}h = T\boldsymbol{\nabla}s + \frac{\boldsymbol{\nabla}P}{\rho}) \\
&= \frac{1}{2}\boldsymbol{\nabla}v^2 + T\boldsymbol{\nabla}s + \frac{\boldsymbol{\nabla}P}{\rho} + \boldsymbol{\nabla}\Phi \\
\text{(using Euler equation} &\quad (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} + \frac{\boldsymbol{\nabla}P}{\rho} + \boldsymbol{\nabla}\Phi = 0) \\
&= \frac{1}{2}\boldsymbol{\nabla}v^2 - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} + T\boldsymbol{\nabla}s
\end{aligned}$$

So we only have to show the cross product of the first two terms of the above expression with $\boldsymbol{\nabla}l$ vanishes. Evaluating them explicitly in cylindrical coordinates

$$\frac{1}{2}\boldsymbol{\nabla}v^2 - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \left(v\frac{\partial v}{\partial\varpi} + \frac{v^2}{\varpi}\right)\mathbf{e}_\varpi + \left(v\frac{\partial v}{\partial z}\right)\mathbf{e}_z$$

and

$$\boldsymbol{\nabla}l = \left(\varpi\frac{\partial v}{\partial\varpi} + v\right)\mathbf{e}_\varpi + \left(\varpi\frac{\partial v}{\partial z}\right)\mathbf{e}_z$$

A straight forward computation shows that the above two expressions crossed into each other do give zero. This completes our proof of $\boldsymbol{\nabla}B\times\boldsymbol{\nabla}l = 0$.

Ex. 12.9 Crocco's theorem [by Xinkai Wu/00]

$$\boldsymbol{\nabla}B = \boldsymbol{\nabla}\left(\frac{1}{2}v^2 + h + \Phi\right)$$

$$\left[\boldsymbol{\nabla}\left(\frac{1}{2}v^2\right)\right]_i = v_j\frac{\partial v_j}{\partial x^i}$$

$$dh = Tds + \frac{dP}{\rho} \Rightarrow \mathbf{\nabla} h = T\mathbf{\nabla} s + \frac{\mathbf{\nabla} P}{\rho}$$

Thus

$$(\mathbf{\nabla} B)_i = v_j \frac{\partial v_j}{\partial x^i} + \left(T\mathbf{\nabla} s + \frac{\mathbf{\nabla} P}{\rho} + \mathbf{\nabla}\Phi \right)_i$$

Now by Euler equation for steady flow

$$\begin{aligned} (\mathbf{v}\cdot\mathbf{\nabla})\mathbf{v} &= -\frac{\mathbf{\nabla} P}{\rho} - \mathbf{\nabla}\Phi \\ \Rightarrow \left(\frac{\mathbf{\nabla} P}{\rho} + \mathbf{\nabla}\Phi \right)_i &= -v_j \frac{\partial v_i}{\partial x^j} \end{aligned}$$

hence

$$(\mathbf{\nabla} B)_i = (T\mathbf{\nabla} s)_i + v_j \frac{\partial v_j}{\partial x^i} - v_j \frac{\partial v_i}{\partial x^j}$$

where the last two terms in the above equation can be readily verified to be equal to $(\mathbf{v}\times\boldsymbol{\omega})_i$. This completes our proof of

$$\mathbf{\nabla} B = T\mathbf{\nabla} s + \mathbf{v}\times\boldsymbol{\omega}$$

C

Ex. 12.11 Cavitation [by Xinkai Wu/02]

We can model this as a steady flow of an ideal fluid and use the conservation of Bernoulli constant along a streamline where we compare the location next to the hydrofoil with that far away from it

$$\begin{aligned} &\frac{1}{2}v^2 + h(s,P) = h(s,P_0) \\ &\Rightarrow v = \sqrt{2(h(s,P_0) - h(s,P))} = \sqrt{2(P_0 - P)/\rho} \\ &\text{setting } P = 0 \text{ gives} \\ &v = \sqrt{2P_0/\rho} \end{aligned}$$

Taking $P_0 = 1\ atmosphere + \rho g_e h$ with $h = 3m$, we get

$$v = 16m/s$$

D

Ex. 12.12 Collapse of a bubble [by Alexei Dvoretskii/99]

Incompressibility implies that $\mathbf{\nabla}\cdot\mathbf{v} = 0$, which in spherical coordinates is

$$\frac{1}{r^2}\frac{d}{dr}(r^2 v) = 0$$

so indeed we see that

$$v = F(t)/r^2$$

The radial component of Euler's equation (gravity is not important in this problem)

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}$$

combined with the fact that $v = F(t)/r^2$ gives

$$\frac{1}{r^2}\frac{dF}{dt} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} = 0$$

Integrating this from R to infinity and using the boundary conditions $v(r = \infty) = 0$, $P(r = \infty) = P_0$, and $P(r = R) = 0$, we get

$$\frac{-1}{R}\frac{dF}{dt} + \frac{1}{2}v^2(R) = \frac{P_0}{\rho}$$

Substituting $F = R^2 v$ and $\frac{d}{dt} = v\frac{d}{dR}$ gives

$$\frac{R}{2}\frac{dv^2}{dR} + \frac{3v^2}{2} + \frac{P_0}{\rho} = 0$$

which can be readily integrated by separation of variables, with the initial condition $v(R_0) = 0$. So we finally get

$$v(R) = \left(\frac{2P_0}{3\rho}\right)^{1/2}\left[\left(\frac{R_0}{R}\right)^3 - 1\right]^{1/2}$$

The stress created by collapsing bubbles will be of order $\rho v^2 \sim P_0\left(\frac{R_0}{R}\right)^3$. Therefore, already fo $R = R_0/10$ the stress is of order 10^3 atmospheres which can inflict damage on the hydrofoil.