Solution for Chapter 5

(compiled by Xinkai Wu, revised by Kip Thorne)

1.

Ex. 5.3 Wiener's Optimal Filter [by Alexei Dvoretskii]

(a). Let's first show $\overline{N(t)} = 0$. One can do this with the help of ergodicity: $\overline{N(t)} =< N(t) >= \int_{-\infty}^{+\infty} K(t-t') < y(t') > dt' = \int_{-\infty}^{+\infty} K(t-t')\overline{y(t')}dt' = 0$. One can also prove this directly by integrating $N(t)$ over time, whose detail we omit here.

Knowing $\overline{N(t)} = 0$, we readily get

$$\overline{N^2(t)} = \sigma_N^2 = \int_0^{+\infty} S_N(f)df = \int_0^{+\infty} |\tilde{K}(f)|^2 S_y(f)df.$$

By ergodicity $\overline{N^2(t)} =< N^2(t) >$. Now let's show $< N^2(t) >=< N^2 >$.

$$\begin{aligned}
< N^2(t) > &= < \int_{-\infty}^{+\infty} K(t-t')y(t')dt' \int_{-\infty}^{+\infty} K(t-t'')y(t'')dt'' > \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dt'dt''K(t-t')K(t-t'') < y(t')y(t'') > \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dt'dt''K(t-t')K(t-t'')f(t'-t'') \\
let \quad & \tilde{t'} = t-t', \ \tilde{t''} = t-t'' \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\tilde{t'}d\tilde{t''}K(\tilde{t'})K(\tilde{t''})f(\tilde{t''}-\tilde{t'}) \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dt'dt''K(t')K(t'')f(t''-t')
\end{aligned}$$

In the above equations we use $f(t'-t'')$ to denote $< y(t')y(t'') >$ because it's a stationary process.

On the other hand, we have

$$\begin{aligned}
< N^2 > &= < \int_{-\infty}^{+\infty} K(t')y(t')dt' \int_{-\infty}^{+\infty} K(t'')y(t'')dt'' > \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dt'dt''K(t')K(t'')f(t'-t'')
\end{aligned}$$

Using the trivial fact that $f(t'-t'') = f(t''-t')$, we see that $< N^2 >=< N^2(t) >= \overline{N^2(t)} = \int_0^{+\infty} |\tilde{K}(f)|^2 S_y(f)df$.

[A much simpler proof by Kip: $N(t) = \int_{-\infty}^{+\infty} K(t'-t)y(t')dt' = \int_{-\infty}^{+\infty} K(t'')y(t''+t)dt''$ where we've made a change of variable $t'' \equiv t'-t$. But y is a stationary random process, so its statistical properties are independent of the origin of time: $p_1(y,t) = p_1(y,0)$. Thus, the statistical properties of $N(t)$ must be the same as those of N. In particular, $< N^2(t) >=< N^2 >$]

(b) Using Parseval's theorem, and the fact $K(t),\ s(t)$ are both real (thus $\tilde{K}(-f) = \tilde{K}^*(f), \tilde{s}(-f) = \tilde{s}^*(f)$), we have

$$S = \int_{-\infty}^{+\infty} K(t)s(t)dt = \frac{1}{2}\int_{-\infty}^{+\infty} (\tilde{K}(f)\tilde{s}^*(f) + c.c)df = \int_{0}^{+\infty} (\tilde{K}(f)\tilde{s}^*(f) + c.c)df$$

We have

$$\frac{S}{< N^2 >^{1/2}} = \frac{\int_0^{+\infty} (\tilde{K}(f)\tilde{s}^*(f) + c.c)df}{\left[\int_0^{+\infty} |\tilde{K}(f)|^2 S_y(f)df\right]^{1/2}}$$

Taking a small variation $\tilde{K}(f) \to \tilde{K}(f) + \delta\tilde{K}(f)$, one readily gets

$$\delta\left(\frac{S}{< N^2 >^{1/2}}\right) = \frac{S}{< N^2 >^{1/2}}\left[\int_0^{+\infty} df\,\delta\tilde{K}(f)\left(\frac{\tilde{s}(f)}{S} - \frac{\tilde{K}(f)S_y(f)}{2 < N^2 >}\right) + c.c.\right]$$

We see that for $\delta\left(\frac{S}{<N^2>^{1/2}}\right)$ to vanish for any $\delta\tilde{K}(f)$, we must have

$$\tilde{K}(f) = const \times \frac{\tilde{s}(f)}{S_y(f)}$$

To show that this choice of $\tilde{K}(f)$ actually delivers a maximum to the ratio $\frac{S}{<N^2>^{1/2}}$, one could do a second variation calculation. Our physical intuition should make it obvious, though, because a filter with such a kernel favors such frequencies for which the signal to noise ratio is high and suppresses theose for which the opposite is true.

2.

Ex. 5.4 Alan Variance of Clocks [by Xinkai Wu]

(a) We just need to find the relation between the Fourier transform of different random processes.

$$\begin{aligned}
\tilde{\Phi}_\tau(f) &= \frac{1}{\sqrt{2}\bar{\omega}\tau}\tilde{\phi}(f)\left(e^{-i2\pi f\cdot 2\tau} - 2e^{-i2\pi f\tau} + 1\right) \\
&= \frac{\sqrt{2}}{\bar{\omega}\tau}\tilde{\phi}(f)e^{-i2\pi f\tau}(cos2\pi f\tau - 1)
\end{aligned}$$

Also, since $\phi(t)$ is obtained by integrating $\omega(t)$ once, we have

$$\tilde{\phi}(f) = \frac{-1}{i2\pi f}\tilde{\omega}(f)$$

Combining the above two results, and using the fact that the spectral density is basically given by the modulus squared Fourier transform, we find

$$\begin{aligned}
S_{\Phi_\tau(f)} &= \frac{2}{\bar{\omega}^2}\left[\frac{cos2\pi f\tau - 1}{2\pi f\tau}\right]^2 S_\omega(f) \\
&\propto f^2 S_\omega(f)\ for\ f << 1/2\pi\tau \\
&\propto f^{-2} S_\omega(f)\ for\ f >> 1/2\pi\tau
\end{aligned}$$

[An alternative way of finding $S_{\Phi_\tau}(f)$: As discussed in Section 5.5 of the text, $\Phi_\tau(t)$ can be regarded as obtained from $\omega(t')$ using a filter $K(t-t')$. Then $S_{\Phi_\tau}(f) = \left|\tilde{K}(f)\right|^2 S_\omega(f)$. To find $\left|\tilde{K}(f)\right|^2$ we feed $\omega(t') = exp[i2\pi f t']$ into our system. We find, using eqn (5.109) and (5.110), $\Phi_\tau(t) = \frac{exp[i2\pi f(t+\tau)]\sqrt{2}(cos2\pi f\tau - 1)}{\bar{\omega}\tau i2\pi f}$, whose modulus square gives $\left|\tilde{K}(f)\right|^2 = \frac{2}{\bar{\omega}^2}\left[\frac{cos2\pi f\tau - 1}{2\pi f\tau}\right]^2$. Thus we arrive at eqn (5.111)]

(b) Using the expression for $S_{\Phi_\tau}(f)$ obtained in the previous part, and making the change of variable $z \equiv 2\pi f\tau$ in the integral, one finds

$$\sigma_\tau = \left[\alpha \frac{S_\omega(1/2\tau)}{\bar{\omega}^2}\frac{1}{2\tau}\right]^{1/2}$$

$$where\ \alpha = \int_0^{+\infty} dz \frac{2}{\pi}\left[\frac{cosz - 1}{z}\right]^2 \frac{S_\omega(z/2\pi\tau)}{S_\omega(1/2\tau)}$$

As one can verify, $\frac{2}{\pi}\left[\frac{cosz-1}{z}\right]^2$ integrates to one, and has a profile shown in

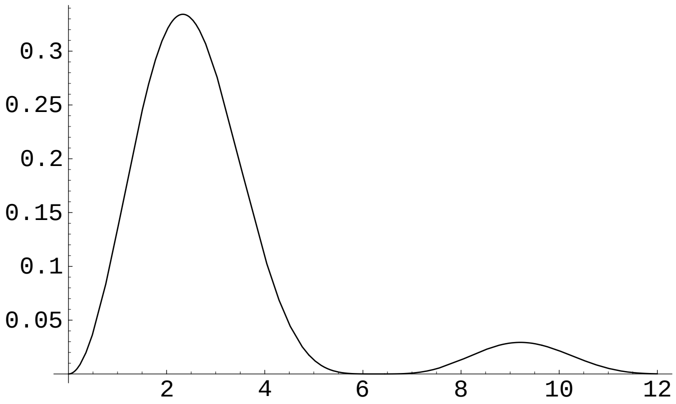

Figure 1: Ex. 5.4 (b)

Fig.1, which isn't too much different from a delta function located at $z = 2.4$ (which isn't too much away from $z = \pi$). Thus we see α is a dimensionless number of order unifty and has very weak τ-dependence which we can ignore.

(c) If ω has a white-noise spectrum $S_\omega(1/2\tau) \sim const$, then $\sigma_\tau \propto 1/\sqrt{\tau}$, and the clock stability is better for long averaging time; if ω has a flicker-noise spectrum $S_\omega(1/2\tau) \propto \tau$, then σ_τ is independent of averaging time; if ω has a random-walk spectrum $S_\omega(1/2\tau) \propto \tau^2$, then $\sigma_\tau \propto \sqrt{\tau}$, and the clock stability is better for short averaging time.

Ex. 5.5 Cosmological Density Fluctuations [by Roger Blandford]

(a) It's quite straightforward to show this,

$$\begin{aligned}
\xi(\mathbf{r}) &= <\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})> = lim_{V\to\infty}\frac{1}{V}\int_V d\mathbf{x}\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r}) \\
&= lim_{V\to\infty}\frac{1}{V}\int_V d\mathbf{x}\frac{1}{(2\pi)^3}\int d\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{x}}\tilde{\delta}_V(\mathbf{k})\frac{1}{(2\pi)^3}\int d\mathbf{k}'e^{-i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{r})}\tilde{\delta}_V(\mathbf{k}') \\
& \quad [performing\ the\ integral\ in\ \mathbf{x}\ gives\ a\ delta\ function\ in\ (\mathbf{k}+\mathbf{k}')] \\
&= lim_{V\to\infty}\frac{1}{V}\frac{1}{(2\pi)^3}\int d\mathbf{k}\ e^{i\mathbf{k}\cdot\mathbf{r}}\tilde{\delta}_V(\mathbf{k})\tilde{\delta}_V(-\mathbf{k}) \\
& \quad [\delta(\mathbf{x})\ is\ real \Rightarrow \tilde{\delta}_V(-\mathbf{k}) = \tilde{\delta}^*_V(\mathbf{k});\ also\ let\ \mathbf{k}\to-\mathbf{k}\ in\ the\ integral] \\
&= \int\frac{d\mathbf{k}}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{r}}lim_{V\to\infty}\frac{|\tilde{\delta}_V(\mathbf{k})|^2}{V} = \int\frac{d\mathbf{k}}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{r}}P(\mathbf{k})
\end{aligned}$$

The universe is isotropic, namely, $\delta(\mathbf{x}) = \delta(|\mathbf{x}|) \Rightarrow \tilde{\delta}_V(\mathbf{k}) = \tilde{\delta}_V(k) \Rightarrow P(\mathbf{k}) = P(k)$. And we can perform the momentum integral in spherical coordinates. Using the fact $\int_0^\pi d\theta sin\theta exp(-ikrcos\theta) = 2sinc(kr)$, one finds

$$\xi(\mathbf{r}) = \int_0^\infty\frac{dk}{2\pi^2}k^2sinc(kr)P(k)$$

(b) The mass measured within a sphere of radius R is given by

$$\begin{aligned}
\delta_R(\mathbf{x}) &= \frac{3}{4\pi R^3}\int_{r<R}d\mathbf{r}\delta(\mathbf{x}+\mathbf{r}) \\
&= \int_V d\mathbf{r}\delta(\mathbf{x}+\mathbf{r})K(\mathbf{r}) \\
where\ K(\mathbf{r}) &= \frac{3}{4\pi R^3}\ for\ \ r<R,\ and\ vanishes\ elsewhere.
\end{aligned}$$

One can regard $K(\mathbf{r})$ as a filter.

By derivation similar to that in the previous part, one finds the variance

$$\sigma^2 = <\delta_R^2(\mathbf{x})> = lim_{V\to\infty}\int\frac{1}{V}\frac{d\mathbf{k}}{(2\pi)^3}|\tilde{\delta}_R(\mathbf{k})|^2$$

By Parseval's theorem, $|\tilde{\delta}_R(\mathbf{k})|^2 = |\tilde{\delta}_V(\mathbf{k})|^2|\tilde{K}(\mathbf{k})|^2$. Using spherical coordinates, one finds $\tilde{K}(\mathbf{k}) = W(kR)$, where $W(x) \equiv \frac{3(sincx-cosx)}{x^2}$. Combining the above results, we get

$$\sigma^2 = \int_0^\infty\frac{dk}{2\pi^2}k^2P(k)W^2(kR)$$

3.

Ex. 5.7 Noise in a L-C-R Circuit [by Alexei Dvoretskii]

(a) $V_{\alpha\beta} = R\dot{q} - F'$, where $\dot{q}$ is the current in the circuit and q is the charge on the capacitor. When we disconnect the resistor R from the rest of the circuit, $\dot{q} = 0$, and thus the spectral density of $V_{\alpha\beta}$ is the same as that of F', namely $S_{\alpha\beta}(f) = 4RkT$.

(b) Now let's place the resistor back into the circuit, and $\dot{q}$ is no longer zero. Using $L\ddot{q} + C^{-1}q = -R\dot{q} + F'$ we can find the Fourier transform of q (and thus that of $V_{\alpha\beta}$) in terms of the Fourier transform of F'. So we get

$$S_{\alpha\beta}(f) = \left| \frac{2\pi i f R}{C^{-1} - L(2\pi f)^2 + 2\pi i f R} - 1 \right|^2 S_{F'}(f)$$

$$[using\ S_{F'}(f) = 4RkT,\ (2\pi f_0)^2 = \frac{1}{LC}]$$

$$\Rightarrow S_{\alpha\beta}(f) = 4RkT \left\{ 1 - \frac{(2\pi f)^2 R^2}{L^2 \left[(2\pi f_0)^2 - (2\pi f)^2\right]^2 + (2\pi f)^2 R^2} \right\}$$

(c) $V_{\alpha\gamma} = \frac{q}{C}$, thus

$$S_{\alpha\gamma}(f) = \frac{1}{C^2} \frac{1}{\left|C^{-1} - L(2\pi f)^2 + 2\pi i f R\right|^2} 4RkT$$

(d) $V_{\beta\gamma} = -L\ddot{q}$, thus

$$S_{\beta\gamma}(f) = L^2 (2\pi f)^4 \frac{1}{\left|C^{-1} - L(2\pi f)^2 + 2\pi i f R\right|^2} 4RkT$$

(e) Similar to Ex 5.3, the number U is given by

$$U = \int_{-\infty}^{+\infty} K(t) V_{\alpha\beta}(t) dt,$$

$$with\ K(t) = \frac{1}{\tau}\ for\ -\tau < t < 0,\ and\ 0\ elsewhere$$

We also introduce the random process $U(t) \equiv \int_{-\infty}^{+\infty} K(t-t') V_{\alpha\beta}(t') dt'$. Then easily seen $\bar{U} = 0$, and

$$(\Delta U)^2 = \overline{U(t)^2} = \int_0^\infty \left| \tilde{K}(f) \right|^2 S_{\alpha\beta}(f) df$$

Using the fact that $\left|\tilde{K}(f)\right|^2 = \left[\frac{sin(\pi f \tau)}{\pi f \tau}\right]^2$, [Again, just like in Ex. 5.4, an alternative way of finding $\left|\tilde{K}(f)\right|^2$ is: note $U(t) = \frac{1}{\tau}\int_t^{t+\tau} V_{\alpha\beta}(t') dt'$, thus $\left|\tilde{K}(f)\right|^2 = \left|\frac{1}{\tau}\int_t^{t+\tau} exp[i2\pi f t'] dt'\right|^2 = \left[\frac{sin(\pi f \tau)}{\pi f \tau}\right]^2$] and plugging in the expression for $S_{\alpha\beta}(f)$ we found in part (b), we get (define $x \equiv \pi f \tau$)

$$
\begin{aligned}
(\Delta U)^2 &= \frac{4RkT}{\pi\tau}\left\{\int_0^\infty \frac{sin^2 x}{x^2}dx - \int_0^\infty \frac{sin^2 x}{\left[x^2-(\pi\tau f_0)^2\right]^2\left(\frac{2L}{\tau R}\right)^2 + x^2}dx\right\} \\
&= \frac{2RkT}{\tau}\left\{1 - \frac{RC}{\tau}\left[1 - e^{-\frac{R\tau}{2L}} cos\left(\tau\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\right)\right]\right\}
\end{aligned}
$$

Note in the above expression $\frac{RC}{\tau} = \frac{1}{2Q\pi\tau f_0}$, where $Q \equiv \frac{2\pi f_0 L}{R}$ is the quality factor of the circuit. In our case, $Q >> 1$, and $\pi\tau f_0 >> 1$, thus $\frac{RC}{\tau} << 1$. So approximately $(\Delta U)^2 \approx \frac{2RkT}{\tau}$.

4.

Ex. 5.8 Thermal Noise in a Sapphire Crystal [by Alexei Dvoretskii]

(a) Assume we are in the classical regime $kT >> hf$. Then $S_{F'}(f) = 4RkT$, with $R = ReZ = Re\left(\frac{\tilde{F}'(f)}{-i2\pi f\tilde{x}(f)}\right) = \frac{2m}{\tau_*}$. Thus $S_{F'}(f) = \frac{8mkT}{\tau_*}$. And

$$
\begin{aligned}
S_x(f) &= \frac{S_{F'}(f)}{\left|m(\omega^2 - (2\pi f)^2) + \frac{4\pi i f m}{\tau_*}\right|^2} \\
&= \frac{\frac{8mkT}{\tau_*}}{\left|m(\omega^2 - (2\pi f)^2) + \frac{4\pi i f m}{\tau_*}\right|^2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
& m(\ddot{x} + \frac{2}{\tau_*}\dot{x} + \omega^2 x) = Re\left(\sqrt{2}F_s e^{i\omega t}\right) \\
& let\ x = Re\left(x_0 e^{i\omega t}\right) \\
& we\ get\ x_0 = \frac{-i\tau_*}{\sqrt{2}m\omega}F_s \\
& thus\ x(t) = \frac{\tau_* F_s}{\sqrt{2}m\omega} sin\omega t
\end{aligned}
$$

(c) After filtering, signal $x(t) = \sqrt{2}x_s sin(\omega t)$ with $x_s = \left|\tilde{K}(\omega/2\pi)\right| \frac{\tau_* F_s}{2m\omega}$; noise $\sigma_x = \left|\tilde{K}(\omega/2\pi)\right| \sqrt{S_x(f)\Delta f}$, where $S_x(f = \omega/2\pi) = \frac{kT\tau_*}{2\pi^2 f^2 m}$. So we see, if $x_s = \sigma_x$, then $F_s = \frac{2m\omega}{\tau_*}\sqrt{\frac{kT\tau_*\Delta f}{2\pi^2 f^2 m}} = \left(\frac{8mkT}{\tau_*\hat{\tau}}\right)^{1/2}$.

(d)

$$
h_s \sim \frac{2}{m\omega^2 l}\left(\frac{2mkT}{\tau_*\hat{\tau}}\right)^{1/2} = 2\sqrt{2}\left(\frac{kT}{ml^2\omega^2}\right)^{1/2}\frac{1}{(\omega\hat{\tau})^{1/2}(\pi Q)^{1/2}} \sim 1.5\times 10^{-26}
$$

Ex. 5.9 Solution of Fokker-Planck Equation for an Oscillator [by Alexander Putilin]

(a) Equation of motion: $\ddot{x}+\frac{2}{\tau_*}\dot{x}+\omega^2 x = F'/m$. Substitute: $x = X_1(t)cos\omega t + X_2(t)sin\omega t$ into this e.o.m., we get

$$cos\omega t\left(\ddot{X}_1 + \frac{2}{\tau_*}\dot{X}_1 + 2\omega\dot{X}_2 + \frac{2\omega}{\tau_*}X_2\right) + sin\omega t\left(\ddot{X}_2 + \frac{2}{\tau_*}\dot{X}_2 - 2\omega\dot{X}_1 - \frac{2\omega}{\tau_*}X_1\right) = \frac{F'}{m}$$

$\{X_1, X_2\}$ evolve on a time scale τ_* means that $\dot{X}_i \sim \frac{X_i}{\tau_*}$, $\ddot{X}_i \sim \frac{X_i}{\tau_*^2}$, with $i = 1, 2$. And since $\tau_* >> \omega^{-1}$, we can neglect the first two terms in the brackets. Thus the Langevin equation reduces to:

$$-2\omega(\dot{X}_1 + X_1/\tau_*)sin\omega t + 2\omega(\dot{X}_2 + X_2/\tau_*)cos\omega t = F'/m$$

(b) Multiply the Langevin equation by $cos\omega t$ and integrate from $t = 0$ to $t = \Delta t$ where $\omega^{-1} << \Delta t << \tau_*$:

$$2\omega\int_0^{\Delta t}\left(\dot{X}_2 + \frac{1}{\tau_*}X_2\right)cos^2\omega t dt - 2\omega\int_0^{\Delta t}\left(\dot{X}_1 + \frac{2}{\tau_*}X_1\right)cos\omega t sin\omega t dt$$
$$= \frac{1}{m}\int_0^{\Delta t} F'(t)cos\omega t dt$$

$cos^2\omega t$ and $cos\omega t sin\omega t$ are fast oscillating functions, so we use their average value inside the integrals, i.e. change $cos^2\omega t \to \overline{cos^2\omega t} = \frac{1}{2}$, $cos\omega t sin\omega t \to \overline{cos\omega t sin\omega t} = 0$. Thus we get

$$\omega\int_0^{\Delta t}\left(\dot{X}_2 + \frac{1}{\tau_*}X_2\right)dt \approx \omega\left(\Delta X_2 + \frac{1}{\tau_*}X_2\Delta t\right) = \frac{1}{m}\int_0^{\Delta t} F'(t)cos\omega t dt$$
$$i.e.\ \Delta X_2 = -\frac{1}{\tau_*}X_2\Delta t + \frac{1}{m\omega}\int_0^{\Delta t} F'(t)cos\omega t dt$$

Similarly, multiplying the Langevin equation by $sin\omega t$ and integrating, we get

$$\Delta X_1 = -\frac{1}{\tau_*}X_1\Delta t - \frac{1}{m\omega}\int_0^{\Delta t} F'(t)sin\omega t dt$$

(c) Taking ensemble average:

$$\overline{\Delta X_1} = \frac{-1}{\tau_*}X_1\Delta t - \frac{1}{m\omega}\int_0^{\Delta t}\overline{F'(t)}sin\omega t dt$$
$$\overline{F'(t)} = 0 \Rightarrow A_1 = \frac{\overline{\Delta X_1}}{\Delta t} = \frac{-X_1}{\tau_*}$$
$$\text{similarly we have } A_2 = \frac{\overline{\Delta X_2}}{\Delta t} = \frac{-X_2}{\tau_*}$$
$$\text{thus } A_j = \frac{-X_j}{\tau_*}$$

$$\text{now consider } B_{jk} = \frac{\overline{\Delta X_j \Delta X_k}}{\Delta t}$$

$$\begin{aligned}
B_{11} = \frac{\overline{(\Delta X_1)^2}}{\Delta t} &= \frac{1}{m^2\omega^2\Delta t}\int_0^{\Delta t} dt_1 dt_2 sin\omega t_1 sin\omega t_2 < F'(t_1)F'(t_2) > +O\left[(\Delta t)\right] \\
&= \frac{1}{m^2\omega^2\Delta t}\int_0^{\Delta t} dt_1 dt_2 sin\omega t_1 sin\omega t_2 C_{F'}(t_1 - t_2) \\
&\quad \text{using Wiener-Khintchine theorem} \\
&= \frac{1}{m^2\omega^2\Delta t}\int_0^{\Delta t} dt_1 dt_2 sin\omega t_1 sin\omega t_2 \int_0^{+\infty} S_{F'}(f) cos[2\pi f(t_1 - t_2)] df \\
&\quad \text{Fluctuation-Dissipation theorem gives } S_{F'}(f) = 4RkT = \frac{8kTm}{\tau_*} \\
&= \frac{8kT}{m\omega^2\tau_*\Delta t}\int_0^{\Delta t} dt_1 dt_2 sin\omega t_1 sin\omega t_2 \int_0^{+\infty} cos[2\pi f(t_1 - t_2)] df \\
&\quad \text{using } \int_0^{+\infty} cos[2\pi f(t_1 - t_2)] df = \frac{1}{2}\delta(t_1 - t_2) \\
&= \frac{4kT}{m\omega^2\tau_*\Delta t}\int_0^{\Delta t} dt_1 sin^2\omega t_1 \approx \frac{4kT}{m\omega^2\tau_*\Delta t}\cdot\frac{\Delta t}{2} = \frac{2kT}{m\omega^2\tau_*}
\end{aligned}$$

Similar calculation gives:

$$\begin{aligned}
B_{22} &= \frac{4kT}{m\omega^2\tau_*\Delta t}\int_0^{\Delta t} dt cos^2\omega t \approx \frac{2kT}{m\omega^2\tau_*} \\
B_{12} = B_{21} &= \frac{4kT}{m\omega^2\tau_*\Delta t}\int_0^{\Delta t} dt cos\omega t sin\omega t \approx 0
\end{aligned}$$

Thus we conclude

$$B_{jk} = \frac{2kT}{m\omega^2\tau_*}\delta_{jk}$$

(d) After plugging the expressions for A_j and B_{jk} into the Fokker-Planck equation for $P_2 = P_2(X_j, t|X_j^{(0)})$, we get

$$\frac{\partial}{\partial t}P_2 = \frac{1}{\tau_*}\frac{\partial}{\partial X_j}(X_j P_2) + \frac{kT}{m\omega^2\tau_*}\nabla^2 P_2$$

If you don't have so much time, you can just verify that Eq. (5.171) is a solution of this equation, and that it has the required form at $t = 0$: $P_2(\mathbf{X}, 0) = \delta(\mathbf{X} - \mathbf{X}^{(0)})$. If you do have some time, you could derive the solution as follows.

Make Fourier transform:

$$P_2(\mathbf{X}, t) = \int \frac{d\mathbf{K}}{(2\pi)^2} e^{-i\mathbf{K}\cdot\mathbf{X}}\tilde{P}_2(\mathbf{K}, t)$$

The Fokker-Planck equation becomes

$$\frac{\partial}{\partial t}\tilde{P}_2 = \frac{-1}{\tau_*}K_j\frac{\partial}{\partial K_j}\tilde{P}_2 - \frac{kT}{m\omega^2\tau_*}\mathbf{K}^2\tilde{P}_2$$

Make a change of variable, $\mathbf{K} \to \mathbf{K}' \equiv e^{-t/\tau_*}\mathbf{K}$. Then

$$K'_j\frac{\partial}{\partial K'_j} = K_j\frac{\partial}{\partial K_j}, \quad \frac{\partial}{\partial t}|_{\mathbf{K}'} = \frac{\partial}{\partial t}|_{\mathbf{K}} + \frac{1}{\tau_*}K'_j\frac{\partial}{\partial K'_j}$$

And the F-P equation reduces to

$$\frac{\partial}{\partial t}\tilde{P}_2(\mathbf{K}',t) = -\frac{kT}{m\omega^2\tau_*}\mathbf{K}'^2 e^{2t/\tau_*}\tilde{P}_2(\mathbf{K}',t)$$

which is easily integrated to give

$$\tilde{P}_2(\mathbf{K}',t) = \tilde{P}_2(\mathbf{K}',0)exp\left[-\frac{kT}{m\omega^2\tau_*}\mathbf{K}'^2\int_0^t dt' e^{2t'/\tau_*}\right]$$

namely, $\tilde{P}_2(\mathbf{K},t) = \tilde{P}_2(e^{-t/\tau_*}\mathbf{K},0)exp\left[-\frac{kT}{2m\omega^2}\left(1-e^{-2t/\tau_*}\right)\mathbf{K}^2\right]$

Using the initial condition $P_2(\mathbf{X},0) = \delta(\mathbf{X}-\mathbf{X}^{(0)})$, i.e. $\tilde{P}_2(\mathbf{K}',0) = e^{i\mathbf{K}'\cdot\mathbf{X}^{(0)}}$, we finally find

$$\tilde{P}_2(\mathbf{K},t) = e^{-\frac{1}{2}\sigma^2\mathbf{K}^2 + i\mathbf{K}\cdot\bar{\mathbf{X}}}$$

where $\sigma^2 \equiv \frac{kT}{m\omega^2}\left(1-e^{-2t/\tau_*}\right)$ and $\bar{\mathbf{X}} \equiv \mathbf{X}^{(0)}e^{-t/\tau_*}$

Making an inverse Fourier transform (which is easy because $\tilde{P}_2(\mathbf{K},t)$ is Gaussian), we find

$$P_2(\mathbf{X},t) = \frac{1}{2\pi\sigma^2}exp\left[-\frac{\left|\mathbf{X}-\bar{\mathbf{X}}\right|^2}{2\sigma^2}\right]$$

(e) At small times $t << \tau_*$, the probability distribution $P_2(\mathbf{X},t)$ is concentrated around the initial value $\mathbf{X} = \mathbf{X}^{(0)}$. The variance of the distribution $\sqrt{\sigma^2} \propto \sqrt{t}$, which is the usual relation for random-walk processes. When t becomes large ($t >> \tau_*$), the system "forgets" about its initial configuration and evolves into the thermal equilibrium state with $\bar{\mathbf{X}} = 0$ and $\sigma^2 = \frac{kT}{m\omega^2}$.

When the signal acts for a short time $\hat{\tau} << \tau_*$, the thermal noise is $\overline{\sigma^2} = \frac{1}{\hat{\tau}}\int_0^{\hat{\tau}} dt\sigma^2(t) = \frac{kT}{m\omega^2}\frac{\hat{\tau}}{\tau_*}$, while the amplitude of the signal is $x_s = \frac{\tau_* F_s}{2m\omega}$, where F_s is the strength of the signal force (see part (b) of the previous exercise). Then the minimal detectable force is given by the relation $x_s = \sqrt{\overline{\sigma^2}}$, namely, $F_s = \sqrt{\frac{4mkT\hat{\tau}}{\tau_*^3}}$. In the opposite limit $\hat{\tau} >> \tau_*$ (Ex 5.8), $F_s = \sqrt{\frac{8mkT}{\tau_*\hat{\tau}}}$.