Solution for Chapter 7

(compiled by Xinkai Wu)

A.

1. Exercise 7.1 Pointillist painting [by unknown author/93]

The idea here is to consider the resolving (or conversely, blurring) ability of the eye. The resolution limit of two point sources is given by Rayleigh's criterion, namely, the two dots begin to look blurred when

$$\theta_A = \theta_s$$

where θ_A is the angular radius of the Airy disk: $\theta_A = 1.22\frac{\lambda}{d} \approx \frac{\lambda}{d}$ with d being the pupil's diameter; and $\theta_s \approx \frac{s}{l}$ is the angular separation between the two dots with s being the size of the dots and l the distance from the painting to the observer.

This gives the necessary distance to see color blending

$$l \approx \frac{ds}{\lambda} \text{ or larger}$$

Taking $d \approx 4\text{mm}$, $s \approx 0.4mm$, and $\lambda \approx 550nm$ (the peak response wavelength of eye), we get

$$l \approx 3m \text{ or larger}$$

2. Exercise 7.2 Thickness of a hair [by Xinkai Wu/00]

Use a laser pointer as the source of coherent illumination, and examine the diffraction pattern formed on the wall.

By Babinet's principle, the diffraction pattern produced by the hair is the same as that produced by a slit, which is given by eq. (7.13)

$$|\psi(\theta)|^2 \propto \left| sinc\left(\frac{ka\theta}{2}\right)\right|^2$$

where a is the thickness of the hair.

Using the above expression, we find that a is related to $\Delta\theta$ (one half the angular difference between the two central zeros of the diffraction pattern corresponding to $\frac{ka\theta}{2} = \pm\pi$) by

$$a = \frac{\lambda}{\Delta\theta}$$

Denoting the linear difference between the first and second zeros as d, the distance from the hair to the wall as L, we have

$$\Delta\theta = \frac{d}{L}$$

And thus

$$a = \frac{\lambda}{(d/L)}$$

Now $\lambda \approx 6 \times 10^{-7} m$, and we measured $d \approx 1.5cm,\ L \approx 2.3m$, thus we find

$$a \approx \frac{6 \times 10^{-7}}{(1.5 \times 10^{-2}/2.3)} \approx 90\mu m$$

B.
1. Exercise 7.3 Diffraction grating [Unknown author/93]
Notation:

$$\begin{aligned} \tilde{A} &\equiv F.T.(A) \equiv \text{Fourier Transform of } A(x) \\ A \otimes B &\equiv \int A(y)B(y-x)dy \\ \tilde{A} \otimes \tilde{B} &\equiv \int \tilde{A}(\theta')\tilde{B}(\theta - \theta')d\theta' \end{aligned}$$

Convolution Theorem:

$$\begin{aligned} F.T.(A \otimes B) &= \tilde{A} \times \tilde{B} \\ F.T.(A \times B) &= \tilde{A} \otimes \tilde{B} \end{aligned}$$

Define

$$f_1 = \sum_{n=-\infty}^{+\infty} \delta(x - 2na)$$
$$f_2 = H(x + a/2) - H(x - a/2)$$
$$f_3 = H(x + Na) - H(x - Na)$$
where $H(x)$ is the step function: $H(x) = 1$ when $x > 0$; $H(x) = 0$ when $x < 0$.

Now note that Fig 7.4b is:

$$F.T.(\{f_1 \otimes f_2\} \times f_3)$$

But it can also be rewritten as

$$t(\theta) \propto F.T.((f_1 \times f_3) \otimes f_2) = F.T.(f_1 \times f_3) \times F.T.(f_2)$$

Now

$$\begin{aligned} F.T.(f_1 \times f_2) \quad &\propto \quad \int_{-\infty}^{+\infty} \sum_{n=-(N-1)/2}^{(N-1)/2} \delta(x - 2na)e^{ikx\theta}dx = \sum_{n=-(N-1)/2}^{(N-1)/2} e^{i2nak\theta} \\ &= \quad \frac{sin(Nak\theta)}{sin(ak\theta)} \end{aligned}$$

and

$$F.T.(f_2) \propto \int_{-a/2}^{a/2} e^{ikx\theta} d\theta \propto sinc\left(\frac{ka\theta}{2}\right) \quad \text{[see eq. 7.13 of text]}$$

Thus

$$I \propto sinc^2\left(\frac{ka\theta}{2}\right) \frac{sin^2(Nak\theta)}{sin^2(ak\theta)}$$

2. Exercise 7.5 Light scattering by particles [by unknown author/93]

(a) Away from the incident direction, the amplitude is the same as that of the diffraction by the aperture (Babinet's principle)

Thus we can use eq. (7.18)

$$\psi(\theta) \propto \int d^2x e^{-ik\mathbf{x}\cdot\theta} \propto jinc\left(\frac{ka\theta}{2}\right) \qquad \text{except at } \theta = 0$$

So we see that the opening angle is

$$\Delta\theta \sim \frac{1.22\lambda}{a} \sim \frac{\lambda}{a} \sim \frac{1}{ka}$$

The light that goes through the aperture (a total power of FA) gets diffracted to form, in the Fraunhofer region, the spreading beam with angular size $\Delta\theta$. Thus the total diffracted power is

$$P_s = FA$$

(b) Return to the particle. The total power that is diffracted into the outgoing spreading beam is $P_s = FA$ (same as for the aperture). In addition, because the particle has $a >> \lambda$, its absorption can be analyzed in the geometric optics limit: it absorbs all the photons that impinge on its area A. So the total power absorbed is $P_{abs} = FA = P_s$. Thus

$$P_{abs} + P_s = 2FA$$

namely, the total 'extinction'' cross section is $2A$.

C.

1. Exercise 7.6 Zone plate [by Alexei Dvoretskii/99]

(a) See Fig. 1. At $z = f/3$ the integral over the first zone $\int_0^\rho 2\pi\rho' d\rho' \, exp[i\pi\rho'^2]$ winds around the circle 3/2 times, giving a net positive real contribution [recall that $\rho \equiv |\mathbf{x}'|/r_F \propto 1/\sqrt{z}$; thus as z is reduced by a factor of 3, ρ^2 is increased by a factor of 3.] Two of the three semicircles in Fig. 1 (A and B) cancel each other, and the net contribution comes from semicircle c. The integral over each successive open(unblocked) zone will give this some contribution until the zone plate ends or irregularities cause a die out of the coherence.

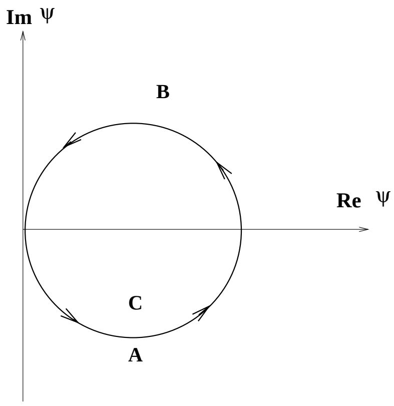

Figure 1: Zone Plate

If we take $z = f/5$, the integral for each open zone winds around the circle 5/2 times and gives a real net positive contribution. Similarly for $z = f/7, f/9, \ldots$.

(b) The shadow is a phenomenon similar to the occultation by the moon for radio waves and near the edge of the disk, the intensity pattern will look like that in Fig. 7.9 in the text.

Now let's consider the formation of the bright spot at the center. The complex wave amplitude at the center is given by

$$\begin{aligned}
\psi_{\mathcal{P}} &= -i \int_{D/2r_F}^{\infty} 2\pi\rho d\rho \; e^{i\pi\rho^2} \psi_{\mathcal{Q}} \\
&= -i \int_0^{\infty} 2\pi\rho d\rho \; e^{i\pi\rho^2} \psi_{\mathcal{Q}} + i \int_0^{D/2r_F} 2\pi\rho d\rho \; e^{i\pi\rho^2} \psi_{\mathcal{Q}} \\
\text{(using the fact:} \quad & -i \int_0^{\infty} 2\pi\rho d\rho \; e^{i\pi\rho^2} \psi_{\mathcal{Q}} = \psi_{\mathcal{Q}}) \\
&= \psi_{\mathcal{Q}} + e^{i\pi\rho^2} \Big|_0^{D/2r_F} \psi_{\mathcal{Q}} \\
&= e^{i\pi \frac{D^2}{4r_F^2}} \psi_{\mathcal{Q}}
\end{aligned}$$

In the above equations, the integrals with ∞ as the upper limit only have a formal meaning, and what is meant is that when integrating to ∞ we must take into account the averaging over rings discussed on page 18 of the text.

Thus we see,

$$|\psi_{\mathcal{P}}|^2 = |\psi_{\mathcal{Q}}|^2$$

i.e. the bright spot at the center has the same energy flux as the original incoming radiation.

Now let's compute the diameter a of this bright spot. It is given by the consideration that waves arriving at its edge from opposite sides of the disk have a $\lambda/2$ difference in their path length.

This path length difference is given by

$$\delta l = \sqrt{z^2 + \left(\frac{D+a}{2}\right)^2} - \sqrt{z^2 + \left(\frac{D-a}{2}\right)^2} \approx \frac{Da}{2z}$$

where z is the distance from the disk to the screen.

Setting $\delta l = \lambda/2$ we find

$$a = \frac{\lambda z}{D}$$

Thus the total power in this central spot is

$$P \approx \pi \left(\frac{\lambda z}{2D}\right)^2 F$$

where F is the incident flux.

2. Exercise 7.11 Wavelength scaling at a caustic [same unknown author as Ex 7.5]

Assuming that the wave is non-dispersive, we have the following expression for the (dimensionless) phase

$$\phi(s, x) = k\left(\frac{As^3}{3} - Bxs\right)$$

where the parameters A and B have no k-dependence. Defining $a = kA$, $b = kB$, we bring $\phi(s, x)$ into the standard form

$$\phi(s, x) = k\left(\frac{as^3}{3} - bxs\right)$$

And we have $a \propto k \propto \lambda^{-1}$, $b \propto k \propto \lambda^{-1}$.

The wave amplitude at the caustic is given by eq. (7.45) and (7.46) of the text:

$$\psi(x = 0) \propto \frac{1}{\lambda r} \int ds\ e^{i\phi(s, x=0)} \propto \frac{1}{\lambda a^{1/3}}$$

where we've suppressed the numerical constants (e.g. π, $Ai(0)$) and other λ-independent quantities(e.g. r).

Thus the peak magnification at the caustic is given by

$$M(x = 0) \propto |\psi(x = 0)|^2 \propto \frac{1}{\lambda^2 a^{2/3}} \propto \frac{1}{\lambda^2 \lambda^{-2/3}} \propto \lambda^{-4/3}$$

Now δx, the spacing of the fringes at a given position x, is given by

$$\begin{aligned}
&\delta\left(2z^{3/2}/3\right) = \pi \\
&\Rightarrow z^{1/2}\delta z = \left(\frac{-bx}{a^{1/3}}\right)^{1/2}\frac{-b\delta x}{a^{1/3}} = \pi \\
&\Rightarrow \delta x \propto \left(\frac{-b}{a^{1/3}}\right)^{-3/2} \propto \left(\frac{\lambda^{-1}}{\lambda^{-1/3}}\right)^{-3/2} \propto \lambda
\end{aligned}$$

D. Exercise 7.10 Convolution via Fourier optics [by Xinkai Wu/00]

From eq. (7.32) of the text, we see that by letting $u = f$, $v \to \infty$, $\psi_F(\mathbf{x}_F) = const \cdot \tilde{\psi}_S(\mathbf{x}_F/f)$. Namely, up to a multiplicative constant, what one gets at the back focal plane is the Fourier transform of the wave amplitude at the front focal plane.

(a) The configuration for computing

$$g \otimes h(x_0, y_0) \equiv \iint g(x,y)h(x+x_0, y+y_0)dxdy$$

is shown in Fig. 2. And here is how it works:

Send in a planar wave $\psi = e^{ikz}$;

Place the sheet with transmission function $t = g(x,y)$ at the front focal plane P_1;

Place the sheet with transmission function $t' = h(x,y)$ also at plane P_1, right next to the first sheet, but displaced in the minus x and y directions by x_0 and y_0 respectively, which gives a transmission function $h(x+x_0, y+y_0)$. Thus after passing through both sheets, the wave amplitude is proportional to the total transmission function $g(x,y)h(x+x_0, y+y_0)$.

At the back focal plane P_2, put a projection plane, with a pinhole on the optical axis. We know that at P_2 we get the the Fourier transform $F.T.[g(x,y)h(x+x_0, y+y_0)]$ which is equal to $\iint g(x,y)h(x+x_0,y+y_0)e^{-i\theta_x x - i\theta_y y}dxdy$. At the pinhole, $\theta_x = \theta_y = 0$, and the wave amplitude there is the desired convolution $\iint g(x,y)h(x+x_0, y+y_0)dxdy$ (we can measure the wave intensity at the pinhole by using a photon detector; also by letting the wave interfere with a reference beam, we can measure its phase. This way we don't lose any information in the convolution).

(b) The configuration is given in Fig. 3. We now have a cylindrical lens L (lens with finite radius of curvature along x direction and infinite radius of curvature along y direction). The following is how we compute

$$g_j \otimes h_j(x_0) \equiv \int g_j(x)h_j(x+x_0)dx \quad j = 1, 2, ...$$

simultaneously.

Send in a planar wave $\psi = e^{ikz}$;

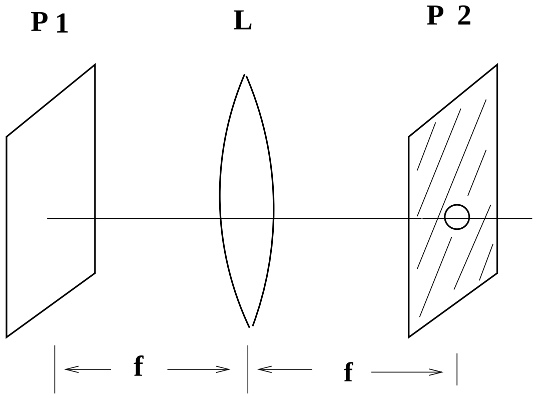

Figure 2: Ex. 7.10 (a)

At the front focal plane P_1, we put a thin sheet with transmission function $t(x, y)$ with $t(x, y_j) \equiv g_j(x)$. Right next to it we put a second sheet with transmission function $t'(x, y)$ where $t'(x, y_j) \equiv h_j(x)$. And we displace it in the minus x direction by x_0 so that $t'_{displaced}(x, y_j) = h_j(x + x_0)$. By the same reasoning as in part (a), after passing through these two sheets, the wave amplitude is

$$\psi(x, y) \sim t(x, y) t'_{displaced}(x, y)$$
$$\text{i.e. } \psi(x, y_j) \sim g_j(x) h_j(x + x_0)$$

At the back focal plane P_2 we put a projection plane with a slit along the y direction at $x = 0$.

The cylindrical lens L Fourier transforms $\psi(x, y)$ in x direction while leaving the y direction unchanged. So the output at plane P_2 along the slit (at $x = 0$) at different values of y_j gives

$$\int g_j(x) h_j(x + x_0) dx \quad j = 1, 2, ...$$

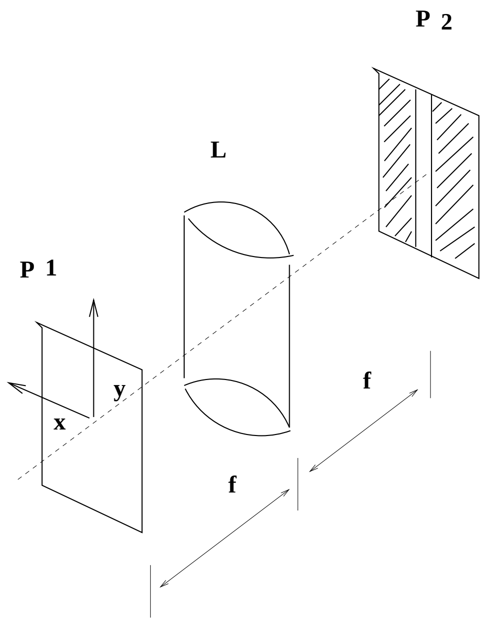

Figure 3: Ex. 7.10 (b)