Solution for Chapter 11

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1 Problem A.(BT-11.1)

[by Alexander Putilin/01]
We have the wave equation (BT-11.12):

$$\rho\frac{\partial^2\xi}{\partial t^2} = (K+\frac{1}{3}\mu)\nabla(\nabla\cdot\xi) + \mu\nabla^2\xi \tag{1}$$

Writing ξ as

$$\xi = \nabla\psi + \nabla\times\mathbf{A} \tag{2}$$

and substituting in (1) we get

$$\nabla\left[\rho\frac{\partial^2}{\partial t^2}\psi - (K+\frac{4}{3}\mu)\nabla^2\psi\right] + \nabla\times\left[\rho\frac{\partial^2}{\partial t^2}\mathbf{A} - \mu\nabla^2\mathbf{A}\right] = 0 \tag{3}$$

Since ψ and $\mathbf{A}$ are independent each of the two terms in (3) vanishes. Thus we get separate equations for scalar and vector potentials.

1.1

$$\frac{\partial^2\psi}{\partial t^2} - c_L^2\nabla^2\psi = \alpha(t) \tag{4}$$

where $c_L = \sqrt{\frac{K+\frac{4}{3}\mu}{\rho}}$ and $\alpha(t)$ is arbitrary function of time.

Notice that ψ is not defined uniquely by (2). If we make the transformation $\psi\to\psi+\omega(t)$ with arbitrary $\omega(t)$, $\nabla\psi$ will not change. Using this gauge freedom we can set $\alpha(t)=0$. To see that it is possible substitute $\psi\to\psi+\omega(t)$ into (4). Then

$$\frac{\partial^2\psi}{\partial t^2} - c_L^2\nabla^2\psi = \alpha(t) - \omega''(t) = 0 \tag{5}$$

if we choose $\omega(t) = \int^t dt'\int^{t'} dt''\alpha(t'')$.

1.2

$$\frac{\partial^2\mathbf{A}}{\partial t^2} - c_T^2\nabla^2\mathbf{A} = \nabla\alpha(t,\mathbf{x}) \tag{6}$$

where $c_T = \sqrt{\frac{\mu}{\rho}}$ and α is arbitrary. As in the previous case we can remove $\nabla\alpha$ term using gauge transformation $\mathbf{A}\to\mathbf{A}+\nabla\omega$. Under this transformation (6) becomes

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - c_T^2 \nabla^2 \mathbf{A} = \nabla \left[\alpha(t, \mathbf{x}) - \frac{\partial^2 \omega}{\partial t^2} + c_T^2 \nabla^2 \omega \right] = 0 \tag{7}$$

We should choose ω such that

$$\left(\frac{\partial^2}{\partial t^2} - c_T^2 \nabla^2 \right) \omega = \alpha \tag{8}$$

(8) is just a wave equation with source term. It always has solution.

2 Problem A.(BT-11.2)

[by Guodong Wang/03]
Including the effect of gravity into equation (BT-11.12),

$$\rho \frac{\partial^2 \xi}{\partial t^2} = (K + \frac{1}{3}\mu)\nabla(\nabla \cdot \xi) + \mu \nabla^2 \xi + \rho \mathbf{g} \tag{9}$$

Consider small pertubations to the displacement and density, Substituing $\rho \to \rho_0 + \delta\rho$, $\xi \to \xi_0 + \xi$ into Eq. (9), Keeping the small pertubations to the first order, we obtain the static equation

$$0 = (K + \frac{1}{3}\mu)\nabla(\nabla \cdot \xi_0) + \mu \nabla^2 \xi_0 + \rho_0 \mathbf{g} \tag{10}$$

which describe the static displacement ξ_0 produced by gravity $\mathbf{g}$, and the wave equation

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = (K + \frac{1}{3}\mu)\nabla(\nabla \cdot \xi) + \mu \nabla^2 \xi + \delta\rho \mathbf{g} \tag{11}$$

Recalling Eq. (BT-11.10), $\delta\rho \simeq -\rho_0 \nabla \cdot \xi$, Eq.(11) can be written as

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = (K + \frac{1}{3}\mu)\nabla(\nabla \cdot \xi) + \mu \nabla^2 \xi - \rho_0 \nabla \cdot \xi \mathbf{g}. \tag{12}$$

The first term and the second term on the right side of Eq.(12) are of the same order of magnitude. The ratio of the third to the second term is of the order of $\frac{\rho_0 g}{\mu k} \sim \frac{\lambda}{\mu/\rho_0 g}$. For short wavelength modes, $\lambda \ll \mu/\rho_0 g$, the third term can be ignored and equation (12) reduces to Eq. (BT-11.12).

Taking $\mu \sim 100GPa$ (see BT table 10.1), $\rho_0 \sim 3g/cm^3$, $\mu/\rho_0 g \sim 1000km$. So only for $\lambda \gtrsim 1000km$, the gravitational terms are important.

3 Problem B.(BT-11.4)

[by Alexander Putilin/01]

We need to find sound velocities of 5 types of elastic waves. Consider them in turn.

3.1 Longitudinal waves along a rod.

The correspoding formula has been derived in BT. (BT-11.28)

$$c_1 = \sqrt{\frac{E}{\rho}} \tag{13}$$

3.2 Longitudinal waves along a sheet

Choose coordinate system so that z-axis is orthogonal to the sheet, x- and y-axes are parallel and wave is propagating in x-direction. Since wave is longitudinal it means

$$\xi_y = 0 \tag{14}$$

No force acts on the face of the sheet, i.e. $T_{zi} = 0$.

$$\Longrightarrow T_{zz} = -K\theta - 2\mu\Sigma_{zz} = 0, \qquad \Sigma_{zz} = -\frac{K}{2\mu}\theta \tag{15}$$

Tracelessness of Σ implies

$$\Sigma_{xx} + \Sigma_{yy} + \Sigma_{zz} = 0 \tag{16}$$

But $\Sigma_{yy} = \xi_{y,y} - \frac{1}{3}\theta = -\frac{1}{3}\theta$. (Cf.(14))
Plugging into (16) we get

$$\Sigma_{xx} = \frac{1}{3}\theta + \frac{K}{2\mu}\theta = \frac{2-\nu}{3(1-2\nu)}\theta \tag{17}$$

Recall that

$$K = \frac{E}{3(1-2\nu)}$$
$$\mu = \frac{E}{2(1+\nu)}$$

Find the relation between θ and $\xi_{x,x}$

$$\Sigma_{xx} = \xi_{x,x} - \frac{1}{3}\theta \Rightarrow \theta = \frac{1-2\nu}{1-\nu}\xi_{x,x} \tag{18}$$

Then

$$T_{xx} = -K\theta - 2\mu\Sigma_{xx} = -\frac{E}{1-\nu^2}\xi_{x,x} \tag{19}$$

Equation of motion is

$$\rho\frac{\partial^2\xi_x}{\partial t^2} = T_{xx,x} + T_{xy,y} + T_{xz,z} = -\frac{E}{1-\nu^2}\frac{\partial^2\xi_x}{\partial x^2} \tag{20}$$

It gives the sound velocity

$$c_2 = \sqrt{\frac{E}{\rho(1-\nu^2)}}, \qquad \frac{c_1}{c_2} = \left(1-\nu^2\right)^{1/2} \tag{21}$$

3.3 Longitudinal waves along a rod embedded in incompressible medium.

Orient the x-axis of a Cartesian coordinate system along the rod. Incompressibility of the medium means that there is no strain in transverse direction, i.e. $\xi_y = \xi_z = 0$.

$$\Rightarrow \theta = \xi_{x,x}, \qquad \Sigma_{xx} = \xi_{x,x} - \frac{1}{3}\theta = \frac{2}{3}\xi_{x,x} \tag{22}$$

Using (22) we get

$$T_{xx} = -K\theta - 2\mu\Sigma_{xx} = -E\frac{1-\nu}{(1+\nu)(1-2\nu)}\xi_{x,x} \tag{23}$$

$$c_3 = \sqrt{\frac{E}{\rho}\frac{1-\nu}{(1+\nu)(1-2\nu)}}, \qquad \frac{c_1}{c_3} = \left(\frac{1-\nu}{(1+\nu)(1-2\nu)}\right)^{-1/2} \tag{24}$$

3.4 Shear waves in an extended solid.

The sound velocity has been calculated in BT. (BT-11.20)

$$c_4 = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{\rho 2(1+\nu)}} \tag{25}$$

$$\frac{c_1}{c_4} = [2(1+\nu)]^{1/2} \tag{26}$$

3.5 Torsional waves along a rod.

This has been also calculated in BT. (BT-11.34)

$$c_5 = c_4 = \sqrt{\frac{\mu}{\rho}}, \qquad \frac{c_1}{c_5} = [2(1+\nu)]^{1/2} \tag{27}$$

4 Problem B.(BT-11.5)

[by Alexander Putilin/01]

We start with the wave equation (BT-11.37)

$$\frac{\partial^4\eta}{\partial z^4} + \frac{\Lambda}{D}\frac{\partial^2\eta}{\partial t^2} = 0 \tag{28}$$

We are looking for the standing-wave solution of the form

$$\eta(t,z) = e^{-i\omega t} f(z) \tag{29}$$

Substituting into (28) we get the O.D.E. for $f(z)$

$$\frac{d^4 f}{dz^4} = \frac{\Lambda \omega^2}{D} f \tag{30}$$

It has the general solution

$$f(z) = A \sin kz + B \cos kz + C \sinh kz + D \cosh kz \tag{31}$$

where A, B, C, D are constants and

$$\omega^2 = \frac{D}{\Lambda} k^4 \tag{32}$$

To define the spectrum of k we should impose boundary conditions. We have two different cases.

4.1 Ends are free.

The appropriate boundary conditions are:

$$f(0) = f(l) = 0 \quad \text{- ends are fixed} \tag{33}$$
$$f''(0) = f''(l) = 0 \quad \text{longitudinal stress vanishes} \tag{34}$$

(because $T_{zz} = -Ex\frac{d^2\eta}{dz^2}$ by Eq. (BT-10.62) with interchange of x and z)

$$\Longrightarrow B = C = D = 0, \qquad kl = \pi n, \qquad n = 1, 2, 3, \ldots \tag{35}$$

So the frequency spectrum is

$$\omega_n = \sqrt{\frac{D}{\Lambda}} k^2 = \sqrt{\frac{D}{\rho A}} \left(\frac{\pi n}{l}\right)^2 \tag{36}$$

4.2 Clamped ends.

In this case the boundary conditions are:

$$f(0) = f(l) = 0 \tag{37}$$
$$f'(0) = f'(l) = 0 \tag{38}$$

They can be nontrivially satisfied only if

$$\cosh(kl)\cos(kl) = 1 \tag{39}$$

so the spectrum is

$$\omega_n = \sqrt{\frac{D}{\rho A}} \left(\frac{\alpha_n}{l}\right)^2, \tag{40}$$

where α_n are solution of the eqn.

$$\cosh(\alpha)\cos(\alpha) = 1 \tag{41}$$

If $\alpha \gg 1$, then $\cosh\alpha \gg 1$ and $\cos\alpha \ll 1$ i.e. $\alpha_n \approx \pi(n+\frac{1}{2})$. Actually this formula gives good approximation for all $n = 1, 2, 3, \ldots$.

$$\Longrightarrow \omega_n \approx \sqrt{\frac{D}{\rho A}} \left[\frac{\pi}{l}(n+\frac{1}{2})\right]^2, \tag{42}$$

The ratios of eigen-frequencies are

$$\omega_1 : \omega_2 : \omega_3 : \cdots \approx 9 : 25 : 49 : \ldots$$

The ratios are noninteger and the frequencies don't make up any harmonic chord. A xylophone with clamped ends would sound hollow.

5 Problem C.(BT-11.7)

[by Alexander Putilin/01]

Assume that the wave is propagating along the z-direction and $z = 0$ is a boundary between two media. ($z < 0$ corresponds to medium 1, and $z > 0$ to medium 2) The longitudinal displacement is given by

$$\xi(z) = \begin{cases} e^{ikz} + re^{-ikz} & z < 0 \\ te^{ik'z} & z > 0 \end{cases} \tag{43}$$

where $k = \frac{\omega}{c_1}$, $k' = \frac{\omega}{c_2}$, ω is frequency, $c_{1,2}$ are sound velocities.

$$c_{1,2} = \sqrt{\frac{K_{1,2} + \frac{4}{3}\mu_{1,2}}{\rho_{1,2}}} \tag{44}$$

r is the ratio of the reflected wave amplitude to the incident amplitude, t is the corresponding ratio for the transmitted wave amplitude. (We've chosen the incident wave to have unit amplitude).

The time dependence is given by the factor $e^{-i\omega t}$ in both cases.

$$\theta = \xi(z)_{,z} = \xi'(z), \qquad \Sigma_{zz} = \xi_{,z} - \frac{1}{3}\theta = \frac{2}{3}\xi' \tag{45}$$

$$T \equiv T_{zz} = -K\theta - 2\mu\Sigma_{zz} = -(K + \frac{4}{3}\mu)\xi'(z) \tag{46}$$

$$T = \begin{cases} -(K_1 + \frac{4}{3}\mu_1)(ike^{ikz} - ikre^{-ikz}) & z < 0 \\ -(K_2 + \frac{4}{3}\mu_2)ik'te^{ik'z} & z > 0 \end{cases} \tag{47}$$

Matching the displacement and the stress on the boundary ($z = 0$) we get the equations:

$$1 + r = t \tag{48}$$

$$(K_1 + \frac{4}{3}\mu_1)ik(1 - r) = (K_2 + \frac{4}{3}\mu_2)ik't \tag{49}$$

which give (after simple algebra):

$$t = \frac{2Z_1}{Z_1 + Z_2} \tag{50}$$

$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{51}$$

$Z_{1,2} = \rho_{1,2}c_{1,2}$

Conservation of energy flux.

We want to show that (Incident Flux)=(Transmitted Flux)+ (Reflected Flux). We have the formula for the flux

$$F = \rho\omega^2 c\langle\xi^2\rangle \tag{52}$$

Thus conservation of energy flux reduces to

$$\rho_1 c_1(1 - |r|^2) = \rho_2 c_2|t|^2, \qquad \text{or} \qquad Z_1(1 - |r|^2) = Z_2|t|^2 \tag{53}$$

Substituting t and r from (50),(51) we see that (53) is an identity.

6 Problem D.(BT-11.8)

[by Alexander Putilin and Kip Thorne/01] Energy release $E = 10^{5.2+1.44M}J$, for $M = 8.5$ it gives

$$E \approx 10^{17.5}J \approx 3 \cdot 10^{17}J \tag{54}$$

This should be equal to the elastic energy

$$E_{el} \approx \mu\Sigma^2(VT)^2\lambda \tag{55}$$

where Σ is the typical strain, $\mu \sim 30GPa$ is the shear modulus, $V \sim 3km/s, T \sim 100s, \lambda \sim \frac{V}{\nu} \sim 3km$ is the wavelength, $\nu \sim 1Hz$ is the frequency and $(VT)^2\lambda$ is the volume of the earthquake region. Equating (54) and (55) we get

$$\Sigma \sim \sqrt{\frac{E}{\mu(VT)^2\lambda}} \sim 10^{-4} \tag{56}$$

This is a reasonable estimate. Since the Earth crust is inhomogenous and has a lot of defects, relatively small strain is sufficient to produce a slippage. (For a perfect crystal the corresponding strain would be $\sim 10^{-2}$.

The typical displacement in the vicinity of the fault is

$$\xi \sim VT\Sigma \sim 30m \tag{57}$$

Now let's estimate the amplitude (and acceleration) of elastic waves far away from the earthquake region. The energy density of the waves is (neglecting the density stratification) $\rho\omega^2\xi^2$, where $\rho \sim 5 \cdot 10^3 kg/m^3$,$\omega = 2\pi\nu \sim 6s^{-1}$, ξ is the wave amplitude. The total energy is $\rho\omega^2\xi^2(Volume)$. There are two types of waves: Rayleigh waves that propagate near the surface and S,P waves propagating in the bulk.

For surface waves: Volume $\sim 2\pi rVT\lambda$ while for bulk waves Volume $\sim r^2VT$. r is the distance between the earthquake and the observation point. Assume quite arbitrary that surface waves carry 1/3 of all the released energy and bulk waves the remaining 2/3. Then the amplitudes of surface and bulk waves and corresponding accelerations can be estimated as

$$\xi_{surf}^2 \sim \frac{E}{6\pi rVT\lambda\rho\omega^2} \qquad a_{surf} \sim \xi_{surf}\omega^2 \tag{58}$$

$$\xi_{bulk}^2 \sim \frac{2E}{3r^2VT\lambda\rho\omega^2} \qquad a_{bulk} \sim \xi_{bulk}\omega^2 \tag{59}$$

If the observer is in the next state

$$r \sim 500km, a_{surf} \sim 40cm/s^2, a_{bulk} \sim 10cm/s^2 \tag{60}$$

If the observer is in the next continent,

$$r \sim 10^4km, a_{surf} \sim 10cm/s^2, a_{bulk} \sim 0.5cm/s^2 \tag{61}$$

Note added by Kip: these conclusions could be modified by attenuation of the waves with distance. The wave amplitude attenuates $\propto e^{-\pi r/\lambda Q}$, where Q is the quality factor. The Q's vary strongly depending on the nature of the rock through which they propagate. They can range from ~ 20 for alluvial materials near the earth's surface to ~ 3000 in the mantle at depths of a few hundred km. Even for the largest $Q \sim 3000$, the intercontinental attenuation is significant, exp[-$\pi 10^4 km/3km \times 3000$] ~ 0.03. The surface waves are typically subject to somewhat greater attenuation than the bulk waves, so they may not dominate at great distances in the manner suggested by the above estimates. For some measurements of Q's see, e.g., http://rses.anu.edu.au/seimology/ar99/atten.html