Solution for Chapter 3

(compiled by Xinkai Wu, revised by Kip Thorne)

1.
Ex. 3.1 Canonical Transformation [by Xinkai Wu]
(a) With the generating function given by eqn. (3.15)

$$p_j = \sum_{i=1}^{W} \frac{\partial f_i}{\partial q^j} P_i, \quad Q^j = f_j$$

(b) Let's first show the useful identity:

$$[Q, P]_{q,p} \equiv \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 1$$

(which says canonical transformations preserve the Poisson bracket). The proof of this identity is

$$\begin{aligned}
\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} &= \left[\left(\frac{\partial Q}{\partial q}\right)_P + \left(\frac{\partial Q}{\partial P}\right)_q \frac{\partial P}{\partial q}\right]\frac{\partial P}{\partial p} \\
&\quad -\frac{\partial P}{\partial q}\left[\left(\frac{\partial Q}{\partial P}\right)_q \frac{\partial P}{\partial p}\right] \\
&= \left(\frac{\partial Q}{\partial q}\right)_P \frac{\partial P}{\partial p} \\
&= \frac{\partial^2 F}{\partial P \partial q}\frac{\partial P}{\partial p} \quad (using\ Q = \left(\frac{\partial F}{\partial P}\right)_q) \\
by\ differentiating\ both\ sides\ of \quad & p = \left(\frac{\partial F}{\partial q}\right)_P \quad w.r.t.\ p,\ we\ get \\
&= 1
\end{aligned}$$

Then regarding (Q, P) as functions of (q, p), noting $H(Q, P) = H(q, p)$, using the Hamilton's equations for the old variables (q, p), and using chain rules, one readily find

$$\begin{aligned}
\frac{dQ}{dt} &= \frac{\partial H}{\partial P}[Q, P]_{q,p} = \frac{\partial H}{\partial P} \\
\frac{dP}{dt} &= -\frac{\partial H}{\partial Q}[Q, P]_{q,p} = -\frac{\partial H}{\partial Q}
\end{aligned}$$

namely, canonical transformations preserve the form of Hamilton's equations.
(c) $dPdQ = |J|dpdq$, where $J = det\left(\frac{\partial(P,Q)}{\partial(p,q)}\right)$ is the Jocobian of the canonical transformation. Easily seen $J = [Q, P]_{q,p} = 1$. Thus $dpdq = dPdQ$.

(d). Consider the "vector field" $(0, p)$ in phase space. Using Stokes theorem, we find

$$\oint pdq = \oint (0,p)\cdot(dp,dq) = \iint \left(\frac{\partial p}{\partial p}\right) dpdq = \iint dpdq$$

and similarly $\oint PdQ = \iint dPdQ$. We've already shown in part (c) that $dpdq = dPdQ$, thus we conclude $\oint pdq = \oint PdQ$.
(e) $d^3q = dxdydz = r^2 sin\theta drd\theta d\phi \neq d^3Q = drd\theta d\phi$; while $d^3p = dp_x dp_y dp_z = dp_r dp_\theta dp_\phi = \frac{1}{r^2 sin\theta} dP_1 dP_2 dP_3 = \frac{1}{r^2 sin\theta} d^3P \neq d^3P$(recall that $P_1 = p_r$, $P_2 = rp_\theta$, $P_3 = rsin\theta p_\phi$). And we see that $d^3qd^3p = d^3Qd^3P$.

Ex. 3.3 Estimating Entropy [by Alexei Dvoretskii]
Let's express the answers in units of the Boltzmann constant k
(a) The electron's energy levels are given by $E_n = -13.6eV/n^2$, with degeneracy $g_n = 2n^2$. The entropy of the electron is given by $S = -k\sum_n g_n \rho_n ln\rho_n$, where $\rho_n = exp(-E_n/kT)/Z$(the ensemble of electrons only exchanges energy with the bath and is a canonical ensemble). At room temperature, $kT \approx 0.025eV$ gives $\rho_n = exp(544/n^2)/Z$. Thus all ρ_n's with $n > 1$ are neglegible compared with ρ_1, and we have $\rho_1 \approx 1/2$, and $S \approx -kg_1\rho_1 ln\rho_1 = kln2 = 0.7k$
(b) The number of states available to each molecule can be estimated as $\Gamma \approx \frac{\Delta V(\Delta P)^3}{h^3} \approx \frac{\frac{1}{n}(mkT)^{3/2}}{h^3}$, with n being the number density of the molecules. For wine(basically water), $n = \rho_{H_2O}/m_{H_2O} \approx 3\times 10^{28}m^{-3}$; at room temperature $(mkT)^{3/2} \approx 10^{-69}kg^3m^3/s^3$. Thus $\Gamma \approx 100$. So we find the entropy per molecule is $S/Nk \approx ln\Gamma \approx 5$. A glass of wine(roughly $200g$) has $N \approx 6\times 10^{24}$, thus its entropy is $S \approx 3\times 10^{25}k$.
(c) Similar to (b), we only need to scale the answer of (b) by the ratio of the volume of the Pacific ocean and the volume of a glass of water. The average depth of the Pacific ocean is roughly 10^3m, its area is about $\frac{1}{3}4\pi R_{earth}^2 \approx 10^{14}m^2$, giving a volume $V_{pacific} \approx 10^{17}m^3$, while $V_{glass} \approx 2\times 10^{-4}m^3$, thus $V_{pacific}/V_{glass} \approx 5\times 10^{20}$. So $S_{pacific} \approx 5\times 10^{20}\cdot 3\times 10^{25}k \approx 10^{46}k$.
(d) Let's use the Debye model of solids. The Debye sphere will contain $3N$ vibrational modes($2N$ transversal and N longitudinal) and therefore the entropy of the ice is (using additivity of entropy) $S = 3N\bar{S}_{mode}$, where $\bar{S}_{mode}$ is the average entropy per mode. $\bar{S}_{mode}$ can be calculated using the Debye mode density spectrum $D(\nu) = \frac{3\nu^2}{\nu_D^3}d\nu$ and the expression for the entropy of a bosonic mode $S = k[(\eta+1)ln(\eta+1) - \eta ln\eta]$, with $\eta = \frac{1}{e^{h\nu/kT}-1}$. Let's estimate the Debye temperature.
$\frac{\hbar\nu_D}{k} = \theta_D = \frac{\hbar c_s}{k}(6\pi^2 n)^{1/3} \approx 170K$, where $c_s \sim 2\times 10^3 m/s$ is the sound speed in ice. $\frac{\theta_D}{T} \approx \frac{170K}{276K} \approx 0.6 < 1$, so we can make a rough estimate $\bar{S}_{mode} \approx kln\eta \approx kln\frac{kT}{\hbar\bar{\nu}}$, where $\bar{\nu}$ is a properly averaged frequency. Take roughly $\bar{\nu} = \frac{1}{2}\nu_D$, then $\bar{S}_{mode} \approx kln3 \approx k$. Thus $S \approx 3Nk$. Take the ice cube to be $2cm$ on each side, we find $N \approx 3\times 10^{23}$ and $S \approx 1\times 10^{24}k$.
(e) The main contribution to the universe's entropy is from the microwave background radiation because there are $\sim 10^9$ times as many of them as protons, neutrons, or electrons(radiation-dominated universe). The theory of the big

bang suggests there should be a comparable amount of neutrinos, but the big bang neutrinos haven't not yet been detected.
Due to the planck exponential cut-off, only the photon modes with $\hbar\omega \sim kT$ ($T \sim 3K$) will be excited. The number of such modes in the universe is roughly $N = D(\omega)\frac{kT}{\hbar}V_{universe}$, where $D(\omega) = \frac{\omega^3}{\pi^2 c^3}$ is the density of modes. Thus $N \approx (\frac{kT}{\hbar c})^3 \frac{1}{\pi^2} V_{universe}$. Taking $V_{universe} \approx \frac{4}{3}\pi R^3$ with $R \approx 10^{10} light\ years$, we find $N \approx 10^{69}$. Each excited mode will on average have a few photons, and each photon will have an entropy $\approx 3.6k$(see. e.g. Cosmological Physics by Peacock). Therefore $S_{universe} \approx 10^{69} \sim 10^{70}k$.

2.
Ex 3.2 Derivation of the Bose-Einstein and Fermi-Dirac Distribution [by Xinkai Wu]

$$\rho_n = const \times exp\left(\frac{\tilde{\mu}n - \tilde{E}_n}{kT}\right) = const \times exp\left(\frac{(\tilde{\mu} - \tilde{E}_s)n}{kT}\right)$$

(a) For a fermion mode,

$$1 = \rho_0 + \rho_1 = const \times \left[1 + exp\left(\frac{\tilde{\mu} - \tilde{E}_s}{kT}\right)\right]$$
$$\Rightarrow const = \frac{1}{1 + exp\left(\frac{\tilde{\mu}-\tilde{E}_s}{kT}\right)}$$

Thus we get

$$\rho_0 = \frac{1}{1 + exp\left(\frac{\tilde{\mu}-\tilde{E}_s}{kT}\right)}, \quad \rho_1 = \frac{exp\left(\frac{\tilde{\mu}-\tilde{E}_s}{kT}\right)}{1 + exp\left(\frac{\tilde{\mu}-\tilde{E}_s}{kT}\right)}$$

And

$$\eta \equiv < n > = \sum_n n\rho_n = 0 \cdot \rho_0 + 1 \cdot \rho_1 = \frac{1}{exp\left(\frac{\tilde{E}_s-\tilde{\mu}}{kT}\right) + 1}$$

(b) For a boson mode,

$$1 = \sum_n \rho_n = const \times \left[\sum_{n=0}^{+\infty} exp\left(\frac{(\tilde{\mu} - \tilde{E}_s)n}{kT}\right)\right]$$
$$= \frac{const}{1 - exp\left(\frac{\tilde{\mu}-\tilde{E}_s}{kT}\right)}$$
$$\Rightarrow const = 1 - exp\left(\frac{\tilde{\mu} - \tilde{E}_s}{kT}\right)$$

Thus

$$\rho_n = \left[1 - exp\left(\frac{\tilde{\mu} - \tilde{E}_s}{kT}\right)\right] exp\left(\frac{n(\tilde{\mu} - \tilde{E}_s)}{kT}\right)$$

And

$$\eta \equiv < n >= \sum_n n\rho_n = \left[1 - exp\left(\frac{\tilde{\mu} - \tilde{E}_s}{kT}\right)\right] \sum_{n=0}^{+\infty} nexp\left[\frac{n(\tilde{\mu} - \tilde{E}_s)}{kT}\right]$$

$$using\ the\ formula\ \sum_{n=0}^{+\infty} ne^{na} = \frac{e^a}{(1 - e^a)^2}$$

$$\Rightarrow \eta = \frac{1}{exp\left(\frac{\tilde{E}_s - \tilde{\mu}}{kT}\right) - 1}$$

3

Ex. 3.4 Additivity of Entropy for Statistically Independent Systems [by Alexei Dvoretskii]

$$S = -k\int \rho ln\rho d\Gamma$$

$$using\ \rho = \prod_a \rho_a$$

$$= -k\int \left(\prod_a \rho_a\right) \sum_b ln\rho_b \left(\prod_a d\Gamma_a\right)$$

$$for\ each\ term\ in\ the\ sum\ let's\ integrate\ out\ all\ the\ other\ subsytems$$

$$= -k\sum_\beta \int \rho_\beta ln\rho_\beta d\Gamma_\beta \cdot \prod_{\alpha\neq\beta} \int \rho_\alpha d\Gamma_\alpha$$

$$but\ \int \rho_\alpha d\Gamma_\alpha = 1,\ so$$

$$= -k\sum_\beta \int \rho_\beta ln\rho_\beta d\Gamma_\beta = \sum_\beta S_\beta$$

4.

Ex. 3.7 Probability Distribution for the Number of Particles in a Cell.[by Alexei Dvoretskii]

(a) For the ergodic hypothesis to hold, we need the measurements to be separated from one another by time intervals τ such that $\tau >> \tau_{ext}$, where τ_{ext} is the characteristic time for the system to exchange particles with the bath.

(b) Let's denote the index n as (N, s), where N labels the number of particles and s labels the state with a given N. The grand partition function is

$$
\begin{aligned}
Z &= \sum_{N,s} exp\left(\frac{-\tilde{E}_{N,s} + \tilde{\mu}N}{kT}\right) = \sum_{N=0}^{+\infty} e^{\tilde{\mu}N/kT} Z_N \\
&where\ Z_N \equiv \sum_s e^{\frac{-\tilde{E}_{N,s}}{kT}}
\end{aligned}
$$

For the system discussed in Ex. 3.6,

$$
Z_N = \frac{a(V,T)^N}{N!},\ with\ a(V,T) = \frac{V}{h^3}\int exp\left(-\frac{(p^2+m^2)^{1/2}}{kT}\right) 4\pi p^2 dp
$$

For Z_N of the above form, we find,

$$
Z = \sum_{N=0}^{+\infty} e^{\tilde{\mu}N/kT} \frac{a(V,T)^N}{N!} = exp\left[a(V,T)e^{\tilde{\mu}/kT}\right]
$$

And thus the mean number of particles is given by (see eqn. (3.70))

$$
\bar{N} = kT\left(\frac{\partial lnZ}{\partial \tilde{\mu}}\right)_{V,T} = a(V,T)e^{\tilde{\mu}/kT}
$$

(note this gives $Z = e^{\bar{N}}$)
Now

$$
\begin{aligned}
p_N &= \sum_s \rho_{N,s} = \sum_s \frac{1}{Z} exp\left(\frac{-\tilde{E}_{N,s} + \tilde{\mu}N}{kT}\right) = \frac{e^{\tilde{\mu}N/kT}}{Z} Z_N \\
&= \frac{e^{\tilde{\mu}N/kT}}{Z}\frac{a(V,T)^N}{N!} = \frac{\left[e^{\tilde{\mu}/kT}a(V,T)\right]^N}{ZN!} = e^{-\bar{N}}\frac{\bar{N}^N}{N!}
\end{aligned}
$$

(c) Those are well-known properties of Poisson distribution:

$$
\begin{aligned}
< N > &= \sum_{N=0}^{+\infty} e^{-\bar{N}}\frac{\bar{N}^N}{N!}N = \sum_{N=1}^{+\infty} e^{-\bar{N}}\frac{\bar{N}^N}{N!}N = \sum_{N=0}^{+\infty} e^{-\bar{N}}\frac{\bar{N}^{N+1}}{N!} = \bar{N} \\
< N^2 > &= \sum_{N=1}^{+\infty} e^{-\bar{N}}\frac{\bar{N}^N}{N!}N^2 = \sum_{N=0}^{+\infty} e^{-\bar{N}}\frac{\bar{N}^{N+1}}{N!}(N+1) \\
&= \sum_{N=0}^{+\infty} p_N \bar{N} N + \sum_{N=0}^{+\infty} p_N \bar{N} = (\bar{N})^2 + \bar{N}
\end{aligned}
$$

And thus

$$
\Delta N \equiv < (N - \bar{N})^2 >^{1/2} = \left(< N^2 > - (\bar{N})^2\right)^{1/2} = \bar{N}^{1/2}
$$

Ex.3.5 Entropy of Thermalized Mode of a Field [by Xinkai Wu]
(a). Recal from Ex. 3.2 that $\eta = 0 \cdot \rho_0 + 1 \cdot \rho_1 = \rho_1$, and thus $\rho_0 = 1 - \rho_1 = 1 - \eta$. Thus $S_S = -k(\rho_0 ln\rho_0 + \rho_1 ln\rho_1) = -k[\eta ln\eta + (1-\eta)ln(1-\eta)]$. In the classical regime $\eta << 1$, $S_S \approx -k[\eta ln\eta + (1-\eta)(-\eta)] \approx -k\eta(ln\eta - 1)$.
(b). From Ex. 3.2 one readily gets $\rho_n = \frac{1}{1+\eta}\left(\frac{\eta}{1+\eta}\right)^n$. And thus

$$S_S = -k\sum_n \rho_n ln\rho_n = -k\sum_n \rho_n \left[-ln(1+\eta) + nln\left(\frac{\eta}{1+\eta}\right)\right]$$
$$= kln(1+\eta) - k\eta ln\left(\frac{\eta}{1+\eta}\right) = k[(\eta+1)ln(\eta+1) - \eta ln\eta]$$

In the classical regime $\eta << 1$, $S_S \approx k[(\eta+1)\eta - \eta ln\eta] \approx -k\eta(ln\eta - 1)$.
(c) See Fig.1 and Fig. 2 (in both figures, x-axis is η and y-axis is $\sigma = S_S/\eta k$).

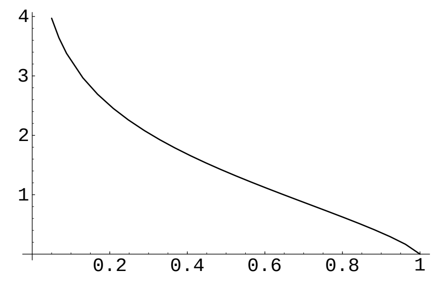

Figure 1: entropy per particle: fermion case

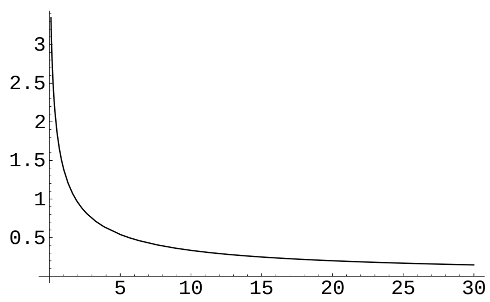

Figure 2: entropy per particle: boson case

For fermions, let $x = 1-\eta$, we get $\sigma_f = -[ln(1-x) + \frac{x}{1-x}lnx]$. In the degenerate regime, $\eta \approx 1$, namely $x \to 0$, we see that $\sigma_f \to 0$.
For bosons, let $x = 1/\eta$, we get $\sigma_b = -xlnx + (1+x)ln(1+x)$. In the classical-wave regime, $\eta >> 1$, namely $x \to 0$, we see that $\sigma_b \to 0$.