

CHAPTER 5

Section 5-1

5-1. First, $f(x,y) \geq 0$. Let R denote the range of (X,Y).

$$\text{Then, } \sum_R f(x,y) = \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} = 1$$

- 5-2 a) $P(X < 2.5, Y < 3) = f(1.5,2)+f(1,1) = 1/8+1/4=3/8$
 b) $P(X < 2.5) = f(1.5, 2) + f(1.5, 3) + f(1,1) = 1/8 + 1/4 + 1/4 = 5/8$
 c) $P(Y < 3) = f(1.5, 2) + f(1,1) = 1/8 + 1/4 = 3/8$
 d) $P(X > 1.8, Y > 4.7) = f(3, 5) = 1/8$

- 5-3. $E(X) = 1(1/4) + 1.5(3/8) + 2.5(1/4) + 3(1/8) = 1.8125$
 $E(Y) = 1(1/4) + 2(1/8) + 3(1/4) + 4(1/4) + 5(1/8) = 2.875$

5-4 a) marginal distribution of X

x	f(x)
1	1/4
1.5	3/8
2.5	1/4
3	1/8

- b) $f_{Y|1.5}(y) = \frac{f_{XY}(1.5, y)}{f_X(1.5)}$ and $f_X(1.5) = 3/8$. Then,

y	$f_{Y 1.5}(y)$
2	$(1/8)/(3/8)=1/3$
3	$(1/4)/(3/8)=2/3$

- c) $f_{X|2}(x) = \frac{f_{XY}(x,2)}{f_Y(2)}$ and $f_Y(2) = 1/8$. Then,

x	$f_{X 2}(y)$
1.5	$(1/8)/(1/8)=1$

- d) $E(Y|X=1.5) = 2(1/3) + 3(2/3) = 2 \frac{1}{3}$
 e) Since $f_{Y|1.5}(y) \neq f_Y(y)$, X and Y are not independent
- 5-5 Let R denote the range of (X,Y). Because
 $\sum_R f(x,y) = c(2+3+4+3+4+5+4+5+6) = 1$, $36c = 1$, and $c = 1/36$
- 5-6. a) $P(X = 1, Y < 4) = f_{XY}(1,1) + f_{XY}(1,2) + f_{XY}(1,3) = \frac{1}{36}(2+3+4) = 1/4$
 b) $P(X = 1)$ is the same as part a. $= 1/4$
 c) $P(Y = 2) = f_{XY}(1,2) + f_{XY}(2,2) + f_{XY}(3,2) = \frac{1}{36}(3+4+5) = 1/3$
 d) $P(X < 2, Y < 2) = f_{XY}(1,1) = \frac{1}{36}(2) = 1/18$

5-7.

$$\begin{aligned}
 E(X) &= 1[f_{XY}(1,1) + f_{XY}(1,2) + f_{XY}(1,3)] + 2[f_{XY}(2,1) + f_{XY}(2,2) + f_{XY}(2,3)] \\
 &\quad + 3[f_{XY}(3,1) + f_{XY}(3,2) + f_{XY}(3,3)] \\
 &= \left(1 \times \frac{9}{36}\right) + \left(2 \times \frac{12}{36}\right) + \left(3 \times \frac{15}{36}\right) = 13/6 = 2.167 \\
 V(X) &= \left(1 - \frac{13}{6}\right)^2 \frac{9}{36} + \left(2 - \frac{13}{6}\right)^2 \frac{12}{36} + \left(3 - \frac{13}{6}\right)^2 \frac{15}{36} = 0.639 \\
 E(Y) &= 2.167 \\
 V(Y) &= 0.639
 \end{aligned}$$

5-8 a) marginal distribution of X

x	$f_X(x) = f_{XY}(x,1) + f_{XY}(x,2) + f_{XY}(x,3)$
1	1/4
2	1/3
3	5/12

b) $f_{Y|X}(y) = \frac{f_{XY}(1,y)}{f_X(1)}$

y	$f_{Y X}(y)$
1	$(2/36)/(1/4)=2/9$
2	$(3/36)/(1/4)=1/3$
3	$(4/36)/(1/4)=4/9$

c) $f_{X|Y}(x) = \frac{f_{XY}(x,2)}{f_Y(2)}$ and $f_Y(2) = f_{XY}(1,2) + f_{XY}(2,2) + f_{XY}(3,2) = \frac{12}{36} = 1/3$

x	$f_{X Y}(x)$
1	$(3/36)/(1/3)=1/4$
2	$(4/36)/(1/3)=1/3$
3	$(5/36)/(1/3)=5/12$

d) $E(Y|X=1) = 1(2/9) + 2(1/3) + 3(4/9) = 20/9$

e) Since $f_{XY}(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

5-9. $f(x,y) \geq 0$ and $\sum_R f(x,y) = 1$

5-10. a) $P(X < 0.5, Y < 1.5) = f_{XY}(-1,-2) + f_{XY}(-0.5,-1) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

b) $P(X < 0.5) = f_{XY}(-1,-2) + f_{XY}(-0.5,-1) = \frac{3}{8}$

c) $P(Y < 1.5) = f_{XY}(-1,-2) + f_{XY}(-0.5,-1) + f_{XY}(0.5,1) = \frac{7}{8}$

d) $P(X > 0.25, Y < 4.5) = f_{XY}(0.5,1) + f_{XY}(1,2) = \frac{5}{8}$

5-11.

$$E(X) = -1\left(\frac{1}{8}\right) - 0.5\left(\frac{1}{4}\right) + 0.5\left(\frac{1}{2}\right) + 1\left(\frac{1}{8}\right) = \frac{1}{8}$$

$$E(Y) = -2\left(\frac{1}{8}\right) - 1\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{8}\right) = \frac{1}{4}$$

5-12 a) marginal distribution of X

x	$f_X(x)$
-1	1/8
-0.5	1/4
0.5	1/2
1	1/8

$$b) f_{Y|X}(y) = \frac{f_{XY}(1,y)}{f_X(1)}$$

y	$f_{Y X}(y)$
2	1/8/(1/8)=1

$$c) f_{X|Y}(x) = \frac{f_{XY}(x,1)}{f_Y(1)}$$

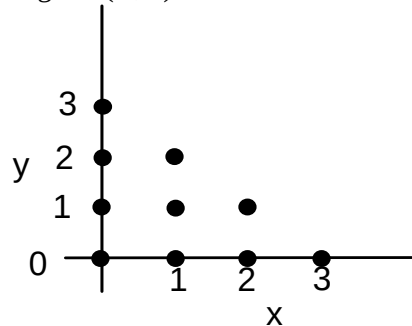
x	$f_{X Y}(x)$
0.5	1/2/(1/2)=1

$$d) E(X|Y=1) = 0.5$$

e) no, X and Y are not independent

5-13. Because X and Y denote the number of printers in each category,
 $X \geq 0$, $Y \geq 0$ and $X + Y = 4$

5-14. a) The range of (X,Y) is



The problem needs probabilities to total one. Modify so that the probability of moderate distortion is 0.04.

x,y	$f_{xy}(x,y)$
0,0	0.857375
0,1	0.1083
0,2	0.00456
0,3	0.000064
1,0	0.027075
1,1	0.00228
1,2	0.000048
2,0	0.000285
2,1	0.000012
3,0	0.000001

b)

x	$f_x(x)$
0	0.970299
1	0.029403
2	0.000297
3	0.000001

c) $E(X)=0(0.970299)+1(0.029403)+2(0.000297)+3*(0.000001)=0.03$
(or $np=3*0.01$)

d) $f_{y|1}(y) = \frac{f_{xy}(1,y)}{f_x(1)}$, $f_x(1) = 0.029403$

y	$f_{y 1}(x)$
0	0.920824
1	0.077543
2	0.001632

e) $E(Y|X=1)=0(.920824)+1(0.077543)+2(0.001632)=0.080807$

g) No, X and Y are not independent because, for example, $f_Y(0) \neq f_{Y|1}(0)$.

5-15 a) The range of (X,Y) is $X \geq 0$, $Y \geq 0$ and $X + Y \leq 4$. X is the number of pages with moderate graphic content and Y is the number of pages with high graphic output out of 4.

	x=0	x=1	x=2	x=3	x=4
y=4	5.35×10^{-05}	0	0	0	0
y=3	0.00184	0.00092	0	0	0
y=2	0.02031	0.02066	0.00499	0	0
y=1	0.08727	0.13542	0.06656	0.01035	0
y=0	0.12436	0.26181	0.19635	0.06212	0.00699

b)

	x=0	x=1	x=2	x=3	x=4
f(x)	0.2338	0.4188	0.2679	0.0725	0.0070

c)

$$E(X) = \sum_{i=0}^4 x_i f(x_i) = 0(0.2338) + 1(0.4188) + 2(0.2679) + 3(0.0725) + 4(0.0070) = 1.2$$

d) $f_{Y|3}(y) = \frac{f_{XY}(3,y)}{f_X(3)}$, $f_X(3) = 0.0725$

y	$f_{Y 3}(y)$
0	0.857
1	0.143
2	0
3	0
4	0

e) $E(Y|X=3) = 0(0.857) + 1(0.143) = 0.143$
f) $V(Y|X=3) = 0^2(0.857) + 1^2(0.143) - 0.143^2 = 0.123$
g) No, X and Y are not independent

- 5-16 a) The range of (X,Y) is $X \geq 0$, $Y \geq 0$ and $X + Y \leq 4$. X is the number of defective items found with inspection device 1 and Y is the number of defective items found with inspection device 2.

	x=0	x=1	x=2	x=3	x=4
y=0	1.94×10^{-19}	1.10×10^{-16}	2.35×10^{-14}	2.22×10^{-12}	7.88×10^{-11}
y=1	2.59×10^{-16}	1.47×10^{-13}	3.12×10^{-11}	2.95×10^{-9}	1.05×10^{-7}
y=2	1.29×10^{-13}	7.31×10^{-11}	1.56×10^{-8}	1.47×10^{-6}	5.22×10^{-5}
y=3	2.86×10^{-11}	1.62×10^{-8}	3.45×10^{-6}	3.26×10^{-4}	0.0116
y=4	2.37×10^{-9}	1.35×10^{-6}	2.86×10^{-4}	0.0271	0.961

$$f(x, y) = \left[\binom{4}{x} (0.993)^x (0.007)^{4-x} \right] \left[\binom{4}{y} (0.997)^y (0.003)^{4-y} \right]$$

For $x = 1, 2, 3, 4$ and $y = 1, 2, 3, 4$

b)

	x=0	x=1	x=2	x=3	x=4
$f(x, y) = \left[\binom{4}{x} (0.993)^x (0.007)^{4-x} \right]$ for $x = 1, 2, 3, 4$					
f(x)	2.40×10^{-9}	1.36×10^{-6}	2.899×10^{-4}	0.0274	0.972

c) since x is binomial $E(X) = n(p) = 4*(0.993)=3.972$

d) $f_{Y|2}(y) = \frac{f_{XY}(2, y)}{f_X(2)} = f(y), f_X(2) = 0.0725$

y	$f_{Y 1}(y)=f(y)$
0	8.1×10^{-11}
1	1.08×10^{-7}
2	5.37×10^{-5}
3	0.0119
4	0.988

e) $E(Y|X=2) = E(Y) = n(p) = 4(0.997)=3.988$

f) $V(Y|X=2) = V(Y) = n(p)(1-p) = 4(0.997)(0.003)=0.0120$

g) Yes, X and Y are independent.

Section 5-2

- 5-17. a) $P(X = 2) = f_{XYZ}(2,1,1) + f_{XYZ}(2,1,2) + f_{XYZ}(2,2,1) + f_{XYZ}(2,2,2) = 0.5$
 b) $P(X = 1, Y = 2) = f_{XYZ}(1,2,1) + f_{XYZ}(1,2,2) = 0.35$
 c) $P(Z < 1.5) = f_{XYZ}(1,1,1) + f_{XYZ}(1,2,1) + f_{XYZ}(2,1,1) + f_{XYZ}(2,2,1) = 0.5$
 d)
 $P(X = 1 \text{ or } Z = 2) = P(X = 1) + P(Z = 2) - P(X = 1, Z = 2) = 0.5 + 0.5 - 0.3 = 0.7$
 e) $E(X) = 1(0.5) + 2(0.5) = 1.5$

- 5-18 a) $P(X = 1 | Y = 1) = \frac{P(X = 1, Y = 1)}{P(Y = 1)} = \frac{0.05 + 0.10}{0.15 + 0.2 + 0.1 + 0.05} = 0.3$
 b) $P(X = 1, Y = 1 | Z = 2) = \frac{P(X = 1, Y = 1, Z = 2)}{P(Z = 2)} = \frac{0.1}{0.1 + 0.2 + 0.15 + 0.05} = 0.2$
 c) $P(X = 1 | Y = 1, Z = 2) = \frac{P(X = 1, Y = 1, Z = 2)}{P(Y = 1, Z = 2)} = \frac{0.10}{0.10 + 0.15} = 0.4$

- 5-19. $f_{X|YZ}(x) = \frac{f_{XYZ}(x,1,2)}{f_{YZ}(1,2)}$ and $f_{YZ}(1,2) = f_{XYZ}(1,1,2) + f_{XYZ}(2,1,2) = 0.25$

x	$f_{X YZ}(x)$
1	$0.10/0.25=0.4$
2	$0.15/0.25=0.6$

- 5-20 a) percentage of slabs classified as high = $p_1 = 0.05$
 percentage of slabs classified as medium = $p_2 = 0.85$
 percentage of slabs classified as low = $p_3 = 0.10$
 b) X is the number of voids independently classified as high $X \geq 0$
 Y is the number of voids independently classified as medium $Y \geq 0$
 Z is the number of with a low number of voids and $Z \geq 0$ and $X+Y+Z = 20$
 c) p_1 is the percentage of slabs classified as high.
 d) $E(X) = np_1 = 20(0.05) = 1$
 $V(X) = np_1(1-p_1) = 20(0.05)(0.95) = 0.95$

5-21. a) $P(X = 1, Y = 17, Z = 3) = 0$ Because the point $(1, 17, 3) \neq 20$ is not in the range of (X, Y, Z) .

b)

$$P(X \leq 1, Y = 17, Z = 3) = P(X = 0, Y = 17, Z = 3) + P(X = 1, Y = 17, Z = 3) \\ = \frac{20!}{0!17!3!} 0.05^0 0.85^{17} 0.10^3 + 0 = 0.07195$$

Because the point $(1, 17, 3) \neq 20$ is not in the range of (X, Y, Z) .

c) Because X is binomial, $P(X \leq 1) = \binom{20}{0} 0.05^0 0.95^{20} + \binom{20}{1} 0.05^1 0.95^{19} = 0.7358$

d.) Because X is binomial $E(X) = np = 20(0.05) = 1$

5-22 a) The probability is 0 because $x+y+z > 20$

b) $P(X = 2 | Y = 17) = \frac{P(X = 2, Y = 17)}{P(Y = 17)}$. Now, because $x+y+z = 20$,

$$P(X=2, Y=17) = P(X=2, Y=17, Z=1) = \frac{20!}{2!17!1!} 0.05^2 0.85^{17} 0.10^1 = 0.0540$$

$$P(X = 2 | Y = 17) = \frac{P(X = 2, Y = 17)}{P(Y = 17)} = \frac{0.0540}{0.2428} = 0.2224$$

c)

$$E(X | Y = 17) = 0 \left(\frac{P(X = 0, Y = 17)}{P(Y = 17)} \right) + 1 \left(\frac{P(X = 1, Y = 17)}{P(Y = 17)} \right) \\ + 2 \left(\frac{P(X = 2, Y = 17)}{P(Y = 17)} \right) + 3 \left(\frac{P(X = 3, Y = 17)}{P(Y = 17)} \right) \\ E(X | Y = 17) = 0 \left(\frac{0.07195}{0.2428} \right) + 1 \left(\frac{0.1079}{0.2428} \right) + 2 \left(\frac{0.05396}{0.2428} \right) + 3 \left(\frac{0.008994}{0.2428} \right) \\ = 1$$

5-23. a) The range consists of nonnegative integers with $x+y+z = 4$.

b) Because the samples are selected without replacement, the trials are not independent and the joint distribution is not multinomial.

$$5-24 \quad P(X = x | Y = 2) = \frac{f_{XY}(x, 2)}{f_Y(2)}$$

$$P(Y = 2) = \frac{\binom{4}{0}\binom{5}{2}\binom{6}{2}}{\binom{15}{4}} + \frac{\binom{4}{1}\binom{5}{2}\binom{6}{1}}{\binom{15}{4}} + \frac{\binom{4}{2}\binom{5}{2}\binom{6}{0}}{\binom{15}{4}} = 0.1098 + 0.1758 + 0.0440 = 0.3296$$

$$P(X = 0 \text{ and } Y = 2) = \frac{\binom{4}{0}\binom{5}{2}\binom{6}{2}}{\binom{15}{4}} = 0.1098$$

$$P(X = 1 \text{ and } Y = 2) = \frac{\binom{4}{1}\binom{5}{2}\binom{6}{1}}{\binom{15}{4}} = 0.1758$$

$$P(X = 2 \text{ and } Y = 2) = \frac{\binom{4}{2}\binom{5}{2}\binom{6}{0}}{\binom{15}{4}} = 0.0440$$

x	$f_{XY}(x, 2)$
0	0.1098/0.3296=0.3331
1	0.1758/0.3296=0.5334
2	0.0440/0.3296=0.1335

5-25. $P(X=x, Y=y, Z=z)$ is the number of subsets of size 4 that contain x printers with graphics enhancements, y printers with extra memory, and z printers with both features divided by the number of subsets of size 4. From the results on the CD material on counting techniques, it can be shown that

$$P(X = x, Y = y, Z = z) = \frac{\binom{4}{x}\binom{5}{y}\binom{6}{z}}{\binom{15}{4}} \quad \text{for } x+y+z = 4.$$

$$a) \quad P(X = 1, Y = 2, Z = 1) = \frac{\binom{4}{1}\binom{5}{2}\binom{6}{1}}{\binom{15}{4}} = 0.1758$$

$$b) \quad P(X = 1, Y = 1) = P(X = 1, Y = 1, Z = 2) = \frac{\binom{4}{1}\binom{5}{1}\binom{6}{2}}{\binom{15}{4}} = 0.2198$$

c) The marginal distribution of X is hypergeometric with $N = 15$, $n = 4$, $K = 4$. Therefore, $E(X) = nK/N = 16/15$ and $V(X) = 4(4/15)(11/15)[11/14] = 0.6146$.

5-26 a)

$$P(X = 1, Y = 2 | Z = 1) = P(X = 1, Y = 2, Z = 1) / P(Z = 1) \\ = \left[\frac{\binom{4}{1} \binom{5}{2} \binom{6}{1}}{\binom{15}{4}} \right] / \left[\frac{\binom{6}{1} \binom{9}{3}}{\binom{15}{4}} \right] = 0.4762$$

b)

$$P(X = 2 | Y = 2) = P(X = 2, Y = 2) / P(Y = 2) \\ = \left[\frac{\binom{4}{2} \binom{5}{2} \binom{6}{0}}{\binom{15}{4}} \right] / \left[\frac{\binom{5}{2} \binom{10}{2}}{\binom{15}{4}} \right] = 0.1334$$

c) Because $X+Y+Z = 4$, if $Y = 0$ and $Z = 3$, then $X = 1$. Because X must equal 1, $f_{X|YZ}(1) = 1$.

5-27. a) The probability distribution is multinomial because the result of each trial (a dropped oven) results in either a major, minor or no defect with probability 0.6, 0.3 and 0.1 respectively. Also, the trials are independent

b) Let X , Y , and Z denote the number of ovens in the sample of four with major, minor, and no defects, respectively.

$$P(X = 2, Y = 2, Z = 0) = \frac{4!}{2!2!0!} 0.6^2 0.3^2 0.1^0 = 0.1944$$

$$c) P(X = 0, Y = 0, Z = 4) = \frac{4!}{0!0!4!} 0.6^0 0.3^0 0.1^4 = 0.0001$$

5-28 a) $f_{XY}(x, y) = \sum_R f_{XYZ}(x, y, z)$ where R is the set of values for z such that $x+y+z = 4$. That is,

R consists of the single value $z = 4-x-y$ and

$$f_{XY}(x, y) = \frac{4!}{x!y!(4-x-y)!} 0.6^x 0.3^y 0.1^{4-x-y} \quad \text{for } x + y \leq 4.$$

$$b) E(X) = np_1 = 4(0.6) = 2.4$$

$$c) E(Y) = np_2 = 4(0.3) = 1.2$$

$$5-29 \quad a) P(X = 2 | Y = 2) = \frac{P(X = 2, Y = 2)}{P(Y = 2)} = \frac{0.1944}{0.2646} = 0.7347$$

$$P(Y = 2) = \binom{4}{2} 0.3^2 0.7^2 = 0.2646 \quad \text{from the binomial marginal distribution of } Y$$

b) Not possible, $x+y+z=4$, the probability is zero.

$$c) P(X | Y = 2) = P(X = 0 | Y = 2), P(X = 1 | Y = 2), P(X = 2 | Y = 2)$$

$$P(X = 0 | Y = 2) = \frac{P(X = 0, Y = 2)}{P(Y = 2)} = \left(\frac{4!}{0!2!2!} 0.6^0 0.3^2 0.1^2 \right) / 0.2646 = 0.0204$$

$$P(X = 1 | Y = 2) = \frac{P(X = 1, Y = 2)}{P(Y = 2)} = \left(\frac{4!}{1!2!1!} 0.6^1 0.3^2 0.1^1 \right) / 0.2646 = 0.2449$$

$$P(X = 2 | Y = 2) = \frac{P(X = 2, Y = 2)}{P(Y = 2)} = \left(\frac{4!}{2!2!0!} 0.6^2 0.3^2 0.1^0 \right) / 0.2646 = 0.7347$$

$$d) E(X|Y=2) = 0(0.0204) + 1(0.2449) + 2(0.7347) = 1.7143$$

5-30 Let X, Y, and Z denote the number of bits with high, moderate, and low distortion. Then, the joint distribution of X, Y, and Z is multinomial with n = 3 and

$$p_1 = 0.01, p_2 = 0.04, \text{ and } p_3 = 0.95.$$

a)

$$\begin{aligned} P(X = 2, Y = 1) &= P(X = 2, Y = 1, Z = 0) \\ &= \frac{3!}{2!1!0!} 0.01^2 0.04^1 0.95^0 = 1.2 \times 10^{-5} \end{aligned}$$

$$b) P(X = 0, Y = 0, Z = 3) = \frac{3!}{0!0!3!} 0.01^0 0.04^0 0.95^3 = 0.8574$$

5-31 a) X has a binomial distribution with n = 3 and p = 0.01. Then, $E(X) = 3(0.01) = 0.03$ and $V(X) = 3(0.01)(0.99) = 0.0297$.

b) First find $P(X | Y = 2)$

$$P(Y = 2) = P(X = 1, Y = 2, Z = 0) + P(X = 0, Y = 2, Z = 1)$$

$$= \frac{3!}{1!2!0!} 0.01(0.04)^2 0.95^0 + \frac{3!}{0!2!1!} 0.01^0 (0.04)^2 0.95^1 = 0.0046$$

$$P(X = 0 | Y = 2) = \frac{P(X = 0, Y = 2)}{P(Y = 2)} = \left(\frac{3!}{0!2!1!} 0.01^0 0.04^2 0.95^1 \right) / 0.004608 = 0.98958$$

$$P(X = 1 | Y = 2) = \frac{P(X = 1, Y = 2)}{P(Y = 2)} = \left(\frac{3!}{1!2!0!} 0.01^1 0.04^2 0.95^0 \right) / 0.004608 = 0.01042$$

$$E(X | Y = 2) = 0(0.98958) + 1(0.01042) = 0.01042$$

$$V(X | Y = 2) = E(X^2) - (E(X))^2 = 0.01042 - (0.01042)^2 = 0.01031$$

- 5-32 a) Let X, Y, and Z denote the risk of new competitors as no risk, moderate risk, and very high risk. Then, the joint distribution of X, Y, and Z is multinomial with $n=12$ and $p_1 = 0.13$, $p_2 = 0.72$, and $p_3 = 0.15$. X, Y and $Z \geq 0$ and $x+y+z=12$
b) $P(X = 1, Y = 3, Z = 1) = 0$, not possible since $x+y+z \neq 12$

c)

$$P(Z \leq 2) = \binom{12}{0} 0.15^0 0.85^{12} + \binom{12}{1} 0.15^1 0.85^{11} + \binom{12}{2} 0.15^2 0.85^{10} \\ = 0.1422 + 0.3012 + 0.2924 = 0.7358$$

- 5-33 a) $P(Z = 2 | Y = 1, X = 10) = 0$

b)

$$P(X = 10) = P(X = 10, Y = 2, Z = 0) + P(X = 10, Y = 1, Z = 1) + P(X = 10, Y = 0, Z = 2) \\ = \frac{12!}{10!2!0!} 0.13^{10} 0.72^2 0.15^0 + \frac{12!}{10!1!1!} 0.13^{10} 0.72^1 0.15^1 + \frac{12!}{10!0!2!} 0.13^{10} 0.72^0 0.15^2 \\ = 4.72 \times 10^{-8} + 1.97 \times 10^{-8} + 2.04 \times 10^{-9} = 6.89 \times 10^{-8} \\ P(Z \leq 1 | X = 10) = \frac{P(Z = 0, Y = 2, X = 10)}{P(X = 10)} + \frac{P(Z = 1, Y = 1, X = 10)}{P(X = 10)} \\ = \frac{12!}{10!2!0!} 0.13^{10} 0.72^2 0.15^0 / 6.89 \times 10^{-8} + \frac{12!}{10!1!1!} 0.13^{10} 0.72^1 0.15^1 / 6.89 \times 10^{-8} \\ = 0.9698$$

c)

$$P(Y \leq 1, Z \leq 1 | X = 10) = \frac{P(Z = 1, Y = 1, X = 10)}{P(X = 10)} \\ = \frac{12!}{10!1!1!} 0.13^{10} 0.72^1 0.15^1 / 6.89 \times 10^{-8} \\ = 0.2852$$

- d) $E(Z | X = 10) = (0(4.72 \times 10^{-8}) + 1(1.97 \times 10^{-8}) + 2(2.04 \times 10^{-9})) / 6.89 \times 10^{-8} \\ = 0.345$

Section 5-3

5-34 Determine c such that $c \int_0^3 \int_0^3 xy dx dy = c \int_0^3 y \frac{x^2}{2} \Big|_0^3 dy = c \left(4.5 \frac{y^2}{2} \Big|_0^3 \right) = \frac{81}{4} c$.

Therefore, $c = 4/81$.

5-35. a) $P(X < 2, Y < 3) = \frac{4}{81} \int_0^3 \int_0^2 xy dx dy = \frac{4}{81} (2) \int_0^3 y dy = \frac{4}{81} (2) \left(\frac{9}{2} \right) = 0.4444$

b) $P(X < 2.5) = P(X < 2.5, Y < 3)$ because the range of Y is from 0 to 3.

$$P(X < 2.5, Y < 3) = \frac{4}{81} \int_0^3 \int_0^{2.5} xy dx dy = \frac{4}{81} (3.125) \int_0^3 y dy = \frac{4}{81} (3.125) \frac{9}{2} = 0.6944$$

c) $P(1 < Y < 2.5) = \frac{4}{81} \int_1^{2.5} \int_0^3 xy dx dy = \frac{4}{81} (4.5) \int_1^{2.5} y dy = \frac{18}{81} \frac{y^2}{2} \Big|_1^{2.5} = 0.5833$

5-35 d)

$$P(X > 1.8, 1 < Y < 2.5) = \frac{4}{81} \int_1^{2.5} \int_{1.8}^3 xy dx dy = \frac{4}{81} (2.88) \int_1^{2.5} y dy = \frac{4}{81} (2.88) \frac{(2.5^2 - 1)}{2} = 0.3733$$

e) $E(X) = \frac{4}{81} \int_0^3 \int_0^3 x^2 y dx dy = \frac{4}{81} \int_0^3 9y dy = \frac{4}{9} \frac{y^2}{2} \Big|_0^3 = 2$

f) $P(X < 0, Y < 4) = \frac{4}{81} \int_0^4 \int_0^0 xy dx dy = 0 \int_0^4 y dy = 0$

5-36 a) $f_X(x) = \int_0^3 f_{XY}(x, y) dy = x \frac{4}{81} \int_0^3 y dy = \frac{4}{81} x (4.5) = \frac{2x}{9}$ for $0 < x < 3$.

b) $f_{Y|1.5}(y) = \frac{f_{XY}(1.5, y)}{f_X(1.5)} = \frac{\frac{4}{81} y (1.5)}{\frac{2}{9} (1.5)} = \frac{2}{9} y$ for $0 < y < 3$.

c) $E(Y|X=1.5) = \int_0^3 y \left(\frac{2}{9} y \right) dy = \frac{2}{9} \int_0^3 y^2 dy = \frac{2y^3}{27} \Big|_0^3 = 2$

d.) $P(Y < 2 | X = 1.5) = f_{Y|1.5}(y) = \int_0^2 \frac{2}{9} y dy = \frac{1}{9} y^2 \Big|_0^2 = \frac{4}{9} - 0 = \frac{4}{9}$

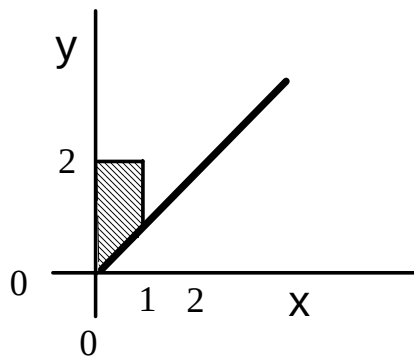
e) $f_{X|2}(x) = \frac{f_{XY}(x, 2)}{f_Y(2)} = \frac{\frac{4}{81} x (2)}{\frac{2}{9} (2)} = \frac{2}{9} x$ for $0 < x < 3$.

5-37.

$$\begin{aligned}
 c \int_0^3 \int_x^{x+2} (x+y) dy dx &= \int_0^3 xy + \frac{y^2}{2} \Big|_x^{x+2} dx \\
 &= \int_0^3 \left[x(x+2) + \frac{(x+2)^2}{2} - x^2 - \frac{x^2}{2} \right] dx \\
 &= c \int_0^3 (4x+2) dx = \left[2x^2 + 2x \right]_0^3 = 24c
 \end{aligned}$$

Therefore, $c = 1/24$.

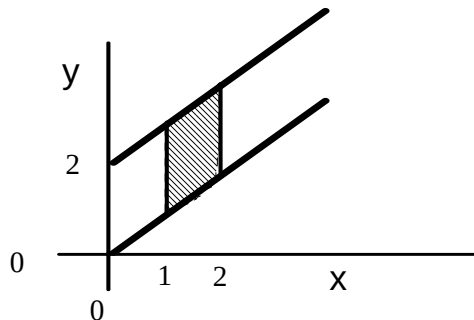
5-38 a) $P(X < 1, Y < 2)$ equals the integral of $f_{XY}(x, y)$ over the following region.



Then,

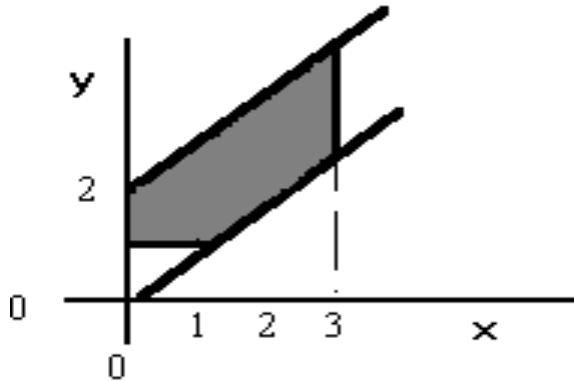
$$\begin{aligned}
 P(X < 1, Y < 2) &= \frac{1}{24} \int_0^1 \int_x^2 (x+y) dy dx = \frac{1}{24} \int_0^1 xy + \frac{y^2}{2} \Big|_x^2 dx = \frac{1}{24} \int_0^1 2x + 2 - \frac{3x^2}{2} dx = \\
 &= \frac{1}{24} \left[x^2 + 2x - \frac{x^3}{2} \right]_0^1 = 0.10417
 \end{aligned}$$

b) $P(1 < X < 2)$ equals the integral of $f_{XY}(x, y)$ over the following region.



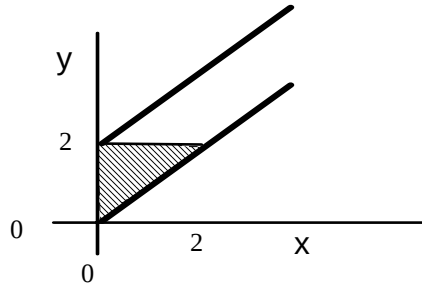
$$\begin{aligned}
 P(1 < X < 2) &= \frac{1}{24} \int_1^2 \int_x^{x+2} (x+y) dy dx = \frac{1}{24} \int_1^2 xy + \frac{y^2}{2} \Big|_x^{x+2} dx \\
 &= \frac{1}{24} \int_1^2 (4x+2) dx = \frac{1}{24} \left[2x^2 + 2x \right]_1^2 = \frac{1}{6}.
 \end{aligned}$$

c) $P(Y > 1)$ is the integral of $f_{XY}(x, y)$ over the following region.



$$\begin{aligned}
 P(Y > 1) &= 1 - P(Y \leq 1) = 1 - \frac{1}{24} \int_0^1 \int_x^1 (x + y) dy dx = 1 - \frac{1}{24} \int_0^1 \left(xy + \frac{y^2}{2} \right) \Big|_x^1 dx \\
 &= 1 - \frac{1}{24} \int_0^1 \left(x + \frac{1}{2} - \frac{3}{2}x^2 \right) dx = 1 - \frac{1}{24} \left(\frac{x^2}{2} + \frac{1}{2} - \frac{1}{2}x^3 \right) \Big|_0^1 \\
 &= 1 - 0.02083 = 0.9792
 \end{aligned}$$

d) $P(X < 2, Y < 2)$ is the integral of $f_{XY}(x, y)$ over the following region.



$$\begin{aligned}
 E(X) &= \frac{1}{24} \int_0^3 \int_x^{x+2} x(x + y) dy dx = \frac{1}{24} \int_0^3 \left(x^2 y + \frac{xy^2}{2} \right) \Big|_x^{x+2} dx \\
 &= \frac{1}{24} \int_0^3 (4x^2 + 2x) dx = \frac{1}{24} \left[\frac{4x^3}{3} + x^2 \right] \Big|_0^3 = \frac{15}{8}
 \end{aligned}$$

e)

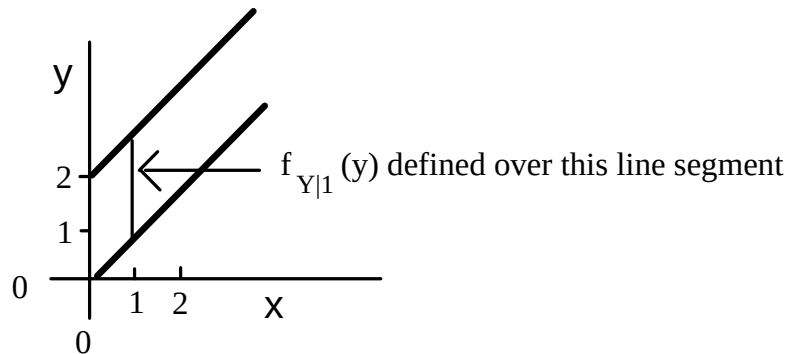
$$\begin{aligned}
 E(X) &= \frac{1}{24} \int_0^3 \int_x^{x+2} x(x + y) dy dx = \frac{1}{24} \int_0^3 \left(x^2 y + \frac{xy^2}{2} \right) \Big|_x^{x+2} dx \\
 &= \frac{1}{24} \int_0^3 (4x^2 + 2x) dx = \frac{1}{24} \left[\frac{4x^3}{3} + x^2 \right] \Big|_0^3 = \frac{15}{8}
 \end{aligned}$$

5-39. a) $f_X(x)$ is the integral of $f_{XY}(x,y)$ over the interval from x to $x+2$. That is,

$$f_X(x) = \frac{1}{24} \int_x^{x+2} (x+y) dy = \frac{1}{24} \left[xy + \frac{y^2}{2} \right]_x^{x+2} = \frac{x}{6} + \frac{1}{12} \quad \text{for } 0 < x < 3.$$

$$\text{b) } f_{Y|1}(y) = \frac{f_{XY}(1,y)}{f_X(1)} = \frac{\frac{1}{24}(1+y)}{\frac{1}{6} + \frac{1}{12}} = \frac{1+y}{6} \quad \text{for } 1 < y < 3.$$

See the following graph,

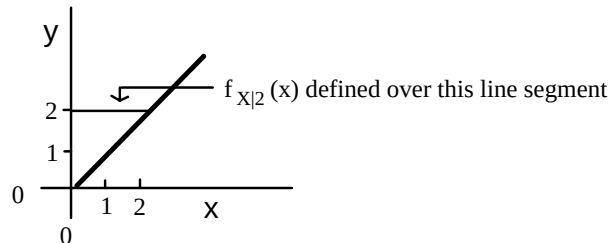


$$\text{c) } E(Y|X=1) = \int_1^3 y \left(\frac{1+y}{6} \right) dy = \frac{1}{6} \int_1^3 (y + y^2) dy = \frac{1}{6} \left(\frac{y^2}{2} + \frac{y^3}{3} \right) \Big|_1^3 = 2.111$$

$$\text{d) } P(Y > 2 | X = 1) = \int_2^3 \left(\frac{1+y}{6} \right) dy = \frac{1}{6} \int_2^3 (1+y) dy = \frac{1}{6} \left(y + \frac{y^2}{2} \right) \Big|_2^3 = 0.5833$$

e) $f_{X|2}(x) = \frac{f_{XY}(x,2)}{f_Y(2)}$. Here $f_Y(y)$ is determined by integrating over x . There are three regions of integration. For $0 < y \leq 2$ the integration is from 0 to y . For $2 < y \leq 3$ the integration is from $y-2$ to y . For $3 < y < 5$ the integration is from y to 3. Because the condition is $y=2$, only the first integration is needed.

$$f_Y(y) = \frac{1}{24} \int_0^y (x+y) dx = \frac{1}{24} \left[\frac{x^2}{2} + xy \right]_0^y = \frac{y^2}{16} \quad \text{for } 0 < y \leq 2.$$



$$\text{Therefore, } f_Y(2) = 1/4 \text{ and } f_{X|2}(x) = \frac{\frac{1}{24}(x+2)}{1/4} = \frac{x+2}{6} \quad \text{for } 0 < x < 2$$

$$5-40 \quad c \int_0^3 \int_0^x xy dy dx = c \int_0^3 x \frac{y^2}{2} \Big|_0^x dx = c \int_0^3 \frac{x^3}{2} dx = \frac{81}{8} c. \text{ Therefore, } c = 8/81$$

$$5-41. \quad a) P(X < 1, Y < 2) = \frac{8}{81} \int_0^1 \int_0^x xy dy dx = \frac{8}{81} \int_0^1 \frac{x^3}{2} dx = \frac{8}{81} \left(\frac{1}{8} \right) = \frac{1}{81}.$$

$$b) P(1 < X < 2) = \frac{8}{81} \int_1^2 \int_0^x xy dy dx = \frac{8}{81} \int_1^2 x \frac{x^2}{2} dx = \left(\frac{8}{81} \right) \frac{x^4}{8} \Big|_1^2 = \left(\frac{8}{81} \right) \frac{(2^4 - 1)}{8} = \frac{5}{27}.$$

c)

$$\begin{aligned} P(Y > 1) &= \frac{8}{81} \int_1^3 \int_1^x xy dy dx = \frac{8}{81} \int_1^3 x \left(\frac{x^2 - 1}{2} \right) dx = \frac{8}{81} \int_1^3 \frac{x^3}{2} - \frac{x}{2} dx = \frac{8}{81} \left(\frac{x^4}{8} - \frac{x^2}{4} \right) \Big|_1^3 \\ &= \frac{8}{81} \left[\left(\frac{3^4}{8} - \frac{3^2}{4} \right) - \left(\frac{1^4}{8} - \frac{1^2}{4} \right) \right] = \frac{1}{81} = 0.01235 \end{aligned}$$

$$d) P(X < 2, Y < 2) = \frac{8}{81} \int_0^2 \int_0^x xy dy dx = \frac{8}{81} \int_0^2 \frac{x^3}{2} dx = \frac{8}{81} \left(\frac{2^4}{8} \right) = \frac{16}{81}.$$

e)

$$\begin{aligned} E(X) &= \frac{8}{81} \int_0^3 \int_0^x x(xy) dy dx = \frac{8}{81} \int_0^3 \int_0^x x^2 y dy dx = \frac{8}{81} \int_0^3 \frac{x^2}{2} x^2 dx = \frac{8}{81} \int_0^3 \frac{x^4}{2} dx \\ &= \left(\frac{8}{81} \right) \left(\frac{3^5}{10} \right) = \frac{12}{5} \end{aligned}$$

f)

$$\begin{aligned} E(Y) &= \frac{8}{81} \int_0^3 \int_0^x y(xy) dy dx = \frac{8}{81} \int_0^3 \int_0^x xy^2 dy dx = \frac{8}{81} \int_0^3 x \frac{x^3}{3} dx \\ &= \frac{8}{81} \int_0^3 \frac{x^4}{3} dx = \left(\frac{8}{81} \right) \left(\frac{3^5}{15} \right) = \frac{8}{5} \end{aligned}$$

$$5-42 \quad a) f(x) = \frac{8}{81} \int_0^x xy dy = \frac{4x^3}{81} \quad 0 < x < 3$$

$$b) f_{Y|X=1}(y) = \frac{f(1, y)}{f(1)} = \frac{\frac{8}{81}(1)y}{\frac{4(1)^3}{81}} = 2y \quad 0 < y < 1$$

$$c) E(Y | X = 1) = \int_0^1 2y dy = y^2 \Big|_0^1 = 1$$

d) $P(Y > 2 | X = 1) = 0$ this isn't possible since the values of y are $0 < y < x$.

$$e) f(y) = \frac{8}{81} \int_0^3 xy dx = \frac{4y}{9}, \text{ therefore}$$

$$f_{X|Y=2}(x) = \frac{f(x, 2)}{f(2)} = \frac{\frac{8}{81}x(2)}{\frac{4(2)}{9}} = \frac{2x}{9} \quad 0 < x < 2$$

5-43. Solve for c

$$c \int_0^\infty \int_0^x e^{-2x-3y} dy dx = \frac{c}{3} \int_0^\infty e^{-2x} (1 - e^{-3x}) dx = \frac{c}{3} \int_0^\infty e^{-2x} - e^{-5x} dx =$$

$$\frac{c}{3} \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{1}{10} c. \quad c = 10$$

5-44 a)

$$\begin{aligned} P(X < 1, Y < 2) &= 10 \int_0^1 \int_0^x e^{-2x-3y} dy dx = \frac{10}{3} \int_0^1 e^{-2x} (1 - e^{-3x}) dx = \frac{10}{3} \int_0^1 e^{-2x} - e^{-5x} dx \\ &= \frac{10}{3} \left(\frac{e^{-5x}}{5} - \frac{e^{-2x}}{2} \right) \Big|_0^1 = 0.77893 \end{aligned}$$

$$P(1 < X < 2) = 10 \int_1^2 \int_0^x e^{-2x-3y} dy dx = \frac{10}{3} \int_1^2 e^{-2x} - e^{-5x} dx$$

$$b) \quad = \frac{10}{3} \left(\frac{e^{-5x}}{5} - \frac{e^{-2x}}{2} \right) \Big|_1^2 = 0.19057$$

c)

$$\begin{aligned} P(Y > 3) &= 10 \int_3^\infty \int_3^x e^{-2x-3y} dy dx = \frac{10}{3} \int_3^\infty e^{-2x} (e^{-9} - e^{-3x}) dx \\ &= \frac{10}{3} \left(\frac{e^{-5x}}{5} - \frac{e^{-9} e^{-2x}}{2} \right) \Big|_3^\infty = 3.059 \times 10^{-7} \end{aligned}$$

d)

$$P(X < 2, Y < 2) = 10 \int_0^2 \int_0^x e^{-2x-3y} dy dx = \frac{10}{3} \int_0^2 e^{-2x} (1 - e^{-3x}) dx = \frac{10}{3} \left(\frac{e^{-10}}{5} - \frac{e^{-4}}{2} \right) \Bigg|_0^2$$

$$= 0.9695$$

$$\text{e) } E(X) = 10 \int_0^\infty \int_0^x x e^{-2x-3y} dy dx = \frac{7}{10}$$

$$\text{f) } E(Y) = 10 \int_0^\infty \int_0^x y e^{-2x-3y} dy dx = \frac{1}{5}$$

$$5-45. \quad \text{a) } f(x) = 10 \int_0^x e^{-2x-3y} dy = \frac{10e^{-2x}}{3} (1 - e^{-3x}) = \frac{10}{3} (e^{-2x} - e^{-5x}) \quad \text{for } 0 < x$$

$$\text{b) } f_{Y|X=1}(y) = \frac{f_{X,Y}(1, y)}{f_X(1)} = \frac{10e^{-2-3y}}{\frac{10}{3}(e^{-2} - e^{-5})} = 3.157e^{-3y} \quad 0 < y < 1$$

$$\text{c) } E(Y|X=1) = 3.157 \int_0^1 y e^{-3y} dy = 0.2809$$

$$\text{d) } f_{X|Y=2}(x) = \frac{f_{X,Y}(x, 2)}{f_Y(2)} = \frac{10e^{-2x-6}}{5e^{-10}} = 2e^{-2x+4} \quad \text{for } 2 < x,$$

where $f(y) = 5e^{-5y}$ for $0 < y$

$$5-46 \quad c \int_0^\infty \int_x^\infty e^{-2x} e^{-3y} dy dx = \frac{c}{3} \int_0^\infty e^{-2x} (e^{-3x}) dx = \frac{c}{3} \int_0^\infty e^{-5x} dx = \frac{1}{15} c \quad c = 15$$

5-47. a)

$$P(X < 1, Y < 2) = 15 \int_0^1 \int_x^2 e^{-2x-3y} dy dx = 5 \int_0^1 e^{-2x} (e^{-3x} - e^{-6}) dx$$

$$= 5 \int_0^1 e^{-5x} dx - 5e^{-6} \int_0^1 e^{-2x} dx = 1 - e^{-5} + \frac{5}{2} e^{-6} (e^{-2} - 1) = 0.9879$$

$$\text{b) } P(1 < X < 2) = 15 \int_1^2 \int_x^\infty e^{-2x-3y} dy dx = 5 \int_1^2 e^{-5x} dx = (e^{-5} - e^{-10}) = 0.0067$$

c)

$$P(Y > 3) = 15 \left(\int_0^3 \int_3^\infty e^{-2x-3y} dy dx + \int_3^\infty \int_x^\infty e^{-2x-3y} dy dx \right) = 5 \int_0^3 e^{-9} e^{-2x} dx + 5 \int_3^\infty e^{-5x} dx$$

$$= -\frac{3}{2} e^{-15} + \frac{5}{2} e^{-9} = 0.000308$$

d)

$$\begin{aligned} P(X < 2, Y < 2) &= 15 \int_0^2 \int_x^2 e^{-2x-3y} dy dx = 5 \int_0^2 e^{-2x} (e^{-3x} - e^{-6}) dx = \\ &= 5 \int_0^2 e^{-5x} dx - 5e^{-6} \int_0^2 e^{-2x} dx = (1 - e^{-10}) + \frac{5}{2} e^{-6} (e^{-4} - 1) = 0.9939 \end{aligned}$$

$$e) E(X) = 15 \int_0^\infty \int_x^\infty x e^{-2x-3y} dy dx = 5 \int_0^\infty x e^{-5x} dx = \frac{1}{5^2} = 0.04$$

f)

$$\begin{aligned} E(Y) &= 15 \int_0^\infty \int_x^\infty y e^{-2x-3y} dy dx = \frac{-3}{2} \int_0^\infty 5y e^{-5y} dy + \frac{5}{2} \int_0^\infty 3y e^{-3y} dy \\ &= -\frac{3}{10} + \frac{5}{6} = \frac{8}{15} \end{aligned}$$

$$5-48 \quad a) f(x) = 15 \int_x^\infty e^{-2x-3y} dy = \frac{15}{3} (e^{-2x-3x}) = 5e^{-5x} \text{ for } x > 0$$

$$b) f_X(1) = 5e^{-5} \quad f_{XY}(1, y) = 15e^{-2-3y}$$

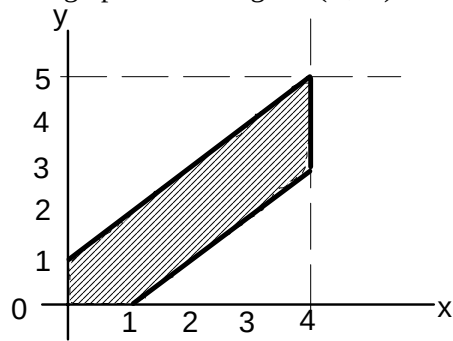
$$f_{Y|X=1}(y) = \frac{15e^{-2-3y}}{5e^{-5}} = 3e^{3-3y} \text{ for } 1 < y$$

$$c) E(Y | X = 1) = \int_1^\infty 3y e^{3-3y} dy = -y e^{3-3y} \Big|_1^\infty + \int_1^\infty e^{3-3y} dy = 4/3$$

$$d) \int_1^2 3e^{3-3y} dy = 1 - e^{-3} = 0.9502 \text{ for } 0 < y, \quad f_Y(2) = \frac{15}{2} e^{-6}$$

$$e) f_{X|Y=2}(y) = \frac{15e^{-2x-6}}{\frac{15}{2}e^{-6}} = 2e^{-2x} \text{ for } 0 < x < 2$$

5-49. The graph of the range of (X, Y) is



$$\int_0^1 \int_0^{x+1} c dy dx + \int_1^4 \int_{x-1}^{x+1} c dy dx = 1$$

$$= c \int_0^1 (x+1) dx + 2c \int_1^4 dx$$

$$= \frac{3}{2}c + 6c = 7.5c = 1$$

Therefore, $c = 1/7.5 = 2/15$

5-50 a) $P(X < 0.5, Y < 0.5) = \int_0^{0.5} \int_0^{0.5} \frac{1}{7.5} dy dx = \frac{1}{30}$

b) $P(X < 0.5) = \int_0^{0.5x+1} \int_0^{0.5} \frac{1}{7.5} dy dx = \frac{1}{7.5} \int_0^{0.5} (x+1) dx = \frac{2}{15} \left(\frac{5}{8}\right) = \frac{1}{12}$

c)

$$\begin{aligned} E(X) &= \int_0^1 \int_0^{x+1} \frac{x}{7.5} dy dx + \int_1^4 \int_{x-1}^{x+1} \frac{x}{7.5} dy dx \\ &= \frac{1}{7.5} \int_0^1 (x^2 + x) dx + \frac{2}{7.5} \int_1^4 (x) dx = \frac{12}{15} \left(\frac{5}{6}\right) + \frac{2}{7.5} (7.5) = \frac{19}{9} \end{aligned}$$

d)

$$\begin{aligned} E(Y) &= \frac{1}{7.5} \int_0^1 \int_0^{x+1} y dy dx + \frac{1}{7.5} \int_1^4 \int_{x-1}^{x+1} y dy dx \\ &= \frac{1}{7.5} \int_0^1 \frac{(x+1)^2}{2} dx + \frac{1}{7.5} \int_1^4 \frac{(x+1)^2 - (x-1)^2}{2} dx \\ &= \frac{1}{15} \int_0^1 (x^2 + 2x + 1) dx + \frac{1}{15} \int_1^4 4x dx \\ &= \frac{1}{15} \left(\frac{7}{3}\right) + \frac{1}{15} (30) = \frac{97}{45} \end{aligned}$$

5-51. a)

$$f(x) = \int_0^{x+1} \frac{1}{7.5} dy = \left(\frac{x+1}{7.5} \right) \quad \text{for } 0 < x < 1,$$

$$f(x) = \int_{x-1}^{x+1} \frac{1}{7.5} dy = \left(\frac{x+1-(x-1)}{7.5} \right) = \frac{2}{7.5} \quad \text{for } 1 < x < 4$$

b)

$$f_{Y|X=1}(y) = \frac{f_{XY}(1, y)}{f_X(1)} = \frac{1/7.5}{2/7.5} = 0.5$$

$$f_{Y|X=1}(y) = 0.5 \quad \text{for } 0 < y < 2$$

$$c) E(Y | X = 1) = \int_0^2 \frac{y}{2} dy = \frac{y^2}{4} \Big|_0^2 = 1$$

$$d) P(Y < 0.5 | X = 1) = \int_0^{0.5} 0.5 dy = 0.5y \Big|_0^{0.5} = 0.25$$

5-52 Let X, Y, and Z denote the time until a problem on line 1, 2, and 3, respectively.

a)

$$P(X > 40, Y > 40, Z > 40) = [P(X > 40)]^3$$

because the random variables are independent with the same distribution. Now,

$$P(X > 40) = \int_{40}^{\infty} \frac{1}{40} e^{-x/40} dx = -e^{-x/40} \Big|_{40}^{\infty} = e^{-1} \quad \text{and the answer is}$$

$$(e^{-1})^3 = e^{-3} = 0.0498.$$

$$b) P(20 < X < 40, 20 < Y < 40, 20 < Z < 40) = [P(20 < X < 40)]^3 \quad \text{and}$$

$$P(20 < X < 40) = -e^{-x/40} \Big|_{20}^{40} = e^{-0.5} - e^{-1} = 0.2387.$$

The answer is $0.2387^3 = 0.0136$.

c) The joint density is not needed because the process is represented by three independent exponential distributions. Therefore, the probabilities may be multiplied.

5-53 a) $\mu=3.2$ $\lambda=1/3.2$

$$P(X > 5, Y > 5) = (1/10.24) \int_5^\infty \int_5^\infty e^{-\frac{x}{3.2} - \frac{y}{3.2}} dy dx = 3.2 \int_5^\infty e^{-\frac{x}{3.2}} \left(e^{-\frac{5}{3.2}} \right) dx$$

$$= \left(e^{-\frac{5}{3.2}} \right) \left(e^{-\frac{5}{3.2}} \right) = 0.0439$$

$$P(X > 10, Y > 10) = (1/10.24) \int_{10}^\infty \int_{10}^\infty e^{-\frac{x}{3.2} - \frac{y}{3.2}} dy dx = 3.2 \int_{10}^\infty e^{-\frac{x}{3.2}} \left(e^{-\frac{10}{3.2}} \right) dx$$

$$= \left(e^{-\frac{10}{3.2}} \right) \left(e^{-\frac{10}{3.2}} \right) = 0.0019$$

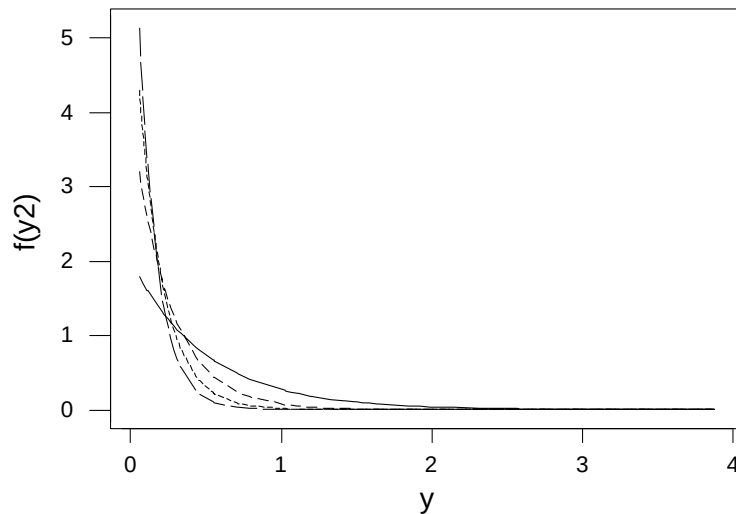
b) Let X denote the number of orders in a 5-minute interval. Then X is a Poisson random variable with $\lambda = 5/3.2 = 1.5625$.

$$P(X = 2) = \frac{e^{-1.5625} (1.5625)^2}{2!} = 0.256$$

For both systems, $P(X = 2)P(Y = 2) = 0.256^2 = 0.0655$

c) The joint probability distribution is not necessary because the two processes are independent and we can just multiply the probabilities.

5-54 a) $f_{Y|X=x}(y)$, for $x = 2, 4, 6, 8$



$$\text{b) } P(Y < 2 | X = 2) = \int_0^2 2e^{-2y} dy = 0.9817$$

$$\text{c) } E(Y | X = 2) = \int_0^\infty 2ye^{-2y} dy = 1/2 \text{ (using integration by parts)}$$

$$\text{d) } E(Y | X = x) = \int_0^\infty xye^{-xy} dy = 1/x \text{ (using integration by parts)}$$

$$\text{e) Use } f_X(x) = \frac{1}{b-a} = \frac{1}{10}, \quad f_{Y|X}(x, y) = xe^{-xy}, \text{ and the relationship } f_{Y|X}(x, y) = \frac{f_{XY}(x, y)}{f_X(x)}$$

$$\text{Therefore, } xe^{-xy} = \frac{f_{XY}(x, y)}{1/10} \quad \text{and} \quad f_{XY}(x, y) = \frac{xe^{-xy}}{10}$$

$$\text{f) } f_Y(y) = \int_0^{10} \frac{xe^{-xy}}{10} dx = \frac{1 - 10ye^{-10y} - e^{-10y}}{10y^2} \text{ (using integration by parts)}$$

Section 5-4

$$\text{5-55. a) } P(X < 0.5) = \int_0^{0.5} \int_0^1 \int_0^1 (8xyz) dz dy dx = \int_0^{0.5} \int_0^1 (4xy) dy dx = \int_0^{0.5} (2x) dx = x^2 \Big|_0^{0.5} = 0.25$$

b)

$$\begin{aligned} P(X < 0.5, Y < 0.5) &= \int_0^{0.5} \int_0^{0.5} \int_0^1 (8xyz) dz dy dx \\ &= \int_0^{0.5} \int_0^{0.5} (4xy) dy dx = \int_0^{0.5} (0.5x) dx = \frac{x^2}{4} \Big|_0^{0.5} = 0.0625 \end{aligned}$$

c) $P(Z < 2) = 1$, because the range of Z is from 0 to 1.

d) $P(X < 0.5 \text{ or } Z < 2) = P(X < 0.5) + P(Z < 2) - P(X < 0.5, Z < 2)$. Now, $P(Z < 2) = 1$ and $P(X < 0.5, Z < 2) = P(X < 0.5)$. Therefore, the answer is 1.

$$\text{e) } E(X) = \int_0^1 \int_0^1 \int_0^1 (8x^2 yz) dz dy dx = \int_0^1 (2x^2) dx = \frac{2x^3}{3} \Big|_0^1 = 2/3$$

5-56 a) $P(X < 0.5 | Y = 0.5)$ is the integral of the conditional density $f_{X|Y}(x)$. Now,

$$f_{X|0.5}(x) = \frac{f_{XY}(x, 0.5)}{f_Y(0.5)} \quad \text{and} \quad f_{XY}(x, 0.5) = \int_0^1 (8xyz) dz = 4xy \text{ for } 0 < x < 1 \text{ and}$$

$$0 < y < 1. \text{ Also, } f_Y(y) = \int_0^1 \int_0^1 (8xyz) dz dx = 2y \text{ for } 0 < y < 1.$$

$$\text{Therefore, } f_{X|0.5}(x) = \frac{2x}{1} = 2x \text{ for } 0 < x < 1.$$

$$\text{Then, } P(X < 0.5 | Y = 0.5) = \int_0^{0.5} 2x dx = 0.25.$$

b) $P(X < 0.5, Y < 0.5 | Z = 0.8)$ is the integral of the conditional density of X and Y.

Now, $f_Z(z) = 2z$ for $0 < z < 1$ as in part a. and

$$f_{XY|Z}(x, y) = \frac{f_{XYZ}(x, y, z)}{f_Z(z)} = \frac{8xy(0.8)}{2(0.8)} = 4xy \text{ for } 0 < x < 1 \text{ and } 0 < y < 1.$$

$$\text{Then, } P(X < 0.5, Y < 0.5 | Z = 0.8) = \int_0^{0.5} \int_0^{0.5} (4xy) dy dx = \int_0^{0.5} (x/2) dx = \frac{1}{16} = 0.0625$$

5-57. a) $f_{YZ}(y, z) = \int_0^1 (8xyz) dx = 4yz$ for $0 < y < 1$ and $0 < z < 1$.

$$\text{Then, } f_{X|YZ}(x) = \frac{f_{XYZ}(x, y, z)}{f_{YZ}(y, z)} = \frac{8x(0.5)(0.8)}{4(0.5)(0.8)} = 2x \text{ for } 0 < x < 1.$$

b) Therefore, $P(X < 0.5 | Y = 0.5, Z = 0.8) = \int_0^{0.5} 2x dx = 0.25$

5-58 a) $\iiint_{x^2+y^2 \leq 4} c dz dy dx$ = the volume of a cylinder with a base of radius 2 and a height of 4 =

$$(\pi 2^2)4 = 16\pi. \text{ Therefore, } c = \frac{1}{16\pi}$$

b) $P(X^2 + Y^2 < 2)$ equals the volume of a cylinder of radius $\sqrt{2}$ and a height of 4 (= 8π) times c . Therefore, the answer is $\frac{8\pi}{16\pi} = 1/2$.

c) $P(Z < 2)$ equals half the volume of the region where $f_{XYZ}(x, y, z)$ is positive times $1/c$. Therefore, the answer is 0.5.

d) $E(X) = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^4 \frac{x}{c} dz dy dx = \frac{1}{c} \int_{-2}^2 \left[4xy \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = \frac{1}{c} \int_{-2}^2 (8x\sqrt{4-x^2}) dx$. Using

substitution, $u = 4 - x^2$, $du = -2x dx$, and $E(X) = \frac{1}{c} \int_4^0 4\sqrt{u} du = \frac{-4}{c} \frac{2}{3} (4 - x^2)^{\frac{3}{2}} \Big|_{-2}^2 = 0$.

5-59. a) $f_{X|1}(x) = \frac{f_{XY}(x, 1)}{f_Y(1)}$ and $f_{XY}(x, y) = \frac{1}{c} \int_0^4 dz = \frac{4}{c} = \frac{1}{4\pi}$ for $x^2 + y^2 < 4$.

$$\text{Also, } f_Y(y) = \frac{1}{c} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^4 dz dx = \frac{8}{c} \sqrt{4-y^2} \text{ for } -2 < y < 2.$$

Then, $f_{X|Y}(x) = \frac{4/c}{\frac{8}{c}\sqrt{4-y^2}}$ evaluated at $y = 1$. That is, $f_{X|1}(x) = \frac{1}{2\sqrt{3}}$ for $-\sqrt{3} < x < \sqrt{3}$.

Therefore, $P(X < 1 | Y < 1) = \int_{-\sqrt{3}}^1 \frac{1}{2\sqrt{3}} dx = \frac{1+\sqrt{3}}{2\sqrt{3}} = 0.7887$

$$b) f_{XY|1}(x,y) = \frac{f_{XYZ}(x,y,1)}{f_Z(1)} \text{ and } f_Z(z) = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{1}{c} dy dx = \int_{-2}^2 \frac{2}{c} \sqrt{4-x^2} dx$$

Because $f_Z(z)$ is a density over the range $0 < z < 4$ that does not depend on Z ,
 $f_Z(z) = 1/4$ for

$$0 < z < 4. \text{ Then, } f_{XY|1}(x,y) = \frac{1/c}{1/4} = \frac{1}{4\pi} \text{ for } x^2 + y^2 < 4.$$

$$\text{Then, } P(x^2 + y^2 < 1 | Z = 1) = \frac{\text{area in } x^2 + y^2 < 1}{4\pi} = 1/4.$$

$$5-60 \quad f_{Z|xy}(z) = \frac{f_{XYZ}(x,y,z)}{f_{XY}(x,y)} \text{ and from part 5-59 a., } f_{XY}(x,y) = \frac{1}{4\pi} \text{ for } x^2 + y^2 < 4.$$

$$\text{Therefore, } f_{Z|xy}(z) = \frac{\frac{1}{16\pi}}{\frac{1}{4\pi}} = 1/4 \text{ for } 0 < z < 4.$$

5-61 Determine c such that $f(x,y,z) = c$ is a joint density probability over the region $x > 0, y > 0$ and $z > 0$ with $x+y+z < 1$

$$\begin{aligned} f(x,y,z) &= c \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx = \int_0^1 \int_0^{1-x} c(1-x-y) dy dx = \int_0^1 \left(c(y - xy - \frac{y^2}{2}) \Big|_0^{1-x} \right) dx \\ &= \int_0^1 c \left((1-x) - x(1-x) - \frac{(1-x)^2}{2} \right) dx = \int_0^1 c \left(\frac{(1-x)^2}{2} \right) dx = c \left(\frac{1}{2} x - \frac{x^2}{2} + \frac{x^3}{6} \right) \Big|_0^1 \\ &= c \frac{1}{6}. \quad \text{Therefore, } c = 6. \end{aligned}$$

$$5-62 \quad a) P(X < 0.5, Y < 0.5, Z < 0.5) = 6 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx \Rightarrow \text{The conditions } x > 0.5, y > 0.5,$$

$z > 0.5$ and $x+y+z < 1$ make a space that is a cube with a volume of 0.125. Therefore the probability of $P(X < 0.5, Y < 0.5, Z < 0.5) = 6(0.125) = 0.75$

b)

$$\begin{aligned} P(X < 0.5, Y < 0.5) &= \int_0^{0.5} \int_0^{0.5} 6(1-x-y) dy dx = \int_0^{0.5} (6y - 6xy - 3y^2) \Big|_0^{0.5} dx \\ &= \int_0^{0.5} \left(\frac{9}{4} - 3x \right) dx = \left(\frac{9}{4} x - \frac{3}{2} x^2 \right) \Big|_0^{0.5} = 3/4 \end{aligned}$$

c)

$$\begin{aligned} P(X < 0.5) &= 6 \int_0^{0.5} \int_0^{1-x} \int_0^{1-x-y} dz dy dx = \int_0^{0.5} \int_0^{1-x} 6(1-x-y) dy dx = \int_0^{0.5} 6 \left(y - xy - \frac{y^2}{2} \right) \Big|_0^{1-x} dx \\ &= \int_0^{0.5} 6 \left(\frac{x^2}{2} - x + \frac{1}{2} \right) dx = \left(x^3 - 3x^2 + 3x \right) \Big|_0^{0.5} = 0.875 \end{aligned}$$

d)

$$E(X) = 6 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x dz dy dx = \int_0^1 \int_0^{1-x} 6x(1-x-y) dy dx = \int_0^{0.5} 6x(y - xy - \frac{y^2}{2}) \Big|_0^{1-x} dx$$

$$= \int_0^1 6(\frac{x^3}{2} - x^2 + \frac{x}{2}) dx = \left(\frac{3x^4}{4} - 2x^3 + \frac{3x^2}{2} \right) \Big|_0^1 = 0.25$$

5-63 a)

$$f(x) = 6 \int_0^{1-x} \int_0^{1-x-y} dz dy = \int_0^{1-x} 6(1-x-y) dy = 6 \left(y - xy - \frac{y^2}{2} \right) \Big|_0^{1-x}$$

$$= 6 \left(\frac{x^2}{2} - x + \frac{1}{2} \right) = 3(x-1)^2 \text{ for } 0 < x < 1$$

b)

$$f(x, y) = 6 \int_0^{1-x-y} dz = 6(1-x-y)$$

for $x > 0, y > 0$ and $x + y < 1$

c)

$$f(x | y = 0.5, z = 0.5) = \frac{f(x, y = 0.5, z = 0.5)}{f(y = 0.5, z = 0.5)} = \frac{6}{6} = 1 \text{ For, } x = 0$$

d) The marginal $f_Y(y)$ is similar to $f_X(x)$ and $f_Y(y) = 3(1-y)^2$ for $0 < y < 1$.

$$f_{X|Y}(x | 0.5) = \frac{f(x, 0.5)}{f_Y(0.5)} = \frac{6(0.5-x)}{3(0.25)} = 4(1-2x) \text{ for } x < 0.5$$

5-64 Let X denote the production yield on a day. Then,

$$P(X > 1400) = P(Z > \frac{1400-1500}{\sqrt{10000}}) = P(Z > -1) = 0.84134.$$

a) Let Y denote the number of days out of five such that the yield exceeds 1400. Then, by independence, Y has a binomial distribution with $n = 5$ and $p = 0.8413$. Therefore, the answer is $P(Y = 5) = \binom{5}{5} 0.8413^5 (1 - 0.8413)^0 = 0.4215$.

b) As in part (a), the answer is

$$P(Y \geq 4) = P(Y = 4) + P(Y = 5)$$

$$= \binom{5}{4} 0.8413^4 (1 - 0.8413)^1 + 0.4215 = 0.8190$$

5-65

a) Let X denote the weight of a brick. Then,

$$P(X > 2.75) = P(Z > \frac{2.75-3}{0.25}) = P(Z > -1) = 0.84134.$$

Let Y denote the number of bricks in the sample of 20 that exceed 2.75 pounds. Then, by independence, Y has a binomial distribution with $n = 20$ and $p = 0.84134$. Therefore, the answer is $P(Y = 20) = \binom{20}{20} 0.84134^{20} = 0.032$.

b) Let A denote the event that the heaviest brick in the sample exceeds 3.75 pounds.

Then, $P(A) = 1 - P(A')$ and A' is the event that all bricks weigh less than 3.75 pounds. As in part a., $P(X < 3.75) = P(Z < 3)$ and

$$P(A) = 1 - [P(Z < 3)]^{20} = 1 - 0.99865^{20} = 0.0267.$$

5-66 a) Let X denote the grams of luminescent ink. Then,

$$P(X < 1.14) = P(Z < \frac{1.14-1.2}{0.3}) = P(Z < -2) = 0.022750.$$

Let Y denote the number of bulbs in the sample of 25 that have less than 1.14 grams.

Then, by independence, Y has a binomial distribution with $n = 25$ and $p = 0.022750$.

Therefore, the answer is

$$P(Y \geq 1) = 1 - P(Y = 0) = \binom{25}{0} 0.02275^0 (0.97725)^{25} = 1 - 0.5625 = 0.4375.$$

b)

$$\begin{aligned} P(Y \leq 5) &= P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) + P(Y = 5) \\ &= \binom{25}{0} 0.02275^0 (0.97725)^{25} + \binom{25}{1} 0.02275^1 (0.97725)^{24} + \binom{25}{2} 0.02275^2 (0.97725)^{23} \\ &\quad + \binom{25}{3} 0.02275^3 (0.97725)^{22} + \binom{25}{4} 0.02275^4 (0.97725)^{21} + \binom{25}{5} 0.02275^5 (0.97725)^{20} \\ &= 0.5625 + 0.3274 + 0.09146 + 0.01632 + 0.002090 + 0.0002043 = 0.99997 \approx 1 \end{aligned}$$

$$c.) P(Y = 0) = \binom{25}{0} 0.02275^0 (0.97725)^{25} = 0.5625$$

d.) The lamps are normally and independently distributed, therefore, the probabilities can be multiplied.

Section 5-5

$$5-67 \quad E(X) = 1(3/8) + 2(1/2) + 4(1/8) = 15/8 = 1.875$$

$$E(Y) = 3(1/8) + 4(1/4) + 5(1/2) + 6(1/8) = 37/8 = 4.625$$

$$\begin{aligned} E(XY) &= [1 \times 3 \times (1/8)] + [1 \times 4 \times (1/4)] + [2 \times 5 \times (1/2)] + [4 \times 6 \times (1/8)] \\ &= 75/8 = 9.375 \end{aligned}$$

$$\sigma_{XY} = E(XY) - E(X)E(Y) = 9.375 - (1.875)(4.625) = 0.703125$$

$$V(X) = 1^2(3/8) + 2^2(1/2) + 4^2(1/8) - (15/8)^2 = 0.8594$$

$$V(Y) = 3^2(1/8) + 4^2(1/4) + 5^2(1/2) + 6^2(1/8) - (37/8)^2 = 0.7344$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{0.703125}{\sqrt{(0.8594)(0.7344)}} = 0.8851$$

$$\begin{aligned}
5-68 \quad E(X) &= -1(1/8) + (-0.5)(1/4) + 0.5(1/2) + 1(1/8) = 0.125 \\
E(Y) &= -2(1/8) + (-1)(1/4) + 1(1/2) + 2(1/8) = 0.25 \\
E(XY) &= [-1 \times -2 \times (1/8)] + [-0.5 \times -1 \times (1/4)] + [0.5 \times 1 \times (1/2)] + [1 \times 2 \times (1/8)] = 0.875 \\
V(X) &= 0.4219 \\
V(Y) &= 1.6875 \\
\sigma_{XY} &= 0.875 - (0.125)(0.25) = 0.8438 \\
\rho_{XY} &= \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{0.8438}{\sqrt{0.4219} \sqrt{1.6875}} = 1
\end{aligned}$$

5-69

$$\begin{aligned}
\sum_{x=1}^3 \sum_{y=1}^3 c(x+y) &= 36c, \quad c = 1/36 \\
E(X) &= \frac{13}{6} \quad E(Y) = \frac{13}{6} \quad E(XY) = \frac{14}{3} \quad \sigma_{xy} = \frac{14}{3} - \left(\frac{13}{6}\right)^2 = \frac{-1}{36} \\
E(X^2) &= \frac{16}{3} \quad E(Y^2) = \frac{16}{3} \quad V(X) = V(Y) = \frac{23}{36} \\
\rho &= \frac{\frac{-1}{36}}{\sqrt{\frac{23}{36}} \sqrt{\frac{23}{36}}} = -0.0435
\end{aligned}$$

$$\begin{aligned}
5-70 \quad E(X) &= 0(0.01) + 1(0.99) = 0.99 \\
E(Y) &= 0(0.02) + 1(0.98) = 0.98 \\
E(XY) &= [0 \times 0 \times (0.002)] + [0 \times 1 \times (0.0098)] + [1 \times 0 \times (0.0198)] + [1 \times 1 \times (0.9702)] = 0.9702 \\
V(X) &= 0.99 - 0.99^2 = 0.0099 \\
V(Y) &= 0.98 - 0.98^2 = 0.0196 \\
\sigma_{XY} &= 0.9702 - (0.99)(0.98) = 0 \\
\rho_{XY} &= \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{0}{\sqrt{0.0099} \sqrt{0.0196}} = 0
\end{aligned}$$

$$\begin{aligned}
5-71 \quad E(X_1) &= np_1 = 20(1/3) = 6.67 \\
E(X_2) &= np_2 = 20(1/3) = 6.67 \\
V(X_1) &= np_1(1-p_1) = 20(1/3)(2/3) = 4.44 \\
V(X_2) &= np_2(1-p_2) = 20(1/3)(2/3) = 4.44 \\
E(X_1 X_2) &= n(n-1)p_1 p_2 = 20(19)(1/3)(1/3) = 42.22 \\
\sigma_{XY} &= 42.22 - 6.67^2 = -2.267 \quad \text{and} \quad \rho_{XY} = \frac{-2.267}{\sqrt{(4.44)(4.44)}} = -0.51
\end{aligned}$$

The sign is negative.

5-72 From Exercise 5-40, $c = 8/81$.

From Exercise 5-41, $E(X) = 12/5$, and $E(Y) = 8/5$

$$E(XY) = \frac{8}{81} \int_0^3 \int_0^x xy(xy) dy dx = \frac{8}{81} \int_0^3 \int_0^x x^2 y^2 dy dx = \frac{8}{81} \int_0^3 \frac{x^3}{3} x^2 dx = \frac{8}{81} \int_0^3 \frac{x^5}{3} dx$$

$$= \left(\frac{8}{81} \right) \left(\frac{3^6}{18} \right) = 4$$

$$\sigma_{xy} = 4 - \left(\frac{12}{5} \right) \left(\frac{8}{5} \right) = 0.16$$

$$E(X^2) = 6 \quad E(Y^2) = 3$$

$$V(X) = 0.24, \quad V(Y) = 0.44$$

$$\rho = \frac{0.16}{\sqrt{0.24}\sqrt{0.44}} = 0.4924$$

5-73. Similarly to 5-49, $c = 2/19$

$$E(X) = \frac{2}{19} \int_0^1 \int_0^{x+1} x dy dx + \frac{2}{19} \int_1^5 \int_{x-1}^{x+1} x dy dx = 2.614$$

$$E(Y) = \frac{2}{19} \int_0^1 \int_0^{x+1} y dy dx + \frac{2}{19} \int_1^5 \int_{x-1}^{x+1} y dy dx = 2.649$$

$$\text{Now, } E(XY) = \frac{2}{19} \int_0^1 \int_0^{x+1} xy dy dx + \frac{2}{19} \int_1^5 \int_{x-1}^{x+1} xy dy dx = 8.7763$$

$$\sigma_{xy} = 8.7763 - (2.614)(2.649) = 1.85181$$

$$E(X^2) = 8.7632 \quad E(Y^2) = 9.11403$$

$$V(X) = 1.930, \quad V(Y) = 2.0968$$

$$\rho = \frac{1.852}{\sqrt{1.930}\sqrt{2.097}} = 0.9206$$

5-74

$$E(X) = \int_0^1 \int_0^{x+1} \frac{x}{7.5} dy dx + \int_1^4 \int_{x-1}^{x+1} \frac{x}{7.5} dy dx$$

$$= \frac{1}{7.5} \int_0^1 (x^2 + x) dx + \frac{2}{7.5} \int_1^4 (x) dx = \frac{12}{15} \left(\frac{5}{6}\right) + \frac{2}{7.5} (7.5) = \frac{19}{9}$$

$$E(X^2) = 222,222.2$$

$$V(X) = 222222.2 - (333.33)^2 = 111,113.31$$

$$E(Y^2) = 1,055,556$$

$$V(Y) = 361,117.11$$

$$E(XY) = 6 \times 10^{-6} \int_0^\infty \int_x^\infty xye^{-.001x-.002y} dy dx = 388,888.9$$

$$\sigma_{xy} = 388,888.9 - (333.33)(833.33) = 111,115.01$$

$$\rho = \frac{111,115.01}{\sqrt{111113.31} \sqrt{361117.11}} = 0.5547$$

5-75 a) $E(X) = 1$ $E(Y) = 1$

$$E(XY) = \int_0^\infty \int_0^\infty xye^{-x-y} dx dy$$

$$= \int_0^\infty xe^{-x} dx \int_0^\infty ye^{-y} dy$$

$$= E(X)E(Y)$$

Therefore, $\sigma_{XY} = \rho_{XY} = 0$.

5-76 Suppose the correlation between X and Y is ρ . for constants a, b, c, and d, what is the correlation between the random variables $U = aX+b$ and $V = cY+d$?

Now, $E(U) = a E(X) + b$ and $E(V) = c E(Y) + d$.

Also, $U - E(U) = a[X - E(X)]$ and $V - E(V) = c[Y - E(Y)]$. Then,

$$\sigma_{UV} = E\{[U - E(U)][V - E(V)]\} = acE\{[X - E(X)][Y - E(Y)]\} = ac\sigma_{XY}$$

Also, $\sigma_U^2 = E[U - E(U)]^2 = a^2 E[X - E(X)]^2 = a^2 \sigma_X^2$ and $\sigma_V^2 = c^2 \sigma_Y^2$. Then,

$$\rho_{UV} = \frac{ac\sigma_{XY}}{\sqrt{a^2\sigma_X^2}\sqrt{c^2\sigma_Y^2}} = \begin{cases} \rho_{XY} & \text{if a and c are of the same sign} \\ -\rho_{XY} & \text{if a and c differ in sign} \end{cases}$$

$$5-77 \quad E(X) = -1(1/4) + 1(1/4) = 0$$

$$E(Y) = -1(1/4) + 1(1/4) = 0$$

$$E(XY) = [-1 \times 0 \times (1/4)] + [-1 \times 0 \times (1/4)] + [1 \times 0 \times (1/4)] + [0 \times 1 \times (1/4)] = 0$$

$$V(X) = 1/2$$

$$V(Y) = 1/2$$

$$\sigma_{XY} = 0 - (0)(0) = 0$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{0}{\sqrt{1/2} \sqrt{1/2}} = 0$$

The correlation is zero, but X and Y are not independent, since, for example, if $y = 0$, X must be -1 or 1 .

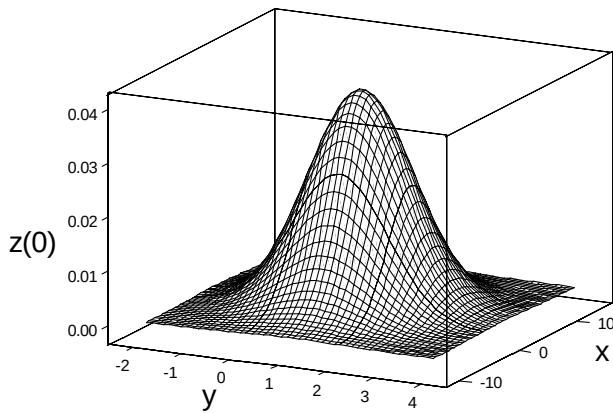
5-78 If X and Y are independent, then $f_{XY}(x, y) = f_X(x) f_Y(y)$ and the range of (X, Y) is rectangular. Therefore,

$$E(XY) = \iint xy f_X(x) f_Y(y) dx dy = \int x f_X(x) dx \int y f_Y(y) dy = E(X)E(Y)$$

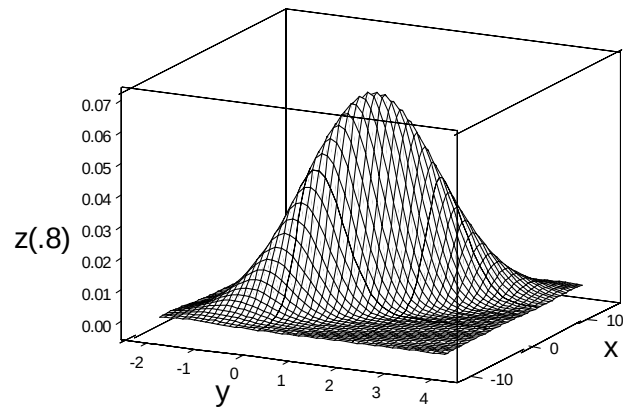
hence $\sigma_{XY} = 0$

Section 5-6

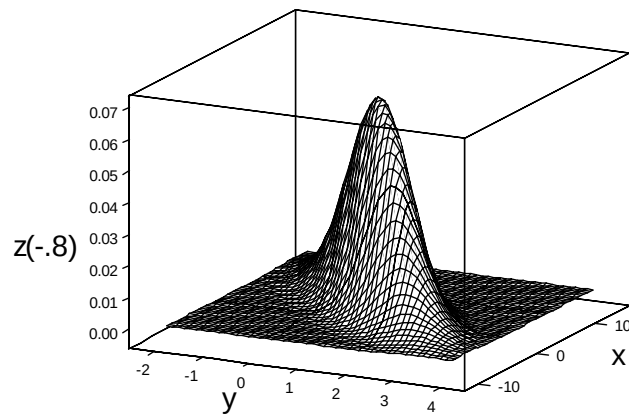
5-79 a)



b)



c)



5-80 Because $\rho = 0$ and X and Y are normally distributed, X and Y are independent.

Therefore,

$$P(2.95 < X < 3.05, 7.60 < Y < 7.80) = P(2.95 < X < 3.05) P(7.60 < Y < 7.80) = P\left(\frac{2.95-3}{0.04} < Z < \frac{3.05-3}{0.04}\right) P\left(\frac{7.60-7.70}{0.08} < Z < \frac{7.80-7.70}{0.08}\right) = 0.7887^2 = 0.6220$$

5-81 Because $\rho = 0$ and X and Y are normally distributed, X and Y are independent.

Therefore, $\mu_X = 0.1$ mm, $\sigma_X = 0.00031$ mm, $\mu_Y = 0.23$ mm, $\sigma_Y = 0.00017$ mm

Probability X is within specification limits is

$$P(0.099535 < X < 0.100465) = P\left(\frac{0.099535 - 0.1}{0.00031} < Z < \frac{0.100465 - 0.1}{0.00031}\right) = P(-1.5 < Z < 1.5) = P(Z < 1.5) - P(Z < -1.5) = 0.8664$$

Probability that Y is within specification limits is

$$P(0.22966 < X < 0.23034) = P\left(\frac{0.22966 - 0.23}{0.00017} < Z < \frac{0.23034 - 0.23}{0.00017}\right) = P(-2 < Z < 2) = P(Z < 2) - P(Z < -2) = 0.9545$$

Probability that a randomly selected lamp is within specification limits is $(0.8664)(0.9545) = 0.8270$

5-82 a) By completing the square in the numerator of the exponent of the bivariate normal PDF, the joint PDF can be written as

$$f_{Y|X=x} = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\left[\frac{1}{\sigma_y^2}\left(y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X))\right)\right]^2 + (1-\rho^2)\left(\frac{x-\mu_X}{\sigma_X}\right)^2}}{2(1-\rho^2)}}}{\frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{\left[\frac{x-\mu_X}{\sigma_X}\right]^2}{2}}}$$

$$\text{Also, } f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{\left[\frac{x-\mu_X}{\sigma_X}\right]^2}{2}}$$

By definition,

$$\begin{aligned} f_{Y|X=x} &= \frac{f_{XY}(x, y)}{f_X(x)} = \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\left[\frac{1}{\sigma_y^2}\left(y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X))\right)\right]^2 + (1-\rho^2)\left(\frac{x-\mu_X}{\sigma_X}\right)^2}}{2(1-\rho^2)}}}{\frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{\left[\frac{x-\mu_X}{\sigma_X}\right]^2}{2}}} \\ &= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\frac{1}{\sigma_y^2}\left[y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X))\right]^2}{2(1-\rho^2)}} \end{aligned}$$

Now $f_{Y|X=x}$ is in the form of a normal distribution.

b) $E(Y|X=x) = \mu_y + \rho \frac{\sigma_y}{\sigma_x}(x - \mu_x)$. This answer can be seen from part 5-82a. Since the PDF is in the form of a normal distribution, then the mean can be obtained from the exponent.

c) $V(Y|X=x) = \sigma_y^2(1 - \rho^2)$. This answer can be seen from part 5-82a. Since the PDF is in the form of a normal distribution, then the variance can be obtained from the exponent.

5-83

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2} \left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} \right]} \right] dx dy = \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2} \left[\frac{(x-\mu_x)^2}{\sigma_x^2} \right]} \right] dx \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{1}{2} \left[\frac{(y-\mu_y)^2}{\sigma_y^2} \right]} \right] dy \end{aligned}$$

and each of the last two integrals is recognized as the integral of a normal probability density function from $-\infty$ to ∞ . That is, each integral equals one. Since $f_{XY}(x, y) = f(x)f(y)$ then X and Y are independent.

5-84 Let
$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\left[\left(\frac{x-\mu_x}{\sigma_x} \right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y} \right)^2 \right]}{2(1-\rho^2)}}$$

Completing the square in the numerator of the exponent we get:

$$\left[\left(\frac{x-\mu_x}{\sigma_x} \right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y} \right)^2 \right] = \left[\left(\frac{y-\mu_y}{\sigma_y} - \rho \left(\frac{x-\mu_x}{\sigma_x} \right) \right)^2 + (1-\rho^2) \left(\frac{x-\mu_x}{\sigma_x} \right)^2 \right]$$

But,

$$\left(\frac{y-\mu_y}{\sigma_y} - \rho \left(\frac{x-\mu_x}{\sigma_x} \right) \right) = \frac{1}{\sigma_y} \left[(y-\mu_y) - \rho \frac{\sigma_y}{\sigma_x} (x-\mu_x) \right] = \frac{1}{\sigma_y} \left[(y - (\mu_y + \rho \frac{\sigma_y}{\sigma_x} (x-\mu_x))) \right]$$

Substituting into $f_{XY}(x, y)$, we get

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\left[\frac{1}{\sigma_y^2} \left[y - (\mu_y + \rho \frac{\sigma_y}{\sigma_x} (x-\mu_x)) \right]^2 + (1-\rho^2) \left(\frac{x-\mu_x}{\sigma_x} \right)^2 \right]}{2(1-\rho^2)}} dy dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2} \left(\frac{x-\mu_x}{\sigma_x} \right)^2} dx \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\left[\left(y - (\mu_y + \rho \frac{\sigma_y}{\sigma_x} (x-\mu_x)) \right)^2 \right]}{2\sigma_x^2(1-\rho^2)}} dy \end{aligned}$$

The integrand in the second integral above is in the form of a normally distributed random variable. By definition of the integral over this function, the second integral is equal to 1:

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\left[\frac{\left(y-(\mu_y+\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x))\right)^2}{2\sigma_x^2(1-\rho^2)}\right]} dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx \times 1
\end{aligned}$$

The remaining integral is also the integral of a normally distributed random variable and therefore, it also integrates to 1, by definition. Therefore,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) = 1$$

5-85

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{\frac{-0.5}{1-\rho^2} \left[\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2} \right]} \right] dy \\
&= \frac{1}{\sqrt{2\pi}\sigma_x} e^{\frac{-0.5(x-\mu_x)^2}{1-\rho^2}\sigma_x^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{\frac{-0.5}{1-\rho^2} \left[\left(\frac{(y-\mu_y)}{\sigma_y} - \frac{\rho(x-\mu_x)}{\sigma_x} \right)^2 - \left(\frac{\rho(x-\mu_x)}{\sigma_x} \right)^2 \right]} dy \\
&= \frac{1}{\sqrt{2\pi}\sigma_x} e^{\frac{-0.5(x-\mu_x)^2}{1-\rho^2}\sigma_x^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{\frac{-0.5}{1-\rho^2} \left[\frac{(y-\mu_y)}{\sigma_y} - \frac{\rho(x-\mu_x)}{\sigma_x} \right]^2} dy
\end{aligned}$$

The last integral is recognized as the integral of a normal probability density with mean $\mu_y + \frac{\sigma_y\rho(x-\mu_x)}{\sigma_x}$ and variance $\sigma_y^2(1-\rho^2)$. Therefore, the last integral equals one and the requested result is obtained.

5-86 $E(X) = \mu_X, E(Y) = \mu_Y, V(X) = \sigma_X^2$, and $V(Y) = \sigma_Y^2$. Also,

$$E(X - \mu_X)(Y - \mu_Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(x - \mu_X)(y - \mu_Y) e^{\frac{-0.5}{1-\rho^2} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} \right]}}{2\pi\sigma_X\sigma_Y(1-\rho^2)^{1/2}} dx dy$$

Let $u = \frac{x-\mu_X}{\sigma_X}$ and $v = \frac{y-\mu_Y}{\sigma_Y}$. Then,

$$\begin{aligned} E(X - \mu_X)(Y - \mu_Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{uve^{\frac{-0.5}{1-\rho^2} [u^2 - 2\rho uv + v^2]}}{2\pi(1-\rho^2)^{1/2}} \sigma_X \sigma_Y du dv \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{uve^{\frac{-0.5}{1-\rho^2} \{ [u-\rho v]^2 + (1-\rho^2)v^2 \}}}{2\pi(1-\rho^2)^{1/2}} \sigma_X \sigma_Y du dv \end{aligned}$$

The integral with respect to u is recognized as a constant times the mean of a normal random variable with mean ρv and variance $1-\rho^2$. Therefore,

$$E(X - \mu_X)(Y - \mu_Y) = \int_{-\infty}^{\infty} \frac{v}{\sqrt{2\pi}} e^{-0.5v^2} \rho v \sigma_X \sigma_Y dv = \rho \sigma_X \sigma_Y \int_{-\infty}^{\infty} \frac{v^2}{\sqrt{2\pi}} e^{-0.5v^2} dv.$$

The last integral is recognized as the variance of a normal random variable with mean 0 and variance 1. Therefore, $E(X - \mu_X)(Y - \mu_Y) = \rho \sigma_X \sigma_Y$ and the correlation between X and Y is ρ .

Section 5-7

- 5-87 a) $E(2X + 3Y) = 2(0) + 3(10) = 30$
 b) $V(2X + 3Y) = 4V(X) + 9V(Y) = 97$
 c) $2X + 3Y$ is normally distributed with mean 30 and variance 97. Therefore,
 $P(2X + 3Y < 30) = P(Z < \frac{30-30}{\sqrt{97}}) = P(Z < 0) = 0.5$
 d) $P(2X + 3Y < 40) = P(Z < \frac{40-30}{\sqrt{97}}) = P(Z < 1.02) = 0.8461$

- 5-88 $Y = 10X$ and $E(Y) = 10E(X) = 50\text{mm}$.
 $V(Y) = 10^2 V(X) = 25\text{mm}^2$

- 5-89 a) Let T denote the total thickness. Then, $T = X + Y$ and $E(T) = 4$ mm,
 $V(T) = 0.1^2 + 0.1^2 = 0.02\text{mm}^2$, and $\sigma_T = 0.1414$ mm.

b)

$$P(T > 4.3) = P\left(Z > \frac{4.3 - 4}{0.1414}\right) = P(Z > 2.12) \\ = 1 - P(Z < 2.12) = 1 - 0.983 = 0.0170$$

- 5-90 a) $X \sim N(0.1, 0.00031)$ and $Y \sim N(0.23, 0.00017)$ Let T denote the total thickness.
Then, $T = X + Y$ and $E(T) = 0.33$ mm,

$$V(T) = 0.00031^2 + 0.00017^2 = 1.25 \times 10^{-7} \text{mm}^2, \text{ and } \sigma_T = 0.000354 \text{ mm.}$$

$$P(T < 0.2337) = P\left(Z < \frac{0.2337 - 0.33}{0.000354}\right) = P(Z < -272) \cong 0$$

b)

$$P(T > 0.2405) = P\left(Z > \frac{0.2405 - 0.33}{0.000345}\right) = P(Z > -253) = 1 - P(Z < 253) \cong 1$$

- 5-91 Let D denote the width of the casing minus the width of the door. Then, D is normally distributed.

$$a) E(D) = 1/8 \quad V(D) = \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 = \frac{5}{256}$$

$$b) P(D > \frac{1}{4}) = P\left(Z > \frac{\frac{1}{4} - \frac{1}{8}}{\sqrt{5/256}}\right) = P(Z > 0.89) = 0.187$$

$$c) P(D < 0) = P\left(Z < \frac{0 - \frac{1}{8}}{\sqrt{5/256}}\right) = P(Z < -0.89) = 0.187$$

- 5-92 $D = A - B - C$

$$a) E(D) = 10 - 2 - 2 = 6 \text{ mm}$$

$$V(D) = 0.1^2 + 0.05^2 + 0.05^2 = 0.015\text{mm}^2$$

$$\sigma_D = 0.1225\text{mm}$$

$$b) P(D < 5.9) = P\left(Z < \frac{5.9 - 6}{0.1225}\right) = P(Z < -0.82) = 0.206.$$

- 5-93 a) Let $\bar{X}$ denote the average fill-volume of 100 cans. $\sigma_{\bar{X}} = \sqrt{0.5^2/100} = 0.05$.

$$b) E(\bar{X}) = 12.1 \text{ and } P(\bar{X} < 12) = P\left(Z < \frac{12 - 12.1}{0.05}\right) = P(Z < -2) = 0.023$$

$$c) P(\bar{X} < 12) = 0.005 \text{ implies that } P\left(Z < \frac{12 - \mu}{0.05}\right) = 0.005.$$

$$\text{Then } \frac{12 - \mu}{0.05} = -2.58 \text{ and } \mu = 12.129.$$

$$d) P(\bar{X} < 12) = 0.005 \text{ implies that } P\left(Z < \frac{12 - 12.1}{\sigma/\sqrt{100}}\right) = 0.005.$$

$$\text{Then } \frac{12 - 12.1}{\sigma/\sqrt{100}} = -2.58 \text{ and } \sigma = 0.388.$$

e) $P(\bar{X} < 12) = 0.01$ implies that $P\left(Z < \frac{12 - 12.1}{0.5/\sqrt{n}}\right) = 0.01$.

Then $\frac{12 - 12.1}{0.5/\sqrt{n}} = -2.33$ and $n = 135.72 \approx 136$.

5-94 Let $\bar{X}$ denote the average thickness of 10 wafers. Then, $E(\bar{X}) = 10$ and $V(\bar{X}) = 0.1$.

a) $P(9 < \bar{X} < 11) = P\left(\frac{9-10}{\sqrt{0.1}} < Z < \frac{11-10}{\sqrt{0.1}}\right) = P(-3.16 < Z < 3.16) = 0.998$.

The answer is $1 - 0.998 = 0.002$

b) $P(\bar{X} > 11) = 0.01$ and $\sigma_{\bar{X}} = 1/\sqrt{n}$.

Therefore, $P(\bar{X} > 11) = P(Z > \frac{11-10}{1/\sqrt{n}}) = 0.01$, $\frac{11-10}{1/\sqrt{n}} = 2.33$ and $n = 5.43$ which is rounded up to 6.

c) $P(\bar{X} > 11) = 0.0005$ and $\sigma_{\bar{X}} = \sigma/\sqrt{10}$.

Therefore, $P(\bar{X} > 11) = P(Z > \frac{11-10}{\sigma/\sqrt{10}}) = 0.0005$, $\frac{11-10}{\sigma/\sqrt{10}} = 3.29$

$\sigma = \sqrt{10} / 3.29 = 0.9612$

5-95 $X \sim N(160, 900)$

a) Let $Y = X_1 + X_2 + \dots + X_{25}$, $E(Y) = 25E(X) = 4000$, $V(Y) = 25^2(900) = 22500$

$P(Y > 4300) =$

$P\left(Z > \frac{4300 - 4000}{\sqrt{22500}}\right) = P(Z > 2) = 1 - P(Z < 2) = 1 - 0.9773 = 0.0227$

b) c) $P(Y > x) = 0.0001$ implies that $P\left(Z > \frac{x - 4000}{\sqrt{22500}}\right) = 0.0001$.

Then $\frac{x - 4000}{150} = 3.72$ and $x = 4558$

Supplemental Exercises

5-96 The sum of $\sum_x \sum_y f(x, y) = 1$, $\left(\frac{1}{4}\right) + \left(\frac{1}{8}\right) + \left(\frac{1}{8}\right) + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right) = 1$

and $f_{XY}(x, y) \geq 0$

5-97 a) $P(X < 0.5, Y < 1.5) = f_{XY}(0,1) + f_{XY}(0,0) = 1/8 + 1/4 = 3/8$.

b) $P(X \leq 1) = f_{XY}(0,0) + f_{XY}(0,1) + f_{XY}(1,0) + f_{XY}(1,1) = 3/4$

c) $P(Y < 1.5) = f_{XY}(0,0) + f_{XY}(0,1) + f_{XY}(1,0) + f_{XY}(1,1) = 3/4$

d) $P(X > 0.5, Y < 1.5) = f_{XY}(1,0) + f_{XY}(1,1) = 3/8$

e) $E(X) = 0(3/8) + 1(3/8) + 2(1/4) = 7/8$.

$V(X) = 0^2(3/8) + 1^2(3/8) + 2^2(1/4) - 7/8^2 = 39/64$

$E(Y) = 1(3/8) + 0(3/8) + 2(1/4) = 7/8$.

$V(Y) = 1^2(3/8) + 0^2(3/8) + 2^2(1/4) - 7/8^2 = 39/64$

5-98 a) $f_X(x) = \sum_y f_{XY}(x,y)$ and $f_X(0) = 3/8$, $f_X(1) = 3/8$, $f_X(2) = 1/4$.

b) $f_{Y|1}(y) = \frac{f_{XY}(1,y)}{f_X(1)}$ and $f_{Y|1}(0) = \frac{1/8}{3/8} = 1/3$, $f_{Y|1}(1) = \frac{1/4}{3/8} = 2/3$.

c) $E(Y | X = 1) = \sum_{y=1} y f_{Y|X=1}(y) = 0(1/3) + 1(2/3) = 2/3$

d) As is discussed after Example 5-19, because the range of (X, Y) is not rectangular, X and Y are not independent.

e) $E(XY) = 1.25$, $E(X) = E(Y) = 0.875$, $V(X) = V(Y) = 0.6094$
 $\text{COV}(X,Y) = E(XY) - E(X)E(Y) = 1.25 - 0.875^2 = 0.4844$

$$\rho_{XY} = \frac{0.4844}{\sqrt{0.6094}\sqrt{0.6094}} = 0.7949$$

5-99 a) $P(X = 2, Y = 4, Z = 14) = \frac{20!}{2!4!14!} 0.10^2 0.20^4 0.70^{14} = 0.0631$

b) $P(X = 0) = 0.10^0 0.90^{20} = 0.1216$

c) $E(X) = np_1 = 20(0.10) = 2$

$V(X) = np_1(1 - p_1) = 20(0.10)(0.9) = 1.8$

d) $f_{X|Z=z}(X | Z = 19) = \frac{f_{XZ}(x, z)}{f_Z(z)}$

$$f_{XZ}(xz) = \frac{20!}{x!z!(20-x-z)!} 0.1^x 0.2^{20-x-z} 0.7^z$$

$$f_Z(z) = \frac{20!}{z!(20-z)!} 0.3^{20-z} 0.7^z$$

$$f_{X|Z=z}(X | Z = 19) = \frac{f_{XZ}(x, z)}{f_Z(z)} = \frac{(20-z)!}{x!(20-x-z)!} \frac{0.1^x 0.2^{20-x-z}}{0.3^{20-z}} = \frac{(20-z)!}{x!(20-x-z)!} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{20-x-z}$$

Therefore, X is a binomial random variable with $n=20-z$ and $p=1/3$. When $z=19$,

$$f_{X|19}(0) = \frac{2}{3} \text{ and } f_{X|19}(1) = \frac{1}{3}.$$

e) $E(X | Z = 19) = 0\left(\frac{2}{3}\right) + 1\left(\frac{1}{3}\right) = \frac{1}{3}$

5-100 Let X, Y, and Z denote the number of bolts rated high, moderate, and low. Then, X, Y, and Z have a multinomial distribution.

a) $P(X = 12, Y = 6, Z = 2) = \frac{20!}{12!6!2!} 0.6^{12} 0.3^6 0.1^2 = 0.0560$.

b) Because X, Y, and Z are multinomial, the marginal distribution of Z is binomial with $n = 20$ and $p = 0.1$.

c) $E(Z) = np = 20(0.1) = 2$.

5-101. a) $f_{Z|16}(z) = \frac{f_{XZ}(16, z)}{f_X(16)}$ and $f_{XZ}(x, z) = \frac{20!}{x!z!(20-x-z)!} 0.6^x 0.3^{(20-x-z)} 0.1^z$ for

$x + z \leq 20$ and $0 \leq x, 0 \leq z$. Then,

$$f_{Z|16}(z) = \frac{\frac{20!}{16!z!(4-z)!} 0.6^{16} 0.3^{(4-z)} 0.1^z}{\frac{20!}{16!4!} 0.6^{16} 0.4^4} = \frac{4!}{z!(4-z)!} \left(\frac{0.3}{0.4}\right)^{4-z} \left(\frac{0.1}{0.4}\right)^z$$

for $0 \leq z \leq 4$. That is the distribution of Z given X = 16 is binomial with n = 4 and p = 0.25.

b) From part a., $E(Z) = 4(0.25) = 1$.

c) Because the conditional distribution of Z given X = 16 does not equal the marginal distribution of Z, X and Z are not independent.

5-102 Let X, Y, and Z denote the number of calls answered in two rings or less, three or four rings, and five rings or more, respectively.

a) $P(X = 8, Y = 1, Z = 1) = \frac{10!}{8!1!1!} 0.7^8 0.25^1 0.05^1 = 0.0649$

b) Let W denote the number of calls answered in four rings or less. Then, W is a binomial random variable with n = 10 and p = 0.95.

Therefore, $P(W = 10) = \binom{10}{10} 0.95^{10} 0.05^0 = 0.5987$.

c) $E(W) = 10(0.95) = 9.5$.

5-103 a) $f_{Z|8}(z) = \frac{f_{XZ}(8, z)}{f_X(8)}$ and $f_{XZ}(x, z) = \frac{10!}{x!z!(10-x-z)!} 0.70^x 0.25^{(10-x-z)} 0.05^z$ for

$x + z \leq 10$ and $0 \leq x, 0 \leq z$. Then,

$$f_{Z|8}(z) = \frac{\frac{10!}{8!z!(2-z)!} 0.70^8 0.25^{(2-z)} 0.05^z}{\frac{10!}{8!2!} 0.70^8 0.30^2} = \frac{2!}{z!(2-z)!} \left(\frac{0.25}{0.30}\right)^{2-z} \left(\frac{0.05}{0.30}\right)^z$$

for $0 \leq z \leq 2$. That is Z is binomial with n = 2 and p = 0.05/0.30 = 1/6.

b) $E(Z)$ given X = 8 is $2(1/6) = 1/3$.

c) Because the conditional distribution of Z given X = 8 does not equal the marginal distribution of Z, X and Z are not independent.

5-104 $\int_0^3 \int_0^2 cx^2 y dy dx = \int_0^3 cx^2 \frac{y^2}{2} \Big|_0^2 dx = 2c \frac{x^3}{3} \Big|_0^3 = 18c$. Therefore, c = 1/18.

$$5-105 \quad a) \quad P(X < 1, Y < 1) = \int_0^1 \int_0^1 \frac{1}{18} x^2 y dy dx = \int_0^1 \frac{1}{18} x^2 \frac{y^2}{2} \Big|_0^1 dx = \frac{1}{36} \frac{x^3}{3} \Big|_0^1 = \frac{1}{108}$$

$$b) \quad P(X < 2.5) = \int_0^{2.5} \int_0^2 \frac{1}{18} x^2 y dy dx = \int_0^{2.5} \frac{1}{18} x^2 \frac{y^2}{2} \Big|_0^2 dx = \frac{1}{9} \frac{x^3}{3} \Big|_0^{2.5} = 0.5787$$

$$c) \quad P(1 < Y < 2.5) = \int_0^3 \int_1^2 \frac{1}{18} x^2 y dy dx = \int_0^3 \frac{1}{18} x^2 \frac{y^2}{2} \Big|_1^2 dx = \frac{1}{12} \frac{x^3}{3} \Big|_0^3 = \frac{3}{4}$$

d)

$$P(X > 2, 1 < Y < 1.5) = \int_2^3 \int_1^{1.5} \frac{1}{18} x^2 y dy dx = \int_2^3 \frac{1}{18} x^2 \frac{y^2}{2} \Big|_1^{1.5} dx = \frac{5}{144} \frac{x^3}{3} \Big|_2^3 = \frac{95}{432} = 0.2199$$

$$e) \quad E(X) = \int_0^3 \int_0^2 \frac{1}{18} x^3 y dy dx = \int_0^3 \frac{1}{18} x^3 2 dx = \frac{1}{9} \frac{x^4}{4} \Big|_0^3 = \frac{9}{4}$$

$$f) \quad E(Y) = \int_0^3 \int_0^2 \frac{1}{18} x^2 y^2 dy dx = \int_0^3 \frac{1}{18} x^2 \frac{8}{3} dx = \frac{4}{27} \frac{x^3}{3} \Big|_0^3 = \frac{4}{3}$$

$$5-106 \quad a) \quad f_X(x) = \int_0^2 \frac{1}{18} x^2 y dy = \frac{1}{9} x^2 \text{ for } 0 < x < 3$$

$$b) \quad f_{Y|X}(y) = \frac{f_{XY}(1, y)}{f_X(1)} = \frac{\frac{1}{18} y}{\frac{1}{9}} = \frac{y}{2} \text{ for } 0 < y < 2.$$

$$c) \quad f_{X|1}(x) = \frac{f_{XY}(x, 1)}{f_Y(1)} = \frac{\frac{1}{18} x^2}{\frac{1}{9}} \text{ and } f_Y(y) = \int_0^3 \frac{1}{18} x^2 y dx = \frac{y}{2} \text{ for } 0 < y < 2.$$

$$\text{Therefore, } f_{X|1}(x) = \frac{\frac{1}{18} x^2}{1/2} = \frac{1}{9} x^2 \text{ for } 0 < x < 3.$$

5-107 The region $x^2 + y^2 \leq 1$ and $0 < z < 4$ is a cylinder of radius 1 (and base area π) and height 4. Therefore, the volume of the cylinder is 4π and $f_{XYZ}(x, y, z) = \frac{1}{4\pi}$ for $x^2 + y^2 \leq 1$ and $0 < z < 4$.

a) The region $x^2 + y^2 \leq 0.5$ is a cylinder of radius $\sqrt{0.5}$ and height 4. Therefore,
 $P(X^2 + Y^2 \leq 0.5) = \frac{4(0.5\pi)}{4\pi} = 1/2$.

b) The region $x^2 + y^2 \leq 0.5$ and $0 < z < 2$ is a cylinder of radius $\sqrt{0.5}$ and height 2. Therefore,

$$P(X^2 + Y^2 \leq 0.5, Z < 2) = \frac{2(0.5\pi)}{4\pi} = 1/4$$

$$c) f_{XY|1}(x, y) = \frac{f_{XYZ}(x, y, 1)}{f_Z(1)} \text{ and } f_Z(z) = \iint_{x^2+y^2 \leq 1} \frac{1}{4\pi} dydx = 1/4$$

$$\text{for } 0 < z < 4. \text{ Then, } f_{XY|1}(x, y) = \frac{1/4\pi}{1/4} = \frac{1}{\pi} \text{ for } x^2 + y^2 \leq 1.$$

$$d) f_X(x) = \int_0^4 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{4\pi} dydz = \int_0^4 \frac{1}{2\pi} \sqrt{1-x^2} dz = \frac{2}{\pi} \sqrt{1-x^2} \text{ for } -1 < x < 1$$

$$5-108 \quad a) f_{Z|0,0}(z) = \frac{f_{XYZ}(0,0,z)}{f_{XY}(0,0)} \text{ and } f_{XY}(x, y) = \int_0^4 \frac{1}{4\pi} dz = 1/\pi \text{ for } x^2 + y^2 \leq 1. \text{ Then,}$$

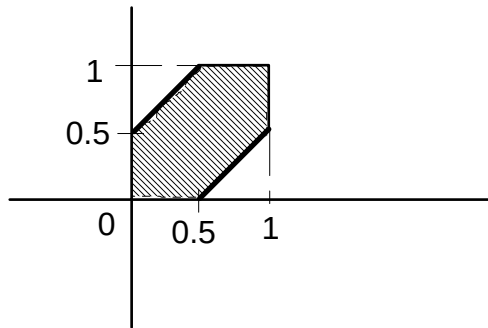
$$f_{Z|0,0}(z) = \frac{1/4\pi}{1/\pi} = 1/4 \text{ for } 0 < z < 4 \text{ and } \mu_{Z|0,0} = 2.$$

$$b) f_{Z|xy}(z) = \frac{f_{XYZ}(x, y, z)}{f_{XY}(x, y)} = \frac{1/4\pi}{1/\pi} = 1/4 \text{ for } 0 < z < 4. \text{ Then, } E(Z) \text{ given } X = x \text{ and } Y = y \text{ is}$$

$$\int_0^4 \frac{z}{4} dz = 2.$$

$$5-109 \quad f_{XY}(x, y) = c \text{ for } 0 < x < 1 \text{ and } 0 < y < 1. \text{ Then, } \int_0^1 \int_0^1 c dx dy = 1 \text{ and } c = 1. \text{ Because}$$

$f_{XY}(x, y)$ is constant, $P(|X - Y| < 0.5)$ is the area of the shaded region below



That is, $P(|X - Y| < 0.5) = 3/4.$

5-110 a) Let $x_1, x_2, \dots, x_6$ denote the lifetimes of the six components, respectively. Because of independence,

$$P(X_1 > 5000, X_2 > 5000, \dots, X_6 > 5000) = P(X_1 > 5000)P(X_2 > 5000) \dots P(X_6 > 5000)$$

If X is exponentially distributed with mean θ , then $\lambda = \frac{1}{\theta}$ and

$$P(X > x) = \int_x^\infty \frac{1}{\theta} e^{-t/\theta} dt = -e^{-t/\theta} \Big|_x^\infty = e^{-x/\theta}. \text{ Therefore, the answer is}$$

$$e^{-5/8} e^{-0.5} e^{-0.25} e^{-0.25} e^{-0.2} e^{-0.2} = e^{-2.325} = 0.0978.$$

b) The probability that at least one component lifetime exceeds 25,000 hours is the same as 1 minus the probability that none of the component lifetimes exceed 25,000 hours. Thus,

$$1 - P(X_1 < 25,000, X_2 < 25,000, \dots, X_6 < 25,000) = 1 - P(X_1 < 25,000) \dots P(X_6 < 25,000) \\ = 1 - (1 - e^{-25/8})(1 - e^{-2.5})(1 - e^{-2.5})(1 - e^{-1.25})(1 - e^{-1.25})(1 - e^{-1}) = 1 - 0.2592 = 0.7408$$

5-111 Let X, Y, and Z denote the number of problems that result in functional, minor, and no defects, respectively.

a) $P(X = 2, Y = 5) = P(X = 2, Y = 5, Z = 3) = \frac{10!}{2!5!3!} 0.2^2 0.5^5 0.3^3 = 0.085$

b) Z is binomial with n = 10 and p = 0.3.

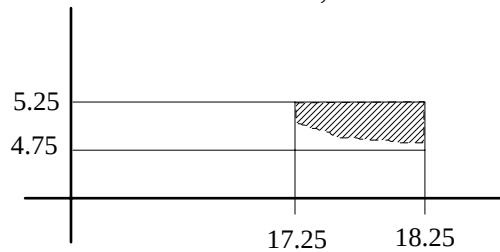
c) $E(Z) = 10(0.3) = 3.$

5-112 a) Let $\bar{X}$ denote the mean weight of the 25 bricks in the sample. Then, $E(\bar{X}) = 3$ and $\sigma_{\bar{X}} = \frac{0.25}{\sqrt{25}} = 0.05$. Then, $P(\bar{X} < 2.95) = P(Z < \frac{2.95 - 3}{0.05}) = P(Z < -1) = 0.159.$

b) $P(\bar{X} > x) = P(Z > \frac{x - 3}{0.05}) = 0.99$. So, $\frac{x - 3}{0.05} = -2.33$ and $x = 2.8835.$

5-113 a)

Because $\int_{17.75}^{18.25} \int_{4.75}^{5.25} c dy dx = 0.25c$, $c = 4$. The area of a panel is XY and $P(XY > 90)$ is the shaded area times 4 below,



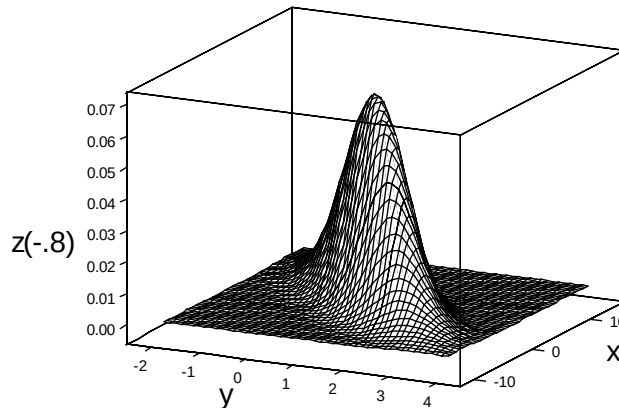
That is, $\int_{17.75}^{18.25} \int_{90/x}^{5.25} 4 dy dx = 4 \int_{17.75}^{18.25} 5.25 - \frac{90}{x} dx = 4 \left(5.25x - 90 \ln x \right) \Big|_{17.75}^{18.25} = 0.499$

b) The perimeter of a panel is $2X + 2Y$ and we want $P(2X + 2Y > 46)$

$$\int_{17.75}^{18.25} \int_{23-x}^{5.25} 4 dy dx = 4 \int_{17.75}^{18.25} 5.25 - (23 - x) dx \\ = 4 \int_{17.75}^{18.25} (-17.75 + x) dx = 4 \left(-17.75x + \frac{x^2}{2} \right) \Big|_{17.75}^{18.25} = 0.5$$

- 5-114 a) Let X denote the weight of a piece of candy and $X \sim N(0.1, 0.01)$. Each package has 16 candies, then P is the total weight of the package with 16 pieces and $E(P) = 16(0.1) = 1.6$ ounces and $V(P) = 16^2(0.01^2) = 0.0256$ ounces²
 b) $P(P < 1.6) = P(Z < \frac{1.6-1.6}{0.16}) = P(Z < 0) = 0.5$.
 c) Let Y equal the total weight of the package with 17 pieces, $E(Y) = 17(0.1) = 1.7$ ounces and $V(Y) = 17^2(0.01^2) = 0.0289$ ounces²
 $P(Y < 1.6) = P(Z < \frac{1.6-1.7}{\sqrt{0.0289}}) = P(Z < -0.59) = 0.2776$.
- 5-115 Let $\bar{X}$ denote the average time to locate 10 parts. Then, $E(\bar{X}) = 45$ and $\sigma_{\bar{X}} = \frac{30}{\sqrt{10}}$
 a) $P(\bar{X} > 60) = P(Z > \frac{60-45}{30/\sqrt{10}}) = P(Z > 1.58) = 0.057$
 b) Let Y denote the total time to locate 10 parts. Then, $Y > 600$ if and only if $\bar{X} > 60$. Therefore, the answer is the same as part a.
- 5-116 a) Let Y denote the weight of an assembly. Then, $E(Y) = 4 + 5.5 + 10 + 8 = 27.5$ and $V(Y) = 0.4^2 + 0.5^2 + 0.2^2 + 0.5^2 = 0.7$.
 $P(Y > 29.5) = P(Z > \frac{29.5-27.5}{\sqrt{0.7}}) = P(Z > 2.39) = 0.0084$
 b) Let $\bar{X}$ denote the mean weight of 8 independent assemblies. Then, $E(\bar{X}) = 27.5$ and $V(\bar{X}) = 0.7/8 = 0.0875$. Also, $P(\bar{X} > 29) = P(Z > \frac{29-27.5}{\sqrt{0.0875}}) = P(Z > 5.07) = 0$.

5-117



5-118

$$f_{XY}(x, y) = \frac{1}{1.2\pi} e^{\left[\frac{-1}{0.72} \{ (x-1)^2 - 1.6(x-1)(y-2) + (y-2)^2 \} \right]}$$

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{.36}} e^{\left[\frac{-1}{2(0.36)} \{ (x-1)^2 - 1.6(x-1)(y-2) + (y-2)^2 \} \right]}$$

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{1-.8^2}} e^{\left[\frac{-1}{2(1-0.8^2)} \{ (x-1)^2 - 2(.8)(x-1)(y-2) + (y-2)^2 \} \right]}$$

$$E(X) = 1, E(Y) = 2 \quad V(X) = 1 \quad V(Y) = 1 \quad \text{and } \rho = 0.8$$

5-119 Let T denote the total thickness. Then, $T = X_1 + X_2$ and

a) $E(T) = 0.5 + 1 = 1.5 \text{ mm}$

a. $V(T) = V(X_1) + V(X_2) + 2\text{Cov}(X_1X_2) = 0.01 + 0.04 + 2(0.014) = 0.078 \text{ mm}^2$

i. where $\text{Cov}(XY) = \rho\sigma_X\sigma_Y = 0.7(0.1)(0.2) = 0.014$

b) $P(T < 1) = P\left(Z < \frac{1-1.5}{\sqrt{0.078}}\right) = P(Z < -1.79) = 0.0367$

c) Let P denote the total thickness. Then, $P = 2X_1 + 3X_2$ and

a. $E(P) = 2(0.5) + 3(1) = 4 \text{ mm}$

$V(P) = 4V(X_1) + 9V(X_2) +$

$2(2)(3)\text{Cov}(X_1X_2) = 4(0.01) + 9(0.04) + 2(2)(3)(0.014) = 0.568 \text{ mm}^2$

where $\text{Cov}(XY) = \rho\sigma_X\sigma_Y = 0.7(0.1)(0.2) = 0.014$

5-120 Let T denote the total thickness. Then, $T = X_1 + X_2 + X_3$ and

a) $E(T) = 0.5 + 1 + 1.5 = 3 \text{ mm}$

$V(T) = V(X_1) + V(X_2) + V(X_3) + 2\text{Cov}(X_1X_2) + 2\text{Cov}(X_2X_3) +$

$2\text{Cov}(X_1X_3) = 0.01 + 0.04 + 0.09 + 2(0.014) + 2(0.03) + 2(0.009) = 0.246 \text{ mm}^2$

where $\text{Cov}(XY) = \rho\sigma_X\sigma_Y$

b) $P(T < 1.5) = P\left(Z < \frac{1.5-3}{0.246}\right) = P(Z < -6.10) \cong 0$

- 5-121 Let X and Y denote the percentage returns for security one and two respectively. If $\frac{1}{2}$ of the total dollars is invested in each then $\frac{1}{2}X + \frac{1}{2}Y$ is the percentage return.
 $E(\frac{1}{2}X + \frac{1}{2}Y) = 0.05$ (or 5 if given in terms of percent)
 $V(\frac{1}{2}X + \frac{1}{2}Y) = \frac{1}{4}V(X) + \frac{1}{4}V(Y) + 2(\frac{1}{2})(\frac{1}{2})\text{Cov}(X, Y)$
 where $\text{Cov}(XY) = \rho\sigma_X\sigma_Y = -0.5(2)(4) = -4$
 $V(\frac{1}{2}X + \frac{1}{2}Y) = \frac{1}{4}(4) + \frac{1}{4}(6) - 2 = 3$
 Also, $E(X) = 5$ and $V(X) = 4$. Therefore, the strategy that splits between the securities has a lower standard deviation of percentage return than investing 2million in the first security.

Mind-Expanding Exercises

- 5-122 By the independence,

$$\begin{aligned} P(X_1 \in A_1, X_2 \in A_2, \dots, X_p \in A_p) &= \int_{A_1} \int_{A_2} \dots \int_{A_p} f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_p}(x_p) dx_1 dx_2 \dots dx_p \\ &= \left[\int_{A_1} f_{X_1}(x_1) dx_1 \right] \left[\int_{A_2} f_{X_2}(x_2) dx_2 \right] \dots \left[\int_{A_p} f_{X_p}(x_p) dx_p \right] \\ &= P(X_1 \in A_1) P(X_2 \in A_2) \dots P(X_p \in A_p) \end{aligned}$$

- 5-123 $E(Y) = c_1\mu_1 + c_2\mu_2 + \dots + c_p\mu_p$.

Also,

$$\begin{aligned} V(Y) &= \int \left[c_1x_1 + c_2x_2 + \dots + c_px_p - (c_1\mu_1 + c_2\mu_2 + \dots + c_p\mu_p) \right]^2 f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_p}(x_p) dx_1 dx_2 \dots dx_p \\ &= \int \left[c_1(x_1 - \mu_1) + \dots + c_p(x_p - \mu_p) \right]^2 f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_p}(x_p) dx_1 dx_2 \dots dx_p \end{aligned}$$

Now, the cross-term

$$\begin{aligned} &\int c_1c_2(x_1 - \mu_1)(x_2 - \mu_2) f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_p}(x_p) dx_1 dx_2 \dots dx_p \\ &= c_1c_2 \left[\int (x_1 - \mu_1) f_{X_1}(x_1) dx_1 \right] \left[\int (x_2 - \mu_2) f_{X_2}(x_2) dx_2 \right] = 0 \end{aligned}$$

from the definition of the mean. Therefore, each cross-term in the last integral for $V(Y)$ is zero and

$$\begin{aligned} V(Y) &= \left[\int c_1^2(x_1 - \mu_1)^2 f_{X_1}(x_1) dx_1 \right] \dots \left[\int c_p^2(x_p - \mu_p)^2 f_{X_p}(x_p) dx_p \right] \\ &= c_1^2V(X_1) + \dots + c_p^2V(X_p). \end{aligned}$$

5-124 $\int_0^a \int_0^b f_{XY}(x, y) dy dx = \int_0^a \int_0^b c dy dx = cab$. Therefore, $c = 1/ab$. Then,

$$f_X(x) = \int_0^b c dy = \frac{1}{a} \text{ for } 0 < x < a, \text{ and } f_Y(y) = \int_0^a c dx = \frac{1}{b} \text{ for } 0 < y < b. \text{ Therefore,}$$

$$f_{XY}(x, y) = f_X(x)f_Y(y) \text{ for all } x \text{ and } y \text{ and } X \text{ and } Y \text{ are independent.}$$

5-125 The marginal density of X is

$$f_X(x) = \int_0^b g(x)h(u)du = g(x) \int_0^b h(u)du = kg(x) \text{ where } k = \int_0^b h(u)du. \text{ Also,}$$

$$f_Y(y) = lh(y) \text{ where } l = \int_0^a g(v)dv. \text{ Because } f_{XY}(x, y) \text{ is a probability density}$$

function, $\int_0^a \int_0^b g(x)h(y)dydx = \left[\int_0^a g(v)dv \right] \left[\int_0^b h(u)du \right] = 1$. Therefore, $kl = 1$ and

$$f_{XY}(x, y) = f_X(x)f_Y(y) \text{ for all } x \text{ and } y.$$

Section 5-8 on CD

S5-1 $f_Y(y) = \frac{1}{4}$ at $y = 3, 5, 7, 9$ from Theorem S5-1.

S5-2 Because $X \geq 0$, the transformation is one-to-one; that is $y = x^2$ and $x = \sqrt{y}$. From Theorem S5-2,

$$f_Y(y) = f_X(\sqrt{y}) = \binom{3}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{3-\sqrt{y}} \text{ for } y = 0, 1, 4, 9.$$

If $p = 0.25$, $f_Y(y) = \binom{3}{\sqrt{y}} (0.25)^{\sqrt{y}} (0.75)^{3-\sqrt{y}} \text{ for } y = 0, 1, 4, 9.$

S5-3 a) $f_Y(y) = f_X\left(\frac{y-10}{2}\right)\left(\frac{1}{2}\right) = \frac{y-10}{72} \text{ for } 10 \leq y \leq 22$

b) $E(Y) = \int_{10}^{22} \frac{y^2 - 10y}{72} dy = \frac{1}{72} \left(\frac{y^3}{3} - \frac{10y^2}{2} \right) \Big|_{10}^{22} = 18$

S5-4 Because $y = -2 \ln x$, $e^{-\frac{y}{2}} = x$. Then, $f_Y(y) = f_X\left(e^{-\frac{y}{2}}\right) \left| -\frac{1}{2} e^{-\frac{y}{2}} \right| = \frac{1}{2} e^{-\frac{y}{2}} \text{ for } 0 \leq e^{-\frac{y}{2}} \leq 1 \text{ or}$
 $y \geq 0$, which is an exponential distribution (which equals a chi-square distribution with $k = 2$ degrees of freedom).

S5-5 a) Let $Q = R$. Then,

$$\begin{aligned} p &= i^2 r & \text{and} & & i &= \sqrt{\frac{p}{q}} \\ q &= r & & & r &= q \end{aligned}$$

$$J = \begin{vmatrix} \frac{\partial}{\partial p} & \frac{\partial}{\partial q} \\ \frac{\partial}{\partial p} & \frac{\partial}{\partial q} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(pq)^{-1/2} & -\frac{1}{2}p^{1/2}q^{-3/2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}(pq)^{-1/2}$$

$$f_{PQ}(p, q) = f_{IR}(\sqrt{\frac{p}{q}}, q) \frac{1}{2}(pq)^{-1/2} = 2\left(\sqrt{\frac{p}{q}}\right)^{\frac{1}{2}}(pq)^{-1/2} = q^{-1}$$

$$\text{for } 0 \leq \sqrt{\frac{p}{q}} \leq 1, \quad 0 \leq q \leq 1$$

That is, for $0 \leq p \leq q, \quad 0 < q \leq 1$.

$$f_P(p) = \int_p^1 q^{-1} dq = -\ln p \quad \text{for } 0 < p \leq 1.$$

$$\text{b) } E(P) = -\int_0^1 p \ln p \, dp. \text{ Let } u = \ln p \text{ and } dv = p \, dp. \text{ Then, } du = 1/p \text{ and}$$

$$v = \frac{p^2}{2}. \text{ Therefore, } E(P) = -(\ln p) \frac{p^2}{2} \Big|_0^1 + \int_0^1 \frac{p}{2} dp = \frac{p^2}{4} \Big|_0^1 = \frac{1}{4}$$

$$\text{S5-6 a) If } y = x^2, \text{ then } x = \sqrt{y} \text{ for } x \geq 0 \text{ and } y \geq 0. \text{ Thus, } f_Y(y) = f_X(\sqrt{y}) \frac{1}{2} y^{-\frac{1}{2}} = \frac{e^{-\sqrt{y}}}{2\sqrt{y}} \text{ for}$$

$$y > 0.$$

$$\text{b) If } y = x^{1/2}, \text{ then } x = y^2 \text{ for } x \geq 0 \text{ and } y \geq 0. \text{ Thus, } f_Y(y) = f_X(y^2) 2y = 2ye^{-y^2} \text{ for } y > 0.$$

$$\text{c) If } y = \ln x, \text{ then } x = e^y \text{ for } x \geq 0. \text{ Thus, } f_Y(y) = f_X(e^y) e^y = e^y e^{-e^y} = e^{y-e^y} \text{ for } -\infty < y < \infty.$$

$$\text{S5-7 a) Now, } \int_0^\infty a v^2 e^{-bv} dv \text{ must equal one. Let } u = bv, \text{ then } 1 = a \int_0^\infty \left(\frac{u}{b}\right)^2 e^{-u} \frac{du}{b} = \frac{a}{b^3} \int_0^\infty u^2 e^{-u} du. \text{ From}$$

$$\text{the definition of the gamma function the last expression is } \frac{a}{b^3} \Gamma(3) = \frac{2a}{b^3}. \text{ Therefore, } a = \frac{b^3}{2}.$$

$$\text{b) If } w = \frac{mv^2}{2}, \text{ then } v = \sqrt{\frac{2w}{m}} \text{ for } v \geq 0, \quad w \geq 0.$$

$$\begin{aligned} f_w(w) &= f_v\left(\sqrt{\frac{2w}{m}}\right) \frac{dv}{dw} = \frac{b^3 2w}{2m} e^{-b\sqrt{\frac{2w}{m}}} (2mw)^{-1/2} \\ &= \frac{b^3 m^{-3/2}}{\sqrt{2}} w^{1/2} e^{-b\sqrt{\frac{2w}{m}}} \end{aligned}$$

$$\text{for } w \geq 0.$$

S5-8 If $y = e^x$, then $x = \ln y$ for $1 \leq x \leq 2$ and $e^1 \leq y \leq e^2$. Thus, $f_Y(y) = f_X(\ln y) \frac{1}{y} = \frac{1}{y}$ for

$$1 \leq \ln y \leq 2. \text{ That is, } f_Y(y) = \frac{1}{y} \text{ for } e \leq y \leq e^2.$$

S5-9 Now $P(Y \leq a) = P(X \geq u(a)) = \int_{u(a)}^{\infty} f_X(x) dx$. By changing the variable of integration from x to y

by using $x = u(y)$, we obtain $P(Y \leq a) = \int_a^{-\infty} f_X(u(y)) u'(y) dy$ because as x tends to ∞ , $y = h(x)$ tends

to $-\infty$. Then, $P(Y \leq a) = \int_{-\infty}^a f_X(u(y)) (-u'(y)) dy$. Because $h(x)$ is decreasing, $u'(y)$ is negative.

Therefore, $|u'(y)| = -u'(y)$ and Theorem S5-1 holds in this case also.

S5-10 If $y = (x-2)^2$, then $x = 2 - \sqrt{y}$ for $0 \leq x \leq 2$ and $x = 2 + \sqrt{y}$ for $2 \leq x \leq 4$. Thus,

$$\begin{aligned} f_Y(y) &= f_X(2 - \sqrt{y}) \left| -\frac{1}{2} y^{-1/2} \right| + f_X(2 + \sqrt{y}) \left| \frac{1}{2} y^{-1/2} \right| \\ &= \frac{2 - \sqrt{y}}{16\sqrt{y}} + \frac{2 + \sqrt{y}}{16\sqrt{y}} \\ &= \left(\frac{1}{4}\right) y^{-1/2} \text{ for } 0 \leq y \leq 4 \end{aligned}$$

S5-11 a) Let $a = S_1 S_2$ and $y = S_1$. Then, $S_1 = y$, $S_2 = \frac{a}{y}$ and

$$J = \begin{vmatrix} \frac{\partial s_1}{\partial a} & \frac{\partial s_1}{\partial y} \\ \frac{\partial s_2}{\partial a} & \frac{\partial s_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{y} & -\frac{a}{2} y^{-3/2} \end{vmatrix} = -\frac{1}{y}. \text{ Then,}$$

$$f_{AY}(a, y) = f_{S_1 S_2}\left(y, \frac{a}{y}\right) \left(\frac{1}{y}\right) = 2y \left(\frac{a}{8y}\right) \left(\frac{1}{y}\right) = \frac{a}{4y} \text{ for } 0 \leq y \leq 1 \text{ and } 0 \leq \frac{a}{y} \leq 4. \text{ That is, for}$$

$$0 \leq y \leq 1 \text{ and } 0 \leq a \leq 4y.$$

$$\text{b) } f_A(a) = \int_{a/4}^1 \frac{a}{4y} dy = -\frac{a}{4} \ln\left(\frac{a}{4}\right) \text{ for } 0 < a \leq 4.$$

S5-12 Let $r = v/i$ and $s = i$. Then, $\dot{i} = S$ and $v = rs$

$$J = \begin{vmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial s} \\ \frac{\partial}{\partial v} & \frac{\partial}{\partial s} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ s & r \end{vmatrix} = s$$

$f_{RS}(r, s) = f_{IV}(s, rs)s = e^{-rs}s$ for $rs \geq 0$ and $1 \leq s \leq 2$. That is,

$f_{RS}(r, s) = se^{-rs}$ for $1 \leq s \leq 2$ and $r \geq 0$.

Then, $f_R(r) = \int_1^2 se^{-rs} ds$. Let $u = s$ and $dv = e^{-rs} ds$. Then, $du = ds$ and $v = \frac{-e^{-rs}}{r}$

Then,

$$\begin{aligned} f_R(r) &= -s \frac{e^{-rs}}{r} \Big|_1^2 + \int_1^2 \frac{e^{-rs}}{r} ds = \frac{e^{-r} - 2e^{-2r}}{r} - \frac{e^{-rs}}{r^2} \Big|_1^2 \\ &= \frac{e^{-r} - 2e^{-2r}}{r} + \frac{e^{-r} - e^{-2r}}{r^2} \\ &= \frac{e^{-r}(r+1) - e^{-2r}(2r+1)}{r^2} \end{aligned}$$

for $r > 0$.

Section 5-9 on CD

S5-13 a) $E(e^{tx}) = \sum_{x=1}^m \frac{e^{tx}}{m} = \frac{1}{m} \sum_{x=1}^m (e^t)^x = \frac{(e^t)^{m+1} - e^t}{m(e^t - 1)} = \frac{e^t(1 - e^{tm})}{m(1 - e^t)}$

b) $M(t) = \frac{1}{m} e^t (1 - e^{tm})(1 - e^t)^{-1}$ and

$$\frac{dM(t)}{dt} = \frac{1}{m} \{ e^t (1 - e^{tm})(1 - e^t)^{-1} + e^t (-me^{tm})(1 - e^t)^{-1} + e^t (1 - e^{tm})(-1)(1 - e^t)^{-2}(-e^t) \}$$

$$\frac{dM(t)}{dt} = \frac{e^t}{m(1 - e^t)} \left\{ 1 - e^{tm} - me^{tm} + \frac{(1 - e^{tm})e^t}{1 - e^t} \right\}$$

$$= \frac{e^t}{m(1 - e^t)^2} \{ 1 - e^{tm} - me^{tm} + me^{(m+1)t} \}$$

Using L'Hospital's rule,

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{dM(t)}{dt} &= \lim_{t \rightarrow 0} \frac{e^t}{m} \lim_{t \rightarrow 0} \frac{-me^{tm} - m^2 e^{tm} + m(m+1)e^{(m+1)t}}{-2(1 - e^t)e^t} \\ &= \lim_{t \rightarrow 0} \frac{e^t}{m} \lim_{t \rightarrow 0} \frac{-m^2 e^{tm} - m^3 e^{tm} + m(m+1)^2 e^{(m+1)t}}{-2(1 - e^t)e^t - 2e^t(-e^t)} \\ &= \frac{1}{m} \times \frac{m(m+1)^2 - m^2 - m^3}{2} = \frac{m^2 + m}{2m} = \frac{m+1}{2} \end{aligned}$$

Therefore, $E(X) = \frac{m+1}{2}$.

$$\begin{aligned}\frac{d^2 M(t)}{dt^2} &= \frac{d^2}{dt^2} \sum_{x=1}^m e^{tx} \frac{1}{m} = \frac{1}{m} \sum_{x=1}^m \frac{d^2}{dt^2} \left(1 + tx + \frac{(tx)^2}{2} + \dots \right) \\ &= \frac{1}{m} \sum_{x=1}^m (x^2 + \text{term involving powers of } t)\end{aligned}$$

Thus,

$$\left. \frac{d^2 M(t)}{dt^2} \right|_{t=0} = \frac{1}{m} \left(\frac{m(m+1)(m+2)}{6} \right) = \frac{(m+1)(2m+2)}{6}$$

Then,

$$\begin{aligned}V(X) &= \frac{2m^2 + 3m + 1}{6} - \frac{(m+1)^2}{4} = \frac{4m^2 + 6m + 2 - 3m^2 - 6m - 3}{12} \\ &= \frac{m^2 - 1}{12}\end{aligned}$$

S5-14 a) $E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$

b) $\frac{dM(t)}{dt} = \lambda e^t e^{\lambda(e^t - 1)}$

$$\left. \frac{dM(t)}{dt} \right|_{t=0} = \lambda = E(X)$$

$$\frac{d^2 M(t)}{dt^2} = \lambda^2 e^{2t} e^{\lambda(e^t - 1)} + \lambda e^t e^{\lambda(e^t - 1)}$$

$$\left. \frac{d^2 M(t)}{dt^2} \right|_{t=0} = \lambda^2 + \lambda$$

$$V(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

S5-15 a) $E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} (1-p)^{x-1} p = \frac{p}{1-p} \sum_{x=1}^{\infty} [e^t (1-p)]^x$

$$= \frac{e^t (1-p)}{1 - (1-p)e^t} \left(\frac{p}{1-p} \right) = \frac{pe^t}{1 - (1-p)e^t}$$

b) $\frac{dM(t)}{dt} = pe^t (1 - (1-p)e^t)^{-2} (1-p)e^t + pe^t (1 - (1-p)e^t)^{-1}$

$$= p(1-p)e^{2t} (1 - (1-p)e^t)^{-2} + pe^t (1 - (1-p)e^t)^{-1}$$

$$\left. \frac{dM(t)}{dt} \right|_{t=0} = \frac{1-p}{p} + 1 = \frac{1}{p} = E(X)$$

$$\frac{d^2 M(t)}{dt^2} = p(1-p)e^{2t} 2(1 - (1-p)e^t)^{-3} (1-p)e^t + p(1-p)(1 - (1-p)e^t)^{-2} 2e^{2t}$$

$$+ pe^t (1 - (1-p)e^t)^{-2} (1-p)e^t + pe^t (1 - (1-p)e^t)^{-1}$$

$$\left. \frac{d^2 M(t)}{dt^2} \right|_{t=0} = \frac{2(1-p)^2}{p^2} + \frac{2(1-p)}{p} + \frac{1-p}{p} + 1 = \frac{2(1-p)^2 + 3p(1-p) + p^2}{p^2}$$

$$= \frac{2 + 2p^2 - 4p + 3p - 3p^2 + p^2}{p^2} = \frac{2-p}{p^2}$$

$$V(X) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

S5-16 $M_Y(t) = Ee^{tY} = Ee^{t(X_1+X_2)} = Ee^{tX_1} Ee^{tX_2}$

$$= (1-2t)^{-k_1/2} (1-2t)^{-k_2/2} = (1-2t)^{-(k_1+k_2)/2}$$

Therefore, Y has a chi-square distribution with $k_1 + k_2$ degrees of freedom.

S5-17 a) $E(e^{tX}) = \int_0^{\infty} e^{tx} 4xe^{-2x} dx = 4 \int_0^{\infty} xe^{(t-2)x} dx$

Using integration by parts with $u = x$ and $dv = e^{(t-2)x} dx$ and $du = dx$,

$$v = \frac{e^{(t-2)x}}{t-2} \text{ we obtain}$$

$$4 \left(\frac{xe^{(t-2)x}}{t-2} \Big|_0^{\infty} - \int_0^{\infty} \frac{e^{(t-2)x}}{t-2} dx \right) = 4 \left(\frac{xe^{(t-2)x}}{t-2} \Big|_0^{\infty} - \frac{e^{(t-2)x}}{(t-2)^2} \Big|_0^{\infty} \right)$$

This integral only exists for $t < 2$. In that case, $E(e^{tX}) = \frac{4}{(t-2)^2}$ for $t < 2$

b) $\frac{dM(t)}{dt} = -8(t-2)^{-3}$ and $\left. \frac{dM(t)}{dt} \right|_{t=0} = -8(-2)^{-3} = 1 = E(X)$

c) $\frac{d^2 M(t)}{dt^2} = 24(t-2)^{-4}$ and $\left. \frac{d^2 M(t)}{dt^2} \right|_{t=0} = \frac{24}{16} = \frac{3}{2}$. Therefore, $V(X) = \frac{3}{2} - 1^2 = \frac{1}{2}$

S5-18 a) $E(e^{tx}) = \int_{\alpha}^{\beta} \frac{e^{tx}}{\beta - \alpha} dx = \frac{e^{tx}}{t(\beta - \alpha)} \Big|_{\alpha}^{\beta} = \frac{e^{t\beta} - e^{t\alpha}}{t(\beta - \alpha)}$

b) $\frac{dM(t)}{dt} = \frac{e^{t\beta} - e^{t\alpha}}{-(\beta - \alpha)t^2} + \frac{\beta e^{t\beta} - \alpha e^{t\alpha}}{t(\beta - \alpha)}$
 $= \frac{(\beta t - 1)e^{t\beta} - (\alpha t - 1)e^{t\alpha}}{t^2(\beta - \alpha)}$

Using L'Hospital's rule,

$$\lim_{t \rightarrow 0} \frac{dM(t)}{dt} = \frac{(\beta t - 1)\beta e^{t\beta} + \beta e^{t\beta} - (\alpha t - 1)\alpha e^{t\alpha} - \alpha e^{t\alpha}}{2t(\beta - \alpha)}$$

$$\lim_{t \rightarrow 0} \frac{dM(t)}{dt} = \frac{\beta^2(\beta t - 1)e^{t\beta} + \beta^2 e^{t\beta} + \beta^2 e^{t\beta} - \alpha^2(\alpha t - 1)e^{t\alpha} - \alpha^2 e^{t\alpha} - \alpha^2 e^{t\alpha}}{2(\beta - \alpha)}$$

$$\lim_{t \rightarrow 0} \frac{dM(t)}{dt} = \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} = \frac{(\beta + \alpha)}{2} = E(X)$$

$$\begin{aligned} \frac{d^2 M(t)}{dt^2} &= \frac{d^2}{dt^2} \int_a^b \frac{1}{b-a} e^{tx} dx = \frac{1}{b-a} \frac{d^2}{dt^2} \left(\frac{e^{tb} - e^{ta}}{t} \right) \\ &= \frac{1}{b-a} \frac{d^2}{dt^2} \left(\frac{tb + \frac{(tb)^2}{2} + \frac{(tb)^3}{3!} - ta - \frac{(ta)^2}{2} - \frac{(ta)^3}{3!} + \dots}{t} \right) \\ &= \frac{1}{b-a} \frac{d^2}{dt^2} \left(b - a + \frac{(b^2 - a^2)t}{2} + \frac{(b^3 - a^3)t^2}{3!} + \dots \right) \\ &= \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ba + a^2}{3} \end{aligned}$$

Thus,

$$V(X) = \frac{b^2 + ba + a^2}{3} - \frac{(b+a)^2}{4} = \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$$

$$\begin{aligned} \text{S5-19 a) } M(t) &= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{(t-\lambda)x} dx \\ &= \lambda \left. \frac{e^{(t-\lambda)x}}{t-\lambda} \right|_0^{\infty} = \frac{-\lambda}{t-\lambda} = \frac{1}{1-\frac{t}{\lambda}} = \left(1 - \frac{t}{\lambda}\right)^{-1} \text{ for } t < \lambda \end{aligned}$$

$$\text{b) } \frac{dM(t)}{dt} = (-1) \left(1 - \frac{t}{\lambda}\right)^{-2} \left(-\frac{1}{\lambda}\right) = \frac{1}{\lambda \left(1 - \frac{t}{\lambda}\right)^2}$$

$$\left. \frac{dM(t)}{dt} \right|_{t=0} = \frac{1}{\lambda}$$

$$\frac{d^2 M(t)}{dt^2} = \frac{2}{\lambda^2 \left(1 - \frac{t}{\lambda}\right)^3}$$

$$\left. \frac{d^2 M(t)}{dt^2} \right|_{t=0} = \frac{2}{\lambda^2}$$

$$V(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$\text{S5-20 a) } M(t) = \int_0^{\infty} e^{tx} \frac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x} dx = \frac{\lambda^r}{\Gamma(r)} \int_0^{\infty} x^{r-1} e^{(t-\lambda)x} dx$$

Let $u = (\lambda - t)x$. Then,

$$M(t) = \frac{\lambda^r}{\Gamma(r)} \int_0^{\infty} \left(\frac{u}{\lambda - t}\right)^{r-1} e^{-u} \frac{du}{\lambda - t} = \frac{\lambda^r \Gamma(r)}{\Gamma(r)(\lambda - t)^r} = \frac{1}{\left(1 - \frac{t}{\lambda}\right)^r} = \left(1 - \frac{t}{\lambda}\right)^{-r} \text{ from}$$

the definition of the gamma function for $t < \lambda$.

$$\text{b) } M'(t) = -r \left(1 - \frac{t}{\lambda}\right)^{-r-1} \left(-\frac{1}{\lambda}\right)$$

$$\left. M'(t) \right|_{t=0} = \frac{r}{\lambda} = E(X)$$

$$M''(t) = \frac{r(r+1)}{\lambda^2} \left(1 - \frac{t}{\lambda}\right)^{-r-2}$$

$$\left. M''(t) \right|_{t=0} = \frac{r(r+1)}{\lambda^2}$$

$$V(X) = \frac{r(r+1)}{\lambda^2} - \left(\frac{r}{\lambda}\right)^2 = \frac{r}{\lambda^2}$$

S5-21 a) $E(e^{tY}) = \prod_{i=1}^n E(e^{tX_i}) = \left(1 - \frac{t}{\lambda}\right)^{-n}$

b) From Exercise S5-20, Y has a gamma distribution with parameter λ and n.

S5-22 a) $M_Y(t) = e^{\mu_1 t + \sigma_1^2 \frac{t^2}{2} + \mu_2 t + \sigma_2^2 \frac{t^2}{2}} = e^{(\mu_1 + \mu_2)t + (\sigma_1^2 + \sigma_2^2) \frac{t^2}{2}}$

b) Y has a normal distribution with mean $\mu_1 + \mu_2$ and variances $\sigma_1^2 + \sigma_2^2$

S5-23 Because a chi-square distribution is a special case of the gamma distribution with $\lambda = \frac{1}{2}$ and $r = \frac{k}{2}$, from

Exercise S5-20.

$$M(t) = (1 - 2t)^{-k/2}$$

$$M'(t) = -\frac{k}{2} (1 - 2t)^{-\frac{k}{2}-1} (-2) = k(1 - 2t)^{-\frac{k}{2}-1}$$

$$M'(t)|_{t=0} = k = E(X)$$

$$M''(t) = 2k(\frac{k}{2} + 1)(1 - 2t)^{-\frac{k}{2}-2}$$

$$M''(t)|_{t=0} = 2k(\frac{k}{2} + 1) = k^2 + 2k$$

$$V(X) = k^2 + 2k - k^2 = 2k$$

S5-24 a) $M(t) = M(0) + M'(0)t + \frac{M''(0)}{2!}t^2 + \dots + \frac{M^{(r)}(0)}{r!}t^r + \dots$ by Taylor's expansion. Now,

$M(0) = 1$ and $M^{(r)}(0) = \mu_r'$ and the result is obtained.

b) From Exercise S5-20, $M(t) = 1 + \frac{r}{\lambda}t + \frac{r(r+1)}{\lambda^2} \frac{t^2}{2!} + \dots$

c) $\mu_1' = \frac{r}{\lambda}$ and $\mu_2' = \frac{r(r+1)}{\lambda^2}$ which agrees with Exercise S5-20.

Section 5-10 on CD

S5-25 Use Chebychev's inequality with $c = 4$. Then, $P(|X - 10| > 4) \leq \frac{1}{16}$.

S5-26 $E(X) = 5$ and $\sigma_X = 2.887$. Then, $P(|X - 5| > 2\sigma_X) \leq \frac{1}{4}$.

The actual probability is $P(|X - 5| > 2\sigma_X) = P(|X - 5| > 5.77) = 0$.

S5-27 $E(X) = 20$ and $V(X) = 400$. Then, $P(|X - 20| > 2\sigma) \leq \frac{1}{4}$ and $P(|X - 20| > 3\sigma) \leq \frac{1}{9}$. The actual probabilities are

$$P(|X - 20| > 2\sigma) = 1 - P(|X - 20| < 40)$$

$$= 1 - \int_0^{60} 0.05e^{-0.05x} dx = 1 - \left[-e^{-0.05x} \right]_0^{60} = 0.0498$$

$$P(|X - 20| > 3\sigma) = 1 - P(|X - 20| < 60)$$

$$= 1 - \int_0^{80} 0.05e^{-0.05x} dx = 1 - \left[-e^{-0.05x} \right]_0^{80} = 0.0183$$

S5-28 $E(X) = 4$ and $\sigma_X = 2$

$P(|X - 4| \geq 4) \leq \frac{1}{4}$ and $P(|X - 4| \geq 6) \leq \frac{1}{9}$. The actual probabilities are

$$P(|X - 4| \geq 4) = 1 - P(|X - 4| < 4) = 1 - \sum_{x=1}^7 \frac{e^{-2} 2^x}{x!} = 1 - 0.8636 = 0.1364$$

$$P(|X - 4| \geq 6) = 1 - P(|X - 4| < 6) = 1 - \sum_{x=1}^9 \frac{e^{-2} 2^x}{x!} = 0.000046$$

S5-29 Let $\bar{X}$ denote the average of 500 diameters. Then, $\sigma_{\bar{X}} = \frac{0.01}{\sqrt{500}} = 4.47 \times 10^{-4}$.

a) $P(|\bar{X} - \mu| \geq 4\sigma_{\bar{X}}) \leq \frac{1}{16}$ and $P(|\bar{X} - \mu| < 0.0018) \geq \frac{15}{16}$. Therefore, the bound is 0.0018.

If $P(|\bar{X} - \mu| < x) = \frac{15}{16}$, then $P(\frac{-x}{\sigma_{\bar{X}}} < \frac{\bar{X} - \mu}{\sigma_{\bar{X}}} < \frac{x}{\sigma_{\bar{X}}}) = 0.9375$. Then,

$$P(\frac{-x}{4.47 \times 10^{-4}} < Z < \frac{x}{4.47 \times 10^{-4}}) = 0.9375. \text{ and } \frac{x}{4.47 \times 10^{-4}} = 1.86. \text{ Therefore, } x = 8.31 \times 10^{-4}.$$

S5-30 a) $E(Y) = P(|X - \mu| \geq c\sigma)$

b) Because $Y \leq 1$, $(X - \mu)^2 \geq (X - \mu)^2 Y$

If $|X - \mu| \geq c\sigma$, then $Y = 1$ and $(X - \mu)^2 Y \geq c^2 \sigma^2 Y$

If $|X - \mu| < c\sigma$, then $Y = 0$ and $(X - \mu)^2 Y = c^2 \sigma^2 Y$.

c) Because $(X - \mu)^2 \geq c^2 \sigma^2 Y$, $E[(X - \mu)^2] \geq c^2 \sigma^2 E(Y)$.

d) From part a., $E(Y) = P(|X - \mu| \geq c\sigma)$. From part c., $\sigma^2 \geq c^2 \sigma^2 P(|X - \mu| \geq c\sigma)$. Therefore,

$$\frac{1}{c^2} \geq P(|X - \mu| \geq c\sigma).$$