CHRISTIAAN HUYGENS'

The Pendulum Clock or Geometrical Demonstration Concerning the Motion of Pendula as Applied to Clocks

TRANSLATED WITH NOTES BY

RICHARD J. BLACKWELL

INTRODUCTION BY

H. J. M. BOS

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THIS first English translation of Christiaan Huygens' (1629–1695) greatest work is a classic in the history of science. Originally published in 1673, *The Pendulum Clock* is the most significant scientific book published in the seventeenth century other than Galileo's *Dialogues* and Newton's *Principia*. It combines both science and technology in a single classic work—what is surely one of the earliest applications of sophisticated scientific theory to the design of a practical device. Huygens' clock is probably the first clock to regulate itself precisely by means of a pendulum and escapement mechanism. The latter parts of the book include analyses of the motions of falling bodies on a cycloid, the mathematics of evolutes, the centers of oscillation of various bodies, and Huygens' first publication of his theorems on centrifugal force.

Christiaan Huygens' The Pendulum Clock is the initial volume in the Iowa State University Press Series in the History of

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THE IOWA STATE UNIVERSITY PRESS SERIES IN THE HISTORY OF TECHNOLOGY AND SCIENCE

Robert E. Schofield
SERIES EDITOR

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The Iowa State University Press

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For Rich, Tom, and Mary

RICHARD J. BLACKWELL is Professor of Philosophy and holder of the Danforth Chair in the Humanities at Saint Louis University where he has taught since 1961. He is the author of *Discovery in the Physical Sciences* (1969), the compiler of *A Bibliography of the Philosophy of Science, 1945–1981* (1983), and the translator of Christian Wolff's *Preliminary Discourse on Philosophy in General* (1963) and of St. Thomas Aquinas' *Commentary on Aristotle's* PHYSICS (1963).

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CONTENTS

SERIES EDITOR'S INTRODUCTION / *Robert E. Schofield* ix

INTRODUCTION / *H. J. M. Bos* xi

TRANSLATOR'S PREFACE / *Richard J. Blackwell* xxxi

THE TRANSLATION

To Louis XIV, Renowned King of France and Navarre
7

License from the King
10

The Pendulum Clock or Geometrical Demonstrations
Concerning the Motion of Pendula
as Applied to Clocks
11

PART I
A Description of the Pendulum Clock
15

PART II
The Falling of Heavy Bodies and Their Motion in a Cycloid
33

PART III
The Evolution and Dimension of Curved Lines
73

PART IV
The Center of Oscillation
105

PART V
Another Construction, Based on the Circular Motion
of Pendula, and the Theorems on Centrifugal Force
173

INDEX
179

SERIES EDITOR'S INTRODUCTION

THERE IS A PARTICULAR PLEASURE in the introducing of this first volume in the Iowa State University Press Series in the History of Technology and Science. The series is designed to enlarge publishing opportunities for books in the history of technology and history of science, which increase public understanding of the nature of technological and scientific creativity and their relationship to one another and to the social, cultural, political, and economic context in which they arise. The terms: history of technology and history of science are to be interpreted in the broadest of senses and the series may include translations (as in this first volume), republications of "classic" works with annotations, memoirs, and biographies of significant figures in engineering, technological, or science's history. The emphasis will, however, be on the scholarly monograph, which may approach its subject from an institutional, social, cultural, or economic view as well as from an "internalist" one. The press and the editor will welcome the submission of manuscripts that fit the plan of the series.

It would be difficult to find another work that embodies as many of the elements this series is intended to represent as does the *Horologium Oscillatorium* (the *Pendulum Clock*) of 1673. Its author, Christiaan Huygens (1629–1695), was the son of Constantijn Huygens, a distinguished Dutch diplomat, poet, and musician, friend and patron of René Descartes. Christiaan was early recognized as a mathematician and student of what, today, would be called mathematical physics. He soon earned an international reputation, becoming a Fellow of the Royal Society of London in 1663 and one of the first Pensionnaires of the Académie Royale des Sciences in Paris in 1666. He also applied his talents to technological problems of optical instruments (lens grinding and telescope design), heat engines, and clock design, attempting to use his invention of the pendulum clock in determination of longitudes at sea.

His *Treatise on Light* (*Traité de Lumière,* Paris, 1678) has long been available in English translation and now, at last, there is this English translation of Huygens' greatest work, the *Horologium Oscillatorium*

(1673). *The Pendulum clock or Geometrical Demonstrations Concerning the Motion of Pendula as Applied to Clocks* combines, in a single classic work, both science and technology in what is surely one of the earliest applications of sophisticated scientific theory to the design of a practical device. The translation, from the critical Latin version of the *Oeuvres Complètes* (The Hague, 1889–1950), vol. 18, by Richard Blackwell is a felicitous as well as accurate rendering of a difficult text. It preserves the structure of Huygens' argument in the geometrical form in which it was originally presented.

Huygens' *Pendulum Clock* was the last major treatise on mechanics to employ traditional geometrical arguments. Within the next decade, publication of Isaac Newton's *Mathematical Principles of Natural Philosophy* (the *Principia,* 1687) was to begin the use of calculus in developing the analytical mechanics which was to dominate the eighteenth and nineteenth centuries. The Preface to the translation, by the distinguished Huygens scholar of Huygens' native country, H. J. M. Bos, leads the reader through some of the more complicated theoretical steps in the work and points out the quality and aesthetics of the geometrical style.

Together, Professors Blackwell and Bos have provided us with the first English version of this scientific/technological classic, with guidance on how it should be read. This is also the last of the major works of the seventeenth-century science of mechanics to appear in English. It is hoped that its availability in that form will encourage more detailed investigation into the life and work of a frequently mentioned, but little-studied giant of science and technology. We can also hope that this auspicious beginning of a new series in the History of Technology and Science will inspire other, and equally valuable, contributions for the scholarly venture.

ROBERT E. SCHOFIELD
Professor, History of Technology and Science
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INTRODUCTION

H. J. M. Bos

1

HUYGENS invented the pendulum clock in the winter of 1656–1657. The idea of using pendulums to count off equal time intervals and thus to measure time, was not new. But Huygens was the first to connect, through handle and fork, a freely suspended pendulum with a clockwork driven by a weight or a spring. In that arrangement the pendulum gives regularity to the clock's motion, while the clock provides the force which keeps the pendulum in motion.

The first clock was produced in 1657; the next year a book (*Horologium,* The Hague, 1658) appeared, describing the invention, patents were granted, and the clock quickly became known. Church clocks at Scheveningen and Utrecht were provided with pendulums in 1658; in Utrecht the clockmaker—Samuel Coster—guaranteed the church officials that the clock would not deviate more than eight minutes in a week—for that time an impressive result.

Fifteen years later, in 1673, Huygens published in Paris his magnum opus *Horologium Oscillatorium,* i.e., *The Pendulum Clock or Geometrical Demonstrations Concerning the Motion of Pendulums as Applied to Clocks.* The pendulum clock itself is described in the first part of the book. Three separate, rather different, and highly abstract mathematical and mechanical treatises follow: Part II on cycloidal motion, Part III on the evolutes of curves, and Part IV on centers of oscillation. A fifth, shorter part is appended; it contains theorems (but no derivations or proofs) on centrifugal force.

It will be good to begin this introduction by sketching the interrelation of the theories and indicating the main results.

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2

PENDULUMS are used in clocks to provide isochronous movement, that is, oscillations such that each back and forth swing always takes the same amount of time, independently of the driving force of the clock. This independence is necessary because it is technically very difficult to keep the driving force constant over longer time spans. A larger force will cause the pendulum to swing to larger amplitudes. Hence for a pendulum to provide isochronous movement it is necessary that its period does not depend on the amplitude to which it swings out.

However, simple pendular motion is not exactly isochronous. Swinging to larger amplitudes takes somewhat more time than to smaller. For small amplitudes the difference is negligible but—as was known at the time Huygens invented his clock—not so for larger ones. In the clocks from 1656–1658 Huygens had adopted two different ways to counter the problem that pendular oscillations are not fully isochronous. The one was to keep the amplitude small. The other was to fix bent metal laminae at both sides of the suspension point of the pendulum (cf., Figure 1, subfigure II, page 14) such that, in swinging out, the thread winds up against the laminae, thus lifting the bob above its normal circular path. The outer parts of the new path are steeper than their counterparts on the circle, and hence—Huygens argued—the bob falls quicker through those parts. Thus the bent laminae produced a motion of the bob which probably was more isochronous than the circular motion. However, when first adopting this solution (e.g., in the earliest clocks produced in 1657) Huygens had no insight in the precise form of the isochronous path or the corresponding form of the laminae.

3

IT is the technical device of the pendulum clock as completed in 1658, with its qualitative answers to the problem of isochronism, that provides the unity of the theoretical parts of the *Horologium Oscillatorium*. The three theories all date in essence from the year 1659—a summit of intensity in Huygens' creative activity. The unity of the book is best illustrated by reviewing how Huygens came to find the main theoretical results. This will also enable us to locate the highlights in the work of 1673.

In presenting Huygens' arguments and results I shall use the following notation:

$d_{fall}(t)$: distance traversed in free fall from rest during time t
$t_{fall}(d)$: time required in free fall from rest through distance d

$T_{pend}(l)$: time in which a pendulum with length l swings from outer to central position (considered for small amplitude oscillation)
$T_{cycl}(l)$: time in which a mass point, released from a point on a cycloid with axis l, reaches the lowest point
l_{sec} : length of a pendulum beating seconds
d_{sec} : distance traversed in the first second of free fall

The period, T, of pendular or cycloidal motion, i.e., the time required for a full back and forth movement, is equal to $4T_{pend}$ or $4T_{cycl}$ respectively. A pendulum beating seconds is one in which $T_{pend} = \frac{1}{2}$ sec.

The sign :: will be used for a proportionality between variables. Thus A :: B means that $A_1 : A_2 = B_1 : B_2$ for any two pairs (A_1, B_1) (A_2, B_2) of corresponding values of A and B.

The symbols introduced here will be used to shorten the formulation; Huygens himself presented his arguments entirely in prose with reference to figures.

The first study Huygens undertook in 1659 concerned the relation between the period of a pendulum and its length in the case of small amplitude oscillations. He found that relation (equivalent to the modern formula $T = 2\pi\sqrt{l/g}$), and he noticed that in deriving it he had used a certain approximation, involving an error that was negligible for small amplitudes but not so for larger ones. He then argued that what for the circular path of the pendulum bob was an approximation, should apply exactly for the isochronous path. He found that the property he had used as approximation indeed defines a curved path, namely the cycloid with vertical axis equal to half the length of the pendulum (cf., Figure 3, page 22). It was in a way an accident of history that Huygens could recognize the property as one pertaining to the cycloid; he had recently studied that curve in connection with problems proposed by Blaise Pascal.

The argument sketched above is not to be found in the *Horologium Oscillatorium;* it has become known only when the relevant manuscripts were edited in Huygens' *Oeuvres Complètes*. Indeed, after having found that the cycloid is the path of isochronic oscillation, Huygens proceeded to prove that result directly. That proof, with the preliminary mechanical and geometrical theory, forms the second part of the *Horologium Oscillatorium*. It is formulated with utmost precision, meeting all standards of rigorous geometrical and mechanical argument. The extreme care which Huygens gave to that proof shows how highly he valued it; elsewhere he described the result as "the most fortunate finding that ever befell me." The culmination of Part II is Proposition 25 which states that a body falling along a cycloid with axis d in a vertical plane will reach the lowest point in a time span $T_{cycl}(d)$ which is independent of the point on the curve where the

body starts its fall. Moreover $T_{cycl}(d)$ has to the time $t_{fall}(d)$ of free fall from rest along the axis of the cycloid the same ratio as the semicircumference of a circle to its diameter (i.e., $\pi : 2$).

If we combine the result of Proposition 25 with the law of fall as expressed in the formula $t_{fall}(d) = \sqrt{2d/g}$ (where g is the constant of gravitational acceleration), we arrive at the formula $T = 2\pi\sqrt{2d/g}$ for the (full) period of cycloidal oscillation. Huygens, however, does not at this point combine his results with the law of fall.

It should be noted that Huygens constructs the theoretical framework of his mechanics here, and also in Part IV, upon a restricted number of axioms or hypotheses, whose validity he deems evident. He does not attempt to deduce these hypotheses from any more fundamental models of nature, as Cartesian vortices or similar mechanistic schemes of explanation. In Hypothesis 2 of Part II, for instance, he states explicitly that he will not inquire into the cause of gravity. This axiomatic approach to mechanics is characteristic for Huygens' early studies in mechanics, of which the *Horologium Oscillatorium* contains the crucial results.

4

HAVING found, in 1659, the exact form of the isochronous path, it was natural for Huygens to ask what form should be given to the laminae that he had used, in his first design, to force the bob into a more isochronous path than the circle. From his knowledge of the cycloid Huygens soon deduced that in order to force the bob to move in a cycloid, the laminae must have the same cycloidal form. This was a truly remarkable and surprising result which inspired Huygens to undertake a general study of pairs of curves that are related to each other as the lamina (the "unwound" curve or the "evolute") and the resulting path of the bob (the "curve described by evolution" or the "evolvent"). The results of that study are in Part III of the *Horologium Oscillatorium*. Its highlights are Propositions 1 and 4, stating that two curves are in the relation of evolute to evolvent if and only if the tangents to the one curve intersect the other curve perpendicularly; Propositions 5 and 6, showing that the evolute of a cycloid is a congruent cycloid; and Proposition 11, giving a method by which the equation of the evolute of any algebraic curve can be calculated. The theory of Part III also relates the evolution of curves to their rectification, that is, to the determination of their arc lengths.

From Propositions 5 and 6 it follows in particular (cf., Figure 3, page 22, and Figure 42, page 80) that the motion of a pendulum with length *l*, performing small amplitude oscillations, can be approximated by cycloidal

oscillation in a cycloid with height ½*l;* this fact enables Huygens, in Part IV, to transfer the results for cycloidal oscillation to the case of real pendulums.

5

In deriving the relation between the period and the length of a pendulum, Huygens had considered a simple pendulum, that is, a mass point suspended on a weightless string. Real pendulums are far from conforming to this abstraction. So, naturally, a further question confronted him in 1659: How to define the length of a pendulum in such a way that it is meaningful to study the relation between period and length for pendulums of arbitrary shape. Huygens' answer was that the length of a real pendulum had to be defined as the length of a simple pendulum oscillating with the same period. Considering a pendulum to swing around a horizontal axis through its suspension point, Huygens gave the name "center line" to the line perpendicular to the axis and passing through the pendulum's center of gravity. He then defined the center of oscillation as the point on the center line whose distance to the axis is equal to the length of the isochronous simple pendulum. Thus the length of a real pendulum is the distance of its center of oscillation to the axis. Put differently, if the mass of a pendulum would be entirely concentrated in its center of oscillation, the resulting simple pendulum would perform exactly the same oscillations as the original pendulum.

Having thus fixed the precise interpretation of the concepts, Huygens was able to develop a full theory of the center of oscillation, which makes up Part IV of the *Horologium Oscillatorium*. He based his theory on the axiom (Hypothesis I) that if the bodies of a system start moving under the influence of gravity alone, the center of gravity of the system cannot rise above its original position. This axiom plays a crucial role in the whole of Huygens' mechanics. It is indeed a very powerful axiom because, in combination with the Galilean laws of fall, it implies the conservation of potential and kinetic energy. The culmination of the theory is in Proposition 5, which gives a general computational rule by which the center of oscillation of any pendulum can, in principle, be calculated. The rule can be rendered as

$$l = \frac{\Sigma a_i^2 g_i}{Gz} \qquad (5.1)$$

where G is the mass of the pendulum, z the distance of its center of gravity to its axis, and $\Sigma a_i^2 g_i$ (later called the "moment of inertia") is the sum of the

products of the weights g_i of the particles in the pendulum and the squares of their distances a_i to the axis. In formulating the rule (5.1) Huygens considers pendulums as consisting of a finite number of mass points. However, he extends the rule to a pendulum of a continuous medium by a limit process, taking the parts of the pendulum to be "minimal" and their number correspondingly large. In the subsequent propositions Huygens develops the geometrical theory that enables him, in Propositions 21–22, to calculate the centers of oscillation of many different shapes of plane and solid homogeneous pendulums. He shows himself able, with precalculus techniques, to reach results equivalent to the integrations involved in calculating the relevant moments of inertia ($\int a^2 dV$) and products $G \cdot z$ ($= \int a dV$). (As Huygens considers homogeneous bodies, the specific gravity is constant and cancels in the fraction (5.1).) Huygens finds in particular (Proposition 22) that the center of oscillation of a sphere with radius r, suspended (by a weightless string) on a point at distance d from its center, is located at distance $2r^2/5d$ below the center. That result is important in Huygens' experiments on pendular motion because he performed these experiments with spheres suspended on threads whose weight was neglected.

6

We now come to the last two propositions of Part IV. In them the lines of argument developed in the three theoretical parts come together to yield the crucial results of Huygens' enquiry into the mechanics of fall and pendular motion. It will therefore be useful to render the contents of these propositions in somewhat more detail.

The main objective of the propositions is to determine two fundamental measures: the length l_{sec} of a pendulum beating seconds and the distance d_{sec} traversed in the first second of free fall. In the course of his arguments about the determination of these measures Huygens also derives the central theoretical result of his book: the precise relation between the period and the length of pendulums.

Huygens was convinced that everywhere on earth the processes of fall and pendular motion are the same. He therefore proposed l_{sec} as a universal measure of length, suggesting that one-third of that measure, which is of the order of length of a foot, should be the universal unit of length, to be called the "pes horarius" ("hour foot"). He suggested that the hour foot could replace all the many different foot measures used in the various towns and regions of Europe, or at least that these measures should all be gauged against the universal hour foot. In view of the results of experiments performed at various places on earth, Huygens later had to acknowledge that

gravity is dependent on the geographical location, so that his suggestion of using l_{sec} as a universal measure is impracticable.

At the time Huygens worked out the theory of the pendulum clock, several attempts had been made to determine d_{sec} directly by experiments with falling bodies. Huygens himself performed such experiments in 1659 and found that they are very unprecise. However, he realized that l_{sec} can be measured with much greater precision, and that by means of the theories he had by then developed d_{sec} can be calculated from l_{sec}, thus providing a much better determination of d_{sec}. In the propositions we are discussing Huygens describes the necessary measurements and calculations.

To determine l_{sec}, Huygens suggests using pendulums consisting of a spherical bob suspended on a thread. The length of the thread is adjusted such that the pendulum (performing small amplitude oscillations) exactly beats seconds. Then l_{sec} is the distance between the suspension point and the center of oscillation. The position of the center of oscillation is known by Proposition 22, so that l_{sec} can be determined from the length of the thread and the radius of the spherical bob. Huygens adds the results of his own measurements (performed in The Hague in 1659): l_{sec} is 3 Parisian feet plus 8½ lines (a line is the 144th part of a foot), and therefore the ratio of the Parisian foot to the "hour foot" is 864 : 881. He stresses the necessity of knowing the precise location of the center of oscillation in the pendulums that are used.

Huygens goes on to explain that the value of l_{sec} can also be determined indirectly by measuring the length and frequency of an arbitrary pendulum and using the fact that the length of a pendulum is proportional to the square of its period and therefore inversely proportional to the square of its frequency. Let l be the length of the pendulum and n the number of swings it performs in an hour. Then $l :: T^2$ implies that

$$l_{sec} : l = 1 : T^2 = n^2 : (3600)^2, \tag{6.1}$$

so that l_{sec} can be calculated from l and n.

At that point (about halfway through the text of Proposition 25) Huygens remarks that until now the proportionality of pendulum length to the square of the period was a fact of experience only. But the theories developed so far imply a proof of that fact, and Huygens gives this proof. He argues as follows. From Propositions 5–6 of Part III on the evolution of curves it follows that, if its amplitude is kept small, a simple pendulum of length l oscillates with the same period as a cycloidal pendulum with axis $\frac{1}{2}l$. Thus

$$(T_{pend}(l))^2 = (T_{cycl}(\tfrac{1}{2}l))^2. \tag{6.2}$$

Now by Proposition 25 of Part II on cycloidal motion,

$$T_{cycl}(\tfrac{1}{2}l) : t_{fall}(\tfrac{1}{2}l)) = \pi : 2, \tag{6.3}$$

that is,

$$(T_{cycl}(\tfrac{1}{2}l))^2 :: (t_{fall}(\tfrac{1}{2}l))^2; \tag{6.4}$$

and by the Galilean law of fall ($d :: (t_{fall}(d))^2$, Proposition 3 of Part II, see below, Section 9)

$$(t_{fall}(\tfrac{1}{2}l))^2 :: \tfrac{1}{2}l; \tag{6.5}$$

and obviously $\tfrac{1}{2}l :: l$. Thus the relations (6.2–6.5) combine to prove Huygens' assertion that

$$l :: (T_{pend}(l))^2. \tag{6.6}$$

The proportionality (6.6) itself does not completely describe the relation between the period and the length of a pendulum because the proportionality factor is not yet determined. For reasons to be explained below (Section 10), Huygens does not give such a factor. However, he has already provided equivalent information earlier, namely the experimentally determined value of l_{sec}. With this value, (6.6) can be evaluated as

$$l : l_{sec} = (2T_{pend}(l))^2 : 1^2; \tag{6.7}$$

so that, given l, $T_{pend}(l)$ is also given, and vice versa.

Thus, at the end of Proposition 25, Huygens has fully determined the relation between the period and the length of pendulums. If we compare this result with the modern version of that relation, as expressed in the formula

$$T = 2\pi\sqrt{l/g}, \tag{6.8}$$

we note that the connection with fall motion, expressed by the constant g of gravity, is not present in Huygens' relations (6.6–6.7). Fall motion does play a role in the proof of relation (6.6), namely in the proportionalities (6.4–6.5), but at this point Huygens leaves the proportionality (6.5) further undetermined; the proportionality factor (which is, in fact, $\tfrac{1}{2}g$), or an equivalent specification, does not enter his arguments.

Huygens himself establishes the precise connection between the relations (6.6–6.7) and fall motion later and in a somewhat different context, namely when he uses the results (6.2–6.6) to determine the value of d_{sec} from the experimentally determined value of l_{sec}. This is the content of Proposition 26. Huygens argues as follows: Combining (6.2) and (6.3) (and using the convenient approximation 355/113 for π) yields

$$2T_{pend}\,(l) : t_{fall}(\tfrac{1}{2}l) = 2\pi : 2 = 355 : 113. \tag{6.9}$$

In particular for the pendulum beating seconds, $2T_{pend}(l_{sec})$ is 1 sec; so that from (6.9) the time $t_{fall}\,(\tfrac{1}{2}l_{sec})$ required to fall through ½ hour feet can be calculated. Huygens finds

$$t_{fall}(\tfrac{1}{2}l_{sec}) = 19'''\,\tfrac{1}{10}. \tag{6.10}$$

Now the proportionality for free fall,

$$(t_{fall})^2 :: d_{fall}, \tag{6.11}$$

can be evaluated by means of (6.10):

$$(t_{fall}(\tfrac{1}{2}l_{sec}))^2 : (t_{fall}(d_{sec}))^2 = \tfrac{1}{2}l_{sec} : d_{sec}, \tag{6.12}$$

or, as $\tfrac{1}{2}l_{sec}$ = ½ hour feet,

$$(19'''\,\tfrac{1}{10})^2 : (60)^2 = \tfrac{1}{2} : d_{sec}; \tag{6.13}$$

which yields

$$d_{sec} = 14 \text{ hour feet, } 9 \text{ inches, and } 6 \text{ lines,} \tag{6.14}$$

which is approximately 15 Parisian feet and one inch. The remainder of the Proposition is devoted to a description of experiments which Huygens had performed earlier to determine d_{sec} directly. I shall not discuss these experiments here; I note only that the experimental values for l_{sec} and d_{sec} which Huygens gives in the *Horologium Oscillatorium* correspond to the value $g = 9.81\ m/sec^2$.

With the determination of d_{sec} Huygens has gathered all the information necessary to establish precisely the connection between pendular motion and fall motion. Thus one may say that at this point finally he attains a result equivalent to the modern formula $T = 2\pi\sqrt{l/g}$. However, in Huygens' presentation the connection remains rather implicit. It will therefore

be instructive here to make the connection explicit by showing that, if we use the value of d_{sec} to evaluate the fundamental proportionality (6.6)

$$(T_{pend}(l))^2 :: l, \tag{6.15}$$

we find a result equivalent to the modern formula (6.8). Indeed, as $t_{fall}(d_{sec}) = 1$ sec, it follows from (6.2–6.3) that $T_{pend}(2d_{sec}) = \frac{1}{2}\pi$ sec; hence

$$(T_{pend}(l))^2 : \pi^2/4 = l : 2d_{sec}. \tag{6.16}$$

Now the full period T is equal to $4T_{pend}$, and, in terms of the gravitational constant g, $2d_{sec} = g$, so we arrive at

$$T^2 : 4\pi^2 = l : g, \tag{6.17}$$

or

$$T = 2\pi\sqrt{l/g}. \tag{6.18}$$

It is worth noting how all the highlights of the earlier parts of Huygens' treatise are brought together here to yield the results of the last two propositions. Part II provides the relation between length and period in the case of cycloidal oscillation. Part III gives the relation between pendular motion and cycloidal oscillation by means of the theory of evolutes, showing that small amplitude pendular oscillation with length *l* behaves as cycloidal oscillation with axis $\frac{1}{2}l$. Part IV provides the determination of the center of oscillation, essential for the experimental determination of l_{sec}.

7

THE *Horologium Oscillatorium* is concluded by a fifth part, much shorter than the others (seven pages in the original edition). It contains the description of another regulating device for clocks, namely the conical pendulum, and a number of theorems concerning centrifugal force. The modern equivalent of these theorems is the familiar formula $F = mv^2/r$, where F is the centrifugal force, m the mass of the body, v its velocity, and r the radius of its orbit. Like most of the other theories in the book, the theorems on centrifugal force date from 1659. At that time Huygens wrote a fully argued theoretical treatise *De Vi Centrifuga,* but he did not publish it; the theorems in Part V are taken from that treatise. Huygens just states these theorems, he does not explain or prove them. Thereby the fifth part con-

trasts strongly with the extensively argued theoretical Parts II–IV. The omission of the proofs blocked the influence which full publication of Huygens' theory might have had at the time the *Horologium Oscillatorium* appeared. The treatise *De Vi Centrifuga* was published posthumously in 1703, but by that time Newton's mechanics had outdated Huygens' approach to the topic of centrifugal force, so that Huygens' derivation of the laws exerted little influence on the further development of mechanics.

8

We have seen how the mechanism of the pendulum clock unifies the technical and the theoretical parts of the *Horologium Oscillatorium*. However, the unity of the work resides not only in the coherence of its subject matter, but also in its style. That style is best characterized as "geometrical physics," physics presented in the language of classical Greek geometry. Huygens' book indeed marks a culmination of geometrical physics. Its style was later superseded by an analytical style of mathematical physics, which arose in the years around 1700, and which used algebra, calculus, and analysis rather than geometry to describe and understand the phenomena of nature.

It is this analytical language of physics that we are now used to, and therefore the same stylistic aspects which provide unity to Huygens' book may also be an obstacle to the modern reader in understanding it. But that obstacle can be overcome, and therefore it may be of use here very briefly to point out some special aspects of the style of geometrical physics.

Huygens presents his theory in the style of the classical geometrical and mechanical works. He explicitly states his starting points in the form of definitions and hypotheses, and he gives his results as propositions with proofs based on the hypotheses and the earlier propositions. Propositions and proofs are formulated almost entirely in prose, referring to geometrical figures that are considered either abstractly or as models of some mechanical object or process. Many of the propositions concern relations between physical quantities such as velocity, time, and distance; these relations are not expressed by means of analytical formulas, but by proportionalities.

9

Let us study the geometrical, rather than analytical, representation of the relations between physical quantities in somewhat more detail, taking as example Propositions 1–3 of Part II. In these propositions Huygens presents the Galilean theory of free fall that is usually summarized in the familiar modern formulas

$$d^2s/dt^2 = dv/dt = g \tag{9.1}$$

$$s = \tfrac{1}{2}vt \tag{9.2}$$

$$s = \tfrac{1}{2}gt^2 = v^2/2g; \tag{9.3}$$

where s is the distance traversed, v the velocity, t the time, and g the constant of gravitational acceleration.

Huygens' Proposition 1 states that in equal time intervals the velocity increases with equal amounts, and that in a sequence of consecutive equal time intervals the increments of the distances traversed increase themselves with a constant amount. The most faithful symbolic rendering of Huygens' expression here is in terms of first- and higher-order differences (Δ, Δ^2):

$$\Delta v \text{ depends only on } \Delta t \tag{9.4}$$

and

$$\text{if } \Delta t \text{ is constant then } \Delta^2 s \text{ is also constant.} \tag{9.5}$$

Huygens knew that (9.4) implies the proportionality

$$\Delta v :: \Delta t. \tag{9.6}$$

Thus Proposition 1 is related to the modern formula (9.1), but with notable differences: neither a derivative nor a physical constant like the acceleration g of gravity occurs in Huygens' formulation. (The proportionality (9.6) can be derived from (9.4) by a process which is essentially the solution of a functional equation; Huygens used this technique, as Galileo had done before him, in the proof of Proposition 3 to derive the proportionalities $s :: t^2$ and $s :: v^2$ from Propositions 1 and 2.)

Proposition 2 states that the space traversed from beginning of fall in a certain time is equal to the space traversed in the same time with a uniform velocity equal to half the velocity reached at the end of the fall. Comparing this formulation with the formula (9.2), it should be noted that no multiplication of velocity by time occurs in Huygens' formulation.

Proposition 3 states that the spaces traversed in free fall from rest are proportional to the squares of the times of fall as well as to the squares of the final velocities, that is:

$$s :: t^2 \text{ and } s :: v^2. \tag{9.7}$$

The related modern formula is (9.3), but, again, no physical constant g is

mentioned in Huygens' formulation, nor is a space put equal to the product of two other-dimensioned physical quantities.

It is important to note how geometrical physics proceeds in those cases where the modern theory uses derivatives (as dv/dt). Instead of derivatives we find statements about the changes occurring in time intervals. In the above propositions, these time intervals are finite; no limit processes are involved. In the further propositions of Part II Huygens is forced to consider the limiting behavior of the changes occurring in the time intervals. He does so in the rigorous geometrical style of classical Greek geometry. When proving, for instance (cf., Proposition 24 of Part II), that the time of fall along a cycloidal arc is equal to a certain value T, he splits the process up into time intervals and corresponding distances that can be chosen as small as one wishes, and he uses predetermined inequalities concerning these intervals to show that the total fall time along the arc can neither be smaller nor greater than T.

Huygens does not always keep to the strict rigor of classical geometry in dealing with limit processes, but his approach to them is always geometrical. In his calculations of centers of oscillation he freely uses "minimal parts" of figures, conceiving, for instance, a circle as divided into minimal equal squares (Part IV, Proposition 9). In these cases he leaves it to the reader to find the rigorous proofs of the results gained in this way.

10

THE three propositions discussed above illustrate another feature of geometrical physics as well: relations between quantities of different dimension are expressed as proportionalities, not as equations. This feature is also evident in the last two propositions of Part IV, rendered in detail in Section 6 above. The arguments of these propositions are almost entirely given in terms of proportionalities. When these proportionalities concern quantities of different dimension (length and time squared in this case), Huygens does not give a proportionality factor. He either leaves the proportionality further unspecified, or he evaluates it by giving two corresponding values for the quantities (as for instance the value of l_{sec} and 1 (sec) to evaluate $(T_{pend}(l))^2 :: l$). The reason why Huygens does not evaluate proportionalities by indicating the proportionality factor is that such a factor would have a new dimension made up by dividing unequal dimensions ($[l]/[T]^2$ for instance). In the geometrical framework in which Huygens expressed his physical theories, such a procedure was conceptually unacceptable.

The absence of dimensioned proportionality factors in geometrical physics has the effect that fundamental physical constants are introduced in

a different way than in modern analytical physics. The analytical expressions of the principal laws of fall and pendular motion are:

$$s = \tfrac{1}{2}gt^2 \tag{10.1}$$

$$T = 2\pi\sqrt{l/g}; \tag{10.2}$$

in both cases the relevant physical constant, g, enters the formula for the law. In geometrical physics the corresponding laws are presented as, further unspecified, proportionalities:

$$s :: t^2 \tag{10.3}$$

$$T^2 :: l. \tag{10.4}$$

The constants necessary for evaluating these proportionalities are provided, if necessary, by the values of two corresponding quantities (cf., e.g., Formulas (6.10–6.12) above). Usually one of the quantities is taken as an appropriate unit; in that case the value of the corresponding quantity serves as a fundamental constant. In the case of the proportionalities (10.3–10.4) these values are d_{sec} (corresponding to $t = 1$ sec) and l_{sec} (corresponding to $T = 2$ sec) respectively. Thus in geometrical physics the fundamental constants are measures of quantities corresponding to some appropriate unit in a fundamental proportionality. Such constants have a primary dimension (length in the case of d_{sec} and l_{sec}). They enter the arguments only if numerical calculations or measurements are involved; they do not enter the theoretical arguments which proceed entirely in terms of proportionalities. This special feature of geometrical physics explains why it was that Huygens only in the last propositions of Part IV, which are themselves devoted to the determination of the fundamental measures d_{sec} and l_{sec}, reached the result on pendular motion equivalent to the modern formula $T = 2\pi\sqrt{l/g}$. It also explains the complicated structure of Huygens' argument in these propositions.

11

Because of the absence of dimensioned proportionality factors, geometrical physics was a physics without formulas, at least without formulas presenting equalities between quantities of different dimension. Indeed the text of the *Horologium Oscillatorium* shows very few formulas, and of these

few, the majority are formulas of analytic geometry, not physics (e.g., the formulas in Part III, Propositions 10 and 11). Still, at one place Huygens comes near to writing down analytical physical formulas, namely in connection with the center of oscillation. Huygens presents his rule for calculating the position of this center (Propositions 5 and 6 of Part IV) as a formula (cf., Formula (5.1) above), involving a quotient of two quantities, the one a sum of products of weights and squares of lengths, the other a product of a weight and a length. Thus he deals with a formula involving quantities of different dimension which are divided. However, Huygens explicitly notes the conceptual difficulty involved in this procedure and he actually avoids it. He explains in Definition 13 of Part IV that, when talking about a product of a weight and a length, he considers the weight to be represented by either a number or a length taken proportional to the weight. The proportionality factor is not specified, and in this particular case that is not necessary, as the weights occur both in the denominator and the numerator of the fraction so that the factors cancel.

The present translation of the *Horologium Oscillatorium* is kept as close as possible to the original, and thus the style of geometrical physics, which so strongly characterizes Huygens' book and which provides its characteristic unity of form, has been kept. At first confrontation it could be an obstacle to the modern reader, but it is hoped that the discussion above may remove much of that obstacle. It should not be too difficult to acquaint oneself with Huygens' style of presentation, to render his statements in abbreviated symbolic notation, and (if one so wishes) subsequently to transform them into the corresponding formulas of modern analytical physics. By studying Huygens' work in this way one will also recognize the aspects of his style for what they are: not cumbersome inadequacies, but explictly stated necessary elements of physical argument. In later times these elements have been hidden in the seemingly self-evident automatisms of the definitions, conventions, and symbolisms of mechanics and differential calculus. But seventeenth-century natural philosophers had to deal with them explicitly, and Huygens' treatise may be enjoyed as a beautiful specimen of this explicit handling of physical concepts and argument.

12

THE discussion of the style of Huygens' *Horologium Oscillatorium* also provides an important clue for understanding the book's later fate. It is no exaggeration to say that from the 1650s till the 1680s Huygens was Europe's most creative mathematician and natural scientist. He was indeed recognized as such. Leibniz and Newton, the men whose achievements Huygens

at the end of his career would come to acknowledge as equal to, if not surpassing his own, admired and praised his work. Yet, in comparison with his intellectual and creative stature, Huygens' influence was remarkably small. Several causes can be discerned for this lack of impact. Huygens tended to publish his results rather late, after much pondering and polishing, or not at all. As we have seen, the style of his writings soon became outdated. And often Huygens' achievements had a certain finality, not opening up new frontiers, nor inviting elaboration or use as starting points for further research.

Late publishing, a style soon to be superseded, and finality of results also apply to the *Horologium Oscillatorium,* and thus the book shared the general fate of Huygens' work: it was admired, recognized as a paradigm of the axiomatic geometrical style of mechanical study, but it exerted little direct influence. Indeed Huygens' own interests in mechanics were already shifting when the book was published. He began to explore the use of springs rather than pendulums as regulators of clocks, and in that connection he developed a general theory of harmonic motion. The concept of force (absent in the book of 1673 apart from the merely stated theorems about centrifugal force) came to be more central in his mechanics. He worked out a mechanistic explanation of gravity (published as an appendix to the *Traité de la Lumiere* in 1690), thus leaving axiomatic mechanics for an endeavor to explain the phenomena of nature within the framework of a mechanistic philosophy.

Even more than those of Huygens, the interests of others working in the field of mechanics shifted away from the style and the subject of the *Horologium Oscillatorium.* With Newton's work the concept of force became central in mechanics, inviting the study of motion under influence of forces, rather than, as in Huygens' treatise, the study of motion based on an axiom equivalent to conservation of energy. Moreover, the decades around 1700 brought the invention of differential and integral calculus, and the swift realization of its power, especially in the formulation of mechanical laws and problems as differential equations, and in the solution of the problems through the integration of these differential equations. This caused the transition to the style of analytical mechanics in which the great eighteenth-century successes of rational mechanics were expressed. These developments combined to block the influence of the *Horologium Oscillatorium.*

13

HOWEVER, the value of a book lies not only in its influence but also in its significance within intellectual history. What, then, is the significance of the

Horologium Oscillatorium? It is, first of all, that it allows us an insight into the working of a very individual and very creative mind. Huygens' way of scientific thinking, sharply formulating the basic concepts and relations in his subject, isolating single powerful ideas, and working these out to their outmost consequences by means of powerful and rigorous mathematical argument, is alive and inspiring regardless of the differences in scientific style between his time and our own. The book is historically significant also because, although conservative in style, it stood, and was recognized, as a solid symbol of one of the main tenets of the seventeenth-century scientific revolution: the power of the mathematical method in understanding and utilizing the phenomena of nature.

14

THE present English edition of the *Horologium Oscillatorium* renders the text of the original edition of 1673 as faithfully as is possible in a translation. As such it offers the reader a first encounter with the text. In that first encounter many aspects of Huygens' book can be understood, valued, and enjoyed: the deep results, the characteristic style, and the brilliance of argument.

But, obviously, there are aspects that will remain hidden when one has only the text as evidence. These are for instance: the prehistory of the theories, their dependence on the work of others, their originality, their influence, and the relation of the Huygensian concepts to those used by others. Also a deeper understanding of Huygens' argument requires more than the text itself offers: the information from his manuscripts, the earlier formulations of his hypotheses, the variants of the derivations, etc. There is, for instance, the first proof of the isochronism of the cycloid, alluded to above, of which no trace is to be found in the *Horologium Oscillatorium*.[1] There is the very complicated question of what experiments Huygens actually performed in 1659 to determine the fundamental measures l_{sec} and d_{sec}.[2] And it has recently been found that there have been earlier versions of parts of Part IV which Huygens rewrote, shortly before the publication, because of criticisms by Roberval.[3]

As the present work is primarily an edition of the text, it cannot go far in explaining these deeper aspects of the work. So for any further study the reader must turn to the standard edition of the original text in Huygens' *Oeuvres Complètes* and to the relevant studies by various scholars. The footnotes to the text provide references to relevant sections in the *Oeuvres*

1. The proof is discussed in the items 13, 14, 20, 22 (pp. 115–144), and 23 of the Bibliography.
2. See item 22 of the Bibliography, esp. pp. 77–101 and 144–59.
3. For further information see the items 16 and 17 of the Bibliography.

Complètes; they also identify the persons which Huygens mentions and the contemporary works he refers to. The Bibliography given at the end of this Introduction is meant as a guide to the literature for those who want to study matters concerning Huygens' *Horologium Oscillatorium* in more detail.

I am sure that the present translation will perform a most useful and valuable function for the scholarly study of the book as well as for the stimulation of a wider interest in Huygens' work. The scholar will find the translation useful for quick orientation in the text and location of the passages of his or her special interest. The teacher will welcome the opportunity offered by this translation to acquaint students with Huygens' thought and seventeenth-century scientific argument in general. And the general reader will benefit from the possibility this translation offers to experience a confrontation with a brilliant scientific mind.

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TRANSLATOR'S PREFACE

CHRISTIAAN HUYGENS' *Horologium oscillatorium sive de motu pendulorum ad horologia aptato demonstrationes geometricae* (Paris: F. Muguet) was first published in April of 1673. It contained 161 pages, plus one unnumbered "Corrigenda" page. A second edition by Guilielmus Jacobus 's Gravesande appeared in 1724 in *Opera varia* (Lugduni Batavorum: Apud Janssonios vander Aa., 15–192). This in turn was republished with the same pagination in *Opera mechanica, geometrica, astronomica et miscellanea* (1751). A modern edition was published in 1934, with an opposite page French translation, in *Oeuvres complètes de Christiaan Huygens,* publiées par la Société Hollandaise des Sciences (La Haye: Martinus Nijhoff) 18: 73–368. A German translation by A. Heckscher and A. v. Oettingen under the title *Die Pendeluhr* appeared in 1913 as vol. 192 in *Ostwalds Klassiker der exakten Wissenschaften* (Leipzig:Verlag von Wilhelm Engelmann). No English translation has been published to date.

Huygens' "Corrigenda" page in the first edition lists twenty-six corrections, mostly of a purely typographical character, in the Latin text and in three of the figures. These changes were made by 's Gravesande in 1724. The editors of the 1724 and 1934 Latin editions have identified a few other typographical errors. All such obvious printing errors have been corrected in the Latin texts used for this translation without indication in the footnotes. On the other hand all changes of wording introduced both by Huygens in his "Corrigenda" and in the margins of his personal copy and later by 's Gravesande in the 1724 edition are carefully and fully described in the footnotes, which give the variant readings for consideration. Such wording changes and addenda are few in number.

In the 1673 edition the geometrical figures appear, without being numbered, at the appropriate places as needed throughout the printed text. In the 1724 edition these figures are printed on twenty-eight plates inserted as separate pages near the proper places in the text. 's Gravesande has corrected the printing errors in the three figures as indicated in Huygens' "Corrigenda." The shading of the solid bodies and of the clock mechanisms in

the figures was also added by 's Gravesande. Because of their superior accuracy and clearer printing in comparison with the figures of the first and second editions, we have chosen to reproduce for this translation the figures found in the 1934 edition, which include 's Gravesande's modifications. In the 1934 edition the sequential numbering of the figures begins with "Figure 17." We have chosen to renumber them beginning with "Figure 1."

In preparing this translation we have remained as close as possible to the original text in rendering the mathematical expressions in the belief that this is important to preserve the spirit of the original work, even though some of Huygens' mathematical expressions would now be considered archaic, or clumsy, or more easily stated in modern notation. Those who seek the latter should consult the German version, mentioned above, in which the translators have chosen to use modern mathematical expressions both in the body of the text and in their extensive notes.

All references found in square brackets within the body of the translation are Huygens' own references, originally printed in the margins of the 1673 edition.

Huygens' Latin style in the *Horologium oscillatorium* is quite laconic and innocent of rhetorical flourishes. A special effort has been made to produce a translation which reads well. However, literal meaning has always been preferred over literary nicety when such a choice was forced upon us.

We wish to acknowledge our indebtedness to the editors and publisher of the 1934 edition of the *Horologium oscillatorium* for their excellent Latin text and for the geometrical figures used in the preparation of this translation. We have benefited considerably from the helpful advice and criticisms of Professors Albert van Heldėn, Alan Gabbey, and H. J. M. Bos, and from the meticulous copy editing of Suzanne Lowitt at the Iowa State University Press. Appreciation must also be extended to the librarians of the Special Collections Department, Northwestern University Library, and of the Vatican Film Library of Saint Louis University's Pius XII Memorial Library, for providing access to the original seventeenth century editions. Finally the publication of this translation was made possible in part by the Beaumont Faculty Development Fund and by the Mellon Faculty Development Fund, both of Saint Louis University, which provided assistance for the costs of preparing the final typescript and for a partial subvention towards publication respectively.

Saint Louis University
6 August 1985

THE TRANSLATION

CHRISTIAAN HUYGENS'

The Pendulum Clock or Geometrical Demonstrations Concerning the Motion of Pendula as Applied to Clocks

This book is divided into five parts.

Part I contains a description of the pendulum clock.

Part II deals with the falling of heavy bodies and their motion on a cycloid.

Part III deals with the evolution and dimension of curved lines.

Part IV deals with the center of rotation or of motion.

Part V displays another construction of the clock in which the motion of the pendulum is circular, and announces theorems on centrifugal force.

TO LOUIS XIV,
RENOWNED KING OF FRANCE AND OF NAVARRE

WE are especially indebted to France, Oh Great King, for the rebirth and restoration of geometry in this century. For it was here that there first appeared those who, by their great and improved efforts, restored and returned to light that lost and almost buried science. Following in their footsteps, very talented men throughout Europe have since then so refined geometry and have so surpassed the findings of the ancients that few things seem to have been left by them for the discovery of posterity. In this science, which I have always admired and loved very much, whenever I have applied my mind I have given preference to the investigation of those things which can be found useful either to making life more comfortable or to the understanding of nature. I believe I have done my best work when I have dealt with things whose usefulness is connected with some subtlety of thought and difficulty of discovery. If it is permitted for us to present a recommendation which does not appear to be unworthy of your greatness, I confess that I have pursued this double goal with greater success in the invention of my clock than anything else.

When Huygens wrote this dedication, his personal political situation was quite delicate. In 1672 French armies had again attacked the Dutch Republic and almost destroyed it. In 1666 Huygens had moved to Paris (where he lived until 1681) at the invitation of King Louis XIV to be a founding member of the Académie Royale des Sciences, of which he had become the preeminent representative. Nationalistic, religious, and philosophical rivalries among his associates at the Académie were balanced by his protection from the King's minister Colbert. The publication of this dedication caused a wave of criticism from Huygens' countrymen in Holland.

For this clock is not only a mechanical invention, but much more importantly it is constructed on geometrical principles. Now what pertains to the latter must be sought in the innermost recesses of that art and is not a light endeavor. Therefore, among all the things to which I have applied myself at great cost in my studies, without doubt I give the first place to this speculation. It is not necessary to say very much to show you, most powerful King, how useful this is. For since my clocks were judged worthy to be placed in the private chambers of your palace, you are aware from daily experience how much better they are in displaying equal hours than other such instruments. Further you are not unaware of the more specialized uses which I intended for them from the beginning. For example, they are especially well suited for celestial observations and for measuring the longitudes of various locations by navigators. By your order, more than once our clocks have been transported by sea. Also under your auspices many clocks intended for astronomical uses can be seen in that outstanding observatory which you recently ordered to be built with a remarkable generosity surpassing all other kings. When I reflect on these matters, I feel very joyful that this invention happened to occur during your reign. No one, I think, who understands how much this invention is due to you, will ask why I was led to dedicate to your august name these laborious studies in which I have explained the full theory and description of my clock. And still less would anyone wonder about this who knew that by your kindness I have enjoyed tranquil leisure for these and other meditations. Moreover it is necessary for me to justify to you this leisure to some degree, and to let it be known in some way that, affected by your many and continuous favors, I have discharged my duty by giving some evidence of being a grateful person. Since you are very busy and must attend to the highest affairs, which is to be expected in one who stands at the summit of things, I realize that you hardly have the liberty to attend to these contemplations, even though you are capable of all things. Nevertheless, I do not think that this work is any less welcome or any less pleasing to you, most

august King. We know that you are most pleased by those things which provide the most public benefit, and that you have no greater concern than that new advances be made in the best sciences and that these sciences be enlightened by new discoveries. This has been clearly established by your exceptional and singular generosity both in promoting the sciences and in remunerating those who are outstanding in their knowledge of the sciences. Neither the costs of wars, which are enormous and much greater than usual expenses, nor the boundaries of your France, limit your generosity. It is quite apparent that you do this so that not only those living under your rule but also the entire world, wherever it is worthy of your favors, will become wiser, happier, and better provided for with you as ruler. Perhaps these literary memorials will also contribute something to your most true and most illustrious glory. And if these writings can prove to posterity that the sciences and the arts flourished at this time, they will also teach them that above all else this was due to your virtue and your magnanimous spirit.[1]

Paris, 25 March 1673

1. In the 1673 edition Huygens included, on six pages immediately following the dedication, a 257 line poem in Latin bearing the heading, "Hadriani Vallii Daphnis Ecloga ad Christianum Hugenium Zulichemium, Constantini F." (reprinted in full in *Oeuvres complètes* 5:292–99). This encomium to Huygens was written in 1665 by Adriaan van der Wall (Hadrianus Vallius). Three of the 257 lines were inserted in 1673, probably by Huygens, in praise of King Louis XIV by suggesting that the recently discovered satellites of Saturn be named the "Bourbon stars."

LICENSE FROM THE KING

LOUIS, by the Grace of God, King of France and of Navarre: To our friends and faithful advisers, members of the session of Parliament, regular masters of petitions of our estate, bailiffs, sinecurists, provosts, their lieutenants, and all other judges and officers to whom this would apply, greetings. Our dear and good friend Francois Muguet, our regular printer, has very humbly brought it to our attention that he has received a book by Christiaan Huygens of Zuilichem, son of Constantijn, entitled *The Pendulum Clock or Geometrical Demonstrations Concerning the Motion of Pendula as Applied to Clocks,* which he wishes to make public, if it pleases us to grant him the permission which he humbly requests, and for these reasons we are moved to deal favorably with the petitioner. We give to him by these presents our permission and agreement as follows: to print, or to have printed, the above mentioned book in such form, character, size, and frequency as he deems proper, for a period of six full and consecutive years, beginning on the day he publishes the book for the first time, making this explicitly known to all persons whatever their quality or condition might be; to sell the book he prints, or has printed, during the above mentioned time in any place in our kingdom, without the satisfaction of the petitioner or of those who may receive rights from him under any circumstances whatsoever, and to pay a fee of fifteen hundred pounds of compensation, to be distributed one-third to us, one-third to the General Hospital of the city of Paris, and the other third to the petitioner; to confiscate all counterfeit copies; and to collect all fees, damages, and interests; provided that he will send two copies to our regular library, one to be deposited in the collection at the Louvre and the other in the collection of our friend and trusted Guardian of the Seals, Mr. Daligre. If you send word that you know that the petitioner is pleased and satisfied with the content of these presents, and that those who may receive rights from him fully and peacefully have ceased, or will cease, from all claims and obstacles to the contrary, agreeing that these presents, or a certification of them, be inserted in each book so that they be held valid and properly notified, then we will immediately order our bailiff or appropriate sergeant to take all actions necessary for the execution of these presents. For such is our pleasure. Done at Versailles on the last day of September in the year of grace one thousand six hundred seventy-two. In the thirtieth year of our reign. Signed: Louis. For the King, Colbert.

Registered in the book of the Communaute des Marchands Libraries et Imprimeurs in Paris, 4 November 1672, in accordance with the decrees of Parliament of 8 April 1653, and of the Privy Council of the King of 27 February 1665. Signed: D. Thierry, Syndic.

Published for the first time on the first day of April, 1673.

The copies have been sent.

The Pendulum Clock or Geometrical Demonstrations Concerning the Motion of Pendula as Applied to Clocks

SIXTEEN years ago we published a pamphlet* in which we publicly described the construction of clocks then recently invented by us. Because we have discovered many things since then relating to the improvement of that instrument, we have decided to explain them in detail in this book. These things are so closely related to the perfection of this invention that they can be considered to be its most important part and to be the foundation, previously neglected, of this whole instrument. For the simple pendulum does not naturally provide an accurate and equal measure of time since its wider motions are observed to be slower than its narrower motions. But by a geometrical method we have found a different and previously unknown way to suspend the pendulum; and we have discovered a line whose curvature is marvelously and quite rationally suited to give the required equality to the pendulum. After applying this line to clocks, we have found that their motion is so accurate and constant that, after many experiments on both land and sea, it is now obvious that they are very useful for investigations in astronomy and for the art of navigation. This line is the path traced out in the air by a nail which is fixed to the circumference of a rotating wheel which revolves continuously. The geometers of the present age have called this line a cycloid and have carefully investigated its many other properties. Of interest to us is what we have called the power of this line to measure time, which we found not by expecting this but only by following in the footsteps of geometry. We made this known some time ago to some friends who understood such things (for we noticed this not long after our first pamphlet about clocks), and now we present the same thing for all to read, supported by a demonstration which we think is most accurate. The main part of this

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**Horologium* (September 1658), in *Oeuvres complètes de Christiaan Huygens,* publiées par la Société Hollandaise des Sciences (La Haye: Martinus Nijhoff, 1888–1950) 17:52–73, in opposite page Latin text and French translation. [This 22 volume modern edition of Huygens' works will be cited hereafter as *Oeuvres.*] For Huygens' early writings on clocks prior to 1673, see *Oeuvres* 17:1–237; 18:1–25.

book will be devoted to presenting this proof. But first it will be necessary to use some new demonstrations to stabilize and expand further the doctrine of the great Galileo concerning the falling of heavy bodies, a doctrine whose most desirable fruit and highest apex is this very property of the cycloid which we have discovered.

Next, in order to apply this line to the use of pendula, it was necessary to add a new consideration of curved lines; namely, those curves which by their evolution generate other curves. From this arises a comparison of the lengths of curved lines and straight lines, which I have pursued further than the present topic requires because it seems to me that this theory is new and elegant.

Furthermore, in order to explain the nature of the composite pendulum, which I demonstrate is useful in the construction of these instruments, it is necessary to add a consideration of the centers of rotation, a topic which has been treated to date by many but with little success. In this consideration are to be found many theorems pertaining to linear, plane, and solid figures, which are worthy of notice, if I am not mistaken. Before all this the mechanical construction of the clock and the application of the pendulum will be presented in a form found to be best adapted to astronomical uses, and all other clocks can easily be related to this sketch by making the required changes.

Since the remarkable success of this invention has brought about what often happens in such cases, and what I predicted; namely, that many would desire to be its creator themselves or would wish to attribute this honor, if not to themselves, to someone else in their country rather than to us, I have decided finally that this is the place to fight against these unfair efforts.[1] Clearly only one thing is needed to fight them; namely, sixteen years ago I discovered the construction of these clocks by my own personal reflection and took the trouble to build them, when at that time no mention was made of such clocks by anyone either orally or in writing, and no rumor about them was heard at all. (I am speaking of the use of the simple pendulum as applied to clocks since I do not believe anyone will try to challenge me regarding the addition of the cycloid.) In the following year, 1658, I published a sketch and a description of this clock and widely distributed copies of both the clock and the pamphlet. These things are so well

1. Huygens' claim in this paragraph to priority in the invention of the pendulum clock is not easy to evaluate, although he is generally credited with this invention. The difficulty resides primarily in determining the precise structure and actual construction of other contemporary clocks for which a prior claim might be made. The key point is the introduction of the idea of regulating the motion of clocks, and with a very high degree of precision, by means of the swings of a simple pendulum. Huygens' chief clockmaker at The Hague was Samuel Coster.

For a review of this matter see *Oeuvres* 18:59–66; John Drummond Robertson, *The Evolution of Clockwork* (London: Cassell, 1931); A. E. Bell, *Christian Huygens and the Development of Science in the Seventeenth Century* (London: Edward Arnold, 1947), 34–40.

known to all that there is no need to confirm them either by the testimony of scholars or by the acts of the state of Holland, which could be done. Hence it is quite clear what judgment should be passed on those who seven years later sold this same construction in their books as if it had been made by them or by their friends.[2] If those who desire to refer the origins of this matter to Galileo say that he tried but failed to make this invention, it seems that they detract praise from him rather than from me, since indeed in that case I would have carried out this same investigation more successfully than he did. On the other hand, if they maintain that this invention was actually made by Galileo, or by his son as one scholar has recently claimed, and that these clocks[3] were exhibited, then I do not know how they could hope to be believed since it is hardly likely that such a useful discovery would remain unnoticed for eight full years prior to my publishing it. And if they say that this invention was intentionally concealed, then they are saying that this was done by someone who wishes to claim the origin of this invention for himself unjustly. Thus they must prove that this is so, and even at that, it would in no way relate to me unless it were also shown that I alone was aware of what was concealed from everyone else. This is what I have found necessary to say in my defense. Let us now proceed to the construction of the clock.

2. It is not known precisely what book or books Huygens has in mind here. Perhaps he refers to Cornelio Malvasia (1603–1666), *Ephemerides novissimae motuum coelestium . . .* (1662).

3. Huygens refers here to a clock made in 1649 by Vincenzio Galilei which is described in ambiguous terms in *Saggi di Naturali Esperienze fatte nell' Accademia del Cimento sotto la Protezione del Serenissimo Principe Leopoldo di Toscane* (Firenze: G. Cocchini, 1667), 22. The relevant passage is reprinted in *Oeuvres* 18:60 n.5.

[Fig. 1: I, II, III, IV]

Fig. I.

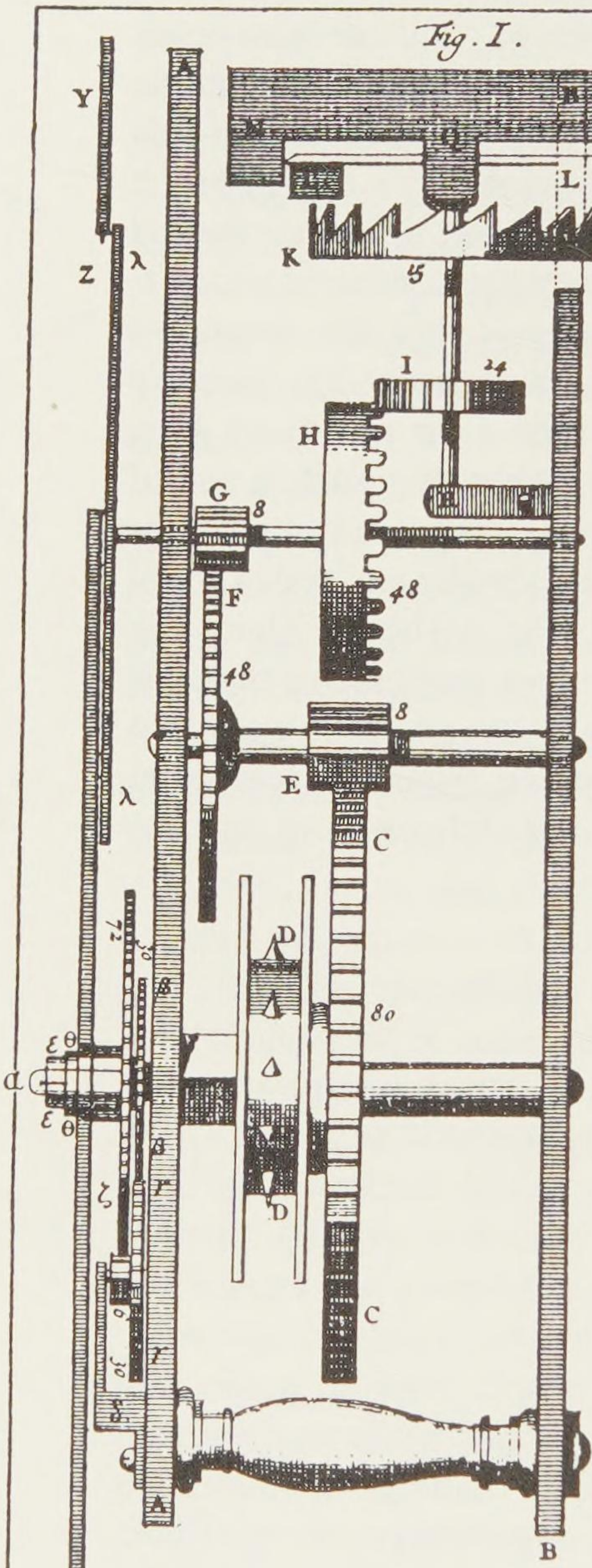
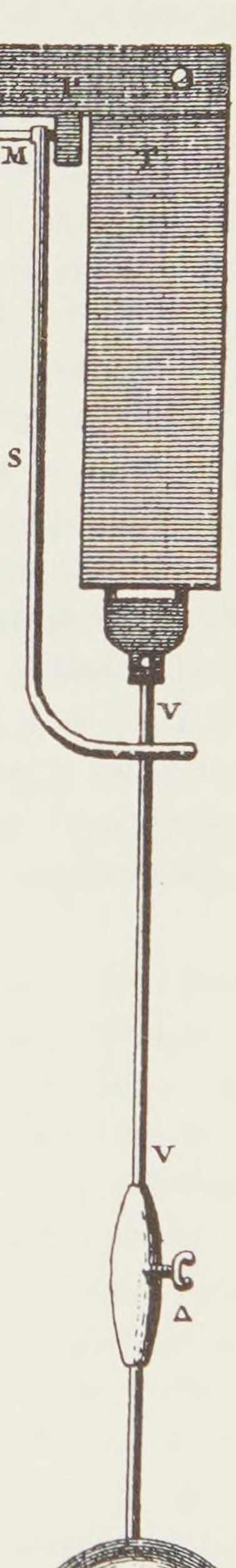


Fig. II.



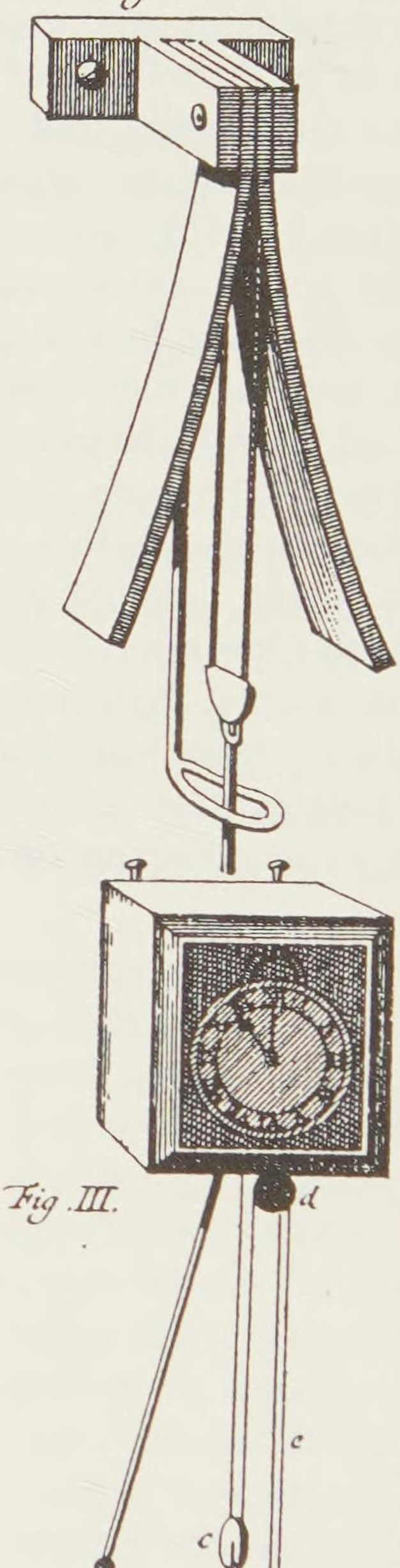
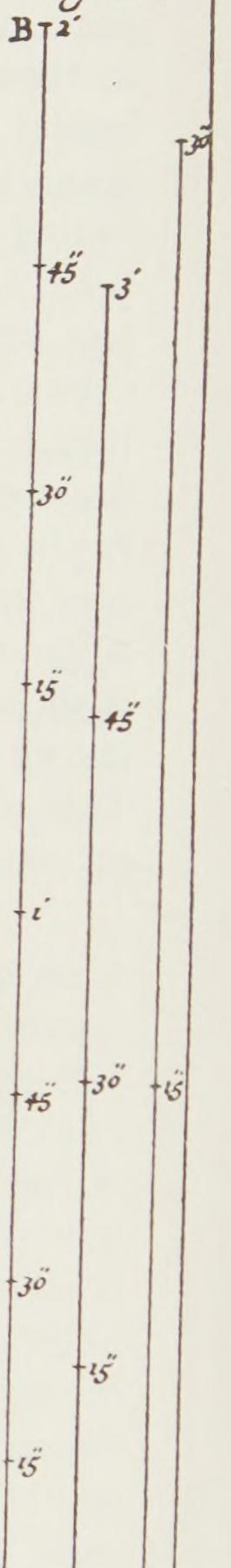


Fig. III.

Fig. IV.



PART I

A Description of the Pendulum Clock

THE given figure [Fig. 1, I] views the clock from the side. AA and BB are two plates, half a foot or a little more in length and two and a half inches wide, whose corners are held together by four small columns so that they are one and one-half inches apart. The axes of the main wheels are inserted into these plates on both sides. The first and most basic wheel is marked C. It has 80 teeth, and to its axis is also attached the smaller wheel D studded with iron spikes so that it can hold a cord with attached weights, the purpose of which will be explained later. The wheel C thus turns by the force of a weight and then moves the pinion E, having 8 teeth, together with the wheel F, which is attached to the same axis and has 48 teeth. The latter moves another pinion G and the wheel H, the number of whose teeth is the same as the previous pinion and wheel. H is a type of wheel which our craftsmen call a crown wheel.[1] Its teeth move the pinion I along with the wheel K, which is attached to the same axis erected on the perpendicular. The pinion I has 24 teeth, and the wheel K has 15 teeth, which are cut like the teeth of a saw. Above the center of the wheel K is placed a horizontal axis LM, which has pallets, and whose ends are held in place by the gnomons NQ and P, which are each attached to the plate BB. Note that in the gnomon NQ the part Q extends downwards. Q has an oblong hole through which the axis LM passes, and also, together with the lower gnomon R, holds the previously mentioned axis, which is common to the wheel K and the pinion I. In the plate BB a large hole has been made, beyond which extends the axis LM, with the pallets, which is inserted in the gnomon P with a short point. It is thus more free to move than if it were to be held by the plate BB and also to extend beyond it. For it is necessary for it to be extended in such a way that the small rod S can be attached to it, and that S can turn together with it. This motion is reciprocal, first in one direction, then in the other,

A French translation of Part I by L. Reverchon was published in *L'Horloger, Revue Générale* (December 1922).

1. For the construction of the crown wheel, see *Oeuvres* 17:15.

because the teeth of the wheel K alternately run against the pallets LL, as is commonly recognized and need not therefore be explained in any more detail.

Further, the small rod S, whose lower part is bent and is pierced with an oblong hole, surrounds the iron rod of the pendulum to which the lead body X is attached. This rod is suspended from above by a double cord between two thin plates, of which only T is visible here. We have drawn an adjacent figure [Fig. 1, II] which shows the form and curvature of each plate and the overall plan for suspending the pendulum. Later we will deal in more detail with the true curvature of these thin plates.

Let us now speak about the motion of the clock and postpone discussion of the other parts of the figure until later. It is quite evident that the motion of the pendulum VX, after it has first been started by hand, is sustained by the force of the wheels which are pulled by the weight, and also that the periodic swings of the pendulum lay down a law and a norm for the motion of all the wheels and of the clock as a whole. For the small rod S, which is moved very slightly by the force of the wheels, not only follows the pendulum which moves it, but also helps its motion for a short time during each swing of the pendulum. It thus perpetuates the motion of the pendulum which otherwise would gradually slow down and come to a stop on its own, or more properly because of air resistance. On the other hand, the nature of the pendulum is such that it always moves in the same path and cannot be changed from that path in any way except by changing its length. Furthermore, after we have attained the equality of motion, mentioned above, which is due to the curvature of the thin plates between which the pendulum is suspended, the wheel K is not permitted to move faster or slower, as it often tends to do in ordinary clocks. Rather each tooth of K must move along in an equal time. From this it is clear that the rotations of all the preceding wheels and ultimately of the hands of the clock are made uniform, since all of them are moved proportionally. Hence, even if there is a defect in the construction of the clock or if the axes of its wheels revolve with difficulty because of a change in the condition of the atmosphere, still, as long as this is not so great as to stop the clock completely, there need be no fear that its motion will be uneven or slowed down. It will always measure the correct time, or else it will measure nothing at all.

Next, the hands of the clock are rotated and coordinated as follows. YY is a third plate, parallel to the first two, and separated about one-quarter of an inch from the plate marked AA. On it are inscribed the hour circles around the point α, which is an extension of the axis of the wheel C. The innermost of these circles is divided into 12 hours, while the other one is divided into 60 minutes. The wheel β is attached to the axis of wheel C, outside the plate AA, and has attached to it a tube which extends across the

plate YY up to the point ϵ. The wheel β lies on this axis in such a way that it rotates with it but still can be turned independently of it, as needed. To ϵ is attached a hand which will rotate in a period of one hour and thus will indicate each minute or one-sixtieth part of an hour. The wheel which we have marked as β drives another wheel having the same number of teeth and also an attached pinion having six teeth. The common small axis of these is held in place by the plate A and the gnomon δ. This pinion moves the wheel ζ, which has 72 teeth, and to which is attached a small tube which extends past the plate Y up to the point θ, which is not extended as far as the tube of the wheel β which it surrounds. To the outside edge of θ is attached an hour hand, which is somewhat shorter than the hand used to indicate the minutes since the hour hand should trace the line of the inner circle. To display the seconds without error, let the axis of the wheel H be extended up to the plate Y, and let the disc λ, on which is inscribed a circle divided into sixty parts, be attached to the end of this axis. Then when a hole is cut in the plate Y at Z, these divisions are marked as they pass by a pointer attached to the top of the hole. The full disposition of the hands and the hour circles is more clearly seen in the small figure [Fig. 1, III] which represents the exterior form of the clock.

Furthermore, when the wheels have been arranged as indicated, it is necessary that the length of the pendulum be such that exactly one second is measured by each swing of the pendulum. This length is three feet and cannot be conveniently indicated in the diagram. When I say "three feet," I am not speaking of "feet" as this term is used in various European countries, but rather in the sense of that exact and eternal measure of a foot as taken from the very length of this pendulum. In what follows I will call this an "hour foot," and the measurement of all other feet must be referred to it if we wish to treat matters exactly in what follows. For example, it will not be still unknown in ages to come that the measure of a Paris foot, when compared to an hour foot, has the ratio of 864 to 881. We will deal in more detail with the topic of the exact constitution of this measure in the pages devoted to the center of rotation.[2] For now, we will indicate, in passing, the times of rotation of the individual wheels and hands so that all of this can be correctly and accurately related to the number of teeth indicated above.

For each revolution of the wheel C, the wheel F rotates ten times, the wheel H 60 times, and the highest wheel K 120 times. Since K has 15 teeth which alternately strike the pallets LL, in one revolution of the wheel K there will be 30 such hits, to which correspond the same number of simple oscillations of the pendulum VX. Thus for 120 rotations of K, there will occur 3600 simple oscillations, which is the number of seconds in one hour.

2. Part IV, Proposition 25.

Hence the wheel C rotates once in one hour, and together with it the hand attached to ϵ, which indicates the minutes. In the same space of time the wheel β, and through it γ, makes one revolution. Since the pinion attached to the latter has six teeth, and since the number of teeth in the wheel ζ is twelve times this, it is obvious that the latter makes one revolution in twelve hours and that the hand attached to it at θ rotates in the same time. Finally, since we have shown that the wheel H makes 60 revolutions for each revolution of the wheel C, then H, together with the disc λ attached to it, will rotate 60 times in one hour, that is, once per minute. Hence the 60 parts of the disc λ will indicate the seconds as they pass by. Thus it is clear that everything is properly arranged. The weight X at the bottom of the pendulum weighs three pounds, and either is made entirely of lead or contains lead under a brass surface. It must be noted (for this is very important) that the pendulum experiences the least resistance from the air not only because of the weight of the metal but also, and more especially, because of its shape. The weight is constructed in the form of an oblong cylinder which comes to a point at both ends, as is shown at *a* in the small drawing of the clock [Fig. 1, III]. For clocks constructed for navigational use, the shape of a vertical lentil seems more appropriate.

Next the same diagram shows the method of suspending the other weight *b* by which the motion of the clock is maintained. Although ignored earlier,[3] we must now explain this so that, when the weight is pulled back up, the running of the clock is not stopped or occasionally interrupted, which must be completely avoided. First let there be constructed a cord which is continuous and which returns back on itself because its ends are exactly joined together. This cord passes over the small disc which is attached to the lowest wheel and which is marked D in the larger diagram. Descending from here, it changes direction by passing under the pulley *c*, to which the weight *b* is attached. From here it ascends up over the small disc *d*, which is attached to the exterior of the clock, which has iron spikes on its circumference, and which is so constructed with serrated teeth that it revolves when the cord *e* is pulled but cannot revolve in the opposite direction. From this small disc the cord descends to the other pulley *f*, to which is attached the small weight *g* which is large enough to sustain the larger weight *b* and prevent it from descending by rotating the disc *d*. From the pulley *f* the cord returns again to that small disc *d* from which it descended. With this arrangement it is clear that the weight *b* always works with half of its weight to rotate the wheels of the clock, and its work does not cease when it rises when the cord *e* has been pulled by hand. Therefore, the motion of the clock is never interrupted and not a moment of time is lost.

3. Huygens first presented this idea in the *Horologium* of 1658. See *Oeuvres* 17:65.

The size of the weight *b* cannot be exactly defined; but insofar as a smaller weight is sufficient to maintain the motion, then the clock must be of better quality and more accurately constructed. In the best clocks which we have to date, this weight has been reduced to six pounds, given that the diameter of the small disc D is about one inch, as the figure shows. Also the weight of the pendulum is three pounds, and its length is three feet. The length of the pendulum, to warn of another point, hangs down through an oblong hole in the casing of the clock which must be as large as needed for the oscillations to be completed. This clock, hung at the height of a man, continued to move for 30 hours.

We must now describe the form of the thin plates between which the pendulum is suspended, and whose most important function is to guarantee the regular motion of the clock. For without them the simple oscillations of the pendulum will not be equal over a long period of time (even though some have thought otherwise),[4] but rather oscillations through smaller arcs will occur in a shorter time. This is easily seen from the following experiment. Take two strings of the same length, attach equal weights at the bottom, suspend each separately, pull one of them back a great distance from the perpendicular and the other only a short distance, and then release them from the hand simultaneously. It will be observed that they do not move in the same direction for a long time, but rather that the pendulum whose oscillations are faster surpasses the other one. Furthermore, the ratios of the times corresponding to any arcs can be defined by numbers based on certain calculations and can be as exact as one could wish. For example, the time of descent through a full quadrant of a circle is related to the time of motion through the smallest possible arc as 34 to 29. This difference[5] should not at all be attributed to air resistance, as some have thought,[6] but is due to the very nature of motion and to the properties of a circle. This can also be concluded from another agrument based on the construction of the isochronous pendulum where the motion diverges considerably from a circular line, as will soon be shown.

It would perhaps seem that the inequality which we have mentioned would be of no significance, and thus no correction of the pendulum is needed, in the case of our clocks since the amplitude of their oscillations is

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4. Galileo, *Discourses Concerning Two New Sciences*. First Day. (*Le Opere di Galileo Galilei* 8:128–29.) "Repeating their goings and comings a good hundred times by themselves, they [two pendula of equal length, one of lead, the other of cork] sensibly showed that the heavy one kept time with the light one so well that not in a hundred oscillations, nor in a thousand, does it get ahead in time even by a moment, but the two travel with equal pace." [S. Drake's translation, 87.]

5. 29:34::1:1.172 . . . Later Huygens discovered that the true value here is 1.180 . . . See *Oeuvres* 18:384–85.

6. This was maintained by Marin Mersenne in the General Preface of his *Cogitata physico-mathematica* (1644).

always the same. This indeed would be the case if the amplitude of all the oscillations were to remain constant and exactly the same. But it is sometimes a little bit larger or smaller, and a sufficiently large difference results from many such small differences. That this is so is clear both from the nature of things and from experiments. For even if the force of the weight is always the same in relation to the wheel closest to it, still not always the same force is brought to bear on the pendulum after it has passed through all the wheels, no matter how much special care is taken. Further, the motion of the wheels is made more difficult by coldness and by the oil, which is applied to them, becoming dirty or disappearing. The oscillations of clocks carried at sea are especially unequal because of the continual heaving of the ship. Hence corrections must be made in all these clocks, in general, and especially in marine clocks, so that the times of wider or narrower oscillations of the pendulum can turn out to be equal.

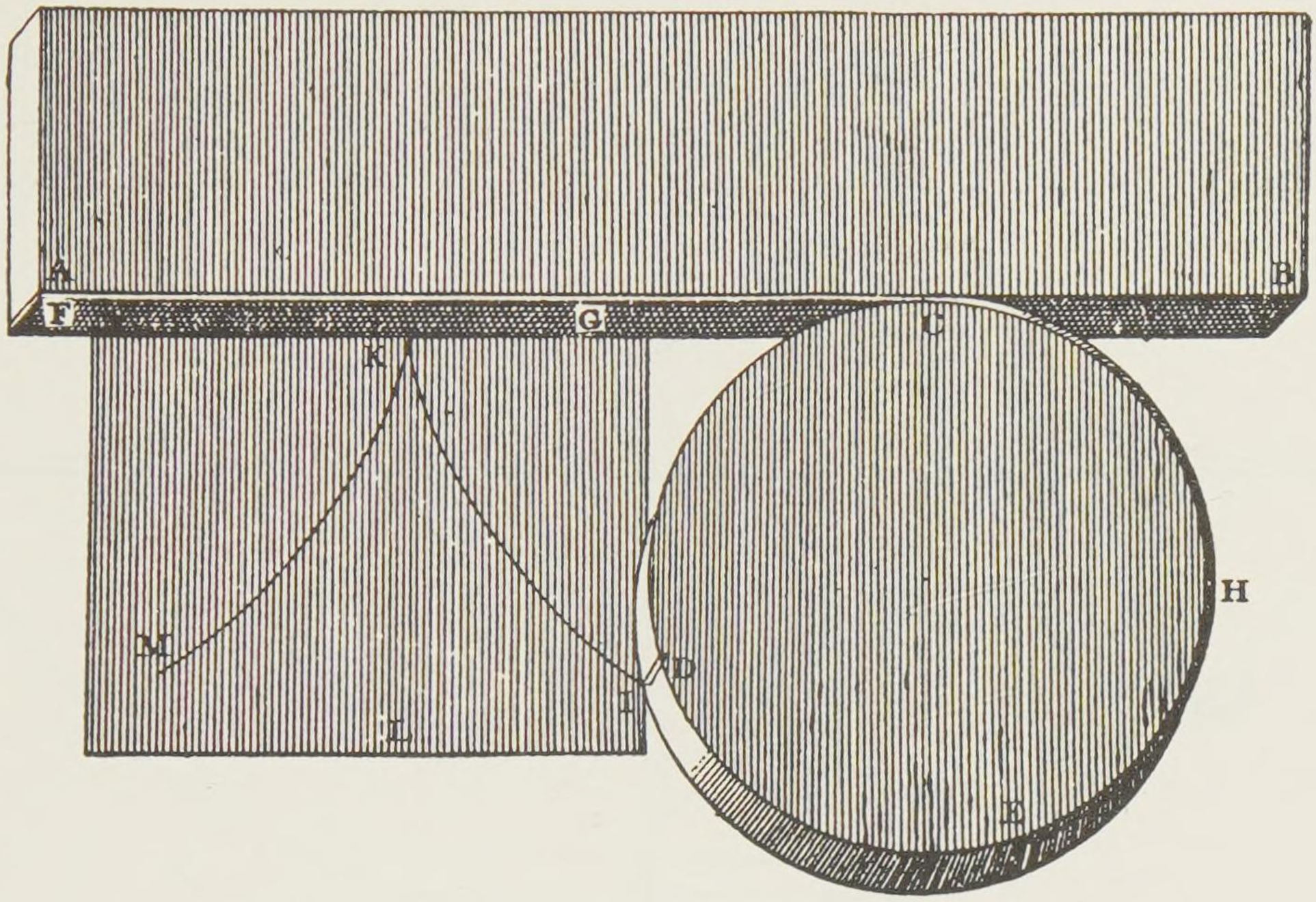
To define the form of the thin plates in which this correction is located, it is necessary first to establish the length of the pendulum, which is easily derived from the fact that the lengths of pendula are related to each other as the squares of the times spent in their individual oscillations. Thus since we have defined the length of the pendulum which measures seconds to be three feet, then a quarter of it, or nine inches, corresponds to a pendulum which would mark off half-seconds. Again, if we seek the length of the pendulum which would make 10000 simple oscillations in one hour, this will be calculated as follows. We know that a three-foot pendulum makes 3600 oscillations in one hour. Hence the period of this pendulum is greater than the period of the pendulum sought by a ratio of 10000 to 3600, or 25 to 9. Now as the square of 25 is related to the square of 9 as 625 to 81, so the length of the 3 feet will be related to the length sought, which is thus $4\frac{66}{100}$ inches.

Given the length of the pendulum, which is three feet in the clocks we propose, the line of the cycloid, which determines the curvature of the thin plates T, is described as follows.

On a flat table let there be attached a rod AB which is one-half inch thick [Fig. 2]. Next let CDE be a cylinder having the same height and having a base whose diameter equals one-half the length of the pendulum. Let FGHE be a band, or rather a thin metal plate, attached to the rod at F and to the cylinder at some point E on its circumference so that it is partly wrapped around the cylinder and is partly extended along the side of the rod AB. Also let DI be an iron point attached to the cylinder and extending a little way below its lower base in such a way that it corresponds exactly to its circumference.

Given this arrangement, let the cylinder roll along the rod AB, the two being separated only by the very small thickness FG and always remaining the same distance apart. Then the curved line KI, which is called a cycloid,

[Fig. 2]

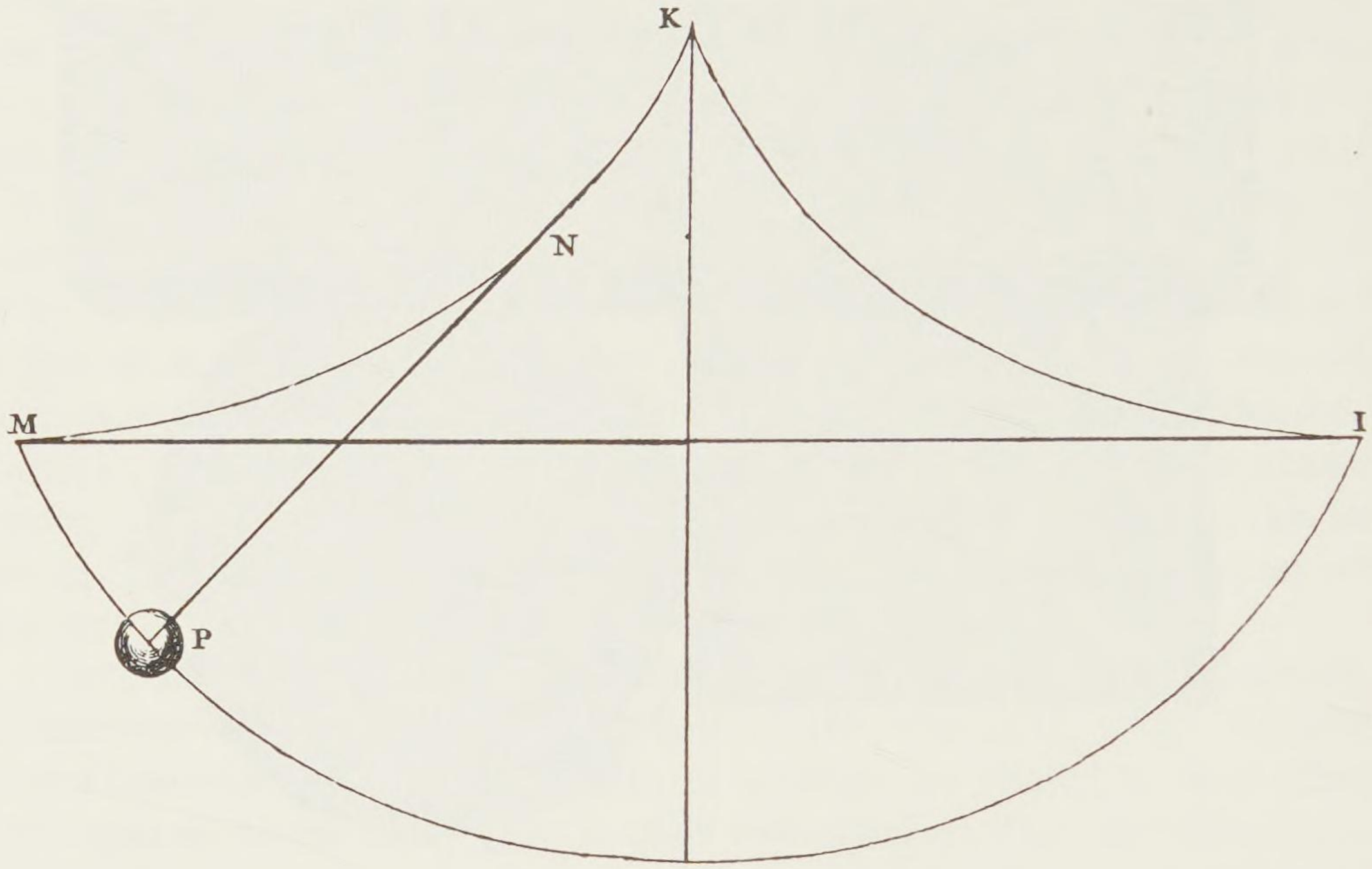


will be described by the point I on the perpendicular plane below the table. We have already attached the thin plate KL to the rod AB. Hence, after the part KI of the cycloid has been traced on that plate, the line is reversed and there is drawn on the opposite side a similar line KM which starts from the same point K. Thus we have accurately formed the figure MKI according to these lines. Now the interval between the thin plates between which the pendulum is suspended must conform to this figure. Only a small part of the arcs KM and KI is needed for clocks, for the remaining parts of the curves are useless since the cord of the pendulum cannot touch them.

So that the nature and effects of this marvelous line can be understood more fully, let the pure semicycloids KM and KI be drawn in another figure [Fig. 3]. Between them is suspended and put in motion the pendulum KNP whose length is twice the diameter of the generating circle. Its oscillations occur in equal periods and have the maximum amplitude through the whole arc MPI. Thus the center P of the attached sphere always rotates on the line MPI, which is itself a pure cycloid. I do not know if any other line has this remarkable property; namely, it describes itself by its own evolution.[7] But each thing which we have said about this will be demonstrated in what follows where we deal with the falling of heavy bodies and with the evolution of curves.

7. In 1678 Huygens determined that the evolution of any epicycloid describes a similar epicycloid. See *Oeuvres* 18:399–405. In May 1692, in *Acta eruditorum,* Jacob Bernoulli proved that the logarithmic spiral has this same property.

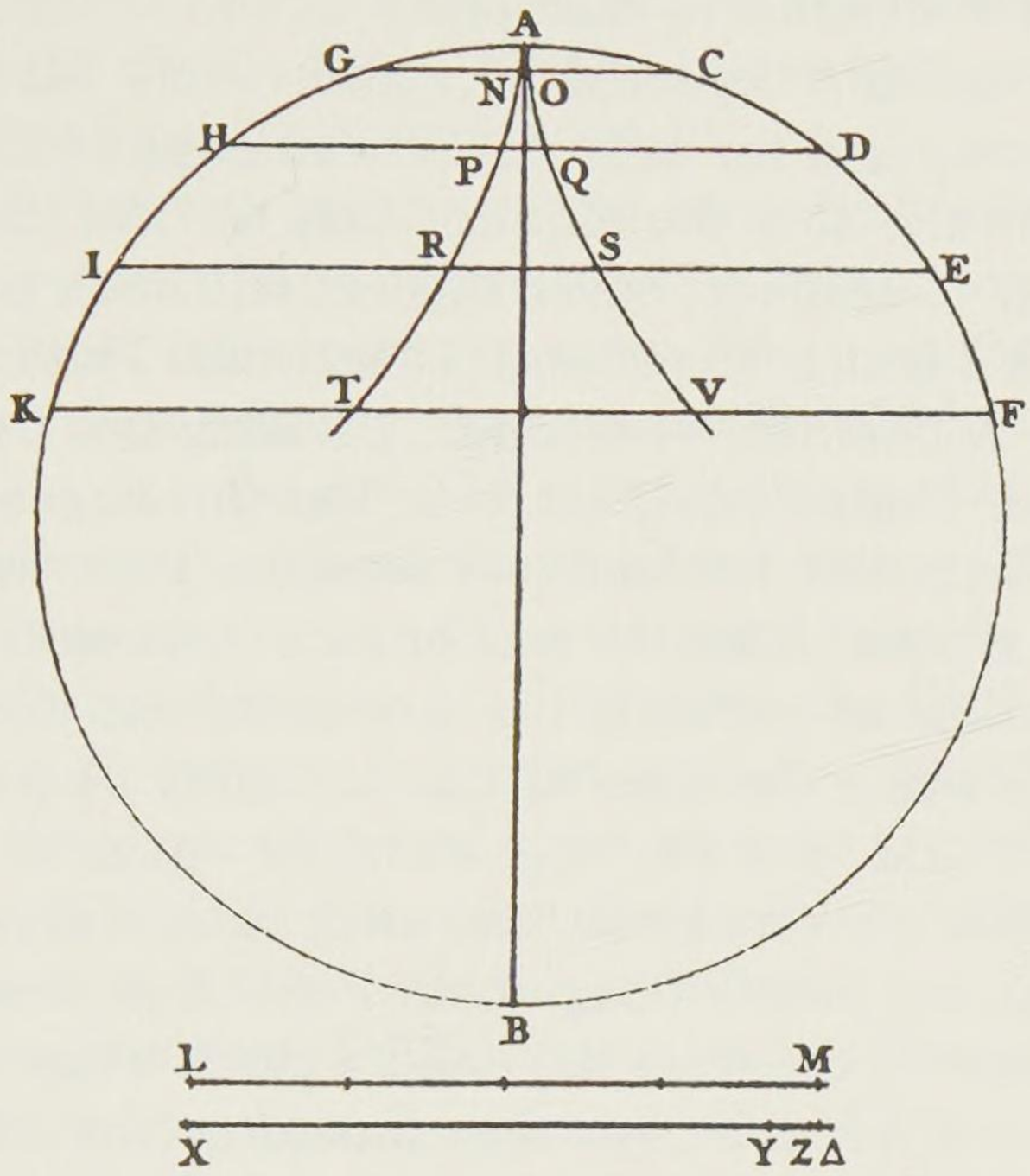
[Fig. 3]



It must be noted that there is another way to describe a cycloid, that is, by drawing a line through designated points. Draw a circle whose diameter AB is equal to half of the length of the pendulum [Fig. 4]. On its circumference mark off any equal arcs AC, CD, DE, EF, AG, GH, HI, IK, and let them be joined by the lines GC, HD, IE, and KF, which are parallel to each other. Next, let the straight line LM be equal to the arc AF, and let it be divided into as many equal parts as are in the arc AF. Then on the straight line CG place the individual lines CN and GO, each equal to one part of LM; on the straight line DH place the individual lines DP and HQ, each equal to two parts of LM; on the straight line EI place the individual lines ER and IS, each equal to three parts of LM; and so forth until all the parts are used. Hence on the last line FK the individual lines FT and KV will be equal to the whole line LM. Now if curves are drawn through the points AOQSV and ANPRT, these curves will be parts of the sought-for cycloid, between which the pendulum must be suspended.

The straight line LM is found to be equal to the arc AF as follows. First let XZ be equal to the sum of the two straight lines which are the cords of half of the arc AF. From the same starting point, let XY equal the cord of the whole arc AF. Then add ZΔ, which is one-third of the difference YZ, to the whole line XZ. Now the whole line XΔ will be very nearly equal to the whole arc AF. For if the arc AF is one-sixth of the circumference (and for our purposes no more than this is ever needed), then the difference in

[Fig. 4]



lengths will be no more than one part in six thousand, as has been demonstrated in what we have written before about the size of circles.[8]

Having explained what relates to the construction of the clock, let us now indicate how it should be adjusted so that it gives a true measurement of the hours. First we will examine as follows whether its motion is accurate.

Select a fixed place for the eye of the observer from which it is possible to see both the stars and the roofs or walls of neighboring houses. Let the latter objects be so located that when one of the fixed stars becomes aligned with one of them, the star instantly ceases to be visible. Next, at the selected place make a hole the same size as the pupil of the eye in such a way that on subsequent days the eye can be placed at the same place without any error. Now note the time indicated by the clock at the instant when some fixed star passes out of view. Do the same thing on the next day, or better after several days have passed. If only one day has passed between the two observations, then the time of the clock at the second observation should fall short of the time noted at the first observation by 3 minutes, 56 seconds. If so, the length of the pendulum has been accurately established, for each revolution of the fixed stars surpasses the mean solar day by that

8. Prob. III, Prop. XII of *De circuli magnitudine inventa* (1654). See *Oeuvres* 12:146–49.

amount. I say the "mean solar day" because the solar days, from meridian to meridian, are not all equal to each other, as will be discussed more fully later. If the observation is repeated only after many days, this much difference must be computed for each day. For example, in the first observation the clock indicates that the moment that the star disappeared was 9 hours, 30 minutes, 18 seconds. Seven days later it indicates that the same star disappeared at 8 hours, 50 minutes, 24 seconds. This time falls short of the prior time by 39 minutes, 54 seconds. Dividing this by seven, we get a daily retardation of 5 minutes, 42 seconds. But this ought to be 3 minutes, 56 seconds; a difference of 1 minute, 46 seconds. Thus each day the clock falls short by this amount from the true or mean measurement of the days.

It is also possible to compare the motion of the clock to the sun in another way. In this case the ratio of the inequality of the natural days is involved. For as I said, not all these days are equal to each other, and although the difference is very small, still after many days it often increases to a point where it can hardly be ignored. For if one were to have both a perfect measurement of the solar days and a clock whose motion gives an exact and true measure of the days, they should not be different. However it necessarily happens that at certain times of the year there is a discrepancy here often of a quarter hour and sometimes even of a half hour, while at other times they both agree on the same time. This can be understood from the table of the equalization of times which we have added below and after we have explained how to use it, as follows.

The equalization in the table is established by assigning to each day what we have found is needed to make our clock agree with the sun or with a sundial. On the other hand the equalization of the day is used to discover how well a clock is adjusted for measuring the days. If a prior equalization is larger than a later one, the time of the clock surpasses the time of a sundial by an amount equal to the difference of these equalizations. But if the equalization of a later day is found to be larger, then this excess will be due to the time of a sundial or the time as observed by the sun. For example, assume that on 5 March the clock and the sundial agree on the same time. The equalization found in the table for that day is 3 minutes, 11 seconds. Assume that we wish to know whether or not the clock accurately measures equal hours on 20 March. The equalization found in the table for that day is 7 minutes, 27 seconds. Since this is larger than the prior equalization by 4 minutes, 16 seconds, the time of the sundial will be that much later than the time indicated by the clock. Hence, if a difference is found, it is easy to decide how much the clock runs faster or slower each day.

In computing this table we have taken account of two factors, both known to astronomers; namely, the obliquity of the ecliptic and the anomaly of the motion of the sun. Reason postulates this equalization, and experience also establishes it by the use of these very clocks without which it

could not be found. For our observations of the sun, which we have repeatedly carried on over many months, noting each day the moment when the sun occupies the circle of the meridian, have been found to agree perfectly with the equalizations here proposed.[9]

This investigation was carried out by using both of the methods mentioned above, but with the emphasis on the former one. If the clock is found to vary a great deal from the mean length of the days, that is, by more than three or four minutes, a correction should be made by lengthening or shortening the pendulum. The rule for this is that the motion of the clock will be accelerated or retarded one minute per day for each 7/10 of an inch subtracted from or added to the pendulum. Since in this way the clock will be very nearly reduced to a true measure, the final correction is best made by adjusting the very small weight Δ which is attached to the rod VV. This weight has the shape of a lentil, whose cross section along the axis is drawn in Figure 1 [Fig. 1, I]. Since this equals only a twentieth or a thirtieth or less [10] of the weight X, then when it is moved a long way from its prior location, it makes a very small difference in the motion of the pendulum. It accelerates the pendulum when it is moved toward the middle of the length of the rod, and it retards it when it is moved upwards or downwards from that point. To save time in finding the place where the truest measure of the day will be given, we have divided the lower half of the rod by using a specific proof based on the laws of motion, assuming that the weight Δ is one-fiftieth of the weight X and that it is equal to the weight of the rod VV. These divisions are shown in Figure 4 [Fig. 1, IV] where the lower portion of the pendulum is shown to be divided into three parts, the lowest of which is AB. Point A is the center of gravity of the weight X, and from the point C, which is the center of oscillation,[11] each part makes a difference of fifteen seconds per day when the lentil Δ has been moved through that interval. The demonstration and discovery of these divisions will be given in the pages dealing with the center of oscillation.

We will now describe the structure of other clocks which are used at sea to determine longitudes, for the clocks which we have examined and determined in the previous pages are very well suited for adaptation to this use. I know that this matter has already been pursued to the point that it appears that little remains to be done to modify the invention for such usage. Still no apology will be needed if we explain what and how this was done, and what remains to be done next.

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9. For Huygens' observations, see *Oeuvres* 17:101–2. For an assessment of the accuracy of his "Table of the Equalization of the Days," see *Oeuvres* 18:50–52.

10. The words "or less" were added in Latin in the "Corrigenda" of the 1673 edition.

11. The words "which is the center of oscillation" were added in Latin in the "Corrigenda" of the 1673 edition.

Table of the Equalization of the Days

(*p.* 15).

Dies.	*Januar.* Min.	Sec.	*Febr.* Min.	Sec.	*Mart.* Min.	Sec.	*Apr.* Min.	Sec.	*Maj.* Min.	Sec.	*Jun.* Min.	Sec.
1	10	40	0	32	2	15	11	18	18	32	18	10
2	10	10	0	24	2	28	11	37	18	39	18	1
3	9	41	0	18	2	42	11	56	18	46	17	51
4	9	13	0	13	2	56	12	15	18	53	17	41
5	8	45	0	9	3	11	12	34	18	59	17	30
6	8	17	0	6	3	26	12	53	19	4	17	19
7	7	50	0	3	3	41	13	12	19	9	17	8
8	7	22	0	1	3	56	13	31	19	14	16	57
9	6	58	0	0	4	12	13	49	19	18	16	46
10	6	34	0	0	4	29	14	6	19	22	16	35
11	6	10	0	0	4	46	14	23	19	25	16	24
12	5	47	0	2	5	4	14	39	19	28	16	13
13	5	24	0	4	5	22	14	55	19	29	16	1
14	5	2	0	8	5	40	15	10	19	29	15	49
15	4	41	0	12	5	58	15	25	19	29	15	37
16	4	21	0	16	6	16	15	39	19	28	15	24
17	4	2	0	21	6	33	15	53	19	26	15	11
18	3	44	0	26	6	51	16	7	19	24	14	58
19	3	27	0	32	7	9	16	21	19	21	14	45
20	3	11	0	40	7	27	16	34	19	18	14	32
21	2	55	0	48	7	45	16	47	19	15	14	9
22	2	39	0	57	8	3	16	59	19	11	14	6
23	2	23	1	6	8	22	17	11	19	7	13	53
24	2	7	1	16	8	41	17	22	19	2	13	40
25	1	52	1	26	9	1	17	33	18	57	13	27
26	1	38	1	37	9	21	17	43	18	51	13	15
27	1	25	1	49	9	41	17	53	18	45	13	3
28	1	13	2	2	10	1	18	3	18	39	12	52
29	1	2			10	21	18	13	18	33	12	41
30	0	51			10	40	18	23	18	26	12	30
31	0	41			10	59			18	18		

Table of the Equalization of the Days

Dies.	*Jul.* Min.	Sec.	*Aug.* Min.	Sec.	*Sept.* Min.	Sec.	*Octob.* Min.	Sec.	*Nov.* Min.	Sec.	*Dec.* Min.	Sec.
1	12	19	10	4	16	23	26	30	31	55	25	34
2	12	8	10	8	16	42	26	49	31	55	25	10
3	11	58	10	13	17	1	27	8	31	54	24	45
4	11	48	10	18	17	21	27	26	31	52	24	20
5	11	38	10	23	17	41	27	43	31	50	23	55
6	11	28	10	28	18	1	28	0	31	47,	53	30
7	11	18	10	34	18	21	28	16	31	43	23	4
8	11	9	10	41	18	41	28	32	31	37	22	38
9	11	0	10	49	19	1	28	47	31	30	22	11
10	10	52	10	58	19	21	29	2	31	22	21	43
11	10	47	11	7	19	41	29	16	31	13	21	14
12	10	38	11	16	20	1	29	30	31	3	20	44
13	10	31	11	25	20	22	29	43	30	53	20	14
14	10	25	11	36	20	43	29	56	30	43	19	44
15	10	19	11	48	21	4	30	9	30	32	19	14
16	10	13	12	1	21	25	30	22	30	20	18	44
17	10	7	12	14	21	47	30	34	30	8	18	14
18	10	2	12	28	22	9	30	45	29	55	17	44
19	9	58	12	42	22	31	30	55	29	40	17	14
20	9	54	12	57	22	52	31	4	29	23	16	44
21	9	51	13	12	23	13	31	12	29	6	16	14
22	9	49	13	27	23	33	31	19	28	48	15	44
23	9	47	13	43	23	53	31	26	28	30	15	14
24	9	46	13	59	24	13	31	32	28	11	14	43
25	9	46	14	16	24	33	31	38	27	51	14	12
26	9	46	14	33	24	53	31	43	27	30	13	41
27	9	47	14	50	25	13	31	47	27	8	13	10
28	9	49	15	8	25	33	31	50	26	45	12	40
29	9	52	15	26	25	52	31	53	26	22	12	10
30	9	56	15	45	26	11	31	55	25	58	11	40
31	10	0	16	4			31	55			11	10

15

The first two such clocks were carried in a British ship in 1664. A Scottish gentleman[12] and friend of ours had them made in accordance with our plans. In place of the weight the clocks had a thin steel plate rolled up into a spring, the force of which rotated the wheels. This was the same type of spring customarily used in small clocks which are portable. So that the clocks could endure the heavings of the ship, he suspended them from steel balls encased in bronze cylinders. He also doubled the weight which continues the motion of the pendulum (which was one-half a foot long), and extended it downward such that it had the shape of an inverted letter F.[13] He did this so that a motion which might otherwise cause the pendulum to stop would be converted into a spinning of the pendulum. After this ship returned to England with three other ships which accompanied it on the voyage, the Commander of the fleet made the following report.[14] After leaving the coast of Guinea, he proceeded to the island called São Tomé, which lies under the circle of the equinox. There he adjusted his clocks in relation to the sun. He then followed a westward course and sailed without interruption for about seven hundred miles. Then he turned and sailed toward the coast of Africa with the help of a south-southwest wind. After holding this course for two or three hundred miles, he was advised by the captains of the other ships, who feared that they would run out of drinkable water before reaching Africa, to change course toward the American islands called Barbados to obtain water. He then conferred with the captains and ordered them to produce their charts and individual computations, from which he found that their calculations differed from his, one by 80 miles, another by 100, and the third by still more. From the evidence of his clocks the Commander concluded that they were no more than about thirty miles from the island of Fuego, which is one of a group of islands not far from Africa which are called the Cape Verdi Islands, and that they could reach it on the next day. Having thus consulted his pendula, he ordered that this course be followed, and at noon the next day that island came into view, and a few hours later provided a stopping place for the ships. This was reported by the Commander himself.

After this these experiments were repeated several times by the Dutch and by the French under the order of the Most Serene King. The results varied, but in such a way that frequently the fault could be attributed to the carelessness of those entrusted with the clocks rather than to the clocks themselves. The best success was attained in the Mediterranean Sea on an

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12. Alexander Bruce, 2nd Earl of Kincardine (c. 1629–1680), who was one of the original Fellows of the Royal Society. For an account of this voyage, and other similar voyages, to test the use of marine clocks in the determination of longitudes, see *Oeuvres* 17:193–98, and Derek Howse, *Greenwich Time and the Discovery of the Longitude* (Oxford: Oxford Univ. Press, 1980).

13. For the construction of this mechanism, see *Oeuvres* 17:166, Fig. 67.

14. The Commander was Capt. Robert Holmes. For his report see *Oeuvres* 17:230–34.

expedition to the island of Crete by the illustrious Duc de Beaufort,[15] who was sent with French troops to bring help to the city of Candia,[16] which was beseiged by the Turks, and where he fell in battle. In his ship he carried clocks designed for such experiments, and he appointed an experienced astronomer[17] to tend to them. From the latter's daily observations we have found that the longitudes of the places where the ships stopped on the voyage, or which could be visually identified in passing, were exactly measured with the help of the clocks. These measurements were found to be as exact as those given in geographical descriptions, which are better known designations of these differences of longitude. For example, between the port of Toulon and the city of Candia a difference of one hour, 22 minutes was found, that is, a difference of longitude of 20°30′ and almost exactly the same difference was found on the return voyage. This agreement is a most certain indication of truth.

Between the same port of Toulon and the island of Maretimo, which lies to the west of the promentory of Sicily formerly called Lilybaeum, the observed temporal difference was 25 minutes, 20 seconds, which corresponds to a 6°20′ difference of longitude. Again from Toulon to the island of Sapienza, which lies west of the Peloponnesus, the difference was 1 hour, 5 minutes, 45 seconds, to which corresponds a 16° 26′ difference of longitude.

The clocks were tested at dawn in relation to the rising sun and at dusk in relation to the setting sun, each moment of time being calculated from the given altitude of the pole. And this method seemed to work best when the ships were at anchor because then these observations were carried out by the eyes alone without the aid of instruments.

In these clocks the length of the pendulum was nine inches and its weight one-half pound. The wheels were rotated by the force of weights and were enclosed together with the weights in a case which was four feet long. At the bottom of the case was added a lead weight of over one hundred pounds so that the instrument would better maintain a perpendicular orientation when suspended in the ship.

Although the motion of the clock was found to be very equal and constant in these experiments, nevertheless we have made an effort to perfect it still further in another way as follows. From the wheel which has serrated teeth and which is next to the pendulum, we suspended a very small weight from a skillfully constructed chain. This weight moves only

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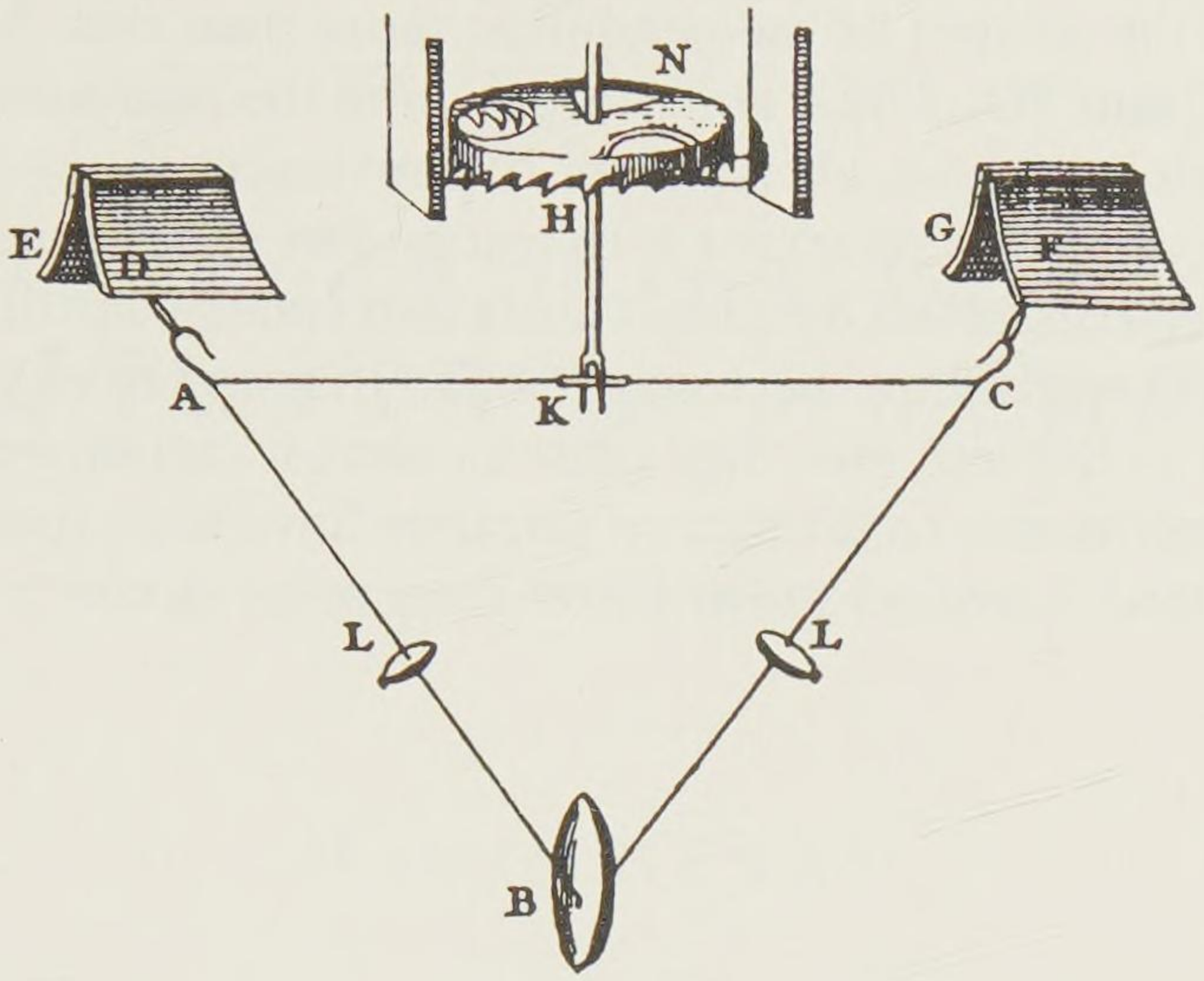
15. François de Bourbon-Vendome, Duc de Beaufort (1616–1669), grandson of Henry IV.
16. Modern name: Iráklion.
17. The "experienced astronomer" apparently was named de la Voye (? –1684), who was admitted to the Académie des Sciences in 1666. Little else is known about him. For an account of this voyage, see *Oeuvres* 6:501–3, and 18:633–35.

that wheel, and the rest of the machine has no other effect than to restore that small lead weight to its prior height each half minute. The method of doing this is almost the same as was explained above in the construction of the clock where a weight is raised with one cord while by another cord it continues to exert its weight on the motion of the clock. With such a construction; namely, that all the force is brought to bear on only that one wheel, the result is a still greater equality of clocks than before. It is quite worth noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motions of each pendulum in opposite swings were so much in agreement that they never receded the least bit from each other and the sound of each was always heard simultaneously. Further, if this agreement was disturbed by some interference, it reestablished itself in a short time. For a long time I was amazed at this unexpected result, but after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible. The cause is that the oscillations of the pendula, in proportion to their weight, communicate some motion to the clocks. This motion, impressed onto the beam, necessarily has the effect of making the pendula come to a state of exactly contrary swings if it happened that they moved otherwise at first, and from this finally the motion of the beam completely ceases. But this cause is not sufficiently powerful unless the opposite motions of the clocks are exactly equal and uniform.

Furthermore, in experiments done on ocean voyages, especially when a storm agitated the seas violently, it was found first that special care must be taken to maintain the motion of the clocks without interruption. It is clear that this is very difficult to do because of the heaving of the ship. For this reason we have, lastly, changed the form of the pendulum according to a new plan and have suspended the clocks themselves differently. This pendulum has a triangular form in whose lower vertex is attached a lentil made of lead. The other two angles made by the cords are suspended between cycloidal plates. At its midpoint the base is held by a split rod by which it is moved, and the rod in turn receives its motion from the the horizontal serrated wheel. The motion of all the wheels does not originate from a weight but from a steel plate rolled into a spring. In the figure [Fig. 5] ABC is the triangular pendulum; B the lead lentil; ED and FG the cycloidal plates; HK the split rod; and N the wheel with serrated teeth, which is below the other wheels of the clock. LL are small lentils used to regulate the motion of the pendulum.

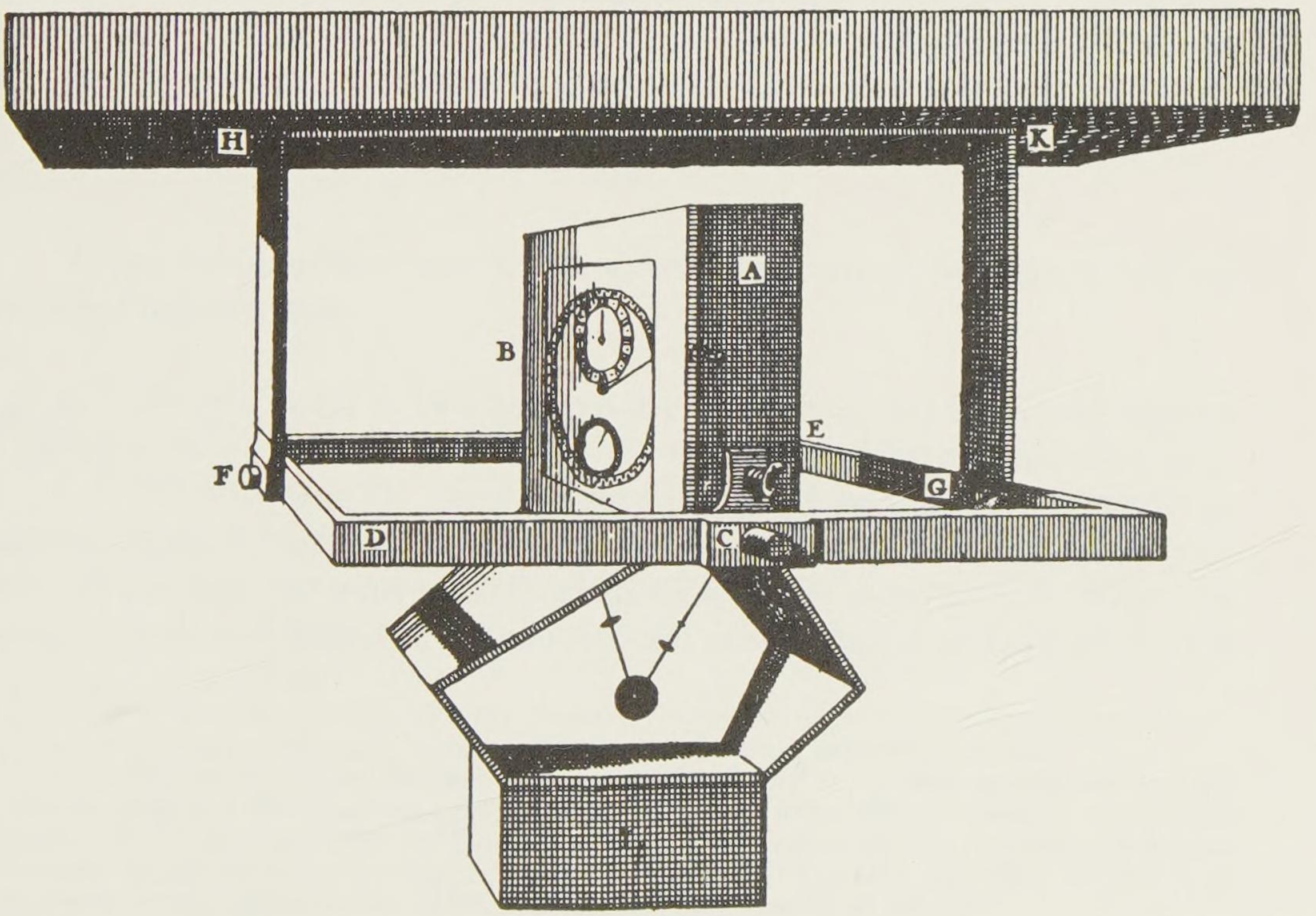
The other figure [Fig. 6] shows the method of suspension. The box AB is inserted into the iron rectangle DE by two axes, of which only C is shown. Next this rectangle is held in turn by its axes FG in the iron gnomon FHKG which is attached immovably to the ceiling joists of the ship. To the bottom of the box a weight of 50 pounds is suspended. With this arrange-

[Fig. 5]



ment the clock maintains a perpendicular position no matter what the inclination of the ship is. Also the axis C, together with the one opposite it, are so located that they correspond in a straight line to the points of suspension of the pendulum just described. From this it follows that the oscilla-

[Fig. 6]



tory motions of the pendulum cannot disturb the instrument, and nothing would destroy the motion of the pendulum more than this. Next the width of the axes CC and FG, which are one inch, and the lead weight suspended below, completely remove all freedom of movement from the clock and have the effect of always returning it to rest and to a perpendicular orientation if it had been disturbed by some strong and sudden shifting of the ship.

Instruments which have been modified in this way have yet to be taken to sea and put to the test. But they offer an almost certain hope of success because, in experiments conducted so far, they have been found to hold up under every kind of motion much better than prior clocks.

PART II

The Falling of Heavy Bodies and Their Motion in a Cycloid

HYPOTHESES

I

If there were no gravity, and if the air did not impede the motion of bodies, then any body will continue its given motion with uniform velocity in a straight line.

II

By the action of gravity, whatever its sources,[1] *it happens that bodies are moved by a motion composed both of a uniform motion in one direction or another and of a motion downward due to gravity.*

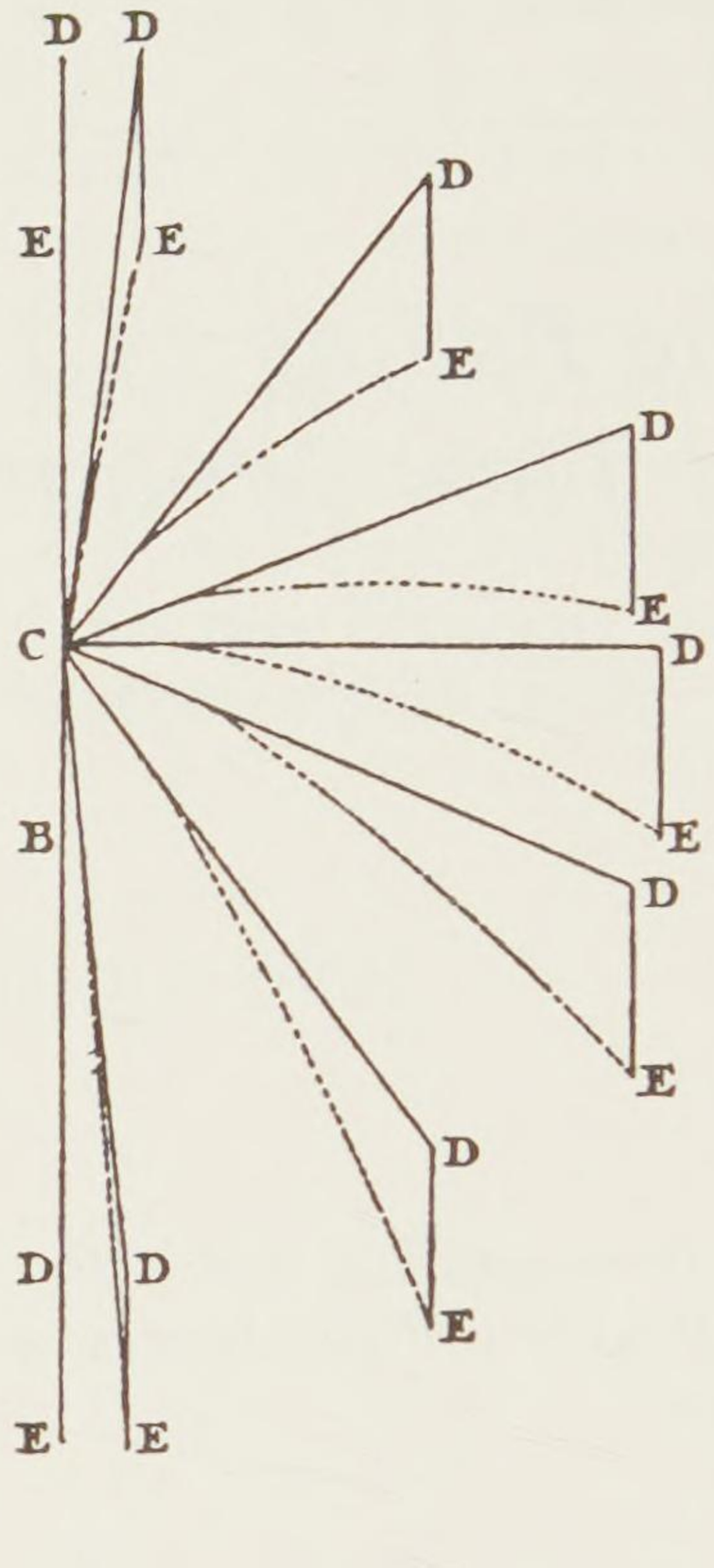
III

These two motions can be considered separately, with neither being impeded by the other.

LET C be a heavy body which, starting from rest, crosses the distance CB by the force of gravity in a certain time F [Fig. 7]. And let the same body be imagined to undergo another motion by which, assuming that gravity does not exist, it crosses the straight line CD with a uniform motion in the same time F. When the force of gravity is added, the body will not move from C to D in the time F

For various earlier writings by Huygens on topics in Part II, see *Oeuvres* 17:125–41.

1. In the *Horologium oscillatorium* Huygens restricts himself to the phenomenological level and avoids any discussion of the cause of gravity. But at that time he was a strong advocate of the Cartesian account of gravity as caused by the vortex motion of subtle matter, a view which he presented and defended against strong opposition at a meeting of the Académie des Sciences in 1668, and which is found in his *Discours de la cause de la pesanteur* (1668; published in 1690). (*Oeuvres* 21:445–88.)



[Fig. 7]

but rather to some point E vertically below D such that the distance DE equals the distance CB. And thus the uniform motion and the motion due to gravity each make a contribution, and neither impedes the other. In what follows later we will define the line in which the body moves with this composite motion when the uniform motion is neither straight up or down but in an oblique direction. But when the uniform motion CD occurs downward on the perpendicular, it is obvious that the line CD is increased by the straight line DE when the motion due to gravity is added. Likewise, when the uniform motion CD is directed upward, CD is decreased by the straight line DE, so that, for example, after the time F the body will always be found at the point E. Thus, if we consider the two motions separately in each case, as we said, and if we recognize that neither motion is in any way impeded by the other, then from this we can discover the cause and the laws of acceleration of heavy falling bodies. And first we will show the following two things.

PROPOSITION I

In equal times equal amounts of velocity are added to a falling body, and in equal times the distances crossed by a body falling from rest are successively increased by an equal amount.

. 8] Let there be a heavy body at rest at A [Fig. 8]. In the first unit of time it falls through the distance AB; and when it has arrived at B, it has acquired a velocity by which it next could cross the distance BD with a uniform velocity in the second unit of time. But we know that in the second unit of time it will cross a distance greater than BD because it would travel the distance BD only if all the action of gravity had ceased at B. Actually it moves with a motion composed of the uniform motion by which it would have crossed the distance BD and of a motion characteristic of falling bodies by which it necessarily falls through a distance equal to AB. Hence by adding DE, which is equal to AB, to BD, we know that in the second unit of time the body will arrive at E.

A
B
D
E
F
G
H
K

But if we inquire what velocity the body has at E at the end of the second unit of time, we find that this ought to be double the velocity which it had at B at the end of the first unit of time. For we said that it is moved by a motion composed of a uniform motion equal to the velocity acquired at B and of a motion due to gravity, which clearly is the same in the second unit of time as in the first. Hence a velocity ought to be added to the falling body in the second unit of time which is equal to the velocity added in the first unit of time. Thus, since it conserves the whole velocity acquired at the end of the first unit of time, it is clear that at the end of the second unit of time it has twice, or double, the velocity which it acquired at the end of the first unit of time.

Now if, after having arrived at E, the body were to be moved with a uniform velocity equal to what it has acquired at E, it is clear that in a third unit of time equal to each of the first two it would cross the distance EF, which is double the distance BD. For we said that the latter was crossed with half this velocity by a uniform motion in an equal time. But by adding again the action of gravity, in the third unit of time the body will cross the distance EF and also the distance FG, which is equal to AB or to DE. And thus at the end of the third unit of time the body will be found at G. It will have here a velocity which is triple that which it had at B at the end of the first unit of time. For in addition to the velocity acquired at E, which we said was double that acquired at B, in the third unit of time of the fall a velocity is added which again is equal to the velocity at the end of the first unit of time. Hence at the end of the third unit of time both velocities add up to triple the velocity found at the end of the first unit of time.

In the same way it can be shown that in the fourth unit of time the body ought to cross both the distance GH, which is triple BD, and the distance HK, which is equal to AB; and the velocity at K, at the end of the fourth unit of time, will be quadruple what it was at B at the end of the first unit of time. Therefore, it is clear that whatever distances we take to be crossed successively in equal times, these distances will each increase by an amount equal to BD, and simultaneously the velocities will also be increased equally in equal times.

PROPOSITION II

The distance crossed in a certain time by a body beginning to fall from rest is one-half the distance which it would cross in an equal time with a uniform motion whose velocity is equal to the velocity acquired at the last moment of the fall.

Assume what was stated in the prior proposition where AB was the distance crossed in a certain time by a body falling from A [Fig. 8]. BD was the distance which in an equal time the body was understood to have crossed with a uniform velocity equal to the velocity which was acquired at the end of the first unit of time or at the end of the distance AB. Now I say that the distance BD is twice AB.

The distances crossed by a falling body in the first four equal times are AB, BE, EG, and GH. There is a certain proportion between these distances. Let us suppose times double these times so that, for example, for the first time we take the two times in which the distances AB and BE were crossed, and for the second time we take the two remaining times in which the distances EG and GK were crossed. It is then necessary that the distances AE and EK, which are crossed from rest in equal times, are related to each other as are the distances AB and BE, which are also crossed from rest in equal times.

From this it follows that AB is to BE as AE is to EK. By conversion KE is to EA as EB or DA is to AB. Also by division DB is to BA as the excess of KE over AE is to EA. From what was proven in the prior proposition, it follows that KE is equal to twice AB plus five times BD. But EA is equal to twice AB plus BD. Thus it is clear that the mentioned excess of KE over EA is equal to four times BD. Hence DB will be to BA as four times DB is to EA. Thus EA will be four times BA. But EA is equal, as we said, to twice AB plus BD. Therefore BD is equal to twice AB. Q.E.D.

PROPOSITION III

If two distances are crossed by a falling body in any times, each of which is measured from the beginning of the fall, these distances are related to each other as the duplicate ratio of these times, or as the squares of the times, or as the squares of the velocities acquired at the end of these times.

For it was shown in the preceding proposition that the distances AB, BE, EG, and GK [Fig. 8], whatever their size, which are crossed in equal times by a falling body, increase by the same amount, which is equal to BD. Now it is clear that BD is twice AB, that BE is triple AB, that EG is five times AB, that GK is seven times AB, and that the remaining distances are

ig. 9] successively increased according to the progression of odd numbers beginning from one, i.e., 1, 3, 5, 7, 9, etc. And since any sum of these numbers, taken consecutively, makes a square whose side equals the number of numbers taken (for example, if the first three are added, they make nine; if four, sixteen), it follows from this that the distances crossed by a falling body, each of which is taken from the beginning of the fall, are related to each other as the duplicate ratio of the times during which the fall occurs, if indeed the times are assumed to be commensurable.

But it is easy to extend this demonstration to incommensurable times. Let there be two such times whose ratio is expressed by the lines AB and CD [Fig. 9], and let the distances crossed in these times be E and F, each taken from the beginning of the fall. Now I say that the distance E is related to F as the square of AB is related to the square of CD.

For if this be denied, the first possibility is that the ratio of E to F is greater than the ratio of the square of AB to the square of CD, that is, it has the ratio of the square of AB to the square of CG, where CG is less than CD. From CD let there be subtracted the part DH, which is smaller than DG, the excess of CD over CG, and let this be done in such a way that the remainder HC is commensurable to AB. It is clear that it is possible to do this. Hence CH will be larger than CG. And the square of the time AB will be related to the square of the time CH as the distance E, which is crossed in the time AB, is related to the distance crossed in the time CH, as shown above. But the distance F which is crossed in the time CD is larger than this distance. Thus the ratio of the distance E to the distance F is less than the ratio of the square of AB to the square of CH. But it was shown that E is to F as the square of AB is to the square of CG. Therefore, the ratio of the square of AB to the square of CG will be less than the ratio of the square of AB to the square of CH. From this it follows that the square of CG is greater than the square of CH, which is absurd because CH was said to be greater than CG. Therefore it is not the case that the ratio of the distance E to the distance F is greater than the ratio of the square of AB to the square of CD.

The second possibility is that the ratio of E to F is less than the ratio of the square of AB to the square of CD. Let the ratio of E to F be the same as the ratio of the square of AB to the square of CL, where CL is greater than CD. From CL let there be subtracted the line LK which is less than the excess LD by which CD is surpassed by CL, and let this be done in such a way that the remainder KC is commensurable to AB. Hence the square of the time AB is related to the square of the time CK as the distance E, which is crossed in the time AB, is related to the distance crossed in the time CK. But the distance F which is crossed in the time CD is less than this distance. Hence the ratio of E to F is greater than the ratio of the square of AB to the square of CK. But it was shown that E is to F as the square of AB is to the

square of CL. Therefore the ratio of the square of AB to the square of CL is the same as the ratio of the square of AB to the square of CK. Thus the square of CL is less than the square of CK, which is absurd because SL is larger than CK. Therefore the ratio of E to F is not less than the ratio of the square of AB to the square of CD, but they necessarily are the same. Finally, since the velocities acquired at the end of the times AB and CD are related to each other in the same way as these times, it is obvious that E is related to F by the same ratio as the squares of the times AB and CD in which they are crossed. Thus the proposition is established.

PROPOSITION IV

If a heavy body begins to move upwards with the same velocity acquired at the end of a descent, then in equal parts of time it will cross the same distances upwards as it did downwards, and it will rise to the same height from which it descended. Also in equal parts of time it will lose equal amounts of velocity.

As in Proposition 2 let there be any number of distances crossed in equal parts of time by a body falling from rest. Let AB [Fig. 8] be the first distance; the second will be composed of BD, which was crossed with the uniform velocity acquired through AB, and of DE which is equal to AB. The third will be composed of EF, which is twice BD, and of FG which is equal to AB; the fourth will be composed of GH, which is triple BD, and of HK which is again equal to AB; and the subsequent distances, as many as wished, will be increased by the same ratio. Now I say that in as many equal times the same individual distances KG, GE, EB, and BA are crossed successively by a body which moves upwards beginning with the velocity acquired at K at the end of the descent.

For the sake of brevity each velocity will be successively designated by the length of the distance crossed by a body in uniform motion with that velocity in one part of time, just as we described it in the case of descent.

From what was shown in the above-mentioned proposition, when the heavy body has arrived at K, it has a velocity of GH plus BD, that is, the velocity KF, because KF equals HG plus BD. For each part, HK and FG, is equal to AB, and together they equal BD, which is twice AB, as was shown in Proposition 2. Thus, if the velocity acquired at K at the end of the descent is directed upwards, and if the body is moved with a uniform motion, it will cross the distance KF in one part of time. But, after taking account of the action of gravity, the ascent KF will be decreased by the distance FG which is equal to AB, as is clear from what was said in our initial hypothesis. Hence in the first part of time the body rises only

through KG, which is the same distance it descended in the previous part of time. Simultaneously the velocity must have decreased by the same amount which it acquired by falling downwards in one part of time, namely, BD. Thus when the body has risen to the point G, its remaining velocity is HG, since at the beginning of the ascent its velocity was HG plus BD. But HG is equal to GD because HG is equal to FE plus DB, that is, HG equals FE plus twice AB, that is, HG equals FE plus FG plus ED. Hence if the body moves upwards from G with the uniform velocity which it has at that point, it will cross the distance GD in one part of time. But, after taking account of the action of gravity, this ascent is decreased by the distance DE, which is equal to AB. Thus in this second part of time it ascends through the distance GE, which it crossed in the same part of time when falling. Simultaneously the velocity must have decreased by the same amount which it acquired by falling in one part of time, namely, BD. Hence when it has risen up to the point E it has only the velocity FE, which is the remainder when BD is subtracted from GD. For, as we said, BD equals DE plus FG.

But EA equals FE, because FE equals BD taken twice; that is, FE equals BD plus twice AB; that is, FE equals BD plus AB plus DE. Therefore if the body moves upwards from E with the uniform velocity which it has at that point, it will cross the distance EA in one part of time. But, after taking account of the action of gravity, this ascent will be decreased by the distance AB. Hence in this part of time it rises only through the distance EB, which it also crossed in the same part of time by falling. Again the velocity at this point must have decreased by the amount which it acquired by falling in one part of time, namely, BD. Thus when the body has risen up to the point B, its remaining velocity is BD, since at E it had the velocity FE which is twice BD. Hence if the body moves upwards from B with the uniform velocity which it has at that point, in one part of time it will cross a distance equal to DB, that is, twice AB. But, after taking account of the action of gravity, this ascent is decreased by a distance equal to AB. Hence in this part of time it will rise only through the distance BA, which was crossed in the first part of time of the descent. And at the end of this last part of time the body necessarily will be found at the point A. But perhaps someone might say that it would rise higher than A, and then fall back to A. But this is absurd since in a motion due only to gravity it is not possible for a body to rise higher than the place from which it fell. For since the velocity which it has at B is again decreased by the amount BD, it is clear that, when the body is at A, it has no velocity left, and thus does not rise higher. Therefore it has been shown that the body rises to the same height from which it fell, and that each distance crossed in equal times of descent is equally measured off in as many equal times of ascent. It is also clear that in equal times equal amounts of velocity are lost. Hence the proposition is established.

In the demonstration of Proposition 2, on which the preceding proof depends, it was assumed that there exists a certain proportion between the distances crossed by a falling body in continuous equal times, and that this proportion is the same no matter what equal times are taken. This indeed must be so because of the nature of the way that things are related to each other, and if this is denied, it must be admitted that it is useless to search for a proportion between these distances. However, this proposition can be demonstrated without that assumption by using Galileo's method. Hence it will be a worthwhile effort to write down here more accurately the demonstration which he gave in a less perfect form.[2] Thus we will demonstrate this again.

PROPOSITION V

The distance crossed in a certain time by a body which begins its fall from rest is half the distance which it would cross in an equal time with a uniform motion having the velocity acquired at the last moment of the fall.

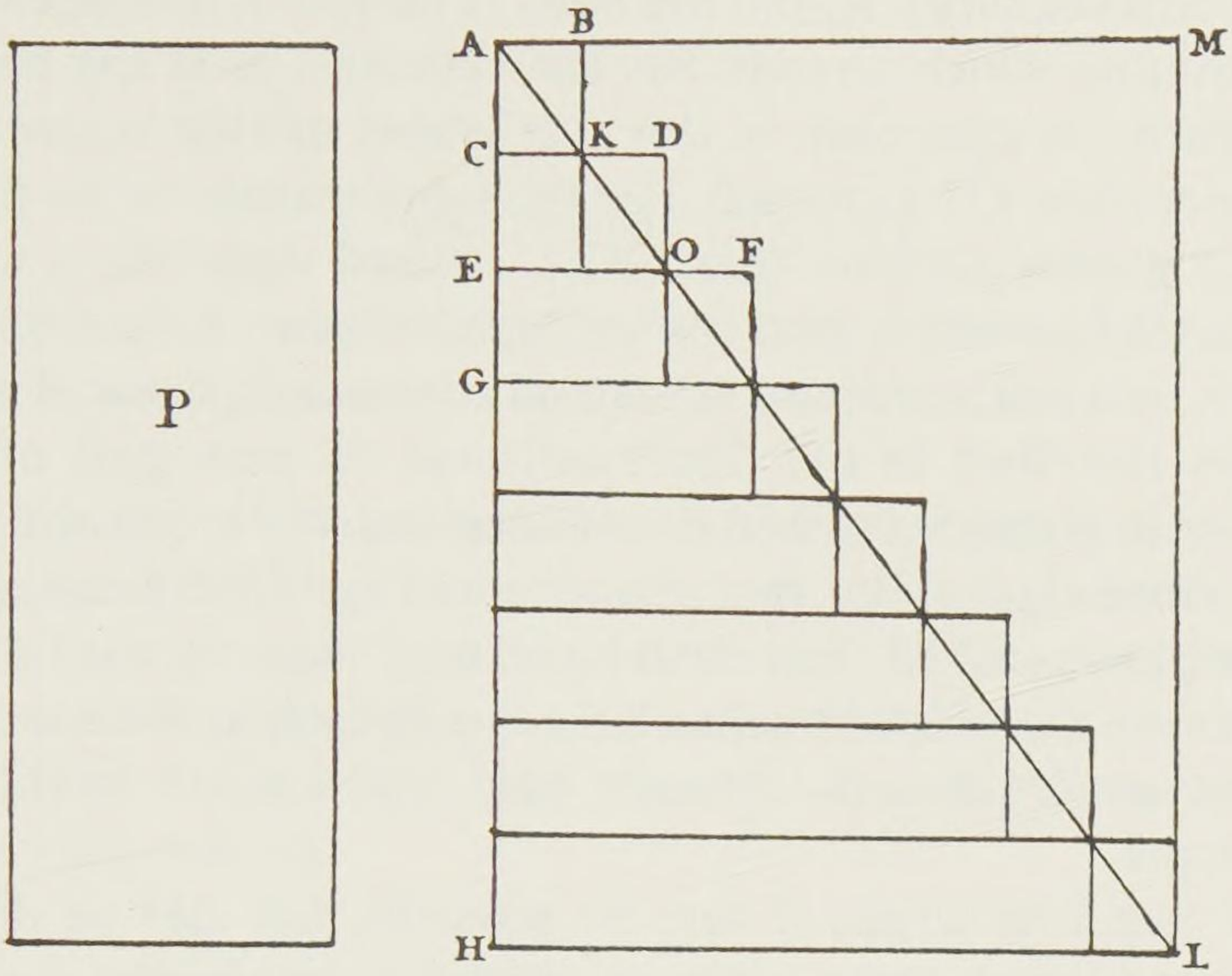
Let AH [Fig. 10] be the total time of fall. In that time the moving body crosses a distance whose quantity is designated by the plane P. Draw HL, of any length, perpendicular to AH, and let it represent the velocity acquired at the end of the fall.[3] Next complete the rectangle AHLM, and let it represent the amount of distance which would be crossed in the time AH with the velocity HL. What must be shown is that the plane P is half of the rectangle MH, or, after drawing the diagonal AL, that the plane P equals the triangle AHL.

If the plane P is not equal to the triangle AHL, then it will be either smaller or larger. First assume that the plane P is smaller than the triangle AHL, if that be possible. Let AH be divided in a number of equal parts AC, CE, EG, etc. And on the triangle AHL let a figure be circumscribed which is composed of rectangles whose altitude equals each part of this division of AH, namely, the rectangles BC, DE, FG. Also within the same triangle let there be inscribed another figure composed of rectangles of the same altitude, namely, KE, OG, etc. Let all this be done in such a way that

2. Galileo, *Discourses Concerning Two New Sciences.* Third Day. Proposition 1, Theorem 1, "On Naturally Accelerated Motion" (*Le Opere di Galileo Galilei* 8:208–9). "The time in which a certain space is traversed by a moveable in uniformly accelerated movement from rest is equal to the time in which the same space would be traversed by the same moveable carried in uniform motion whose degree of speed is one-half the maximum and final degree of speed of the previous, uniformly accelerated, motion." [S. Drake's translation, 165.]

3. In the margin of his copy of the 1673 edition Huygens wrote in Latin the following, almost illegible, words: "The distance designated by the plane P should be given after the distance designated by the rectangle since HL is drawn to any length."

[Fig. 10]



the excess of the former figure over the latter figure is less than the excess of the triangle AHL over the plane P. It is clear that this can be done since the entire excess of the circumscribed figure over the inscribed figure equals the lowest rectangle, which has HL as a base. Therefore the whole excess of the triangle AHL over the inscribed figure will be less than its excess over the plane P, and thus the figure inscribed in the triangle is larger than the plane P.

Further, since the straight line AH represents the entire time of fall, its equal parts AC, CE, EG, represent equal parts of that time. And since the velocities of falling bodies increase in the same proportion as the times of fall [Proposition 1], and since HL is the velocity acquired at the end of the whole time, CK will be the velocity acquired at the end of the first part of the time, for AH is to AC as HL is to CK. Likewise EO will be the velocity acquired at the end of the second part of the time, and so forth. But it is clear that in the first time AC some distance larger than zero is crossed by the moving body. And in the second time CE the distance crossed is greater than KE since the distance KE would have been crossed in the time CE by a uniform motion with the velocity CK. For the distances crossed by a uniform motion have a ratio composed of the ratios of the times and of the velocities. And hence, since in the time AH the distance MH would have been crossed with the uniform velocity HL, it follows that in the time CE the distance KE would be crossed with the velocity CK, since the ratio of the rectangle MH to the rectangle KE is composed of the ratios of AH to CE and of HL to CK.

Now, as was said, the distance KE would be crossed in the time CE with the uniform velocity CK, but the body is moved in the time CE with an accelerated motion which already has the velocity CK at the beginning of this time. Hence it is clear that, in this accelerated motion in the time CE, a greater distance than KE is crossed. For the same reason, in the third period of time EG, a greater distance than OG is crossed since this is what would be crossed in that time EG with the uniform velocity EO. And so successively, in each part of the time AH, the body crosses a greater distance than the rectangles inscribed in the figure adjacent to each part of the time. Hence the whole distance crossed by an accelerated motion will be greater than the inscribed figure. But that distance was assumed to be equal to the plane P. Thus the inscribed figure will be smaller than the distance P, which is absurd because this was shown to be larger than that distance. Thus the plane P is not smaller than the triangle AHL. Next it will be shown that it also is not larger.

Assume that it is larger, if that be possible. Let AH be divided into equal parts, and using these parts as altitudes, let figures composed of rectangles be inscribed and circumscribed again on the triangle AHL, as before. Let this be done in such a way that the one exceeds the other by an amount less than the amount by which the plane P exceeds the triangle AHL. Hence the circumscribed figure will necessarily be less than the plane P. Now it is clear that in the first part of the time AC the distance crossed by the body is less than BC, since this would be crossed in the same time with the uniform velocity CK which the body acquires only at the end of the time AC. Likewise in the second part of the time CE the distance crossed by an accelerated motion is less than DE since this would be crossed in the same time CE with the uniform velocity EO which it acquires only at the end of the time CE. And in like fashion, in each part of the time AH, less distance is crossed by the moving body than is represented by the rectangles circumscribed on the figure adjacent to each part of the time. Hence the whole distance crossed by an accelerated motion will be less than this circumscribed figure. But that distance was assumed to be equal to the plane P. Thus the plane P will be less than the circumscribed figure, which is absurd since this figure was shown to be less than the plane P. Therefore the plane P is not larger than the triangle AHL. But it was also shown that it is not less than that triangle. Therefore it must be equal. Q.E.D.

It should be noted that everything that has been demonstrated up to this point is equally true both of bodies falling and rising on inclined planes and of bodies moved on the perpendicular. For what was said about the effect of gravity must apply to both cases for the same reason.

From this it will not be difficult to prove the following proposition which Galileo asked that we accept as in a sense being self-evident.[4] For the demonstration which he tried to give later and which appears in the later

edition of his works does not seem to me to be too strong. The proposition is the following.

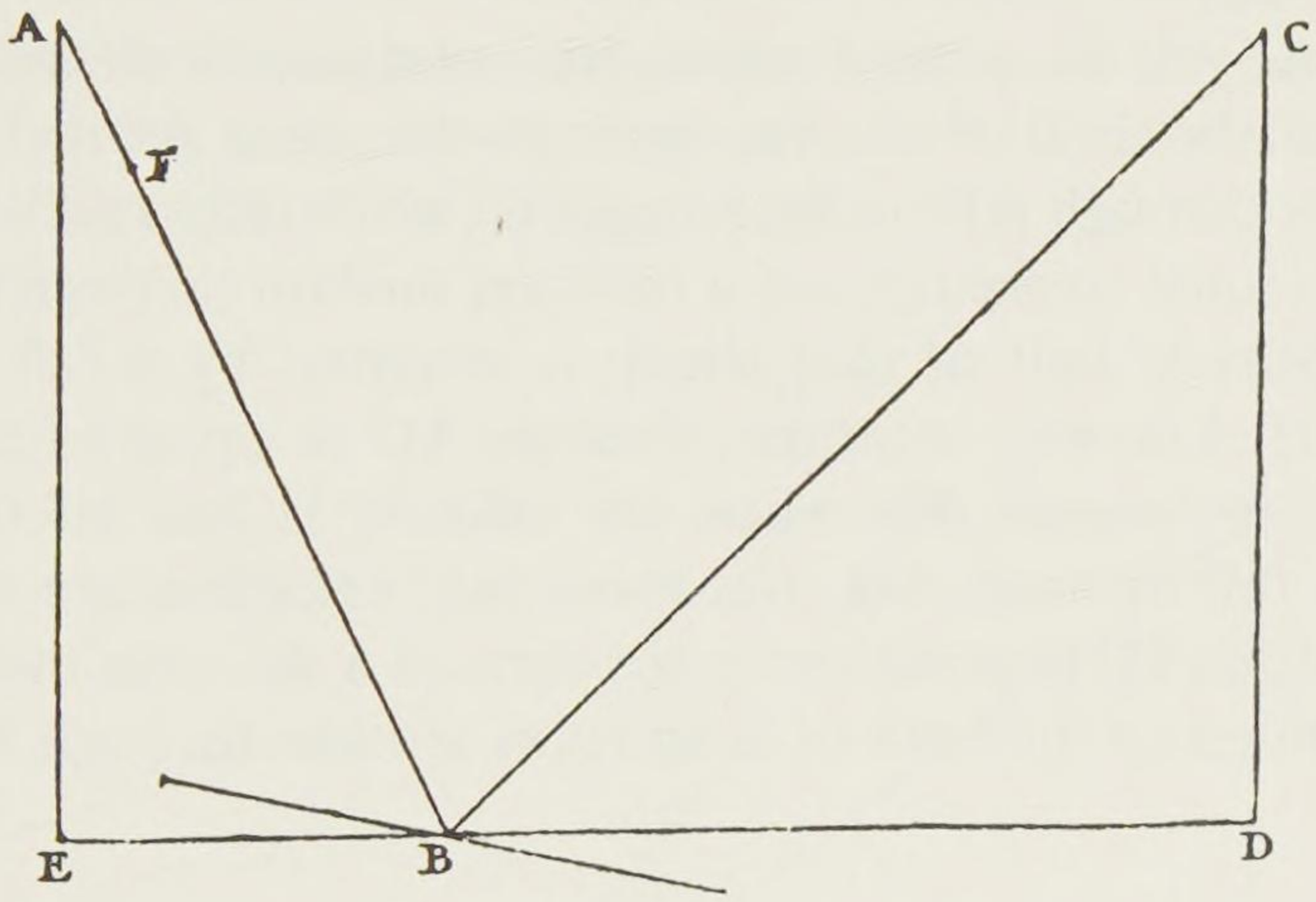
PROPOSITION VI

The velocities acquired by bodies falling through variably inclined planes are equal if the elevations of the planes are equal.

The elevation of the plane will be called its height on the perpendicular.

Let AB and CB [Fig. 11] be sections of inclined planes extended to the horizontal plane, and let their heights AE and CD be equal. Then let a body fall from A through the plane AB, and a body fall from C through the plane CB. Now I say that in each case the same degree of velocity will be acquired at the point B.

[Fig. 11]



For if we were to assume that the body falling through CB were to acquire less velocity than the body falling through AB, it would follow that the body falling through CB would acquire exactly the same velocity as a body falling only through FB, where FB is less than AB. But the body falling through CB acquires a velocity by which it could ascend again through the whole of BC [Proposition 4]. Therefore, if the body falling

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4. Galileo, *Discourses Concerning Two New Sciences.* Third Day. (*Le Opere di Galileo Galilei* 8:205.) ". . . the Author requires and takes as true one single assumption; that is, [Postulate] I assume that the degrees of speed acquired by the same moveable over different inclinations of planes are equal whenever the heights of those planes are equal." [S. Drake's translation, 162.]

from F to B were to continue its motion through BC, which it could do by reflection in the oblique direction, it would rise up to C, that is, up to a point which is higher than the place from which it fell, which is absurd.

If in the place of either plane a perpendicular is substituted which has an equal height, and if a body falls through that distance, it is clear that it acquires the same velocity as the body falling on the inclined planes. For the demonstration is the same.

From this there properly follows the demonstration of another of Galileo's theorems on which are built all the other theorems which he presented concerning motions along inclined planes. This proposition is the following.

PROPOSITION VII

The times of descent on variably inclined planes whose elevations are equal are related to each other as the lengths of the planes.

Let AC and AD be inclined planes having the same elevation [Fig. 12]. Now I say that the time of descent through the plane AC is related to the time of descent through AD as the length AC is related to AD. For the time through AC is equal to the time of a uniform motion through the same AC when the velocity is half of that which is acquired by a fall through AC [Proposition 2]. Likewise the time through AD is equal to the time of a uniform motion through AD when the velocity is half of that which is acquired by a fall through AD. But these half velocities are equal to each other [Proposition 6]; thus the indicated time of a uniform motion through AC will be related to the time of a uniform motion through AD as AC to

[Fig. 12]

AD. Hence, since the times are respectively equal to these latter times, it is clear that the time of descent through AC will have the same ratio to the time of descent through AD as AC has to AD. Q.E.D.

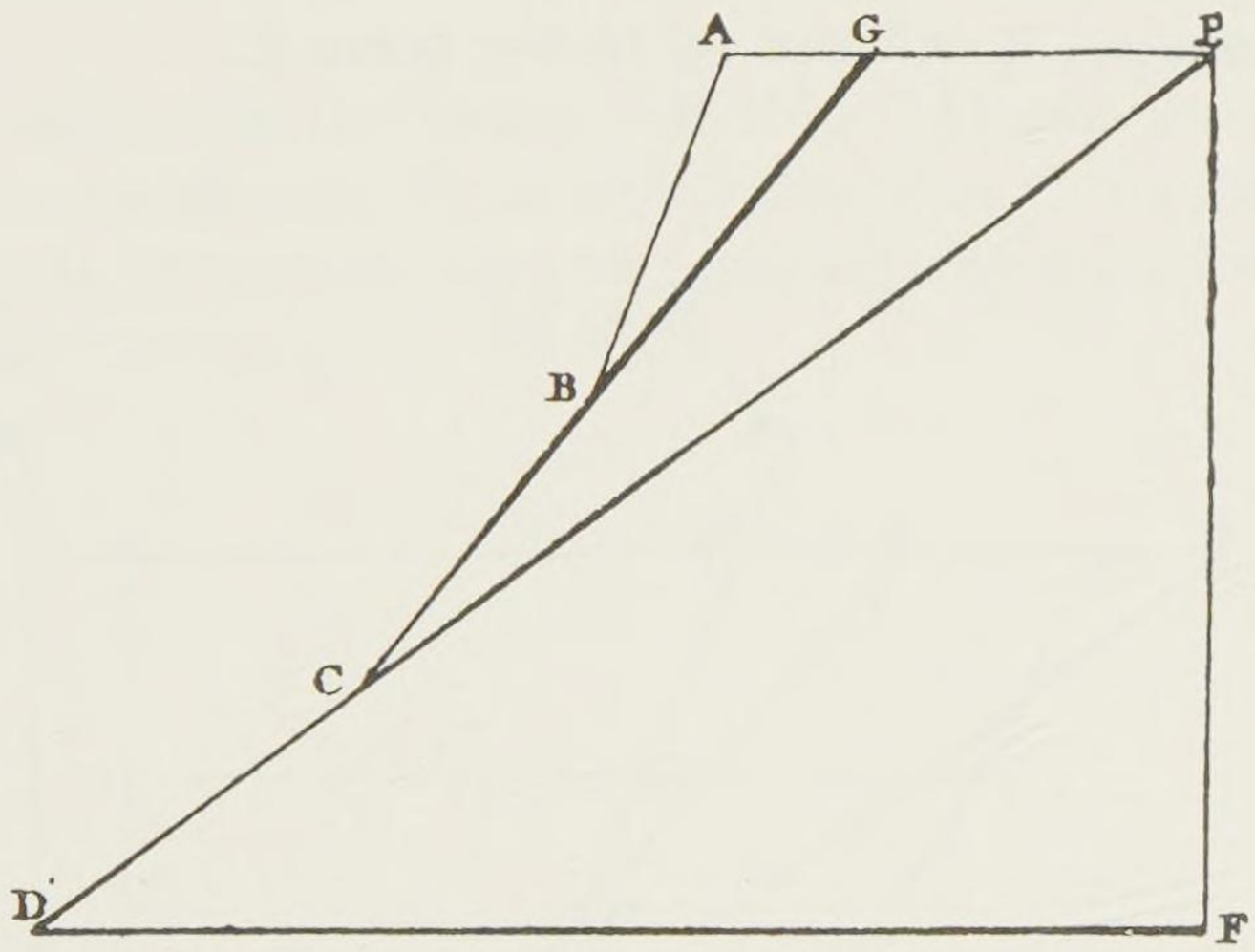
In the same way it is also shown that the time of descent through AC is related to the time of fall through the perpendicular AB as AC is related to the length AB.

PROPOSITION VIII

If from the same height a body descends by a continuous motion through any number of contiguous planes having any inclinations whatsoever, it will always acquire at the end the same velocity; namely, a velocity equal to that which would be acquired by falling perpendicularly from the same height.

Let AB, BC, and CD be contiguous planes [Fig. 13] whose terminus A has a height above the horizontal line DF, drawn through the lower terminus D, equal to the perpendicular EF. And let a body descend through these planes from A to D. Now I say that at D it will have the same velocity which it would have at F by falling from E.

[Fig. 13]



For when CB is extended it cuts AE at G, and likewise DC, when extended, cuts AE at E. Now a body descending through AB will acquire the same velocity at B as a body descending through GB [Proposition 6]. Hence it is clear that, if the change of direction at B has no effect, the body

will have the same velocity when it arrives at C as if it had descended through the plane GC, which is the same as it would have by descending through EC. Hence, in the same way it will cross the remaining plane CD as if it had passed through EC. And thus at D it will have the same velocity as if it had descended through the plane ED, that is, the same velocity acquired by a fall through the perpendicular EF. Q.E.D.

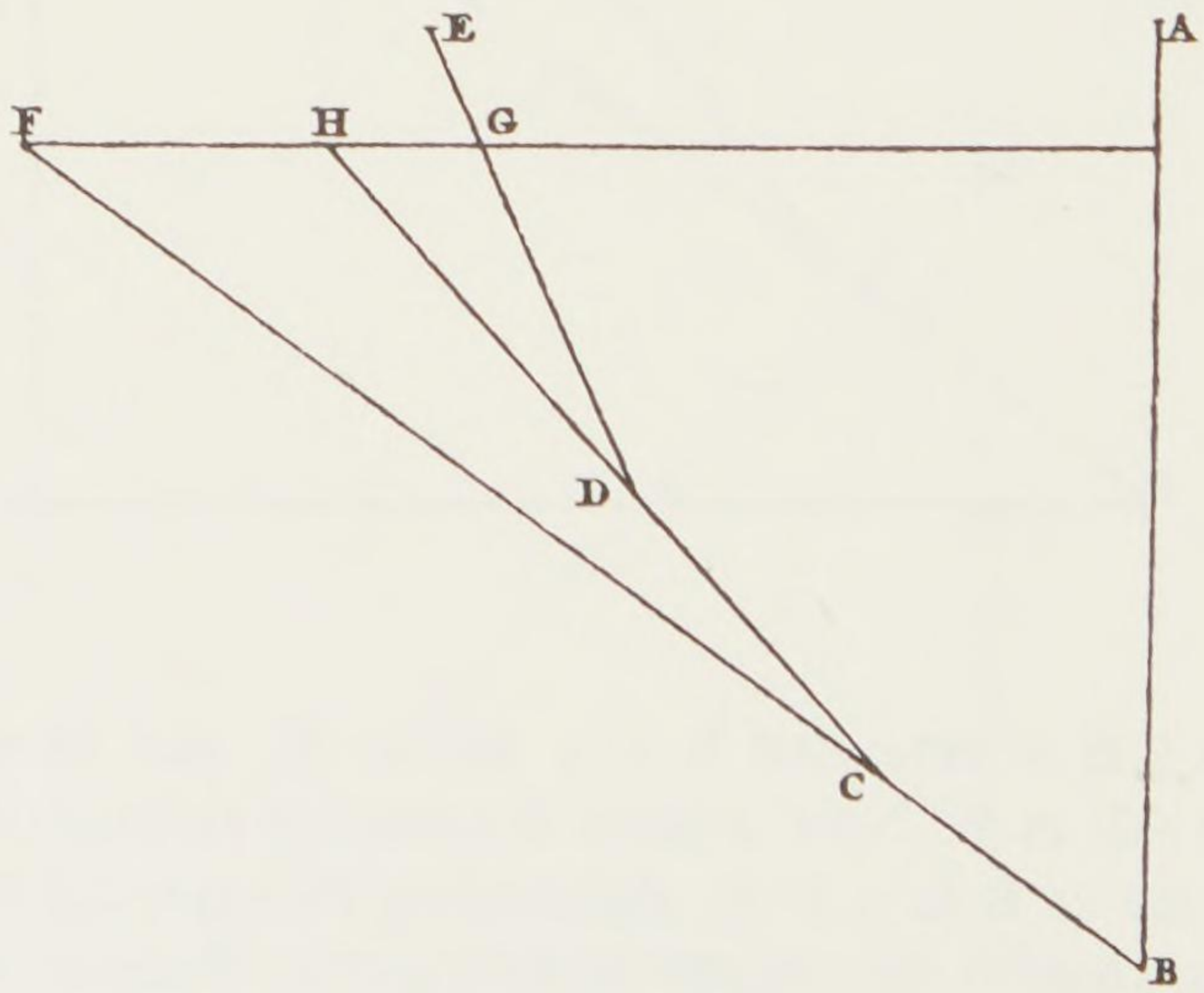
From this it also follows that a body descending through the circumference of a circle or through any curved line (for here we can consider any curve to be composed of an infinite number of straight lines) will always acquire a velocity equal to what it would acquire by descending from an equal height. This velocity will be equal to the velocity acquired in a perpendicular fall from the same height.

PROPOSITION IX

If, after falling, a body converts its motion upwards, it will rise to the same height from which it came, no matter how many contiguous plane surfaces it may have crossed, and no matter what their inclinations are.

Let a body fall from the height AB [Fig. 14]. From the point B let the planes BC, CD, and DE be inclined upwards such that their extremity E has the same height as the point A. Now I say that if a body, after falling through AB, converts its motion so that it continues to be moved through these inclined planes, it will rise up to the point E.

[Fig. 14]



For suppose, if this be possible, that the body rises only to point G. Let BC and CD be extended until they cut the horizontal GF at F and H. Now after the body has crossed the planes BC and CD, it would have only enough velocity for it to ascend through DG, or through DH, for these two velocities are equal, as was shown in Proposition 6. Hence, after having crossed the plane BC, it would only have enough velocity for it to ascend through CH, or through CF. Thus at B it has only enough velocity to ascend through BF, that is, the same velocity which it would acquire by descending through FB. But at B it has enough velocity to ascend all the way up to A. Therefore, with the velocity which it acquires by descending through FB it would be able to ascend through BA, which is higher than the place from which it descended. But this is impossible.

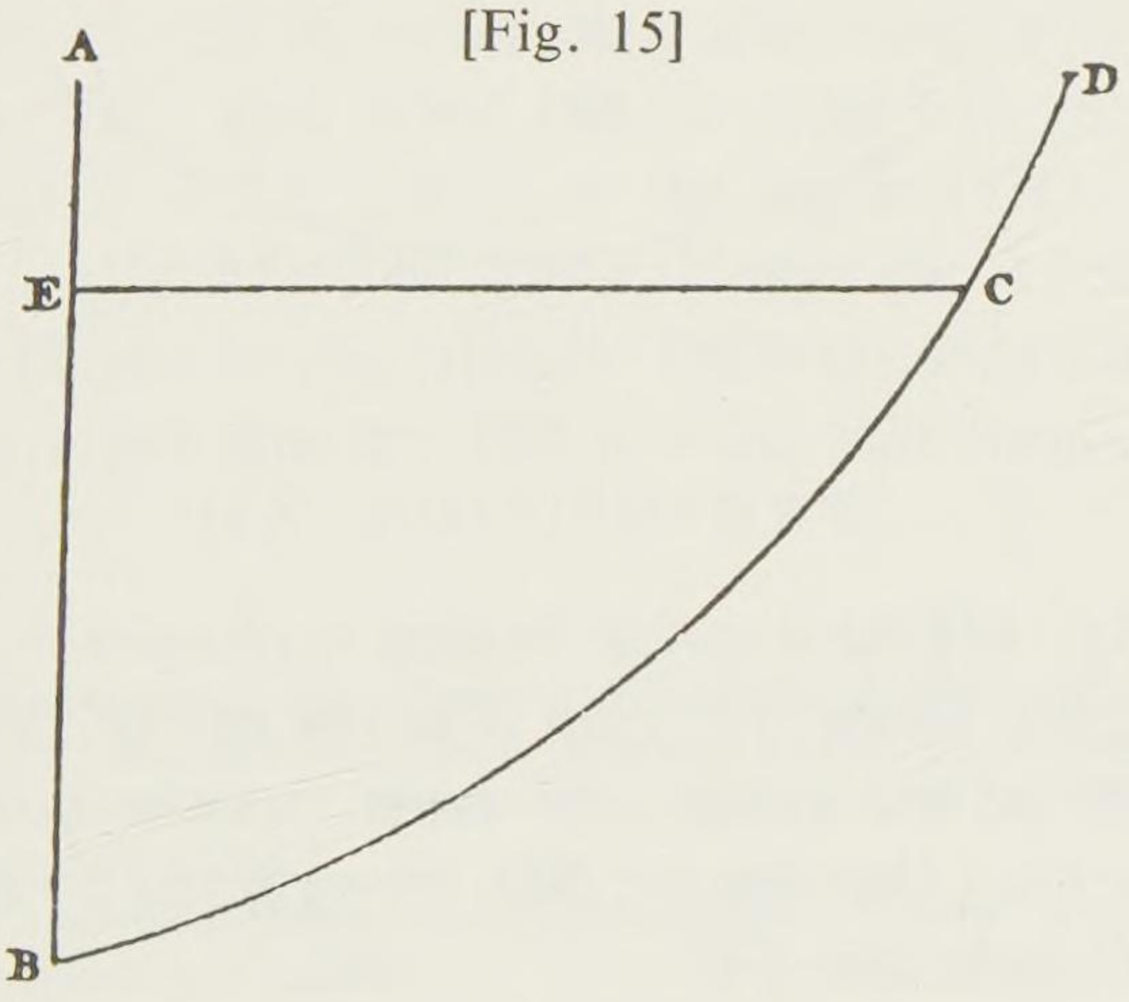
Exactly the same demonstration applies no matter how many planes are crossed by the descending body. Hence, if there be an infinite number of planes, that is, if any curved surface is given, the body will rise through it to the same height from which it came.

PROPOSITION X

If a body falls perpendicularly or through any surface, and if it later moves upwards by the acquired impetus through any other surface, then it will always have the same velocity at points of equal height in its descent and ascent.

Let a body fall from the height AB [Fig. 15] and then continue its motion through the surface BCD, in which the point C has the same height as the point E in AB. Now I say that the same velocity is present in the body at C as was present in it at E.

[Fig. 15]



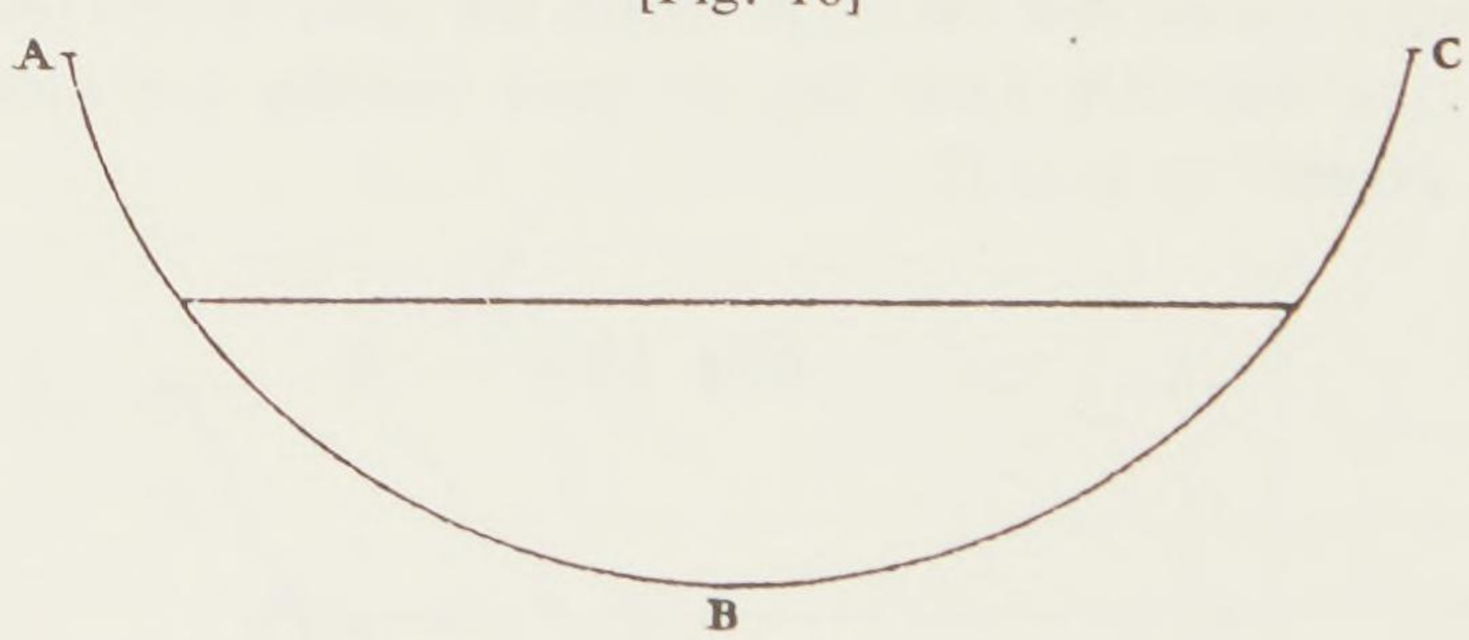
For at C it has an additional velocity by which it could ascend up to the point D, which is the same height as A [Proposition 9]. And in descending through AE it acquires a velocity by which, when the direction is reversed, it will ascend through CD [Proposition 9]. Hence it is clear that when it arrives at C in its ascent, the body has the same velocity which it had at E in its descent. Q.E.D.

PROPOSITION XI

If a body moves downwards through some surface and then reverses its motion and moves upwards through the same surface or through another similar surface similarly situated, then it will descend and ascend through the same distance in equal times.

For example, let a body descend through the surface AB [Fig. 16], and when it has arrived at B, let it reverse its motion and ascend upwards through the same surface AB or through the surface BC which is similar to AB and which is similarly situated in respect to the horizontal plane. Now it is clear from what was demonstrated earlier that it will rise to the same height from which it came. For it always has the same velocity at points of equal height in its descent and ascent [Proposition 10]. Thus it is clear that the same line is crossed twice with the same velocity in each part of the motion. Hence the times of both motions are necessarily equal. Q.E.D.

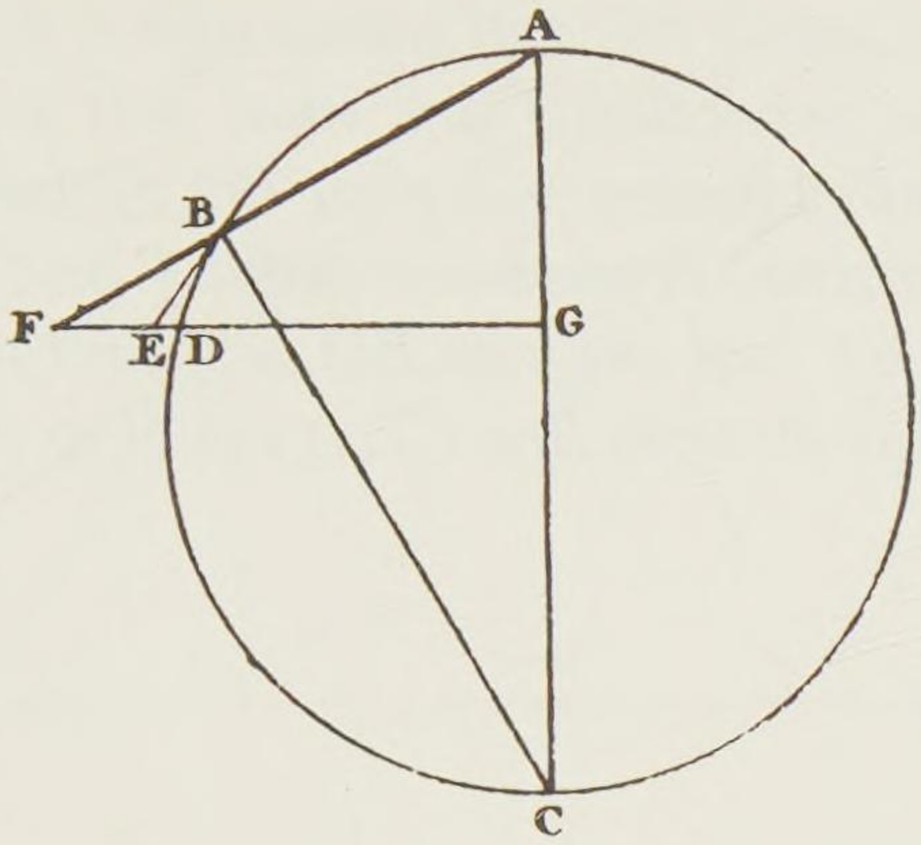
[Fig. 16]



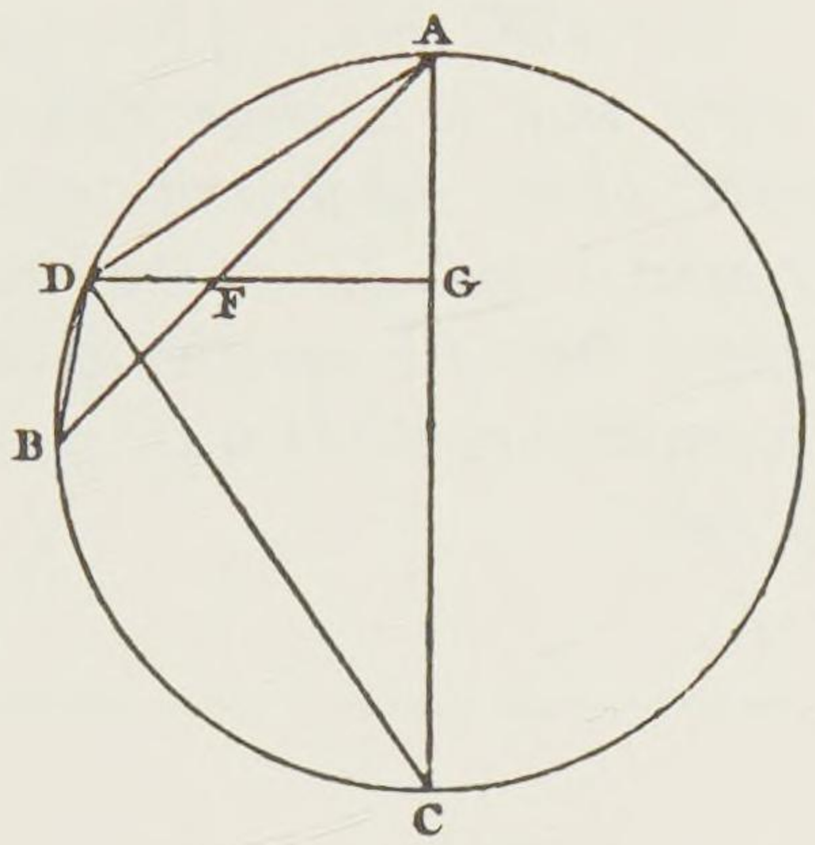
PROPOSITION XII

Let ABC [Fig. 17] be a circle having a diameter AC to which FG is drawn at right angles. Draw AF from A at the end of the diameter such that it cuts GF outside of the circle. AF must cut the circumference, for instance, at B. Now I say that the arc BD, intercepted by the lines GF and AF, is less than the straight line DF.

[Fig. 17]



[Fig. 18]



Draw the line BC, and from the point B draw the straight line BE tangent to the circumference. BE necessarily cuts the straight line FG between F and D. Now the angle BAC in the circle is equal to the angle EBC.[5] Hence the angle FBE, which together with EBC forms the right angle FBC, will be equal to BCA. But since ABC and AGF are similar triangles, the angle F will be equal to the angle ACB. Thus this same angle F is equal to the angle FBE. Hence FEB is an isosceles triangle whose sides FE and EB are equal. If ED is added to each of these lines, then FD is equal to the sum of BE and ED. But it is clear that these latter two lines are larger than the arc BD, which is intercepted at the same two points and which is concave in the same direction. Therefore FD is larger than this same arc BD. Q.E.D.

PROPOSITION XIII

Given the same construction, if the straight line AB cuts DG inside the circle [Fig. 18], then I say that the arc BD, which is intercepted by the straight lines GD and AB, is larger than the straight line DF.

Draw the line DC, and draw DB as a cord to the arc DB. Now the angle ABD is equal to ACD, that is, to the angle ADG. And the angle DFB is larger than the angle ADF, or ADG. Hence the same angle DFB is also larger than DBF. Hence in the triangle DFB the side DB is larger than the side DF. And therefore the arc DB is even still larger than the line DF. Q.E.D.

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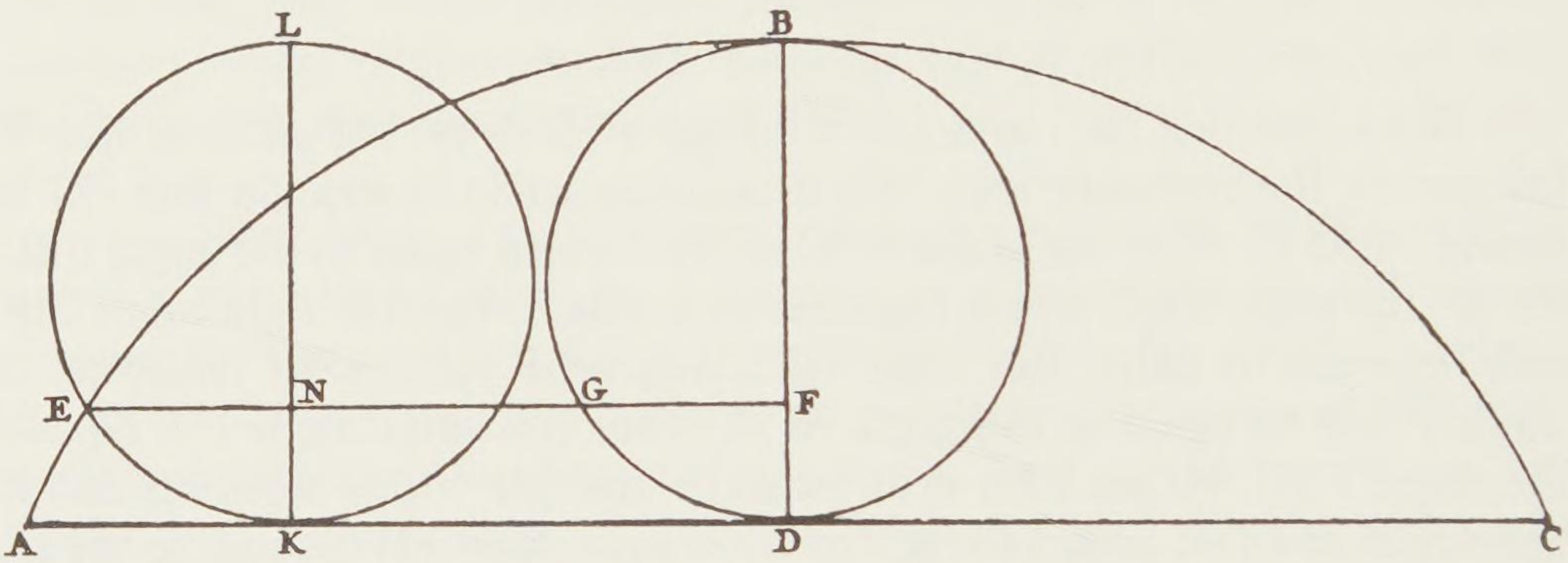
5. 's Gravesande has noted that this is Proposition 32, Book 3, of Euclid's *Elements*. "If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle." [T. L. Heath's translation 2:65.]

Since an asterisk for a reference appears in the text of the 1673 edition at this point, Huygens no doubt intended to include this reference to Euclid.

PROPOSITION XIV

Let ABC [Fig. 19] be a cycloid whose base is AC and whose axis is BD. I think that it is clear how this figure is generated from what was said above[6] *about the definition and the mechanical description of the cycloid. Around its axis BD let the circle BGD be drawn. From some point E on the cycloid draw EF parallel to the base AC. EF cuts the axis BD at F and the circumference BGD at G. Now I say that the straight line GE is equal to the arc GB.*[7]

[Fig. 19]



Through the point E draw the circle LEK which is equal to BGD and which is tangent to the base of the cycloid at K. Draw the diameter KL. Now the straight line AK is equal to the arc EK, and the whole line AD is equal to the semicircumference KEL. Therefore, KD is equal to the arc EL or GB. But KD or NF is equal to EG, since EN equals GF, and NF is common to both. Thus it is proven that GE is equal to the arc GB.

PROPOSITION XV

Given a point on a cycloid, through that point to draw a straight line which is tangent to the cycloid.

Let ABC [Fig. 20] be a cycloid, and let B be the given point through which the tangent must be drawn.

Around the axis AD of the cycloid draw the generating circle AED.

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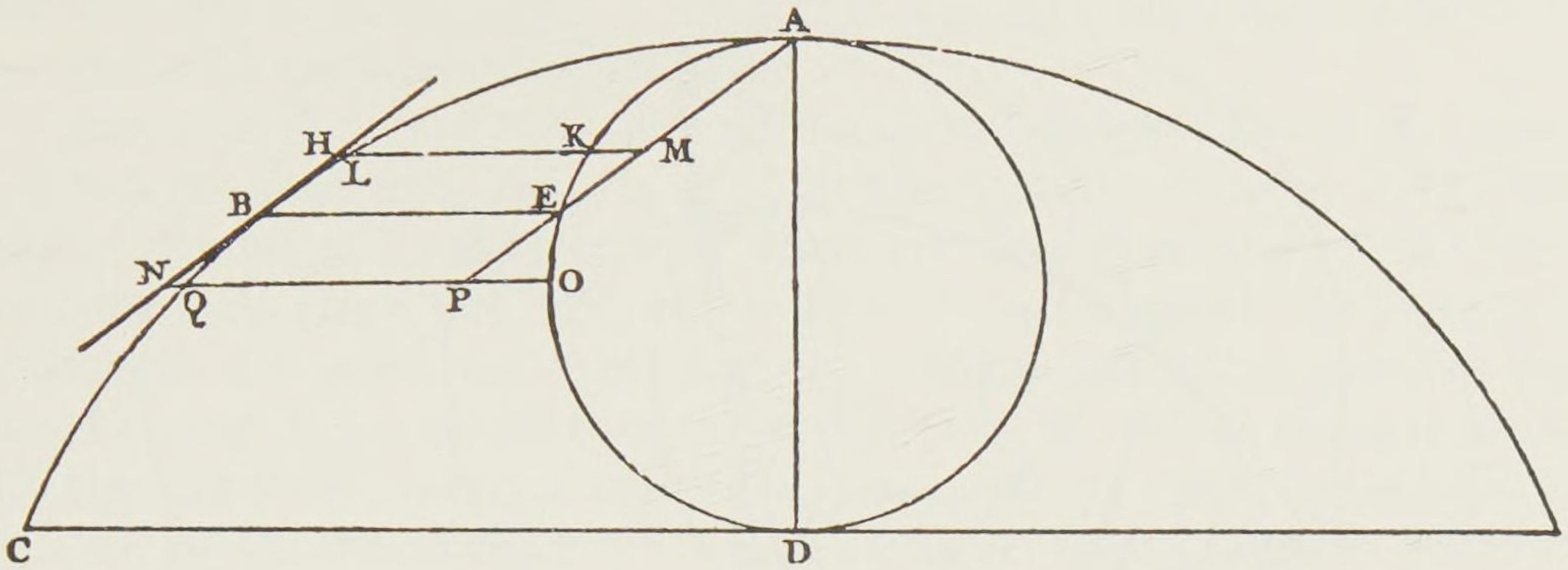
6. Part I, paragraphs 12ff.
7. Huygens discovered this proposition in 1658. See *Oeuvres* 14:347.

[Fig. 20]



Draw BE which is parallel to the base of the cycloid and which cuts the indicated circle at E. Draw AE, and finally through B draw HBN parallel to AE. Now I say that HBN is tangent to the cycloid at B.

On the line HBN take any point other than B, and first take some point H above B. Through H draw a straight line parallel to the base which cuts the cycloid at L, the circle AED at K, and the line AE at M. Now KL is equal to the arc KA, and the straight line KM is less than the arc KE. Hence the straight line ML will be less than the arc AE, that is, less than the line EB, or MH. From this it is clear that point H lies outside the cycloid.

Next on the line HN take some point N below B. Through N draw, as before, a parallel to the base, which cuts the cycloid at Q, the circle AED at O, and the extended straight line AE at P. Now OQ is equal to the arc OA, but OP is larger than the arc OE. Hence PQ will be less than the arc EA, that is, less than the straight line EB, or PN. From this it is clear again that the point N lies outside the cycloid. Therefore, since every point except B on the straight line HBN lies outside the cycloid, it is proven that HBN touches the cycloid at the point B. Q.E.D.

I had some doubts as to whether I should provide a special place for this demonstration since I have found that a very similar proof by Mr. Wren has been published in a book by Mr. Wallis[8] on the cycloid. However, it is possible to establish the proposition by using a universal method which applies not only to cycloids but to other curves generated by the rolling of a figure, as long as that figure is concave in the same direction and is one of those figures which are called geometrical.

Let NAB [Fig. 21] be a curve generated by the rolling of the figure OL on the line LD and traced out by the point N taken in the circumference of

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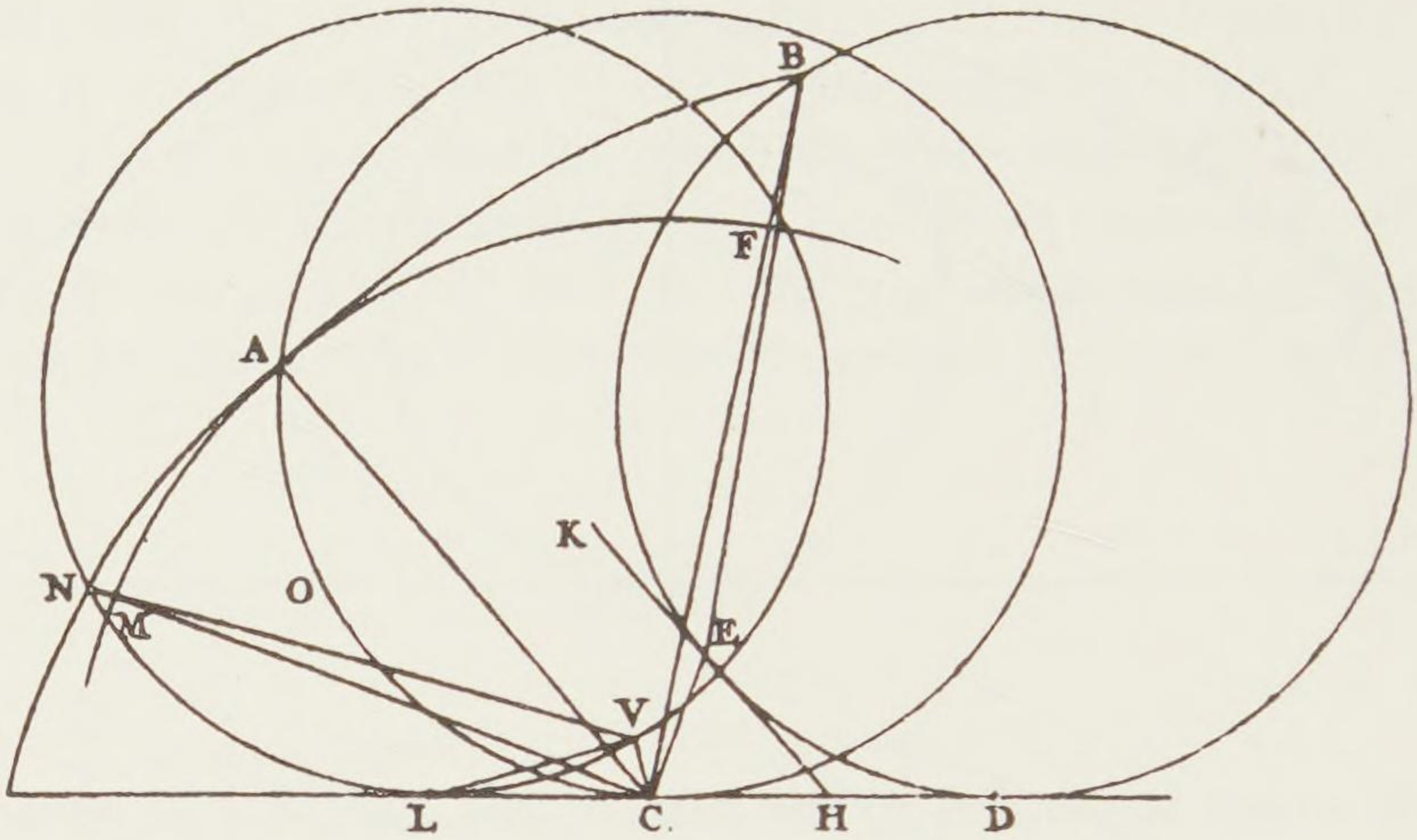
8. John Wallis, *Tractatus duo, prior de cycloide et corporibus inde genitis; posterior, epistolaris in quo agitur de cissoide . . .* (Oxoniae: Typ. Ac. Lichfieldianis, 1659). "De recta tangente cycloidem primariam," 71.

[Fig. 21]



the figure OL.[9] One could draw a tangent at point A of the curve. Draw the straight line CA from the point C where the figure touched the line LD when the generating point was at A. This point of contact can always be found since the problem reduces to drawing two straight parallel lines, of which one is tangent to the generated figure and the other passes through the generating point given in the rotating figure such that these parallels are separated from each other by the same distance by which the point A is separated from the line LD. Now I say that CA cuts the curve at right angles; in other words, the circumference MAF described by the radius CA around the center C touches the curve at the point A; hence a perpendicular to AC drawn through the point A is tangent to the curve at this point.

First draw CB to the point B of the curve which is located further above the line LD than the point A. Consider the figure to be located at BED where the generating point would be at B, the point of contact with LD at D, and the point of the curve which was at C when the generating point was at A has now risen to E. Draw EC and EB, and draw a tangent to the figure at E which cuts LD at H.

Now the straight line CD is equal to the curve ED. And the sum of EH and HD is larger than this curve. Thus EH is larger than CH. Hence the angle ECH is larger than CEH, and thus ECL is smaller than CEK. By adding the angle KEB, which is equal to LCA, to KEC, we get the angle CEB. And by subtracting the angle LCB from ECL, we get ECB. Hence the angle CEB is very much larger than the angle ECB. Thus in the triangle CEB, the side CB is larger than EB. But EB is clearly equal to CA since this

9. Huygens here develops an idea borrowed from Descartes and van Schooten. See *Oeuvres* 18:388–94.

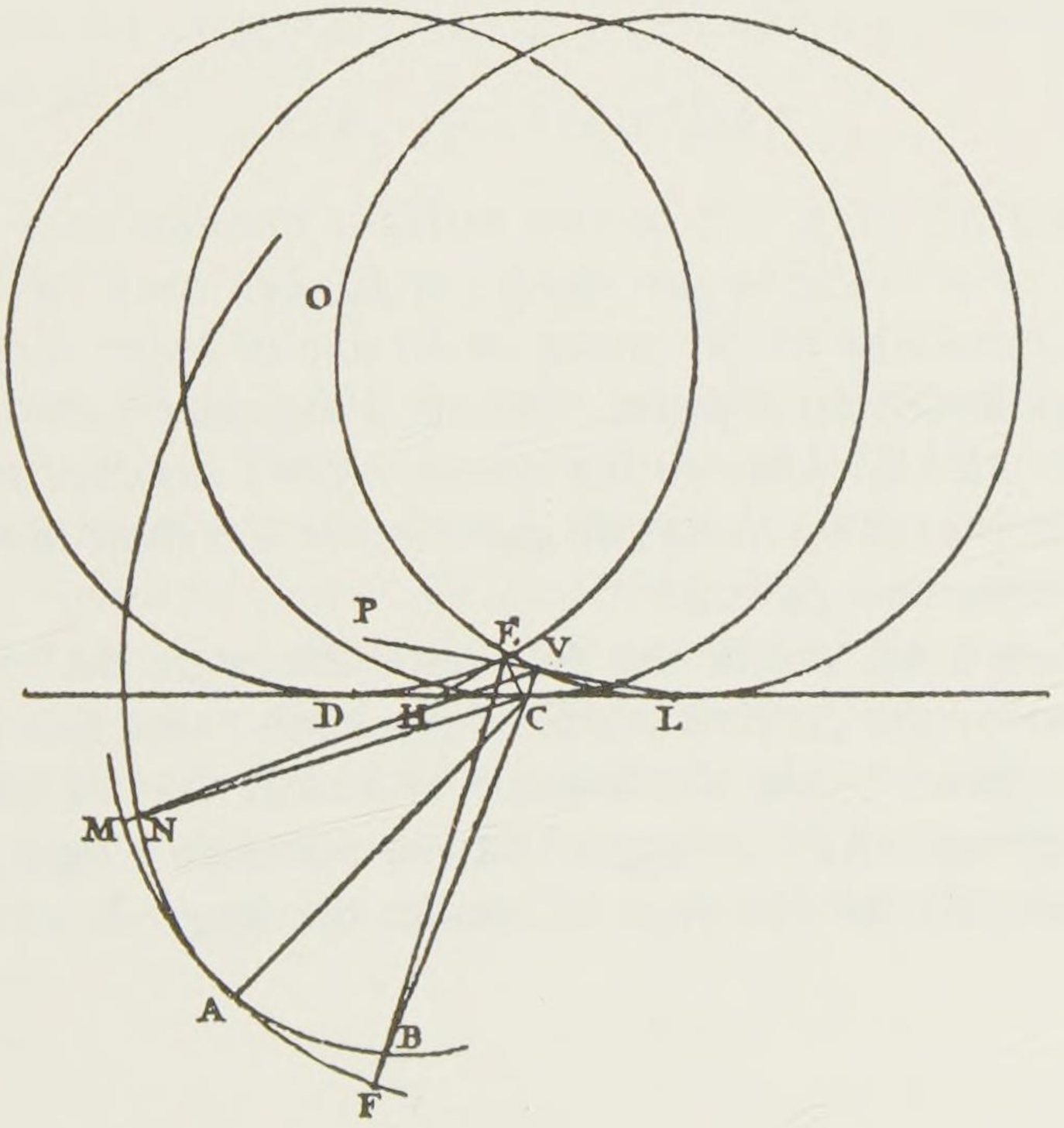
is the same line which was rotated with the figure. Hence CB is also larger than CA, that is, larger than CF. Therefore it is established that the point B lies outside of the circumference MAF.

Next take the point N located on the curve between LD and the point A. When the generating point is at N, let the figure be situated at VL, the point of contact at L, and the point which previously touched LD at C has now risen to V. Draw CN, NV, VC, and VL. Now VN equals CA since it is identical to CA rotated to VN. And the straight line LC is equal to the curve LV, and thus is larger than the straight line LV. Hence in the triangle CLV the angle LVC is larger than LCV. If we add the upper angle LVN to LVC, we get the whole angle NVC which is larger than LCV, and which is thus very much larger than the angle NCV which is part of LCV. Therefore in the triangle CVN the side CN is larger than the side VN, which is equal to CA. Thus CN is larger than CA, that is, larger than CM. From this it is apparent that the point N lies outside the circle MAF, which touches the curve at the point A. Q.E.D.

The construction and the demonstration are the same whether the curve is produced by a generating point located either inside or outside the boundary of the rotating figure. But in the latter case part of the curve descends below LD, and because of this some differences arise in the demonstration.

Let the point A, through which a tangent could be drawn, be taken on the curve NAB which descends below the line CL [Fig. 22]. NAB is described by a point N located outside the rotating figure, but it has a

[Fig. 22]



definite position in the same plane. Having found the point C where the rotating figure touches CD when the generating point was at A, draw the straight line CA. Now I say that CA cuts the curve NAB at right angles; in other words, the circumference described by the radius CA around the center C is tangent to the curve NAB at the point A. But it will be shown that this circumference touches the curve from the outside, whereas in the part of the curve located above the line CD, it touches it from the inside.

All things being given and described as before, it is clear again that the angle ECH is larger than CEH. By adding HCB to ECH we get the angle ECB, and by subtracting HEB, which is equal to DCA, from CEH, we get the angle CEB. Hence ECB is much larger than CEB. Thus in the triangle ECB the side EB is larger than CB. But EB is equal to CA or CF. Hence CF is also larger than CB. Therefore the point F of the circumference lies outside the curve as compared to the center.

Next it is clear that the angle LVC is larger than LCV. Thus CVP, which together with LVC equals two right angles, is less than VCD. By adding angle DCN to VCD, we get VCN, and by subtracting angle PVN from CVP, we get CVN. Hence the angle VCN is much larger than CVN. Thus in the triangle CVN the side VN is larger than CN. But VN is equal to CA or CM. Thus CM is also larger than CN, and therefore the point M of the circumference lies outside the curve NAB as compared to the center. Thus it is established that the circumference touches the curve at point A. Q.E.D.

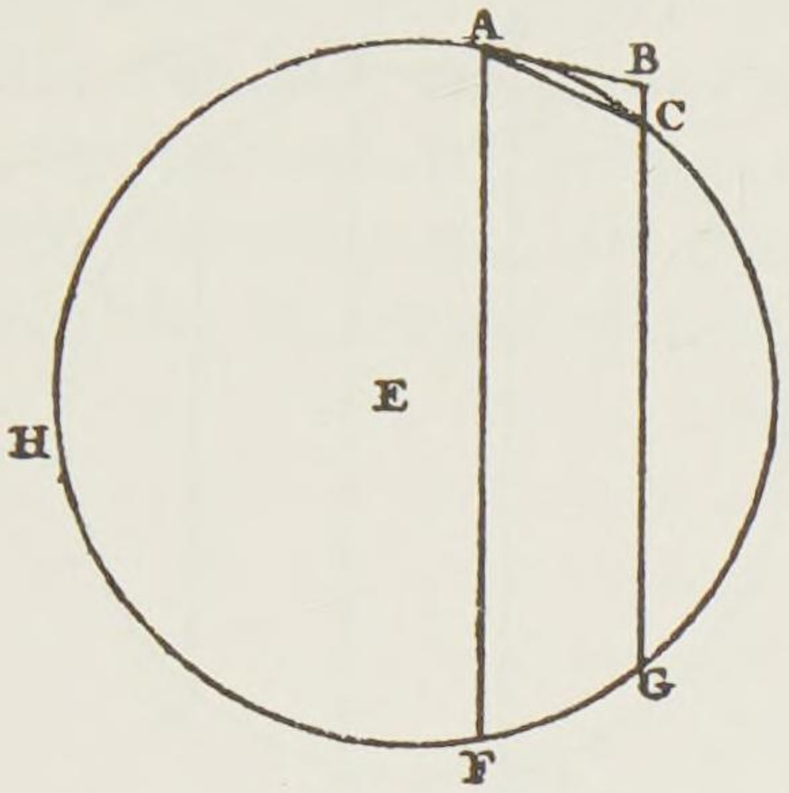
Further, if the point of the curve through which the tangent is to be drawn is the same point at which the radius cuts the curve, then the sought-for tangent will always be perpendicular to the radius, as is easily shown.

PROPOSITION XVI

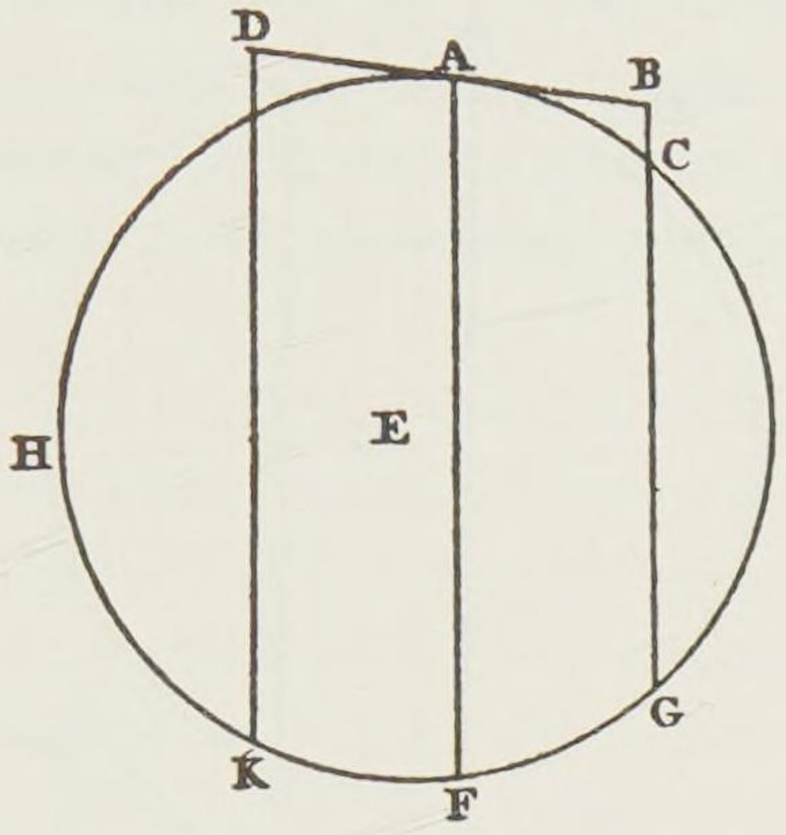
Let AF and BG [*Fig. 23*] *be two straight parallel lines which cut the circumference of a circle whose center is E. Let both of these lines be located on the same side of the center, or let one of them, AF, pass through the center. Draw a tangent at point A which is located on the line cutting the circumference which is closer to the center. Now I say that the part AB of the tangent intercepted by these two parallels is less than the arc AC intercepted by the same two parallels.*

Draw the cord AC on the arc AC. Now the angle BAF is equal to the angle contained by the portion AHF of the circle, which is larger than or equal to a semicircle. Hence the angle BAF is less than or equal to a right angle. Thus the angle ABC is larger than or equal to a right angle. Therefore in the triangle ABC the side AC under the angle B is larger than the

[Fig. 23]



[Fig. 24]



side AB. But this side AC is smaller than the arc AC. Hence AB is clearly smaller than the arc AC.

PROPOSITION XVII

Given the same circumstances, let there be a third parallel line DK [*Fig. 24*] *which cuts the circle and which is the same distance from AF, which was closer to the center, as AF is from BG. Now I say that the part of the tangent at A intercepted by this added parallel and the middle parallel, that is, the line AD, is less than the arc AC which is intercepted by the first two parallels.*

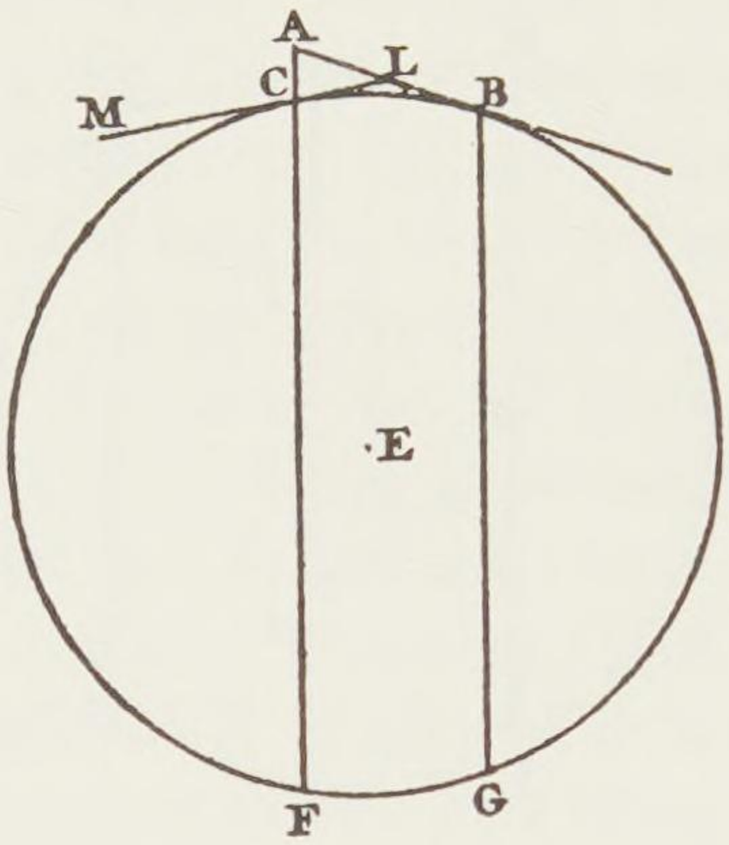
This is obvious since AD is equal to AB which we have already shown is less than the arc AC.

PROPOSITION XVIII

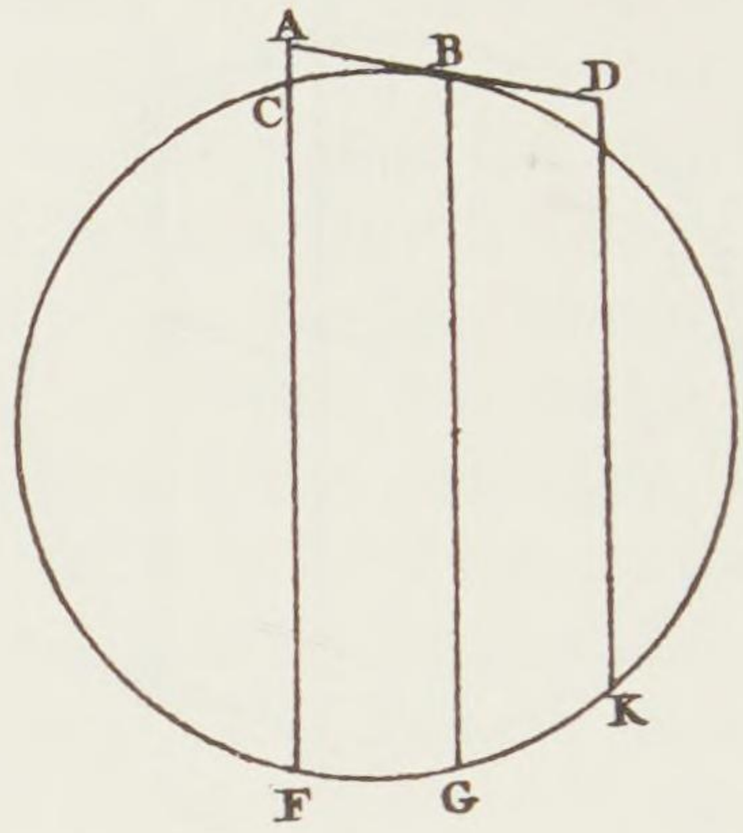
Let AF and BG [*Fig. 25*] *be two parallel lines which cut through a circle whose center is E. Draw a straight line tangent to the circumference at B which is the point where the circumference is cut by the parallel which is either further removed from the center, or the same distance from the center, than the other parallel. Now the part BA of this tangent intercepted by the parallels is larger than the arc BC intercepted by the same parallels.*

At the point C draw the tangent MCL to the circumference. This cuts the tangent BA at L. In the triangle ACL the angle C is equal to the angle MCF, that is, to the angle contained in the portion CBF of the circle. But

[Fig. 25]



[Fig. 26]



the angle A is equal to the angle contained by the portion BCG of the circle, which portion is larger than or equal to the portion CBG since BG is either more distance from the center, or the same distance from the center, than CF. Thus in triangle ACL the angle A is either less than or equal to the angle C. Hence the side CL is either less than or equal to the side AL. But CL plus LB is larger than the arc CB. Therefore AL plus LB, that is, the tangent AB, will be larger than the arc CB. Q.E.D.

PROPOSITION XIX

Given the same circumstances, let there be a third straight line DK [*Fig. 26*] *which is parallel to the first two, which cuts the circle, and which is the same distance from the parallel which is further from the center as the latter is from the other parallel AF. Now the part of the tangent at B intercepted by the middle parallel and the added line DK, that is, the line BD, will be larger than the arc BC.*

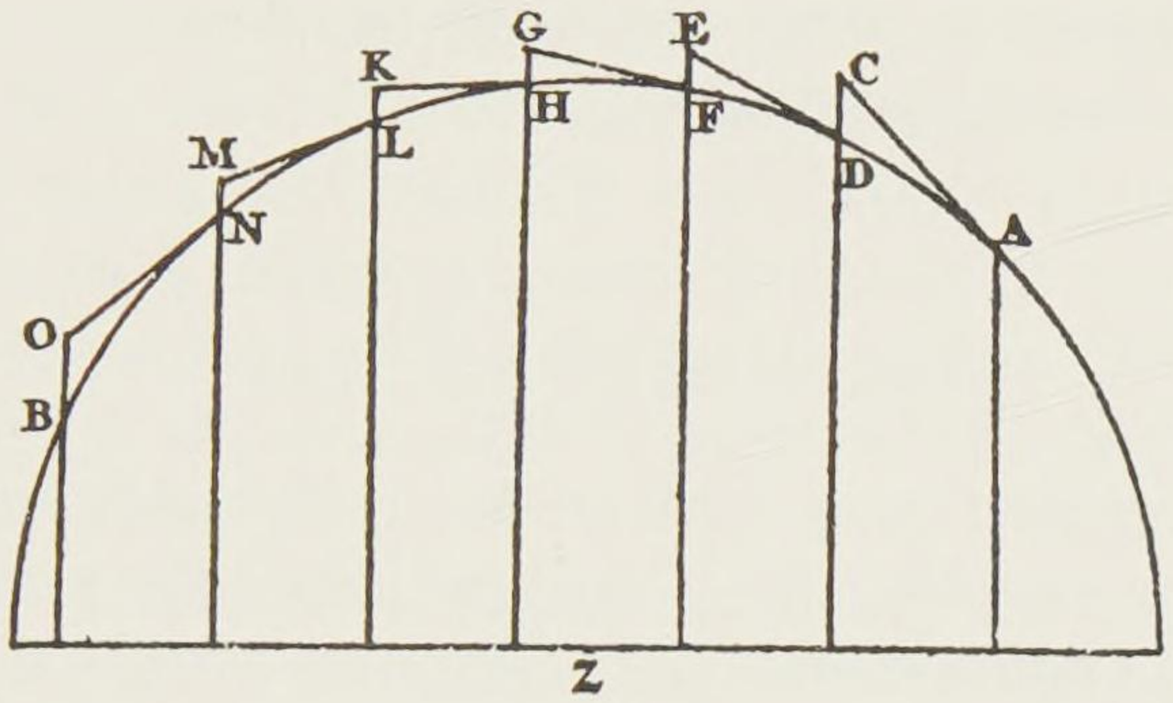
This is obvious since BD is equal to BA which we have already shown is larger than the arc BC.

PROPOSITION XX

Let AB [*Fig. 27*] *be an arc of a circle which is less than a semicircumference and which is cut into any number of parts by parallel straight lines. Let there be an equal distance between the parallels and between the mutually parallel lines drawn through the ends of the arcs; namely, CD, EF, GH, KL, etc. Draw tangents to the circumference in the same direction at*

the point A at the end of one arc and at all the other points of the other sections, and let each tangent cut the next parallel line, as do the tangents AC, DE, FG, HK, etc. Now I say that these tangents taken together, but minus the first one AC, are less than the given arc AB; and that these same tangents, including AC, are greater than the arc AB decreased by its last part NB, that is, greater than the arc AN.

[Fig. 27]



First let us grant that some of the parallels are located on both sides of the center Z. Among those located on the side B, let GH be either the closest to the center or let it pass through the center. Thus each of the tangents located between GH and BO, that is, HK, LM, and NO, is less than its corresponding arc [Proposition 16]. Also the tangent GF is less than the next arc FD [Proposition 17], and likewise the tangent ED is less than the arc DA. Thus all the tangents located between BO and CD are less than the arcs BH plus FA, and hence are clearly less than the arcs BH plus HA, or less than the arc BA. Q.E.D.

Next we will prove that all the tangents between BO and A are larger than the arc AN. For the parallel GH will be either closer to the center Z than is the parallel EF, which I assume is the closest to the center of the parallels on the side of A, or it is further away, or it is equidistant.

Now if EF is further from the center than GH, or is equidistant, then the tangent FG will be larger than its arc FH, and the remaining tangents on the side A, that is, ED and CA, will also each be larger than their arcs [Proposition 18]. And thus the sum of GF, ED, and CA is larger than the arc HA. But the tangent LM will be larger than the arc HL [Proposition 19], and the tangent NO larger than the arc LN, and thus the sum of all the tangents, except HK, will be larger than the arc AN. Therefore very clearly, when HK is added, all the tangents located between A and B will be larger than the arc AN.

But if GH is further from the center than EF, then the tangent KH will

be larger than the arc HF [Proposition 19], the tangent ML as before is larger than the arc LH, and the tangent ON is larger than the arc HL; and thus the sum of the tangents ON, ML and KH is larger than the arc NF. And the tangent ED is larger than its own arc FD [Proposition 18], and the tangent CA likewise is larger than its arc DA. Thus the sum of the tangents between BO and A, excepting GF, is larger than the arc NA. Thus clearly, when GF is added, these tangents, that is, all which are located between BO and A, will be larger than the arc NA.

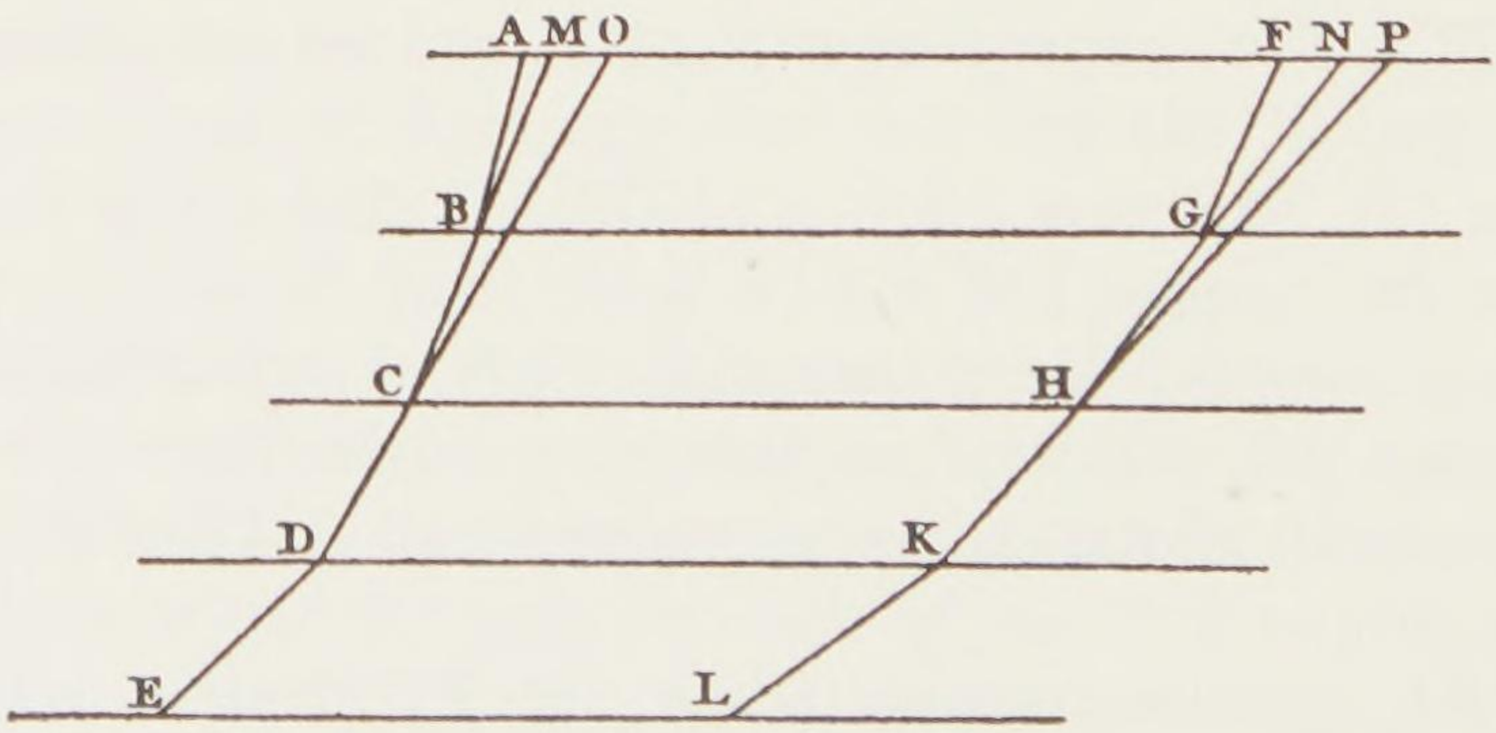
From this the demonstration in other cases is clear, no mattter what arc of a semicircumference is taken, for it will be either the same as, or only a part of, the preceding demonstration.

PROPOSITION XXI

Let a body descend by a continuous motion through any number of contiguous planes, and later let it descend from the same height through another series of an equal number of contiguous planes. Let the latter series be constructed in such a way that each plane corresponds in height to another plane in the first series, but let the planes in the second series have a larger inclination than those in the first series. Now I say that the time of descent through the less inclined planes will be less than the time of descent through the more inclined planes.

Let there be two series of planes ABCDE and FGHKL located between the same horizontal parallels [Fig. 28]. Let these planes correspond to each other two by two, and let the planes of each series be contained within the same horizontal parallels. Also let each plane of the series FGHKL be more inclined to the horizontal than the plane of corresponding height in the series ABCDE. Now I say that the descent through ABCDE will occur in a shorter time than the descent through FGHKL.

[Fig. 28]



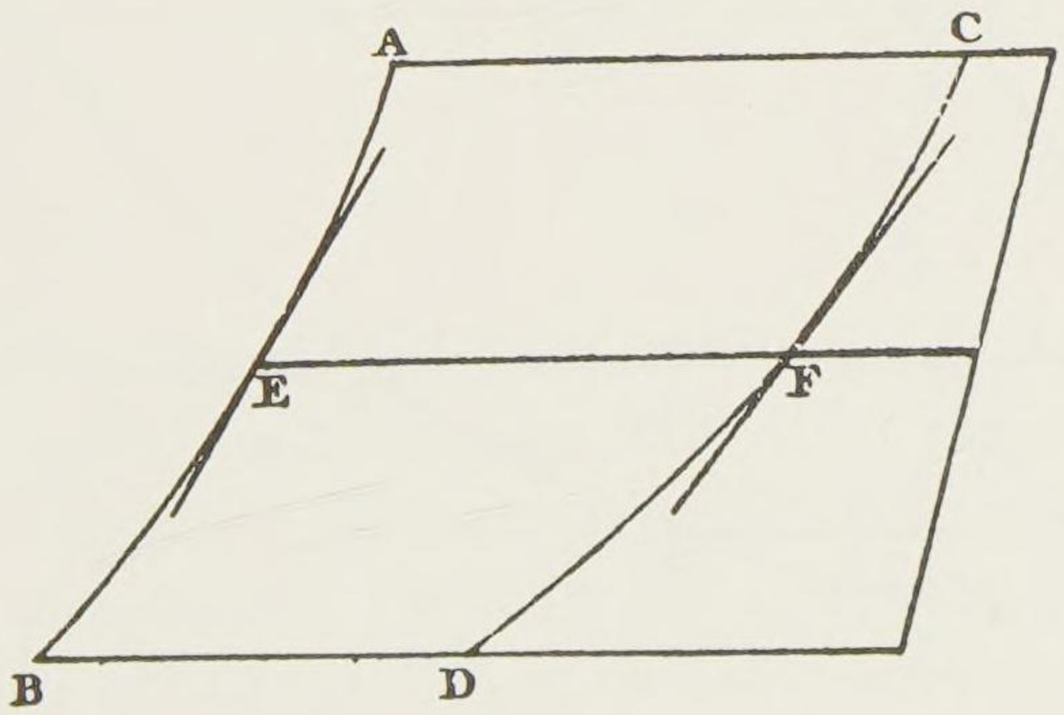
First it is clear that the time of descent through AB is shorter than the time of descent through FG, since the ratio of these times is equal to the ratio of AB to FG [Proposition 7], and AB is less than FG because of its smaller inclination. Next let the lines CB and HG be extended upwards, and let them cut the horizontal AF at M and N. Now the time through BC after AB is equal to the time through this same BC after MB because at the point B the body has the same velocity whether it descended through AB or through MB [Proposition 6]; and likewise the time through GH after FG is equal to the time through this same GH after NG. But the time through BC after MB is related to the time through GH after NG as BC is related to GH in length, or as CM to HN. For the same ratio is found between the times through the whole lengths MC and NH, and through the partial lengths MB and NG [Proposition 7], and hence also between the remaining times. But BC is less than GH because of its smaller inclination. Therefore it is clear that the time through BC after MB or after AB is shorter than the time through GH after NG or after FG.

By extending DC and KH upwards until they cut the horizontal AF at O and P, it is proven in the same way that the time through CD after ABC or after OC is shorter than the time through HK after FGH or after PH. And finally the time through DE after ABCD is shorter than the time through KL after FGHK. Therefore the whole time of descent through ABCDE will be shorter than the time of descent through FGHKL. Q.E.D.

Next, if we consider curved lines to be composed of innumerable straight lines, then from the above it is clear that if there be two inclined surfaces which correspond to curved lines of the same height, and if at every point of the same height the inclination of the one surface is always larger than the other, then a body will fall through the less inclined surface in a shorter time than through the greater inclined surface.

For example, let there be two inclined surfaces [Fig. 29] corresponding to the curves AB and CD, which have the same height, and which are such that at any points E and F of equal height CD is more inclined than AB,

[Fig. 29]



that is, the tangent to the curve at F is more inclined to the horizontal than the tangent to the curve at E. Then the time of descent through AB will be shorter than through CD.

The same thing occurs if one of the lines is a straight line, provided that the inclination of the straight line, which is the same everywhere, is larger or smaller than the inclination of the curve at every one of its points.

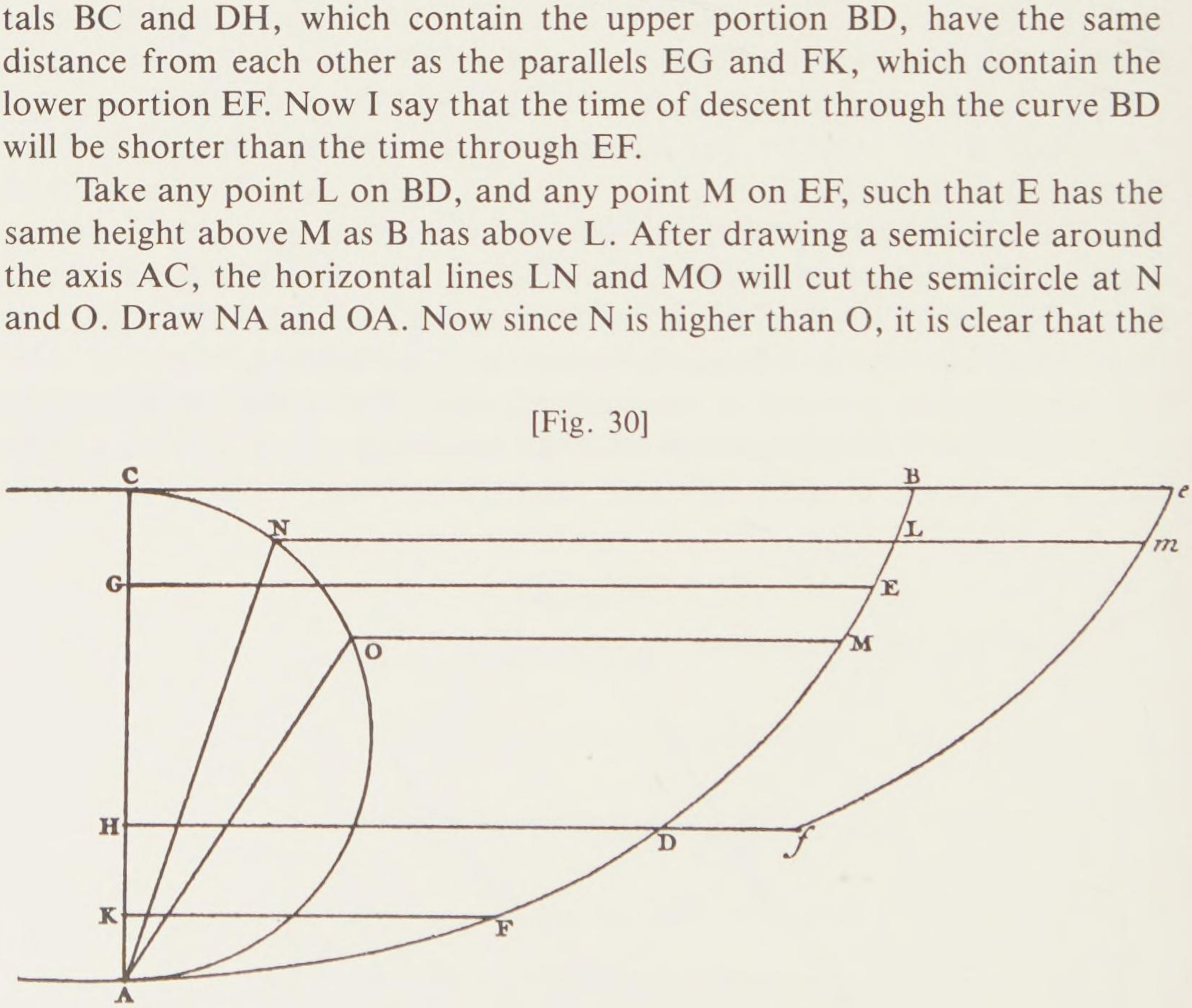
PROPOSITION XXII

Let there be a cycloid whose axis is erected as a perpendicular with its vertex located at the bottom. And let there be taken two portions of the curve having equal height but with the one being closer than the other to the vertex. Then the time of descent through the upper portion of the curve will be shorter than the time through the lower portion.

Let AB be a cycloid whose axis AC is erected on the perpendicular and whose vertex A is located at the bottom [Fig. 30]. On the cycloid mark off the portions BD and EF having equal heights, that is, the parallel horizontals BC and DH, which contain the upper portion BD, have the same distance from each other as the parallels EG and FK, which contain the lower portion EF. Now I say that the time of descent through the curve BD will be shorter than the time through EF.

Take any point L on BD, and any point M on EF, such that E has the same height above M as B has above L. After drawing a semicircle around the axis AC, the horizontal lines LN and MO will cut the semicircle at N and O. Draw NA and OA. Now since N is higher than O, it is clear that the

[Fig. 30]

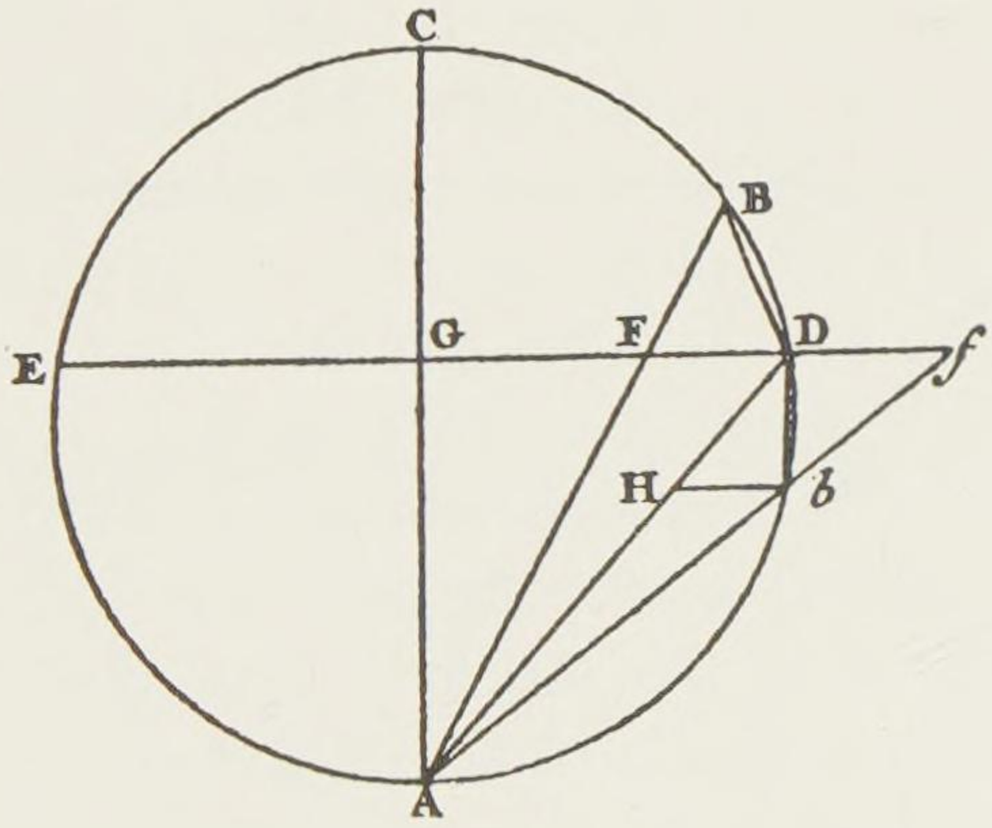


line NA is less inclined to the horizontal than is OA. But the tangent to the curve at L is parallel to NA [Proposition 15], and the tangent to the curve at M is parallel to OA. Therefore the curve BD at the point L is less inclined than the curve EF at the point M. Now let the portion EF, its inclination remaining invariant, be understood to be moved higher to the position *ef,* such that it is contained between the same parallels as the portion BD. Then the point M will be found at *m*, at a height equal to the point L. Therefore the inclination of the curve *ef* at the point *m,* which is the same as the inclination of the curve EF at M, will be larger than the inclination of the curve BD at L. In the same way it is shown that, at any other point of the curve *ef,* the inclination is greater than the inclination at an equally high point in the curve BD. Hence the time of descent through BD will be shorter than the time through *ef* [Proposition 21], or which is the same thing, through EF. Q.E.D.

LEMMA

Let AC [*Fig. 31*] *be the diameter of a circle which cuts the line DE at right angles. From the end A of the diameter draw the line AB which cuts the circumference at B and the line DE at F. Now I say that these three lines, AB, AD, and AF, are proportional.*

[Fig. 31]



First assume that the point of intersection F lies within the circle; and draw the cord corresponding to the arc BD. Since the arcs AE and AD are equal, the angles EDA and ABD, which are contained in these portions of the circumference, are equal. Thus in the triangles ABD and ADF, the angles ABD and ADF are equal. But each has a common angle at A. Hence

the indicated triangles will be similar, and therefore BA is related to AD as AD is to AF.

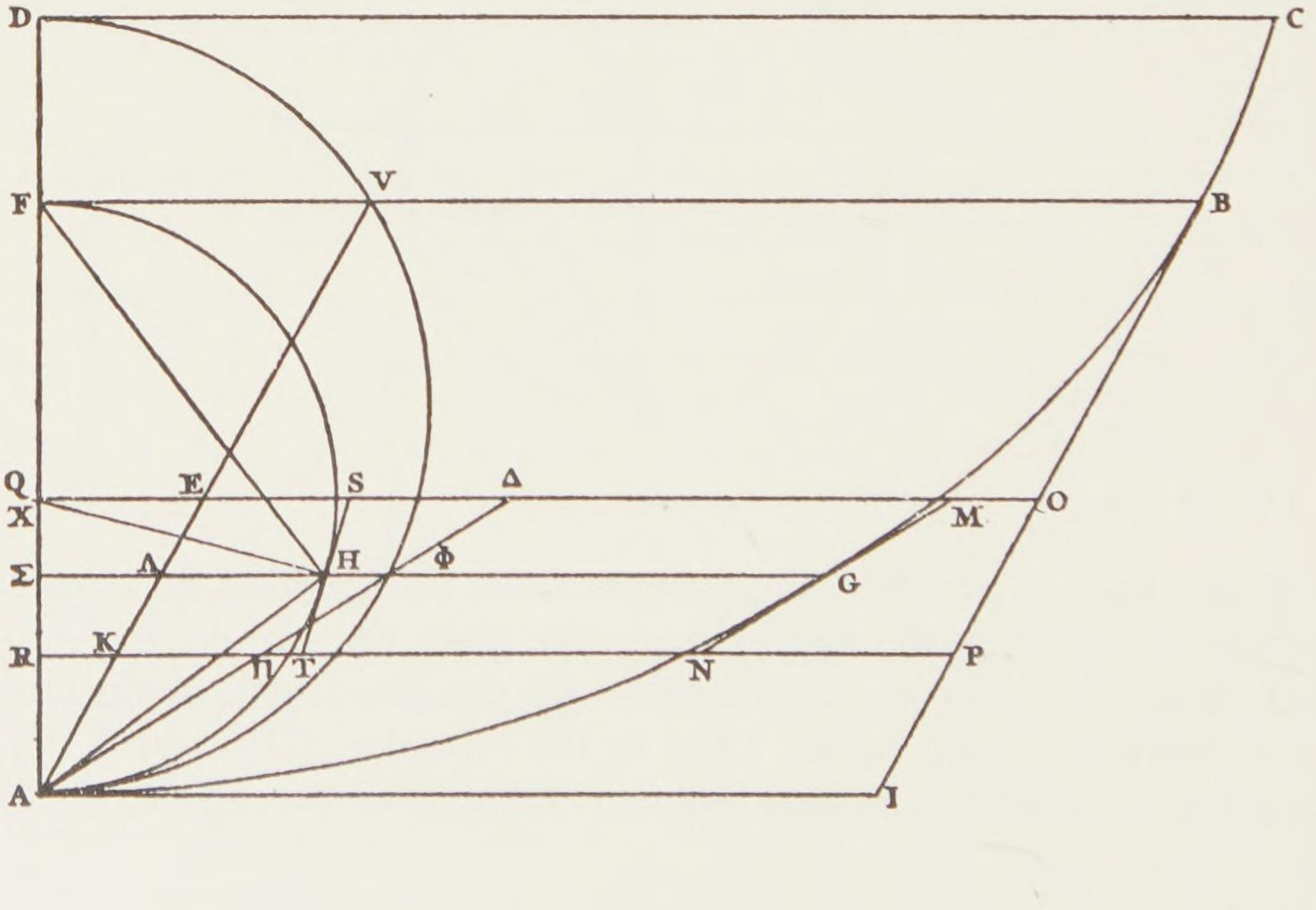
But if the point of intersection *f* lies outside the circle, draw *b*H parallel to DE. This line cuts AD as H. Thus, according to what was just demonstrated, DA will be related to A*b* as A*b* is to AH, that is, as A*f* to AD. Therefore again the lines A*f*, AD, and A*b* will be proportional. Thus the proposition is established.

PROPOSITION XXIII

Let ABC [*Fig. 32*] *be a cycloid whose vertex A is located at the bottom and whose axis AD is erected on the perpendicular. Select any point B on the cycloid, and draw from it downwards the line BI which is tangent to the cycloid and which ends at the horizontal line AI. Draw BF perpendicular to the axis. Divide FA in half at X and draw on it the semicircle FHA. Next through any point G on the curve BA draw the line ΣG parallel to BF. This line cuts the circumference FHA at H and the axis AD at Σ. Through the points G and H imagine that tangents are drawn to each curve, and let MN and ST be parts of these tangents intercepted by the two horizontal lines MS and NT. Between these same lines MS and NT are contained the part OP of the tangent BI and the part QR of the axis DA.*

Given this construction, I say that the time in which a body crosses the line MN, with the uniform velocity acquired after it has fallen through the

[Fig. 32]



arc BG of the cycloid, will be related to the time in which it would cross the line OP, with half of the uniform velocity which it would acquire by falling through the whole tangent BI, as the tangent ST is related to the part QR of the axis.

On the axis AD draw the semicircle DVA, which cuts BF at V and ΣG at Φ. Draw AV which cuts OQ at E, PR at K, and GΣ at Λ. Draw HF, HA, HX, and AΦ, the latter of which cuts OQ at Δ and PR at Π.

The previously mentioned time through MN is related to the time through OP by a ratio composed of the ratio of these same lines MN to OP plus the inverse ratio of the velocities by which they are crossed,[10] this latter being the ratio of half of the velocity from BI or from FA to the velocity from BG or from FΣ [Proposition 8]. But the whole velocity from FA is related to the velocity from FΣ as the ratio of the square roots of the lengths FA to FΣ [Proposition 3], and thus the same as FA to FH. Therefore half the velocity from FA will be related to the velocity from FΣ as FX to FH. Thus the indicated time through MN is related to the time through OP by a ratio composed of the ratios of MN to OP and of FX to FH. But we will show that the first of these ratios, that is, MN to OP, is the same as FH to HΣ.

For the tangent BI of the cycloid is parallel to VA, and likewise the tangent MGN is parallel to ΦA. And thus MN equals ΔΠ, and OP equals EK. Therefore the indicated ratio of MN to OP is the same as ΔΠ to EK; that is, as ΔA to EA; that is, as ΦA to ΛA; that is, as VA to ΦA [preceding Lemma]. But VA is to AΦ as FA is to AH. For since the square of VA equals the rectangle DAF, and since the square of AΦ equals the rectangle DAΣ, which rectangles are related to each other as FA to ΣA, that is, as the square of FA to the square of AH, then from this the square of VA is to the square of ΦA as the square of FA is to the square of AH, and also VA is to AΦ in length as FA is to AH. Thus the ratio of MN to OP is equal to the ratio of FA to AH, that is, it is the same as the ratio of FH to HΣ, as was said, because FAH and FHΣ are similar triangles. Thus the indicated ratio of time through MN to the time through OP is composed of the ratios of FX to FH and of FH to HΣ, and therefore will be the same as the ratio of FX or XH to HΣ. But the radius XH is to HΣ as the tangent ST is to the line QR, as is easily seen. Therefore, it is clear that the time of motion which we mentioned through MN is related to the time through OP as ST to QR. Q.E.D.

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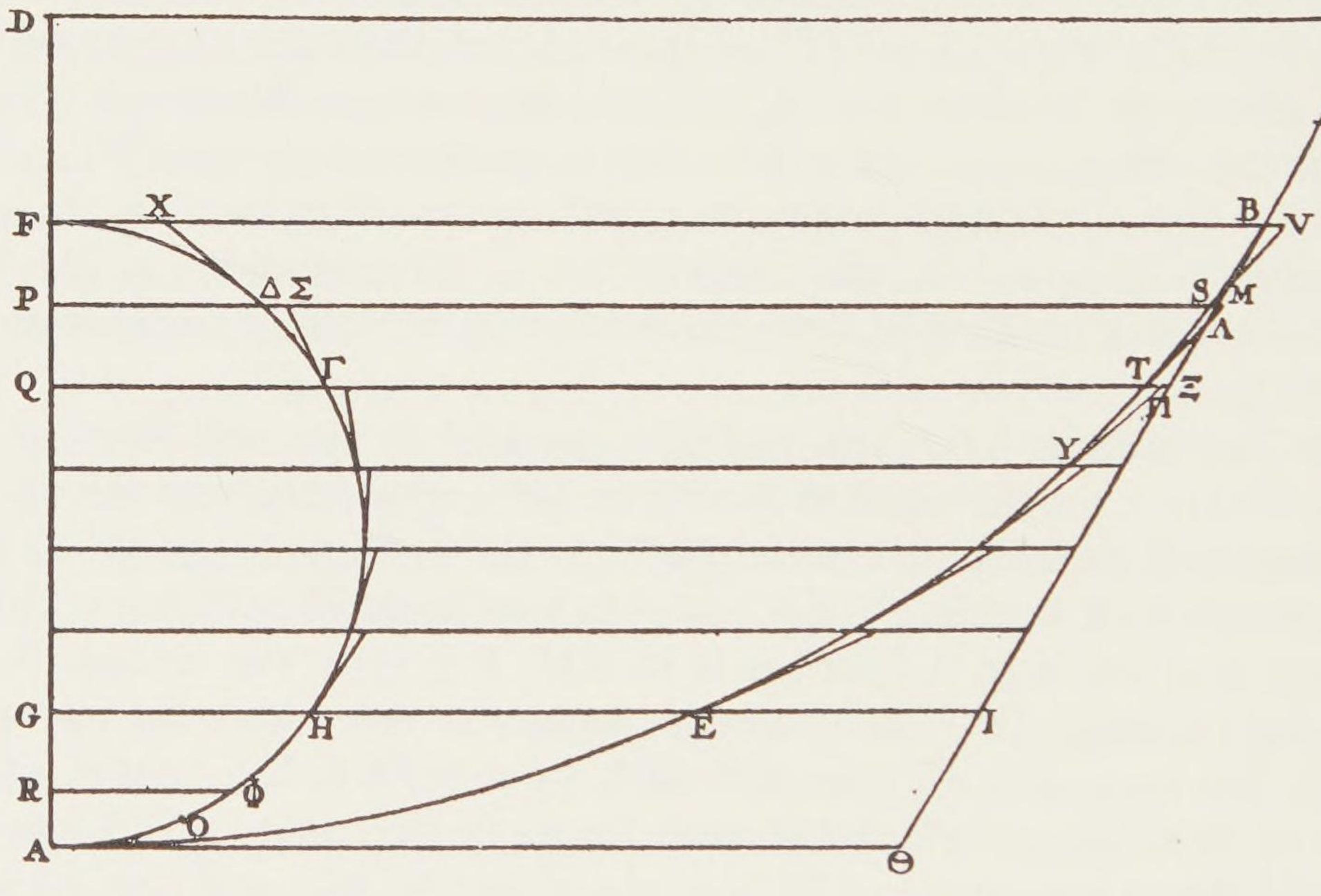
10. In the 1673 edition Huygens included a reference at this point to Galileo, *Discourses Concerning Two New Sciences.* Third Day. Proposition 5, "On Equable Motion" (*Le Opere di Galileo Galilei* 8:195). "If two moveables are carried in equable motion but with unequal speeds, and unequal spaces are run through, then the ratio of the times will be compounded from the ratio of the spaces and from the inverse ratio of speeds." [S. Drake's translation, 151.]

PROPOSITION XXIV

Again as in the preceding proposition, let ABC [*Fig. 33*] *be a cycloid whose vertex A is located at the bottom and whose axis AD is erected on the perpendicular. Having taken any point B on the cycloid, draw from it downwards the straight line BΘ which is a tangent to the cycloid and which cuts the horizontal AΘ at Θ. Draw BF perpendicular to the axis, and around FA draw the semicircle FHA. Finally draw another straight line GE, parallel to FB, which cuts the cycloid at E, the line BΘ at I, the circumference FHA at H, and the axis DA at G.*

[Fig. 33]



Now I say that the time of descent through the arc BE of the cycloid is related to the time of descent through the tangent BI with half the velocity acquired from a fall through BΘ as the arc FH is related to the straight line FG.

For if this is not true, then the ratio of the time through the arc BE to the indicated time through BI will be either greater or smaller than the ratio of the arc FH to the straight line FG. Let us assume first that it is greater, if that be possible.

If so, then there will be some time shorter than the time through BE (let this be time Z) which will be related to the indicated time through BI as the arc FH is related to the straight line FG. Now if we take some other point N above B on the cycloid, then the time through BE after NB will be

shorter than the time through BE. But it is clear that the point N can be taken so close to B that the difference of these times is as small as one wishes, and hence would be less than the time by which Z is exceeded by the time through BE. Let the point N be so chosen. It follows then that the time through BE after NB will be larger than the time Z, and thus will have a ratio to the indicated time through BI, with half the velocity from BΘ, which is larger than the ratio of the arc FH to the line FG. Hence this ratio would be equal to the ratio of the arc FHO to the line FG.

Divide FG into the equal parts FP, PQ, etc., each of which is less than the height of the line NB and also less than the height of the arc HO. It is clear that this can be done. Next from the points of these divisions draw the straight lines PΛ, QΞ, etc., parallel to the base DC and ending at the tangent BΘ. From the points where these lines cut the circumference FH, and also from H, draw the tangents ΔX, ΓΣ, etc., upwards as far as the next parallel. Likewise, from the points where the indicated parallels cut the cycloid, draw the tangents SV, TM, etc., upwards. After adding to the line FG one part GR equal to the parts of the division, draw RΦ parallel to DC. It is clear that this line cuts the circumference FHA between H and O because GR is less than the height of the point H above O. We will next argue as follows.

The time through the tangent VS with the uniform velocity acquired from BS is larger than the time of a continuously accelerated motion through the arc BS after NB. For the velocity from BS is less than the velocity from NB because the height of BS is less than the height of NB. But the velocity from BS is assumed to be uniformly continued through the tangent VS since the velocity acquired from NB is continuously accelerated through the arc BS, and this arc is less than the tangent VS, and all of its parts are more vertical than any part of the tangent VS. Therefore the time through the tangent VS with the velocity from BS will be clearly larger than the time through the arc BS after NB. Likewise the time through the tangent MT with the velocity from BT will be larger than the time through the arc ST after NS. And the time through the tangent ΠY with the velocity from BY will be larger than the time through the arc TY after NT. And thus the sum of the times of the uniform motions through all the tangents down to the lowest one which touches the cycloid at E, with velocities through each one being equal to the velocity acquired by falling from B up to these points of contact, will be larger than the time through the arc BE after NB. But these same times will also be smaller, as we will now show.

Consider again these same times of the uniform motions through the tangents of the cycloid. The time through the tangent VS with the velocity from BS is related to the time through the line BΛ, with half the velocity from FA, as the tangent to the circumference ΔX is related to the part FP of the axis [Proposition 23]. Likewise the time through the tangent MT with

the velocity from BT is related to the time through the line ΛΞ, with half the velocity from FA, as the tangent ΓΣ is related to the line PQ. And so forth, each of the times through tangents of the cycloid, which are the same as mentioned above, will be related to the times of uniform motion through the corresponding parts of the line BI, with half the velocity from BΘ, as the tangents to the circumference FH, contained between the same parallels, are related to the corresponding parts of the line FG.

Therefore, we have certain quantities in the straight lines FP, PQ, etc., and an equal number of other quantities in the times in which the lines BΛ, ΛΞ, etc., are crossed with a uniform motion with half the velocity from BΘ. And each quantity in the former series is related to its successor in the same proportion that each quantity of the second series is related to its successor. For the quantities in each series are equal to each other. Furthermore the quantities in the first series are related to other quantities; namely, to the tangents to the circle ΔX, ΓΣ, etc., in the same proportions and in the same order as the quantities in the second series are related to other quantities; namely, to the times of the indicated motions through the tangents of the cycloid VS, MT, etc. Therefore, the sum of all the first quantities is related to the sum of the quantities to which they are related, that is, the whole line FG is related to all the tangents XΔ, ΓΣ, etc. as the time in which the whole of BI is crossed, with half the velocity from BΘ, will be related to the sum of the times of all the indicated motions through the tangents of the cycloid VS, MT, etc.[11] Thus, by inversion, the ratio of the times of the indicated motions through the tangents of the cycloid to the time through the line BI, with half the velocity from BΘ, will be the same as the ratio of all the indicated tangents to the circumference FH to the straight line FG. But this ratio is less than the ratio of the arc FO to the same line FG, because the arc FΦ, and thus also the arc FO, is larger than all of the indicated tangents to the arc FH [Proposition 20]. But the time through BE after NB was assumed to be related to the time through BI, with half the velocity from BΘ, as the arc FO is related to the straight line FG. Therefore the sum of the indicated times through the tangents of the cycloid will be less than the time through BE after NB, while previously it was shown that it is larger. This is absurd. Therefore the time through the arc BE of the cycloid has a ratio to the time through the tangent BI, with half the velocity from BΘ or from FA, which is not larger than the ratio of the arc FH of the circumference to the straight line FG.

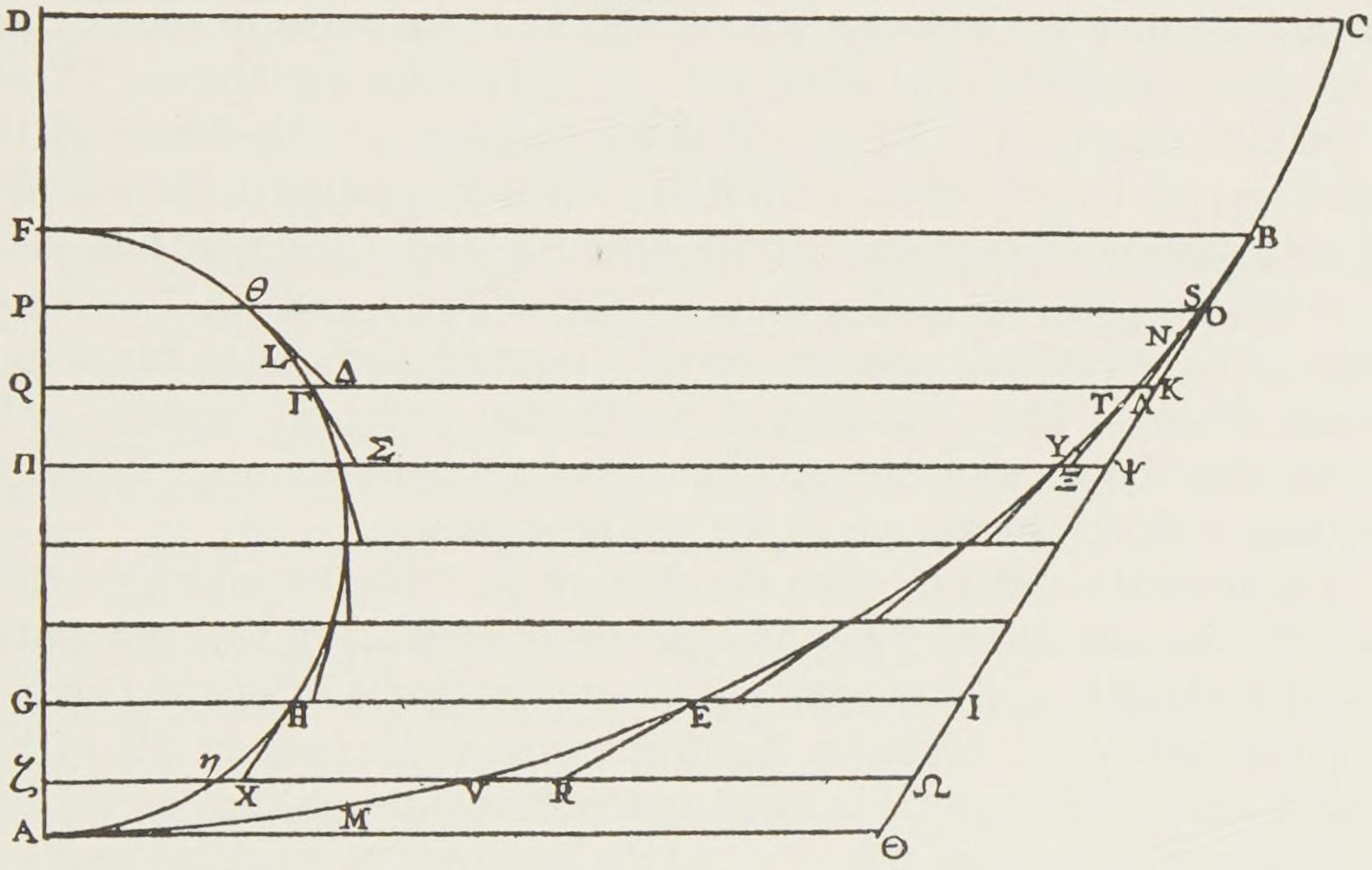
Next, assume that this ratio is smaller, if that be possible. If so, there will be some time greater than the time through the arc BE (let this time be

11. In the 1673 edition Huygens included a reference at this point to Proposition 2 of Archimedes' *On Spheroids and Conoids.* See T. L. Heath, ed. *The Works of Archimedes* (New York: Dover, n.d.) 110.

Z) which will be related to the indicated time through BI as the arc FH is related to the straight line FG.

Let the arc NM [Fig. 34] be equal in height to the arc BE, and let its upper terminus point N be lower than the point B. Then the time through the arc NM will be shorter than the time through the arc BE [Proposition 22]. Now it is clear that the point N can be taken so close to the point B that the difference between these times is as small as one wishes, and hence would be less than the time by which Z exceeds the time through the arc BE. Let the point N be so chosen. On this basis the time through NM will be shorter than the time Z, and will have a ratio to the indicated time through BI, with half the velocity from BΘ, which is less than the ratio of the arc FH to the straight line FG. Hence this ratio would be equal to the ratio of the arc LH to the straight line FG.

[Fig. 34]



Divide FG into equal parts FP, PQ, etc., each of which is less in height than the arc BN of the cycloid and also less in height than the arc FL of the circumference. After adding to FG one more of these parts Gζ, draw from the points of division the lines PO, QK, etc., which are parallel to the base DC and which terminate at the tangent BΘ. Next from the point ζ draw the straight line ζΩ which cuts the cycloid at V and the circumference at η. From each of the points where the drawn parallels cut the circumference FH, draw tangents downwards as far as the next parallel, for example, ΘΔ, ΓΣ. From the lowest of these points H draw the tangent which cuts the line

ζΩ at X. Likewise, from the points where the indicated parallels cut the cycloid, draw the same number of tangents downwards, for example, SΛ, TΞ. The lowest of these, namely, the tangent drawn from the point E, cuts the line ζΩ at R.

Now Pζ is equal to FG, which is the height of the arc BE. And the altitude of the arc NM is equal to that of the arc BE from the construction. Thus Pζ will be equal to the altitude of the arc NM. But from the construction the line PO is above the terminal point N. Therefore, the line ζΩ, and in it the point V, is above the terminal point M. Hence, since the arc SV has the same height as the arc NM but its terminus S is higher than N, the time through SV will be shorter than the time through NM [Proposition 22].

But the time through the tangent SΛ with the uniform velocity from BS is shorter than the time of an accelerated fall through the arc ST beginning at S. For the velocity from BS, by which it is assumed that the whole of SΛ is crossed, is equal to the velocity from ST [Proposition 8], which is acquired at the end of a motion through the arc ST; and SΛ is less than ST. Likewise the time through the tangent TΞ, with the uniform velocity from BT, is shorter than the time of an accelerated fall through the arc TY after ST. For the velocity from BT, by which it is assumed that the whole of TΞ is crossed, is equal to the velocity from SY, which is acquired at the end of the indicated motion through the arc TY after ST, and TΞ is less than the arc TY. And thus the sum of the times of the uniform motions through the tangents of the cycloid, with the velocity through each being equal to the velocity acquired by a fall from B up to that point of contact, will be shorter than the time of an accelerated fall through the arc SV. But this same sum will also be longer, as we will now show.

The indicated time through the tangent SΛ with the uniform velocity from BS is related to the time through the line OK, with half the uniform velocity from BΘ, as the tangent ΘΔ of the semicircle is related to the line PQ [Proposition 23]. Likewise the time through the tangent TΞ with the uniform velocity from BT is related to the time through the line KΨ, with half the uniform velocity from BΘ, as the tangent ΓΣ is related to the line QΠ. And thus successively each of the times through the tangents of the cycloid, which are the same as mentioned above, will be related to the times of uniform motion through the corresponding parts of the line OΩ, with half the velocity from BΘ, as the tangents to the circumference Θη, contained within these same parallels, are related to the corresponding parts of the line Pζ. Hence, as in the first part of this demonstration, we conclude that the sum of the lines PQ, QΠ, etc., that is, the whole line Pζ, is related to the sum of the tangents ΘΔ, ΓΣ, etc., as the time in which the whole of OΩ is crossed, with half the velocity from BΘ, is related to the sum of the times of the indicated motions through the tangents of the cycloid SΛ, TΞ, etc. And thus, by conversion, the sum of the times through the tangents to

the cycloid will have the same ratio to the indicated time of uniform motion through the line OΩ, or through the line BI, as the sum of the indicated tangents to the arc Θη has to the straight line Pζ or FG. As a result this ratio of the times is larger than the ratio of the arc LH to the straight line FG. For the arc ΘH, and thus clearly also the arc LH, is less than the indicated tangents of the arc Θη [Proposition 20]. But we postulated at the beginning that the time through NM has the same ratio to the time through BI as the arc LH has to the straight line FG. Therefore the time through NM, and much more the time through SV, will be less than the time through the tangents to the cycloid. But this is absurd, since this time was shown above to be less than the time through the arc SV. Therefore it is clear that the ratio of the time through the arc BE of the cycloid to the time through the tangent BI, with half the uniform velocity from BΘ, is not less than the ratio of the arc FH to the straight line FG.

But it was also shown that this ratio is not larger. Therefore it must be the same. Q.E.D.

PROPOSITION XXV

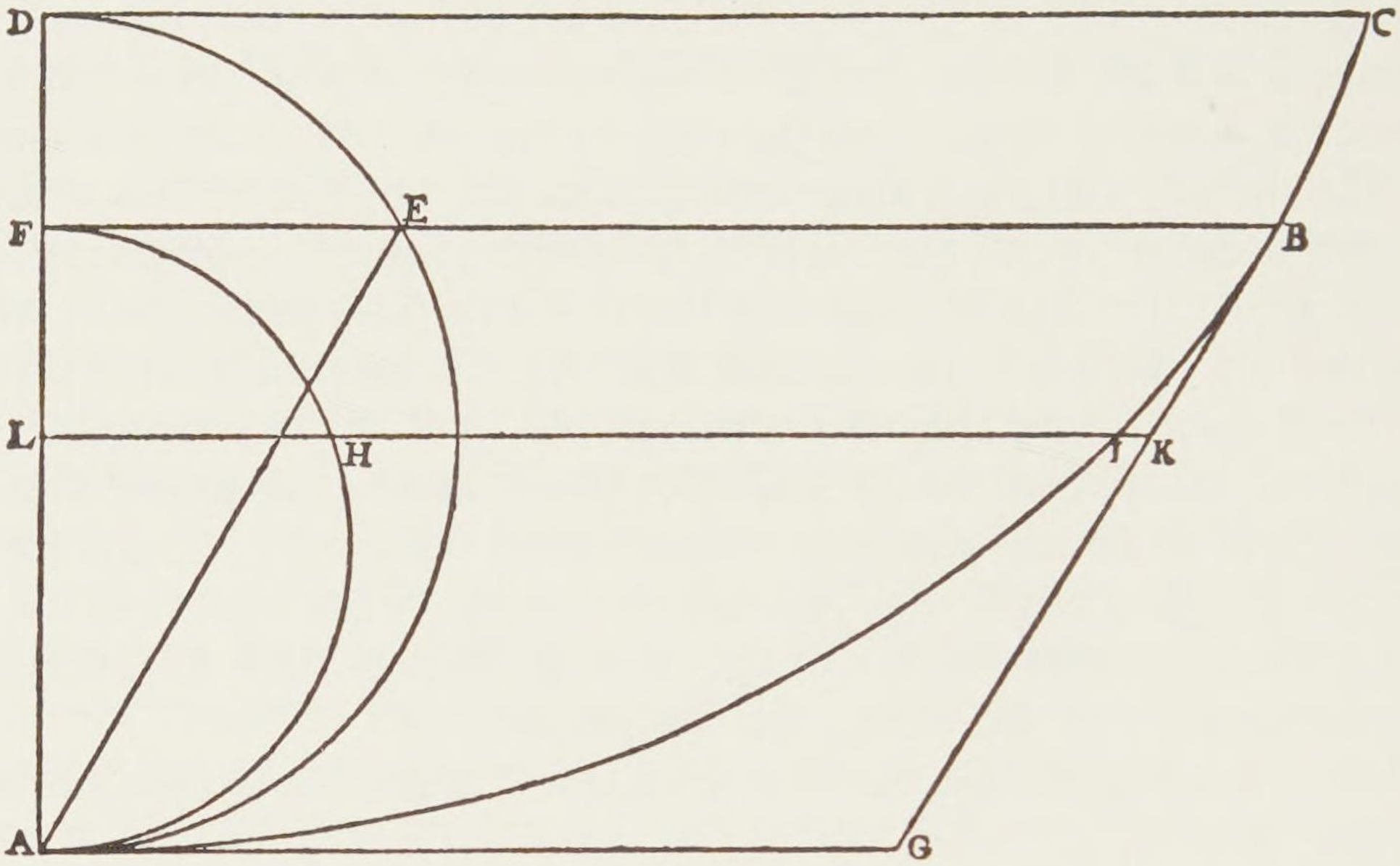
On a cycloid whose axis is erected on the perpendicular and whose vertex is located at the bottom, the times of descent, in which a body arrives at the lowest point at the vertex after having departed from any point on the cycloid, are equal to each other; and these times are related to the time of a perpendicular fall through the whole axis of the cycloid with the same ratio by which the semicircumference of a circle is related to its diameter.

Let ABC [Fig. 35] be a cycloid whose vertex A is located at the bottom and whose axis AD is erected on the perpendicular. Select any point on the cycloid, for example B, and let a body descend by its natural impetus through the arc BA, or through a surface so curved. Now I say that the time of this descent is related to the time of a fall through the axis DA as the semicircumference of a circle is related to its diameter. When this has been demonstrated, it will also be established that the times of descent through all arcs of the cycloid terminating at A are equal to each other.

On the axis DA draw a semicircle whose circumference cuts the straight line BF, which is parallel to the base DC, at E. Draw the line EA, and parallel to it draw the line BG which is tangent to the cycloid at B. BG will cut the horizontal straight line drawn through A at G. Also on FA draw the semicircle FHA.

Now according to the preceding proposition, the time of descent through the arc BA of the cycloid is related to the time of a uniform motion through the line BG, with half the velocity from BG, as the semicircular arc

[Fig. 35]



FHA is related to the straight line FA. But the time of the indicated uniform motion through BG is equal to the time of a naturally accelerated fall through BG, or through EA which is parallel to it, and thus is equal to the time of an accelerated fall through the axis DA.[12] Hence the time through the arc BA will be related to the time of a fall through the axis DA as the semicircular circumference FHA is related to the diameter FA. Q.E.D.

Now if it is assumed that the whole cavity of the cycloid has been constructed, it is clear that, after a body has descended through the arc BA, it will continue its motion and ascend through another arc equal to BA [Proposition 9], and the same amount of time will be consumed in the ascent as in the descent [Proposition 11]. After this, it will return again through A to B; and the times of each of these oscillations, completed in either large or small arcs of the cycloid, will be related to the time of a perpendicular fall through the axis DA as the whole circumference of a circle is related to its diameter.

12. In the 1673 edition Huygens included a reference at this point to Galileo, *Discourses Concerning Two New Sciences*. Third Day. Proposition 6, Theorem 6, "On Naturally Accelerated Motion" (*Le Opere di Galileo Galilei* 8:221). "If, from the highest or lowest point of a vertical circle, any inclined planes whatever are drawn to its circumference, the times of descent through these will be equal." [S. Drake's translation, 178.]

PROPOSITION XXVI

Given the same construction, draw the additional horizontal straight line HI [Fig. 35], which cuts the arc BA at I and the circumference FHA at H. Now I say that the ratio of the time through the arc BI to the time through the arc IA after BI is equal to the ratio of the arc FH of the circumference to HA.

For the line HI cuts the tangent BG at K and the axis DA at L. And thus the time through the arc BA is related to the time of a uniform motion through BG, with half the velocity from BG, as the arc FHA is related to the straight line FA [Proposition 24]. But the time of the indicated uniform motion through BG is related to the time of a uniform motion through BK, with half the same velocity from BG, as BG is related to BK in length, that is, as FA to FL. And further the time of a uniform motion, with the indicated velocity, through BK is related to the time through the arc BI as FL is related to the arc FH [Proposition 24]. Therefore, by equality,[13] the time through the arc BA is to the time through BI as the arc FHA is to FH. And by division and by conversion, the time through BI is to the time through IA after BI as the arc FH is to HA. Q.E.D.

13. This refers to Euclid's *Elements* 5:23. "If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, they will also be in the same ratio *ex aequali.*" [T. L. Heath's translation, 2:181.] See *Oeuvres* 16:396.

PART III

The Evolution and Dimension of Curved Lines

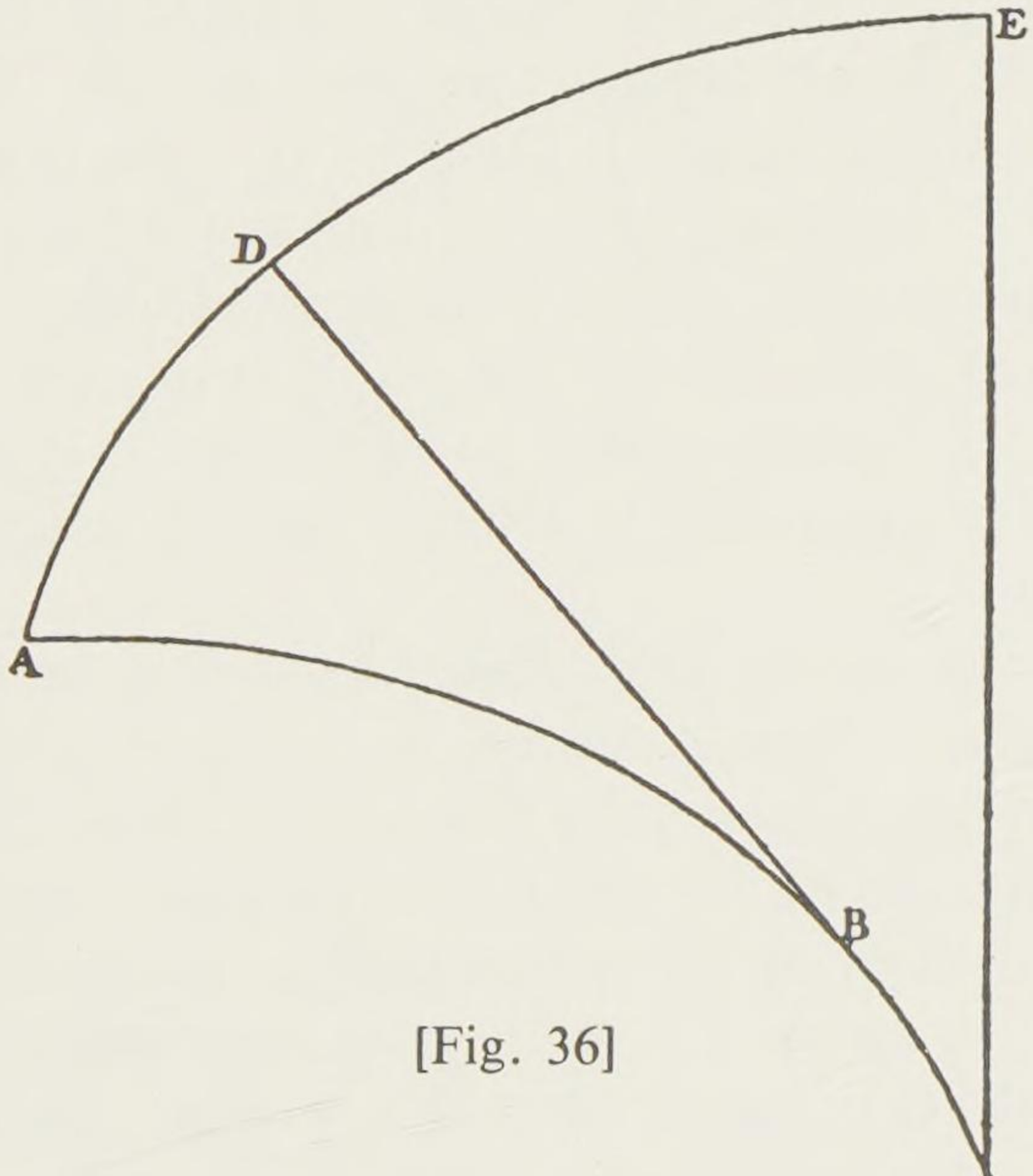
DEFINITIONS

I

A line is said to be curved in one direction when all the tangents touch on the same side. Moreover, if some portions of the curve are straight lines, then these latter when extended become tangents.

II

When two such lines extend from the same point such that the convexity of the one faces the concavity of the other, such as the curves ABC and ADE shown in the figure [Fig. 36], both are said to be curved in the same direction.



[Fig. 36]

For various earlier writings by Huygens on topics in Part III, see *Oeuvres* 17:142–48.

III

If a string or flexible line is understood to be stretched around a line curved in one direction and if one end of the string remains fixed while the other end is pulled away such that the freed part of the line always remains taut, then it is clear that some other curve is described by this end of the string. This latter is called the curve "described by evolution."

IV

The curve around which the string had been stretched is called the evolute. In the above figure [Fig. 36] ABC is the evolute, and ADE is described by evolution from ABC such that when the end of the string arrives at D from A, the taut part is the straight line DB and the remaining part BC is still attached to the curve ABC. It is also clear that DB is tangent to the evolute at B.

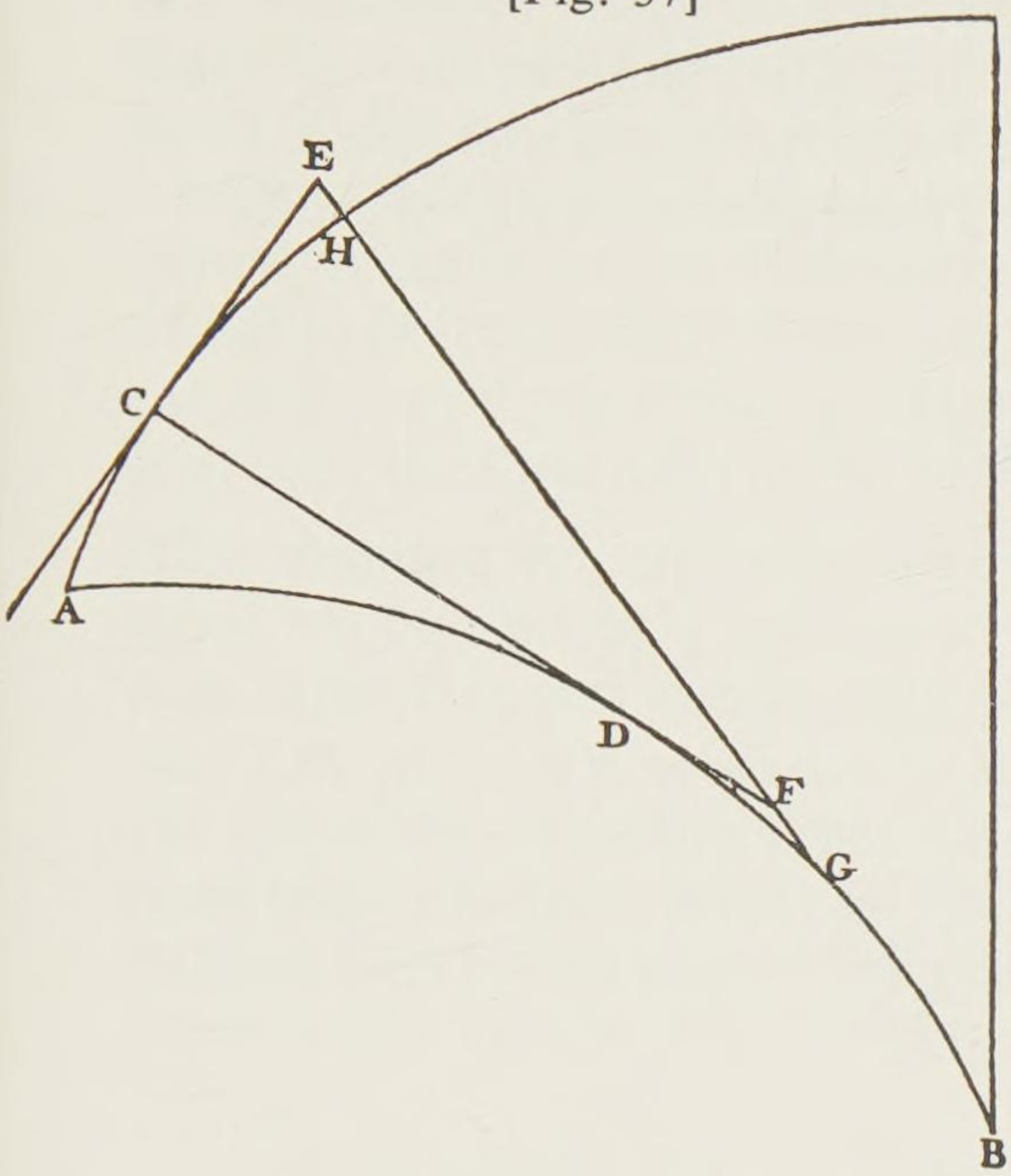
PROPOSITION I

Every straight line tangent to an evolute meets a line described by evolution at right angles.

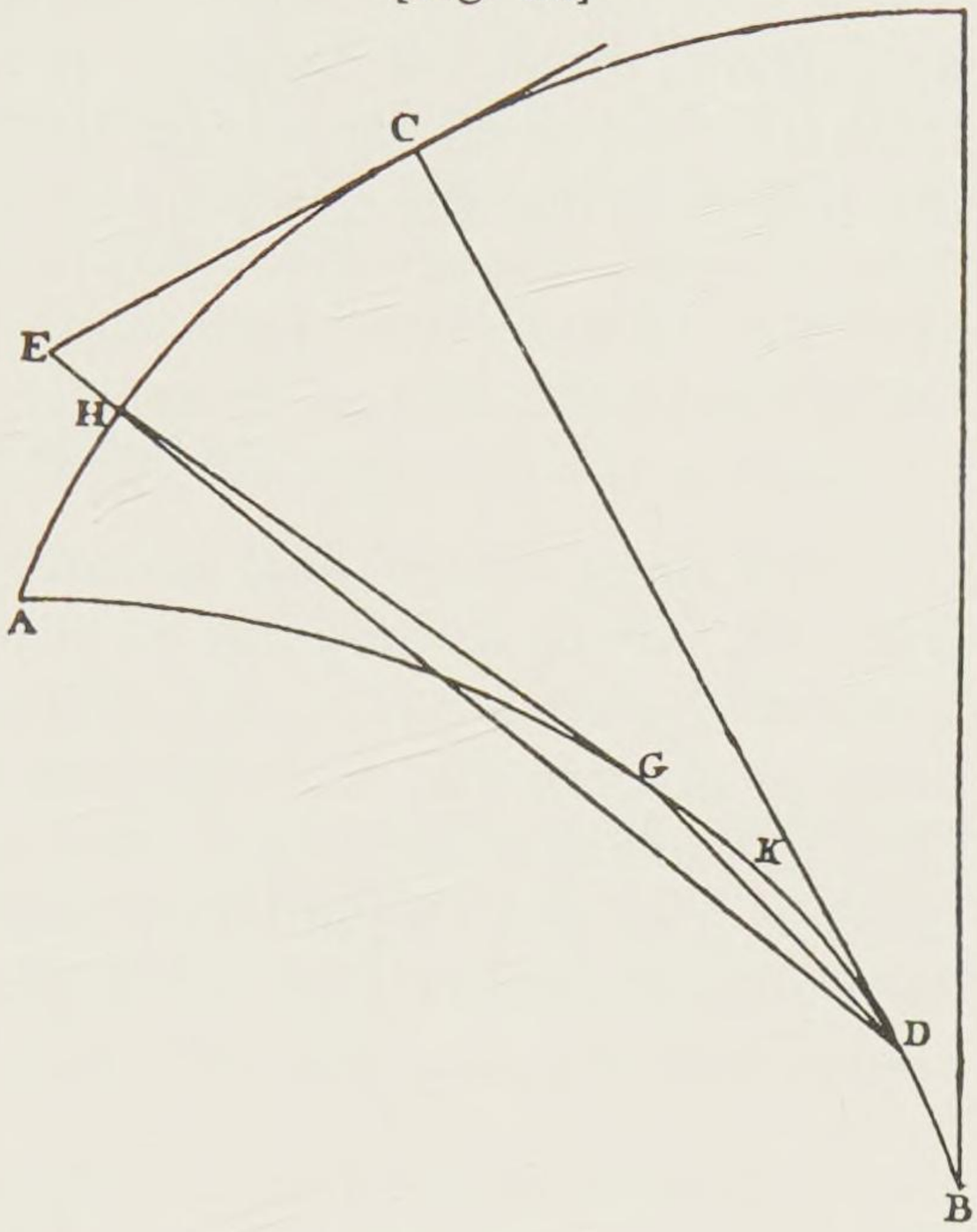
Let AB be an evolute, and let AH be described by its evolution [Fig. 37]. And let the straight line FDC, which is tangent to the curve at D, meet the curve ACH at C. Now I say that they meet at right angles, that is, if the straight line CE is drawn perpendicular to CD, I say that CE is tangent to the curve ACH at C. For since DC is tangent to the evolute at D, it is clear that it represents the position of the string precisely when its end has arrived at C. Thus if we can show that, in the whole remaining description of the curve ACH, the string never extends as far as the straight line CE except at the point C, it will be clear that the straight line CE touches the curve ACH at that same point.

Let H be a point on AC other than C. At first let H be more remote than C from the start of the evolution at A. Let HG be taken as the free part of the string when its end has arrived at H. Then HG is tangent to the line AB at G. Meanwhile as the part CH of the curve was described, the arc DG was evolved. And when CD is extended in the direction of D, it meets HG at some point, let us say F. Let GH meet the straight line CE at E. Now the sum of DF and FG is greater than DG, whether the latter be curved or straight. Thus by adding the straight line DC to each, it follows that the sum of the straight lines CF and FG is greater than the sum of the straight lines CD and DG. But because of the evolution, it is obvious that the

[Fig. 37]



[Fig. 38]



straight line HG is equal to the sum of the straight line CD and the line DG. Hence the sum of CF and FG is greater than the straight line HG, and by subtracting the common element FG, CF is greater than HF. But FE is greater than FC since the angle C of triangle FCE is a right angle. Therefore FE is obviously greater than FH. Hence it follows that on this side of the point C, the end of the string does not extend to the straight line CE.

Next let the point H be closer than the point C to the start of the evolution at A [Fig. 38]. And let HG be the position of the string just when its end is at H. Draw the lines DG and DH, and let the latter meet the straight line CE at E. Now it is apparent that the straight line DG cannot lie in the same direction as HG, and thus HGD is a triangle. The straight line DG is either smaller than DKG, or equal to it if the line DG is part of the evolute. If GH is added to each, then the sum of the lines DG and GH is less than or equal to DKG and GH, or to the line DC which is equal to DKG plus GH. But the line DH is less than the sum of DG and GH. Hence DH is also less than the line DC. But DE is larger than DC because, in the triangle DCE, C is a right angle. Therefore DH is much smaller than DE. Hence the point H, that is, the end of the string GH, is located inside the angle DCE. From this it is clear that between A and C the string does not extend as far as the line CE. Thus CE is a tangent to the curve at C, and hence DC, to which CE was drawn as a perpendicular, meets the curve at right angles. Q.E.D.

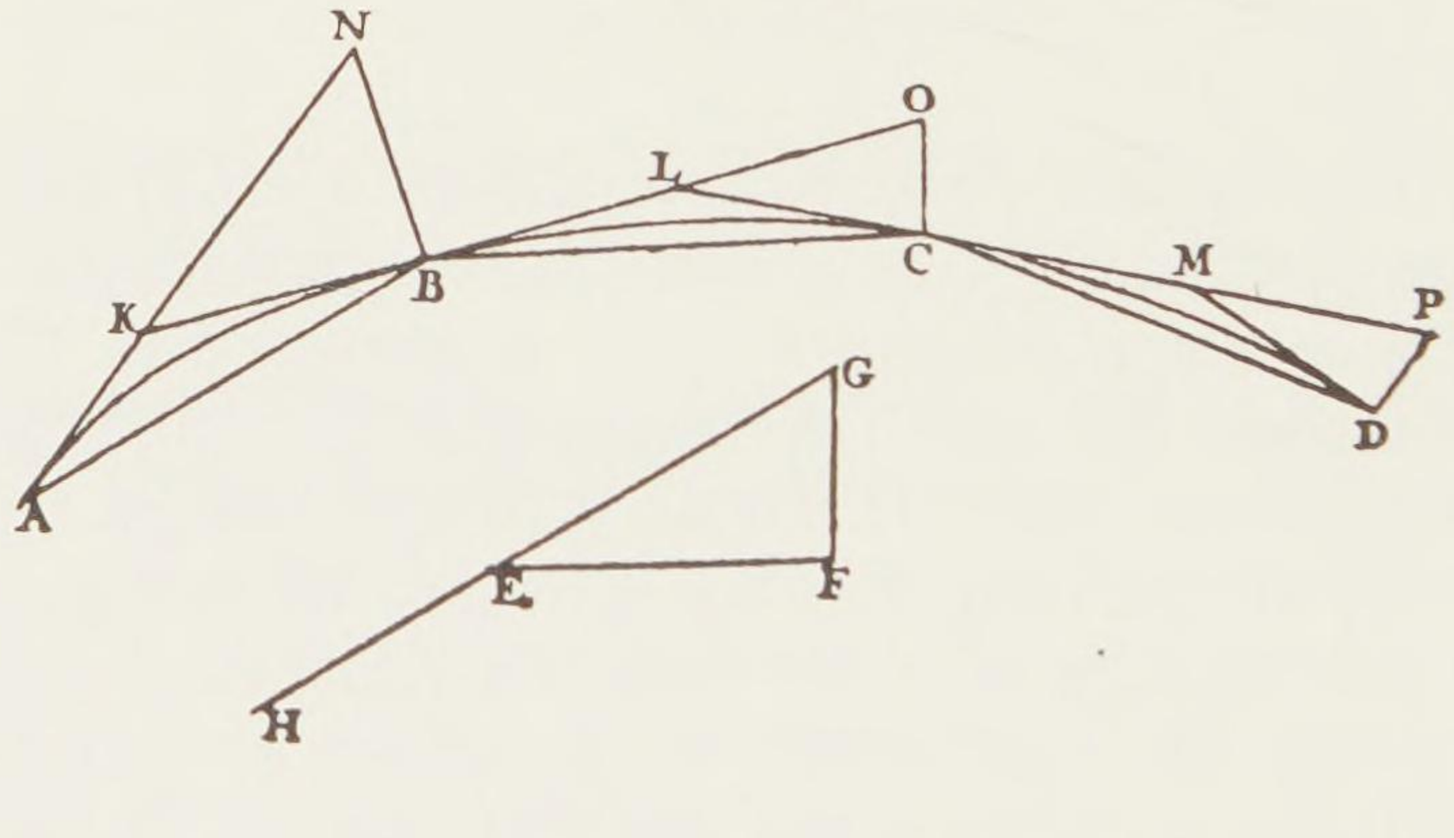
From this it is also clear that AHC is curved in only one direction, and

that it is concave in the same direction as AGB by whose evolution it was described. For all the tangents to the line AHC fall outside the space DGAHC, and all the tangents to the line AGD fall inside that space. From this it is clear that the concavity of AHC faces the convexity of AGD.

PROPOSITION II

Any finite curved line, concave in one direction, for example ABD [*Fig. 39*], *can be divided into so many parts that, if one draws the cords subtending each arc (i.e., AB, BC, and CD), draws tangents from each point of division and from the first end of the curve (i.e., AN, BO, and CP), and draws at each next point of division a normal which meets the tangents (i.e., BN, CO, and DP), then I say that each cord has a ratio to its corresponding normal (i.e., AB to BN, BC to CO, and CD to DP) which is greater than any ratio one may wish to give.*

[Fig. 39]



Let there be given the ratio of lines EF to FG [Fig. 39], which are joined by a right angle at F, and draw the straight line GEH.

First let us assume that the curve ABD is divided at B and C into such small parts that the tangents which touch the curve at any two successive points intersect each other at angles each of which is greater than the angle FEH. Such are the angles AKB, BLC, and CMD. That this can be done is so evident that it needs no demonstration. Next draw the cords AB, BC, and CD; and draw BN, CO, and DP as normals to the curve which intersect the extensions of AK, BL, and CM at N, O, and P. Now I say that each ratio of lines, i.e., of AB to BN, of BC to CO, and of CD to DP, is greater than the ratio of EF to FG.

For since the angle AKB is greater than the angle HEF, the angle NKB is less than the angle GEF because the sum of the angles NKB and AKB is

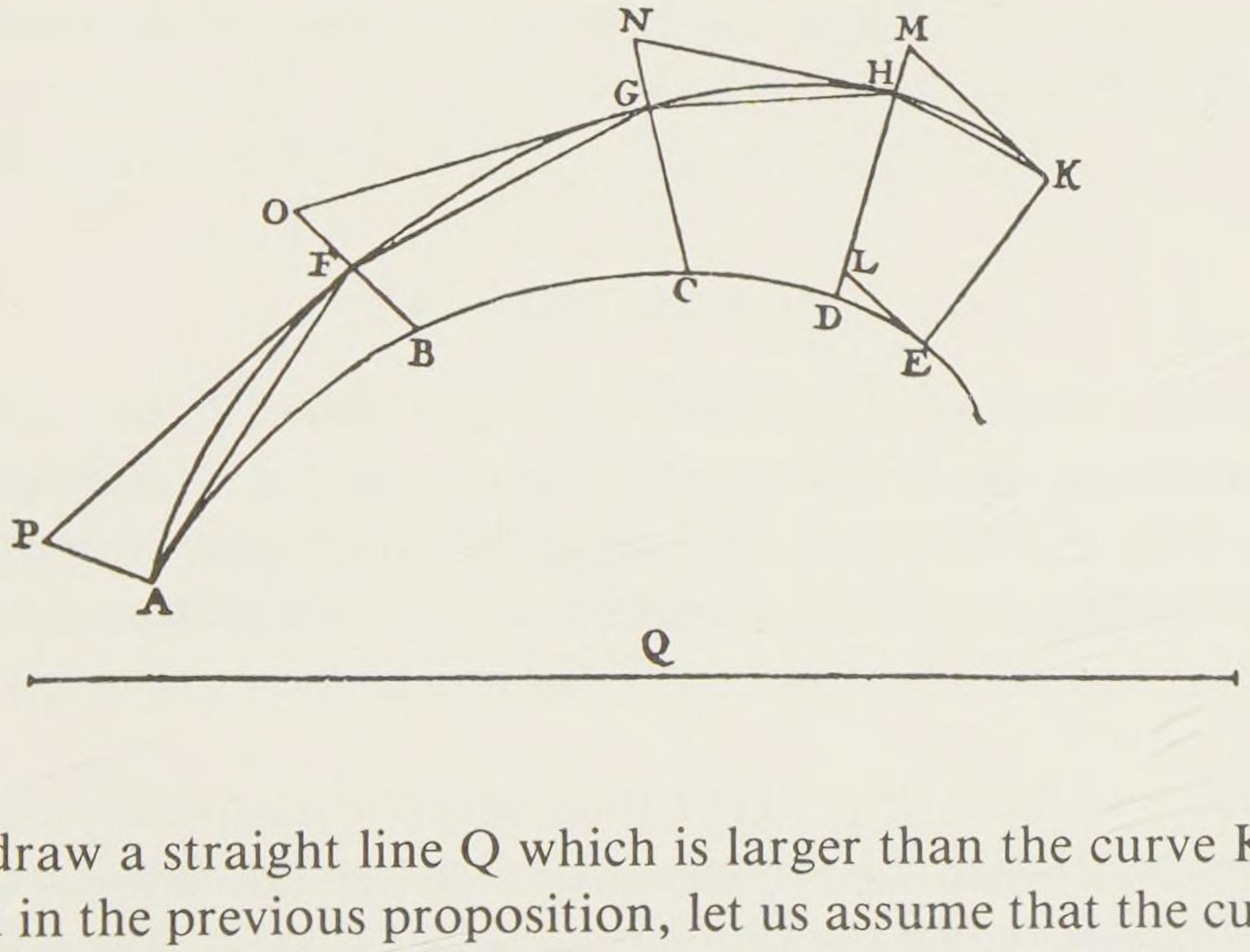
two right angles. But angle B of triangle KBN is a right angle, and so is angle F in triangle EFG. Hence the ratio of KB to BN is greater than the ratio of EF to FG. But AB is greater than KB because the angle K in triangle AKB is obtuse and is also larger than the angle HEF which was obtuse by construction. Thus the ratio of AB to BN is greater than the ratio of KB to BN, and hence is obviously greater than the ratio of EF to FG. In the same way it can be shown that the ratio of BC to CO, and of CD to DP, is greater than the ratio of EF to FG. Thus the proposition is proven.

PROPOSITION III

Let there be two curves which are bent in one direction, which are concave in the same direction, and which start from the same point. These curves cannot be so related that each straight line which is normal to one of them is also normal to the other.

Let us assume that ACE and AGK are such curves, having A as a common starting point [Fig. 40]. From some point K in the outside line, draw the straight line KE as a normal to the curve AGK and also as a normal to the curve ACE, as if this were possible.

[Fig. 40]



Next draw a straight line Q which is larger than the curve KGA. Also, as was said in the previous proposition, let us assume that the curve KGA is divided at the points H, G, and F into so many parts that the ratio of each cord KH, HG, GF, and FA, to each adjacent normal HM, GN, FO, and AP, is greater than the ratio of the line Q to the line KE. Hence also the

ratio of the sum of all these cords to the sum of all these normals is greater than the ratio of Q to KE. Next extend these normals until they meet the curve ACE at D, C, and B, hypothetically at right angles. Now KE will be less than MD. For, having drawn EL perpendicular to KE, EL is a tangent to the curve ACE because KE meets the curve ECA at right angles, and thus KE must meet the line MD between D and M. Now since KE is the shortest line between the parallels EL and KM, it will be less than ML, and hence certainly less than MD. It can be shown in the same way that HD is less than NC, that GC is less than OB, and that FB is less than PA. Therefore, since PA is greater than FB, the sum of PA and OF will be greater than OB. Again, since OB is greater than GC, the sum of OB and NG will be greater than NC. But the sum of PA and OF was greater than OB. Thus the sum of PA, OF, and NG will clearly be greater than NC. On the other hand, since NC is greater than HD, the sum of NC and MH will be greater than MD. Hence, if we replace NC by the sum of PA, OF, and NG, which is greater, then the sum of PA, OF, NG, and MH will be greater than MD. And this sum will also certainly be greater than the line KE since MD was greater than KE. But we have said that the ratio of the sum of all the cords AF, FG, GH, and HK, to the sum of all the normals PA, OF, NG, and MN, is greater than the ratio of the line Q to KE. Therefore, since KE is less than the sum of the normals, the ratio of the cords to KE is clearly greater than the ratio of Q to KE. Hence the sum of the cords will be greater than the line Q. But Q was assumed to be greater than the curve AGK. Therefore the sum of the cords AF, FG, GH, and HK will be greater than the curve AGK of whose parts they are cords. This is absurd, since each cord is less than its arc. Hence it is not possible for two curved lines to be related as first indicated. Q.E.D.

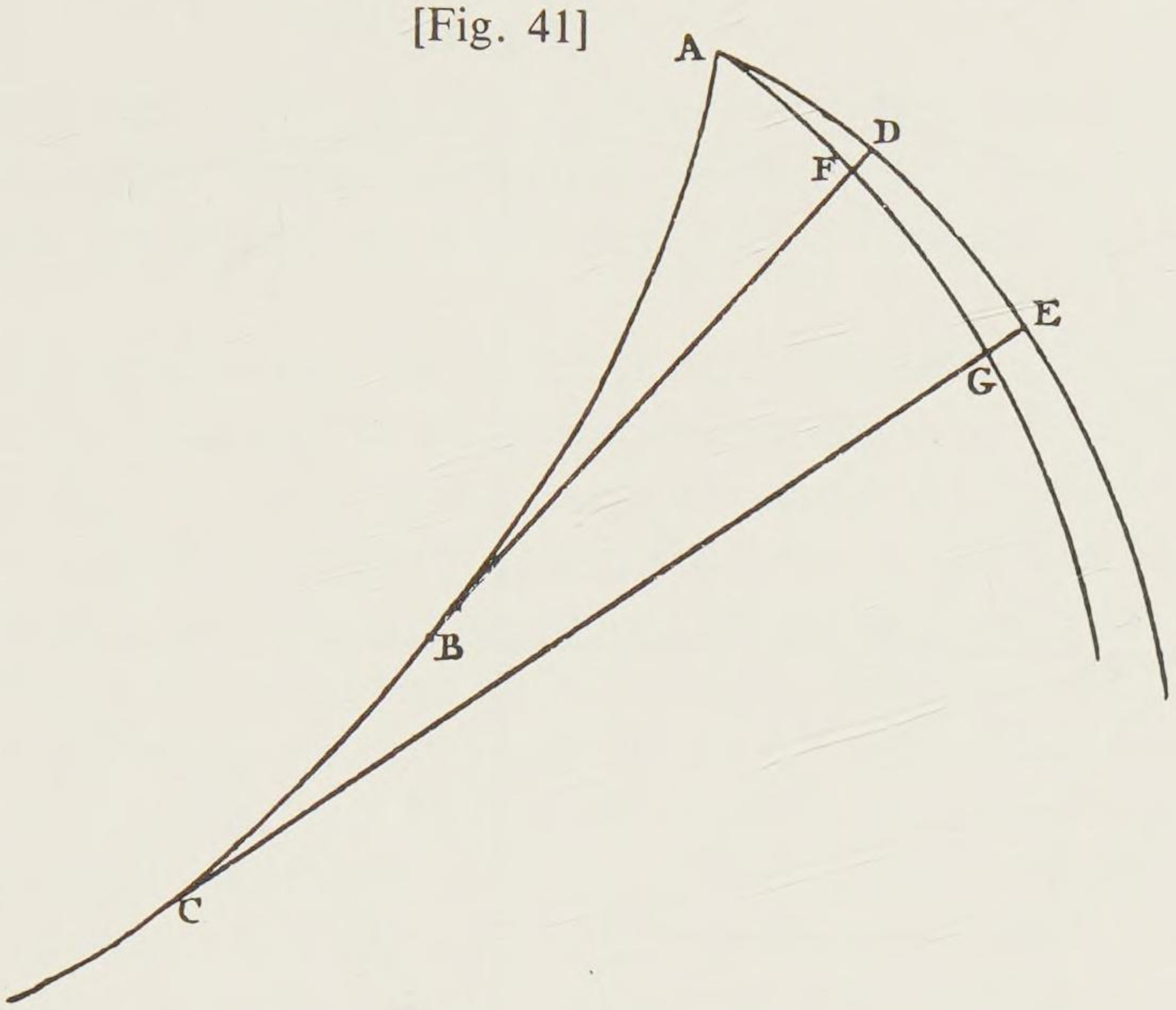
PROPOSITION IV

If two lines starting at the same point are curved in one direction, and if they are concave on the same side, and if they are so related that all the tangents to one of them meet the other at right angles, then the latter line will be described by evolution of the former line, beginning at the common point.

Let ABC and ADE [Fig. 41] be lines which are curved in one direction, which are each concave in the same direction, and which have a common end point at A. Moreover let all the lines tangent to the line ABC, i.e., BD and CE, meet the line ADE at right angles. Now I say that ADE is described by the evolution of ABC, beginning at the end A.

Let us assume that some other curve AFG is described by the above

[Fig. 41]



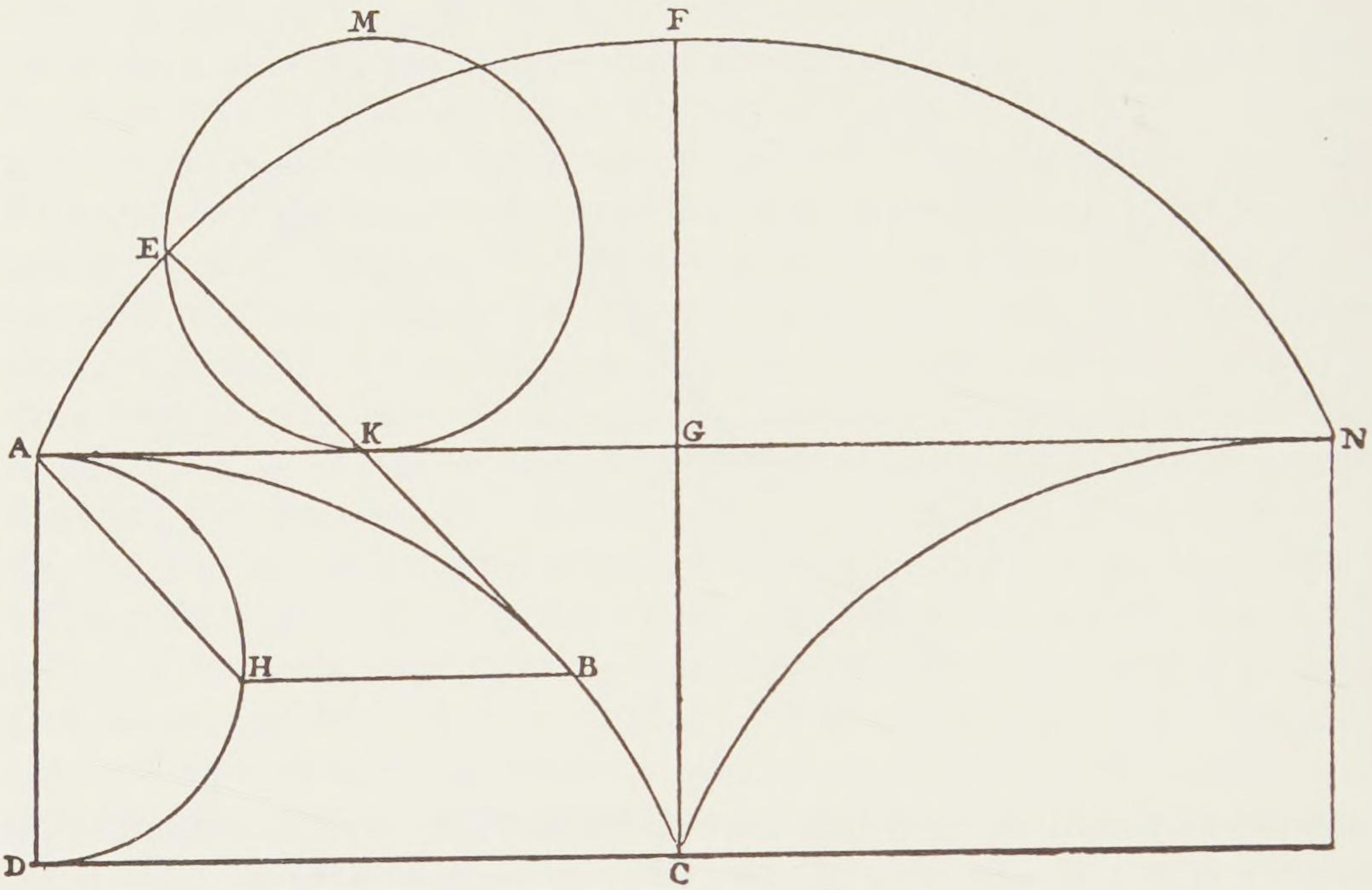
mentioned evolution, as if that were possible. Then any line tangent to the evolute ABC, for example, BD and CE, would meet AFG at right angles, for example, at F and G [Proposition 1]. But it was granted that these same tangents also meet the line ADE at right angles. Therefore the curves ADE and AFG, beginning at the same point A, are both curved in one direction, and are both concave in the same direction, indeed both are concave in the same direction as ABC itself. For this is true of ADE by assumption, and of AFG because of Proposition 1, Part III. And every line which meets one of them at right angles also meets the other at right angles. But it was shown above [Proposition 3] that this is impossible. Thus it follows that ADE will be described by the evolution of the line ABC. Q.E.D.

PROPOSITION V

If a straight line is tangent to a cycloid at its apex, and if on that line as a base another cycloid similar and equal to the first is constructed starting from the point of the apex just mentioned, then any straight line which is tangent to the lower cycloid will meet the arc of the higher cycloid at right angles.

Let the straight line AG be a tangent to the cycloid ABC at the apex A [Fig. 42]. On AG as a base construct another similar cycloid AEF whose apex is F. Moreover let the line BK be a tangent to the cycloid ABC at B. Now I say that the extension of BK will meet the cycloid AEF at right angles.

[Fig. 42]



Let the generative circle AHD be drawn around AD, which is the axis of the cycloid ABC. Let BH, which is parallel to the base, meet the circle at H, and draw HA. Since BK is tangent to the cycloid at B, it follows that it is parallel to HA [Proposition 15, Part II]. Thus AHBK is a parallelogram, and so AK is equal to HB, i.e., to the arc AH [Proposition 14, Part II]. Next construct the circle KM equal to the generative circle AHD, making it tangent to the base AG at K, and extend BK to the point E. Now since AH is parallel to BKE, and since the angle EKA thus equals KAH, it is clear that the extension of BK cuts an arc from the circle KM equal to the arc which AH cuts from the circle AHD. Thus the arc KE is equal to the arc AH, that is, to the line HB and to the line KA. From this it follows, according to the properties of a cycloid, that since the generative circle touches the base at K, the point describing the cycloid would be at E. Thus the line KE meets the cycloid at E at right angles [Proposition 15, Part II]. But KE is the extension of BK. Therefore it is clear that BK when extended meets the cycloid at right angles. Q.E.D.

PROPOSITION VI

By the evolution of a semicycloid, starting at the apex, another semicycloid is described, which is equal and similar to its evolute, and whose base is the line which is tangent to the evolute cycloid at its apex.

Let ABC be a semicycloid on which is superimposed another similar semicycloid AEF, as explained in the preceding proposition [Fig. 42]. I say that if a flexible line stretched along the semicycloid ABC were evolved, beginning at A, it would describe by its end point the semicycloid AEF. For since the semicycloids ABC and AEF proceed from the same point A, since they are curved in one direction and both are concave in the same direction, and furthermore since they are so related that all the tangents of the semicycloid ABC meet the semicycloid AEF at right angles, it follows that AEF is described by the evolution of ABC, beginning at A [Proposition 4]. Q.E.D.

Also, if we construct the half-cycloid CN as a twin to ABC, situated on the other side of the line CG, then it is apparent that another semicycloid FN will be described by the end of the string either by the evolution of CN or by rolling the string now extended to CF back around CN. And FN taken together with AEF constitutes a whole cycloid.

From this and from Proposition 25 concerning the falling of heavy bodies [Part II], what was said above in the construction of the clock [Part I] about the equal motion of the pendulum is now obvious. For it is clear that a pendulum which is suspended and set in motion between two plates curved in the shape of a semicycloid describes an arc of a cycloid by its motion, and thus its oscillations, whatever their magnitude, are completed in equal times. It makes no difference whether the moving body is guided by a surface curved in the shape of a cycloid or whether it is tied to a string which traces that line in the air, as long as in each case it has the same freedom of motion and has the same inclination to move at every point of the curve.

PROPOSITION VII

A cycloid is four times longer than its axis, and four times longer than the diameter of its generating circle.

Let us return to the preceding figure [Fig. 42]. After the whole semicycloid ABC has been evolved, its string occupies the line CF. CF is twice AD because the axes of the cycloids ABC and AEF are equal. Hence it is apparent that the semicycloid ABC, which is equal to the string wrapped around it, is twice its axis AD. And thus the whole cycloid is four times its axis.

It is also clear that the tangent BE, which refers to the extended part of the string which was previously applied to part of the curve BA, is equal in length to BA. But BE is twice BH or AH, since it was shown in Proposition 5 that KE is equal to AH. Thus the part AB of the cycloid will be twice AH or BK, granting that BH is parallel to the base of the cycloid. This is true no matter where the point B is located on the cycloid.

The outstanding English geometer Christopher Wren was the first to discover this dimension of the cycloid, although in a quite different way;[1] and he later confirmed this by an elegant demonstration which was published in a book on the cycloid by the illustrious Mr. John Wallis.[2] Many other most beautiful discoveries about this line have been made by mathematicians of our day, who were given the occasion especially by certain problems posed by Blaise Pascal, a Frenchman who excelled in these studies. In reviewing both his own discoveries and those of others, Pascal says that Mersenne was the first to draw attention to this line in nature,[3] and that Roberval was the first to define its tangents and to measure plane and solid cycloids. He himself discovered the center of gravity both of a plane cycloid and of its parts. He says that Wren was the first to find a straight line equal to the curve of a cycloid, and that I was the first to discover the absolute dimension of that portion of a cycloid which is cut off by a line parallel to the base through a point on the axis below the vertex by one-quarter of its length; namely this portion is equal to one-half of an equilateral hexagon inscribed within the generating circle.[4] Finally Pascal says that he found the centers of gravity of solid amd semisolid cycloids and their parts, both around the base and around the axis. He also says that he discovered the center of gravity of this line, i.e., the cycloid (but only after learning of its dimension from Wren), the dimension of convex surfaces, which include these solids and their parts, the center of gravity of such surfaces, and finally the dimension of curves of cycloids which are either elongated or shortened, that is, of curves which are described by a point taken either inside or outside of the circumference of the generating circle. Demonstrations of the above were published by Pascal. After this Mr. Wallis reported his own most subtle meditations concerning this line, contending that he had discovered all these same things by his own effort and that he had solved the problems posed by Pascal. The same claim was made

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1. Huygens first learned of this from Blaise Pascal's *Historia trochoidis sive cycloidis; gallice la roulette* (Paris, 1658) and from a letter from Pascal dated 6 January 1659, printed in *Oeuvres* 2:309–10.

2. John Wallis. *Tractatus duo, prior de cycloide et corporibus inde genitis; posterior, epistolaris in quo agitur de cissoide . . .* (Oxoniae: Typ. Ac. Lichfieldianis, 1659).

3. This occurred in 1615. See C. de Waard, ed., *Correspondance du P. Marin Mersenne* (Paris: G. Beauchesne, 1933) 1:25. "Note sur la vie de Mersenne"; also C. de Waard, "Une lettre inédite de Roberval du 6 janvier 1637 contenant le premier énoncé de la cycloïde." *Bulletin des sciences mathematiques,* E. Picard and P. Appell, eds. Ser. 2. Vol. 45 (Paris: Gauthier-Villars, 1921).

4. This occurred in 1658. See *Oeuvres* 14:350–51.

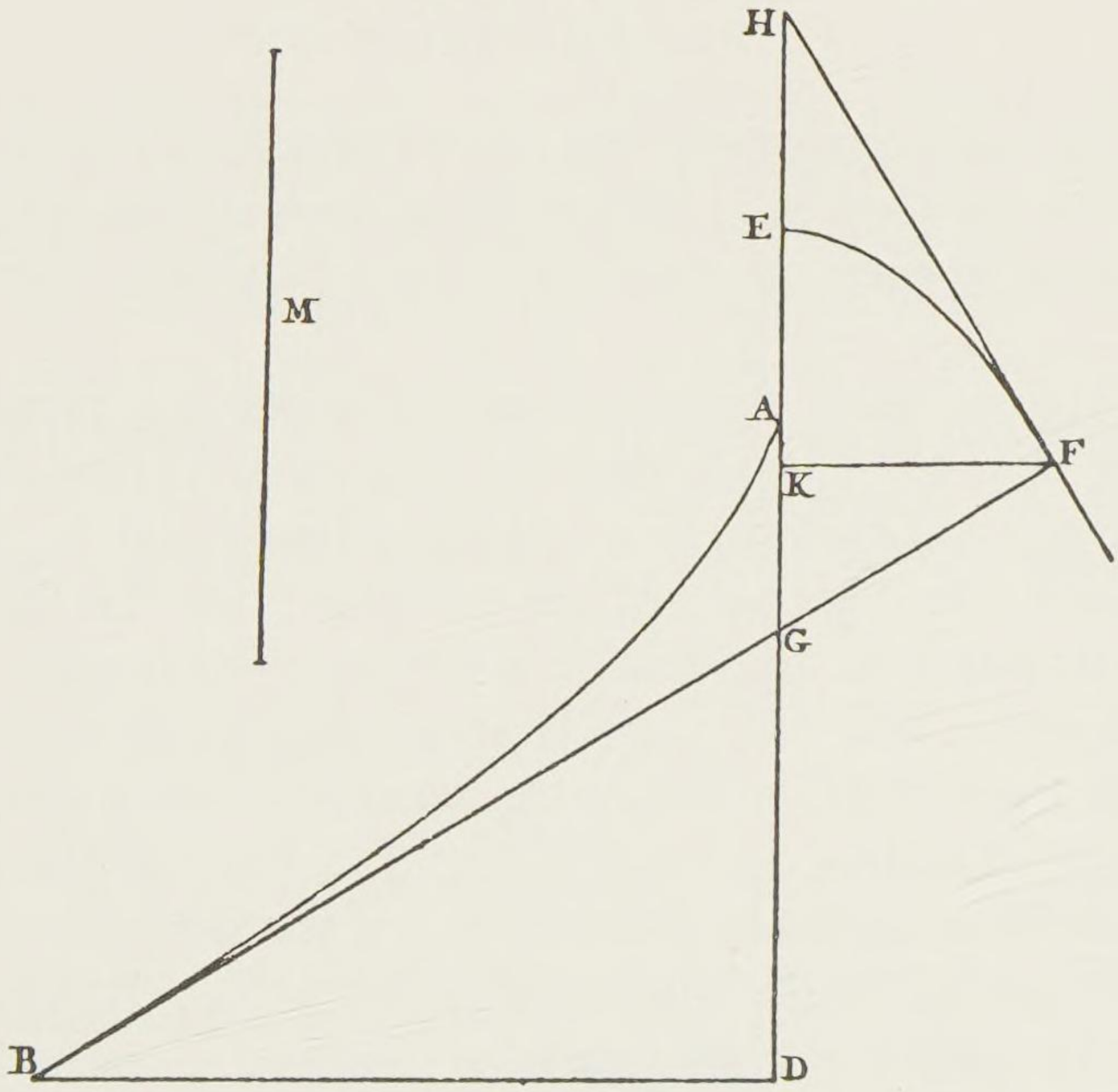
for himself by the most learned A. de la Loubère.[5] How much is owed to each of them should be judged by scholars from their writings. We have reported all of the above because, if we had passed by in silence, these discoveries might not seem very illustrious when actually they have established the fact that of all lines none is understood better or at greater depth than the cycloid. Further it is appropriate to apply to other lines the method we used to measure this one. Let us now do this.

PROPOSITION VIII

Find the line whose evolution describes a parabola.

Let AB be a paraboloid whose axis is AD and whose vertex is A [Fig. 43]. Draw BD perpendicular to the axis. It is a property of this figure that the cube of the abscissa to the vertex, DA, is equal to a solid whose base is the square of DB and whose altitude is equal to a certain given line M. This curve has been known to geometers for some time. Extend the axis upwards to E, and let AE be equal to $\frac{8}{27}$ of M. Now if a string is continuously stretched out along EAB and begins to be evolved from E, I say that its evolution will describe the parabola EF which has the axis EAG, the vertex E, and a *latus rectum* equal to twice EA.

[Fig. 43]



5. Antoine de la Loubère, SJ (Lalovera) (1600–1664).

Selecting any point B on the curve AB, draw the line BG tangent to curve at B and meeting the axis EA at G. Next from G draw GF to meet the parabola EF at right angles at F. FH, which is tangent to the parabola at F, is perpendicular to GF. Finally let FK be perpendicular to the axis EG.

Then KG is equal to half of the *latus rectum,* that is, to EA. Next, by adding or subtracting AK to both sides, EK equals AG. But AG is ⅓AD because BG is a tangent to the paraboloid at B; this can be easily proven from the nature of this curve. Thus EK equals ⅓AD; and KH, which is twice KE from the nature of a parabola, will be equal to ⅔AD. Thus the cube of KH is equal to ⅧAD, i.e., having the square of DB as a base and an altitude equal to ⅧM, that is, to AE. Hence the square of DB is to the square of KH as the length KH is to AE, that is, to KG. But KH was equal to ⅔AD, that is, to GD. Thus the square of BD is to the square of DG as HK to KG. But HK is to KG as the square of FK is to the square of KG. Therefore the square of BD is to the square of DG as the square of FK is to the square of KG. Next BD is to DG in length as FK is to KG. From this it follows that BGF is a straight line. But GF meets the parabola EF at right angles. Hence it is clear that the extension of BG, a tangent of the paraboloid, meets the parabola at right angles. The same can be demonstrated in the same way of any other tangent. Therefore it follows that the parabola EF is described by the evolution of the line EAB, beginning at E [Proposition 4]. Q.E.D.

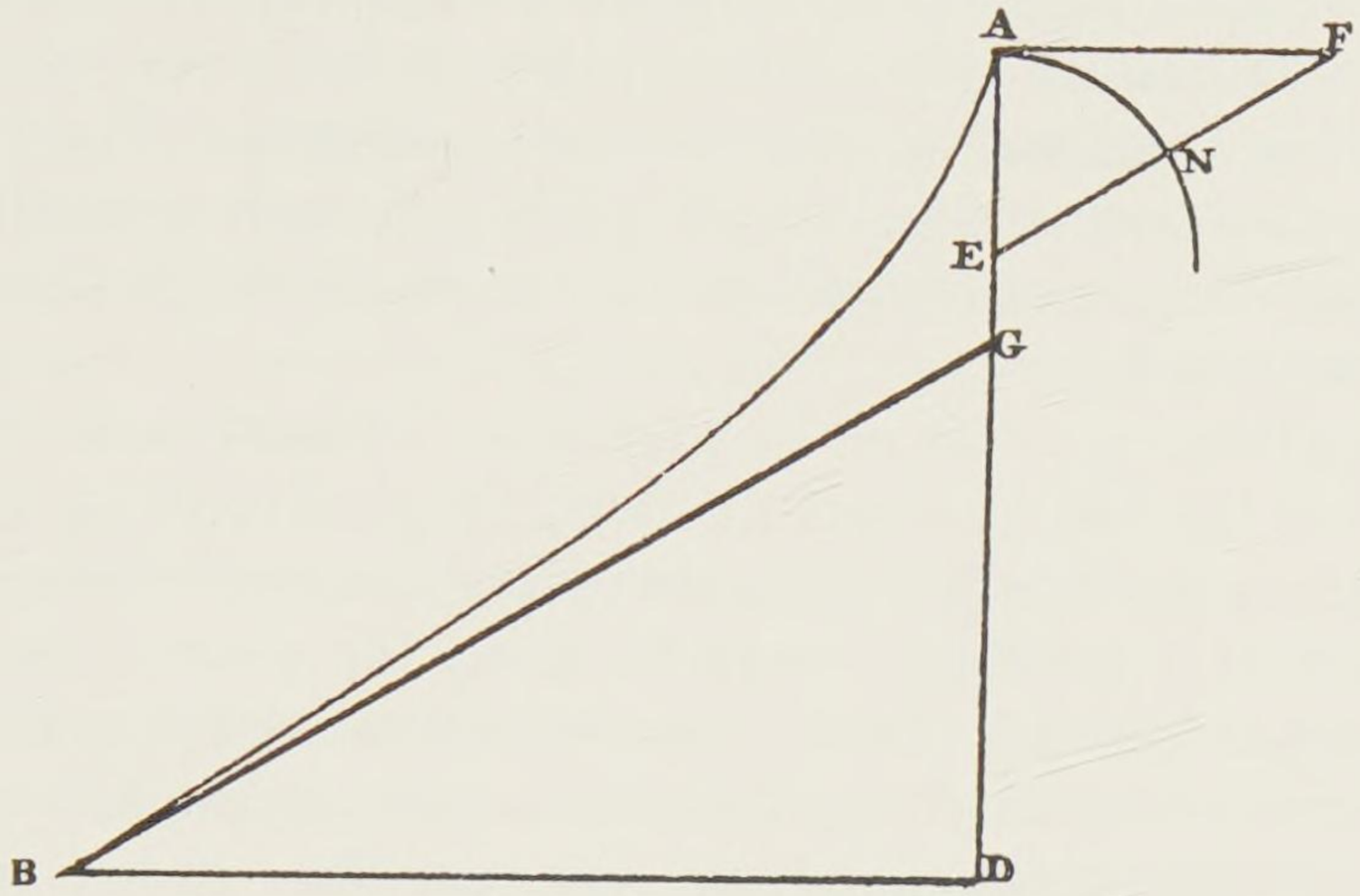
PROPOSITION IX

Find a straight line equal to a given portion of the curve of a paraboloid; namely a paraboloid in which the squares of the perpendicular to the axis are related to each other as the cubes of the abscissas to the vertex.

How this can be is evident from the preceding proposition. But the parabola EF [Fig 43] is not needed for this construction, which is made as follows. We are required to find a straight line equal to any given part of the paraboloid AB [Fig. 44]. At the point B draw the tangent BG, which meets the axis AG at G. Since BG is a tangent, AG equals ⅓AD, which lies between the vertex and BD which is perpendicular to the axis. Next let AE equal ⅧM, which is the *latus rectum* of the paraboloid AB. Draw EF parallel to BG, and let it meet the line AF, which is parallel to BD, at F. Now if NF is added to BG, NF being the difference of EF over EA, then we will have a straight line equal to the curve AB. The proof of this is easily seen from what was said earlier.

Therefore the curve AB always exceeds the tangent BG in the same amount that EF exceeds EA.

[Fig. 44]



Once again we have come across a line whose length has already been measured by others. In 1659 Johannes van Heuraet[6] of Haarlem showed that this curve is equal to this straight line. His demonstration was added to the commentaries of Johannes van Schooten[7] on the *Geometry* of Descartes, which was published that same year. Van Heuraet was the first person to provide a straight line measurement of a curved line of which some points are defined geometrically, while at the same time Wren determined the length of a cycloid, using a no less ingenious proof by an epicheirema.

After van Heuraet's discovery was published, I know that the very learned Mr. Wallis in his book on the cissoid wished to attribute the same discovery to William Neile,[8] a gentleman from his own country. But after carefully examining what he presents there, it seems to me that Neile was very close to that discovery, but he clearly did not arrive at it.[9] For from his demonstration, as presented by Wallis, it appears that he did not adequately grasp what sort of curve it was whose measure he saw was given in the construction. If he had realized that this curve had been known to geome-

6. Johannes Heuratius (Hendrik van Heuraet) (1633–1660).
7. Johannes Schotenius (Franz van Schooten) (c. 1615–1660).
8. Wilhelmus Nelius (William Neil or Neile) (1637–1670).
9. In a later letter to Wallis, which has been lost, Huygens retracted this opinion. We know this from Wallis's letter to Leibniz, 1 December 1696 [Wallis, *Opera mathematica* (Oxford: 1699) 3:653]. "Concerning the discovery of Neile, who (as I firmly stated in Proposition 38 of my *Arithmetica infinitorum*) was the very first to present a straight line equal to that curve: what Huygens had said (i.e., that he was not very far away but did not completely arrive at the goal), he later retracted (in his letters to me) and he requested that I repeat this to Neile. Although Neile knew immediately which of all the curves this curve was, I was not certain. But Brounker [William, Viscount Brounker or Brouncker, c. 1620–1684] immediately determined that it was a paraboloid, to which I gave the name 'semi-cubical.' "

ters for a long time, it is credible that either he or others in his name would have communicated such a noble discovery to geometers in a more developed form. If things had been different, this discovery was worthy enough for them to shout "eureka," as Archimedes once did. To be sure Fermat,[10] a councillor at Toulouse and an expert geometer, wrote demonstrations of this same discovery, as he formulated it. The year 1660 is claimed for this, but it was certainly later.

While on this topic, permit us to indicate what we contributed to the development of this extraordinary discovery. We provided van Heuraet with the occasion to make the discovery, and before him and all others we found the dimension of a parabolic curve by using the given quadrature of a hyperbola, which was part of van Heuraet's discovery. For towards the end of 1657 we found two things simultaneously; namely, the above mentioned dimension of a parabolic curve, and the reduction of the surface of a parabolic conoid to a circle. In letters to van Schooten and other friends we announced that we had made two important discoveries about the parabola, and that one of them was the reduction of the surface of a conoid to a circle. Van Schooten shared this letter with van Heuraet, who was a close friend. To this most acute man, it was not difficult to recognize that the dimension of this parabolic curve was closely related to the dimension of the surface of a conoid. Given both of these findings and pushing his investigation further, he came upon these other paraboloidal curves, for which absolutely equal straight lines can be found.

If by chance anyone would wish to have testimony concerning the reduction of the surface of a conoid to a plane surface, it seems well to quote a few lines from a letter by the illustrious Mr. Francois Sluse,[11] who is regarded as one of the outstanding geometers of the day. In a letter dated that same year he congratulated me for that discovery, more prolixly indeed than I deserved. Writing on 24 December 1657, he says,

> I add only two things. The first is. . . . The other is that I regard all these curves and their linear loci as worth almost nothing in comparison with your discovery in which you have demonstrated the relation between the surface of a parabolic conoid and the circle which is its base. I freely give preference to this most beautiful induction about the quadrature of the circle over all the many things which I have previously deduced concerning linear loci, and which, given the occasion, I will communicate to you if you so wish.[12]

The following year I also discovered how to reduce the surfaces of

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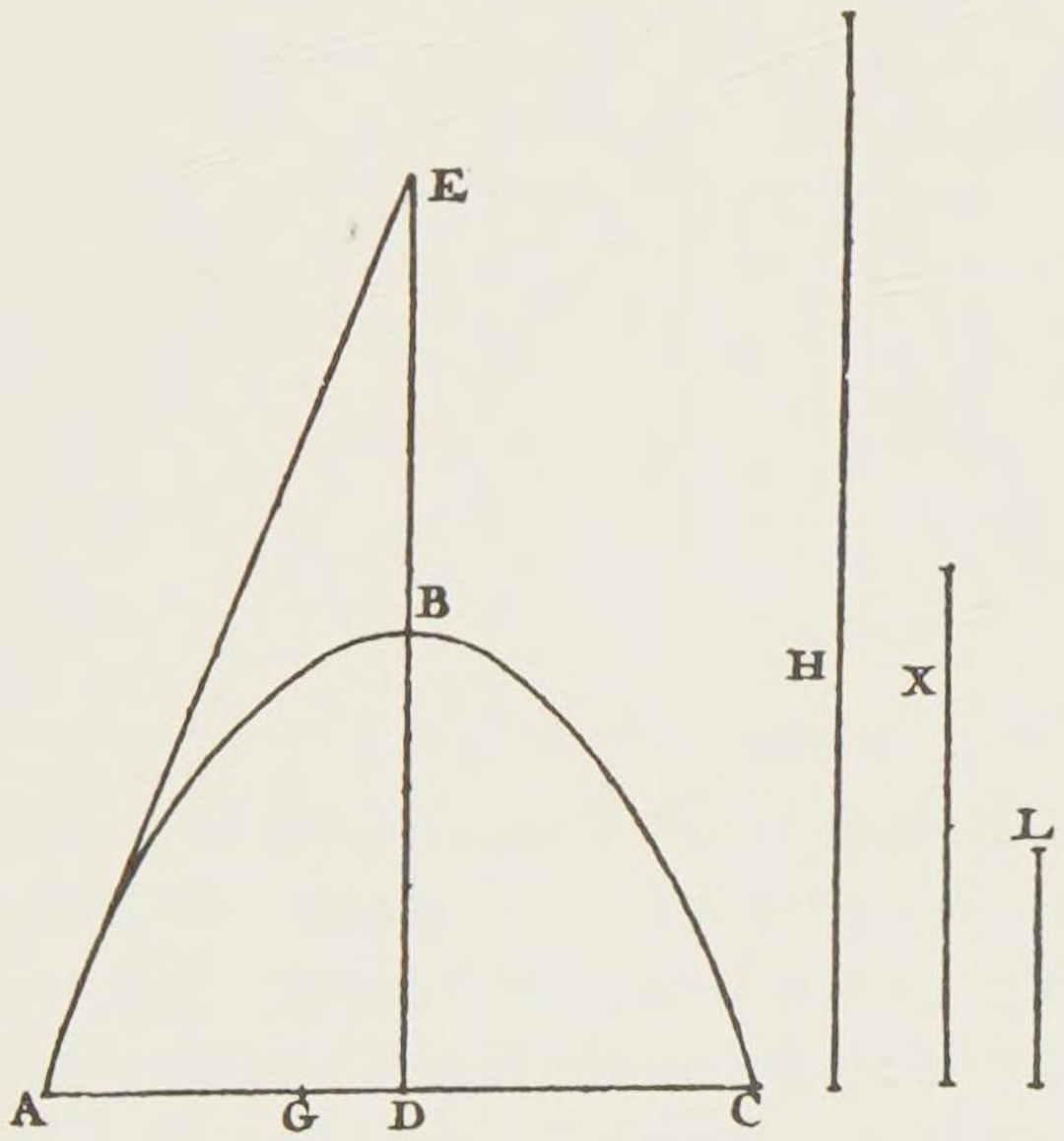
10. Pierre de Fermat (c. 1608–1665).
11. Franciscus Slusius (René François Walter, Baron de Sluze or Sluse) (1622–1685).
12. *Oeuvres* 2:107.

hyperbolic and spheroidal conoids to circles. I communicated the constructions of these problems, but not their demonstrations, to geometers with whom I was then in correspondence, i.e., to Pascal and others in France and to Wallis in England. Not much later Wallis published his own demonstrations of these points, along with many other subtle discoveries. As a result I should omit my own complete demonstrations.[13] But since our constructions are not without elegance, and since they are still unpublished, I wish to give them here.

Find a circle equal to the curved surface of a parabolic conoid.

Let there be given a conoid [Fig. 45][14] whose section through the axis is the parabola ABC, whose axis is BD. The vertex is B, and the diameter of the base is AC, which meets the axis BD at right angles. We are required to find a circle equal to the curved surface of the conoid.

[Fig. 45]



Extending the axis above the vertex, let BE equal BD. Draw EA, which is tangent to the parabola ABC at A. Next cut AD at G so that AG is to GD as EA is to AD. Let the line H be equal to the sum of AE and DG. Let the line L be equal to one-third of the base AC. Let K be a mean proportional between H and L. Draw a circle of radius K. This circle will be equal to the

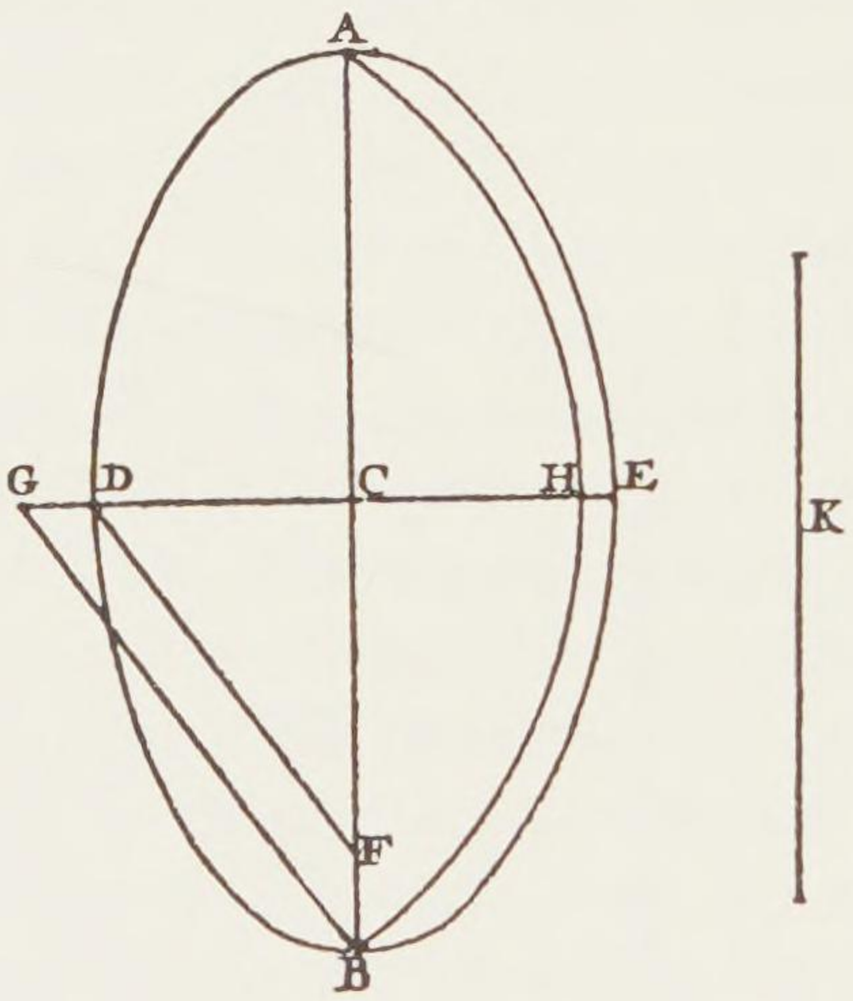
13. For Huygens' earlier work on the rectification of various curves, see *Oeuvres* 14:234ff.
14. In Fig. 45 read "K" for "X."

curved surface of the conoid ABC. From this it follows that if AE is twice AD, then the curved surface of the conoid is related to the circle on the base as 14 to 9. If AE is triple AD, the relation is 13 to 6; if AE is four times AD, the relation is 14 to 5. These numbers will always be so related if AE has this same ratio to AD.

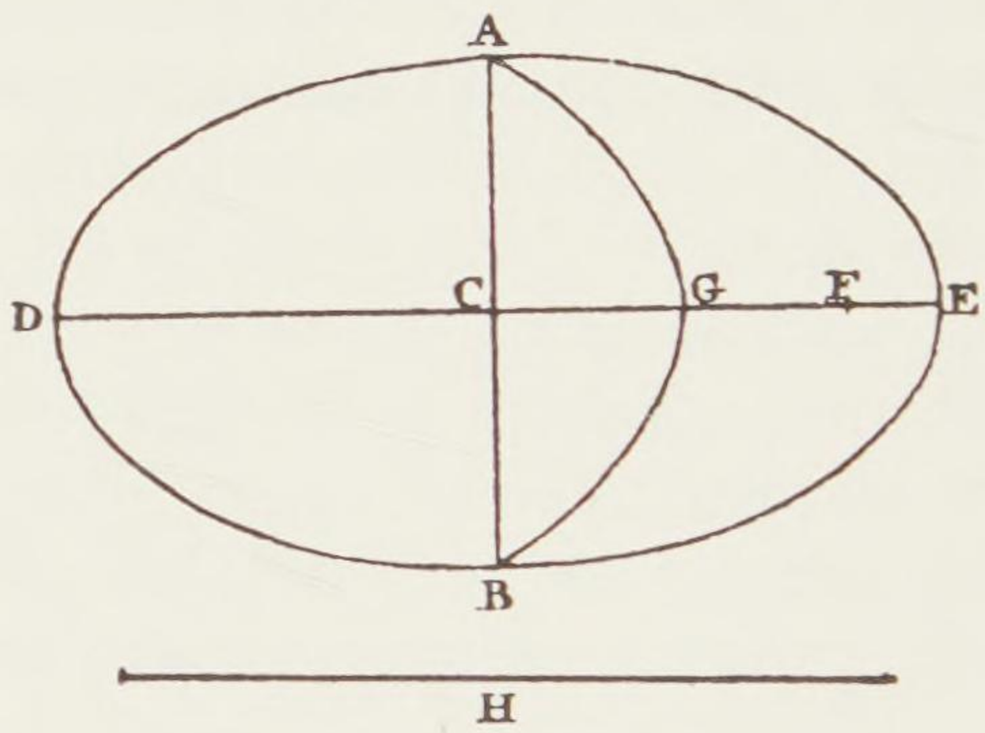
Find a circle equal to the surface of an oblong spheroid.

Let there be an oblong spheroid [Fig. 46] having AB as its axis and C as its center. Let ADBE be an ellipse produced by a section through the axis, with DE being the minor diameter.

[Fig. 46]



[Fig. 47]



Let DF equal CB, that is, let F be one of the foci of the ellipse ADBE. Draw BG parallel to FD, meeting the extension of ED at G. Using G as the center and GB as the radius, draw on the axis AB the arc of the circumference BHA. Let there be a straight line K which is the mean proportional between the semidiameter CD and the straight line equal to the sum of the arc AHB and the diameter DE. K will then be the radius of the circle which is equal to the surface of the spheroid ADBE.

Find a circle equal to a wide or compressed spheroid.

Let there be a wide spheroid [Fig. 47] whose axis is AB and whose center is C. Let ADBE be an ellipse produced by a section through the axis.

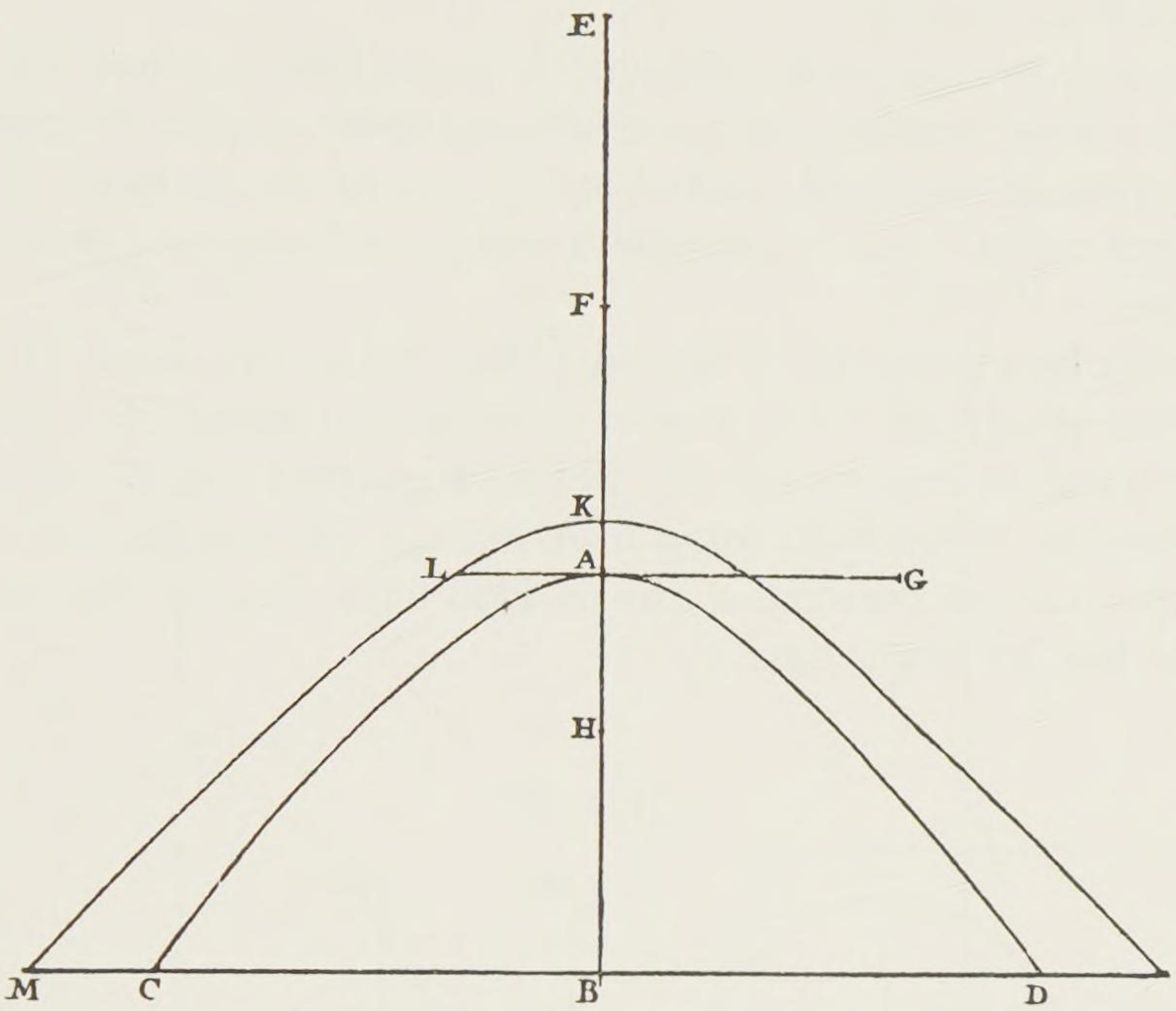
Again let F be one of the foci. Divide FC in half at G. AGB is a parabola, having the axis BA as its base and G as its vertex. Let H be a mean proportional between the diameter DE and the straight line equal to

the parabolic curve AGB. H will then be the radius of the circle which is equal to the surface of the spheroid under consideration.

Find a circle equal to the curved surface of a hyperbolic conoid.

Let there be a hyperbolic conoid [Fig. 48] whose axis is AB. CAD is a hyperbola produced by a section through the axis; EA is its *latus transversum,* F its center, and AG its *latus rectum.*

[Fig. 48]



On the axis let AH equal half of the *latus rectum* AG, and let HF be related to AF in length as the square of AF is related to the square of FK. Then construct another hyperbola KLM having the vertex K, the same axis, and the same center F as before. Let its *latus rectum* and *latus transversum* be inversely proportional to those of the prior hyperbola. Let the extension of BC meet the hyperbola at M, and let AL be parallel to BC. Now the curved surface of the conoid will be related to the circle at its base having the diameter CD, as the space ALMB, constructed of three straight lines and a hyperbolic curve, is related to half of the square of BC. From this the remainder of the construction is easily completed, assuming the quadrature of the hyperbola.

Therefore, from the known rules of geometry, the surface of a parabolic conoid and also the surface of a sphere are reduced to a circle. To

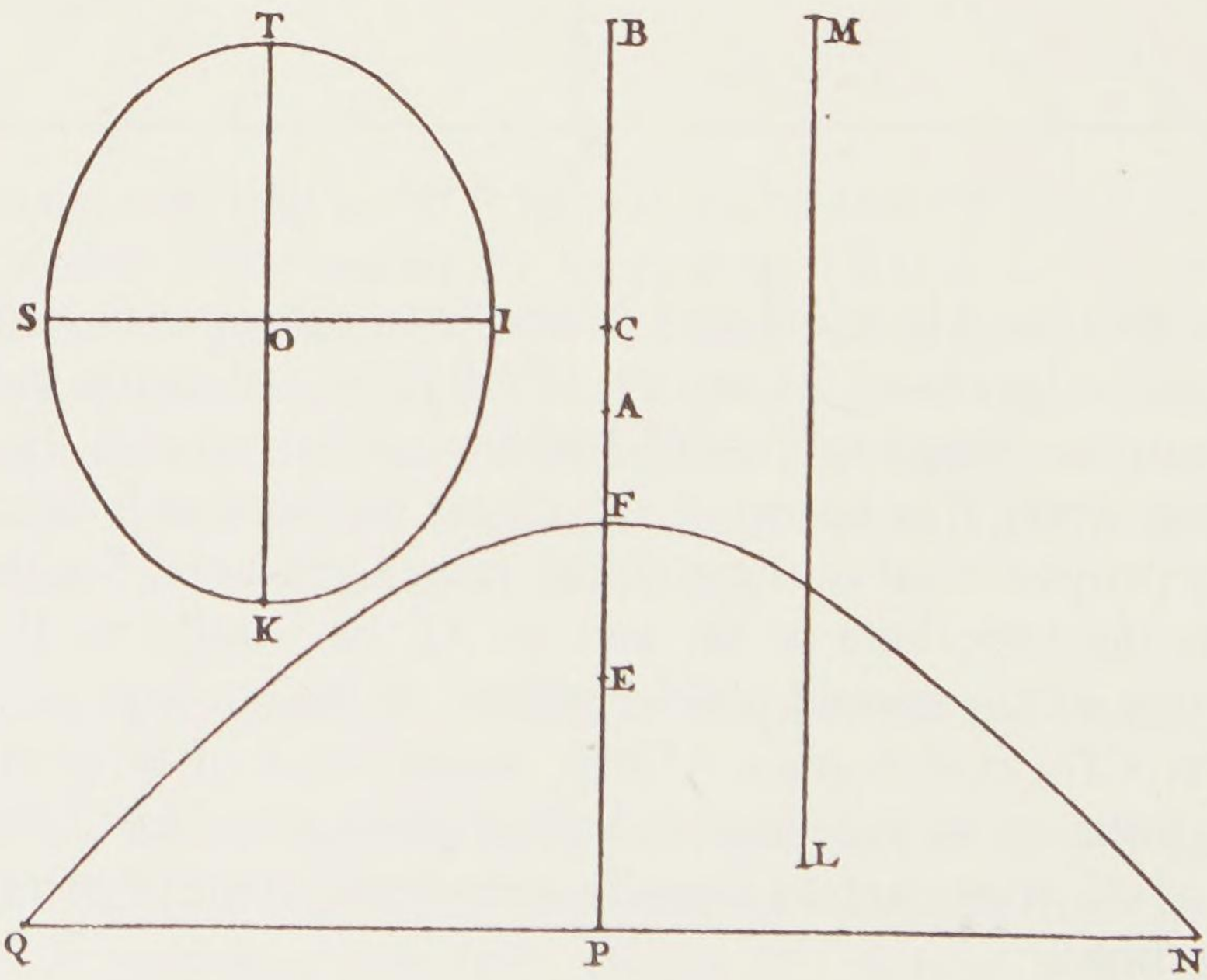
accomplish the same for the surface of an oblong spheroid, it must be granted that the length of an arc of a circumference can be equated with a straight line. Further, for the surfaces of a compressed spheroid or of a hyperbolic conoid to be reduced to a plane by the same argument, the quadrature of the hyperbola is required. For the length of a parabolic line, which notion we used in the case of this spheroid, depends on the quadrature of the hyperbola, as we will soon show.

Further it is not unimportant to add that we have discovered how, without any use of the supposition of the quadrature of the hyperbola, to construct a circle equal to the sum of the surfaces of a compressed spheroid and a hyperbolic conoid.

Given any compressed spheroid, it is possible to find a hyperbolic conoid, or on the contrary, given any hyperbolic conoid, it is possible to find a compressed spheroid, such that the sum of the surfaces of both will be equal to a circle.[15] As an example it will be sufficient to present only the simpler case.

Let there be a compressed spheroid [Fig. 49] whose axis is SI. Let STIK be an ellipse produced by a section through the axis. The center of the ellipse is O, and its major axis TK. Let it be granted that an ellipse is such that the relation between its *latus transversum* TK and its *latus rectum* is equal to the relation between a line divided according to the extreme and mean ratio and its larger part.

[Fig. 49]



15. See *Oeuvres* 14:192–95, 324–26, 329–34.

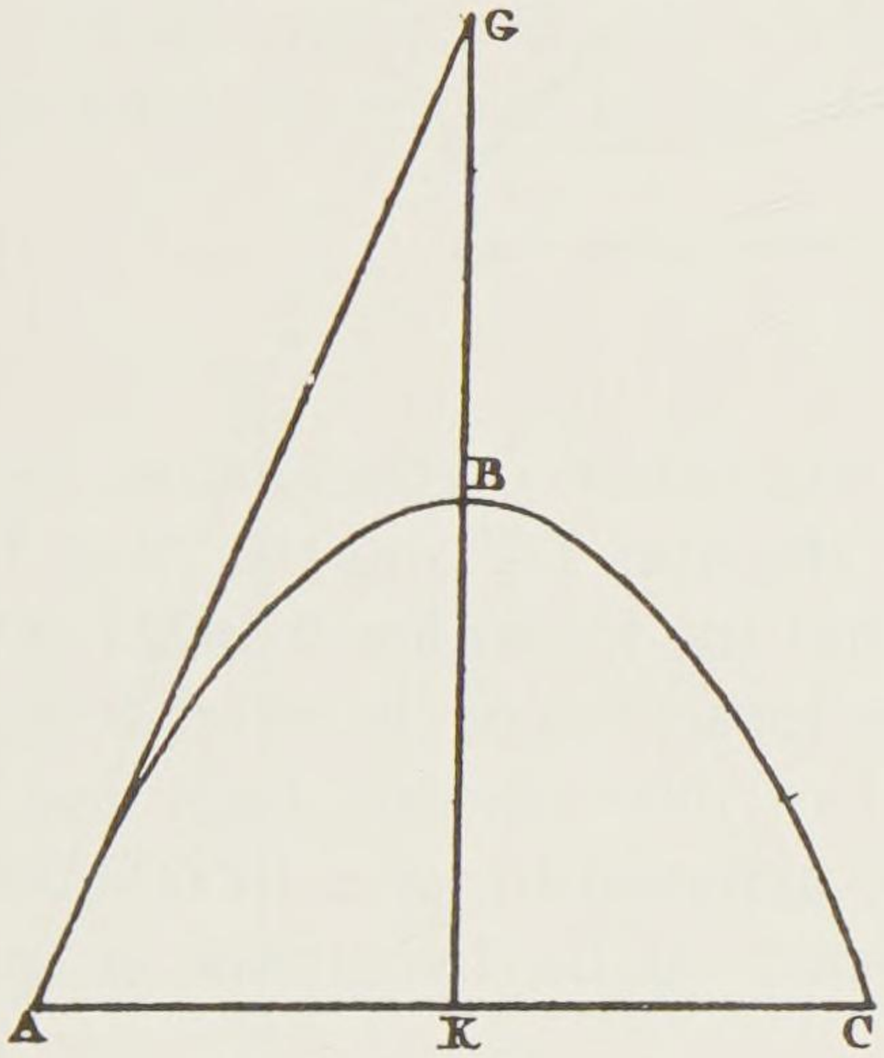
Let the square of BC be equal to twice the square of SO, and let the square of BA be equal to twice the square of OK. Let these four lines, BC, BA, BF, and BE, be continuous proportionals, and let EP equal EA. Next construct the hyperbolic conoid QFN, whose axis is FP. To the axis add FB, or half of the *latus transversum*. BC equals half of the *latus rectum*.

The curved surface of this conoid, plus the surface of the spheroid SI, will be equal to a circle of radius ML, where the square of ML equals the square of TK plus twice the square of SI.

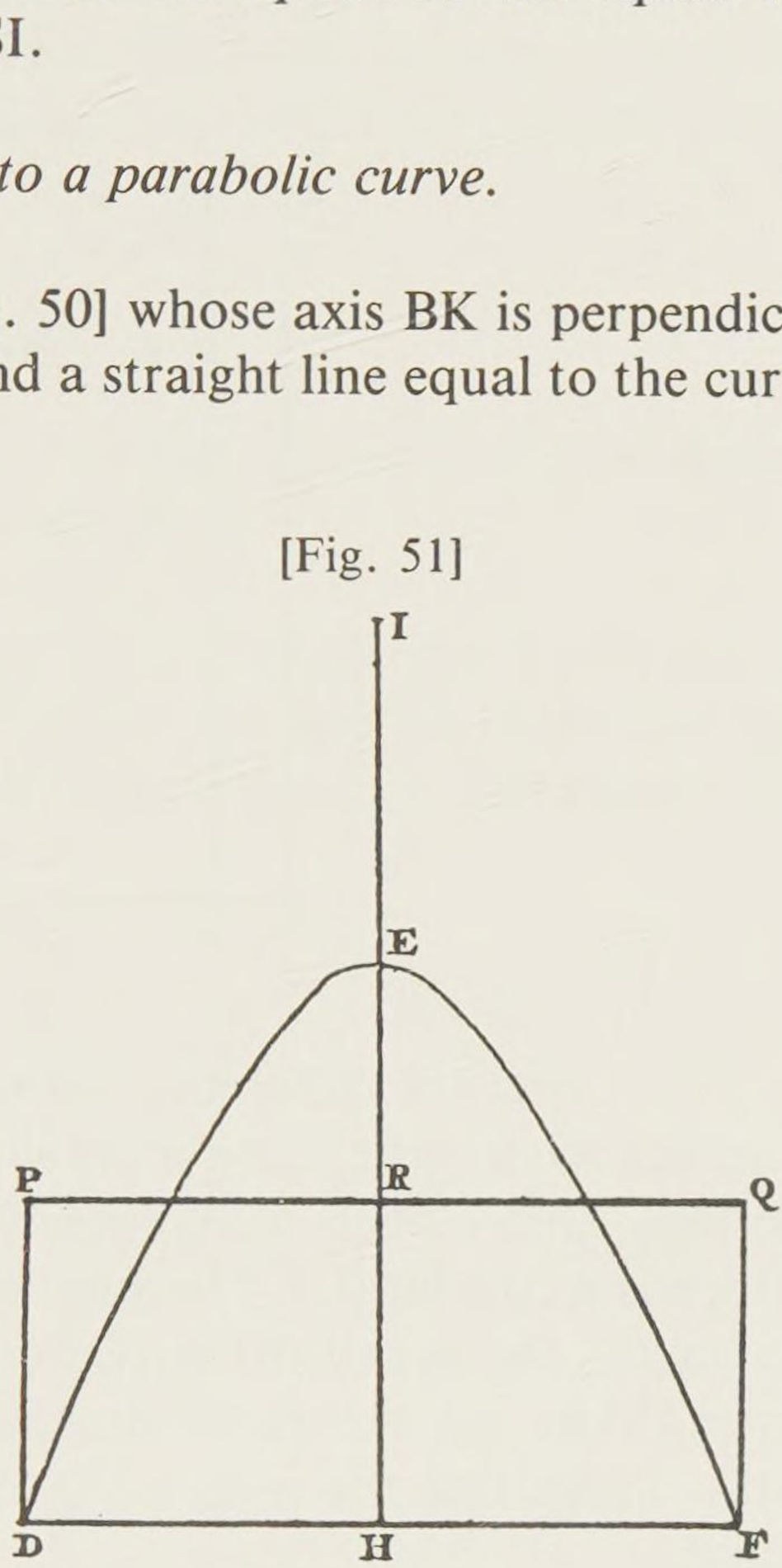
Find a straight line equal to a parabolic curve.

Let ABC be part of a parabola [Fig. 50] whose axis BK is perpendicular to its base AC. We are required to find a straight line equal to the curve ABC.

[Fig. 50]



[Fig. 51]



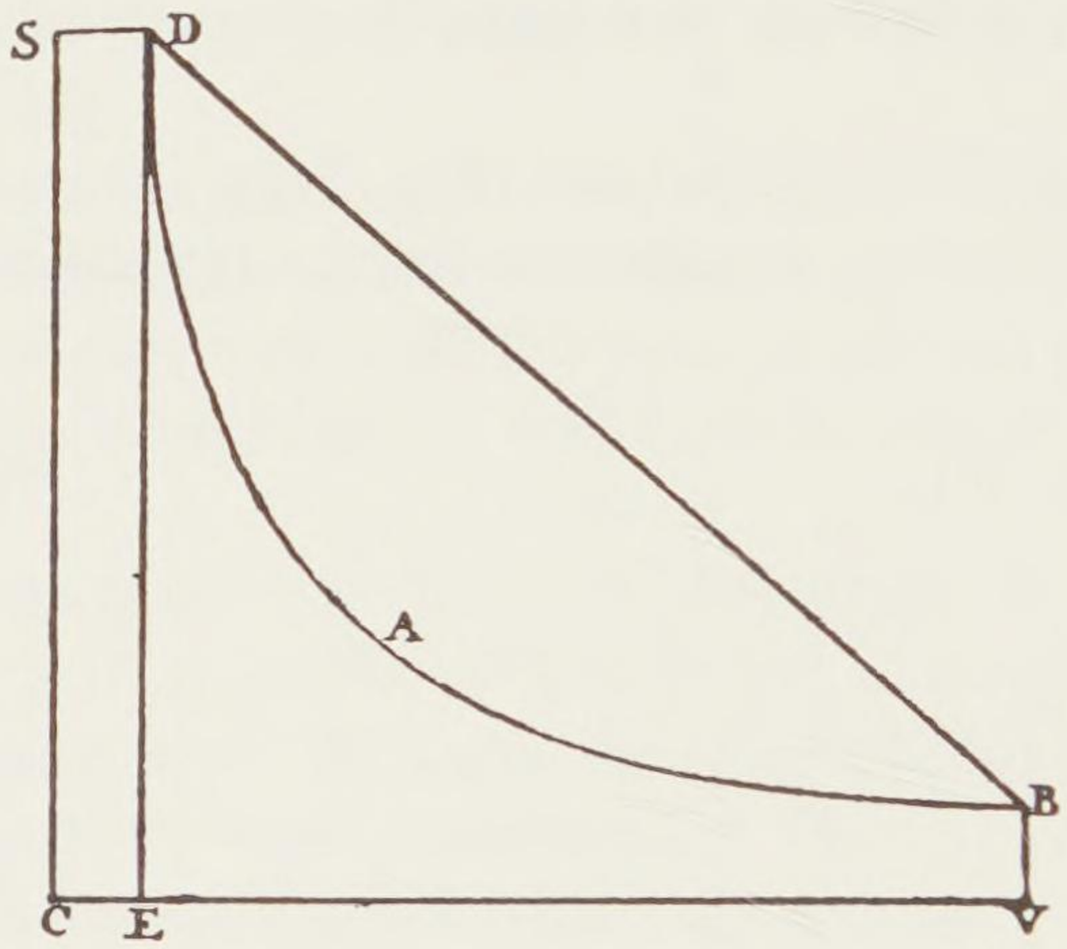
Let the straight line IE be equal to half of the base, AK [Fig. 51]. Extend IE to H such that IH equals AG, where AG is the tangent to the parabola at the base A and which, when extended, meets the axis at G. Next draw the portion of a hyperbola DEF having E as vertex, I as center, EH as diameter, and base DHF as a perpendicular to the diameter. Its *latus rectum* can be taken arbitrarily. Now on the base DF construct the parallelogram DPQF, which is equal to the area DEF. Its side PQ will cut the diameter of the hyperbola at R in such a way that RI is equal to the parobolic curve AB, which is half of ABC.

From this it is clear how the measure of a parabolic curve depends on the quadrature of a hyperbola, and vice versa.

Any problem which reduces to one of these two cases is capable of a numerical solution to any degree of accuracy by means of that admirable invention, logarithms. By using the latter we have recently found that the quadrature of the hyperbola can be explained numerically to any degree of accuracy. Our rule is as follows.

Let DAB be a portion of a hyperbola [Fig. 52] whose asymptotes are CS and CV. Draw DE and BV parallel to the asymptote SC.

[Fig. 52]



First find the difference between the logarithms corresponding to the numbers having the same mutual ratio as the lines DE and BV. Next find the logarithm of this difference. To this add the logarithm 0,36221,56887 (which is a constant).[16] The sum will be the logarithm of the number which designates the area DEVBAD, composed of three straight lines and the curve DAB. Let the parallelogram DC be assumed to have 100000,00000 parts. From this the area of DAB, a portion of the hyperbola, is easily found.

For example, let the ratio of DE to BV be 36 to 5. Then:

	1,55630,25008	(logarithm of 36)
−	0,69897,00043	(logarithm of 5)
	0,85733,24965[17]	(the difference of the logarithms)
	9,93314,92856	(the logarithm of the difference)
+	0,36221,56887	(a constant logarithm)
	10,29536,49743	(the logarithm of the area DEVBAD)

16. This number is -log log *e*. See *Oeuvres* 14:477.

17. In the 1673 edition this number is printed incorrectly as "0,85733,14965." 's Gravesande corrected this in the 1724 edition, changing the "1" to "2." Huygens gives the correct number in *Oeuvres* 14:477.

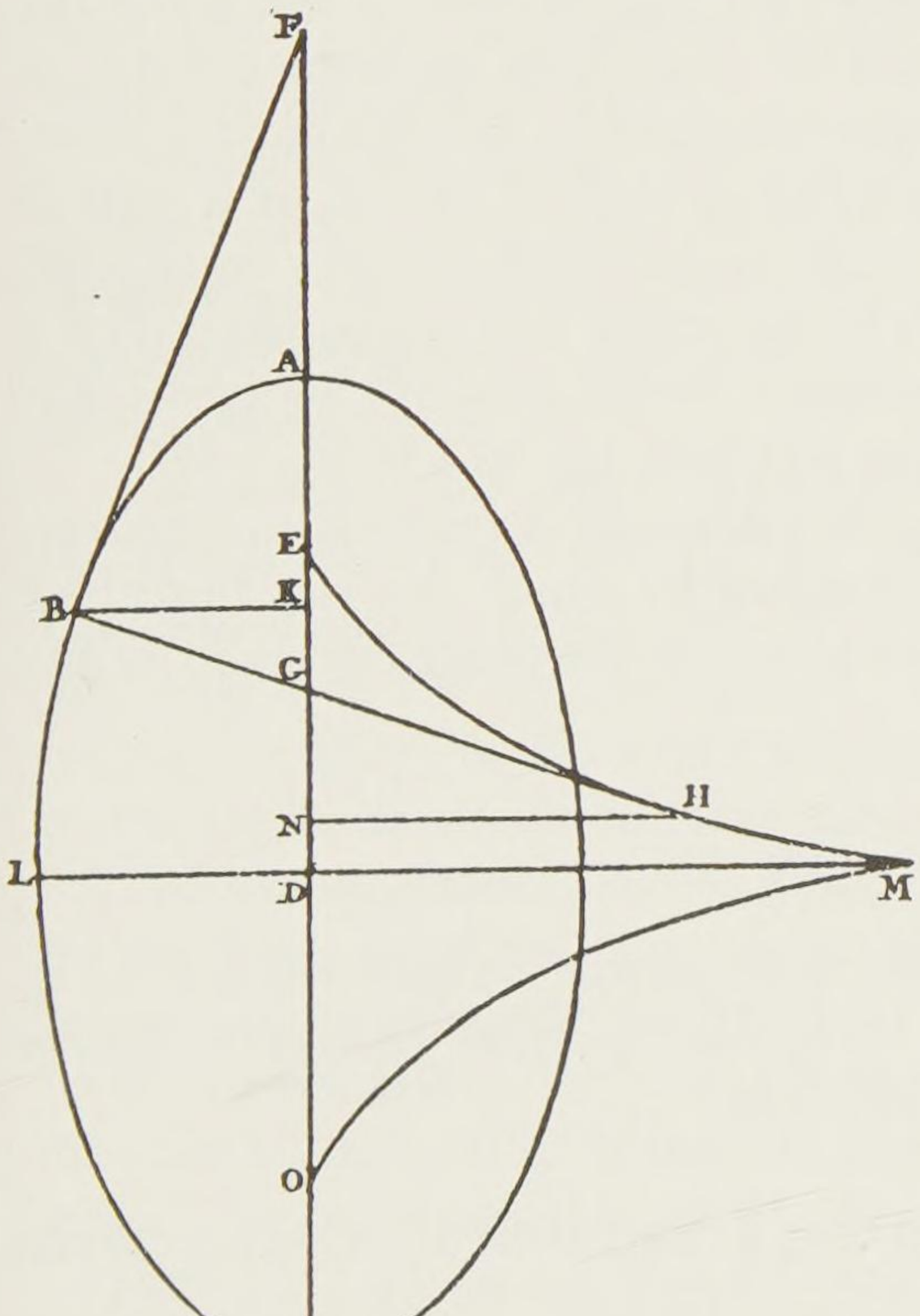
The number corresponding to this logarithm will have eleven figures, since the characteristic is 10. First find the number which corresponds most closely to the logarithm found above. This number is 19740. Next from the difference between this logarithm and the one immediately following it in the table, the remaining figures 81026 are determined. Write them after the first figures, giving the result 197408,10260, having added a zero at the end since the number of figures is eleven. Therefore the area of DEVBAD will equal approximately 197408, 10260 parts, i.e., the same size parts which we assumed when we said that the parallelogram DC had 100000,00000 parts.

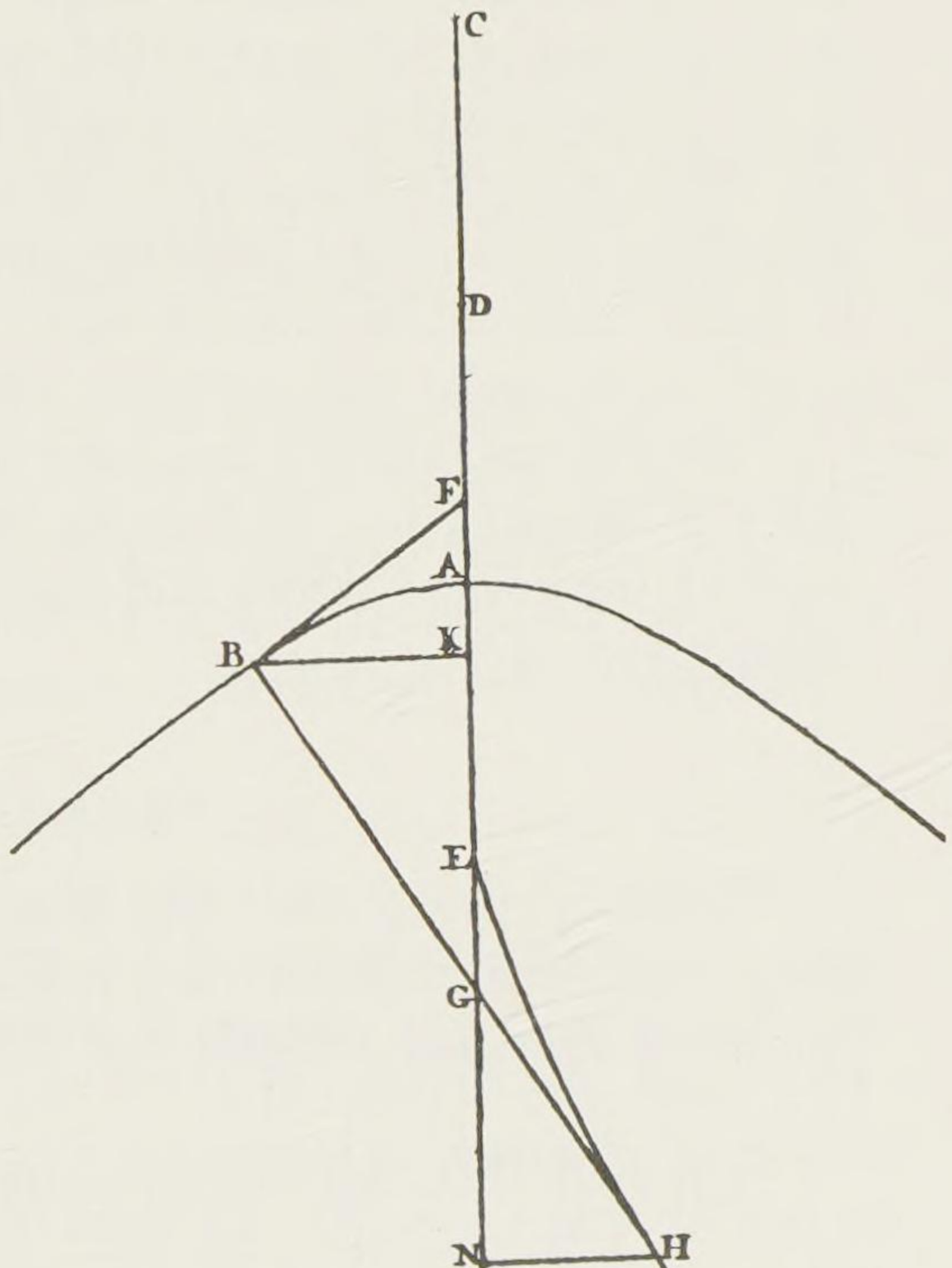
PROPOSITION X

Find curved lines whose evolutions describe ellipses and hyperbolas. And find straight lines equal to those curved lines.

Let AB be any ellipse or hyperbola [Fig. 53] whose transverse axis is AC, whose center is D, and whose *latus rectum* is twice AE. Selecting any point B on the section, let BK be perpendicular to the axis, and draw from

[Fig. 53]





B a tangent which meets the axis at F. Let BG be perpendicular to FB, meeting the axis at G. Extend BG to H such that BH has a ratio to HG equal to the ratio composed of the ratio of GF to FK and of AD to DE.

Now I say that the curve EHM, all of whose points can be found in the same way as H, is the curve whose evolution, together with EA, describes the arc AB. For BH is tangent to the curve at H and is equal to the whole line HEA. Hence, if EA is subtracted from HB, the remainder of HB is equal to the curve HE. But since each and every point of that curve can be found in the same certain way, the curve in both cases is among those which are determined by purely geometrical means. As a result the relation of all these points to the points of the axis AC can be expressed by a certain equation, which I find has as many as six terms, but which has the least terms when AB is a hyperbola whose *latus rectum* and *latus transversum* are equal. In that case, from any point on the curve, e.g., H, draw HN perpendicular to the axis CAN. Let a = AC, x = DN, and y = NH. Then the cube of $xx - yy - aa$ will always equal $27xxyyaa$. In this case the points of the curve EHM can be found much more quickly than in the previously mentioned construction, as we will show in what follows.

Furthermore it should be noted that for the ellipse each quadrant is described by the evolution of its own line; for example, the quadrant ABL by evolution of the line AEHM, and the quadrant CL by evolution of the similar opposite line COM. But in these two figures there is the following diversity. Although both the ellipse and the hyperbola have the point E as the beginning of the curve EHM, taking AE equal to half the *latus rectum,* in the hyperbola this curve extends to infinity, while in the ellipse it ends at the point M of the minor axis, taking LM equal to half of the *latus rectum,* according to which the perpendiculars to the minor axis are ordered. For it is easily apparent to anyone who considers the genesis of this curve that these are its ends, because in an ellipse AD is to DE as LM is to MD.

We will not take the time here to give a demonstration of these matters, but will instead go on to provide the method by which one can find these curves from conic sections and can find innumerable other curves from any other given curve.

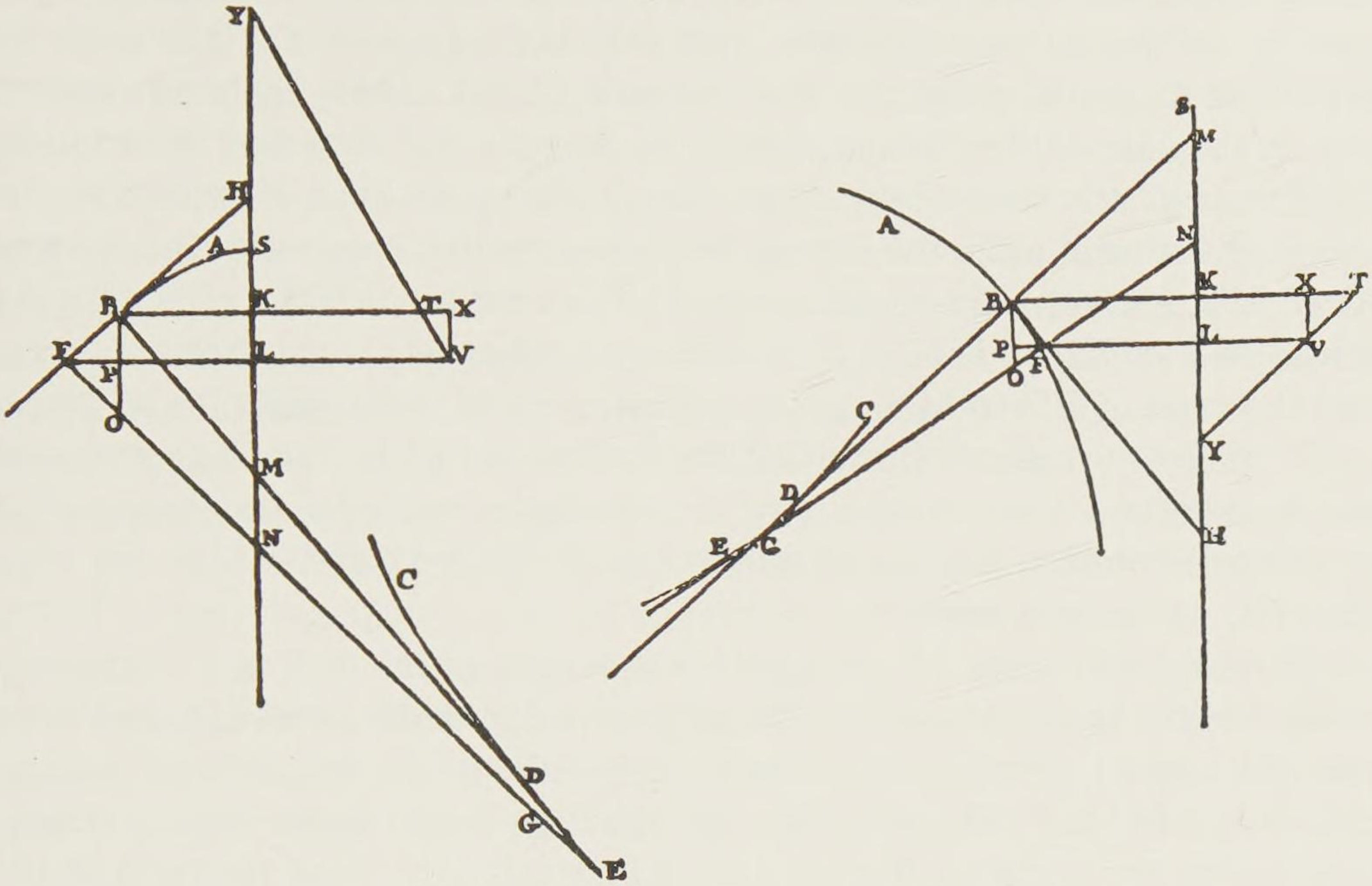
PROPOSITION XI

Given a curved line, find another curve whose evolution describes it. Show that for any geometrical curve, there exists another geometrical curve for which an equal straight line can be given.

Let ABF [Fig. 54][18] be any curved line, or part thereof, which is curved

18. In his copy of the 1673 edition Huygens wrote in the letter "C" which had been omitted in Fig. 54.

[Fig. 54]



in one direction. And let KL be a straight line to which all points are referred. We are required to find another curve, for example DE, whose evolution will describe ABF.

Assume that such a line has already been found. Now since all the tangents to the curve DE must meet at right angles with the line ABF, which is described by evolution, it is also clear in the reverse relation that lines which are perpendicular to ABF, for example, BD and FE, will be tangents to the evolute CDE.

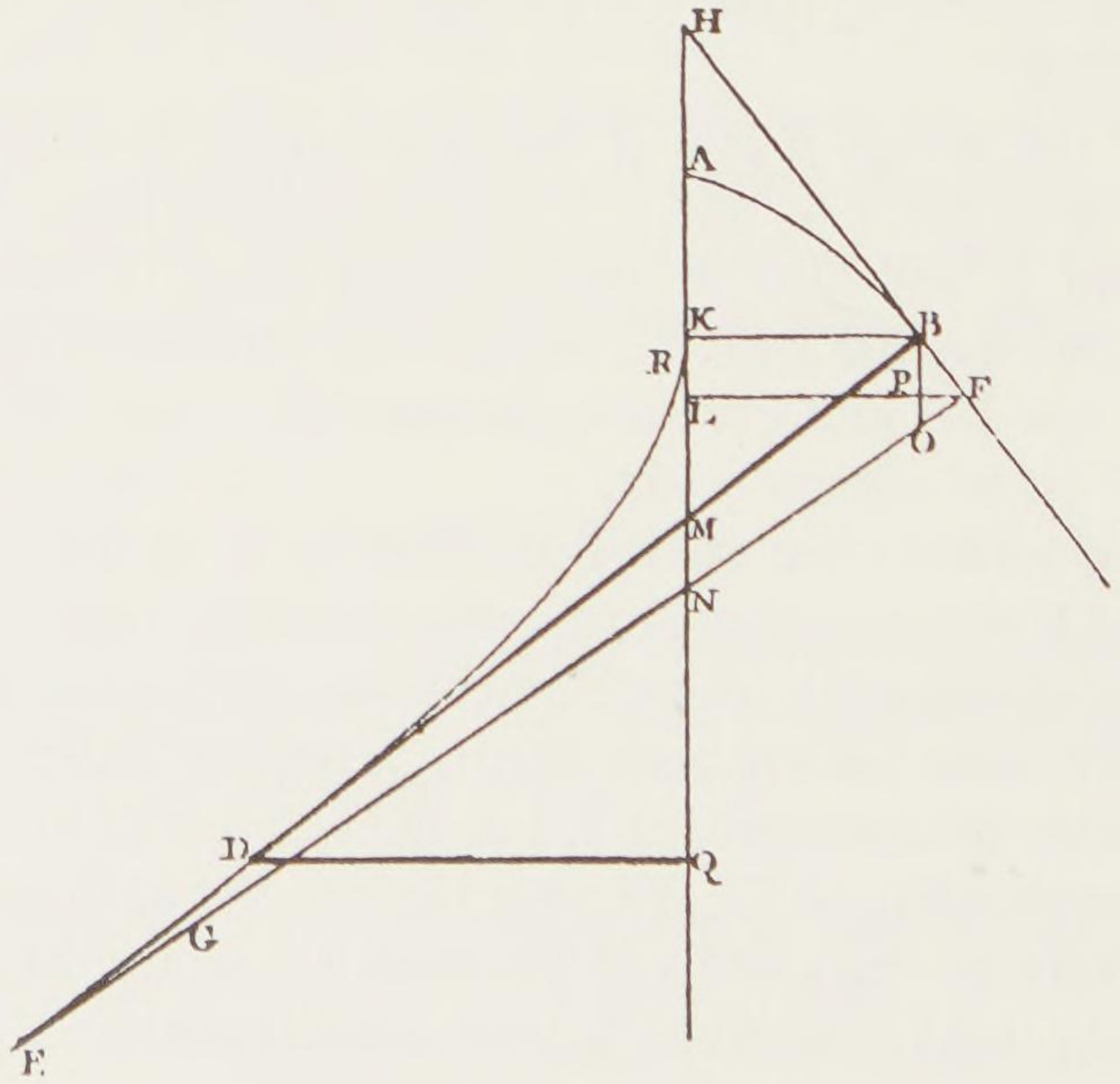
Next select the points B and F, which are close to each other. Now if the evolution begins from A, and if F is more distant than B from A, then the contact point E will also be more distant than D from A. And the intersection of the lines BD and FE, which is G, will fall beyond the point D on the line BD. For BD and FE must intersect since they are perpendicular to the curve BF on its concave side.

Moreover, insofar as the point F is closer to B, to the same extent the points D, G, and E will also appear to come together. And if the interval BF is taken to be infinitely small, these three points can be treated as one. As a result the line BH, after having been drawn, is tangent to the curve at B and also can be thought of as tangent at F. Let BO be parallel to KL, and let BK and FL be perpendiculars to KL. FL cuts the line BO at P, and let M and N be the points where the lines BD and FE meet KL. Since the ratio of BG to GM is the same as that of BO to MN, then when the latter is given, so is the

former. And when the line BM is given in length and in position, so is the point G on the extension of BM, and also D on the curve CDE, since we have taken G and D to be one. But the ratio of BO to MN is already known both in the case of the cycloid, which we investigated first and found to be 2 to 1, and in the case of the other curves which we have examined so far where we found it to be the sum of two given ratios. Now since the ratio of BO to MN is composed of the ratio of BO to BP or of NH to LH plus the ratio of BP or KL to MN, it is clear that if either of these latter are given, then the ratio of BO to MN, which is composed of them, will also be given. It will be clear in what follows that the former are given for all geometrical curves. And as a result by their use it is always possible to designate curves by whose evolution the given curves are described, and which are thus reducible to straight lines.

First let ABF [Fig. 55] be a parabola whose vertex is A and whose axis is AQ. Let the lines BM and FN be perpendicular to the parabola, and draw lines BK and FL perpendicular to the axis AQ. It is a property of a parabola that MK and NL will each be equal to half of the *latus rectum*. Subtracting the common element LM, KL equals MN. Now the ratio of BG to GM is composed of the ratios of NH to HL and of KL to MN, as was said, and the latter is equal to a ratio of one to one. Hence it follows that the ratio of BG to GM is equal to the ratio of NH to HL. By dividing, BM is to MG as NL is to LH, or as MK is to KH, for LH and KH can be taken as the same line because of the closeness of the points B and F. But if point B is given, so is the ratio of MK to KH, since the length of both MK and

[Fig. 55]



KH are known, that is, MK equals half of the *latus rectum* and KH is twice KA. The line BM is also given in position and length. Hence MG is known, and thus so is the point G, or D, on the curve RDE, by extending BM up to G such that BM is to MG as half the *latus rectum* is to twice KA.

And thus, by taking as many other points besides B on the parabola ABF as we wish, the same number of points can be found on the line RDE, using the same argument. From this it is established that RDE is a geometrical line, and this is a property from which other conclusions can be deduced. For example, let us try to find the equation which expresses the relation of all the points of the curve RDE to the straight line AQ. Draw DQ perpendicular to AQ, and let a = the *latus rectum* of the parabola ABF, b = AK, x = AQ, and y = QD. Since the ratio of BM to MD, that is, of KM to MQ, equals the ratio of $\frac{1}{2}a$ to $2b$, and since KM = $\frac{1}{2}a$, MQ will equal $2b$. But MA = $\frac{1}{2}a + b$. Hence AQ or x equals $3b + \frac{1}{2}a$. Thus $b = \frac{1}{3}x - \frac{1}{6}a$. Next since the square of MK, that is, $\frac{1}{4}aa$, is related to the square of KB, that is ab, as the square of MQ, that is $4bb$, is related to the square of QD, then the square of QD, or yy, equals $\frac{16b^3}{a}$. For b let us substitute $\frac{1}{3}x - \frac{1}{6}a$, to which it has been found equal. Then $yy = 16(\frac{1}{3}x - \frac{1}{6}a)^3$ divided by a. And thus $\frac{27ayy}{16} = (x - \frac{1}{2}a)^3$. Taking AR on the axis of the parabola equal to $\frac{1}{2}a$, then RQ = $x - \frac{1}{2}a$. Therefore it follows that the nature of this curve RDE is such that the cube of the line RQ will always be equal to a parallelepiped whose base is the square of QD and whose height is $\frac{27}{16}a$. And thus this is the same paraboloid whose evolution, as we have shown above, describes the parabola AB. Indeed the *latus rectum* of this paraboloid equals $\frac{27}{16}$ of the *latus rectum* of the parabola AB. For the *latus rectum* of the latter is equal to $\frac{16}{27}$ of the *latus rectum* of the paraboloid, as has been determined here.

Furthermore it is clear how the ratio of OB to BF, or of NH to HL, can always be found, not only for the parabola ABF but also for any other geometrical curve. Draw the straight line FH as a tangent to the curve at some selected point F [Fig. 54], and let FN be perpendicular to FH. From this NH and HL will be given, and hence also their ratio.

But it is not equally clear how the ratio of KL to MN can be determined. So let us show how this can also always be found.

Let the straight lines KT and LV be perpendicular to KL. Let KT equal KM, and LV equal LN. Draw VX parallel to LN and let it meet KT at X. Now the difference between LK and NM will always be the same as the difference between LN and KM, that is, between LV and KT. But the difference between LV and KT equals XT. And XV equals LK. Therefore NM is equal either to the sum of VX and XT, or to the excess of VX over XT. And thus, if the ratio of VX to XT is given, then there will also be given the ratio of VX either to the sum of VX and XT or to the excess of VX over XT; in other words the ratio of VX or LK to NM will be given.

But it should be noticed that, since we assumed that KT equals KM and that LV equals LN, the points T and V could be located on either a straight or a curved line, as we will soon show. It is a straight line if ABF happens to be a conic section whose axis is KL. In that case the ratio of VX to XT is given when the position of the line VT, on which the points V and T are located, is given. This ratio is always the same, whatever the interval KL might be.

But if the line is a curve, the ratio of VX to XT varies insofar as the interval KL is larger or smaller. It is necessary to inquire what this ratio will be when KL is imagined to be infinitely small, since we also assumed that the points B and F are very close to each other. Likewise we must also think of the points V and T as marking off the smallest part of a curved line, and hence the line VT coincides with a tangent to the curve at T. Let TY be that tangent, for it is possible to draw TY since the curve on which T and V are located is a geometrical curve.[19] Hence the ratio of YK to KT will be given, and thus also the ratio of VX to XT, from which we have shown that the ratio of LK to NM is given.

To find the kind of line on which T and V are located, select a point S on the line KL. Let x = SK, and y = KT. The curve ABF was given, and BM was drawn at right angles to it. Therefore the length of the line KM will be found, by use of the method of tangents provided by Descartes,[20] to be equal to KT or y. From this equation one can determine the nature of the curve to which the tangent was drawn. All of this will be clearer in the following example.

Let ABF [Fig. 56] be the same paraboloid which we found above to be equal to a straight line. In this case the cubes of the perpendiculars to the line SK are related to each other as the squares of the abscissas on SK. We are required to find a curve CDE by whose evolution the paraboloid SBF is described.

First the ratio of BO to BP is easily found here since we know that the tangent to the paraboloid at B was drawn by assuming that SH equals half of SK. And since BM is perpendicular to this tangent, the lines MH and HK are given, and thus so is their ratio, which is equal to the ratio of OB to BP.

Next, to determine the ratio of BP or KL to MN, draw perpendicular to KL the lines KT and LV, which are equal respectively to KM and LN, of which LN is the larger.[21] Let VX be parallel to LK. Now from the sum of

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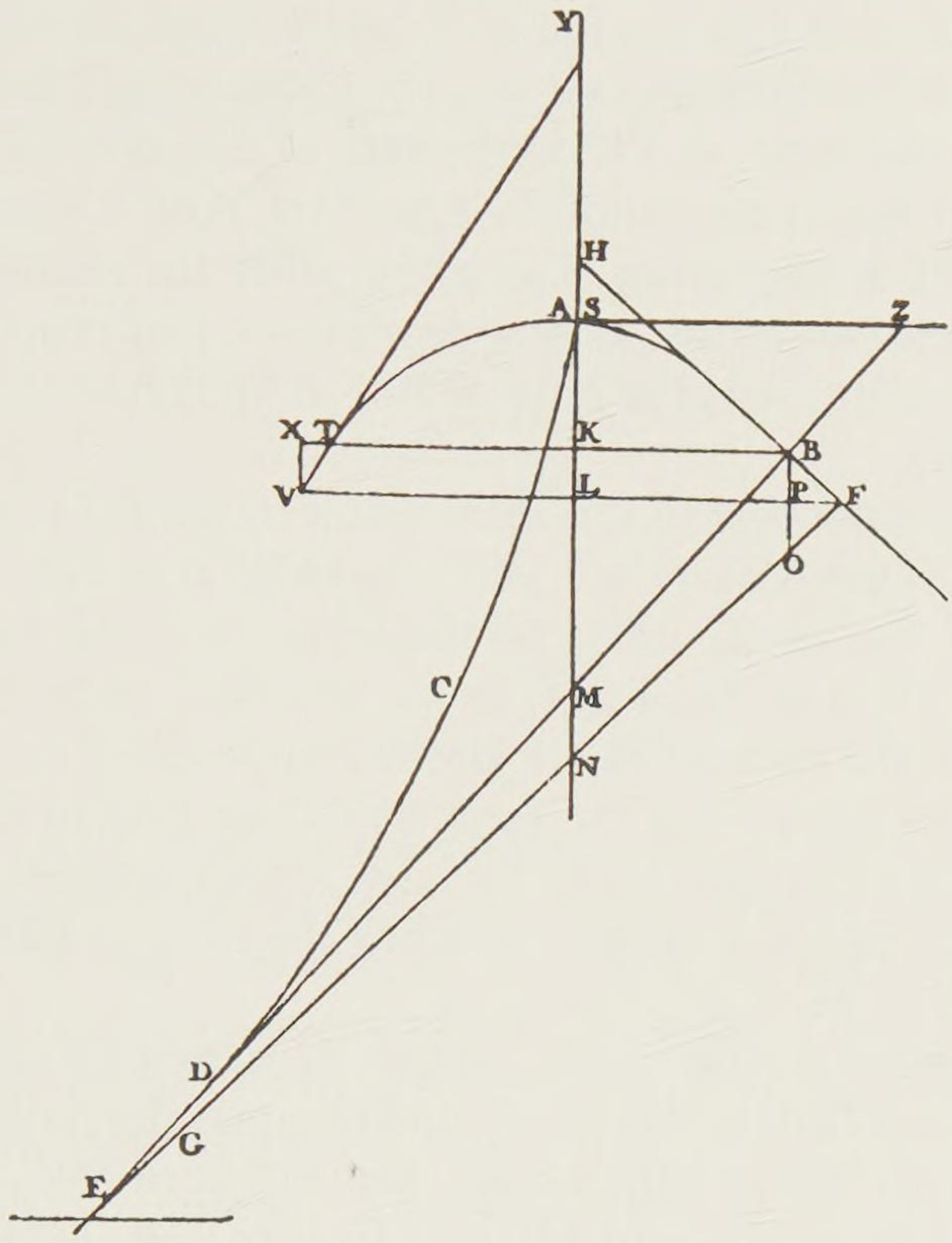
19. See *Oeuvres* 14:442–48.

20. Descartes, *Geometry,* Book II. Adam and Tannery, eds. 6:413–24.

21. The words "of which LN is the larger" were added in Latin in the "Corrigenda" of the 1673 edition. In his copy Huygens wrote in Latin in the margin, "It is assumed here that the line LN is larger than KM, which it would have been better to have proven; still it is true."

In a footnote on page 112 of the 1724 edition, 's Gravesande added in Latin: "This demonstration is not difficult. Let x = the abscissa SK, u = the perpendicular KB, and a = the *latus rectum* of the paraboloid.

[Fig. 56]



KL and LN subtract KM, leaving MN. In other words from the sum of XV and VL, or the sum of XV and XK, subtract KT, which clearly leaves the sum of VX and XT. Therefore the sum of VX and XT equals MN. Hence also the ratio of KL to MN is the same as the ratio of VX to the sum of VX and XT. In order to determine this ratio when the interval KL is very small, it is necessary, according to what was said above, to determine the *locus* or the line on which the points T and V exist. To do this, let a = the *latus rectum* of the paraboloid ABF, let x = SK, and y = KT.

Since KH, KB, and KM are proportional, since HK = $\frac{3}{2}x$, and since from the nature of a paraboloid, KB equals the cube root of axx, then KM or KT equals $\frac{2}{3}$ of the cube root of aax, which equals y. From this $y^3 = \frac{8}{27}aax$. Hence it is clear that the points T and V are located on a paraboloid

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Since SH = ½ SK, HK = ¾ SK ($\frac{3}{2}x$). And since HBM is a right angle, HBK and KBM are similar right triangles, and HK ($\frac{3}{2}x$), KB (u), and KM are in continuous proportion. Therefore KM = $\frac{2uu}{3x}$, and its square $\frac{4u^4}{9xx} = \frac{4au^4}{9axx}$. But the author has already pointed out that, from the nature of the paraboloid ABF, $u^3 = axx$. Hence the square of the line KM = $\frac{4au^4}{9axx} = \frac{4au^4}{9u^3} = \frac{4}{9}au$. From this it follows that KM is increased if BK (u) is increased. But since LF is larger than BK, then LN will be larger than KM. Q.E.D.

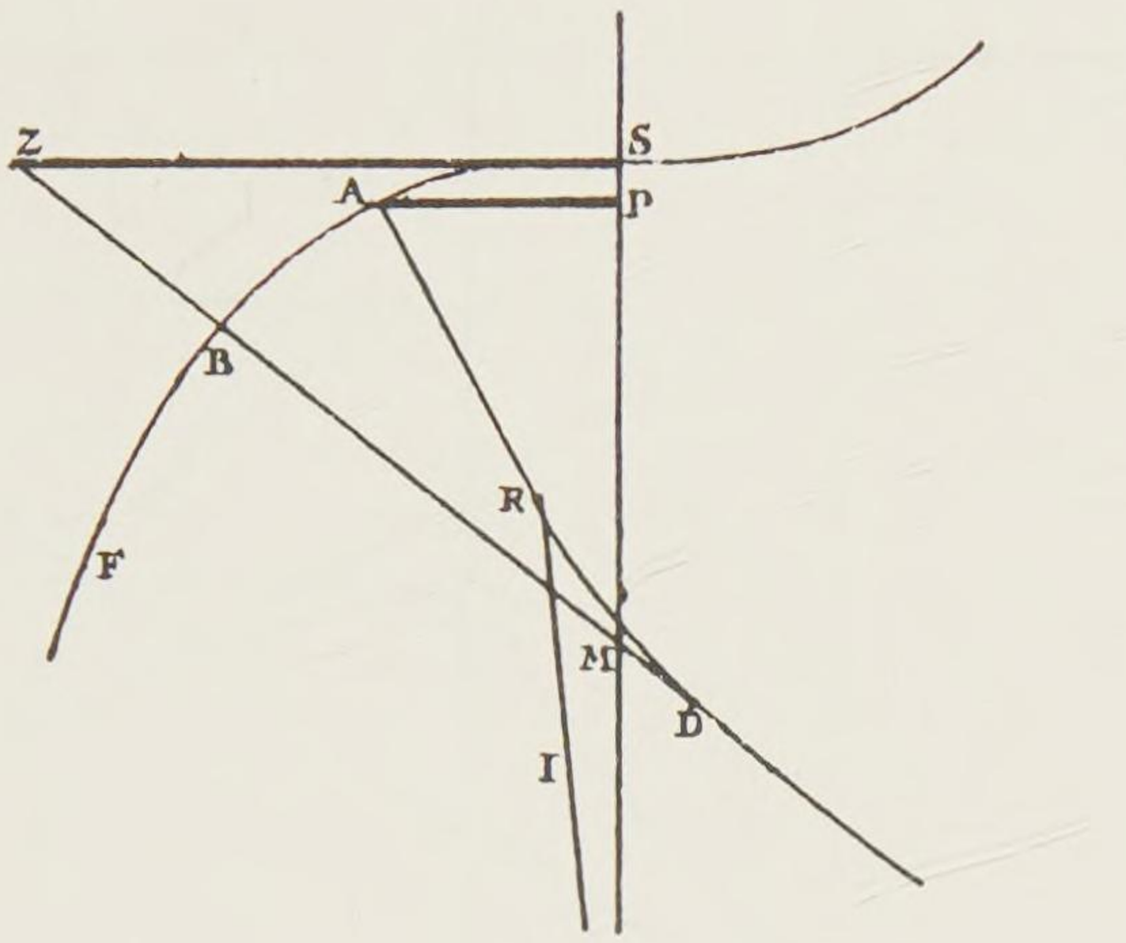
which geometers call cubical. To this curve draw a tangent at T, taking SY equal to twice SK, and join the points Y and T. Now the ratio of VX to the sum of VX and XT, which we said was the same as the ratio of KL to MN, will be equal to the ratio of YK to the sum of YK and KT. But when this ratio is given, so also is the ratio of KL to MN. And it was shown that the ratio of OB to PB is also given. Therefore, since the ratio of BD to DM is composed of these two ratios, as was clear above, this ratio of BD to DM is given, and by division, so is the ratio of BM to MD. And thus the point D is on the curve DE.

However we can come to this same result by a most brief construction. y = KT or KM. Thus MH = $y + \frac{3}{2}x$. And MH is to HK, or OB is to BP, as $y + \frac{3}{2}x$ is to $\frac{3}{2}x$, or, doubling all the terms, as $2y + 3x$ is to $3x$. Next, since YK = $3x$, YK is to YK + KT, or KL is to MN, as $3x$ is to $3x + y$. But we have said that the ratio of BD to DM is composed of the ratios of OB to BP and of KL to MN. Thus the ratio of BD to DM is composed of the ratios of $2y + 3x$ to $3x$, and of $3x$ to $3x + y$, and hence is the same as the ratio of $2y + 3x$ to $3x + y$. And by dividing the ratio of BM to MD is equal to the ratio of y to $3x + y$.

Let SZ be perpendicular to SK, and let the extension of MD meet SZ at Z. Now the ratio of BM to MD was found to equal the ratio of y to $y + 3x$, that is, the ratio of MK to MK + 3KS. But MK is to MK + 3KS as MB is to MB + 3BZ. Hence MB is to MD as MB is to MB + 3BZ. From this it follows that MD can be taken to be equal to MB + 3BZ. And in this same way as many points of the curve CDE can be found as one wishes. Thus any arc of this curve, for example, DS, will be equal to a straight line, DB, which is perpendicular to the paraboloid SAB. Therefore it is established that this is a geometric curve, and if we wish, we can express in an equation the relation of all of its points to the points of the axis SK.

In the same way, if we extend our investigation to that paraboloid or cubical parabola [Fig. 57] in which the cubes of the perpendiculars to the axis are related as the portions of the abscissas on the axis, we will find that the curve by whose evolution it is described, and which can be equated with a straight line, is determined by its points through a construction which is no more difficult. Let SAB be such a curve, and let SM be its axis (even though it is improper to speak of an axis in this curve since its form is such that, when SZ is drawn perpendicular to SM, similar parts of the curve are located on each side of SZ). Through any point B on the paraboloid, draw the line BD which cuts the curve at right angles and which meets the axis at M and the line SZ at Z. Next let BD equal half of BM plus three-halves of BZ. D will be one of the points of the sought-for curve RD or RI by whose evolution, together with the straight line RA, the paraboloid SAB was described. However it is worth noting in this case, and also in some other cases of paraboloids of this type, that there are two evolutions in opposite

[Fig. 57]



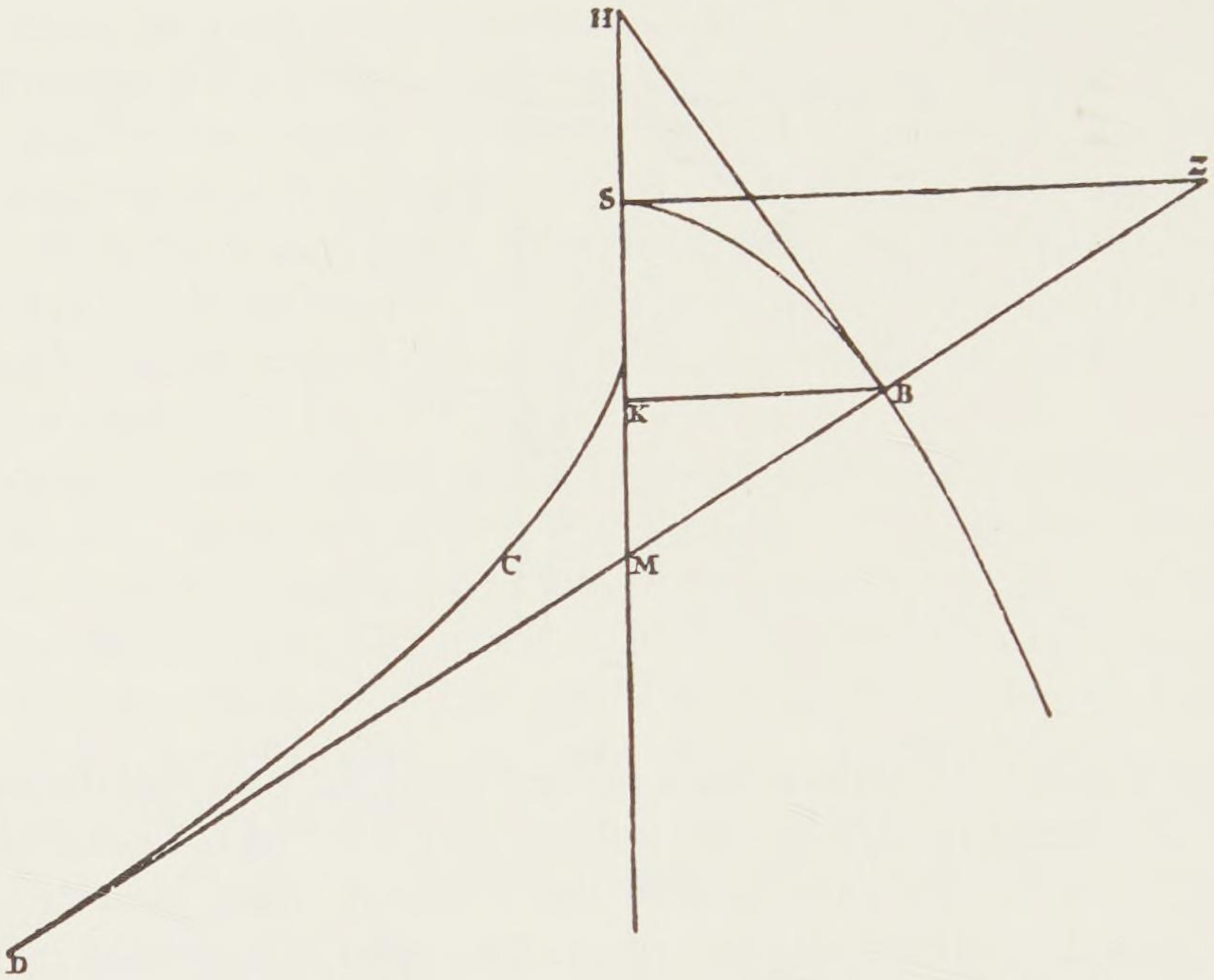
directions which both start from a certain point A. First, by the evolution of ARD, extended to infinity, the infinite part ABF of the paraboloid was described; then by the evolution of the whole of ARI, also extended to infinity, only the part AS was described. The point A is defined by drawing the perpendicular PA after letting SP be related to the *latus rectum* of the paraboloid as one is related to the eighth root of 91125 (this number is the cube of 45). And the point R, the common end of the two curves RD and RI, is found in the same way as all the other points of these curves, for example, as the point D was just found.

Finally, no matter what the curve SAB is in the genus of paraboloids, we have seen that it is always easy to find through its points another curve by whose evolution it was described and which therefore can be equated with a straight line. As a result we will display a universal construction in the following table, which can be extended as far as one wishes.

$$\text{If} \left\{ \begin{array}{ll} ax = y^2 & \\ a^2x = y^3 & \\ ax^2 = y^3, & \text{then} \\ ax^3 = y^4 & \\ a^3x = y^4 & \end{array} \right. \left\{ \begin{array}{r} \text{BM} + 2\,\text{BZ} \\ \frac{1}{2}\text{BM} + \frac{3}{2}\text{BZ} \\ 2\,\text{BM} + 3\,\text{BZ} \\ 3\,\text{BM} + 4\,\text{BZ} \\ \frac{1}{3}\text{BM} + \frac{4}{3}\text{BZ} \end{array} \right\} = \text{BD}$$

Let SB [Fig. 58] be a parabola or a paraboloid, whose vertex is S. SK is either the axis or a perpendicular to the axis to which the points of the paraboloid are referred by an equation. SK is to be understood as drawn always on the concave side of SB, and SZ is perpendicular to SK. Now let x = SK, let y = BK, which is a perpendicular to SK from any point on the curve, and let a = the *latus rectum* of the curve. The first part of the table on the left indicates the nature of each paraboloid by its own equation. To

[Fig. 58]



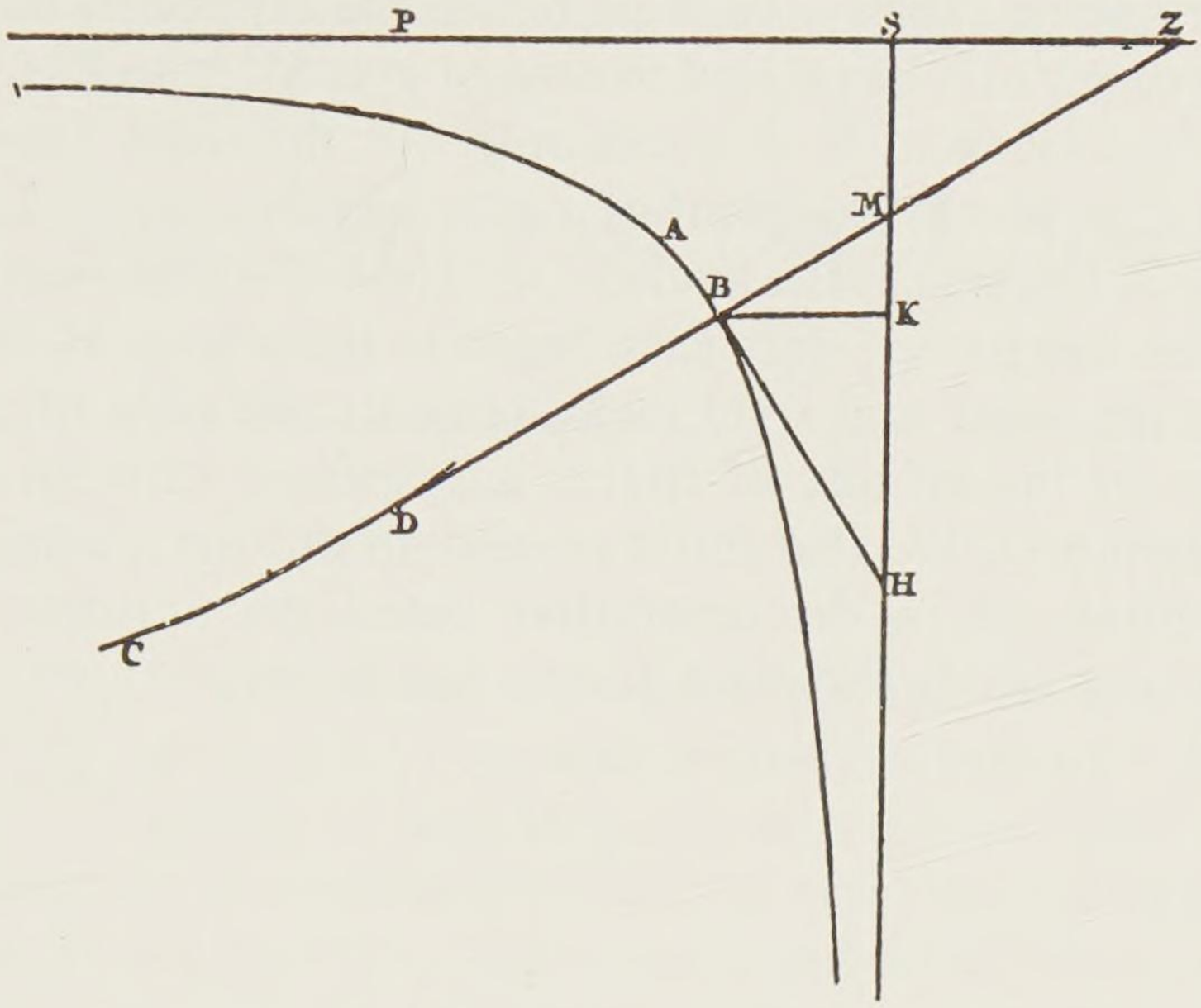
these there correspond on the right side the lengths of the line BD, which determine the point D on the sought-for curve DC, granting that BD meets the curve SB at right angles. For example, if SB is a parabola, which is a conic section, we know that its equation is the first one in the table, $ax = y^2$. To this there corresponds on the other side BM + 2 BZ = BD. From this the length of the line BD is known, and thus the means of finding any point of the curve DC. This latter curve, in this case, is a paraboloid, as demonstrated above, whose equation is the third one in the table.

The table is constructed as follows. The coefficient of BM is the exponent of x in the equation, and the coefficient of BZ is the exponent of y. The expression composed of both is then divided by the exponent of a.

In addition to these paraboloidal lines, we have found other lines to which curves comparable to straight lines can be reduced by using a similar construction. These lines are similar to the hyperbola in that they have asymptotes which always meet at right angles. The first of these, which we have already dealt with, is the hyperbola itself, which is a conic section.

In order to give an account of the nature of the other lines, let PS and SK [Fig. 59] be asymptotes of the curve AB which meet at right angles. From any point of the curve B, draw BK parallel to PS. Let x = SK, and y = KB. If AB is a hyperbola, we know that the rectangle formed by the lines SK and KB, that is, the rectangle xy, will always be equal to a square, which we call aa.

[Fig. 59]



The next hyperboloid is the one in which the solid formed by the square of the line SK and KB as its height, that is, the solid *xxy,* will be equal to a certain cube, which we call a^3. And thus there exist innumerable other hyperboloids of this type. The following table expresses the properties of each by an equation and also the means of constructing the curve DC by whose evolution each is generated.

$$\text{If} \begin{cases} xy = a^2 \\ x^2y = a^3 \\ xy^2 = a^3, \\ x^3y = a^4 \\ xy^3 = a^4 \end{cases} \text{then} \begin{cases} \frac{1}{2}\text{BM} + \frac{1}{2}\text{BZ} \\ \frac{2}{3}\text{BM} + \frac{1}{3}\text{BZ} \\ \frac{1}{3}\text{BM} + \frac{2}{3}\text{BZ} \\ \frac{3}{4}\text{BM} + \frac{1}{4}\text{BZ} \\ \frac{1}{4}\text{BM} + \frac{3}{4}\text{BZ} \end{cases} = \text{BD}$$

As before, the line DBMZ cuts the curve AB at right angles and meets the asymptotes SK and SP at M and Z. For example, if AB is a hyperbola, whose equation is $xy = a^2$, then BD = ½BM + ½BZ, as the table indicates. Point D will be located on the sought-for curve DC, of which any number of other points can be found in the same way, and any arc of which can be equated with a straight line. DC is the same curve whose relation to the axis of the hyperbola we have expressed above in an equation. The construction of this table is clearly the same as the previous table.

Furthermore, since for the construction of both these curves and the curves generated by paraboloids it was necessary to draw the lines DBZ which at a given point B cut the curves AB, or their tangents BH, at right angles, let us explain universally how these tangents are found. In the

equation which explains the nature of each curve—the equations are given in the two preceding tables—we must notice the exponents of x and y, and make the exponent of x be related to that of y as SK is to KH. Draw HB as a tangent to the curve at B. For example, for the third hyperbola whose equation is $xy^2 = a^3$, the exponent of x is 1, and that of y is 2. Hence SK is to KH as 1 to 2. Experts in the analytic art know the demonstration of this. For a long time has passed since they began to investigate all these lines and to determine the plane and solid measurements not only of these paraboloids but also of the infinity of spaces interspersed between hyperboloids and their asymptotes. We also could present all this here, using an easy and universal method for a demonstration based on a single property of tangents. But this is not the place for such matters.

PART IV

The Center of Oscillation

WHEN I was still quite young, the learned Mersenne proposed to me and to many others an inquiry into the centers of oscillation or of motion.[1] This problem was still a celebrated one among the geometers of that time, as I gather from the letters he sent to me and from the recently published letters of Descartes, which contain a response to Mersenne's letters on these matters. He asked that I try to find such centers in the case of sectors of a circle which are suspended either at the angle or from the center of the arc and then moved laterally, and also in the case of segments of a circle and of triangles which are hung either from the vertex or from the middle of the base. This reduces to finding a simple pendulum, that is, a weight attached to a string, whose length is such that it completes its oscillations in the same time as the suspended figures mentioned above. At the same time he also promised a quite large and enviable reward for my work if I could satisfactorily provide what he sought. But no one could then offer what he desired. As for me, I could find nothing on my first attempt which could open up this investigation; being stymied at the threshold, as it were, I dropped any further inquiry at that time. Descartes,[2] Honoré Fabri,[3] and other famous men[4] who held the promise of success, also failed to hit the target except in a few of the easier cases. But even there, it seems to me, they offered no appropriate demonstrations. In the light of what we will present here, I hope that this will be clear to anyone who compares their work and ours. The latter, I think, is demonstrated by more certain principles and will be found to be entirely in agreement with experience. The occasion of dealing

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For various earlier writings by Huygens on topics in Part IV, see *Oeuvres* 17:149–52, and 393–555.

1. Letters of M. Mersenne to Constantyn Huygens (Sept. 1646 and 12 Oct. 1646); Letter of Ch. Huygens to M. Mersenne (28 Oct. 1646); and Letter of M. Mersenne to Ch. Huygens (8 Dec. 1646), *Oeuvres* 1:19–28, 45–47.

2. For Descartes's contribution, see Descartes, *Oeuvres,* Adam and Tannery, eds. 4:366, and the editor's discussion on p. 370.

3. Honoratus Fabrius (Honoré Fabri, SJ) (1606–1688).

4. "Other" refers to Roberval and to William, Lord Brounker.

with these topics again arose from the need to regulate the time of the pendula of our clocks when a movable weight is added to them in addition to the weight located at the lowest point, as was explained in our description of the clock. As a result, by starting from the first origin and by using a better approach, I overcame all the difficulties, and found not only the solution to Mersenne's problem but also other more difficult things. I also found a method by which one can determine with certainty the center of oscillation in lines, surfaces, and solid bodies.[5] Thus, in addition to the pleasure of finding something which others have sought for a long time and of knowing the laws and decrees of nature in these matters, I also realized its usefulness for finding an easy and sure way to regulate the time of clocks, which was my original reason for studying these topics. Another result, which I think will be helpful to many, is that by this means I can offer a most accurate definition of length which is certain and which will last for all ages. This has been added at the end of the following considerations.

DEFINITIONS

I

A pendulum is any figure endowed with gravity, either a line or a surface or a solid, which is suspended such that it can continue its reciprocal motion, by the force of its own gravity, around a point or better around an axis which is understood to be parallel to the plane of the horizon.

II

The axis which is parallel to the plane of the horizon, and around which the motion of the pendulum is understood to occur, is called the axis of oscillation.

III

A simple pendulum is one which is understood to consist of a string, or an inflexible line devoid of gravity, and of a weight which is attached to the lowest part of the string. The gravity of the weight is understood to be located at one point.

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5. For a discussion of Huygens' methods of calculation in Part IV, see *Oeuvres* 16:361–73.

IV

A composite pendulum is one which consists of many weights which maintain a fixed distance from each other and from the axis of oscillation. It follows that any suspended figure endowed with gravity can be said to be a composite pendulum since it is divisible by thought into as many parts as one wishes.

V

Pendula are called isochronous when their oscillations through similar arcs are completed in equal times.

VI

The plane of oscillation is the plane which is perpendicular to the axis of oscillation and which is understood to run through the center of gravity of the suspended figure.

VII

The center line is a straight line which is perpendicular to the axis of oscillation and which is drawn through the center of gravity of the figure.

VIII

The perpendicular line is a straight line in the plane of oscillation which is drawn from the axis of oscillation perpendicular to the plane of the horizon.

IX

The center of oscillation or of rotation of any figure is that point in the center line whose distance from the axis of oscillation is equal to the length of a simple pendulum which is isochronous with that figure.

X

The axis of gravity is any straight line which passes through the center of gravity of the figure.

XI

A plane figure, or a line situated in a plane, is said to be moved in the plane when the axis of oscillation is in the same plane as that figure or line.

XII

A plane figure, or a line situated in a plane, is said to be moved laterally when the axis of oscillation is perpendicular to the plane of that figure or line.

XIII

When we say that weights are multiplied in straight lines, this is to be understood to refer to numbers or lines which represent the quantities of weights and their mutual ratios.

HYPOTHESES

I

If any number of weights begin to move by the force of their own gravity, their center of gravity cannot rise higher than the place at which it was located at the beginning of the motion.

In these cases the height is to be taken as the distance from a horizontal plane, and the heavy bodies are to be taken as trying to descend to this plane along straight lines perpendicular to the plane. This has been either expressly assumed by all who have treated of the center of gravity or has been added by their readers. For without this, there is no sense in a discussion of the center of gravity.

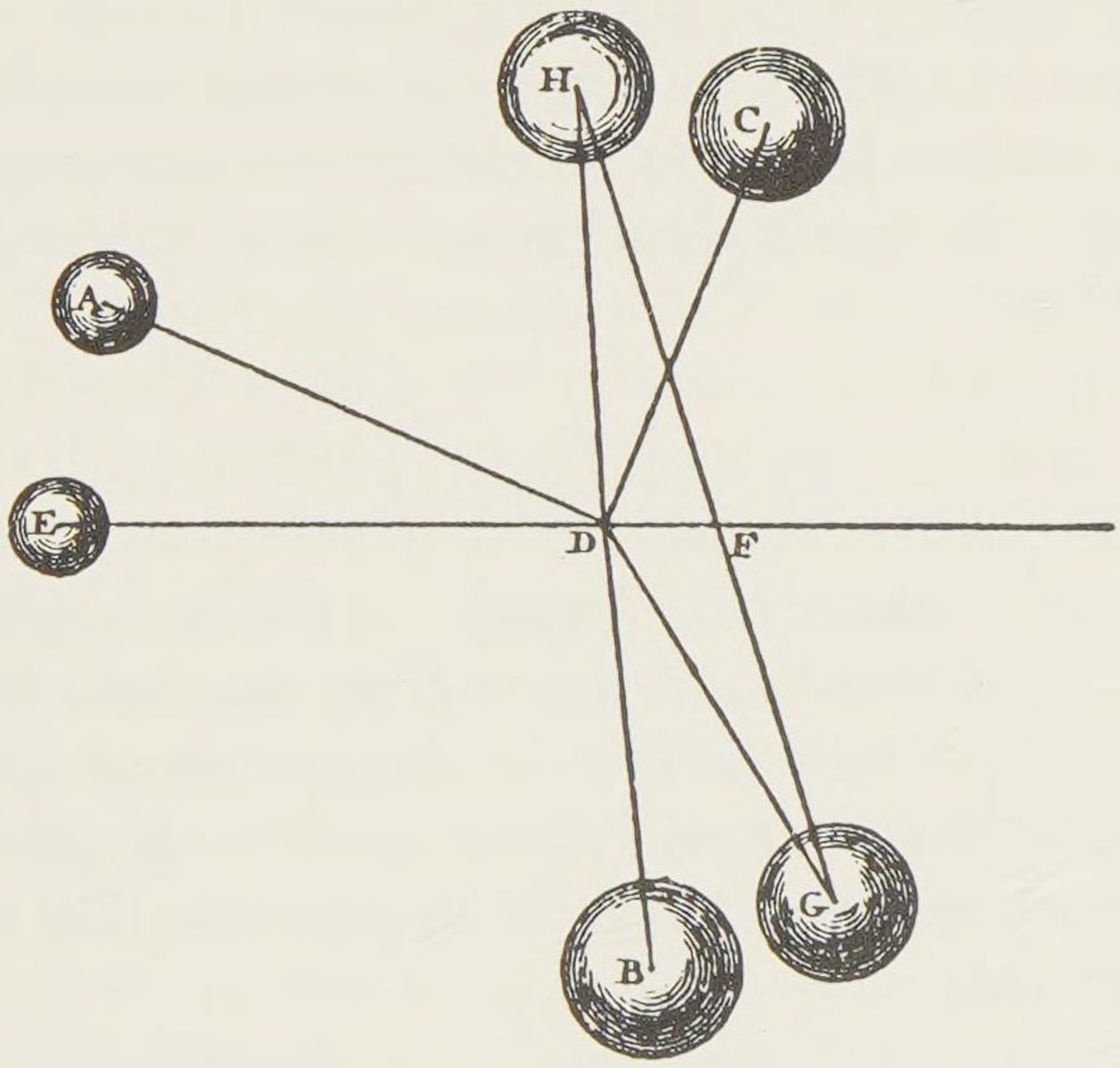
But lest our hypothesis create a doubt, we will show that it states only what no one has ever denied; namely, that heavy bodies do not move upwards. For first, if we consider only one heavy body, it is beyond doubt that it cannot ascend higher by the force of its own gravity, where "ascend" is taken to mean that its center of gravity ascends. Next the same thing must be conceded in the case of any number of weights joined to each other by inflexible lines, for there is no reason not to consider this as only one body. And thus their common center of gravity cannot rise higher.

Next, if we consider the case of any number of weights not joined to each other, we know that they also have a common center of gravity. The height of this center, I say, should be taken as equal to the height of the

center of gravity composed of all the weights. For all the weights can be brought to this height of the center of gravity with no other power being added than that located in the weights themselves. The only requirement is that the weights be arbitrarily joined by inflexible lines and moved around the center of gravity; no other force or determinate power is needed. Hence, just as it is not possible for the various weights located in the same horizontal plane to be all raised equally above that plane by the force of their own gravity, likewise it is not possible for the center of gravity of various weights, wherever they may be located, to rise to a greater height than it had. We will show as follows that any number of weights can be brought, without the addition of any force, to the horizontal plane which passes through their common center of gravity.

Let A, B, and C [Fig. 60] be weights in the given locations. Let their common center of gravity be D, through which passes a horizontal plane of which the line EF is a section. Let DA, DB, and DC be inflexible lines which connect these weights in fixed relations to each other. Finally the weights move until A is in the plane EF at E. Since all the rods rotated through equal angles, B will now be at G, and C at H.

[Fig. 60]



Next imagine that B and C are connected by the rod HG, which cuts the plane EF at F. This is also where the center of gravity of these two connected weights must be located. For D is the center of gravity of the three bodies located at E, G, and H; and the center of gravity of the body at E is also in the plane EDF. Next let the weights H and G be put in motion without any force around the point F as an axis. Both will arrive simulta-

neously at the plane EF. Hence it is clear that the three weights, which previously were at A, B, and C, have now been moved to the height of their center of gravity D by their own equilibrium. Q.E.D. The demonstration is the same for any other number of weights.

Furthermore, our hypothesis is also valid for liquid bodies. By its means it is possible to demonstrate not only all the things which Archimedes maintained about floating bodies but also many other theorems of mechanics. Indeed, if those builders of new machines who tried in vain to produce perpetual motion had known how to use this hypothesis, they would have easily seen their errors and would have understood that this is in no way possible through mechanical means.

II

Abstracting from the air and from every other manifest impediment, which we wish to be understood in the following demonstrations, the center of gravity of a rotating pendulum crosses through equal arcs in descending and in ascending.

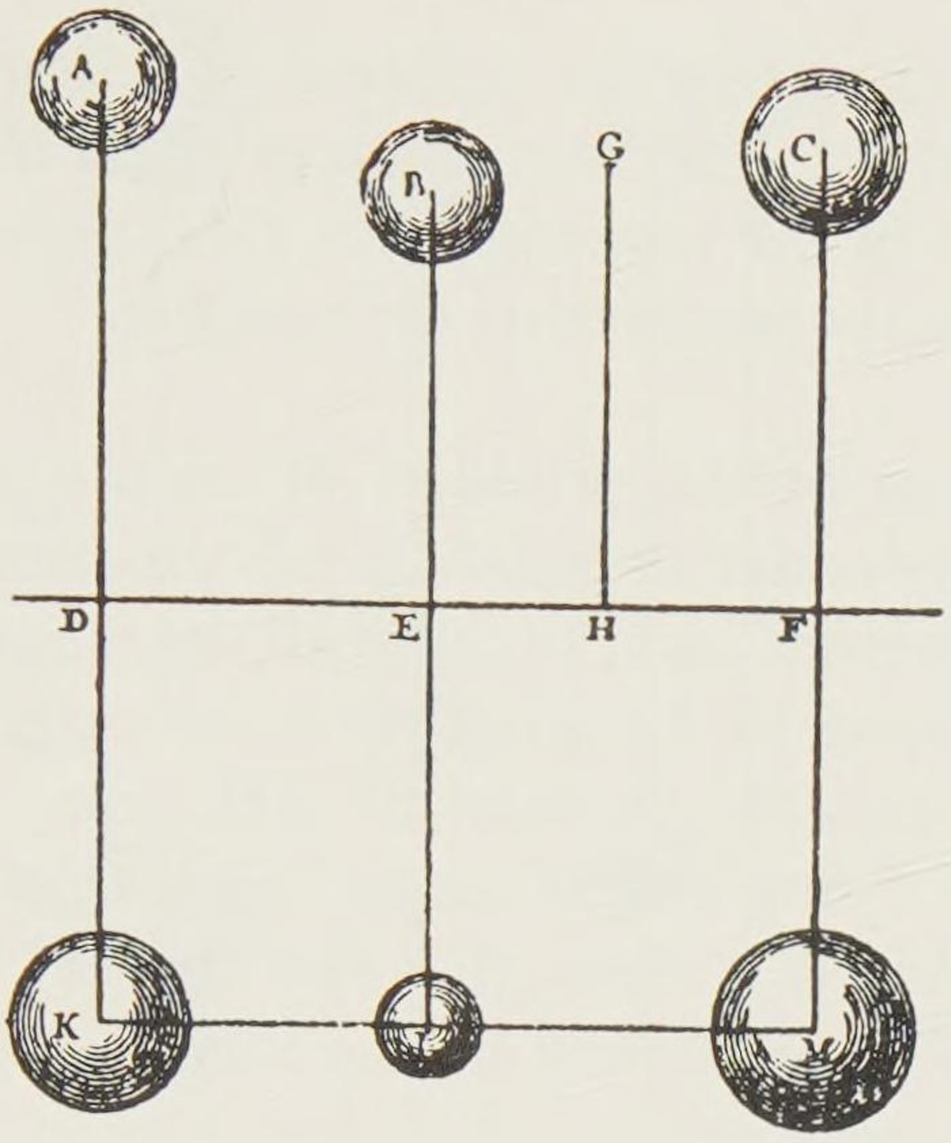
This has already been demonstrated for the simple pendulum in Proposition 9 of the Falling of Heavy Bodies [Part II]. And experience dictates that the same must be said of composite pendula. For whatever the figure of the pendulum is, it is found to be equally able to continue its motion, except in so far as it is impeded by the resistance of the air.

PROPOSITION I

If any number of weights are located on the same side of a plane, and if perpendiculars are drawn to that plane from the center of gravity of each weight, then the sum of the products of these perpendiculars multiplied by the corresponding weights is equal to the sum of the products of the perpendicular to that plane from the center of gravity of all the weights multiplied by all the weights.

Let A, B, and C [Fig. 61] be weights located on the same side of a plane, whose section is the line DF. Draw AD, BE, and CF from the corresponding weights perpendicular to that plane. Let the point G be the center of gravity of all the weights A, B, and C. From G draw GH perpendicular to the plane. Now I say that the sum of the products obtained by multiplying each weight by its corresponding perpendicular is equal to the sum of the products of all the weights multiplied by the line GH.

[Fig. 61]



Let the perpendiculars drawn from each weight be extended to the other side of the plane DF. And let DK, EL, and FM each be equal to HG. Each of these lines represents an inflexible rod parallel to the horizon. Assume that weights are located at K, L, and M such that they are in equilibrium around the plane DEF with each of the opposing weights A, B, and C. Hence the sum of K, L, and M will be in equilibrium with the sum of A, B, and C. Therefore the length of AD is to DK as the weight of K is to the weight A. Thus DA, multiplied by the magnitude of A, is equal to DK, or GH, multiplied by K. Likewise EB times B equals EL, or GH, times L. And FC times C equals FM, or GH, times M. Hence the sum of the products of AD times A, BE times B, and CF times F equals the sum of the products of GH times K and L and M. But since K, L, and M are in equilibrium with A, B, and C, they will also be in equilibrium with A, B, and C when the latter are suspended from their center of gravity G. Therefore, since DK, EL, and FM each is equal to the distance GH, the sum of the magnitudes of A, B, and C must be equal to the sum of K, L, and M. Hence the sum of the products of GH times A and B and C will be equal to the sum of the products of DA times A, EB times B, and FC times C. Q.E.D.

In this demonstration the lines AK, BL, and CM were drawn parallel to the horizon, and the plane was drawn perpendicular to the horizon. However, it is clear that if everything is rotated together to any other position, the same equality of the products will be preserved since all the lines are the same as before. Thus the proposition is established.

PROPOSITION II

Given the same conditions as before except that all the weights A, B, and C [Fig. 61] are equal, I say that the sum of all the perpendiculars AD, BE, and CF is equal to the perpendicular GH, drawn from the center of gravity, multiplied by the number of the weights.

For since the sum of the products of each weight multiplied by its corresponding perpendicular is equal to the product of GH times all the weights, then in the present case, because all the weights are equal, the sum of those products is equal to the product of one weight times the sum of all the perpendiculars. Also the product of GH times all the weights is the same as the product of one weight times GH multiplied by the number of the weights. From this it is clear that the sum of the perpendiculars is necessarily equal to GH multiplied by the number of the weights. Q.E.D.

PROPOSITION III

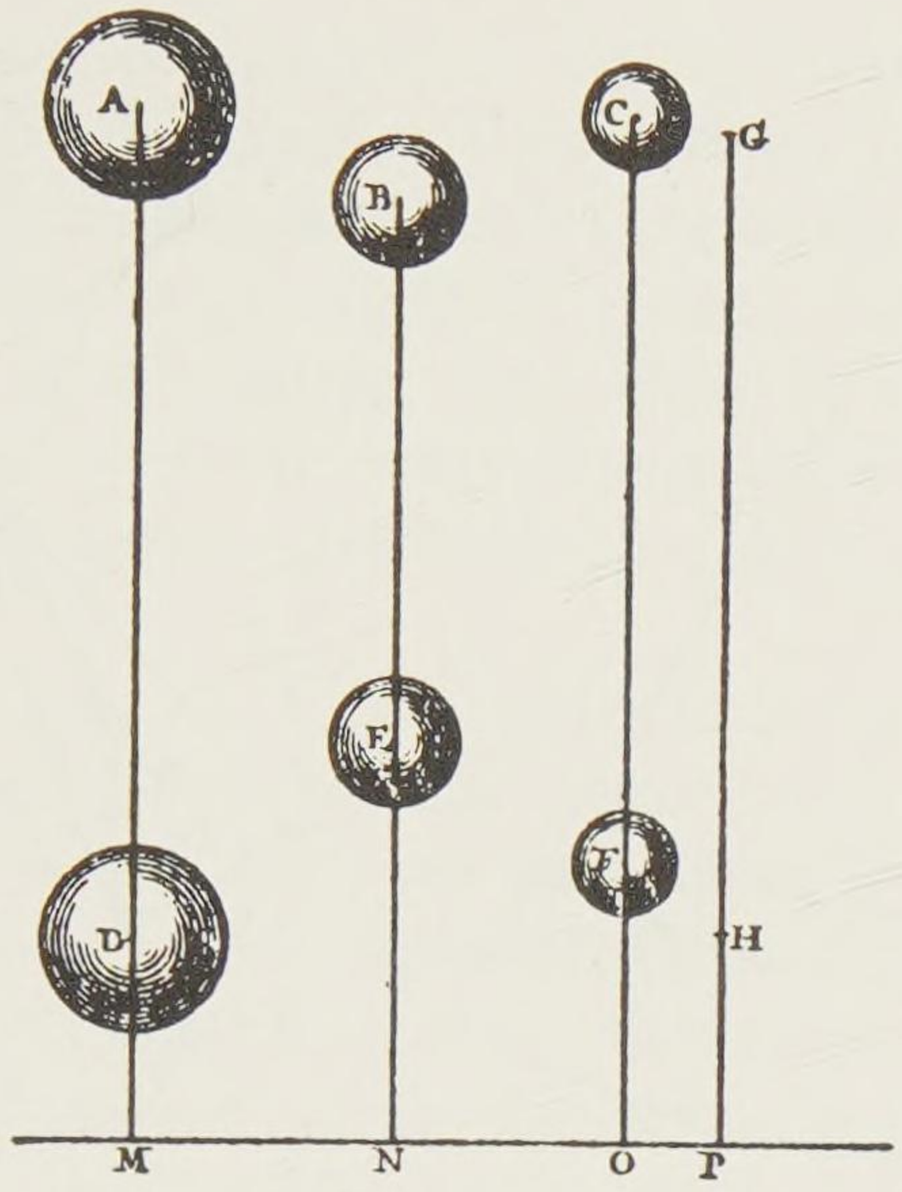
If any number of bodies all fall or rise, but through unequal distances, the sum of the products of the height of the descent or ascent of each, multiplied by its corresponding magnitude, is equal to the product of the height of the descent or ascent of the center of gravity of all the bodies, multiplied by the sum of their magnitudes.

Let A, B, and C [Fig. 62] be bodies which either fall from A, B, and C to D, E, and F, or else ascend from D, E, and F to A, B, and C. Let their common center of gravity, when they are at A, B, and C, have the same height as the point G; and when they are at D, E, and F, let it have the same height as the point H. I say that the sum of the products of the height AD times A, BE times B, and CF times C, is equal to the product of GH times A, B, and C.

Imagine a horizontal plane whose section is the line MP, and extend the lines AD, BE, CF, and GH so that they meet the plane at M, N, O, and P.

Now the sum of the products of AM times A, BN times B, and CO times C, is equal to the product of GP times the sum of A, B, and C [Proposition 1]. Likewise the sum of the products of DM times A, EN times B, and FO times C, is equal to the product of HP times the sum of A, B, and C. Therefore, the difference between the first and second products is equal to the product of GH times the sum of the magnitudes A, B, and C. But it is clear that this difference is equal to the products of AD times A, BE times B, and CF times C. Therefore the sum of these last products is equal to the product of GH times the sum of A, B, and C. Q.E.D.

[Fig. 62]



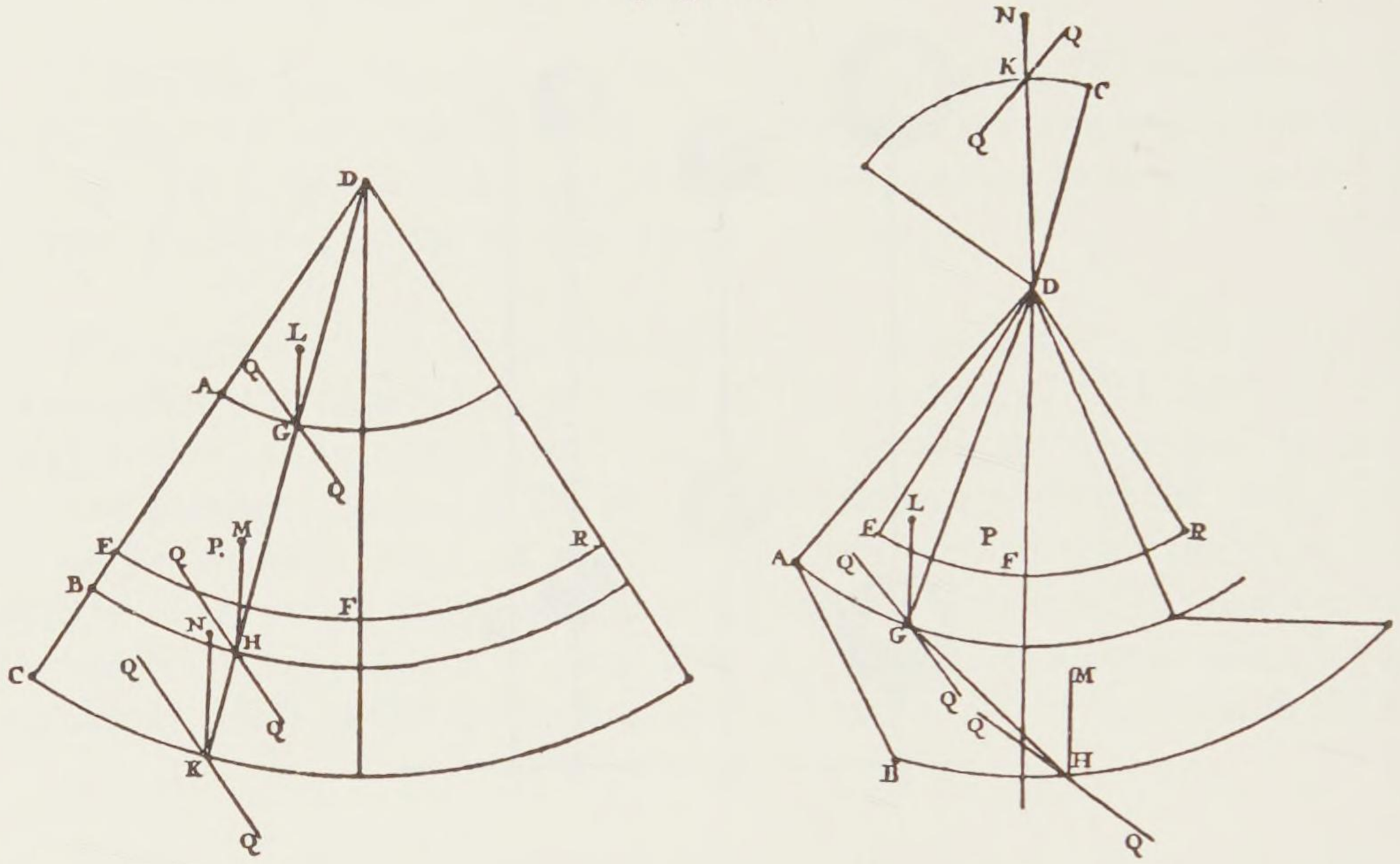
PROPOSITION IV

Assume that a pendulum is composed of many weights, and beginning from rest, has completed any part of its whole oscillation. Imagine next that the common bond between the weights has been broken and that each weight converts its acquired velocity upwards and rises as high as it can. Granting all this, the common center of gravity will return to the same height which it had before the oscillation began.

Let there be a pendulum [Fig. 63] which is composed of any number of weights attached to a rod or to a weightless surface. Let his pendulum be suspended from an axis which is drawn through the point D and which is understood to be perpendicular to the plane which is seen in the diagram. In this same plane let E be the center of gravity of the weights A, B, and C. Also let the center line DE be inclined to the perpendicular line DF by the angle EDF, since the pendulum has been pulled back this far. From this point it begins to move and completes any part of its oscillation such that the weights A, B, and C arrive at the points G, H, and K. Next, imagine that the common bond between the weights is broken and that each converts its acquired velocity upwards (which could occur if each encountered an inclined plane) and rises as high as it can; namely, to L, M, and N. When they have arrived there, the common center of gravity is at the point P. I say that P has the same height as the point E.

For first it is clear from the first of our assumed hypotheses that P is not higher than E. But that it is not lower we will show as follows. Assume,

[Fig. 63]



as if it were possible, that P is lower than E. And imagine that the weights fall again from the heights LG, MH, and NK, to which they had ascended. From this it is clear that they would acquire the same velocity which they had in order to ascend to those heights [Proposition 4, Part II], that is, the velocity they acquired from the motion of the pendulum from CBAD to KHGD. Consequently, if the weights are now returned with those velocities to the rod or surface to which they had been attached, and if they are reattached to it and continue their motion along the original arc—which could occur if before they touch the rod they are imagined to rebound from the inclined planes QQ—then the pendulum restored in this way will complete the remaining part of its oscillation just as if it had continued its motion without any interruption. Thus the center of gravity, E, of the pendulum crosses equal arcs EF and FR in falling and ascending, and hence is found to have the same height at R and at E. But it was assumed that E was higher than P, which is the center of gravity of the weights located at L, M, and N. Therefore R is higher than P. Consequently, the center of gravity of the weights falling from L, M, and N rises higher than the place from which it descended, which is absurd [Hypothesis I]. Therefore, the center of gravity P is not lower than E. Nor is it higher than E. Hence it must have the same height. Q.E.D.

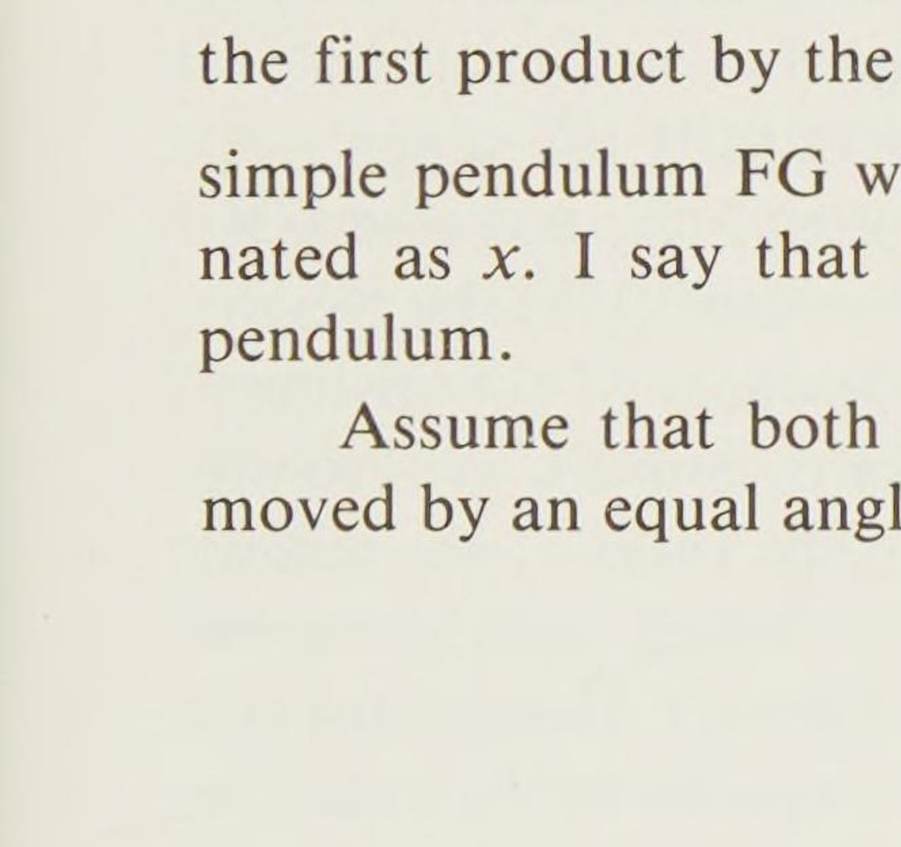
PROPOSITION V

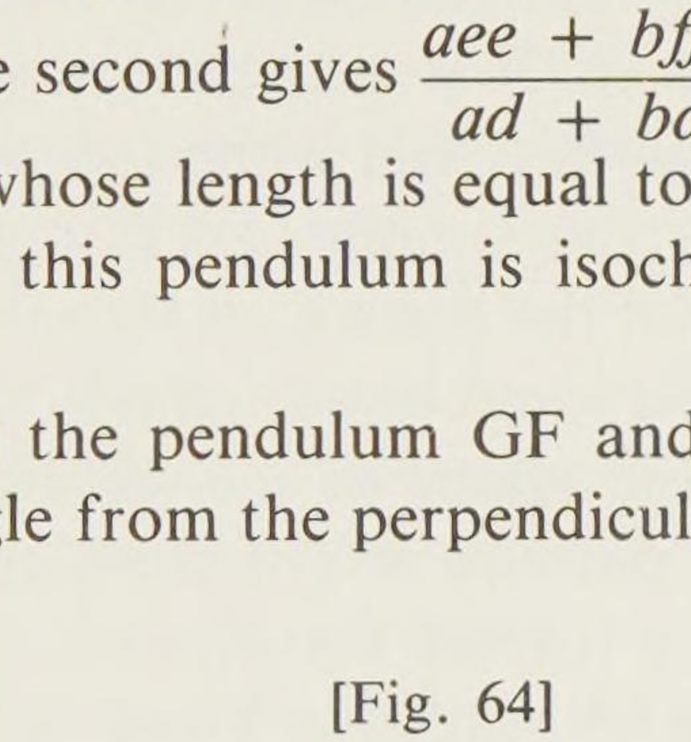
Given a pendulum composed of any number of weights, let each weight be multiplied by the square of its distance from the axis of oscillation, and let the sum of these products be divided by the product of the sum of the weights multiplied by the distance of their common center of gravity from the same axis of oscillation. The result is the length of a simple pendulum isochronous with the composite pendulum, or the distance between the axis and the center of oscillation of the composite pendulum.

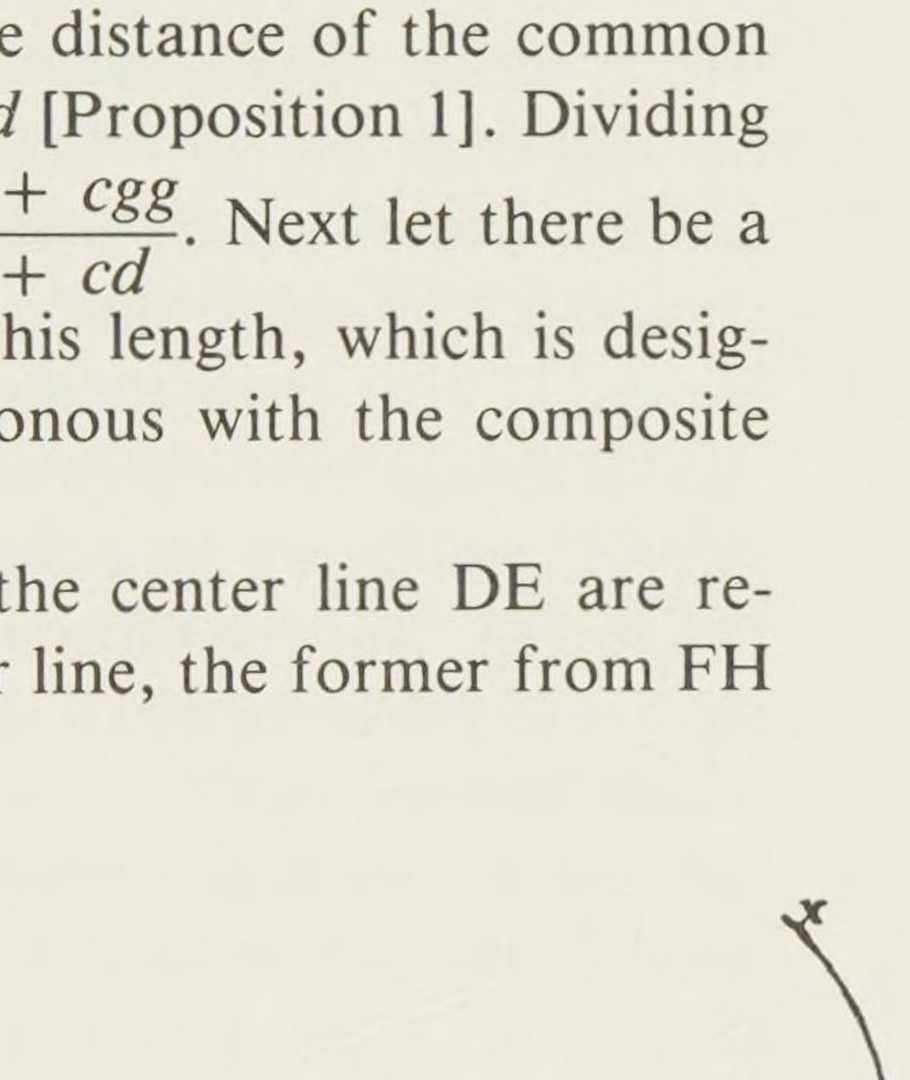
Let the weights A, B, and C [Fig. 64] compose a pendulum (we need to consider only the weight of the pendulum and not its figure or magnitude.) Let the pendulum be suspended from an axis which goes through point D and which is understood to be perpendicular to the plane of the diagram. The common center of gravity E is also in that plane; it makes no difference if the weights are in different planes. Let d be the distance of the point E from the axis, that is, the line ED. Let e be the distance AD of the weight A; f the distance BD; and g the distance CD. Multiplying each weight by the square of its distance gives the sum of the products $aee + bff + cgg$. Next, multiplying the sum of the weights by the distance of the common center of gravity gives the product $ad + bd + cd$ [Proposition 1]. Dividing the first product by the second gives $\frac{aee + bff + cgg}{ad + bd + cd}$. Next let there be a simple pendulum FG whose length is equal to this length, which is designated as x. I say that this pendulum is isochronous with the composite pendulum.

Assume that both the pendulum GF and the center line DE are removed by an equal angle from the perpendicular line, the former from FH

[Fig. 64]







and the latter from DK. Assume also that they are free to move from these positions. On the line DE let DL equal FG. And as the weight G of the pendulum FG in one oscillation crosses through the arc GM, which is cut in half by the perpendicular line FH, likewise the point L crosses through the similar and equal arc LN, which is cut in half by DK. Also the center of gravity E crosses through the similar arc EI. Having taken any points O and P on the arcs GM and NL which divide those arcs similarly, then if it can be shown that the velocity of the weight G at O is equal to the velocity of L at P, then it will be established that both arcs are crossed in equal times, and thus the pendulum FG will be isochronous with the pendulum composed of A, B, and C. This is shown as follows.

Assume first, as if this were possible, that the velocity of L at P is greater than the velocity of the weight G at O. Now it is clear that while the point L crosses the arc LP, the center of gravity E also crosses the similar arc EQ. From the points Q, P, and O draw perpendiculars upwards to meet the cords of the arcs EI, LN, and GM at R, S, and Y. Let SP be called *y*. Therefore, since the ratio of LD or *x*, to ED or *d*, equals the ratio of SP or *y*, to RQ, then RQ equals $\frac{dy}{x}$. Now since the weight G has a velocity at O which allows it to rise to the same height from which it fell; namely, through the arc OM or the perpendicular OY which is equal to PS, it follows that when L arrives at P it will have a greater velocity than is needed to ascend to the height PS. And while L moves to P, the weights A, B, and C also cross through arcs similar to LP; namely, AT, BV, and CX. And the ratio of the velocity of the point L at P to the velocity of the weight A at T, which are held by the same bond, equals the ratio of the distance DL to DA. But the ratio of the square of the velocity of point L at P to the square of the velocity of the point A at T is equal to the ratio of the height which can be reached by the former velocity to the height which can be reached by the latter velocity [Proposition 3 and 4, Part II]. Hence also the ratio of the square of the distance DL, or *xx*, to the square of the distance DA, or *ee*, is equal to the ratio of the height which can be reached by the velocity of the point L at P (which was said to be greater than PS or *y*) to the height which can be reached by the velocity of the weight A at T, assuming that after A arrives at T it leaves the pendulum and converts its motion upwards. This height will be greater than $\frac{eey}{xx}$.

By the same reasoning the height to which the weight B ascends, with the velocity acquired through the arc BV, will be greater than $\frac{ffy}{xx}$. And the height to which the weight C ascends, with the velocity acquired through the arc CX, will be greater than $\frac{ggy}{xx}$. Multiplying each of these heights by the corresponding weights, the sum of the products will be

greater than $\frac{aeey + bffy + cggy}{xx}$. This was proven to be also greater than $\frac{ady + bdy + cdy}{x}$. For since it was granted that the length x equals $\frac{aee + bff + cgg}{ad + bd + cd}$, then $adx + bdx + cdx$ equals $aee + bff + cgg$. Multiplying everything by y and dividing by xx, then $\frac{ady + bdy + cdy}{x}$ equals $\frac{aeey + bffy + cggy}{xx}$. Thus what was said follows. But this sum of products is equal to the product of the height to which the common center of gravity of the weights A, B, and C rises, multiplied by the sum of those weights, $a + b + c$, if, as was said, each moves upwards as far as it can. But the quantity $\frac{ady + bdy + cdy}{x}$ is obtained as the product of distance of fall of the center of gravity of those weights (which distance is RQ, or $\frac{dy}{x}$, as was found above), multiplied by the same sum of weights, $a + b + c$. But it has been shown that the first product is greater than this second product. Therefore it follows that the ascent of the center of gravity of the weights A, B, and C, assuming that they leave the pendulum when they arrive at T, V, and X, and convert their acquired velocity upwards, will be greater than the descent of this same center of gravity when they move from A, B, and C to T, V, and X. This is absurd, since the indicated ascent should be equal to the descent, according to the preceding proposition.

Next, if it is assumed that the velocity of the point L, when it arrives at P, is less than the velocity of the weight G, when it arrives at O, then we can show in the same way that the possible ascent of the center of gravity of the weights A, B, and C is less than its descent. This again contradicts the preceding proposition. Therefore it follows that the velocity of the point L, when moved to P, is equal to the velocity of the weight G at O. Hence, as said above, it follows that the simple pendulum FG is isochronous with the pendulum composed of A, B, and C.

PROPOSITION VI

Given a pendulum composed of any number of equal weights, let the sum of the squares of the distances of each weight from the axis of oscillation be divided by the product of the distance of the common center of gravity from the same axis of oscillation multiplied by the number of weights. The result is the length of a simple pendulum isochronous with the composite pendulum.

Assume the same situation as before except that all the weights are

equal to each other. Let each weight be called *a*. Again no consideration need be given of their magnitude except that their extension is taken to be very small.

According to the previous proposition, the length of the simple, isochronous pendulum will be $\frac{aee + aff + agg}{ad + ad + ad}$. But since both the numerator and the denominator can be divided by *a*, this same length equals $\frac{ee + ff + gg}{3d}$. This signifies that the sum of the squares of the distances of the weights from the axis of oscillation is divided by the product of the distance of the common center of gravity from the same axis of oscillation, multiplied by the number of weights, which here is 3. It is easy to see that the number by which the distance *d* is multiplied must correspond to the number of the weights. Hence the proposition is proven.

If we assume that the equal weights are connected in one straight line and suspended from the top of the line, then it has already been established that the distance of their common center of gravity from the axis of oscillation, multiplied by the number of weights, is equal to the sum of the distances of all the weights from the same axis of oscillation [Propositon 2]. As a result, in this case the length of the simple pendulum, which is isochronous with a composite pendulum, is found if the sum of the squares of the distance of each weight from the axis of oscillation is divided by the sum of all these distances.

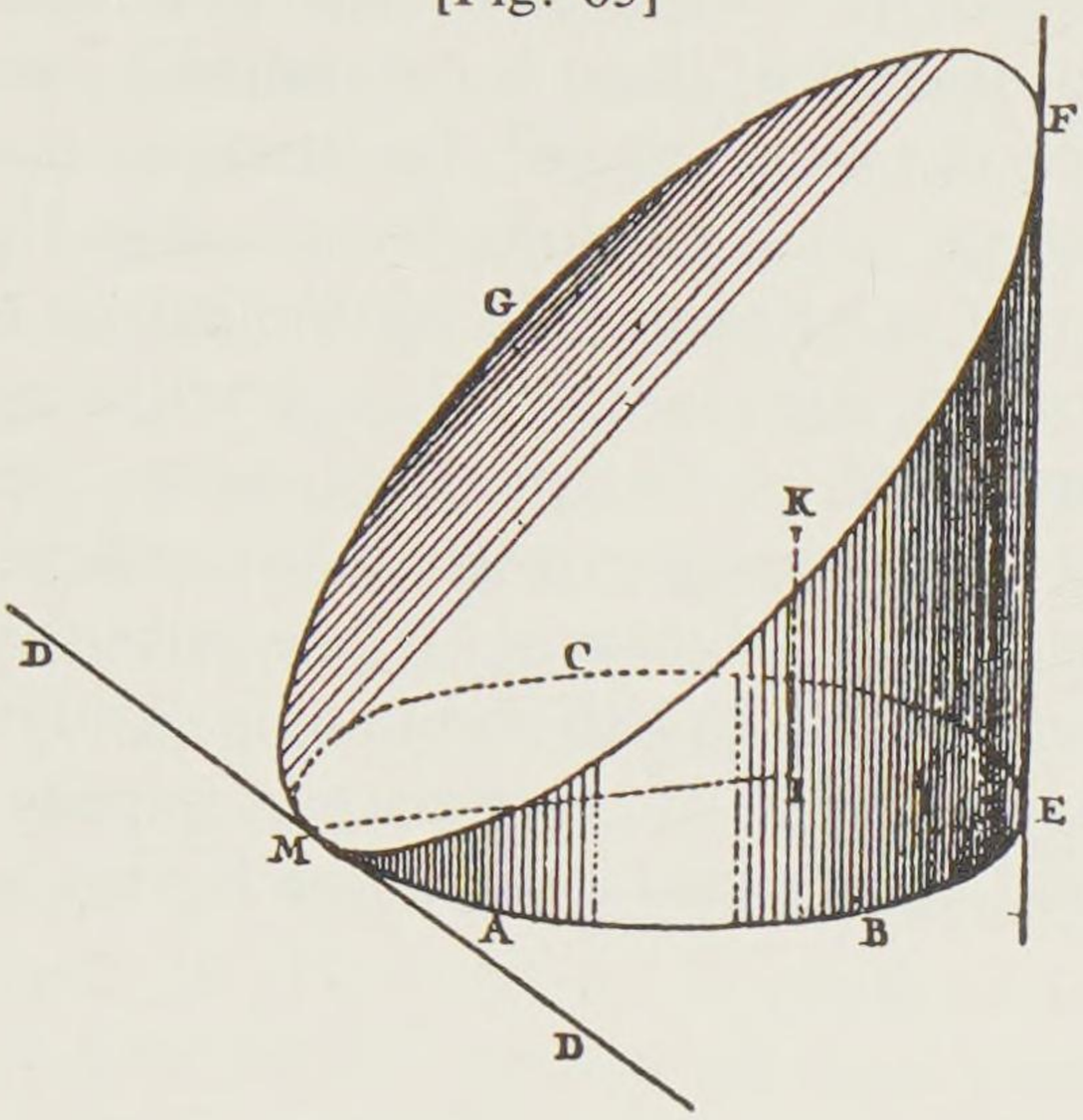
DEFINITION XIV

Let there be a figure and a straight line in the same plane which touches the figure externally. Let another straight line, which is perpendicular to the plane, move through the perimeter of the figure. It describes a surface, which is then cut by a plane which is drawn through the indicated tangent and which is inclined to the plane of the given figure. The solid bounded by these two planes and by that part of the described surface which is intercepted between the two planes is called a wedge,[6] *which is cut off above the given figure as its base.*

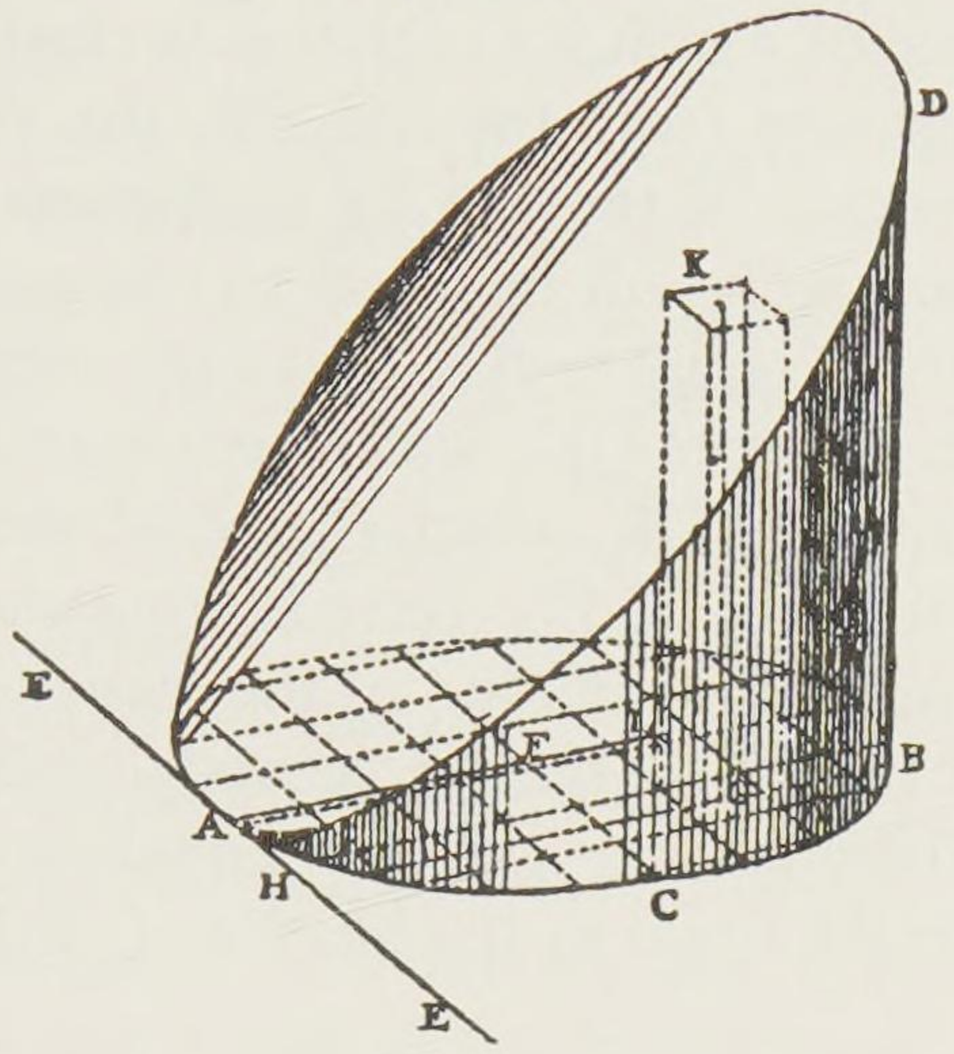
In the adjacent figure [Fig. 65] ABEC is the given figure; the line MD is the tangent; EF is the line which is moved around through the perimeter of the figure; and the wedge is the solid figure bounded by the planes ABEC and MFG and by part of the surface described by the line EF.

6. The Latin word used here is "cuneus," already used by Huygens in 1650 in his *De iis quae liquido supernatant, Oeuvres* 11:159–60, 204–10.

[Fig. 65]



[Fig. 66]



DEFINITION XV

The distance between the line through which a wedge is cut and the point of its base at which falls the perpendicular from its center of gravity is called the subcentrix of the wedge.

Thus in the same figure [Fig. 65], if K is the center of gravity of the wedge, and if the line KI is drawn perpendicular to its base ABEC, and if IM is perpendicular to MD, then IM is the line which we call the subcentrix.

PROPOSITION VII

A wedge cut above any plane figure by a plane inclined at half a right angle is equal to the solid produced by multiplying that figure by a height equal to the distance of the center of gravity of the figure from the line through which the wedge is cut.[7]

Above the plane figure ACB [Fig. 66] let the wedge ABD be cut off by a plane which is inclined at half a right angle and which passes through EE, which is tangent to the figure ACB and which is in its plane. Let F be the center of gravity of the figure, and from F draw FA perpendicular to EE. I say that the wedge ACB is equal to the solid produced by multiplying the figure ACB by a height equal to FA.

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7. For a more general version of this Proposition, i.e., without the restriction to 45°, see *Oeuvres* 16:499.

Imagine that the figure ACB is divided into very small, equal portions, one of which is G. Now it is clear that, if each of these is multiplied by its distance from the line EE, the sum of the products will be equal to the product of the line AF multiplied by all the small portions [Proposition I], that is, it will be equal to the product of the figure ACB multiplied by a height equal to AF. And each small portion, for example, G, multiplied by its distance GH, will be equal to a very small parallelepiped or prism, for example, GK, which is erected above it and which terminates at the oblique surface AD. For their heights are equal to the distances GH because the angle of inclination of the planes AD and ACB is a half right angle. And it is clear that the whole wedge ABD is composed of these parallelepipeds. Therefore the wedge is equal to a solid constructed on its base having a height equal to the line FA. Q.E.D.

PROPOSITION VIII

Let a straight line touch a plane figure, imagine that the figure is divided into very small, equal portions, and let perpendiculars be drawn from each small portion to the straight line. The sum of the squares of all these latter distances will be equal to a certain rectangle multiplied by the number of these small portions. The rectangle has as one side the distance of the center of gravity of the figure from the given straight line and for the other side the subcentrix of the wedge which is cut off above the figure through that line.

Granting everything which was assumed in the previous construction, let LA [Fig. 66] be the subcentrix to the line EE in the wedge ABD. It must be shown that the sum of the squares of all the distances of the small portions of the figure ACB is equal to a rectangle of sides FA and LA, multiplied by the number of the small portions.

It was established in the preceding demonstration that the height of each parallelepiped GK is equal to the distance of each small portion G, which is its base, from the line AE. Therefore, if we now multiply the parallelepiped GK by the distance GH, this is the same as multiplying the small portion G by the square of the distance GH. The same is true for all the other parallelepipeds. And the products of all the parallelepipeds, multiplied by their distances from the line AE, is equal to the product of the wedge ABD, multiplied by the distance LA [Proposition 1], because the center of gravity of the wedge is above the point L. Hence, the sum of the products of each small portion G, multiplied by its distance from the line AE, is also equal to the product of the wedge ABD, multiplied by the line LA, that is, it is equal to the product of the figure ACB, multiplied by a

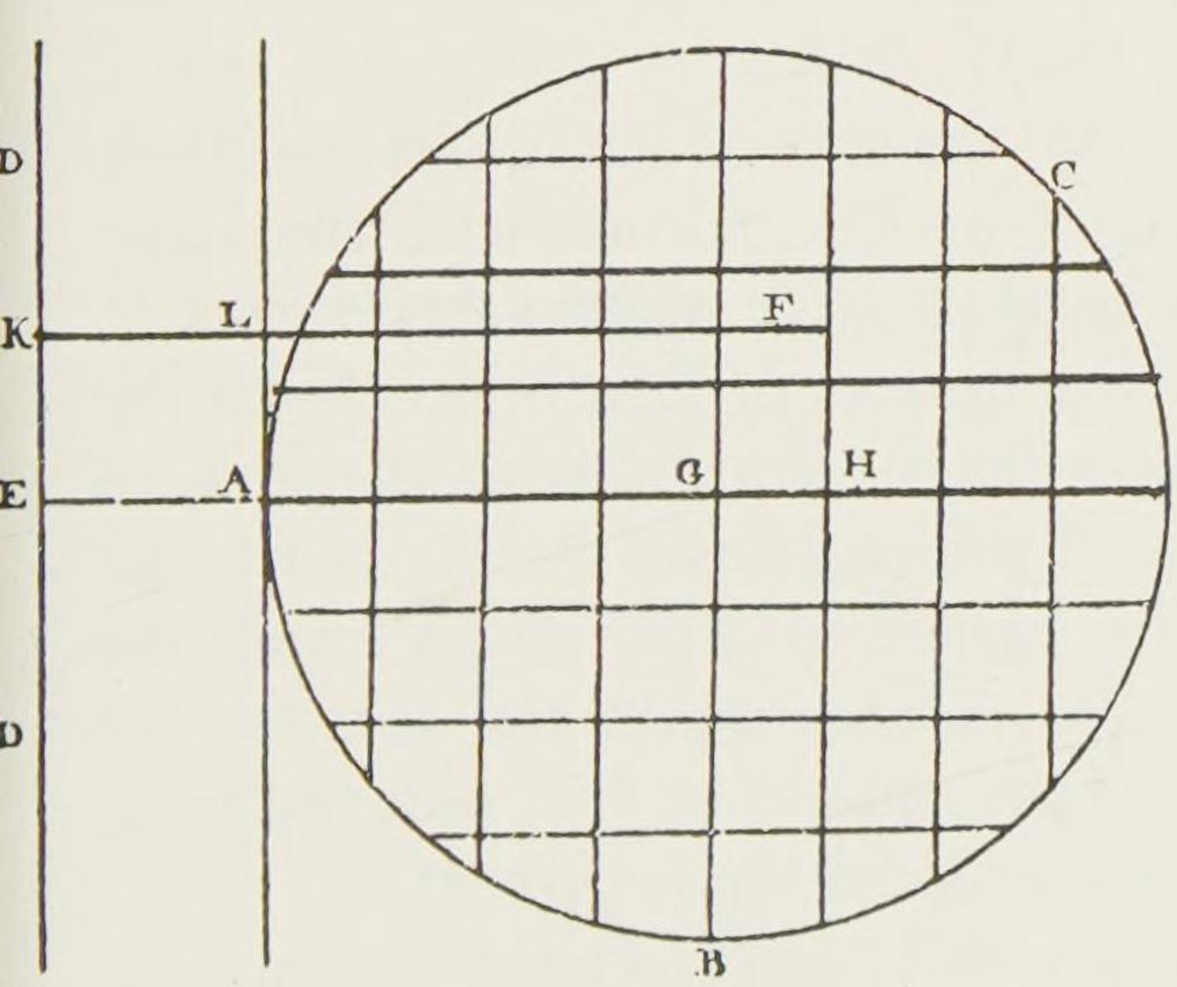
rectangle of sides FA and LA. For the wedge ABD is equal to the figure ACB, multiplied by the line FA [Proposition 7]. Furthermore, since the figure ACB is equal to the product of one small portion G, multiplied by the number of small portions, it necessarily follows that the above mentioned product of the figure ACB, multiplied by the rectangle of sides FA and LA, is equal to the product of the small portion G, multiplied by the rectangle of sides FA and LA, multiplied by the number of small portions G. This however is also equal to the sum of the products of each small portion G, multiplied by the square of its distance from the line AE, or to the product of one small portion G, multiplied by the sum of the squares of all the distances. Therefore, after dropping the multiplication by the small portion G from both products, it necessarily follows that the sum of the squares of the distances is equal to the rectangle of sides FA and LA, multiplied by the number of small portions into which it was understood that the figure ACB was divided. Q.E.D.

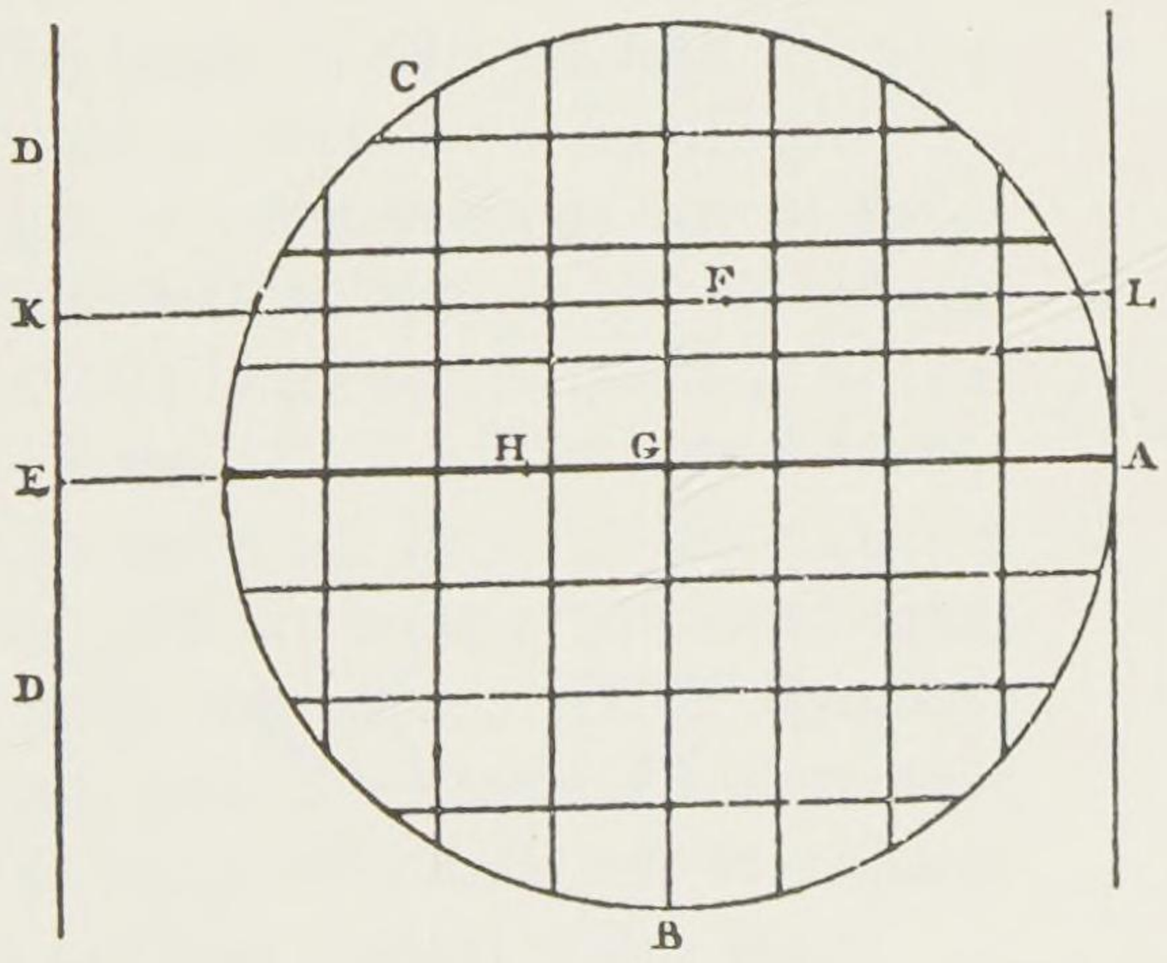
PROPOSITION IX

Let there be given a plane figure and, in the same plane, a straight line which either does or does not cut the figure. Let the figure be understood to be divided into many, very small, equal portions from each of which a perpendicular is drawn to the given line. Find the sum of the squares of all these perpendiculars, or find a plane which, when multiplied by the number of the small portions, is equal to the indicated sum of squares.

Let ABC [Fig. 67] be a given plane figure, and let ED be a straight line in the same plane. Letting the figure be divided by thought into very many,

[Fig. 67]





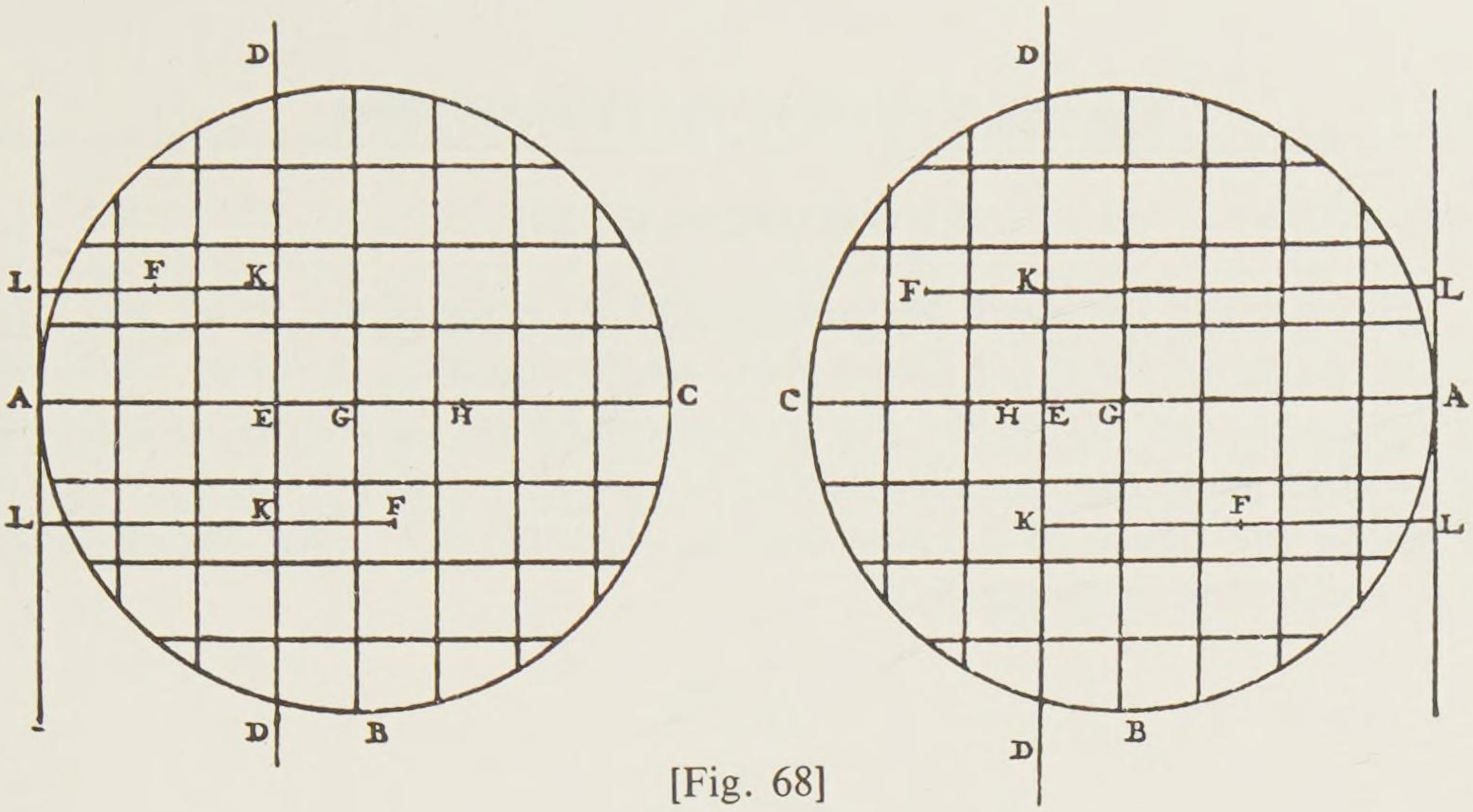
equal, small portions, imagine that from each small portion a perpendicular is drawn to the line ED, for example, the perpendicular FK from the small portion F. It is required to find the sum of the squares of all these perpendiculars.

Let the straight line AL be a tangent to the figure, and let AL be distinct from but parallel to the given line ED. AL can lie either on the same side as ED or on the other side of the figure. Let GA, which meets ED at E, be the distance of the center of gravity of the figure from the line AL. And let HA be the subcentrix of the wedge which is cut off above the figure by a plane through the line AL. Now I say that the sum of the squares which is sought is equal to the sum of the rectangle AGH and the square of EG, multiplied by the number of small portions into which the figure is understood to be divided.

Let FK be extended as far as needed so that it meets the tangent AL at the point L. Consider first the case in which the line ED is separated from the figure and the tangent AL is drawn on the same side of the figure. The proposition is then proven as follows. The sum of all of the squares of FK equals the same number of the squares of KL, plus twice that number of the rectangles KLF, plus the same number of the squares of LF. But the squares of KL equal the same number of squares of EA. And it is clear that the rectangles KLF are equal to the same number of rectangles EAG, since the sum of all the FLs is equal to the same number of GAs [Proposition 2]. And finally the squares of LF are equal to the same number of the rectangles HAG [Proposition 8], that is, are equal to the same number of squares of AG plus the same number of rectangles AGH. Therefore the sum of the squares of FK will be equal to the same number of the squares of EA, plus twice the same number of the rectangles EAG, plus the same number of the squares of AG and the same number of the rectangles AGH. But these three; namely, the square of EA, plus twice the rectangle EAG, plus the square of AG, constitute the square of EG. Hence it is clear that the sum of the squares of FK is equal to the same number of the squares of EG, plus the same number of rectangles AGH. Q.E.D.

Next in all the remaining cases [Fig. 68] the sum of the squares of FK is equal to the same number of the squares of KL, minus twice the same number of the rectangles KLF, plus the same number of the squares of LF; that is, is equal to the same number of the squares of EA, minus twice the same number of the rectangles EAG, plus the same number of the squares of AG together with the same number of the rectangles AGH. But in all these cases the square of EA, plus the square of AG, minus twice the rectangle EAG, equals the square of EG. Hence again the sum of the squares of FK equals the same number of the squares of EG, plus the same number of the rectangles AGH. Thus the proposition is proven.

From this it follows that the rectangle AGH has the same size in both



[Fig. 68]

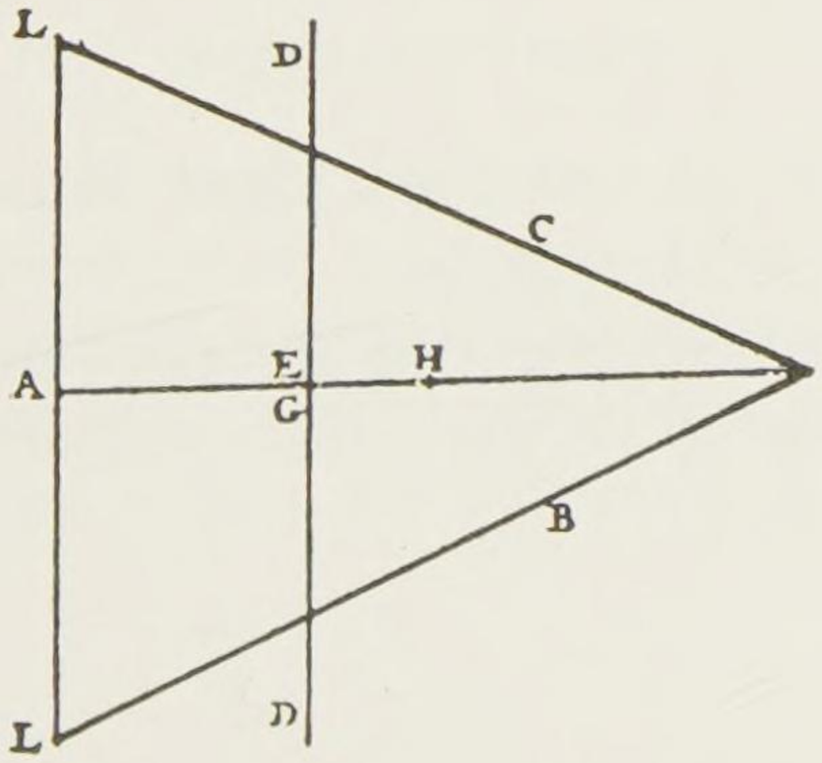
wedges, that is, in the two cases in which AH is the subcentrix of a wedge cut off by a plane passing through either parallel tangent. And thus AG in the first case is related to AG in the second case as HG in the second case is related to HG in the first case. And as the two lines AG are related to each other, so also in each case are related the wedges cut off by a plane through AL, as follows from Proposition 7. Hence also the two lines GH are related reciprocally.

It is also clear that, if we are given the center of gravity G of the plane figure and also the subcentrix of a wedge cut off by a plane passing through one of the tangents, then we also know the subcentrix of the wedge cut off by a plane passing through the other tangent.

PROPOSITION X

Given the same presuppositions of the preceding proposition except that the line ED [*Fig. 69*] *passes through G, which is the center of gravity of the figure ABC, then the sum of the squares of the distances from ED of the small portions, into which the figure is understood to be divided, is equal to only the rectangle AGH, multiplied by the number of the small portions.*

This is obvious, since the square of EG is then zero.

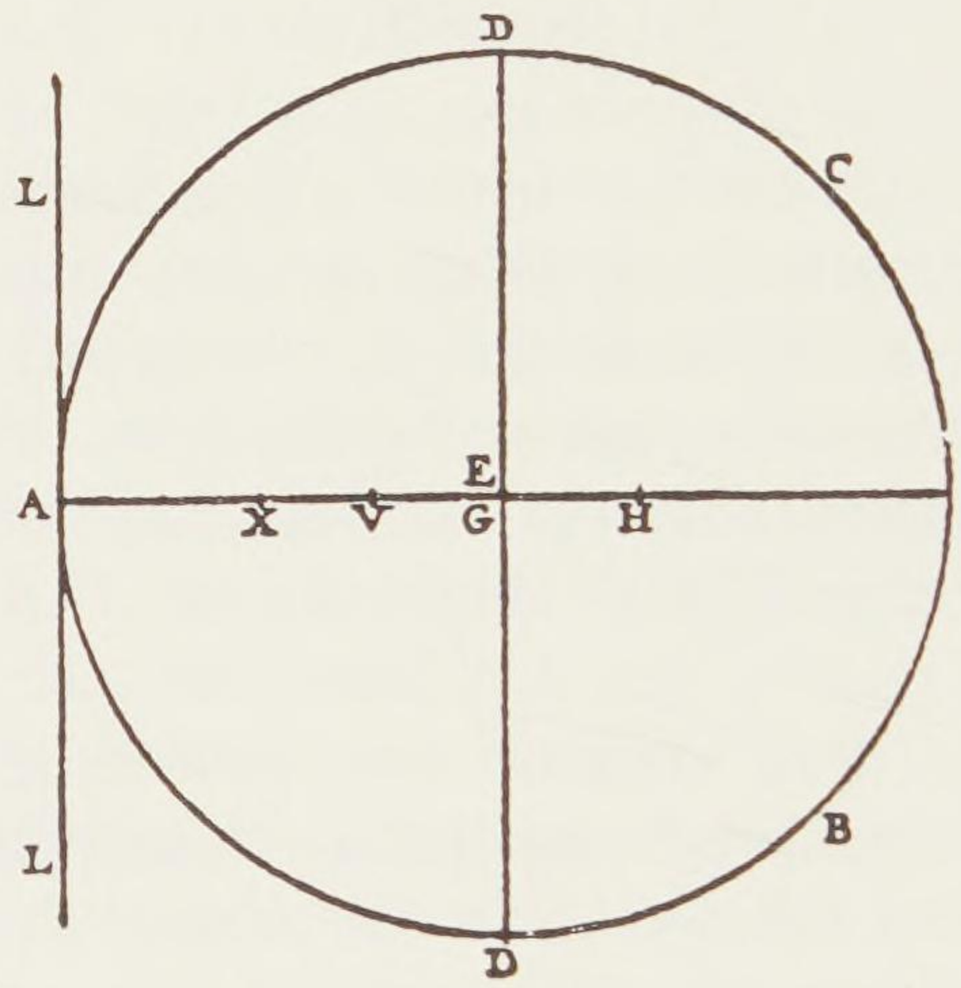


[Fig. 69]

PROPOSITION XI

Given again the same presuppositions of Proposition 9, let DE [*Fig. 70*] *be an axis of the plane figure ABC which divides it into two equal and similar parts. Let VG be the distance of the center of gravity of the half-figure DAD from the line ED. And let GX be the subcentrix of a wedge cut off above the figure by a plane passing through ED. Then the rectangle XGV equals the rectangle AGH.*

[Fig. 70]



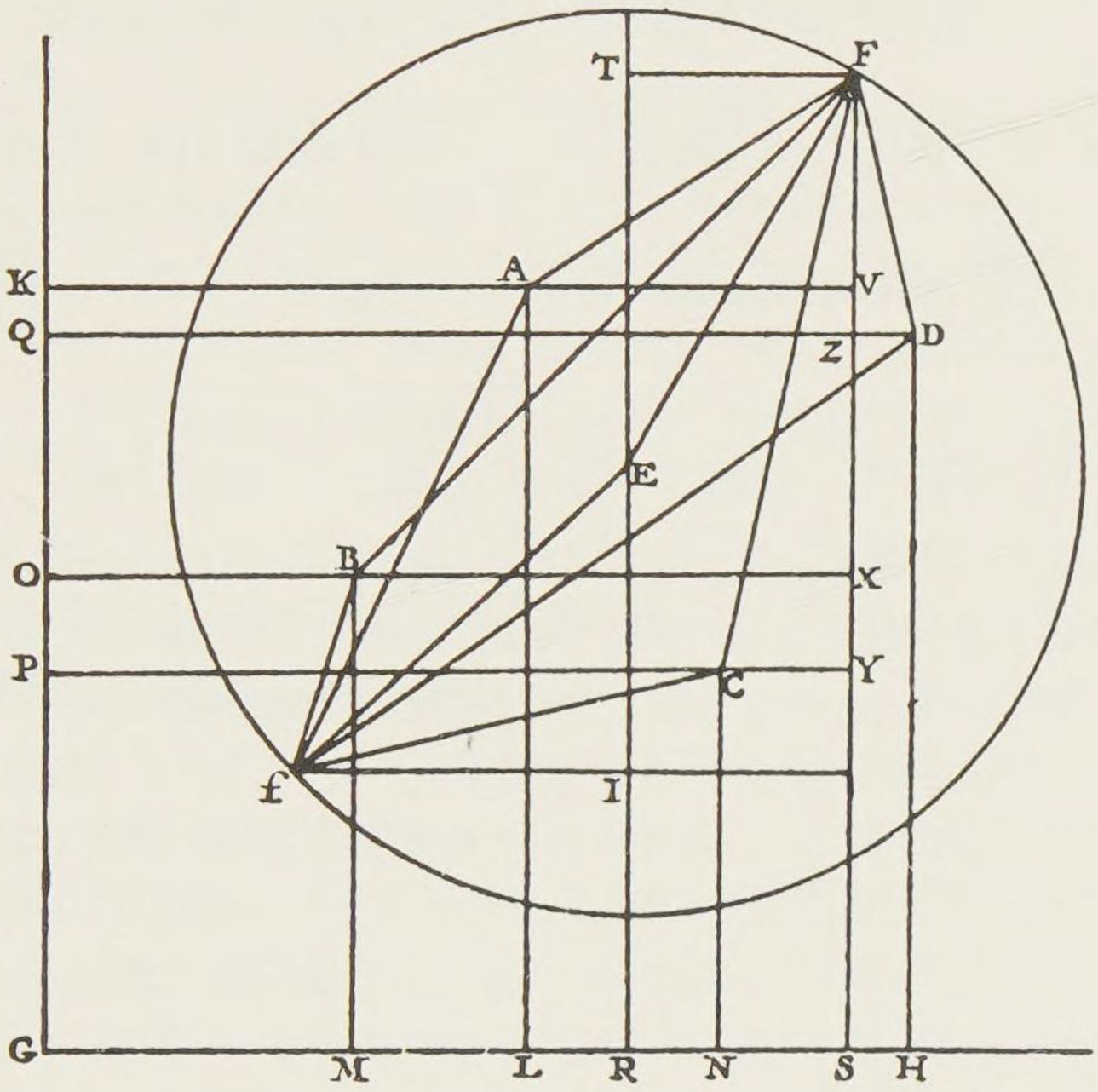
The rectangle XGV, multiplied by the number of the small portions of the figure DAD, is equal to the sum of the squares of the perpendiculars drawn from the small portions of this half-figure to the line ED [Proposition 8]. And consequently this same rectangle XGV, multiplied by the number of the small portions of the whole figure ABC, is equal to the sum of the squares of the perpendiculars drawn from all the small portions of the figure ABC to the line ED; that is, it is equal to the rectangle AGH, multiplied by the same number of small portions, as was established in the preceding proposition. From this it follows that the rectangles XGV and AGH are equal to each other. Q.E.D.

PROPOSITION XII

Given any number of points in a plane, draw any circle whose center is the center of gravity of these points. Next, from each of the given points, draw a straight line to any point on the circumference of the circle. Then the sum of the squares of all these lines will always be equal to the same plane.

Let A, B, C, and D [Fig. 71] be the given points. Let E be their center of gravity, or the center of equal magnitudes suspended at the points. Around the center E draw any circle F*f*; on its circumference select some point F; and draw the straight lines AF, BF, CF, and DF to F from the given points. Now I say that the sum of the squares of all these lines is equal to a certain determined plane which is always the same, wherever the point F was selected on the circumference.

[Fig. 71]



Draw the lines GH and GK at right angles and such that all the given points fall on the same side of each of the lines. Draw the perpendiculars AL, AK; BM, BO; CN, CP; and DH, DQ from each point to the lines GH and GK. Draw the perpendiculars ER and FS from the center of gravity and from the point F to one or the other of the lines GH or GK. Also draw the perpendiculars AV, BX, CY, and DZ from the given points to the line FS. Next let

a = AL	e = AK	z = the radius EF
b = BM	f = BO	x = GS
c = CN	g = CP	
d = DH	h = DQ	

Now since E is the center of gravity of the points A, B, C, and D, then if the perpendiculars are added together and if this sum is divided into as many parts as there are given points, ER will be equal to one of these parts

[Proposition 2]. Likewise, by dividing the sum of the perpendiculars AK, BO, CP, and DQ into the same number of parts, one of these parts will be equal to the perpendicular from E to the line GK, or to RG [Proposition 2]. Let the sum of AL, BM, CN, and DH, or $a + b + c + d$, be called l. Let the sum of AK, BO, CP, and DQ, or $e + f + g + h$, be called m. Let the number of the given points be called θ. Then ER $= \frac{l}{\theta}$, and RG $= \frac{m}{\theta}$.

Since x = GS, RS or FT $= x - \frac{m}{\theta}$, or $\frac{m}{\theta} - x$ if GR is greater than GS. But the square of FT always equals $xx - \frac{2xm}{\theta} + \frac{mm}{\theta\theta}$. Subtracting this from the square of FE or zz gives the result that the square of TE $= zz - xx + \frac{2xm}{\theta} - \frac{mm}{\theta\theta}$. Consequently TE $= \sqrt{zz - xx + \frac{2xm}{\theta} - \frac{mm}{\theta\theta}}$. But ER $= \frac{l}{\theta}$. Thus TR $= \frac{l}{\theta} \pm \sqrt{zz - xx + \frac{2xm}{\theta} - \frac{mm}{\theta\theta}}$. For the sake of brevity let this value for TR be called y. Next let us bring together the sum of all the squares of FA, FB, FC, and FD. The square of AF equals the squares of AV and of VF. But AV is equal to the difference between VK and AK, or between SG and AK. Therefore AV equals $x - e$ or $e - x$; and the square of AV equals $xx - 2ex + ee$. And VF is equal to the difference between FS and VS, or between FS and AL. Therefore VF equals $y - a$ or $a - y$; and the square of VF equals $yy - 2ay + aa$. By adding the squares of AV and VF, the square of FA equals $xx - 2ex + ee + yy - 2ay + aa$. The squares of the remaining lines FB, FC, and FD, are found in the same way. Putting all of them in order, we then have the following:

The square of FA $= xx - 2ex + ee + yy - 2ay + aa.$
The square of FB $= xx - 2fx + ff + yy - 2by + bb.$
The square of FC $= xx - 2gx + gg + yy - 2cy + cc.$
The square of FD $= xx - 2hx + hh + yy - 2dy + dd.$

The sum of these squares will be $\theta xx - 2mx + nn + \theta yy - 2ly + kk$, if we assume that the squares $ee + ff + gg + hh = nn$, and that the squares $aa + bb + cc + dd = kk$. For θ was the number of the given points, and thus also the number of the squares, and we have assumed that $e + f + g + h = m$ and that $a + b + c + d = l$.

In the terms θyy and $2ly$ of this sum, substitute for y the expression which it represents, namely: $\frac{l}{\theta} \pm \sqrt{zz - xx + \frac{2xm}{\theta} - \frac{mm}{\theta\theta}}$. The result is

$$\theta yy = \frac{ll}{\theta} + 2l\sqrt{zz - xx + \frac{2xm}{\theta} - \frac{mm}{\theta\theta}} + \theta zz - \theta xx + 2xm - \frac{mm}{\theta},$$

and $-2ly = \frac{2ll}{\theta} - 2l\sqrt{zz - xx + \frac{2xm}{\theta} - \frac{mm}{\theta\theta}}$; or $\theta yy = \frac{ll}{\theta} - 2l\sqrt{zz - xx + \frac{2xm}{\theta} - \frac{mm}{\theta\theta}} + \theta zz - \theta xx + 2xm - \frac{mm}{\theta\theta}$, and $-2ly = \frac{-2ll}{\theta} + 2l\sqrt{zz - xx + \frac{2xm}{\theta} - \frac{mm}{\theta\theta}}$. Consequently, in both cases, for $\theta yy - 2ly$ we have $-\frac{ll}{\theta} + \theta zz - \theta xx + 2xm - \frac{mm}{\theta}$. By adding the remaining quantities contained in the previously mentioned sum; namely, $\theta xx - 2xm + nn + kk$, the sum total of the squares FA, FB, FC, and FD, equals $\theta zz + nn + kk - \frac{mm - ll}{\theta}$. It is apparent that this is a determined plane since all these quantities are known; and it will be found to be always the same wherever the point F is selected on the circumference. Q.E.D.

If it is assumed that the given points have different, but mutually commensurable, weights, for example, point A has a weight of 2, B is 3, C is 4, and D is 7, then after finding their center of gravity, draw another circle around it, draw lines from the given points to any point on the circumference, and multiply the square of each line by the number of the weight at that point, for example multiply AF by 2, BF by 3, CF by 4, and DF by 7. And again I say that the sum of all these squares is equal to a determinate space, which is always the same wherever the point was selected on the circumference. This is clear from the preceding demonstration if we imagine that the points are multiplied by the number of the weight attributed to each, as if two points were joined at A, three at B, four at C, and seven at D, and all are equally heavy.

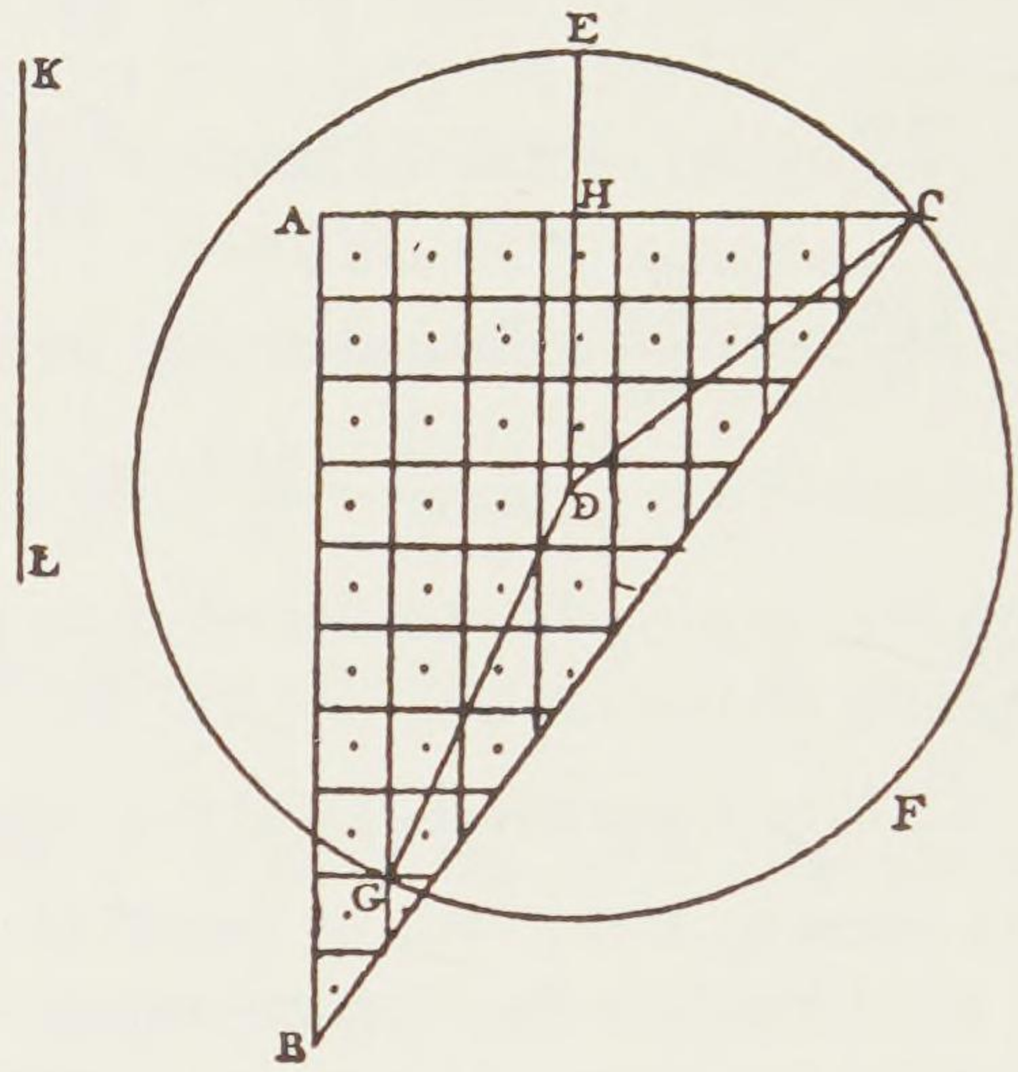
PROPOSITION XIII

If a plane figure, or a line in a plane, is alternately suspended from points which exist in the same plane and which are equally distant from its center of gravity, then this figure or line is isochronous with itself when its oscillation is lateral.

Let ABC [Fig. 72] be a plane figure, or a line existing in a plane, whose center of gravity is D. Around this as a center, draw the circumference of the circle ECF in the same plane. I say that, if from any point on that circumference, for example, E, C, or G, the suspended figure is oscillated laterally, then it will be isochronous with itself or with the same simple pendulum.

First let the figure be suspended from the point E. When this point lies outside the figure, as in this case, it is necessary to think of the line EH,

[Fig. 72]



from which the figure hangs, as rigid and as fixed immovably to the figure.

Imagine that the figure ABC is divided into many equal, very small portions, and that from the centers of gravity of all of them, lines are drawn to the point E. It is clear that these lines are perpendicular to the axis of oscillation when the figure oscillates laterally. Therefore the sum of the squares of these perpendiculars, divided by the line ED, and multiplied by the number of small portions into which the figure is divided, gives the length KL of a simple pendulum isochronous with the figure [Proposition 6]. If the figure is suspended from the point G, again the length of the simple, isochronous pendulum is found by dividing the sum of the squares of the lines drawn from the small portions of the figure to the point G, by the line GD, and multiplying by the number of small portions [Proposition 6]. Therefore, when the points G and E are on the circumference of the circle whose center is D, which is the center of gravity of the figure ABC or the center of gravity of all the points which are the centers of the equal small portions of the figure, then the sum of the squares of the lines drawn from the indicated small portions to the point G is equal to the sum of the squares of the lines drawn from those same small portions to the point E [Proposition 12]. In each suspension these sums of squares are applied to equal magnitudes; namely, in the suspension from E, to the line ED, multiplied by the number of all the small portions; and in the suspension from G, to the line DG, multiplied by the number of these same small portions. Consequently it is clear that the length of the isochronous pendulum, that is, KL, is the same in the latter case; namely, the suspension from G, as it was in the former case.

If the figure is suspended from C or from any other point of the

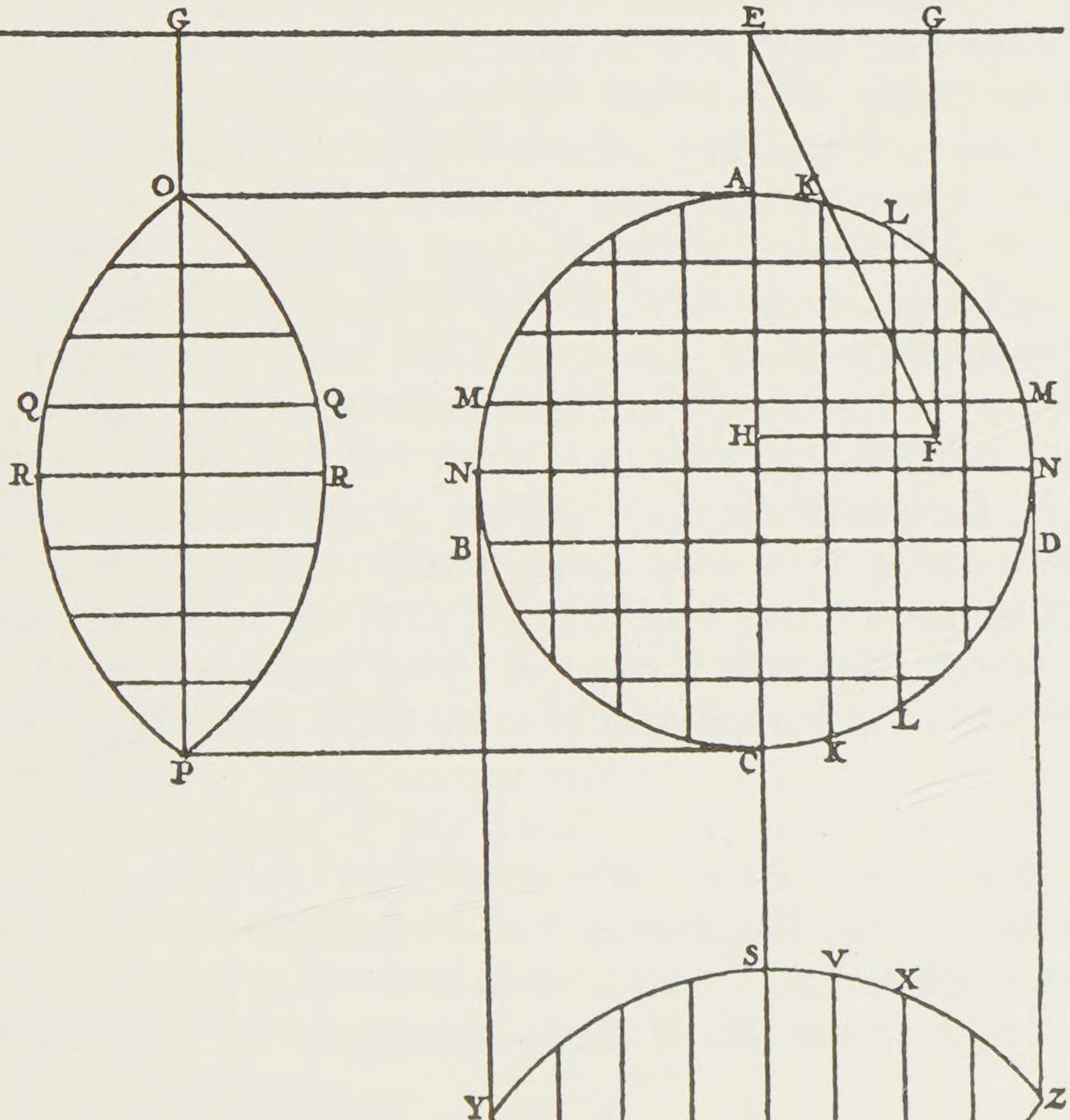
circumference ECF, we can prove in the same way that it is isochronous with the same pendulum KL. Thus the proposition is established.

PROPOSITION XIV

Let there be given a solid figure and a straight line of any length which either falls outside the figure or passes through it. Let the figure be divided by thought into many equal, very small portions, from each of which a perpendicular is understood to be drawn to the given line. Find the sum of the squares of all these perpendiculars, or find a plane which, when multiplied by the number of small portions, is equal to this same sum of squares.

Let ABCD [Fig. 73] be given as a solid figure. And let there be a straight line which passes through the point E and is understood to be perpendicular to this page. This line either cuts through the figure or falls completely outside of it. Imagine that the solid ABCD is composed of very many equal, small portions, such as F, and that perpendicular lines are drawn from each small portion to the given line which passes through E, such as the line FE. We are required to find the sum of all of the squares of the FEs.

[Fig. 73]



Let the figure be cut by the plane EAC which passes through the given line and through the center of gravity of the figure. Also imagine another plane which passes through the given line and through EG, which is perpendicular to it.

Now it is already known that the square of any perpendicular line drawn from any small portion to the given line passing through E, for example FE, is equal to the sum of the squares of FG and FH, which are perpendiculars from the same small portion to the planes passing through EG and EC, which were mentioned above [Euclid, I, 47]. Consequently, if we can determine the sum of the squares of the perpendiculars from all of the small portions to the planes passing through EG and EC, then we will have as equal to this the sum of the squares of the perpendiculars from all of the same small portions to the given line which passes through the point E.

The first of these sums of squares is calculated as follows. Assume first that the plane figure OQP is given to the side of the solid figure ABCD. The two figures have the same height. The plane figure is cut by the lines QQ and RR which correspond to, and are parallel to, the planes MM and NN which cut the solid figure ABCD. Also the ratio of these lines to each other equals the ratio of the planes to each other, when both are taken in the proper corresponding order. For example, the line RR is related to QQ as the plane NN is related to MM. Therefore, if the plane figure OQP is understood to be divided into the same number of equal, very small portions as is the solid figure ABCD, then in each segment of the plane figure, for example QQRR, there will be the same number of small portions as are located in the corresponding segment of the solid figure, MMNN. Consequently, the sum of the squares of the perpendiculars from all of the small portions of the figure OQP to the plane EG will equal the sum of the squares of the perpendiculars from all of the small portions of the solid figure drawn to the same plane EG. The former sum of squares will be known if for the figure OQP and its wedge we can determine what we said was required in Proposition 9. And when that is known, we will also have determined the sum of the squares of the perpendiculars drawn from all of the small portions of the solid figure ABCD to the plane EG.

Next assume another plane figure SYTZ, which has the same width as the solid ABCD, that is, it is enclosed by the planes BY and DZ which are tangent to the solid and are parallel to the plane EAC. This plane figure is cut by the lines VV, XX, etc., which correspond to, and are parallel to, the planes KK and LL which cut the figure ABCD. Also the ratio of these lines to each other equals the ratio of those planes to each other, when both are taken in the proper corresponding order. Once again the sum of the squares of all the perpendiculars from the small portions of the figure SYTZ to the line ST is equal to the sum of the squares of all the perpendiculars drawn

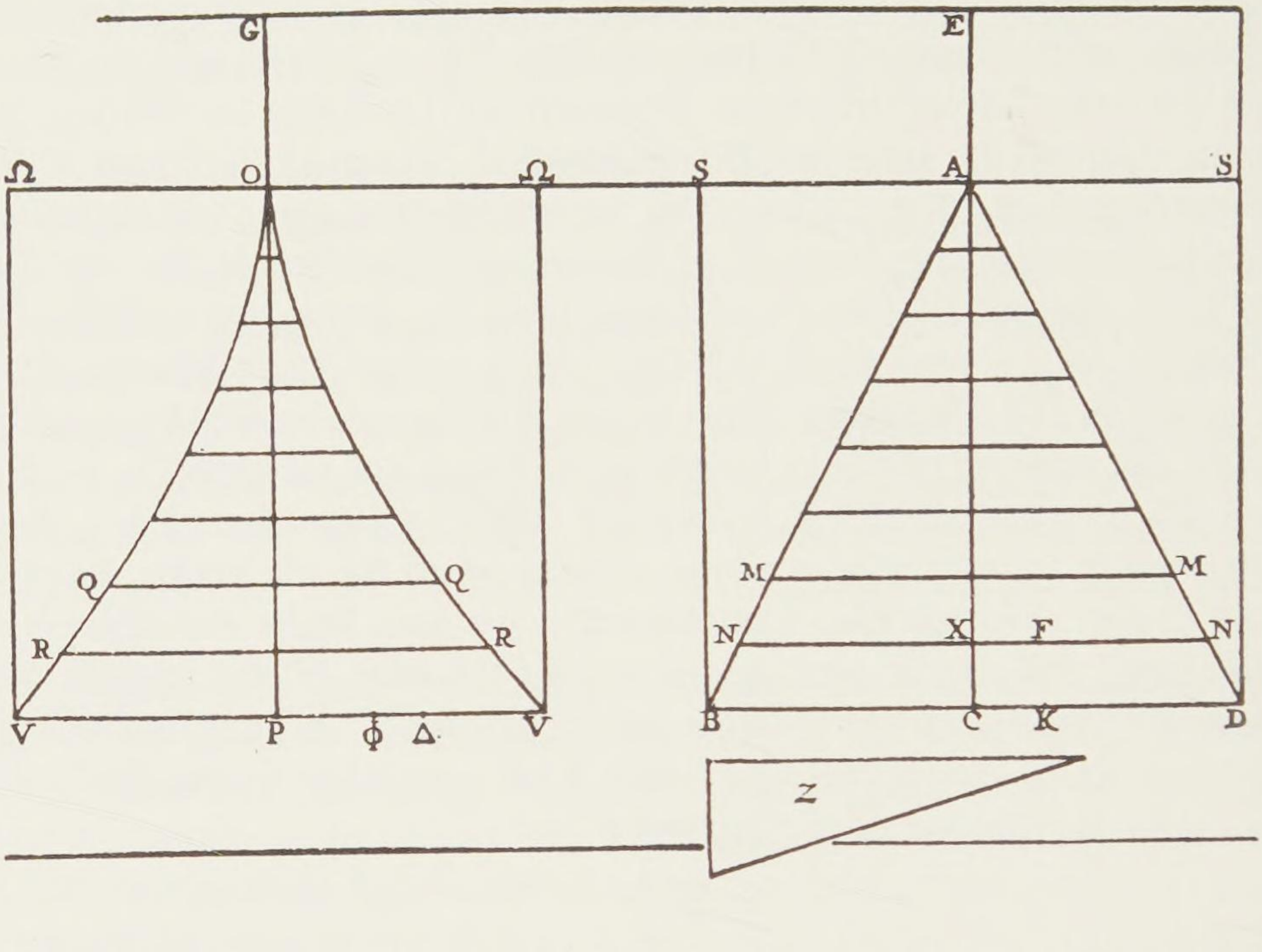
from the small portions of the solid ABCD to the plane AC. But the former sum of squares will be known if we can determine the distance of the center of gravity of the figure SYTZ from the lines BY or DZ and also the distance from the same line of the center of gravity of the wedge cut off by a plane passing through the same line [Proposition 9]. But since the figure SYTZ is symmetrical with ST as its axis, this same sum of squares will be known if we can determine the distance of the center of gravity of the half-figure SZT from the axis ST, and the distance from ST of the center of gravity of the wedge cut off above the half-figure by a plane drawn through its axis [Proposition 11]. Therefore, these things being given, we will know also the sum of the squares of the perpendiculars which are imagined to be drawn from all the small portions of the solid ABCD to the plane EAC. But we have already found the sum of the squares of all the perpendiculars to the plane drawn through EG. Consequently, we also know the aggregate of both sums, that is, as was shown above, the sum of the squares of the perpendiculars drawn from all the small portions of the solid ABCD to the given line which passes through E and is perpendicular to the plane of this page. This is what we were required to do.

PROPOSITION XV

Given the same assumptions, let the solid ABCD [*Fig. 73*] *be of such a nature that in the plane figure SYTX, which is proportioned to it, the distance of the center of gravity from the tangents BY or DZ is not known, or of such a nature that the subcentrix of the wedges, cut off above it by a plane passing through BY or DZ, is not known. But in the proportional figure OQP, which is to the side, we do know the distance ΦP* [*Fig. 74*] *of the center of gravity of the half-figure OPV from the axis OP. From this it is possible to find the sum of the squares of the distances of the small portions of the solid ABCD from the plane ED. However it is necessary that all the sections NN and MM be similar planes and that the plane EC pass through the center of gravity of all of them, as happens in the prism, pyramid, cone, conoid, and many other figures. It is also necessary to know the distances of the center of gravity of these planes from the tangents parallel to the axis of oscillation and to know the subcentrices of the wedges cut off above them by planes drawn through those same tangents.*

For example, if BD is the largest of the indicated sections, and if we imagine a straight line through B parallel to the axis E, that is, perpendicular to the plane seen here, then it is necessary to know the distance BC of the center of gravity of the section BD from the indicated line through B,

[Fig. 74]



and also the subcentrix BK of the wedge cut off above the section BD by a plane drawn through the same line at B.

Given this, let PV be cut in half at Δ such that ΔP is to PΦ as the rectangle BCK is to a certain space Z. I say that this, multiplied by the number of small portions of the solid ABCD, is equal to the sought-for sum of squares of the distances of these same small portions from the plane EC.

It has already been established that the squares of the distances of the small portions of the plane section BD from the plane EC, which passes through its center of gravity, or the squares of the distances of the solid small portions of the segment BNND from the same plane, are equal to the rectangle BCK, multiplied by the number of the indicated small portions [Proposition 10]. Likewise, if NX is the distance of the center of gravity of the plane section NN from the line which is imagined to pass through N parallel to the axis E, and if NF is the subcentrix of the wedge cut off above NN by a plane passing through the same line, then the squares of the distances of the plane small portions of the section NN from the plane EC, or the squares of the distances of the solid small portions of the segment NMMN from the same plane, will be equal to the rectangle NXF, multiplied by the number of the small portions of the section NN or of the segment NMMN. But BD is divided at C and K in the same way that NN is divided at X and F. Hence the rectangle BCK is related to the rectangle NXF as the square of BD is related to the square of NN.

But the number of small portions of the section BD is related to the number of small portions of the section NN as the sections themselves are related, that is, as the square of BD is related to the square of NN. Hence the ratio of the rectangle BCK, multiplied by the number of small portions of the section BD, to the rectangle NXF, multiplied by the number of small portions of the section NN, is a ratio equal to the square of the ratio of the square of BD to the square of NN, that is, the ratio of the square of VV to the square of RR in the proportional figure. Therefore, the first mentioned sum of the squares of the distances of the small portions of the segment BNND from the plane EC is related to the other sum of the squares of the distances of the small portions of the segment NMMN, as the square of VV is related to the square of RR. In the same way it can be shown that the sums of the squares of the distances of the small portions in the remaining segments of the solid ABCD are related to each other by the ratio of the squares of the lines in the figure OVV which correspond to the base of each segment. As a result, the sum of the squares of the distances of the small portions of all the segments of the solid ABCD from the plane EC, will be related to the sum of the squares of the distances of the small portions of the same number of segments, all of which equal the largest segment (that is, of a cylinder or prism BDSS having the same base and height as the solid ABCD), in the same way as the sum of the squares of VV, RR, QQ, etc., is related to the same number of squares, all of which are equal to the largest one VV (that is, as the solid produced by revolving OVV around the axis OP is related to the cylinder VVΩΩ which has the same base and height.) It is clear that this ratio of the solid OVV to the cylinder VVΩΩ is composed by the ratio of the planes by whose revolution they were generated, that is, by the ratio of the plane OPV to the rectangle PΩ, and by the ratio of the distances of the centers of gravity of these planes from the axis OP, that is, by the ratio of PΦ to PΔ. This first of these ratios, namely, of the plane OPV to the rectangle PΩ, is the same as the ratio of the solid ABCD to the cylinder or prism BDSS, that is, the same as the ratio of the number of small portions of the solid ABCD to the number of small portions of the cylinder or prism BDSS. The other ratio, namely, PΦ to PΔ, is the same, by construction, as the ratio of the space Z to the rectangle BCK. Thus the above mentioned ratio of the sum of the squares of the distances of all the small portions of the solid ABCD from the plane EC, to the sum of the squares of the distances of all of the small portions of the cylinder or prism BDSS from the same plane, is composed of the ratio of the number of the small portions of the solid ABCD to the number of the small portions of the cylinder or prism BDSS, and of the ratio of the space Z to the rectangle BCK. In other words, it is equal to the ratio of the rectangle Z, multiplied by the number of small portions of the solid ABCD, to the rectangle BCK, multiplied by the number of small portions of the cylinder or prism BDSS.

But the fourth of these magnitudes is equal to the second. That is, the rectangle BCK, multiplied by the number of small portions of the cylinder or prism BDSS, is equal to the sum of the squares of the distances of the small portions of the same prism or cylinder BDSS from the plane EC. For the rectangle BCK, multiplied by the number of small portions of the segment BNND, is equal to the squares of the distances of the small portions of this same segment from the plane EC [Proposition 10]. Therefore, the third term equals the first term. That is, the plane Z, multiplied by the number of small portions of the solid ABCD, equals the sum of the squares of the distances of the small portions of this same solid ABCD from the plane EC [Euclid, 5, 14].[8] Q.E.D.

It should be noted that, when the solid ABD is rotated around the axis AC, the rectangle BCK is always equal to one-fourth of the square of BC. For the subcentrix of the wedge cut off above the circle BD by a plane passing through the tangent at B, namely, the line BK, equals 5/4 of the radius BC. Therefore, if PV is assumed to equal BC, and if we establish that PΔ is related to PΦ as the rectangle BCK, that is, 1/4 of the square of BC, that is, the square of PΔ, is related to another plane Z, then it follows that Z is equal to the rectangle ΔPΦ. And therefore this same rectangle, multiplied by the number of small portions of the solid ABD, is equal to the sought-for sum of the squares of all the perpendiculars drawn from these same small portions to the plane EC.

PROPOSITION XVI

If any figure, whether it be a line, surface, or solid, is suspended from different places, and is then oscillated around axes, which are parallel to each other and which are equally distant from the center of gravity of the figure, then such a figure will be isochronous with itself.

Assume any magnitude [Fig. 75], whose center of gravity is the point E. It is suspended first from an axis which is imagined to pass through F, which is perpendicular to the plane of this page, which is thus also the plane of oscillation. In that plane draw the circumference FHG around E as the center and with EF as the radius. After selecting any point on this circumference, for instance, H, let the magnitude be understood to be suspended a second time from an axis fixed at that point, and let it be oscillated in the same plane of oscillation. I say that this latter will be isochronous with itself oscillating around the axis F.

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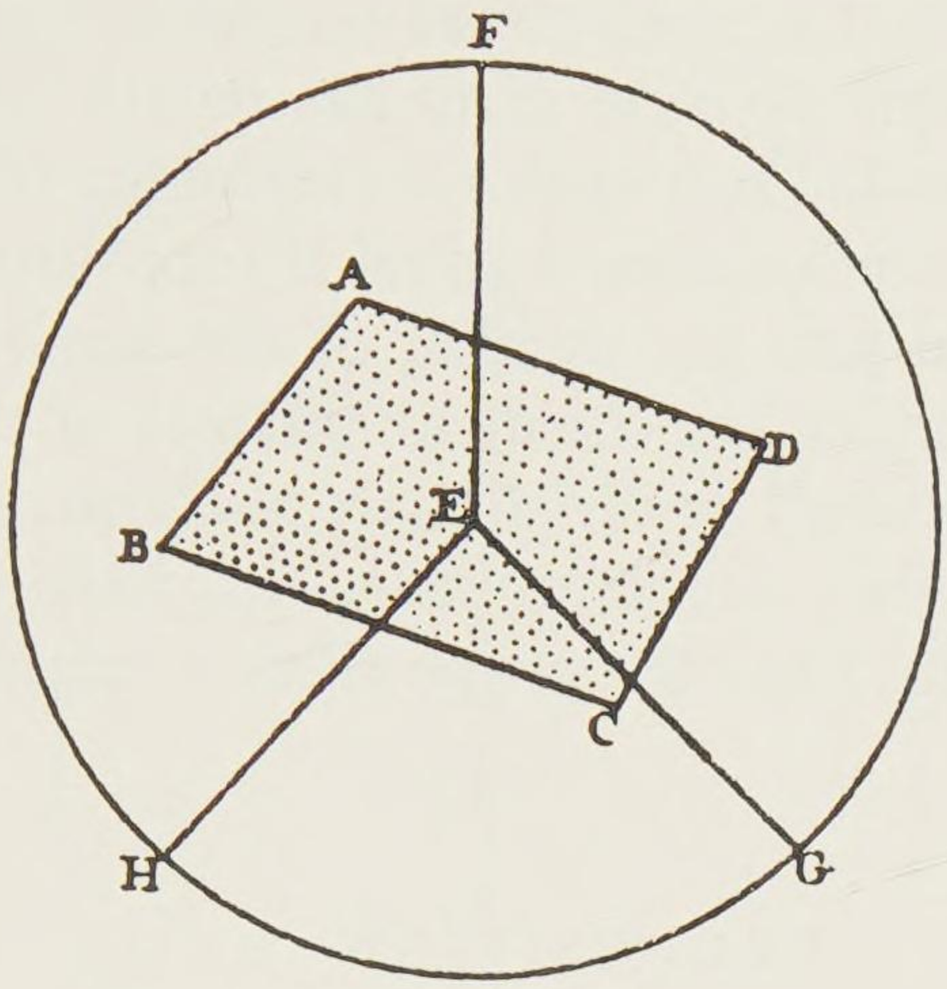
8. Proposition 14, Book 5, Euclid's *Elements:* "If a first magnitude have to a second the same ratio as a third has to a fourth, and the first be greater than the third, the second will also be greater than the fourth; if equal, equal; and if less, less." [T. L. Heath's translation, 2:162.]

[Fig. 75]



Imagine that the proposed magnitude is divided into many, equal, very small portions. Then, since in both suspensions the plane of oscillation remains the same in relation to the parts of the magnitude, it is obvious that, if perpendiculars are drawn from all the small portions, into which the magnitude is divided, to the plane of oscillation, in both cases they will meet that plane at the same points. Let these points be the ones marked in the space ABCD.

Now since E is the center of gravity of the proposed magnitude, then in any position it will maintain its equilibrium around the axis which passes through E and which is perpendicular to the plane ABCD. Consequently it is easy to see that, if an equal weight is assigned to all the previously mentioned points marked in the space ABCD, the point E will be the center of gravity of all these weights. And if, as can happen, many perpendiculars coincide at various points, those points should be considered to be doubles, and the weights assigned to them to be multiplied a number of times. When things are so considered, it is again clear that E is the center of gravity.

Furthermore it is clear that the sum of the squares of the lines drawn from all the indicated points to the point F is equal to the sum of the squares of the perpendiculars drawn from each small portion of the proposed magnitude to the axis of oscillation passing through F, for the lines whose squares are being considered have the same length in both cases. Similarly, in the case of the suspension from the axis through H, it is clear that the sum of the squares of the lines from all of the points indicated in the space ABCD to the point H is equal to the sum of the squares of the perpendiculars drawn from all of the small portions of the proposed magnitude to the axis of oscillation passing through H. Therefore in each case, if the sum of the squares of the lines drawn from all of the indicated points to

the point F or H, is divided by the lines EF or EH, multiplied by the number of small portions into which the proposed magnitude is understood to be divided, then this division gives the length of a simple pendulum which is isochronous with the magnitude suspended from F or H. But in the two cases the sum of the squares is equal [Proposition 12], as also are the lines EF and EH, and also the number of the small portions. Therefore, since both the numerators and the denominators of the above division are equal, the lengths produced by division are also equal, that is, the length of the isochronous pendulum of the proposed magnitude as suspended from F or from H is the same. Thus the proposition is established.

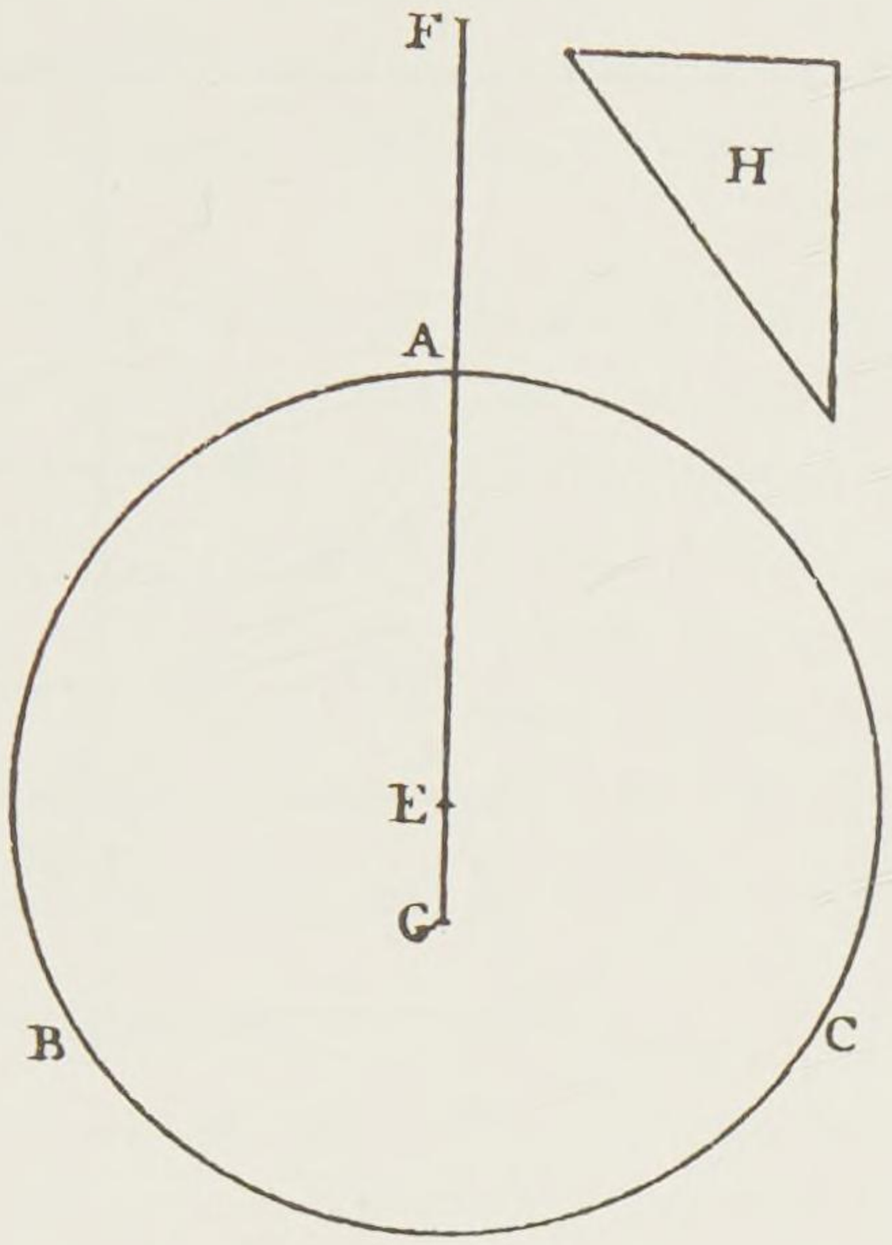
PROPOSITION XVII

Let a given plane, multiplied by the number of small portions into which a suspended figure is understood to be divided, be equal to the sum of the squares of the distances from the axis of oscillation. And let that plane be divided by a line equal to the distance between the axis of oscillation and the center of gravity of the suspended magnitude. This gives the length of the simple pendulum isochronous with that suspended figure.

Let ABC [Fig. 76] be a figure whose center of gravity is E, and which is suspended from an axis which passes through F and is perpendicular to the plane which is seen here. Assume that this figure is divided into many, equal, small portions, from each of which perpendiculars are understood to be drawn to the indicated axis. Using what was shown earlier, find the plane H which, when multiplied by the number of the indicated small portions, equals the sum of the squares of the indicated perpendiculars. Dividing the plane H by the line FE gives the length FG. I say that this is the length of a simple pendulum having oscillations isochronous with the magnitude ABC oscillating around the axis through F.

For the sum of the squares of the distances from axis F, divided by the distance FE multiplied by the number of the small portions, gives the length of the simple, isochronous pendulum [Proposition 6]. But this sum of squares was assumed to be equal to the plane H multiplied by the same number of small portions. Consequently, the plane H, multiplied by the same number of small portions, divided by the distance FE, multiplied by the number of small portions, or, omitting the common factor, the plane H divided by the distance FE gives the length of the simple isochronous pendulum. It is thus shown that this length is FG. Q.E.D.

[Fig. 76]



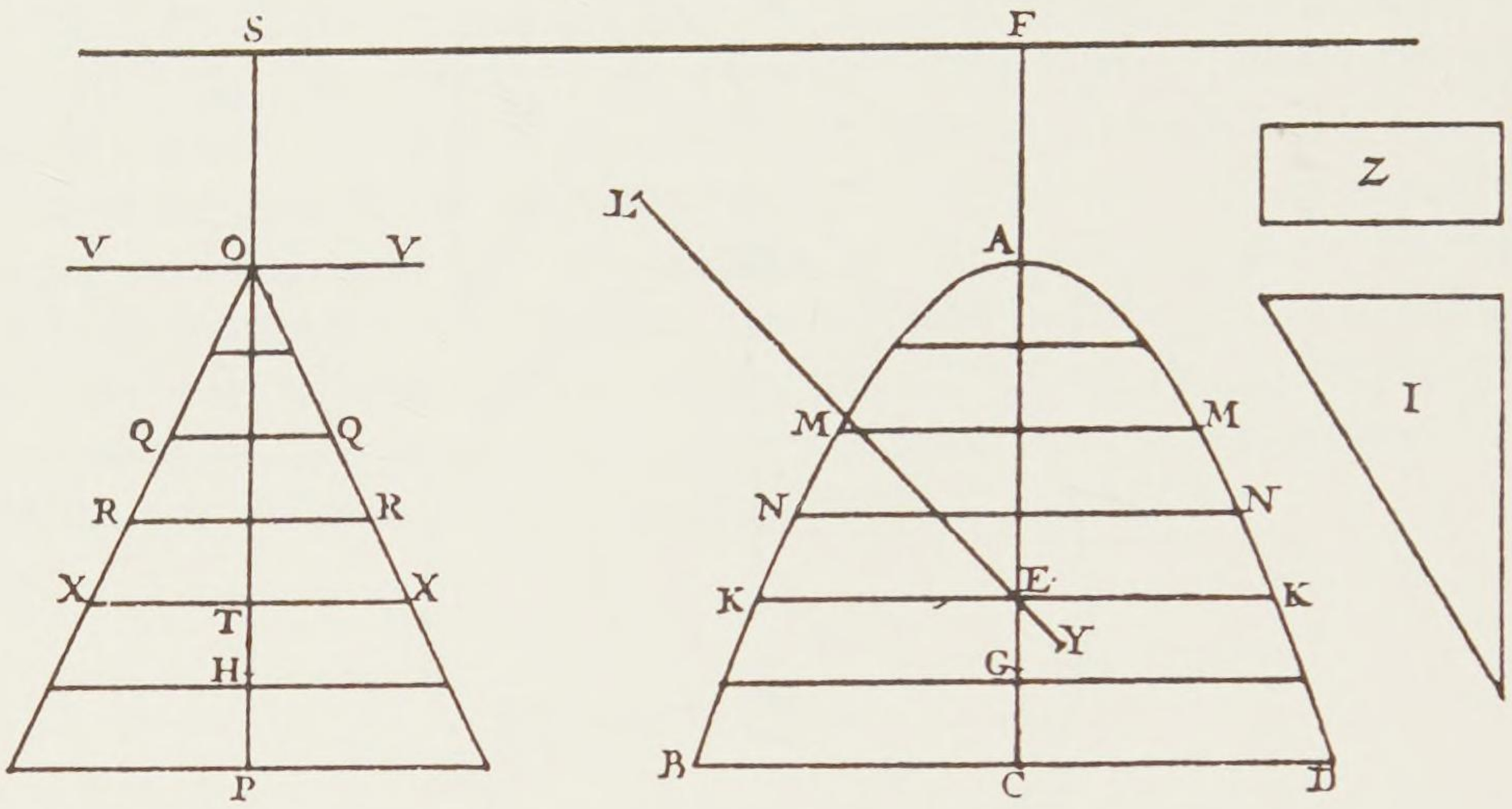
PROPOSITION XVIII

Let a plane space, multiplied by the number of small portions of a suspended magnitude, be equal to the sum of the squares of the distances from the axis of gravity, which is parallel to the axis of oscillation. And let that space be divided by a line equal to the distance between the two indicated axes. This gives a line equal to the interval by which the center of oscillation is lower than the center of gravity of the same magnitude.

Let ABCD [Fig. 77] be a magnitude whose center of gravity is E, and whose center of oscillation is G when it is suspended from the axis which passes through the point F and which is understood to be perpendicular to the plane of this page. Next, in addition to the axis through F, imagine another parallel axis passing through the center of gravity E. Let the magnitude be divided by thought into many, equal, very small portions. Let the plane I, multiplied by the number of these small portions, be equal to the sum of the squares of the distances from the axis through E. Divide the plane I by the distance FE, giving a certain straight line. I say that this line is equal to the interval EG by which the center of oscillation is lower than the center of gravity of the magnitude ABCD.

In order that we might give a universal demonstration of this proposition, imagine a plane figure OQP which is analogous to the magnitude ABCD and located to its side. Since OQP is cut by the same horizontal planes as the magnitude ABCD, its segments intercepted between any two planes are related to each other in the same ratio as the segments of the

[Fig. 77]



magnitude ACBD to which they correspond. Let each segment of the figure OQP be divided into as many equal, small portions as are contained in the corresponding segment of the figure ABCD. This can be understood to occur no matter what the magnitude ABCD is, that is, a line or a surface or a solid. The center of gravity of the figure OQP, which is T, obviously always has the same height as the center of gravity of the magnitude ABCD. Therefore, if a horizontal plane drawn through F cuts the center line[9] of the figure OQP, in this case at S, the distances ST and FE are equal.

Further it has already been established that the squares of the distances from the axis of oscillation F, divided by the distance FE multiplied by the number of small portions, gives the length of the isochronous pendulum [Proposition 6]. This length was represented as FG. And it is obvious that the sum of those squares is equal to the squares of the distances from the horizontal plane through F, plus the squares of the distances from the vertical plane FE drawn through the axis F and the center of gravity E [Euclid, 1, 47]. But the squares of the distances of the magnitude ABCD from the horizontal plane through F equal the squares of the distances of the figure OQP from the line SF. These latter squares (if O is the highest point of the figure OQP, and if OH is the subcentrix of a wedge cut off above it by a plane passing the line OV, which is parallel to SF) are equal to the sum of the rectangle OTH and the square of ST, multiplied by the number of small portions of the indicated figure [Proposition 9], that is, the magnitude ABCD. But the squares of the distances of the magnitude

9. See Part IV, Definition VII.

ABCD from the plane FE are always the same, no matter what the distance is between the axis of oscillation F and the center of gravity E. Let those squares be equal to the space Z, multiplied by the number of small portions of the magnitude ABCD.

Therefore, since the squares of the distances of the magnitude ABCD from the axis of oscillation F are equal to the sum of the square of ST, plus the rectangle OTH, plus the plane Z, all multiplied by the number of small portions of the same magnitude, then if this whole expression is divided by the distance FE or ST, this gives the length FG of the pendulum isochronous with the magnitude ABCD [Propositon 6]. But the division of the square of ST by its own side ST gives ST, or FE. Therefore the remainder EG is given by the division of the sum of the rectangle OTH, plus the plane Z, by ST or FE.

As a result what remains for us to prove is that the rectangle OTH, plus the plane Z, equals the plane I. For when this is established, we have also proven that the plane I, divided by the distance FE, gives a length equal to EG. The former is proven as follows. The rectangle OTH, multiplied by the number of small portions of the figure OQP, or of the magnitude ABCD, equals the square of the distances of the figure from the line XT [Proposition 10], which is drawn through the center of gravity T and is parallel to SF. And thus it is also equal to the squares of the distances of the magnitude ABCD from the horizontal plane KK drawn through the center of gravity E. For the distances are the same in both cases. But the plane Z, multiplied by the same factor, was taken to be equal to the squares of the distances of the magnitude ABCD from the vertical plane FE. And it is clear that these squares of the distances from the plane FE, plus the above mentioned squares of the distances from the horizontal plane through E, are equal to the squares of the distances from the axis of gravity through E, which is parallel to the axis F [Euclid, 1, 47]. And thus the rectangle OTH, plus the plane Z, multiplied by the number of small portions of the magnitude ABCD, will be equal to the squares of the distances of this same magnitude from the axis through E. But the plane I, multiplied by the same number of small portions, was also taken to be equal to the same squares of distances. Therefore the plane I equals the sum of the rectangle OTH and the plane Z. This is what remained to be proven.

From this it is again clear what was demonstrated in Proposition 16; namely, that any magnitude is isochronous with itself if it is suspended from many places and then oscillated around parallel axes which are equally distant from its center of gravity.

Whether the magnitude ABCD be suspended from the axis F or from the axis L, which is parallel to F, it is manifest that in both cases the squares of the distances from the axis through E, which is parallel to the axes F and L, are equal. And the plane I, which, when multiplied by the number of

small portions, equals the sum of the squares, will also be the same in both cases. But this plane, divided by the distance of the center of gravity from the axis of oscillation, which was assumed to be the same in both cases, gives the distance by which the center of oscillation lies below the center of gravity. Therefore this distance will also be the same in both cases. For example, if the suspension is made at L, this distance is EY and is equal to EG. And the whole of YL equals GF. Therefore in each case the same simple pendulum is isochronous with the magnitude ABCD.

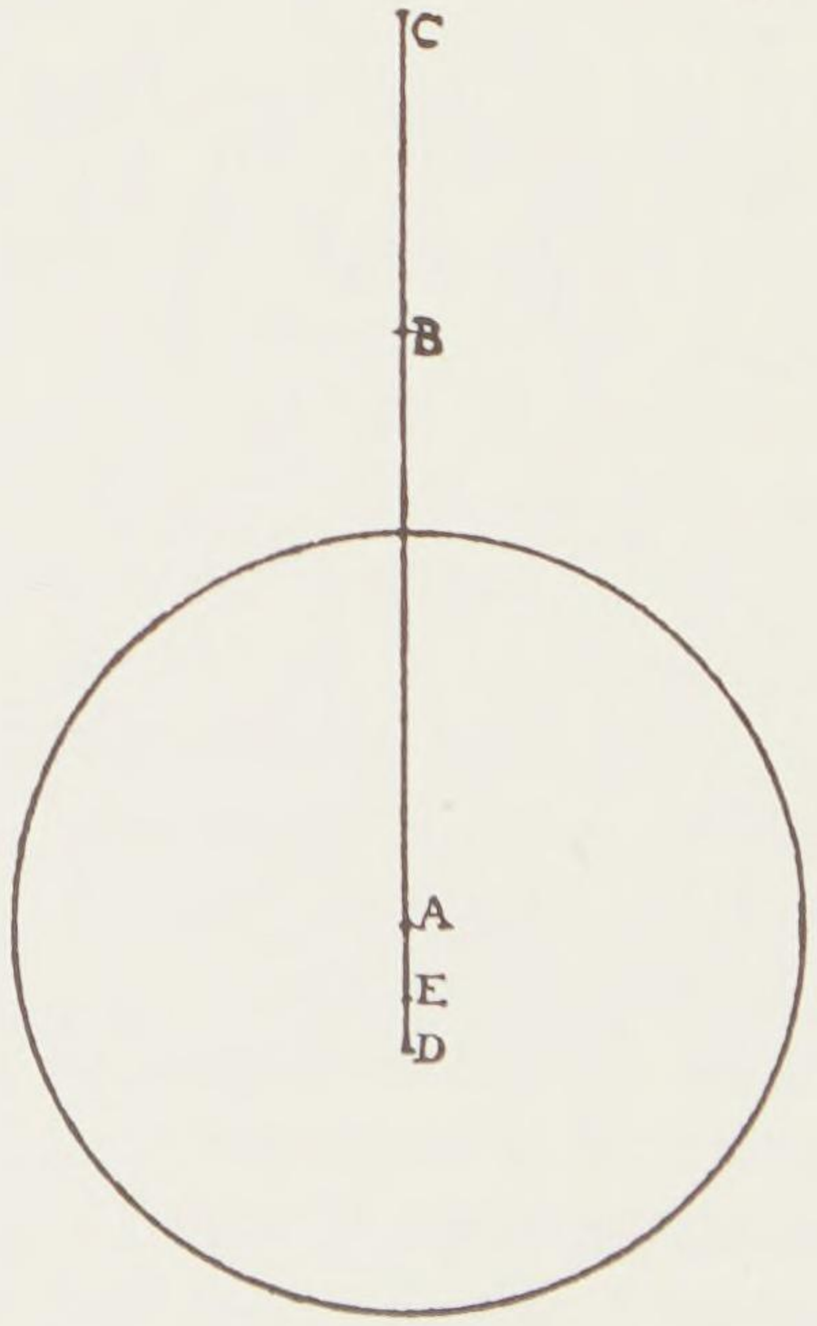
PROPOSITION XIX

If the same oscillating magnitude is suspended from a shorter, and then from a longer, length, then the distances of the axes of oscillation from the center of gravity are inversely related to the distances of the centers of oscillation from the same center of gravity.

Let an oscillating magnitude, whose center of gravity is A [Fig. 78], be suspended first from the axis at B and then from the axis at C. In the first case let D be the center of oscillation, and in the second case let E be the center of oscillation. I say that BA is to CA as EA is to DA.

For in the suspension from B the distance AD, by which the center of oscillation lies below the center of gravity, is obtained by dividing the dis-

[Fig. 78]



tance BA into a certain space which, when multiplied by the number of equal, very small portions into which the magnitude is imagined to be divided, equals the squares of the distances from the axis through A, which is parallel to the axis at B [Proposition 18]. Hence the rectangle BAD will be equal to the indicated space. Likewise in the suspension from C, the distance AE is obtained by dividing that same space by the distance CA. Hence the rectangle CAE is also equal to that same space. Therefore the rectangles BAD and CAE are equal to each other, and thus the ratio of BA to CA equals the ratio of AE to AD. Q.E.D.

From this it is clear that, if we are given a simple pendulum which is isochronous with a suspended magnitude in one case, and if we know its center of gravity, then we also know the length of the isochronous pendulum in all other suspensions, whether they be longer or shorter, as long as the plane of oscillation remains the same.

PROPOSITION XX

The center of oscillation and the point of suspension are inversely related.

In the same figure [Fig. 78] D is the center of oscillation when the suspension occurs from B. And if everything is inverted and the suspension occurs from D, then B is the center of oscillation. This is obvious from the preceding proposition.

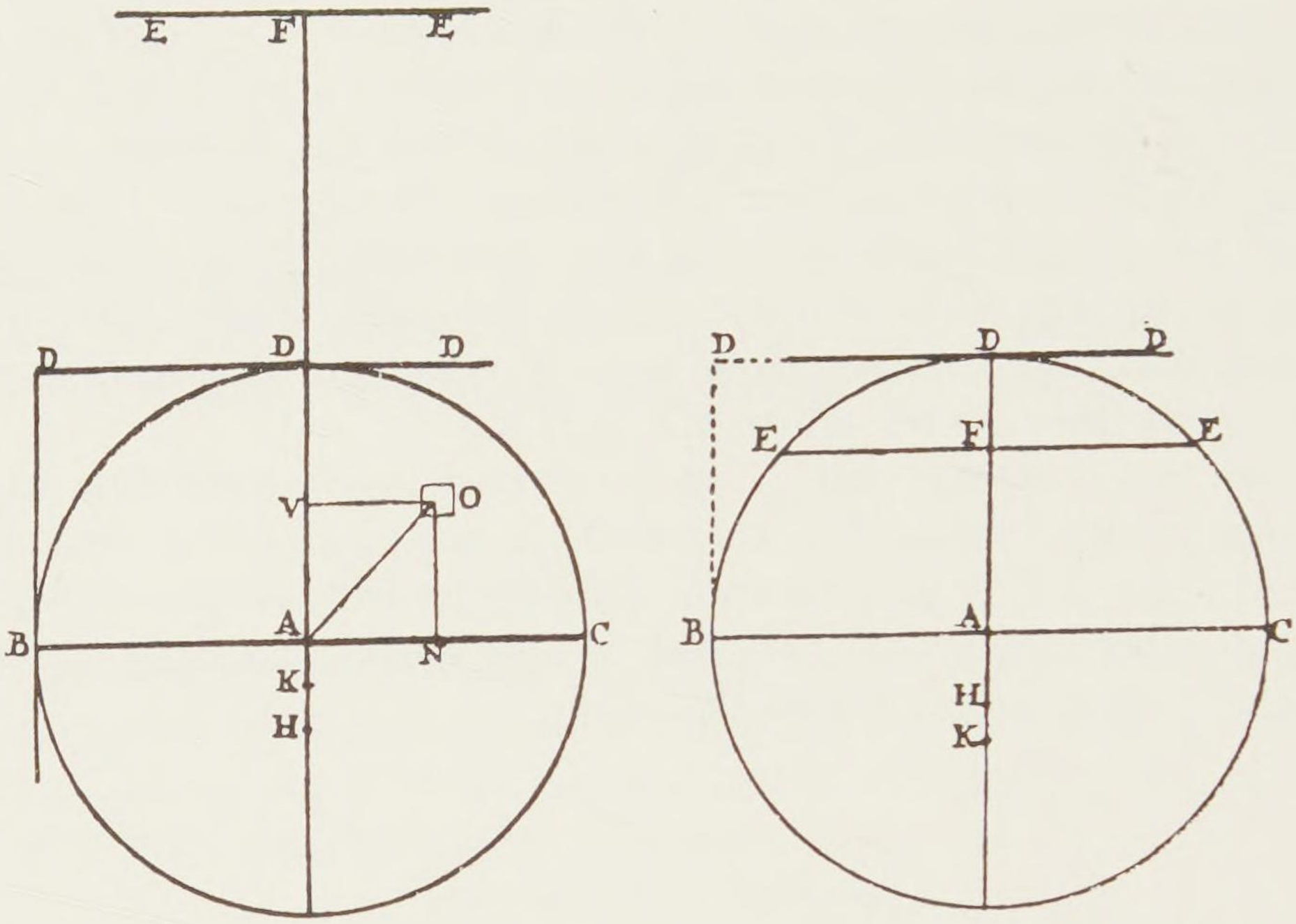
PROPOSITION XXI

How to find the centers of oscillation in plane figures.

If one has understood what has been demonstrated to this point, then it will be easy to define the centers of oscillation for many figures which are usually considered in geometry. To speak first of plane figures, we have defined for them above a double motion of oscillation; namely, around an axis located in the same plane as the figure, and around an axis which is perpendicular to the plane of the figure. The former we have called oscillation in the plane; the latter we have called lateral oscillation.

Let us consider the former first; namely, a motion around an axis located in the same plane, for example, the figure BCD around the axis EF [Fig. 79]. The center of oscillation K is found as follows. Imagine a wedge cut off above the figure by a plane which cuts the plane of the figure in such a way that its intersection DD is parallel to the axis of oscillation. Let AD

[Fig. 79]



be given as the distance of the center of gravity from this intersection. And let DH be the subcentrix of that wedge in relation to the same intersection. The center of oscillation K of the figure BDC is found by dividing the rectangle DAH by the distance FA, for this division gives the distance AK, by which the center of oscillation is located below the center of gravity. For the rectangle DAH, multiplied by the number of small portions of the figure BCD, equals the squares of the distances from the line BAC, which is drawn through the center of gravity A and is parallel to the axis of oscillation EF [Proposition 10]. Hence the division of this same rectangle by the distance FA gives the distance AK, by which the center of oscillation is located below the center of gravity A [Proposition 18].

From this it is clear that, if the axis of oscillation is DD, then the center of oscillation is located at the point H. And hence the length DH of a simple pendulum isochronous with the figure BCD is then identical with the subcentrix of the wedge, in respect to DD, cut off by a plane through DD. This has been noted previously by others,[10] but not demonstrated.

It is not our intention here to examine how to find the centers of gravity of wedges cut off above plane figures. This is already known in many cases. For example, if the figure BCD is a circle, DH equals 5/8 of the

10. This apparently refers to Pierre Mousnier and Honoré Fabri, SJ, in *Tractatus physicus de motu locali, in quo effectus omnes, qui ad impetum, motum naturalem, violentum, & mixtum pertinent, explicantur & ex principiis physicis demonstrantur.* Auctore Petro Mousnerio Doctore Medico: Cuncta excerpta ex praelectionibus R. P. Honorati Fabry, SJ. (1646). See *Oeuvres* 18:52–58.

diameter. And if it is a rectangle, DH equals ⅔ of the diameter. This explains why a rod or a line endowed with weight, suspended from one end, is isochronous with a pendulum whose length is ⅔ of the rod's length. For such a line can be considered to be a very narrow rectangle.

If the figure is a triangle, and if its vertex points upwards, DH equals ¾ of the diameter; if the vertex points downwards, DH equals ½ of the diameter.

It should be noted that what was demonstrated in Proposition 16 applies to the motion of plane figures as follows. If we give the figure BCD various positions by turning it around the axis BAC so that it lies parallel to the horizon or is inclined obliquely to it, and if the axis of oscillation FE remains permanent, then the length FK of the isochronous pendulum also stays the same. This is obvious from that proposition.

Next let us consider a plane figure which oscillates around an axis which is perpendicular to the plane of the figure. This we have called lateral oscillation. For example, let the figure BCD [Fig. 80] be moved around an axis which is understood to run through the point F perpendicular to the plane DHC. In this case, in addition to the wedge which is cut off above the figure by a plane drawn through DD, which is tangent to the figure at its highest point, we must also consider the wedge which is cut off by a plane through BD, which is tangent to the side of the figure, and which is perpendicular to the tangent DD.[11] And in addition to the center of gravity A of the figure and the subcentrix HD of the first wedge, we must also know the subcentrix LB of the second wedge. For in this way we will know the rectangles DAH and BAL, the sum of which constitute in this case the space to be divided, which henceforth will be called the rectangle of oscillation. This space, divided by the distance FA, gives the distance AK, by which the center of oscillation K lies below the center of gravity A.

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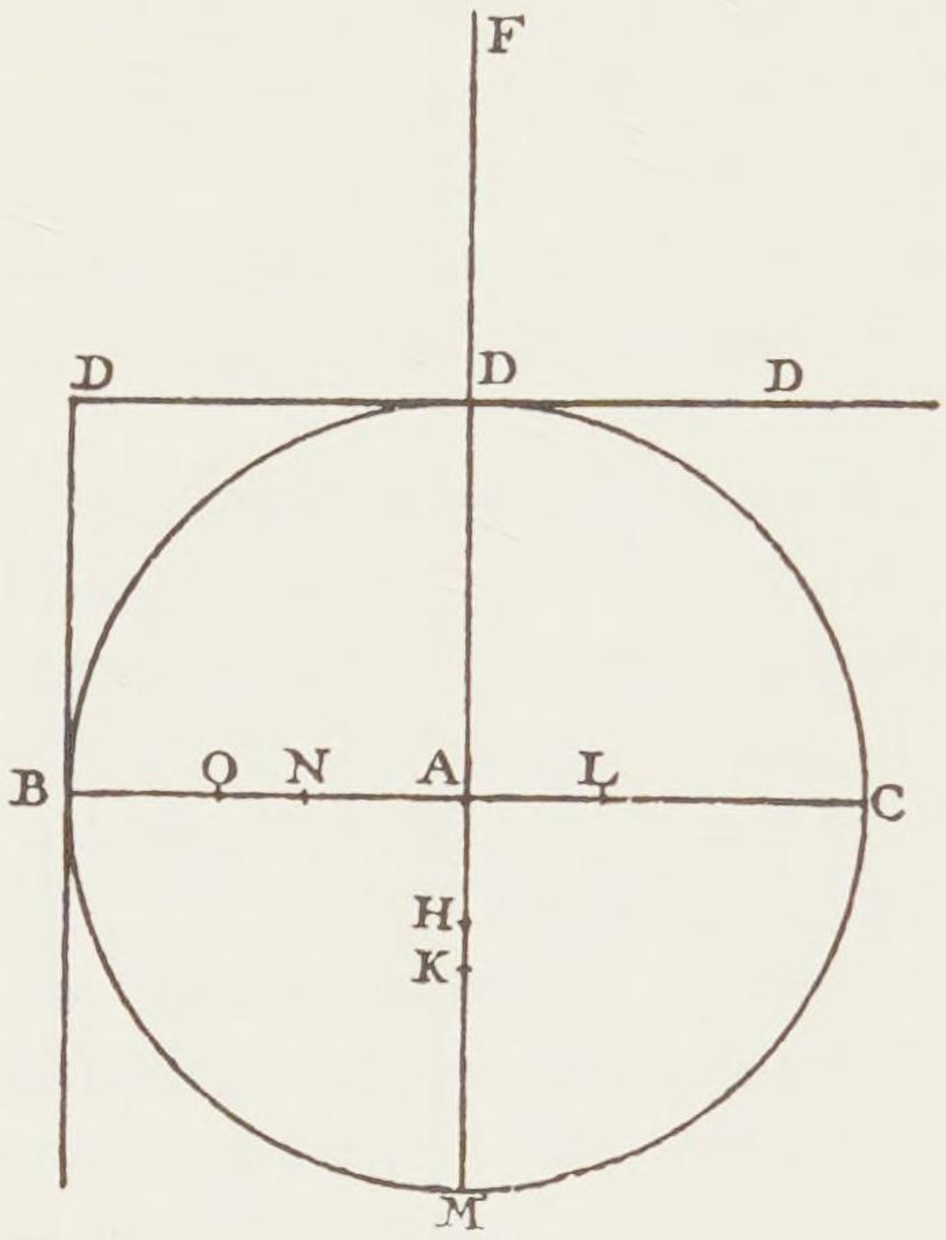
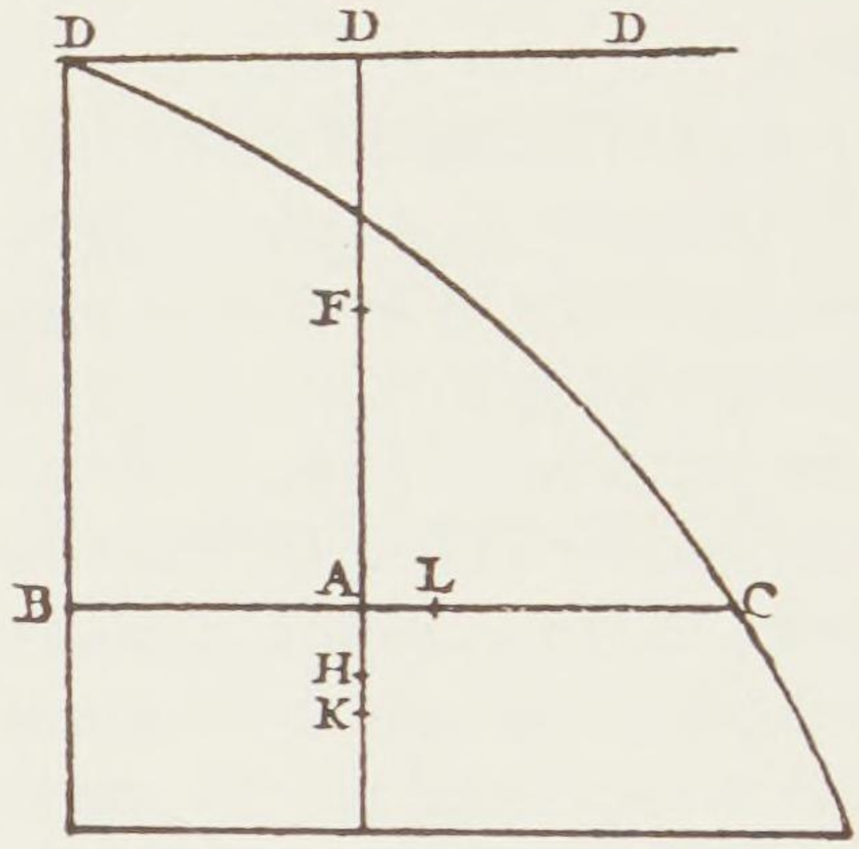
11. In his copy of the 1673 edition Huygens replaced this sentence with the following, more detailed, statement in Latin of his method of calculation:

> In this case it is necessary to determine the sum of the squares of the distances of all the particles from the line which is imagined to run through the center of gravity A and to be parallel to the axis of oscillation, as was explained in Proposition 18. This is equal to the sum of the squares of the distances from the center of gravity A itself, since this figure is a plane figure. This is also the same as the sum of the squares of the distances from both the line BAC and from the line DA. For it is clear that the square of the line OA, which I take to be the distance of any given particle from the center A, is equal to the squares of the distances ON and OV by which that same particle is separated from the lines BAC and DA (*Elements* 1:47). And the sum of the squares of the distances from the line BAC is equal to the rectangle DAH, since DH is the subcentrix of the wedge which is cut off above the figure by the tangent DD, which is parallel to BA (Proposition 10). Likewise the sum of the squares of the distances from the line DA is equal to the rectangle BAL, since BL is the subcentrix of the wedge which is cut off by the tangent BD, which is parallel to AD.

In the 1724 edition 's Gravesande included this passage in the body of the Latin text (as suggested by Huygens in a note added to the "Corrigenda" page in his copy of the 1673 edition) as a replacement for the indicated sentence.

[Fig. 80]





But if FA is the axis of the figure BCD [Fig. 80-I], the wedge cut off above the half-figure DBM by a plane through DM can be substituted for the wedge cut off above the whole figure by a plane through BD. For if OA is the subcentrix in relation to DM of the former wedge, and if NA is the distance of the center of gravity of the plane figure DBM from that same line DM, then it is clear that the rectangle OAN is equal to the rectangle BAL [Proposition 11]. Therefore the rectangle OAN, added to the rectangle DAH, also constitutes a plane which, when divided by the distance FA, gives the distance AK.

The demonstration of these matters is evident from what has preceded. For the rectangles DAH plus BAL, or DAH plus OAN, multiplied by the number of small portions of the figure, are equal to the squares of the distances from the center of gravity A, or what is the same thing in this case, are equal to the squares of the distances from the axis of gravity, which is parallel to the axis of oscillation. Therefore the indicated rectangles, divided by the distance FA, gives the length of the interval AK [Proposition 18].

The Center of Oscillation of a Circle

In a circle the rectangles DAH and BAL are clearly equal to each other, and their sum equals half the square of the radius. Hence, if FA is to the radius AB as AB is to some other length, then half of this latter length is the distance AK from the center of gravity to the center of oscillation. There-

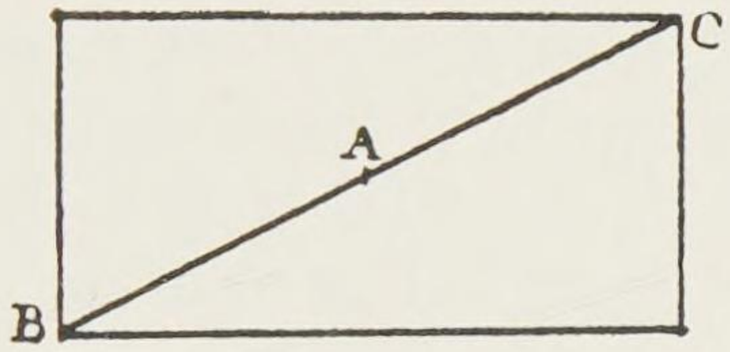
fore, if the circle oscillates around the axis D located in its circumference, DK will equal three-fourths of the diameter DM.

Using this method[12] we have found the centers of oscillation for the following plane figures. It is sufficient simply to state the results as follows.

The Center of Oscillation of a Rectangle

In every rectangle, for example, CB [Fig. 81], the space to be divided, or what we have called the rectangle of oscillation, is found to be equal to one-third of the square of the half-diagonal AC. Hence it follows that, if a rectangle is suspended from one of its angles and is oscillated laterally, the pendulum which is isochronous with it will be ⅔ of the entire diagonal.

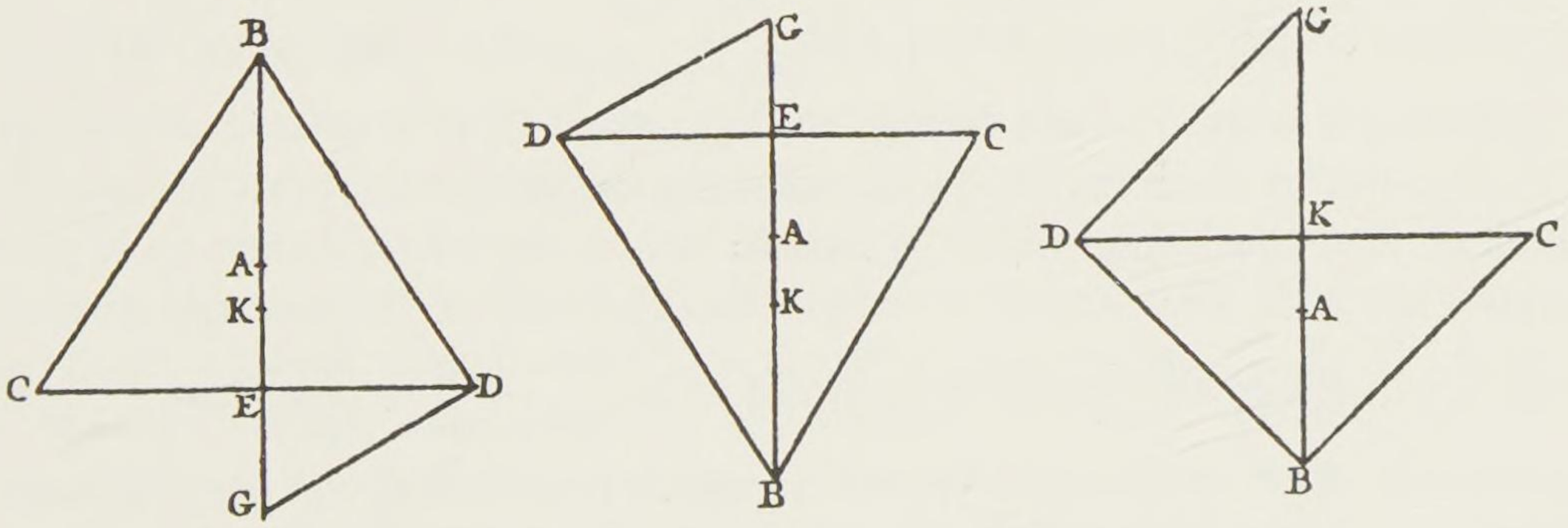
[Fig. 81]



The Center of Oscillation of an Isoceles Triangle

In an isosceles triangle such as CBD [Fig. 82] the space to be divided is equal to one-eighteenth of the square of the diameter BE, plus one-twenty-fourth of the square of the base CD. Hence, if from an angle at the base the line DG is drawn, which is perpendicular to the side DB and which meets

[Fig. 82]



12. Actually Huygens discovered the above by using a quite different method. See *Oeuvres* 16:455. For his calculations of the centers of oscillation which follow, see *Oeuvres* 16:385, 393–555.

the extension of the diameter BE at G, and if A is the center of gravity of the triangle, and if the interval GA is divided into four equal parts of which one, AK, is added to BA, then BK will be the length of the isochronous pendulum, when the triangle is suspended from the vertex B. But if it is suspended from the midpoint E of the base, the length EK of the isochronous pendulum will equal half of BG.

From this it follows that, if an isosceles right triangle is suspended from the midpoint of its base, it is isochronous with a pendulum whose length equals its diameter. Likewise, if it is suspended from its right angle, it is isochronous with the same pendulum.

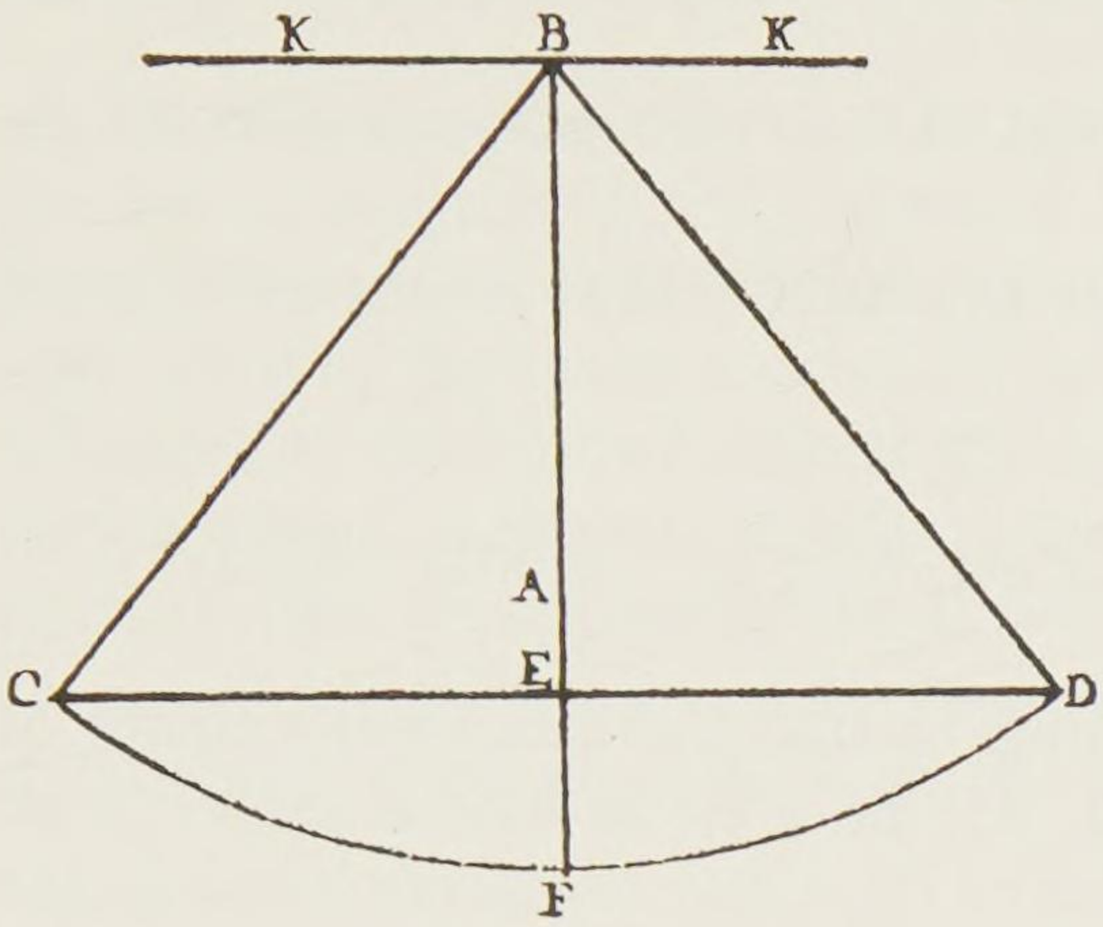
The Center of Oscillation of a Parabola

For a segment of a parabola cut off by a perpendicular to its axis, the space to be divided is equal to $^{12}/_{175}$ times the square of the axis, plus one-fifth of the square of half of the base. When a parabola is suspended from the point of its vertex, the length of the isochronous pendulum is found to be $^{5}/_{7}$ of the axis, plus $^{1}/_{3}$ of the *latus rectum*. When suspended from the midpoint of the base, this length will be $^{4}/_{7}$ of the axis, plus half of the *latus rectum*.

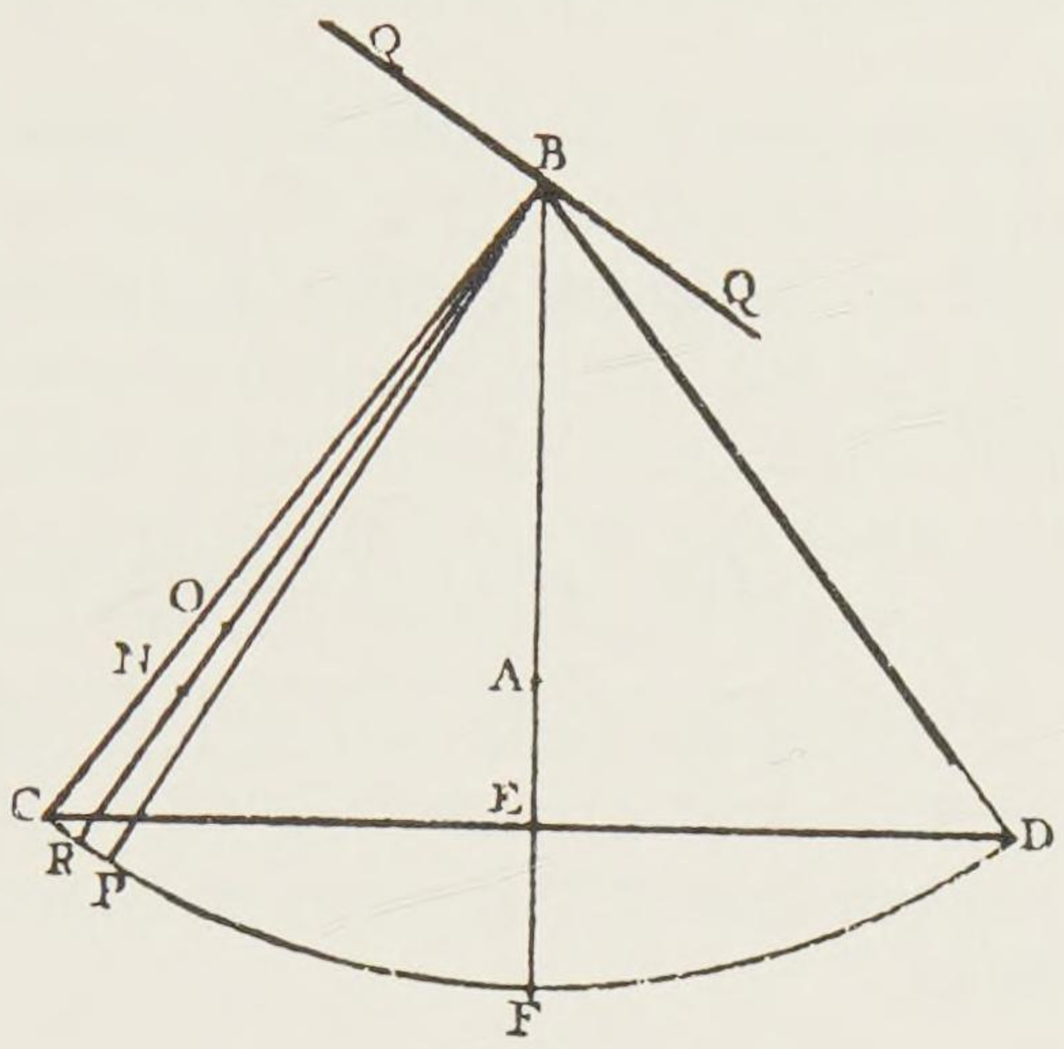
The Center of Oscillation of a Sector of a Circle

In the sector BCD of a circle [Fig. 83] let r be the radius BC; p the semiarc CF; and b the half-cord CE. The space to be divided is $\frac{1}{2}rr - \frac{4bbrr}{9pp}$, that is, half of the square of BC, minus the square of BC, minus the square of BA, A being the center of gravity of the sector. For then BA $= \frac{2br}{3p}$. When the sector is suspended from B, the center of the circle, the pendulum which is isochronous with it is $\frac{3pr}{4b}$, that is, three-quarters of the line which is related to the radius BF, as the arc CFD is related to the cord CD. We found these results by considering the subcentrices of the wedges; namely, first the wedge which is cut off above the whole sector by a plane drawn through BK, which is parallel to the cord CD (the subcentrix in relation to BK of this wedge was found to be $\frac{3r}{8} - \frac{3a}{8} + \frac{3pr}{8b}$, where a is the versed sine EF), and second the wedge which is cut off above the half-sector BFC by a plane through BF (the subcentrix in relation to BF of this wedge was found to be $\frac{3b}{8} - \frac{3br}{8a} + \frac{3pr}{8a}$).

[Fig. 83]



[Fig. 84]



But the center of oscillation of a sector is found more easily by using another method as follows. Imagine that the sector BCP [Fig. 84], which can be treated as a triangle, is a very small part of the sector BCD. The squares of the distances of its very small portions from the point B are equal to the squares of the distances from the line BR, which divides the sector in half, plus the squares of the distances from the line BQ, which is perpendicular to BR. But the ratio of these latter squares to the former ones is greater than any given ratio, because the angle CBP is extremely small. Hence the former squares can be considered as equal to zero.

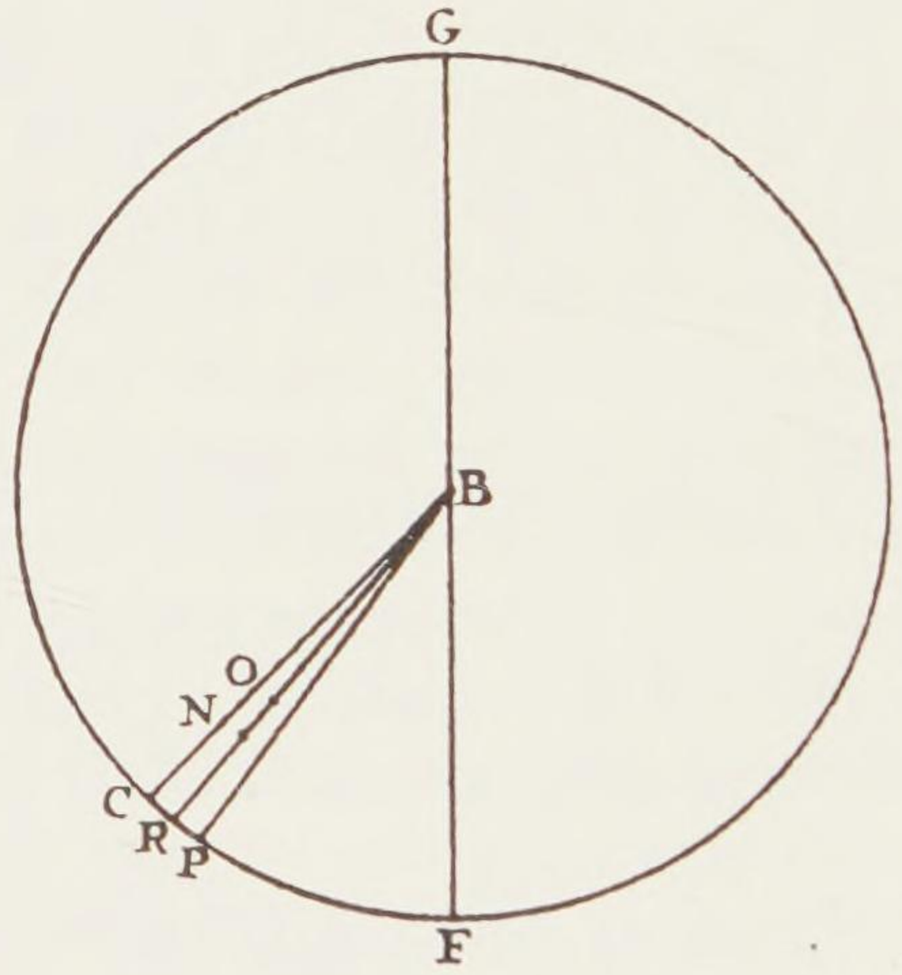
Let BO be two-thirds of BR, that is, O is the center of gravity of the triangle BCP. Let BN be three-quarters of BR; thus N is the center of gravity of a wedge cut off above the triangle BCP by a plane through BQ. Granting this, it is clear that the squares of the distances of the small portions of the triangle BCP from the line BQ, are equal to the rectangle NBO, multiplied by the number of small portions of the same triangle. And thus the rectangle NBO, so multiplied, can be considered to be equal to the squares of the distances of the small portions of the triangle BCP from the point B. But the squares of these latter distances are related to the squares of the distances of the whole sector, as the sector BCP is related to the sector BCD, that is, as the number of small portions of the sector BCP is related to the number of small portions of the sector BCD. This is easy to understand because the sector BCD is divisible into sectors like BCP. Hence the rectangle NBO, multiplied by the number of small portions of the sector BCD, is equal to the squares of the distances of its small portions from the point B. Therefore the rectangle NBO, divided by BA, which is the distance between the point of suspension and the center of gravity of the sector, gives the length of the isochronous pendulum, when the sector is

suspended from B [Proposition 17]. But the rectangle NBO equals $\frac{1}{2}rr$, and the distance BA equals $\frac{2br}{3p}$, as was said above. Hence by division we get $\frac{3pr}{4b}$ as the length of the isochronous pendulum, the same result found above.

The Center of Oscillation of a Circle, by Another Method

Using the same method it is possible to find quite easily the center of oscillation of a circle. Let GCF [Fig. 85] be a circle whose center is B. Imagine that BCP is a very small sector of the circle, as above in the sector BCD.

[Fig. 85]



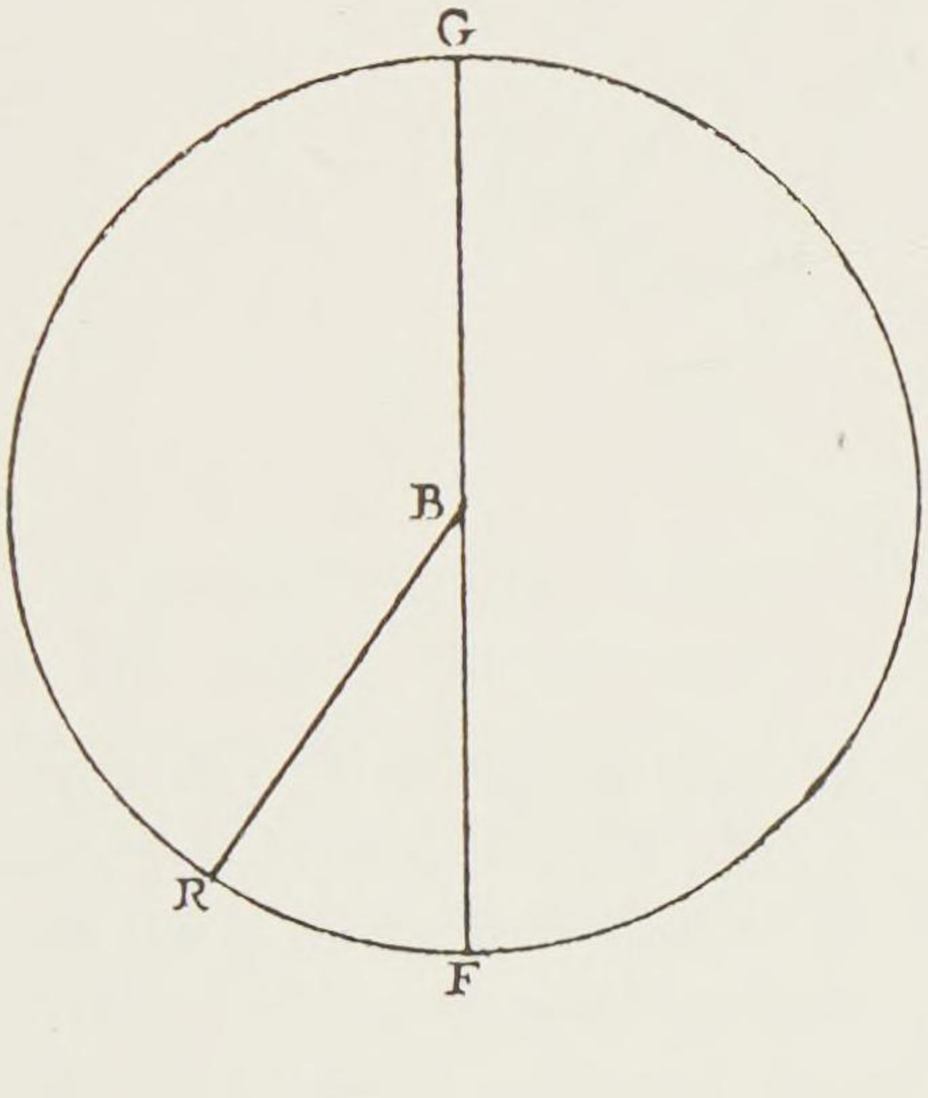
Following the method used above, the squares of the distances of the small portions of the sector BCP from the center B, are equal to the rectangle NBO, that is, to half the square of the radius, multiplied by the number of small portions of that sector. But the circle is composed of such sectors. Hence the squares of the distances of the small portions of the whole circle from the center B, are equal to half of the square of the radius, multiplied by the number of the small portions of the circle.

But B is the center of gravity of the circle. Therefore the indicated half of the square of the radius will be in this case the area to be divided by the distance between the point of suspension and the center B, in order to obtain the interval by which the center of oscillation lies below the center B [Proposition 18]. This result is the same as was shown above.

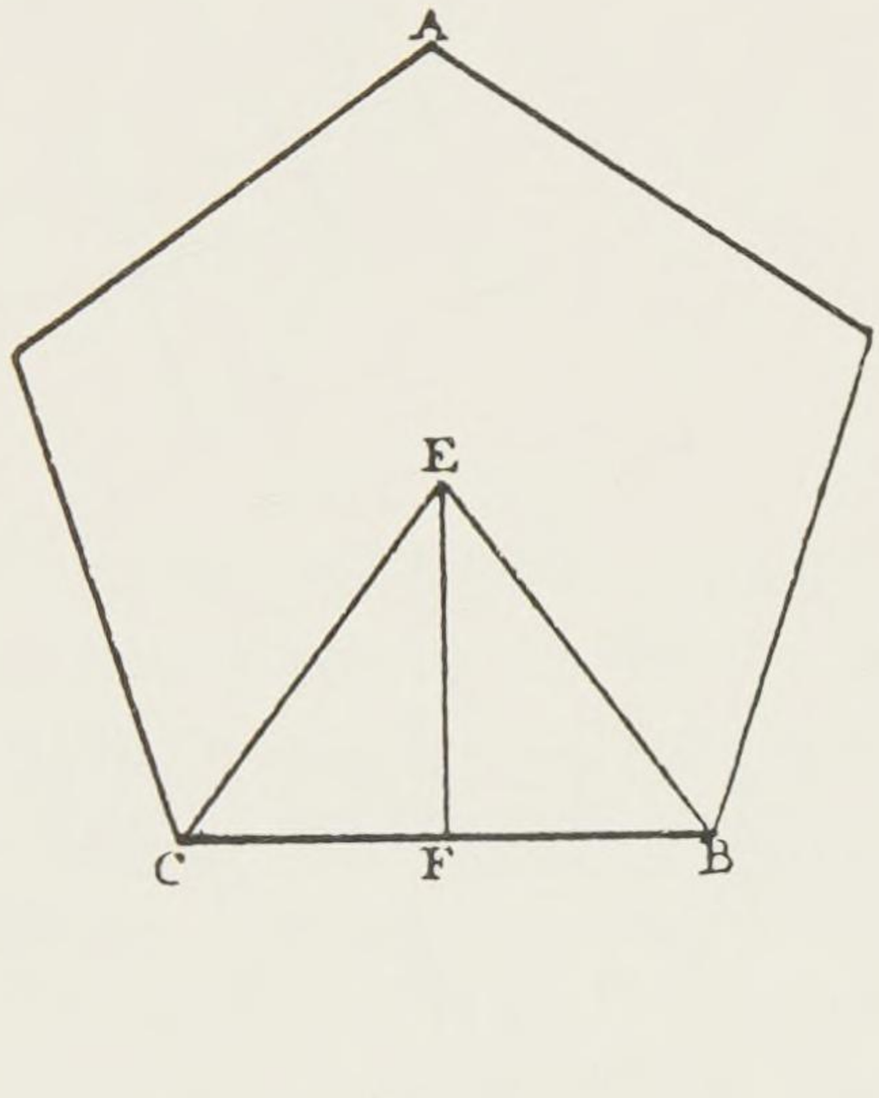
The Center of Oscillation of the Circumference of a Circle

It is also quite easy to find the center of oscillation of the circumference of a circle by this method. Let there be a circumference drawn around the center B with the radius BR [Fig. 86]. Then the square of BR, multiplied by the number of small portions into which the circumference is imagined to be divided, equals the squares of the distances of all its small portions from the center B. Hence the square of BR is in this case the space to be divided [Proposition 18]. It is also clear from this that, if the circumference is suspended from the point G, the length of the isochronous pendulum will be equal to the diameter GF.

[Fig. 86]



[Fig. 87]



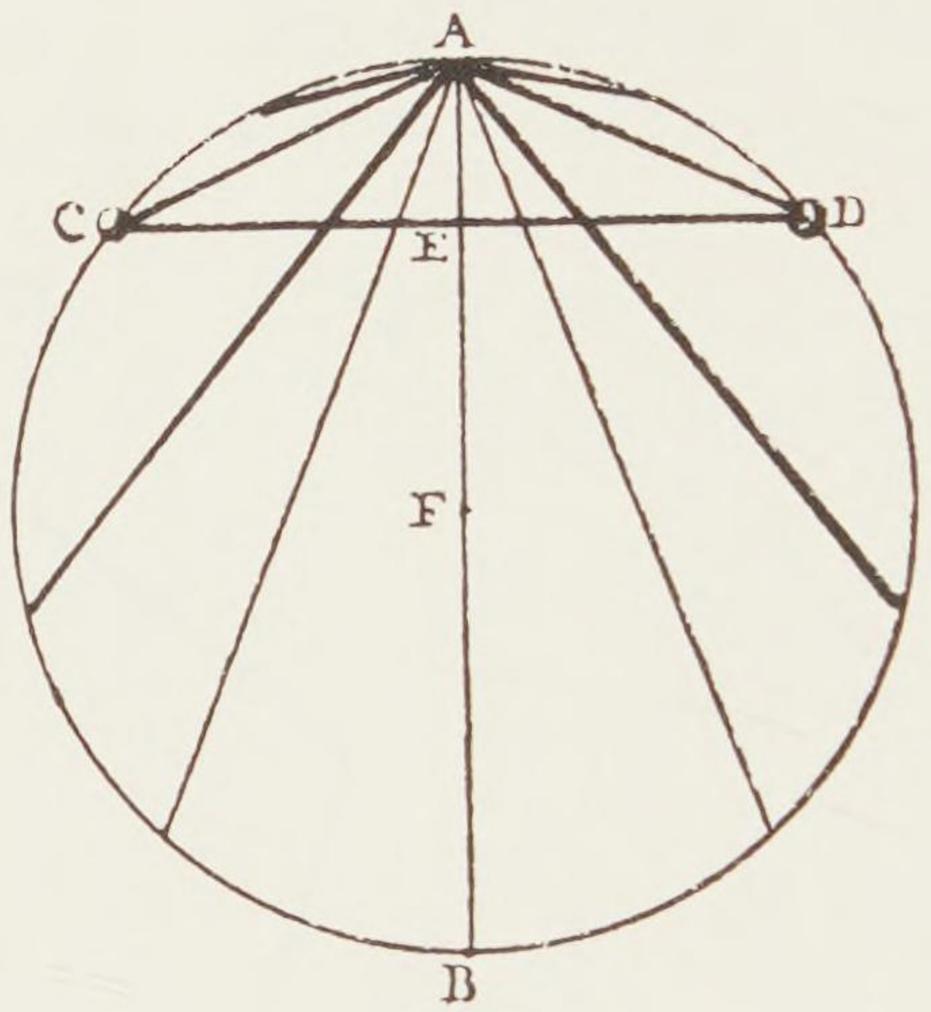
The Center of Oscillation of Regular Polygons

In a similar way one can find a pendulum which is isochronous with any regular polygon, for example, ABC [Fig. 87]. For the space to be divided is equal to half of the square of a perpendicular from the center to the side of the polygon, plus one-twenty-fourth of the square of the side. And if one wishes to find the pendulum which is isochronous with the perimeter of a polygon, the space to be divided is equal to the square of the perpendicular from the center to the side, plus one-twelfth of the square of the side.

The Use of Plane and Solid Loci in This Theory

Furthermore a consideration of loci will not be unwelcomed at this point. For example, given a point of suspension A and the length AB [Fig. 88], we are asked to find the locus of two weights, equally distant from A and from the perpendicular AB, which, when oscillated around the axis at A perpendicular to the plane through ACD, are isochronous with a simple pendulum of length AB.

[Fig. 88]



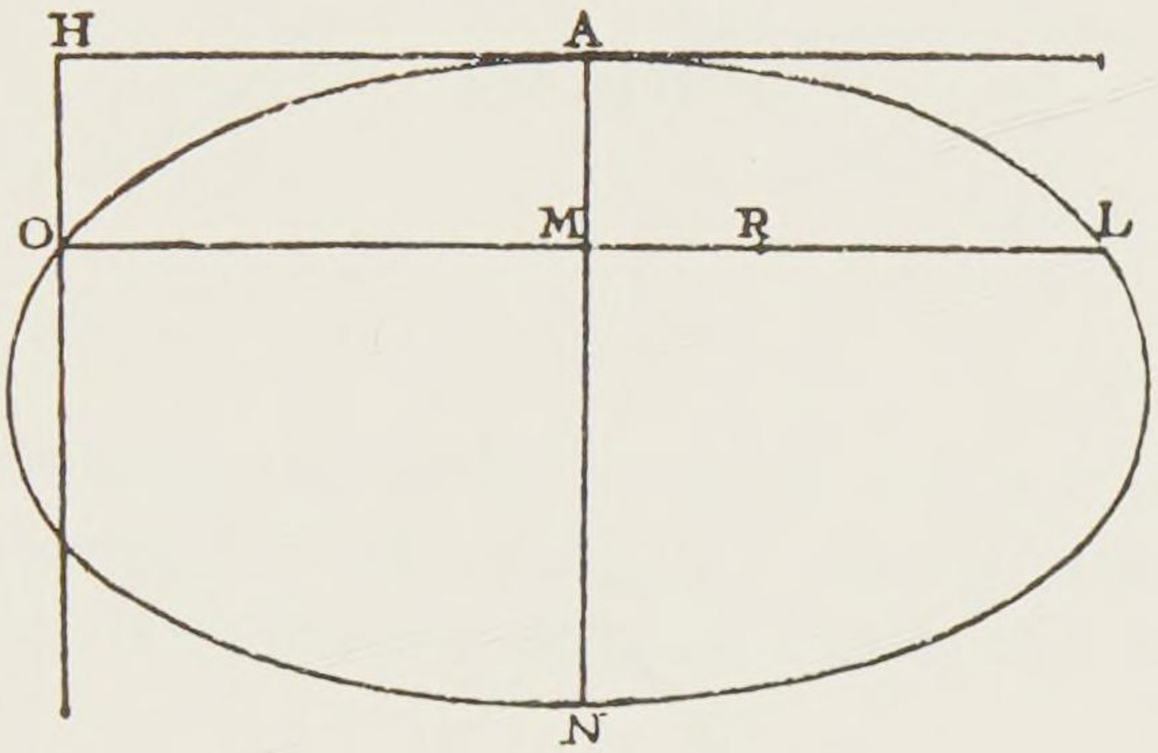
Let a = AB. Draw CD which cuts AB at right angles at E. Let x be an indeterminate length AE. And let y = EC or ED. Consequently the square of AC = $xx + yy$. But this, multiplied by the number of small portions of the weights C and D, which are understood here to be very small, equals the squares of the distances of these same small portions from the axis of suspension A. Hence the square of AC, or $xx + yy$, divided by the distance AE between the axis of suspension and the center of gravity of the weights C and D, gives $\frac{xx + yy}{x}$ as the length of the isochronous pendulum [Proposition 17]. And this is equal to AB or a. Thus $a = \frac{xx + yy}{x}$. And $yy = ax - xx$. Hence it is clear that the locus of the points C and D is on the circumference of a circle, whose center F is located at the middle of AB, and whose radius equals $\frac{1}{2}a$, or FA. Therefore, wherever two equal weights, equally distant from A, are placed on the circumference ACBD, when they are oscillated they are isochronous with a pendulum having a length equal to the diameter AB.

From this it is also obvious that, if weight be attributed to the circumference ACBD, and if any portion of it is equally divided at A or at B, and

if it is suspended from an axis through A, then it will be isochronous with the same pendulum AB.

An example of a solid locus is as follows. Let AN [Fig. 89] be an inflexible line without weight. We are asked to attach at some point on AN, for example, M, a perpendicular line or rod OML, which is endowed with weight and divided in half at M, such that, when it is oscillated laterally around A as the point of suspension, it will be isochronous with a simple pendulum of length AN.

[Fig. 89]



Draw OH parallel to AN; AH parallel to OM; and let OR equal ⅔OL. Thus OR will be the subcentrix of a wedge cut off above the line OL by a plane drawn through OH. And AM will be the subcentrix of another wedge cut off above the same line OL by a plane through the line AH (in this case the wedge is a rectangle). Hence that rectangle, which above we called the rectangle of oscillation, is simply the rectangle OMR, which, when divided by the length AM, gives the distance of the center of oscillation of the line OL, which is suspended from A, below the point M.

Next let a = AN; x = AM; and y = MO or ML. Thus the rectangle OMR = $\frac{1}{3}yy$, which, when divided by AM, gives $\frac{yy}{3x}$. This length is equal to MN, since we wished to have the center of oscillation of the rod OL located at N. Thus the equation becomes $\frac{yy}{3x} + x = a$. Hence $y = \sqrt{3ax - 3xx}$. This signifies that the points O and L are on an ellipse whose minor axis is AN and whose *latus rectum,* which determines the perpendiculars to this axis, is triple AN.

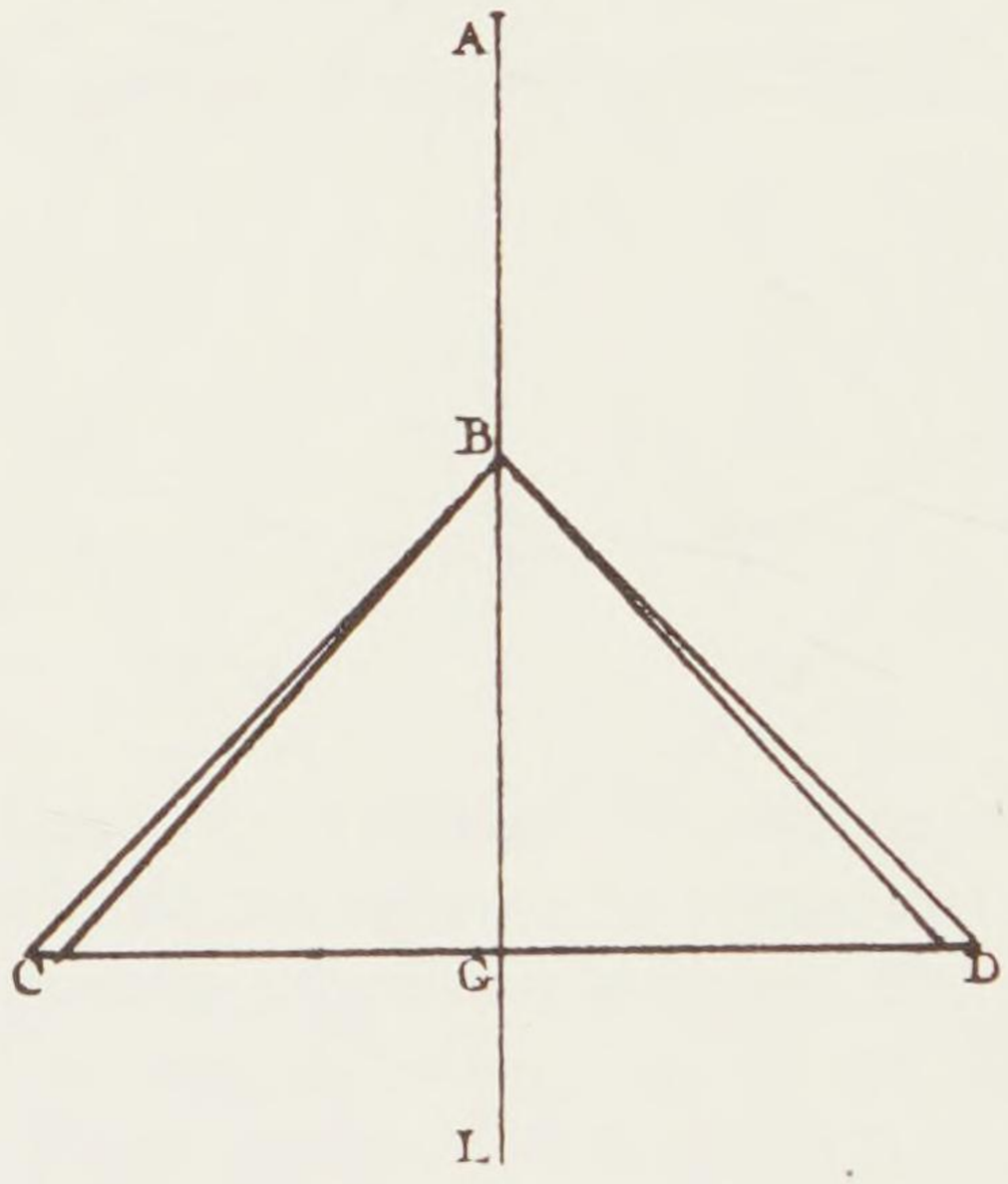
From this it is clear that since the whole rod is parallel to OL and ends at the ellipse, and since its oscillations are isochronous with the simple pendulum AN, then when the whole plane of the ellipse is suspended from A and oscillated laterally, it also is isochronous with the pendulum AN. The

same is true of any part of the ellipse which is cut off by one or two lines perpendicular to AN.

We will add another example of a plane locus which contains something worth noting.

Let the weightless rod AB be suspended from A [Fig. 90]. We are asked to attach at its point B two equal triangles (which make an equal angle with the axis AB and which are separated at B by an angle which is considered to be very small or infinitely small) such that, when suspended from A, they make oscillations isochronous with a simple pendulum of the given length AL.

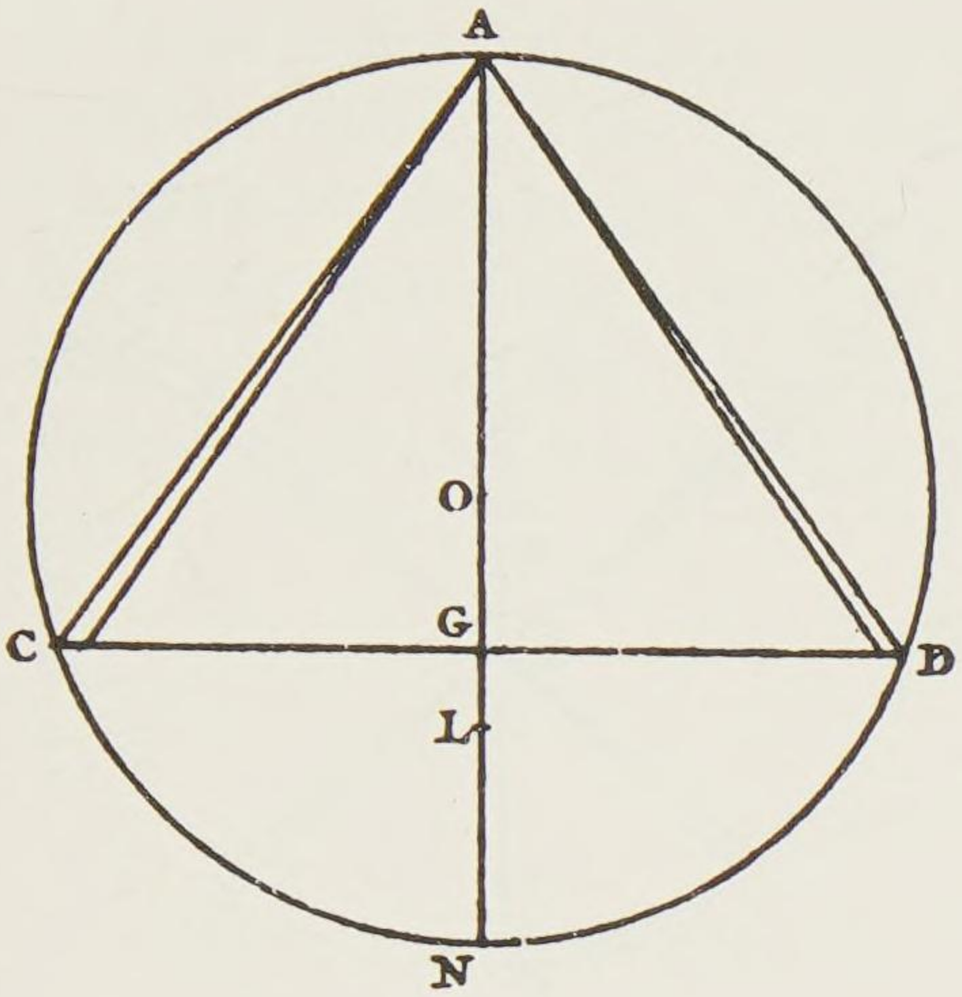
[Fig. 90]



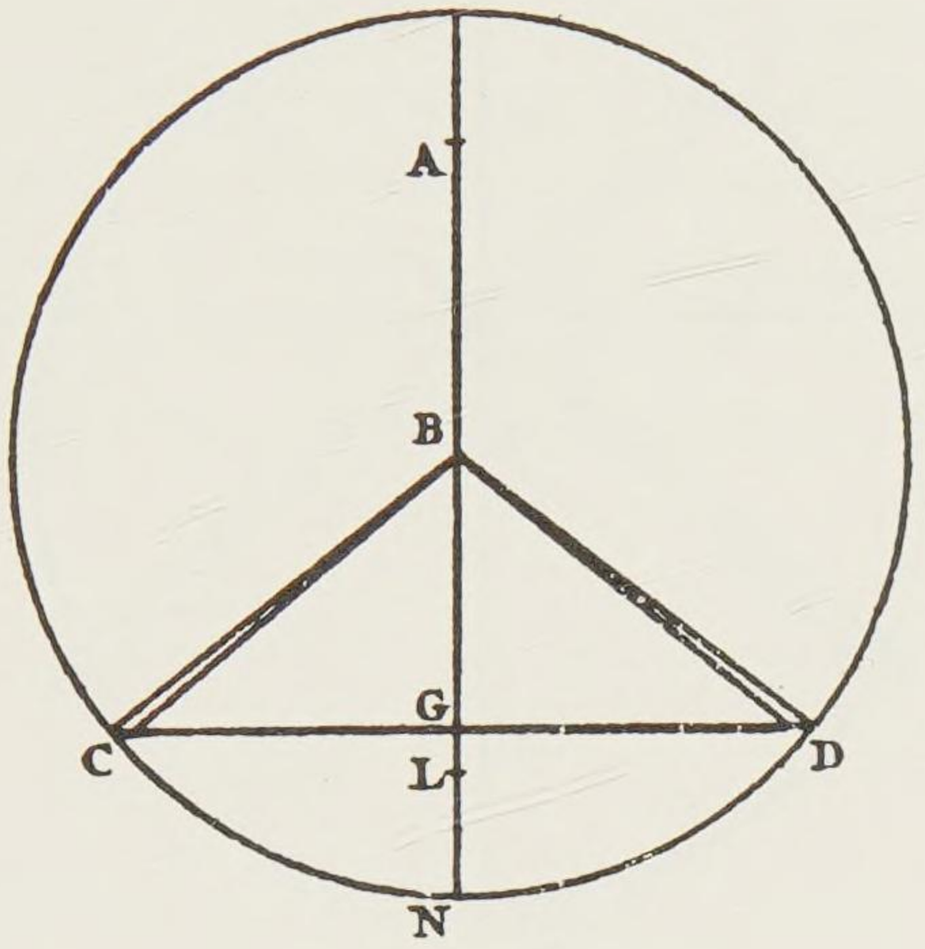
Draw CG perpendicular to BG; let a = AB; b = AL; x = BG; and y = CG. This gives us the equation $y = \sqrt{2ab - 2aa - \frac{8}{3}ax + \frac{4}{3}bx - xx}$. From this it is clear that the bases C and D of the triangles, which bases are considered here as points, are located on the circumference of a circle [Fig. 91]. For the equation contains the simple term $- xx$.

But it should be noted here that, if a is equal to zero, that is, if the point where the triangles BC and BD are attached is identical with the point A, then the equation will be $y = \sqrt{\frac{4}{3}\,bx - xx}$. Hence in this case, if we let AO = $\frac{2}{3}b$, that is AO = $\frac{2}{3}$AL, and if a circle ADN is drawn through A around the center O, then the bases of the triangles AC and AD are on that circumference. Therefore, since any two very acute triangles, which are contained between A and the circumference ACND and which correspond in size and location, have the point L as the center of oscillation when AL = $\frac{3}{4}$ of the diameter AN, and since the whole circle is composed of such

[Fig. 91]



[Fig. 92]



pairs of triangles, and since the same is true of any portion of the circle ACND which has sides equal to AC and AD, it clearly follows that L is the center of oscillation of both the whole circle and of any such portion as indicated.

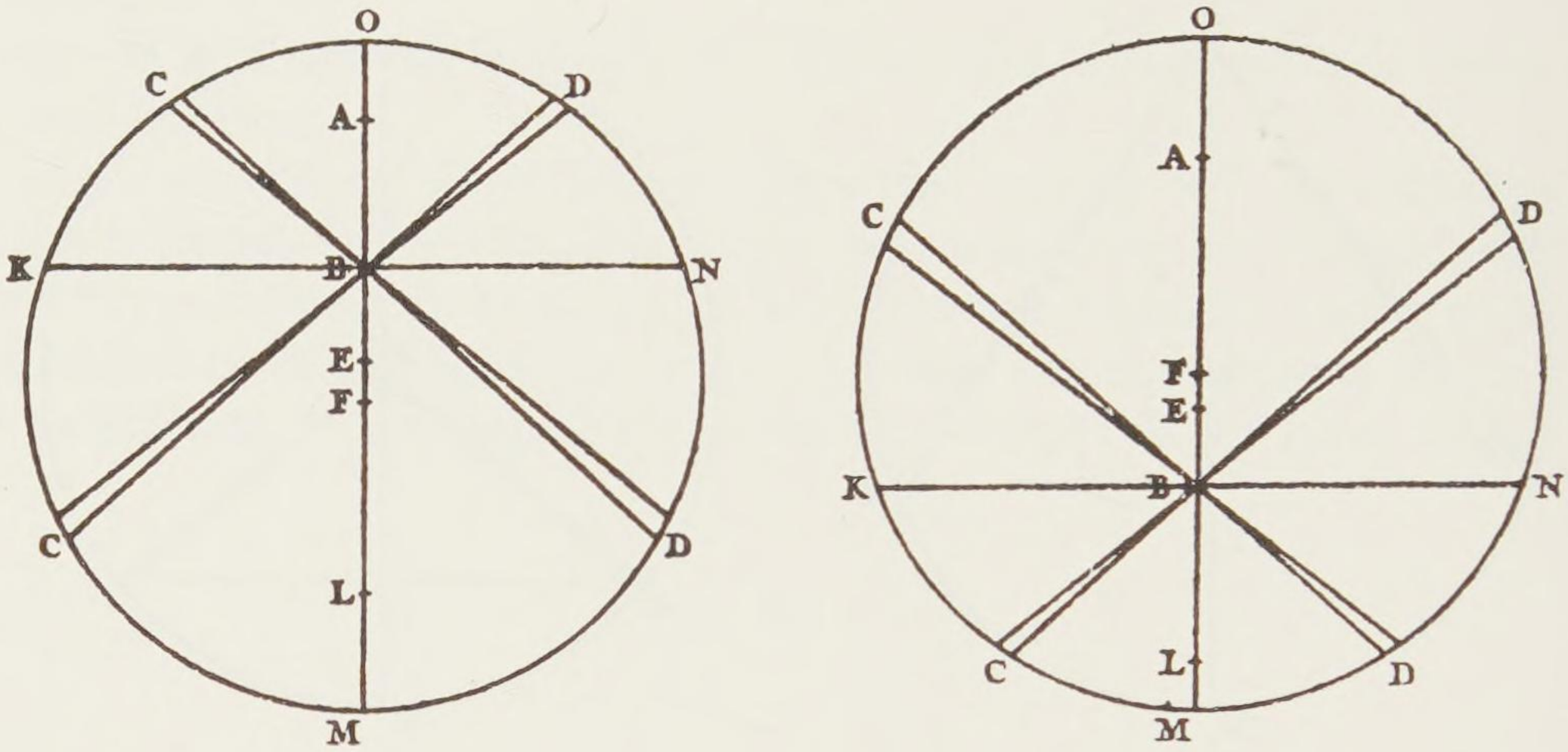
Further, if in the established equation we let $\frac{8}{3}a = \frac{4}{3}b$, or $2a = b$, that is, if the triangles are understood to be attached at B, which cuts the length AL in half [Fig. 92], then $y = \sqrt{2aa - xx}$. This equation shows that, if a circumference is drawn with B as its center and with a radius whose square is twice the square of BA, this circumference will be the locus of the very acute triangles BC and BD, of which the point L will be the center of oscillation when the suspension is from A. And since the whole circle, and any sector of it whose axis is on the line AL, are both composed of such pairs of triangles, it is clear that the point L is the center of their oscillations when the suspension is from A.

Consequently any sector of a circle, suspended from a point which is removed from the center of the circle by half of the side of the square inscribed in the circle, will have an isochronous pendulum equal to the whole of the side. And thus, in this one case, the pendulum isochronous with the sector is found without determining the dimension of the arc.

Next, in order to give a universal construction of the first equation, $y = \sqrt{2ab - 2aa - \frac{8}{3}ax + \frac{4}{3}bx - xx}$, let AL be divided in half at E [Fig. 93], and add to BE one-third of itself, EF. F will be the center of a circle which is described by taking as the radius FO, whose square is equal to twice the difference of the squares AE and EF.

Thus, if two equal, very acute triangles, such as BC and BD, are constructed between the point B and the described circumference, then L will be the center of oscillation when the suspension is from A. Assuming

[Fig. 93]



the same suspension, it follows that the same point L is also the center of oscillation for any portion of the described circle, when its vertex is at B and its axis is on the line AL, as are the pairs CBD. The same is true of the segments of the circle KON and KMN, which are determined by the line KBN which is perpendicular to AB.

These observations about the lateral motion of planes and lines are sufficient. We add only the following point. If one has found the centers of oscillation of regular figures, that is, of figures which are symmetrical relative to the axis, for example, isosceles triangles or regular sections of a parabola, then one also has determined the centers of oscillation of figures which are produced, as it were, by their dislocation, for example, scalene triangles and oblique parabolas. For example, imagine that the isosceles triangle BAC [Fig. 94], whose axis is AD, is suspended from the point E, and then imagine another scalene triangle FAG, which has the same axis AD and whose base FG is equal to the base BC. I say that this latter triangle, when suspended from E, is isochronous with the former triangle BAC.

[Fig. 94]

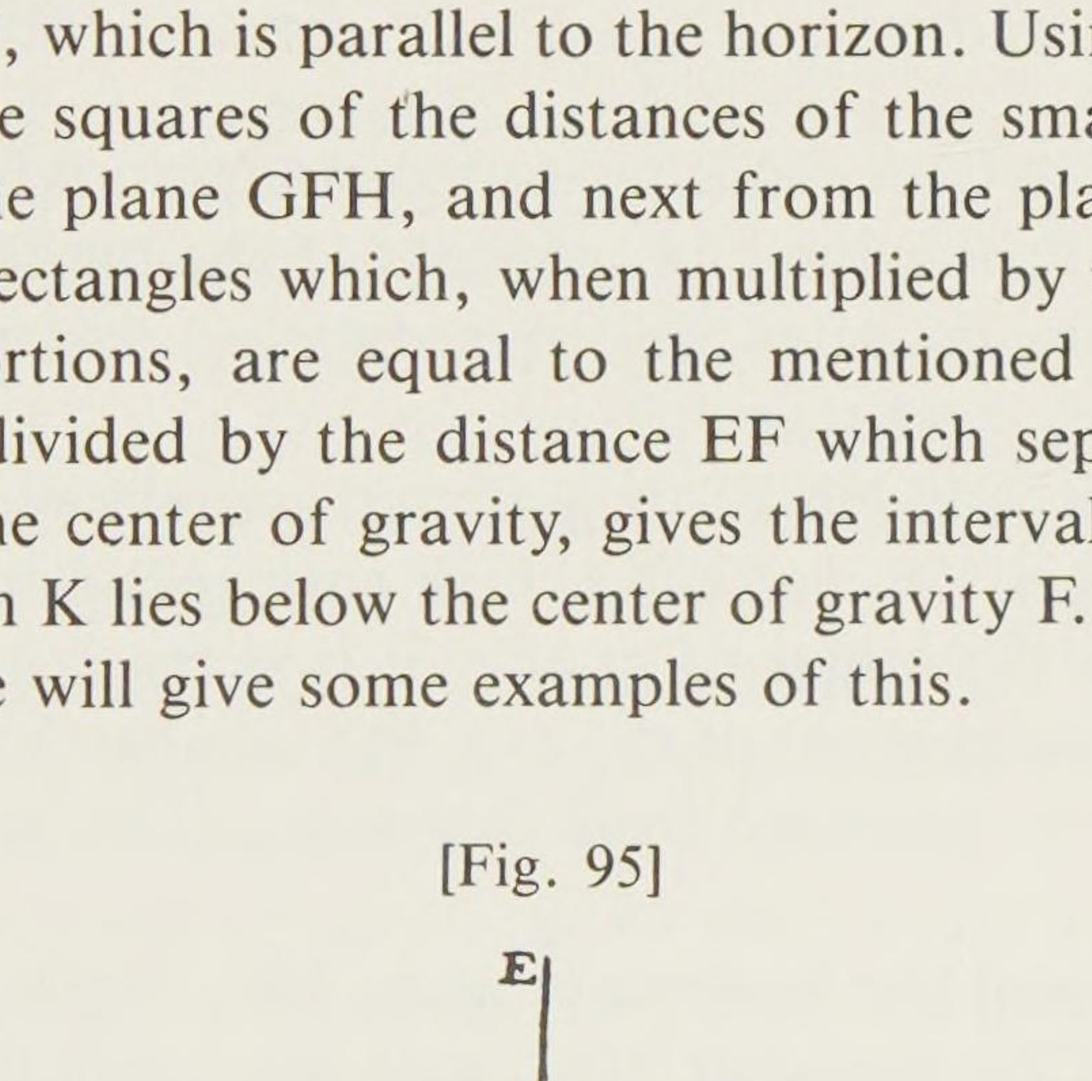
For if the rod or heavy line FG is attached to the weightless rod ED at D in an oblique position, and is then suspended from E, it is isochronous with the rod BC when it is attached at D in a similar way [Proposition 16]. The same is true of the other rods of both triangles which cut the axis AD at the same points and which are equal to each other. For it is necessary that both whole triangles, which can be understood to be composed of the same lines or rods, are isochronous. For other figures the demonstration is similar.

PROPOSITION XXII

How to find the centers of oscillation in solid figures.

Using what was demonstrated above, it is also possible to find easily the center of oscillation for solid figures. For if ABC [Fig. 95] is a solid figure, suspended from an axis which is understood to pass through the point E and to be perpendicular to the plane of this page, F is its center of gravity. Draw through F the plane EFD, which passes through the axis E, and the plane GFH, which is parallel to the horizon. Using Proposition 14, find the sum of the squares of the distances of the small portions of the solid ABC from the plane GFH, and next from the plane EFD. In other words, find both rectangles which, when multiplied by the number of the indicated small portions, are equal to the mentioned sums of squares. These rectangles, divided by the distance EF which separates the axis of suspension from the center of gravity, gives the interval FK by which the center of oscillation K lies below the center of gravity F. This is clear from Proposition 18. We will give some examples of this.

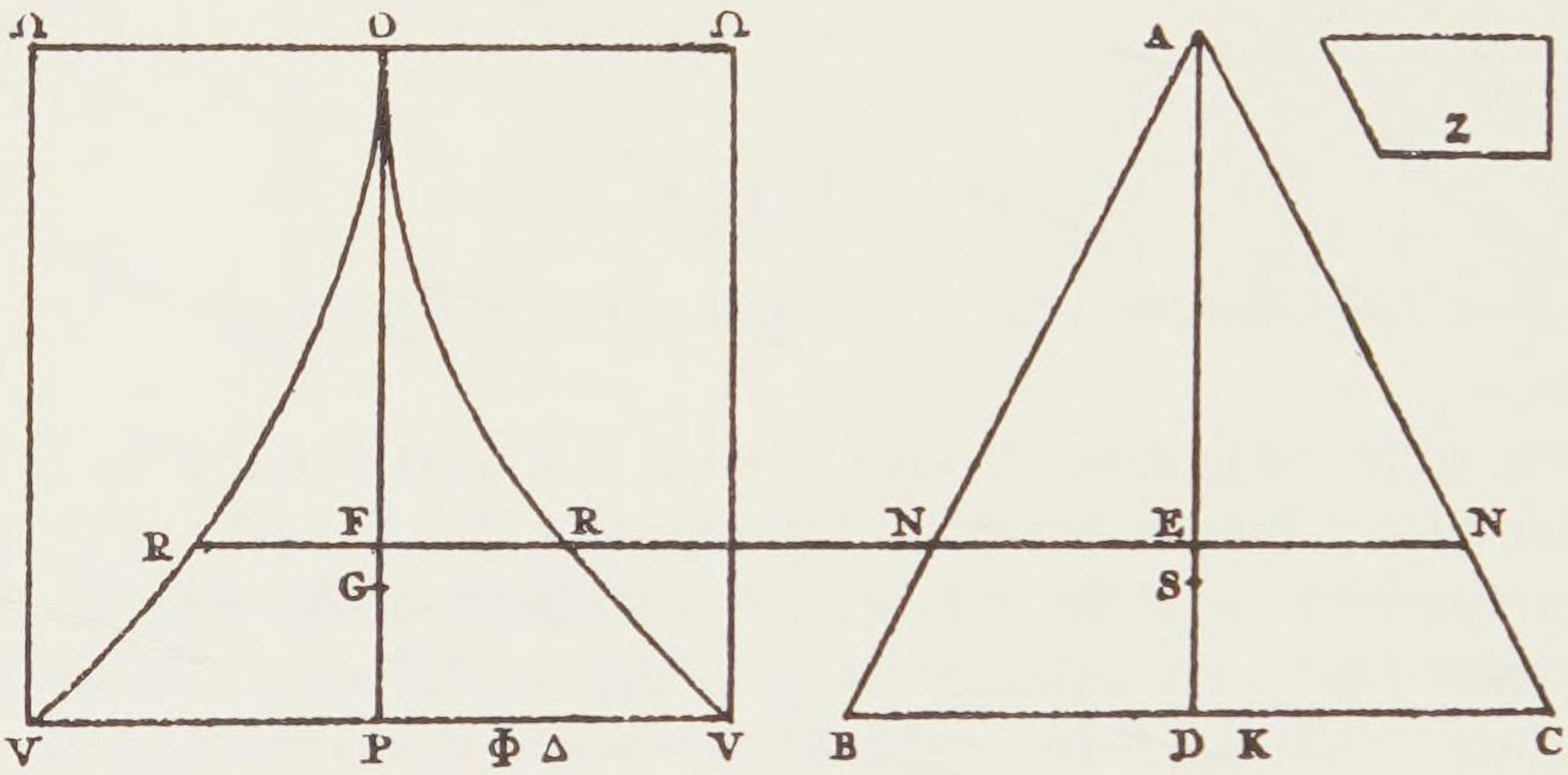
[Fig. 95]



The Center of Oscillation in a Pyramid

First let ABC [Fig. 96] be a pyramid, having the vertex A, the axis AD, and a square base whose side is BC. Assume that it is oscillated around an axis which passes through A and which is perpendicular to the plane of this page.

[Fig. 96]



The plane figure OVV, which is proportional to ABC and placed at its side, according to Proposition 14, consists of the parabolic residues OPV; namely, of what remains after the semiparabolas, having the vertex O, are subtracted from the rectangles ΩP.

The sections BC and NN of the pyramid are related to each other in the same way as the lines VV and RR, which correspond to them in the plane figure. And just as the center of gravity E is separated from the vertex of the pyramid by three-quarters of the axis AD, so also the center of gravity F of the figure OVV is separated from the vertex O by three-quarters of the diameter OP.

Next imagine that the horizontal plane NE, which passes through the center of gravity of the pyramid ABC, cuts the figure OVV along RF. And find the subcentrix of the wedge above the figure OVV which is cut off by a plane through OΩ. This subcentrix is OG (which is ⅘ of the diameter OP). The rectangle OFG, multiplied by the number of the small portions of the figure OVV, will then be equal to the squares of the distances from the line RF [Proposition 10], and thus is also equal to the squares of the distances of the small portions of the solid ABC from the plane NE. From this the rectangle OFG is equal to ³⁄₈₀ of the square of OP or of AD.

Next, in order to find the sum of the squares of the distances from the plane AD, one must know first the subcentrix of the wedge cut off above the square base BC of the pyramid by a plane passing through the line at B

which is understood to be parallel to the axis A. This subcentrix is BK, and is ⅔BC. One must also know the distance of the center of gravity of the half-figure OPV from OP. This is ΦP, and is 3/10PV. Hence, dividing PV in half at Δ, then ΔP is to PΦ, that is, 5 is to 3, as the rectangle BDK, which is 1/12 of the square of BC, is to another space Z. The latter, multiplied by the number of the small portions of the solid ABC, equals the squares of the distances from the plane AD [Proposition 15]. It is clear that the space Z is equal to 1/20 of the square of BC.

And thus the whole space to be divided equals in this case 3/80 of the square of AD, plus 1/20 of the square of BC. Hence, if the suspension is made from the vertex of the pyramid, as was assumed, then the distance AE, which serves as the divisor, equals ¼ AD. Therefore ES, which is the interval by which the center of oscillation lies below the center of gravity, is equal to 1/20AD, plus 1/15 of the third proportional between AD and BC, and the whole distance AS is equal to ⅘AD, plus 1/15 of the indicated third proportional.

The Center of Oscillation of a Cone

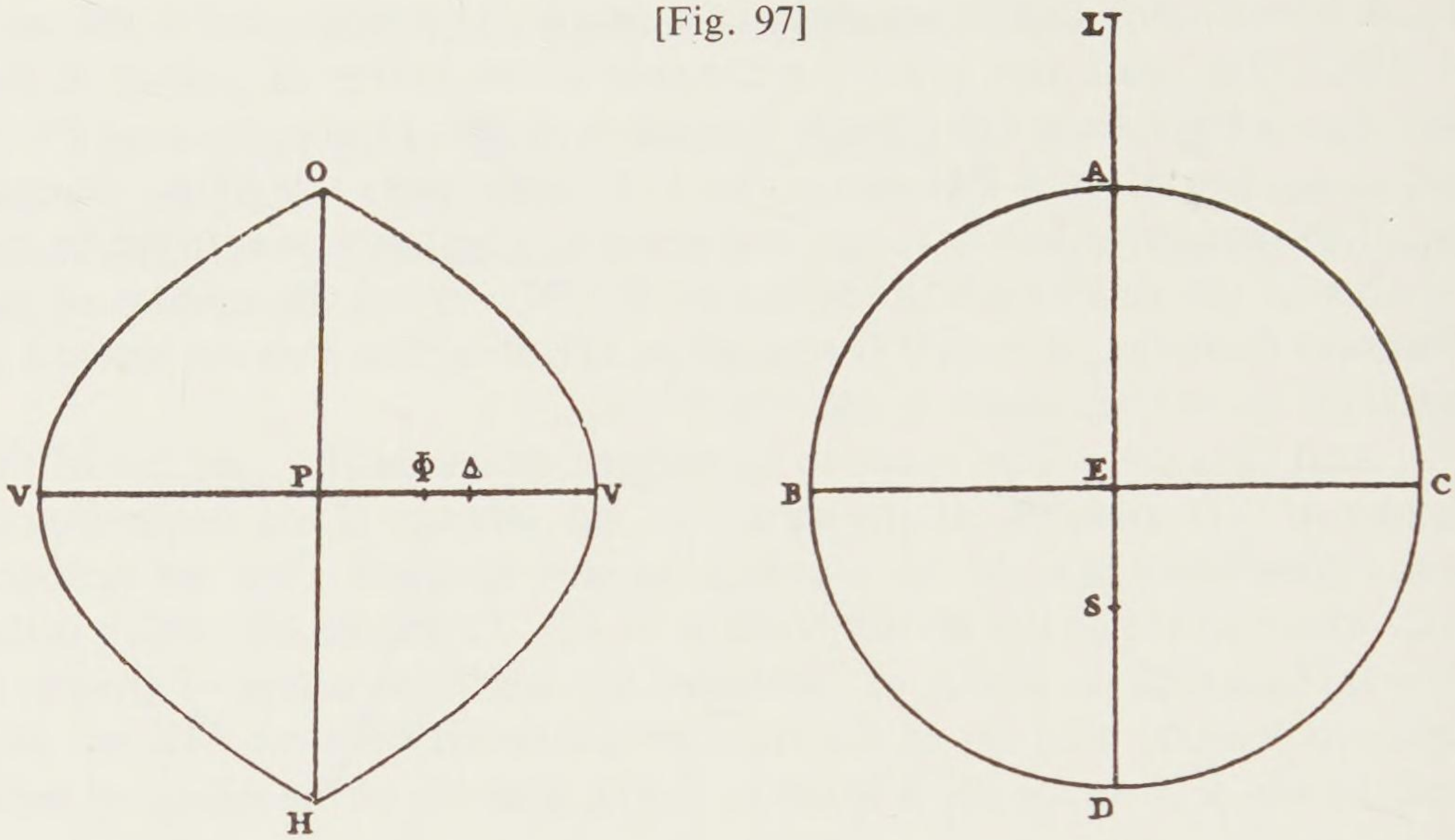
If ABC [Fig. 96] is a cone, then everything is the same, except that the space Z is now equal to the rectangle ΔPΦ [Proposition 15], that is, is equal to 3/20 of the square of PV or of BD, or 3/80 of the square of BC. This is because for the cone the whole space to be divided is 3/80 of the square of AD, plus 3/80 of the square of BC. Therefore, assuming that the suspension is from the vertex A, then ES, the distance by which the center of oscillation lies below the center of gravity, is equal to 1/20AD, plus 1/20 of the third proportional between AD and BC. And the whole distance AS equals ⅘AD, plus ⅕ of the third proportional between AD and DB. And from this it is clear that, if AD and DB are equal, that is, if ABC is a right-angled cone, then AS equals the axis AD.

It also follows from Proposition 20 that if this right-angled cone is suspended from the center D of its base, then it will be isochronous with itself when suspended from the vertex A, as is also true of a right triangle, as was shown above.

The Center of Oscillation of a Sphere

Let ABC [Fig. 97] be a sphere. OVH will be the plane figure which is proportioned to the sphere and placed to its side. OVH is composed of parabolas whose common base OH is equal to the diameter AD of the sphere. Let the sphere be cut by planes through its center E, of which BC is parallel to the horizon and AD is vertical. To find the sum of the squares of the distances from the plane AD, one must know the distance of the center

[Fig. 97]



of gravity of the parabola OVH from OH. This is ΦP, which is ⅖VP. Finally, dividing in half at Δ, it is clear that the rectangle ΔPΦ, multiplied by the number of the small portions of the sphere ABC, is equal to the squares of the distances from the plane AD [Proposition 15]. This rectangle ΔPΦ is equal to ⅕ of the square of PV or of BE.

But the squares of the distances from the plane BC are clearly equal to the squares of the distances from the plane AD, and hence are also equal to the rectangle ΔPΦ, multiplied by the indicated number of small portions. Therefore, for the sphere ABC the space to be divided will be twice the rectangle ΔPΦ, and hence is equal to ⅖ of the square of the radius EB.

Consequently, if the sphere is suspended from the point A on its surface, then ES, the distance from the center E of the sphere to the center of oscillation S, equals ⅖ of the radius AE. And the whole distance AS equals ⅞ of the diameter AD. But if the sphere is suspended from some other point L, ES will equal ⅖ of the third proportional between LE and EB.

The Center of Oscillation of a Cylinder

For a cylinder we have found that the space to be divided equals 1/12 of the square of its height, plus ¼ of the square of the radius of its base. Hence, if a cylinder is suspended from the center of its base which points upwards, the length of the isochronous pendulum equals ⅔ of its height, plus half of the distance which is related to the radius of its base as the latter is related to its height.

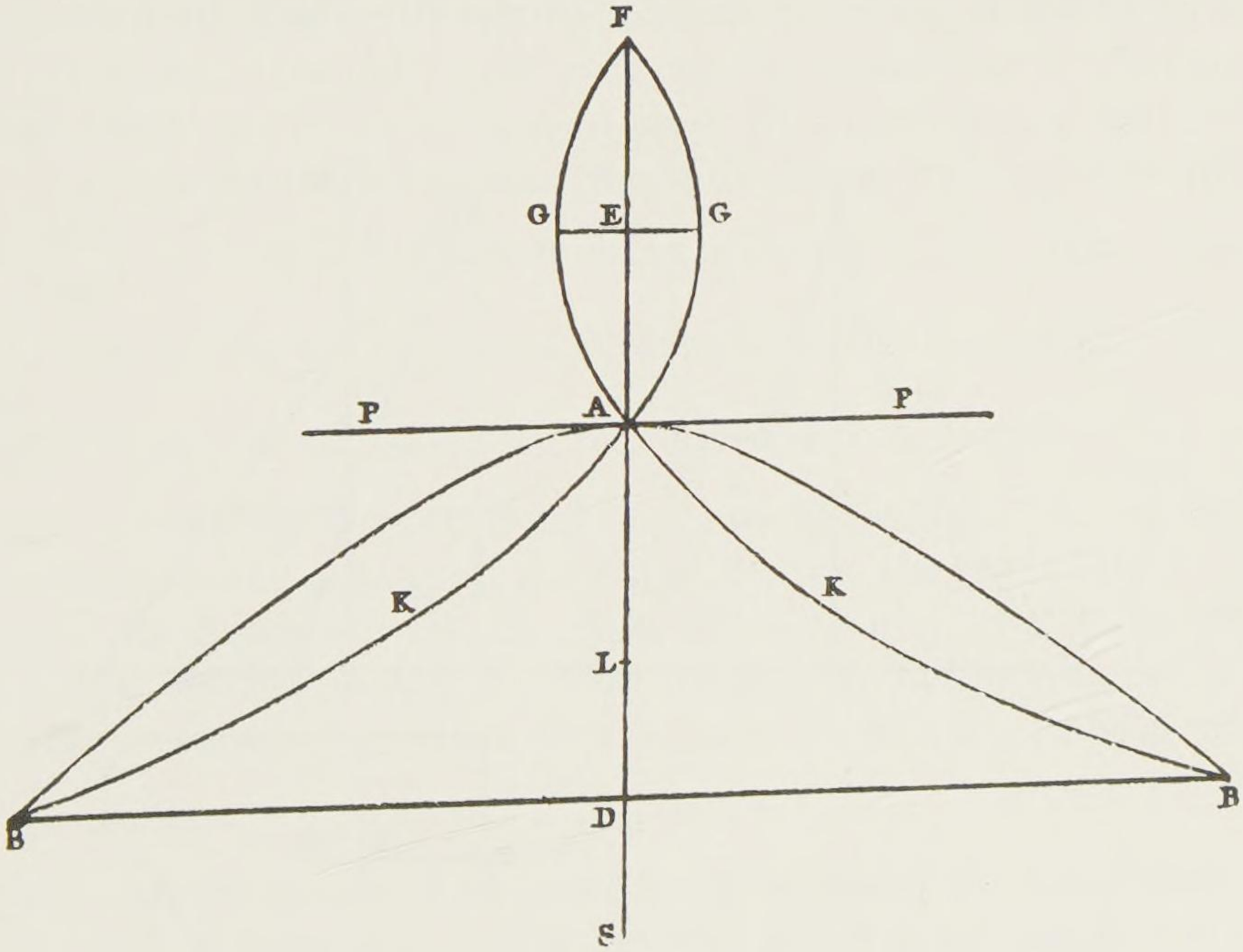
The Center of Oscillation of a Parabolic Conoid

For a parabolic conoid the rectangle of oscillation is 1/18 of the square of its height, plus 1/4 of the square of its base. Hence, if it is suspended from the point of its vertex, the length of the isochronous pendulum is 3/4 of the axis, plus 1/4 of the distance which is related to the radius of the base as this latter is related to the axis, that is, plus 1/4 of the *latus rectum* of the generating parabola.

The Center of Oscillation of a Hyperbolic Conoid

It is also possible to find the center of oscillation of a hyperbolic conoid. For example, let there be a conoid [Fig. 98] whose section through its axis is the hyperbola BAB, whose axis is AD, and whose *latus transversum* is AF. Let BKAKB be a plane figure which is proportional to the conoid. BKAKB is contained between the base BB and the similar portions AKB of the parabolic lines. These portions AKB pass through the vertex A of the parabola and have GE as their axis, which divides the *latus transversum* in half and is parallel to the base BB. L, which is the center of gravity of this figure BKAKB, is separated from the vertex A by the same distance as the center of gravity of the conoid ABB is separated from A. And the

[Fig. 98]



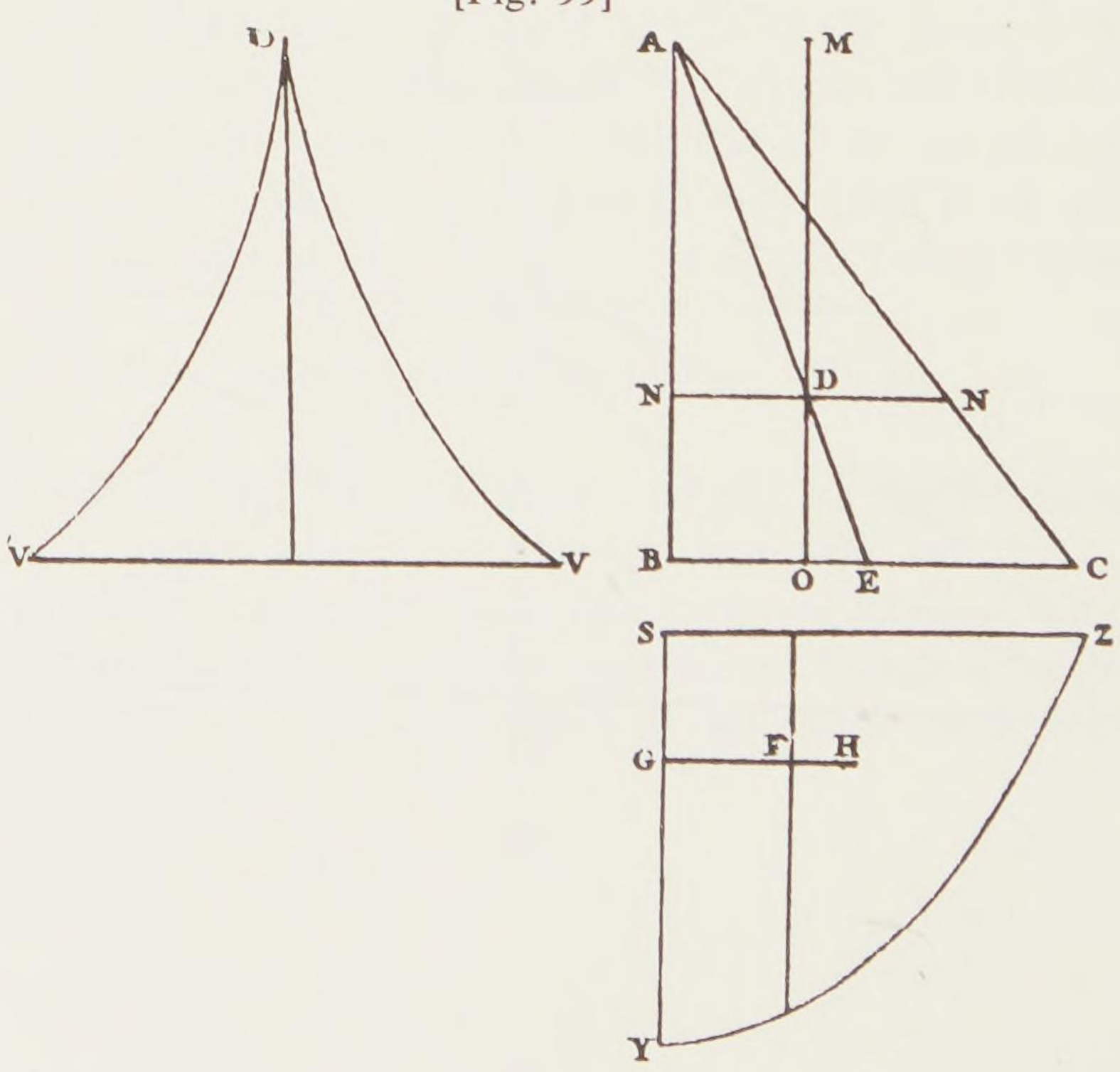
axis AD is related to AL as three times FA, plus twice AD, is related to twice FA, plus one and a half times AD. Finally it is possible to find the distance of the center of gravity of the half-figure ADBK from AD; and it also is possible to find the subcentrix of the wedge above the figure BKAKB, cut off by a plane through AP which is parallel to BB, in other words, the subcentrix of the wedge in relation to AP. From all of this the center of oscillation of the conoid can be determined for any suspension, provided that the axis around which it is moved is parallel to the base of the conoid. I have found that, if the axis AD is taken to be equal to the *latus transversum* AF, then the space to be divided is equal to 1/20 of the square of AB, plus 31/200 of the square of DB. And AL is then 7/10AD.

Hence, if such a conoid is suspended from the vertex A, the length AS of the isochronous pendulum is equal to 27/35AD, plus 31/140 of the third proportional between AD and DB.

The Center of Oscillation of a Half-Cone

Finally it is possible to find the center of oscillation in certain half-solids which are cut by a section through the axis. For example, let ABC [Fig. 99] be a half-cone; A is its vertex, and BC is the radius of its base. Its center of gravity D is known, because AD is 3/4 of the line AE which divides BC at E in such a way that a quarter of the circumference of a circle is related to its radius as 2/3CB is related to BE. E is then the center of gravity of the semicircular base, and on AE are located the centers of gravity of all the segments of the half-cone ABD which are parallel to the base.

[Fig. 99]



Next let OVV be a proportional figure, placed to the side. It is the same figure which was described above for the whole cone. By its means one can find the sum of the squares of the distances of the small portions of the half-cone from the horizontal plane ND, which is drawn through its center of gravity. But in order to determine the squares of the distances from the vertical plane MDO, one must use another proportional figure SYZ, as above in Proposition 14. Its vertical sections represent the proportional lines in the corresponding sections of the half-cone ABC. Regarding this figure it is necessary to know the distance of its center of gravity from SY, which is clearly equal to the distance DN of the center of gravity of the half-cone from the plane of the triangle AB. Granting that HG is the subcentrix of a wedge cut off above the figure SZY by a plane drawn through SY, it is necessary to know the rectangle GFH which, when multiplied by the number of small portions of the half-cone ABC, will equal the squares of the distances of the half-cone from the plane MDO. But it is possible to determine that rectangle GFH as follows without knowing the length of the subcentrix HG.

We said above, when dealing with the cone, that the squares of the distances from the plane through its axis, are equal to 3/80 of the square of the diameter of the base, or 3/20 of the square of the radius, multiplied by the number of small portions of the whole cone. And hence here also for the half-cone ABC, the squares of the distances from the plane AB are equal to 3/20 of the square of BC, multiplied by the number of small portions of that half-cone. But the rectangle HGF, multiplied by the number of small portions of the half-cone ABC, is equal to the squares of the distances from the plane AB, as is clear from Proposition 9. Therefore the rectangle HGF equals 3/20 of the square of BC. Letting a = AB; b = BC; and q = a quarter of the circumference described by the radius BC; then EG = $\frac{2bb}{3q}$. Since ND is equal to three-quarters of this, then ND or GF = $\frac{1bb}{2q}$. Subtracting the square of this from the rectangle HGF, which was 3/20 of the square of BC, gives the rectangle GFH = $3/20bb - \frac{1b^4}{4qq}$. But this rectangle, multiplied by the number of small portions of the half-cone ABC, is equal to the squares of the distances from the plane MDO. And the squares of the distances from the plane ND are equal, as also for the cone, to 3/80 of the square of AB, or $3/80aa$, multiplied by the number of small portions of the half-cone ABC. And thus the whole space to be divided will be equal to $3/80aa + 3/20bb - \frac{1b^4}{4qq}$.

From this the center of oscillation can be found for any suspension of the half-cone, as long as it is from an axis which is parallel to the base of the triangle at the section AB. It should be noted that, although the nature

of the figure SZY was entirely unknown, the subcentrix GH, of the wedge cut off above it by a plane through SY, has been found by this method. For since the rectangle HGF was equal to 3/20*bb*, or 3/20 of the square of BC, and since GF equals $\frac{1bb}{2q}$, then GH equals 3/10*q*.

Furthermore, it is possible to find the centers of oscillation for a half-cylinder and a parabolic half-conoid, in addition to other half-solids. We leave this investigation to others.

What we have said about the centers of oscillation of oblique plane figures, which are produced as it were by the dislocation of regular figures, applies in the same way to oblique solid figures; namely, their centers of oscillation do not differ from the centers of oscillation of regular figures. For example, let ABC and AFG [Fig. 94] be two cones, one right-angled, the other scalene. Their diameters and bases are equal. If they are suspended from the vertex or from any axes which are equally distant from their center of gravity, they will be isochronous, provided that the axis from which the scalene cone is suspended is perpendicular to the plane of the triangle through the diameter which in turn is at right angles to the plane of the base.

PROPOSITION XXIII

How to regulate the motion of clocks by adding a small secondary weight which can be moved upwards and downwards on the rod of the pendulum, which is divided in a certain way.

In order to do this, we must first find the center of oscillation of a pendulum composed of a rod endowed with weight and of a weight attached at its lowest part. [Fig.

Let AC be a rod with an attached weight, and let *a* = its length [Fig. 100]. Imagine that both the rod and its attached weight C are divided into equal, very small portions, of which *b* = the number in the rod, and *c* = the number in the weight C. Thus the ratio of *b* to *c* is the ratio of the weight of the rod to the weight of the attached body. Now the length of a simple pendulum isochronous with the given pendulum will be obtained if the sum of the squares of the distances of all the small portions from the point of suspension A is divided by the sum of these distances [Proposition 6]. Let AC be cut in half at M, and also cut at T such that AT is twice TC. Now M is the center of gravity of the line AC. And AT is the subcentrix of a wedge cut off above AC by a plane through AD, which is perpendicular to AC. This wedge is actually a triangle. As a result the sum of the squares of the distances of the small portions of the rod from the point A is equal to

the rectangle AMT, plus the square of AM, that is, the rectangle TAM, multiplied by the number of small portions *b,* that is, $\frac{1}{3}aab$. For MA $= \frac{1}{2}a$; TA $= \frac{2}{3}a$; and the rectangle TAM $= \frac{1}{3}aa$. And the sum of the squares of the distances of the small portions of the weight C from the same point A is equal to the square of AC, multiplied by the number of small portions of the weight C, that is, *aac*. Therefore the sum of all the squares of the distances of the small portions of both the rod and the weight C will be $\frac{1}{3}aab + aac$.

Further, the sum of the distances of the small portions of the rod AC from the point A is equal to $\frac{1}{2}ba$, that is, to the length of the rod, which is *a,* multiplied by half of the number of the small portions which it contains. And the sum of the distances of the small portions of the weight C from the same point A is *ac*. Thus the sum of both sets of distances is $\frac{1}{2}ab + ac$. Dividing this into the sum of squares previously found; namely, $\frac{1}{3}aab + aac$, gives for the length of the isochronous pendulum $\frac{\frac{1}{3}aab + aac}{\frac{1}{2}ab + ac}$ or $\frac{\frac{1}{3}ab + ac}{\frac{1}{2}b + c}$. This is to be interpreted as follows: half of the weight of the rod, plus the weight of the attached body, is to a third of the weight of the rod, plus the weight of the same attached body, as the length AC is to another length. It is necessary to measure the length AC from the point of suspension A to the center of gravity of the weight C. For the size of the attached weight is not taken into account, it being considered to be extremely small.

[Fig. A

Next imagine that in addition to the weight C another weight D is attached to the rod [Fig. 101]. Let d = its weight, or the number of its small portions. And let f = the distance AD. In order to find the simple pendulum which is isochronous with a pendulum constructed in this way, one must add to the above sum of squares the squares of the distances of the small portions of the weight D from the point A. These squares are *dff*. Thus the sum of all the squares now is $\frac{1}{3}aab + aac + ffd$. Also to the sum of the distances one must add the distances of the small portions of the weight D. This is *df*. Thus the sum of all the distances is now $\frac{1}{2}ba + ca + df$. When this is divided into the sum of the squares, the length of the isochronous pendulum is $\frac{\frac{1}{3}aab + aac + ffd}{\frac{1}{2}ab + ac + fd}$.

Now if we let p = this length of the pendulum which is isochronous with the given pendulum; and if all else remains the same except the distance AD or f which determines the position of the weight D; and if we are asked to find this distance f; then this is done as follows.

Granted that $p = \frac{\frac{1}{3}aab + aac + ffd}{\frac{1}{2}ab + ac + fd}$, from this we get

$$ff = pf + \frac{\frac{1}{2}abp + cap - \frac{1}{3}aab - aac}{d},$$

$$\text{and } f = \frac{1}{2}p \pm \sqrt{\frac{1}{4}pp + \frac{\frac{1}{2}abp + cap - \frac{1}{3}aab - aac}{d}}.$$

It should be observed that this has two true roots when $\frac{1}{2}abp + cap$ is less than $\frac{1}{3}aab + aac$; that is, when the length p is less than $\frac{\frac{1}{3}ab + ac}{\frac{1}{2}b + c}$. This latter was found above to be the length of the isochronous pendulum, or the distance of the center of oscillation from the point of suspension, for the pendulum composed of the rod AC and the weight C.

From this it is clear that if we wish to accelerate the motion of the pendulum by adding the weight D, the latter can be placed in two positions, such as D or E, between A and C. Each of these gives the same speed to the pendulum. These positions are equally distant from the point N, which is separated from A by half of the length p, that is, by half of the length of the simple pendulum which was taken to be isochronous with this composite pendulum. And it is clear that, when this length p is given as a very little bit less than AC, the point N is also a very little bit above the midpoint of the rod AC.

Next, from the preceding equation,

$$f = \frac{1}{2}p \pm \sqrt{\frac{1}{4}pp + \frac{\frac{1}{2}abp + acp - \frac{1}{3}aab - aac}{d}}$$

the length p can be determined. For it is clear that $\frac{1}{4}pp + \frac{\frac{1}{2}abp + acp}{d}$ should not be less than $\frac{\frac{1}{3}aab + aac}{d}$. Hence it should not be less than $\frac{a}{d}\sqrt{\frac{4}{3}bd + 4cd + bb + 4bc + 4cc} - \frac{ab + 2ac}{d}$. If p is equal to this quantity, that is, if $\frac{1}{4}pp + \frac{\frac{1}{2}abp + acp}{d} = \frac{\frac{1}{3}aab + aac}{d}$, then in the same preceding equation, $f = \frac{1}{2}p$, that is,

$$f = \frac{a}{2d}\sqrt{\frac{4}{3}bd + 4cd + bb + 4bc + 4cc} - \frac{ab + 2ac}{2d}$$[13]

This determines the distance of the weight D from the point A at which it would accelerate the motion of the pendulum as much as possible.

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13. Here and immediately above, the partial expression $\frac{ab + 2ac}{2d}$ erroneously contains a negative sign both in the first edition of 1673 and in 's Gravesande's edition of 1724.

These results apply to the regulation of clocks as follows. For example, consider a pendulum clock which marks off one second with each oscillation. The weight of the rod is 1/50 of the weight of the body attached at the lowest end of the pendulum. In addition there is another small weight which can be moved along the rod and whose weight is assumed to be the same as the weight of the rod. We are asked to find where to place this weight on the rod so that the motion of the clock is accelerated by one minute in 24 hours. Also, where it should be placed to give an acceleration of two, or three, or four, etc., minutes.

Multiplying twenty-four hours by sixty, gives 1440 as the number of minutes in one day. Subtract one from this when the desired acceleration is one minute, giving 1439. But the ratio of the square of 1440 to the square of 1439 is very nearly the same as the ratio of the square of 1440 to the square of 1438. Therefore, if one imagines that the length of a simple pendulum, which marks off seconds, is divided into 1440 equal parts, and if 1438 of these parts are attributed to another pendulum, the latter will run ahead of the former by one minute in 24 hours. Hence in this case p has 1438 parts.

But since the pendulum clock, which is composed of a metal rod and an attached weight, should be isochronous with a simple pendulum having 1440 parts, we must first find the length of that rod by using the above equation. The expression $\frac{1/3ab + ac}{1/2b + c}$ was found to be equal to the length of a simple pendulum which is isochronous with the pendulum composed of a rod, having the length a and the weight b, and of an attached body whose weight is c. Hence if we let s = the length of the simple isochronous pendulum, then $a = \frac{1/2bs + cs}{1/3b + c}$. And since in this case $c = 50$, $b = 1$, and $s = 1440$, then $a = 1444\frac{4}{5}$ for the length of the rod.

$$\text{Now since } f = 1/2p \pm \sqrt{1/4pp + \frac{1/2abp + acp - 1/3aab - aac}{d}}\text{, then } f$$

$= 1/2p \pm \sqrt{1/4pp + 72962p - 105061210}$. Hence, since p equals 1438 parts, as we said, then we find that $f = 1331\frac{1}{2}$ parts, which are of the same length as the 1440 parts contained in s, i.e., in the simple pendulum whose oscillations mark off seconds. If the length of s is taken to be three hour feet, a name we have used before, then f will be 33 inches, plus 3 twelfths of an inch, which are called lines. And in order to determine the location of the weight D to produce an acceleration of one minute in 24 hours, one must subtract this length f from the total length of three feet, which gives the length of 2 inches, 9 lines, which is to be taken above the center of oscillation of the composite pendulum. Using the same method, we have calculated the other distances at which the rod is to be divided when various other lengths are attributed to p. These are given in the table below. The

divisions of the rod of the pendulum which were used above in our description of the clock [Part I] are also based on these numbers. As was already indicated there, these daily accelerations increase by 15 seconds, or a quarter of a minute. For example, if the movable weight D is attached at part 73,4, and if one finds that the clock runs slow by 15 seconds in 24 hours, then the weight D should be moved upwards to the number 85,6 to correct this.

The center of oscillation is located above the center of gravity C by 1,4 parts.

Acceleration of a Clock in 24 Hours	*The Parts to Be Taken Upwards from the Center of Oscillation*
Minutes, Seconds	*Lines and tenths of lines of an hour foot*
0,15	7,0
0,30	15,2
0,45	23,7
1,0	32,6
1,15	41,9
1,30	51,7
1,45	62,2
2,0	73,4
2,15	85,6
2,30	99,0
2,45	114,1
3,0	131,8
3,15	154,3
3,30	192,6

PROPOSITION XXIV

It is not possible to determine the center of oscillation for pendula suspended between cycloids, and how to overcome the difficulty which this causes.

If one very closely compares what we have demonstrated above concerning pendula suspended between cycloids with our discussions of the center of oscillation, he will see that these oscillations fall short of the perfect equality which we would prefer. First he will have doubts, in determining the generating circle of a cycloid, as to whether the length of the pendulum should be measured from the point of suspension to the center of gravity of the attached lead weight or to the center of oscillation. Between these two there is often a sensible interval, and it increases as the size of the lead sphere or lentil increases. What would be the case if the diameter of the

sphere were equal to a quarter or a third of the length of the pendulum? If we say that that length should be measured to the center of oscillation, then it will not be clear how what was proven about the center of oscillation applies to a pendulum which is continually changing its length, as is the case for a pendulum which moves between cycloids. For it would seem that its center of oscillation changes for each different length. But this should not be understood in this way. The matter is clearly most difficult to explain if we demand complete precision. For in our demonstrations concerning the equality of times on a cycloid, we considered the body moving through it as if it were a point devoid of weight. But when we consider what actually occurs, this difficulty should not be taken as very important. For in clocks the size of the weight, which keeps the pendulum constant, need not be so large (although, the larger the weight, the better the clock) that the difference between the center of gravity and the center of oscillation can cause any trouble. However, if we wish to escape these problems completely, we can succeed if we make the sphere or lentil of the pendulum move around its own horizontal axis. This is done by inserting both ends of that axis into the bottom of the rod of the pendulum; the rod having been split in half for this purpose. For in this way the nature of motion is such that the sphere of the pendulum will maintain perpetually the same position in respect to the horizontal plane; and any point in it, as well as its center, will cross the same cycloids. Hence a consideration of the centers of oscillation is no longer relevant. Such a pendulum will maintain an equality of times which is no less perfect than if all of its weight were contained in one point.

PROPOSITION XXV

A method of establishing a universal and perpetual measure.

A certain and permanent measure of magnitudes, which is not subject to chance modifications and which cannot be abolished, corrupted, or damaged by the passage of time, is a most useful thing which many have sought for a long time. If this had been found in ancient times, we would not now be so perplexed by the disputes over the measurement of the old Roman and Hebrew foot. However, this measure is easily established by means of our clock, without which this either could not be done or else could be done only with great difficulty. Indeed some have maintained that they have done this by using the simple oscillation of pendula to number the oscillations contained in one revolution of the whole heavens, or part of a revolution by determining the distances of fixed stars according to right ascension. But this method does not give the same degree of certitude attainable by the use of clocks, and this lengthy work is most difficult and

tedious because of the care that must be taken in counting. However, in addition to the clock, our conception of the centers of oscillation also makes a contribution to a most precise investigation of this measure. Hence we will deal with this subject here at the end after having given our theory of clocks.

The most appropriate clocks for this purpose are ones in which each oscillation marks off one second, or one-half second, and which have a pointer to show this. First let such a clock be regulated by observations of the fixed stars over a period of several days, using the method explained in our description of the clock. Then next to it suspend another simple pendulum, that is, one composed of a lead sphere, or some other material of constant weight, tied to a thin string, and set it in motion with a small push. Then increase or decrease the length of the string until its oscillations, over a quarter of an hour or half an hour, occur in unison with the oscillations of the adjusted clock. I said that the pendulum should be set in motion with a small push because small oscillations, for example, 5 or 6 degrees, are sufficient to give equal times, but not a large number of degrees. Then measure the distance from the point of suspension to the center of the simple pendulum. For the case in which each oscillation marks off one second, divide this distance into three parts. Each of these is the length of an hour foot, a name we have used above.[14] By doing all this, the hour foot can be established not only in all nations, but also can be reestablished for ages to come. Also, all other measurements of a foot can be expressed once and for all by their proportion to the hour foot, and can thus be known with certainty for posterity. For example, we have already said above[15] that the Parisian foot is related to the hour foot as 864 to 881. In other words, if the Parisian foot is given, then we would say that a simple pendulum, whose oscillations mark off seconds of an hour, has a length equal to three of these feet, plus eight and one-half lines. And the Parisian foot is related to the Rhenish foot, which is used in my country, as 144 to 139, that is, the former exceeds the latter by five Parisian lines. And thus this foot, and all others, acquire a permanent measure for all time.

We have shown above how to find the center of oscillation of a sphere suspended from a string of any length. Namely, if the distance between the point of suspension and the center of the sphere is related to its radius as that radius is related to some other length, then two-fifths of that length, measuring down from the center of the sphere, ends at the center of oscillation to be found. It is easy to see why a consideration of this center of oscillation is necessary to determine an hour foot accurately. For if the

14. Paragraph 5 of Part I.
15. *Ibid.*

distance from the point of suspension to the center of the sphere is known, but the size of the sphere is not defined in relation to the length of the string, then the measurement of a pendulum whose oscillations mark off seconds will not be certain. The larger the sphere, the smaller will be the distance between the center of the sphere and the point of suspension. In isochronous pendula the centers of oscillation are equally distant from the points of suspension. But the center of oscillation descends further below the center of a larger sphere than a smaller one.

Before this determination of the center of oscillation, it was necessary for those who sought a method of establishing a universal measure—this task was proposed by the illustrious British Royal Society at the time I first invented my clock, and more recently by Gabriel Mouton,[16] the learned astronomer from Lyon—as I was saying, they found it necessary either to designate the diameter of the suspended globe, or to equate it to a definite proportion of the length of the string, for example, one-thirtieth or other part, or to use some known measure, like a finger or a thumb. But this latter method takes as certain something which should be questioned; even though I realize that there is hardly a sensible error as long as the sphere is not much larger than what I have said. The prior method could succeed in a given attempt but great care must be taken in counting the oscillations and in making the calculations. As a result it is preferable to follow the certain method of using centers of oscillation, and thus not to be obligated by any laws beyond what is necessary. Also, as in this method, it is better to use larger spheres than smaller ones, since the former are less influenced by air resistance.

Furthermore, to determine this measure, one can use not only a sphere suspended from a string, but also a cone, a cylinder, and all the other solids and planes whose centers of oscillation we have defined above. For in all isochronous pendula the distance from the point of suspension to the center of oscillation is exact and equal. Nor for this purpose are we limited to using only clocks in which the oscillations of the pendulum mark off seconds or half seconds. The same results will be obtained by the use of a pendulum of any other length, provided that one knows the proportion of the wheels or number of teeth in relation to the number of oscillations completed in a certain time. For example, first find a simple pendulum in which each oscillation corresponds to one, or two, or three oscillations of the clock. From this establish the number of the oscillations of that pendulum completed in one hour. Let that number be squared. Then the square

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16. Gabriel Moutonus or Mouton (1618–1694), *Observationes diametrorum solis et lunae apparentium* . . . (Lugduni: Matth. Liberal, 1670) 427–48. Mouton's suggestion was to divide one degree of a great circle of the earth on a decimal scale system for the basic unit of length (similar to the contemporary metric system), and the latter was then associated with the number of oscillations of simple pendula of the unit length.

of 3600, which is the number of seconds in one hour, will be related to the square of that number, as the length of the discovered simple pendulum (this length must always be measured from the point of suspension to the center of oscillation) is related to the length of a pendulum equal to three hour feet, of which we have spoken. This establishes the fact that the lengths of any two pendula are related to each other as the squares of the times in which they complete their oscillations. Consequently these lengths are inversely proportional to the squares of the numbers of oscillations completed in equal intervals of time. Up to now only experience confirmed this theorem about the lengths of pendula; namely, that they are related as the squares of the times in which the oscillations are completed. But now the demonstration of this is clear from what was established earlier. For we have shown that each oscillation of a pendulum suspended between cycloids has a definite ratio to a perpendicular fall from a height of half of the length of the pendulum. This ratio is that of the circumference of a circle to its diameter. From this it is easily concluded that the times of oscillation of two pendula are related to each other as the times of perpendicular fall from heights of half the lengths of the pendula. These half heights, and also the whole heights, are related to each other as the squares of the times in which they are crossed by a perpendicular fall [Proposition 3, Part II]. They are thus also related as the squares of the times which measure the oscillations. But the extremely small oscillations of a simple pendulum do not sensibly differ from the extremely small oscillations of a pendulum suspended between cycloids and having the same length. Therefore the lengths of simple pendula are related to each other as the square of the times in which very small oscillations are completed.

In summary, the work of counting the oscillations which occur in an hour or a half hour is not too burdensome. And clocks are available which have a pointer to indicate seconds. And no matter what the length of the simple pendulum is taken to be, the number of its oscillations occurring in one hour can be determined by our method. And as a result it is possible to calculate, as shown above, the length of a pendulum of three feet which marks off seconds.

PROPOSITION XXVI

How to define the space which is crossed by a heavy body falling perpendicularly in a given time.

Those who have previously studied this measure agree that it is necessary to consult experiments. But the experiments which have been conducted so far do not easily give an exact determination of this measure,

because of the speed which the falling body has acquired at the end of its motion. But if one uses our Proposition 25 in The Falling of Heavy Bodies [Part II], and if one knows the length of a pendulum which marks off seconds, then he can explain this matter as a certain derivation and without an experiment. First one determines the space which a heavy body crosses in one second; then from this it is possible to determine the space in any other second. For we have said that the length of a pendulum which marks off seconds is 3 hour feet. But the time of one small oscillation is related to the time of perpendicular fall from half the height of the pendulum as the circumference of a circle is related to its diameter, that is, as 355 to 113. Thus, if the first of these numbers is related to the other as the time of one second, or sixty thirds, is related to the other time, the latter then is 19⅑0′′′.[17] This is the time of descent through half of the height of the pendulum, that is, through 18 inches of a foot. But the squares of the times are related to each other as the spaces crossed in those times, as was shown in the previous proposition. Hence the square of 19⅑0′′′ is related to the square of 60′′′, that is, 36481 is to 360000, as 18 inches are related to another distance, which is 14 feet, 9 inches, 6 lines. This is the height of perpendicular fall in one second. But since an hour foot is related to a Parisian foot as 881 to 864, then this same height, reduced to that measure, is very nearly 15 feet, one inch. This agrees completely with our most accurate experiments, in which the precise point of time at which the fall ends is not identified by the judgment of the eye or the ear. Neither of these is completely satisfactory; but the space crossed during a fall can be determined without any error by another method which we will try to show here.

A half oscillation of a pendulum suspended from a wall or an erect board indicates the lapse of time used by a falling body. In order that the sphere of this pendulum be released at the same moment as the lead weight which is going to fall, each is tied to, and held by, a thin string, which is cut by a flame. But before this, another thin cord has been attached to the lead weight which is to fall. Its length is such that when all of it is stretched out by the falling weight, the pendulum has not yet hit the wall. The other end of this thin cord is attached to a prepared scale made on paper or on a thin membrane. This scale is attached to the wall or board in such a way that it is easy for it to follow the cord which pulls it, and to descend in a straight line along the length of the cord, passing the place where the sphere of the pendulum will hit the wall. When the whole length of the cord has been used up, the upper part of the scale will be pulled downwards by the falling lead weight before the pendulum hits the board. The size of this part is indicated when the sphere, which has been lightly covered with soot, makes

17. The symbol ′ ′ ′ means 1/60th of a second.

its mark on the scale which is sliding by. If the length of the cord is added to this, then one has an exact definition of the space of the fall.

Even though we have assumed that air resistance has no effect in these matters, the measure we have assigned to falling bodies agrees exactly with experiments. Certainly air resistance is not large enough to make a sensible difference at the heights to which we can ascend, provided that the solid bodies are made of metal, or are of a quite large size if they are made of lighter material. For lightness of material in bodies which cut through the air in falling is compensated for by the size of the body. Thus a sphere made of wood, or even of cork, becomes equal to a sphere of lead, when the diameter of the former is related to the diameter of the latter, as the proper weight of lead is related to the weight of wood or cork. For then the weights of the spheres are related to each other as their surfaces. Nevertheless, when bodies, which differ a great deal in their intrinsic weight, fall with equal speed, as far as the senses are able to perceive, it is not necessary to consider the proportion of their diameters. For they could have equal speed because both of them are sufficiently large, or because they did not fall from too great a height. What must be primarily noted is that the height could be so great, or if the height is moderate, the lightness of the projected body could be so great, that air resistance would change the acceleration of the motion quite considerably from what we have proven above. In general, for any body which falls through the air or some other fluid, there exists a definite degree of speed, determined by its weight and its surface, which it cannot exceed, or rather which it cannot attain. This is the speed which the air or some other fluid, moving upwards, would need to have to sustain the given body so that it is floating and suspended in it. But perhaps there will be another occasion for us to treat these matters more fully.

PART V

Another Construction, Based on the Circular Motion of Pendula, and the Theorems on Centrifugal Force

THERE is another kind of oscillation in addition to the one we have examined up to this point; namely, a motion in which a suspended weight is moved around through the circumference of a circle. From this we were led to the construction of another clock at about the same time we invented the first one. It is also based on a certain principle for the equality of times, but its use is less widespread because the other clock is so much easier and simpler to construct. Nevertheless many clocks of the type of which we are now speaking have also been constructed, and not without success. One thing special about them is that the last pointer, which indicates the seconds, turns with a continuous and equal motion instead of in discrete jumps, as happens in our first clock and in all others. Another special feature is that clocks constructed in this way move without any noise or sound, although for astronomical observations a sound repeated at each second is not without its uses. I originally intended to publish here a lengthy description of these clocks, along with matters pertaining to circular motion and centrifugal force, as it might be called, a subject about which I have more to say than I am able to do at present. But, in order that those interested in these things can sooner enjoy these new and not useless speculations, and in order that their publication not be prevented by some accident, I have decided, contrary to my plan, to add this fifth part in which the structure of this clock is briefly explained and in which the theorems pertaining to centrifugal force are presented, although their demonstration will be postponed to another time.[1]

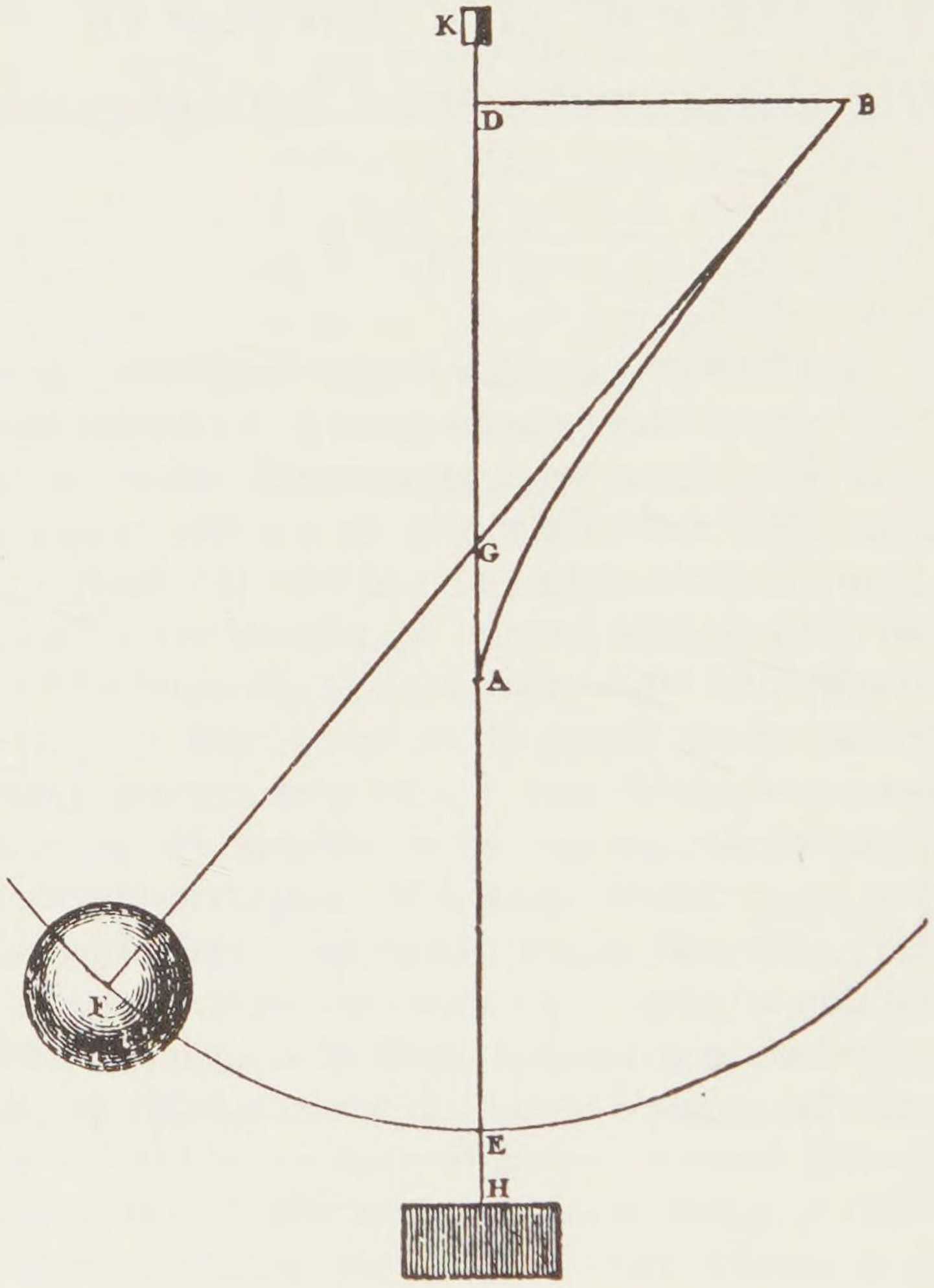
The Construction of the Second Clock

I do not think it is necessary to give here the arrangement of the wheels which comprise the interior of the clock, since they can be easily arranged

1. At this point in the 1724 edition 's Gravesande added the following reference note: "See the author's *Opera posthuma,* 401ff." For these demonstrations, see Huygens' *De vi centrifuga, Oeuvres* 16:253–301.

by artisans and can be modified in various ways. Rather it is sufficient to explain that part of the clock which regulates its motion in a determined way. This is shown here in the figure [Fig. 102].

[Fig. 102]



Let the axis DH be perpendicular to the horizon and moveable on two poles. To it at A is attached a thin plate having some width. The latter is curved along the line AB, which is a paraboloid whose evolution, after a straight line has been added to it, describes a parabola, as was shown in Proposition 8, Part III. AE is the straight line which is added to AB. The line EF represents the parabola described by the evolution of BAE. BGF is the string which is applied to the curve BA and whose endpoint describes the parabola. F is a weight attached to the string. While the axis DH turns on itself, the string BGF, stretched out in a straight line, rotates the small sphere F in such a way that it passes through circles parallel to the horizon.

These circles will be larger or smaller as the axis DH is moved with a larger or smaller force by the wheels of the clock acting on the small drum K. It moves in such a way that all the circles are contained on the surface of a parabolic conoid. As a result the times of rotation will always come out to be equal, as will be clear from what we will say shortly about this motion.

If we wish each revolution to mark off a half second, then the *latus rectum* of the parabola EF must be 4½ inches of our hour foot, that is, it must be half of the length of a pendulum in which each oscillation marks off a half second. The size of the *latus rectum* of the paraboloid AB depends on the *latus rectum* of the parabola EF; namely, the former contains the latter $^{27}/_{16}$ times. The same is true of the length AE, which is half of the *latus rectum* of the parabola. But if we wish that each revolution marks off one second, then both the *latera recta* and the line AE must be taken to be four times what they were.

Up to this point we have described the string BGF as unique and simple. But it must be noted that it is much more preferable if in its upper part it is a double string, making an angle of 20 or 30 degrees pointed towards F. To this end the thin plate AB should be wide enough at B to accommodate this spreading of the strings, or it could be made to split into two parts. When this is done, the circular motion of the weight F will continue without any assistance, and it will stretch both of the strings connected to it in a straight line, which would not happen if it were held by only one string. One must also note that the force needed to continue this circular motion is derived from the wheels of the clock, which are moved by the weight or some other power. This force is communicated to the axis KH from the small drum K, and by a very small pressure it preserves the motion of the sphere F, once it has begun.

For this to happen more easily, the revolution of the axis KH must be as free as possible. We have found that the best way of doing this is to construct the lower part of the axis out of hard steel, and to place under it a level surface of diamond. Any very small particle of the latter can be located under the perforated metal plate of the axis.

Moreover, in place of that part of the string BGF which corresponds to the curve AB, one could substitute a very thin chain, made of gold or some other metal, whose length is more apt to remain invariant. We have tried this in our first clock where the pendulum was suspended between cycloids. But in that case the continuous bending of the chain interfered considerably with the free oscillation of the pendulum because of the friction between the links, no matter how small they were.

Theorems on Centrifugal Force Arising from Circular Motion[2]

I

If two equal bodies move through unequal circumferences in equal times, the centrifugal force on the larger circumference will be related to that on the smaller one, as the circumferences themselves, or their diameters, are related to each other.

II

If two equal bodies move with equal speed on unequal circumferences, their centrifugal forces will be inversely proportional to the diameters.

III

If two equal bodies move with unequal speed on equal circumferences, but each maintains a constant motion as we wish this to be understood in these theorems, then the centrifugal force of the faster will be related to the centrifugal force of the slower as the squares of the speeds.

IV

If two equal bodies rotating on unequal circumferences have equal centrifugal force, then the time of revolution on the larger circumference will be related to the time of revolution on the smaller one as are the square roots of the diameters.

V

If a body moves on the circumference of a circle with the same speed which it acquires in falling from a height equal to one-quarter of the diameter, then it will have a centrifugal force equal to its weight, that is, it will pull the string by which it is held at the center with the same force as when it is suspended from the center.

---

2. Huygens' work on these theorems predates the *Horologium oscillatorium* by many years, and can be found in his *De vi centrifuga* (1659; first published posthumously in 1703). See *Oeuvres* 16:253–301. In the earlier treatise there are four additional theorems (nos. 6, 9, 11, and 17) and the wording is not exactly the same. Huygens was curiously guarded in publishing these important theorems and his proofs of them. In a letter dated 4 September 1669, to Henry Oldenburg, Secretary of the Royal Society, Huygens sent fourteen anagrams in the form of numerical ciphers for safekeeping "to avoid disputes and to render to each what is due in the discovery of new things." Among their topics were the isochronism of pendula and centrifugal forces, including some of the theorems first publicly announced here four years later at the end of the *Horologium oscillatorium*. For these anagrams and their decoded Latin messages, see *Oeuvres* 6:485–90.

VI

On the concave surface of a parabolic conoid which is perpendicular to the axis, all the revolutions of a body moving through circumferences parallel to the horizon, whether they be small or large, will be completed in equal times; each time will be equal to the double oscillation of a pendulum whose length is half of the "latus rectum" of the generating parabola.

VII

If two bodies, suspended from unequal strings, rotate so that they move through circumferences parallel to the horizon while the top of the string remains fixed, and if the heights of the cones whose surfaces are described by this motion of the strings are equal, then the times of revolution will also be equal.

VIII

If two bodies, suspended from equal or unequal strings, rotate with a conical motion as above, and if the heights of the cones are unequal, then the times of revolution will be related to each other as the square roots of the heights.

IX

If a pendulum rotating with a conical motion makes very small revolutions, then the time of each revolution will be related to the time of perpendicular fall from twice the height of the pendulum, in the same way as the circumference of a circle is related to its diameter; and hence it is equal to the time of two, very small, lateral oscillations of the same pendulum.

X

If a body moves on a circumference, and if each revolution is completed in the same time in which a pendulum, whose length is the radius of the circle, completes a very small revolution of a conical motion or completes two, very small, lateral oscillations, then it will have a centrifugal force equal to its weight.

XI

The time of revolution of any pendulum having a conical motion will be equal to the time of perpendicular fall from a height equal to the string

of the pendulum, when the angle of inclination of the string to the plane of the horizon is approximately 2 degrees, 54 minutes. It is exactly equal when the sine of the indicated angle is related to the radius as the square inscribed in the circle is related to the square of its circumference.

XII

If two pendula of equal weight have strings of unequal length, if they rotate with a conical motion, and if the heights of the cones are equal, then the forces by which they pull on their strings will have the same ratio as the lengths of the strings.

XIII

If a simple pendulum is moved by a maximum lateral oscillation, that is, if it descends through a full quarter of a circle, then when it arrives at the bottom point of the circumference, it will pull its string by a force three times larger than if it were simply suspended at that point.

INDEX OF SUBJECTS AND PERSONS

Page numbers for definitions are in italics

Académie Royale des Sciences, 7n, 33n
Acceleration, 34, 42, 65, 68, 70. *See also* Falling bodies
Africa, 28
Archimedes, 66n, 86, 110

Barbados, 28
Bell, A. E., 12n
Bernoulli, Jacob, 21n
Bourbon stars (satellites of Saturn), 9n
Bourbon-Vendome, François de, Duc de Beaufort, 29
British Royal Society, 169, 176n
Brounker, Viscount William, 85n, 105n
Bruce, Alexander, 28n

Candia, 29
Cape Verdi Islands, 28
Center line, *107,* 115, 138
Center of gravity, 25, 82, 107–67
Center of oscillation, 12, 17, 25, *107,* 137–70
of circle, 144, 148
of circumference of circle, 149
of cone, 157
of cylinder, 158
of ellipse, 151
of half-cone, 160
of half-cylinder, 162
of half-solid, 160, 162
of hyperbolic conoid, 159
of isosceles right triangle, 146
of isosceles triangle, 145
of oblique parabola, 154
of oblique plane figure, 162
of oblique solid figure, 162
of parabola, 146
of parabolic conoid, 159
of parabolic half-conoid, 162
and plane loci, 150–55
of pyramid, 156
of rectangle, 145
of regular polygon, 149
of right-angled cone, 162
of scalene cone, 162
of scalene triangle, 154
of sector of circle, 146–48
of solid figure, 155
and solid loci, 150–55
of sphere, 157
Centrifugal force, 173
theorems on, 176–78
Clocks, 11–32, 106, 162–69, 173–75
adjustment of, 23–27, 162–66
and astronomy, 8, 11, 12, 173
construction of, 11, 12, 15–23, 29–32, 173–75
discovery of, 12–13
invention of, 7–9, 11–13
isochronism of. *See* Isochronism

Clocks (*continued*)
and longitudes, measuring of, 8, 11, 18, 25
marine, 20, 28–32
regulation table for, 166
spring, 28, 30
Colbert, Jean Baptiste, 7n, 10
Coster, Samuel, 12n
Crete, 29
Crown wheel, 15
Curve "described by evolution," *74*
Curved in one direction, *73*
Cycloid, 11, 12, 21, 22, 30, 50–54, 60–71, 79–83, 166, 167, 170
construction of, 20–22
generating circle of, 80–82, 166
generating point of, 52–54

Daligre, M., 10
Day, mean solar, 23–25
inequality of, 24–27
Descartes, R., 52n, 85, 98, 105
Drake, S., 19n, 40n, 43n, 63n, 70n

Ecliptic, obliquity of, 24
Elevation of a plane, *43*
Epicycloid, 21n
Equalization of days, 24–25
table of, 26–27
Euclid, 49n, 71n, 130, 134, 139
Evolute, *74*
Evolution (of curves), 12, 21, *74*, 75–84, 93–104, 174
of cycloid, 79–80
of parabola, 83–84
of semicycloid, 81

Fabri, Honoré, 105, 142n
Falling bodies, 33–48, 58–71, 170–72
distance traversed, one second, 171
Fermat, Pierre de, 86
Foot
Hebrew, 167
hour, 17, 20, 165, 166, *168*, 170, 171, 175
Parisian, 17, 168, 171
Rhenish, 168
Roman, 167
Force, centrifugal. *See* Centrifugal force
Fuego, 28

Galilei, Galileo, 13, 19n, 43n, 63n
on falling bodies, 12, 40, 42, 44, 70n
Galilei, Vincenzio, 13
Geometrical method, 8, 11
Geometry, rebirth of, 7
Gravity, 33, 42, 106, 107
action of, 33, 35, 38, 39
axis of, *107*, 137, 144
center of. *See* Center of gravity
force of, 108, 109
Guinea, 28

Heath, T. L., 49n, 66n, 71n, 134n
Heuraet, Johannes van, 85, 86
Holmes, Capt. Robert, 28n
Horologium (1658), 11n, 12, 18n
Hour foot. *See* Foot, hour
Hours, 17
Howse, Derek, 28n
Huygens, Constantyn, 105n

Iráklion, 29n
Isochronism, 16, 9, 20, 30, 81, 106, *107*, 115–18, 127–29, 134–67, 176n

Leibniz, G., 85n
Length, definition of, 106
universal measure of, 167–70

License to publish, 10
Lilybaeum, 29
Logarithmic spiral, 21n
Logarithms, 92, 93
Longitudes, measurement of, 8, 28–29
Loubère (Lalovera), Antoine de la, 83
Louis XIV, 7–10, 28
Louvre, 10

Malvasia, Cornelio, 13n
Maretimo, 29
Mersenne, Marin, 19n, 82, 105, 106
Minutes, 17
Motion, 33–48, 58–71
  accelerated, 34, 42, 65, 68, 70
  due to gravity, 34, 35, 38, 39
  perpetual, 110
  uniform, 33–48, 65–69
Mousnier, Pierre, 142n
Mouton, Gabriel, 169
Moved in the plane, *108*
Moved laterally, *108*
Muguet, François, 10
Multiplied in straight lines, *108*

Neile, Willliams, 85

Oldenburg, Henry, 176n
Oscillation
  axis of, *106,* 107, 108, 115, 117, 118, 128, 131, 135–44
  center of. *See* Center of oscillation
  lateral, *141,* 143
  in the plane, *141*
  plane of, *107*
  rectangle of, *143,* 145, 151

Pascal, Blaise, 82, 87
Peloponnesus, 29
Pendulum, 16, 18, 21, 25, 28, *106*
  composite, 12, *107,* 110, 115–18, 164, 165
  isochronous. *See* Isochronism
  length of, 19–23, 29, 115–18, 128, 136–41, 162–67, 170, 171
  simple, 11, 12, 105, *106,* 107, 110, 115, 117, 118, 128, 136, 139–42, 162–65, 168, 170
  triangular, 30–32
Perpendicular line, *107*

Quadrature of the hyperbola, 90–92

Rectification (of curves), 12, 22, 23, 84–104
  of compressed spheroid, 88, 90
  of cycloid, 82
  of ellipse, 93–94
  of geometric curve, 94, 97
  of hyperbola, 93, 102–3
  of hyperbolic conoid, 89–90
  of oblong spheroid, 88, 90
  of parabola, 91, 96–97, 100–102
  of parabolic conoid, 87
  of paraboloid, 84, 98–102
Reverchon, L., 15n
Robertson, John Drummond, 12n
Roberval, G. P. de, 82, 105n

São Tomé, 28
Sapienza, 29
Schooten, Johannes van, 52n, 85, 86
Seconds, 17, 173
  's Gravesande, 49n, 92n, 98n, 143n, 164n, 173n
Sicily, 29
Sluse, François, 86

Stars, fixed, 23, 24, 167, 168
Sun, 24, 25, 28, 29
Sundial, 24

Tables
  of equalization of the days, 26–27
  for regulation of clocks, 166
Thierry, D., 10
Thirds, 171
Toulon, 29

Vallius, Hadrianus. *See* Wall, Adriaan van der
Voye, de la, 29n

Waard, C. de, 82n
Wall, Adriaan van der, 9n
Wallis, John, 51, 82, 85, 87
Wedge, *118*
  subcentrix of, *119*
Wren, Christopher, 51, 82, 85

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