# CHOICE AND CHANCE.

John Wilson

# CHOICE AND CHANCE



BY THE REV.

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# PREFACE.

In this second edition I have enlarged the appendices so as to meet the wants of advanced students. I have also added a collection of upwards of one hundred miscellaneous examples, which I think will add very much to the utility of the book.

It should be observed that the two chapters headed respectively Choice and Chance are simply arithmetic, and ought not to be beyond the comprehension of the ordinary reader who has never seen an algebraical symbol. But while expressly written for unscientific readers, they have been found very helpful to the young mathematician, when he was about to read in his algebra the hitherto difficult and embarrassing chapters on permutations and combinations, or on probability.

The appendices are addressed entirely to algebraical students. In the first appendix the usual theorems respecting permutations and combinations are established by new proofs, the same reasoning which was pursued with as little technicality as possible in the body of the work, being here expressed in algebraical language.

In the second and third appendices, which are newly added in this edition, a series of propositions are given which are not usually found in text books of algebra.

But I can see no reason why examples of such simple propositions as the xiiith and xxvth should be excluded from elementary treatises in which more complex but essentially less important theorems generally find place.

The classification of a variety of propositions under the titles of Distribution and Derangement will contribute (it is hoped) to disentangle the confusion in which all questions involving selection or arrangement are commonly massed together, and will facilitate in some degree that precision of language and clearness of expression which ought always to be aimed at in mathematics.

In the fourth appendix I have exhibited the seeming paradox that a wager which is mathematically fair is mathematically disadvantageous to both contracting parties. And I have endeavoured to cast into a simple and intelligible form the principles upon which the difficulties of the celebrated Petersburg problem are explained.

W. ALLEN WHITWORTH.

ST. JOHN'S COLLEGE,
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## PREFACE TO THE FIRST EDITION.

The following pages are a reproduction of lectures on Arithmetic, given in Queen's College, Liverpool, in the Michaelmas Term, 1866. Many of the students to whom the lectures were addressed were just entering upon the study of algebra, and it seemed well, while the greater part of their time was devoted to the somewhat mechanical solution of examples necessary to give them a practical facility in algebraical work, that their logical faculties should be meanwhile exercised in the thoughtful applications of the arithmetical art with which they were already familiar.

I had already discovered, that the usual method of treating questions of selection and arrangement was capable of modification and so great simplification, that the subject might be placed on a purely arithmetical basis; and I deemed that nothing would better serve to furnish the exercise which I desired for my classes, and to elicit and encourage a habit of exact reasoning, than to set before them, and establish as an application of arithmetic, the principles upon which such questions of "choice and chance" might be solved.

The success of my experiment has induced me to publish the present work, in the hope that the expositions already accepted by a limited audience may prove of service in a wider sphere, in conducing to a more thoughtful study of arithmetic than is common at present; extending the perception and recognition of the important truth, that arithmetic, or the art of counting, demands no more science than good and exact common sense.

In the first chapter I have set down and established as arithmetical rules all the principles usually required in estimating the choice which is open to us in making a selection or arrangement out of a number of given articles under given conditions. In the second chapter I have explained how different degress of probability are expressed arithmetically, and how the principles of the preceding chapter are applied to the calculation of chances. These two chapters will prove intelligible to any one who understands the first principles of arithmetic, provided he will consider each step as he goes on; not content with the mere statement of any rule, but careful to follow the explanations given and to recognise the reason of each successive principle.

For the sake of mathematical students I have added, as an appendix, a new treatment of permutations and combinations with algebraical symbols. In my experience as a teacher I have found the proofs here set forth more intelligible to younger students than those given in the text books in common use.

LIVERPOOL, 1st *February*, 1867.

# CONTENTS.

# ERRATA.

Page 22, line 6. For $\underline{|24}$, read 24.

Page 89, the 8th and following lines. Read "there are *ten* which give a result greater than 8. Hence the required chance is $\frac{10}{36}$ or $\frac{5}{18}$."

Page 103, line 8. For $\times$ read $+$.

At the head of pages 114, 118, 126. For CHOICE, read CHANCE.

## EXPLANATIONS.

The sign $=$ is used to signify that the two expressions between which it is placed are *equal* to one another, or represent the same arithmetical value.

The *sum* of any given numbers is that which is obtained by adding the given numbers together. The sign $+$ *(plus)* is used to denote that the number which follows it is added to that which precedes. Thus—

$$5 + 3 = 8,$$

or the sum of five and three is eight.

The *difference* of two given numbers is that which is obtained by subtracting the smaller of them from the greater. The sign $-$ *(minus)* is used to denote that the number which follows it is subtracted from that which precedes. Thus—

$$5 - 3 = 2,$$

or the difference of five and three is two.

The *product* of two given numbers is that which is obtained by multiplying the given numbers together. The *continued product* of three or more numbers is obtained by multiplying any two of them together, and then multiplying the result by a third, and so on until all the given numbers have been used

The sign $\times$ is used to denote that the number which precedes it is multiplied by that which follows. Thus—

$$5 \times 3 = 15; \quad 5 \times 3 \times 2 = 30;$$

or the product of five and three is fifteen: the continued product of five, three, and two, is thirty.

A full point (.) is often used instead of the sign $\times$ of multiplication. Thus—

$$5 . 4 . 3 . 2 . 1 = 120.$$

The *quotient* of two given numbers is that which is obtained by dividing the former of them by the latter. The sign $\div$ is used to denote that the number which precedes it is divided by that which follows. Thus—

$$15 \div 5 = 3,$$

or the quotient of fifteen and five is three.

Instead of using the sign $\div$, the quotient of the two numbers is often expressed by writing the former above the latter in the form of a fraction. Thus—

$$\frac{15}{5} = 15 \div 5 = 3.$$

The few other signs which are used in the body of the work are explained wherever they are first introduced. But the appendices, being addressed to mathematical readers, involve a more technical notation, for the explanation of which the student must be referred to treatises on algebra.

# CHOICE AND CHANCE.

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## CHAPTER I.

### CHOICE.

We have continually to make our choice among different courses of action open to us, and upon the discretion with which we make it, in little matters and in great, depends our prosperity and our happiness. Of this discretion a higher philosophy treats, and it is not to be supposed that Arithmetic has anything to do with it; but it is the province of Arithmetic, under given circumstances, to *measure* the choice which we have to exercise, or to determine precisely the number of courses open to us.

Suppose, for instance, that a member is to be returned to parliament for a certain borough, and that four candidates present themselves. Arithmetic has nothing to do with the manner in which we shall exercise our privilege as a voter, which depends on our discretion in judging the qualifications of the different candidates; but it belongs to Arithmetic, as the science of counting and calculation, to tell us that the number of ways in which (if we vote at all) we can exercise our choice, is *four*.

The operation is, indeed, in this case so simple that we scarcely recognise its arithmetical character at all; but if we pass on to a more complicated case, we shall observe that some thought or calculation is required to determine the number of courses open to us: and thought about numbers is Arithmetic.

Suppose, then, that the borough has to return *two* members instead of one. And still suppose that we have the same four candidates, whom we will distinguish by names, as *A, B, C, D*. If we try to note down all the ways in which it is possible for us to vote, we shall find them to be six in number; thus we may vote for any of the following:—

| | | |
|---|---|---|
| *A* and *B*, | *A* and *C*, | *A* and *D*, |
| *C* and *D*, | *B* and *D*, | *B* and *C*. |

But we can hardly make this experiment without perceiving that the resulting number, *six*, must in some way depend arithmetically upon the number of candidates and the number of members to be returned, or without suspecting that on some of the principles of arithmetic we ought to be able to arrive at that result without the labour of noting all the possible courses open to us, and then counting them up; a labour which we may observe would be very great if eight or ten candidates offered themselves, instead of four.

In the present chapter we shall establish and explain the principles upon which such calculations are made arithmetically. It will be found that they are very simple in nature as well as few in number. In the next chapter we shall apply the same principles to the solution of problems in Probability, a subject of very great interest, and some practical importance.

We found, by experiment or trial, that there were six ways of voting for two out of four candidates. So we may say that, out of any four given articles, six selections of two articles may be made. But we call special attention to the sense in which we use the words "six selections." We do not mean that a man can select two articles, and having taken them can select two more, and then two more, and so on till he has made six selections altogether; for it is obvious that the four articles would be exhausted by the second selection, but when we speak of six selections being possible, we mean that there are six different ways of making one selection, just as among four candidates there are six ways of selecting two to vote for.

This language may appear at first to be arbitrary and unnecessary, but as we proceed with the subject we shall find that it simplifies the expression of many of our results.

In making the selection of two candidates out of four, in the case just considered, it was immaterial which of the two selected ones we took first; the selection of *A* first, and then *B*, was to every intent and purpose the same thing as the selection of *B* first, and then *A*.

But if we alter the question a little, and ask in how many ways a society can select a president and vice-president out of four candidates for office, the order of selection becomes of importance. To elect *A* and *B* as president and vice-president respectively, is not the same thing as to elect *B* and *A* for those two offices respectively. Hence there are twice as many ways as before of making the election, viz.,—

| | | |
|---|---|---|
| *A* and *B*, | *A* and *C*, | *A* and *D*, |
| *C* and *D*, | *B* and *D*, | *B* and *C*, |
| *B* and *A*, | *C* and *A*, | *D* and *A*, |
| *D* and *C*, | *D* and *B*, | *C* and *B*. |

So if four articles of any kind are given us, there will be *twelve* ways of choosing two of them in a particular order; or, as we may more briefly express it, out of four given articles, twelve arrangements of two articles can be made. But it must be observed that the same remarks apply here, which we made on the use of the phrase "six selections" on page 3. We do not mean that twelve arrangements or six selections can be successively made; but that if one arrangement or one selection of two articles have to be made out of the four given articles, we have the choice of twelve ways of making the arrangement, and of six ways of making the selection.

We may give the following formal definitions of the words *selection* and *arrangement*, in the sense in which we have used them:—

DEF. I.—A *selection* (or *combination*) of any number of articles, means a group of that number of articles classed together, but not regarded as having any particular order among themselves.

DEF. II.—An *arrangement* (or *permutation*) of any number of articles, means a group of that number of articles, not only classed together, but regarded as having a particular order among themselves.

Thus the six groups,—

| | | |
|---|---|---|
| $A\ B\ C$, | $B\ C\ A$, | $C\ A\ B$, |
| $A\ C\ B$, | $B\ A\ C$, | $C\ B\ A$, |

are all *the same* selection (or combination) of three letters, but they are all *different* arrangements (or permutations) of three letters.

So, out of the four letters $A$, $B$, $C$, $D$, we can make four selections of three letters, viz.—

$B\ C\ D$,
$C\ D\ A$,
$D\ A\ B$,
$A\ B\ C$;

but out of the same four letters we can make twenty-four arrangements of three letters, viz.—

| | | | | | |
|---|---|---|---|---|---|
| *BCD*, | *BDC*, | *CDB*, | *CBD*, | *DBC*, | *DCB*, |
| *CDA*, | *CAD*, | *DAC*, | *DCA*, | *ACD*, | *ADC*, |
| *DAB*, | *DBA*, | *ABD*, | *ADB*, | *BDA*, | *BAD*, |
| *ABC*, | *ACB*, | *BCA*, | *BAC*, | *CAB*, | *CBA*. |

Having thus explained the language we shall have to employ, we may now proceed to establish the principles on which all calculations of choice must be founded.

The great principle upon which we shall base all our reasoning throughout our work, may be stated as follows:—

*If one thing can be done in a given number of different ways, and then another thing in another given number of different ways, the number of different ways in which both things can be done is obtained by multiplying together the two given numbers.*

We shall first illustrate this principle, and then proceed to prove it.

Suppose we have a box containing five capital letters, *A*, *B*, *C*, *D*, *E*, and three small letters, *x*, *y*, *z*.

| | | | | |
|---|---|---|---|---|
| *A* | *B* | *C* | *D* | *E* |
| | *x* | *y* | *z* | |

The number of ways in which we can select a capital letter out of the box is *five;* the number of ways in which we can select a small letter is *three;* therefore, by the principle we have just stated, the number of different ways in which we can select a capital letter and a small one is *fifteen*, which we find on trial to be correct, all the possible selections being as follows:—

| | | | | |
|---|---|---|---|---|
| $Ax$, | $Bx$, | $Cx$, | $Dx$, | $Ex$, |
| $Ay$, | $By$, | $Cy$, | $Dy$, | $Ey$, |
| $Az$, | $Bz$, | $Cz$, | $Dz$, | $Ez$. |

Again, suppose there are four paths to the top of a mountain, the principle asserts that we have the choice of sixteen ways of ascending and descending. For there are

4 ways up,
4 ways down,

and $4 \times 4 = 16$.

We can verify this: for if $P$, $Q$, $R$, $S$ be the names of the four paths, we can make our choice among the following sixteen plans, the first-mentioned path being the way up, and the second the way down:—

| | | | |
|---|---|---|---|
| $P$ and $P$, | $P$ and $Q$, | $P$ and $R$, | $P$ and $S$, |
| $Q$ and $P$, | $Q$ and $Q$, | $Q$ and $R$, | $Q$ and $S$, |
| $R$ and $P$, | $R$ and $Q$, | $R$ and $R$, | $R$ and $S$, |
| $S$ and $P$, | $S$ and $Q$, | $S$ and $R$, | $S$ and $S$. |

Or, if we had desired to ascertain what choice we had of going up and down by different paths, we might still have applied the principle, reasoning thus:

There are *four* ways of going up, and when we are at the top we have the choice of *three* ways of descending (since we are not to come down by the same path that brought us up). Hence the number of ways of ascending and descending is $4 \times 3$, or *twelve*.

These twelve ways will be obtained from the sixteen described in the former case, by omitting the four ineligible ways,

$P$ and $P$, $Q$ and $Q$, $R$ and $R$, $S$ and $S$.

The foregoing examples will suffice to illustrate the meaning and application of our fundamental proposition. We will now give a formal proof of it. We shall henceforth refer to it as Rule I.

## RULE I.

*If one thing can be done in a given number of different ways, and when it is done in any way another thing can be done in another given number of different ways, then the number of different ways in which the two things can be done is the product of the two given numbers.*

For let $A, B, C, D, E$, &c. represent the different ways in which the first thing can be done (taking as many letters as may be necessary to represent all the different ways), and similarly let $a, b, c, d$, &c. represent the different ways of doing the second thing. Then, if we form a table as below, having the letters

*A, B, C, D, E* &c. at the head of the several columns, and the letters *a, b, c, d,* &c. at the end of the several horizontal rows, we may regard each square in the table as representing the case in which the first thing is done, in the way marked at the head of the column in which the square is taken; and the second thing in the way marked at the end of the row.

| WAYS OF DOING THE SECOND THING. | WAYS OF DOING THE FIRST THING. | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *E* | *F* | *G* | *&c.* |
| *a* | | | | | | | | |
| *b* | | | * | | | | | |
| *c* | | | | | † | | | |
| *d* | | | | | | | | |
| *&c.* | | | | | | | | |

Thus the square marked with the asterisk (*) will denote the case in which the first thing is done in the way which we called *C*, and the second thing in the way which we called *b*; and the square marked with the dagger (†) will denote the case in which the first thing is done in the way $E$, and the second in the way $c$, and so on.

Now it will be readily seen that all the squares represent different cases, and that every case is represented by some square or other. Hence the number of possible cases is the same as the number of squares. But there are as many columns as there are ways of doing the first thing, and each column contains as many squares as there are ways of doing the second thing. Therefore the number of squares is the product of the number of ways of doing the two several things, and therefore, this product expresses also the whole number of possible cases, or the whole number of ways in which the two events can be done.

This proves the rule.

*Question.*—If a halfpenny and a penny be tossed, in how many ways can they fall?

*Answer.*—The halfpenny can fall in two ways, and the penny in two ways, and $2 \times 2 = 4$, therefore they can fall in four ways.

The four ways, of course, are as follows:—

(1) both heads.
(2) both tails.
(3) halfpenny head and penny tail.
(4) halfpenny tail and penny head.

*Question.*—If two dice be thrown together, in how many ways can they fall?

*Answer.*—The first can fall in six ways, and the second in six ways, and $6 \times 6 = 36$; therefore there are thirty-six ways in which the two dice can fall.

The thirty-six ways may be represented as follows:—

| | | | | | |
|---|---|---|---|---|---|
| 1 and 1, | 1 and 2, | 1 and 3, | 1 and 4, | 1 and 5, | 1 and 6, |
| 2 and 1, | 2 and 2, | 2 and 3, | 2 and 4, | 2 and 5, | 2 and 6, |
| 3 and 1, | 3 and 2, | 3 and 3, | 3 and 4, | 3 and 5, | 3 and 6, |
| 4 and 1, | 4 and 2, | 4 and 3, | 4 and 4, | 4 and 5, | 4 and 6, |
| 5 and 1, | 5 and 2, | 5 and 3, | 5 and 4, | 5 and 5, | 5 and 6, |
| 6 and 1, | 6 and 2, | 6 and 3, | 6 and 4, | 6 and 5, | 6 and 6. |

*Question.*—In how many ways can two prizes be given to a class of ten boys, without giving both to the same boy?

*Answer.*—The first prize can be given in ten ways, and when it is given the second can be given in nine ways, and $10 \times 9 = 90$; therefore we have the choice of ninety ways of giving the two prizes.

*Question.*—In how many ways can two prizes be given to a class of ten boys, it being permitted to give both to the same boy?

*Answer.*—The first prize can be given in ten ways, and when it is given the second can be given in ten ways; therefore both can be given in $10 \times 10$, or 100 ways.

*Question.*—Two persons get into a railway carriage where there are six vacant seats. In how many different ways can they seat themselves.

*Answer.*—The first person can take any of the vacant seats; therefore he can seat himself in six different ways. Then there are five seats left, and therefore the other person has the choice of five different ways of seating himself. Hence there are $6 \times 5$, or 30 different ways in which they can take their seats.

*Question.*—In how many ways can we make a two-lettered word out of an alphabet of twenty-six letters, the two letters in the word being different?

*Answer.*—We can choose our first letter in twenty-six ways, and when it is chosen we can choose the second in twenty-five ways. Therefore we have the choice of $26 \times 25$, or 650 ways.

*Question.*—In how many ways can we select a consonant and a vowel out of an alphabet of twenty consonants and six vowels?

*Answer.*—We can choose the consonant in twenty ways, the vowel in six ways, both in one hundred and twenty ways.

*Question.*—In how many ways can we make a two-lettered word, consisting of one consonant and one vowel?

*Answer.*—By the last answer, we can choose our two letters in one hundred and twenty ways, and when we have chosen them we can arrange them in two ways. Hence we can make the word in $120 \times 2$, or 240 different ways.

*Question.*—There are twelve ladies and ten gentlemen, of whom three ladies and two gentlemen are sisters and brothers, the rest being unrelated: in how many ways might a marriage be effected?

*Answer.*—If all were unrelated we might make the match in $12 \times 10$, or 120 ways; but this will include the $3 \times 2$, or 6 ways in which the selected lady and gentlemen are sister and brother. Therefore the number of eligible ways is $120 - 6$, or 114.

## RULE II.

*If a series of things can be done successively in given numbers of ways, the number of ways in which all the things can be done is the continued product of all the given numbers.*

This rule is only an extension of the former one, and needs not a separate proof. Its correctness will be sufficiently evident from considering an example.

Suppose the first thing can be done in four ways, and the second in three, then the first and second together form an event or operation, which can happen (by Rule I.) in $4 \times 3$, or 12 ways. Now suppose the third thing can be done in five ways. Then, since the first and second together can happen in twelve ways, and the third in five ways, it follows from Rule I. that the first and second and the third, can be done in $12 \times 5$, or 60 ways; that is, *all three* can be done in $4 \times 3 \times 5$ ways.

So if a fourth thing can be done in seven ways, then, since the first three can be done in sixty ways, and the fourth in seven ways, the first three and the fourth can be done (by Rule I.) in $60 \times 7$, or 420 ways; that is, *all four* can be done in $4 \times 3 \times 5 \times 7$ ways. And so on, however many things there may be.

The applications of this proposition are very numerous and very important; the solution of almost every question concerning permutations or combinations depending upon it, as will presently be seen.

As an example, suppose I have six letters to be delivered in different parts of the town, and two boys offer their services to deliver them. To determine in how many different ways I have the choice of sending the letters we may reason as follows. The first letter may be sent in either of two ways; so may the second; so may the third, and so on. Hence the whole number of ways is, by the rule, $2 \times 2 \times 2 \times 2 \times 2 \times 2$ or 64. So, if there were three boys, the choice would lie among $3 \times 3 \times 3 \times 3 \times 3 \times 3$ or 729 ways.

The question, in how many ways can six things be divided between two boys, will be seen to be almost identical with the question of the six notes sent by the two boys. The only difference is, that among the 64 ways of sending the notes were included the two ways in which either boy carried them all. Now six things cannot be said to be divided among two boys if they all are given to one. Hence these two ways must be rejected, and there will only be 62 ways of dividing six things between two boys.

But, again, suppose we are asked in how many ways can six things be divided into two parcels, the question seems at first to be identical with the last. But, on consideration, we observe that if *a, b, c, d, e, f* represent the six things, one of the ways of dividing them between the two boys would be to give

*a, b,* to the *first* boy,
*c, d, e, f,* to the *second*;

and another different way would be to give

*a, b,* to the *second* boy,
*c, d, e, f,* to the *first*;

but if the question be merely of dividing the six things into two parcels, with no distinction between them, corresponding to the two ways noted for the previous question, we have now only the one way, viz., to put

*a, b,* into *one* parcel,
*c, d, e, f,* into the *other*.

Hence for every two ways of dividing the things between two different boys, there is only one way of dividing them into two indifferent parcels; and, therefore, we have the choice in this last case of only thirty-one ways.

The correctness of this result may be more clearly understood by the following consideration. Suppose we have six articles to divide between two boys. We may resolve the operation into the two operations of (1) dividing the articles into two parcels, and (2) when these parcels are made, giving them to the two boys. Now we can form our two parcels in thirty-one ways, and when the two parcels are made, we can give them, one to each boy, in two ways; hence by Rule I., we can make the parcels and dispense them in 31 × 2 or 62 ways.

*Question.*—Twenty competitors run a race for three prizes, in how many different ways is it possible that the prizes may be given?

*Answer.*—The first prize can be given in twenty ways; when it is given, the second may be given in nineteen; then the third can be given in eighteen ways. Hence the whole number of ways of giving the three prizes is 20 × 19 × 18, or 6840.

*Question.*—In how many ways can four letters be put into four envelopes, one into each?

*Answer.*—For the first envelope we have the choice of all the letters, or there are four ways of filling the first envelope; then there are three letters left, and therefore three ways of filling the second envelope; then there are two letters left, or two ways of filling the third envelope; so there is only one way of filling the last. Hence there are 4 × 3 × 2 × 1, or 24 ways of doing the whole.

*Question.*—How many different sums may be formed with a sovereign, a half-sovereign, a crown, a half-crown, a shilling, a sixpence, a penny, and a half-penny?

*Answer.*—Each coin may be either taken or left, that is, it may be disposed of in two ways, and there are eight coins. Hence (by Rule II.) all may be disposed of in

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2, \text{ or } 256$$

ways. One of these ways would, however, consist in the rejection of *all* the coins, which would not be a way of taking any sum. Therefore the number of different sums that can be made is 255.

*Question.*—There are twenty candidates for an office, and seven electors. In how many ways can the votes be given?

*Answer.*—Each man can vote in twenty ways, and there are seven men to vote. Therefore all the votes can be given (by Rule II.) in

$$20 \times 20 \times 20 \times 20 \times 20 \times 20 \times 20 \text{ or } 1280000000$$

different ways.

*Question.*—In how many ways can the following letters be divided between two persons:—

$$a, a, a, a, b, b, b, c, c, d?$$

*Answer.*—Of the $a, a, a, a$, the first person can take either *none*, or *one*, or *two*, or *three*, or *four*. That is, the $a, a, a, a$ can be divided in five different ways; so also the $b, b, b$ can be divided in four ways; the $c, c$ in three ways; and the $d$ can be disposed of in two ways. Hence (by Rule II.) the whole division can be made in

$$5 \times 4 \times 3 \times 2, \text{ or } 120$$

different ways, including, however, the ways in which either person gets *none* and the other gets *all*. Excluding these two ways, the number of eligible ways is 118.

*Question.*— In the ordinary system of notation, how many numbers are there which consist of five digits?

*Answer.*— The first digit may be any of the ten except 0. We have, therefore, the choice of nine ways of determining this digit. Each of the other four digits may be any whatever, and therefore there are ten ways of determining each of them. Hence, altogether (by Rule II.) the number can be formed in

$$9 \times 10 \times 10 \times 10 \times 10, \text{ or } 90000$$

different ways.

Of course these are all the numbers from 10000 to 99999 inclusive.

*Question.*— The cylinder of a letter-lock contains four rings, each marked with twenty-six different letters; how many different attempts to open the lock may be made by a person ignorant of the key-word?

*Answer.*— The first ring can be placed in twenty-six different positions; so may the second; so may the third; so may the fourth. Hence (by Rule II.) there are

$$26 \times 26 \times 26 \times 26, \text{ or } 456976$$

different positions possible, and *one* of these is the right one. Hence it is possible to make 456975 unsuccessful trials.

## RULE III.

*The number of ways in which a given number of things can be arranged is the continued product of the given number, and all whole numbers less than it.*

Thus, three things can be arranged in $3 \times 2 \times 1$, or 6 ways; four things in $4 \times 3 \times 2 \times 1$, or 24 ways; five things in $5 \times 4 \times 3 \times 2 \times 1$, or 120 ways.

It will be sufficient to shew the reason of this rule in a particular case. The reasoning will be of a sufficiently general character to apply to any other case.

Take for example the case of five things. We have then a choice of five ways of filling the first place in order. When that place is filled there remain four things, and therefore we have a choice of four ways of filling the second place. Then there are three things left, and we can fill the third place in three ways. So we can fill the fourth place in two ways, and the last place in only one way, since we must give to it the one thing that is now left. Hence (by Rule II.) all the places can be filled in $5 \times 4 \times 3 \times 2 \times 1$ ways, or the whole set of five things can be arranged in $5 \times 4 \times 3 \times 2 \times 1$ ways, which shews that Rule III. is true in this case.

By exactly similar reasoning, we can shew that the rule is true in any other case. Hence we may accept it universally.

It is usual to put the mark $\underline{|\ \ }$ round a number to denote the continued product of *that number and all lesser numbers*. Thus —

$\underline{|2}$ denotes $2 \times 1$, or 2;

$\underline{|3}$ denotes $3 \times 2 \times 1$, or 6;

$\underline{|4}$ denotes $4 \times 3 \times 2 \times 1$, or 24;

$\underline{|5}$ denotes $5 \times 4 \times 3 \times 2 \times 1$, or 120;

$\underline{|6}$ denotes $6 \times 5 \times 4 \times 3 \times 2 \times 1$, or 720;

and so on.

A great number of questions will be seen, on a little consideration, to be particular applications of Rule III.

Suppose, for instance, that we have to place six statues in six niches, it seems, at first sight, that as the statues and the niches can each of them separately be taken in any order, we should have to consider the order of both to determine what choice of arrangement we have.

But, on consideration, it will be seen that even though we take the niches in any stated order, yet any possible result whatsoever may be attained by varying the order of the statues. We may, in fact, regard the niches as forming a row in fixed order, and we have only to consider in how many different orders the six statues may be taken so as to fill the six niches in order. Consequently, the number of ways in which it is possible to arrange the six statues in the six niches is the same as the number of orders in which the six statues can themselves be taken, which by the rule is $\underline{|6}$, or 720.

This explanation will be the better understood by comparing the next two questions.

*Question.* — In how many ways can twelve ladies and twelve gentlemen form themselves into couples for a dance?

*Answer.* — $\underline{|12}$. For the first gentleman can choose a partner in twelve ways; then the second has choice of eleven; the third has choice of ten, and so on. Therefore they can take partners altogether in

$$12.11.10.9.8.7.6.5.4.3.2.1, \text{ or } \underline{|12}$$

ways.

*Question.* — There are twelve ladies and twelve gentlemen in a ball-room; in how many ways can they take their places for a contre-danse?

*Answer.* — The couples can be formed in $\underline{|12}$ ways, (last question) and when formed, the couples can be arranged in $\underline{|12}$ different orders (Rule III.) Therefore the twelve ladies and twelve gentlemen can arrange themselves in $\underline{|12} \times \underline{|12}$ different ways.

Or we may reason thus:

The ladies can take their places in $\underline{|12}$ different ways, (by Rule III.) and so the gentlemen can take theirs in $\underline{|12}$ different ways. Therefore (by Rule I.)

the ladies and gentlemen can arrange themselves in $\underline{|12} \times \underline{|12}$ different ways, as before.

*Question.* — In how many different orders can the letters *a*, *b*, *c*, *d*, *e*, *f* be arranged so as to begin with *ab*?

*Answer* $\underline{|4}$. — For our only choice lies in the arrangement of the remaining four letters, which can be put in $\underline{|4}$ or 24 different orders (by Rule III.)

*Question.* — A shelf contains five volumes of Latin, six of Greek, and eight of English. In how many ways can the nineteen books be arranged, keeping all the Latin together, all the Greek together, and all the English together?

*Answer.* — The volumes of Latin can be arranged among themselves (by Rule III.) in $\underline{|5}$ ways, the volumes of Greek among themselves in $\underline{|6}$ ways, and the volumes of English among themselves in $\underline{|8}$ ways. Also, when each set is thus prepared, the three sets can be placed on the shelf in $\underline{|3}$ different orders. Therefore, by Rule II., the number of ways in which the whole can be done is

$$\underline{|5} \times \underline{|6} \times \underline{|8} \times \underline{|3}, \text{ or } 20901888000.$$

*Question.* — In how many ways could the same books be arranged indiscriminately on the shelf?

*Answer.* — $\underline{|19}$, or 121645100408832000 ways.

It often requires considerable thought to determine what is meant by "different ways" of forming a ring. The next three questions suggest three meanings which the words in several circumstances will bear. It will be well to consider them, and compare them carefully, that the distinctions among them may be thoroughly recognised.

*Question.*—A table being laid for six persons, in how many ways can they take their places?

*Answer.*—By Rule III., the number of ways is $|\underline{6}$ or 720.

*Question.*—In how many ways can six children form themselves into a ring, to dance round a may-pole.

*Answer.*—In this case we have not to assign the six children to particular places absolutely, but only to arrange them relatively to one another. We may, in fact, make all possible arrangements, by placing the first child, **A**, in any fixed position, and disposing the others, *B*, *C*, *D*, *E*, *F*, in different ways with respect to him. Thus there is no essential difference between the three arrangements —

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | D | | | B | | | E | |
| E | | C | C | | **A** | F | | D |
| | * | | | * | | | * | |
| F | | B | D | | F | **A** | | C |
| | **A** | | | E | | | B | |

but two different arrangements would be —

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | D | | | | | D | |
| E | | C | | | C | | E |
| | * | | | | | * | |
| F | | B | | | B | | F |
| | **A** | | | | | **A** | |

And any other essentially different arrangement might be obtained without disturbing **A**, since absolute position is not taken into account. Now the five children *B, C, D, E, F* can be arranged, by Rule III., in $\underline{|5}$ or 120 ways. This, therefore, is the whole number of ways in which such a ring can be formed.

*Question.*—In how many ways can six stones be strung on an elastic band to form a bracelet?

*Answer.*—This question is not equivalent to the preceding one, for if we examine the last two arrangements, which we marked down as examples of the different ways in which the ring could be made, we shall observe that though they would count as different arrangements of children round a may-pole, they would count as the same arrangement of stones in a bracelet, presenting only opposite views of the same bracelet; each being, in fact, the arrangement that would be presented by turning the other completely over. So the 120 arrangements which we could make according to the last question, might be disposed into 60 pairs, each pair presenting only opposite views of the same ring, and not representing more than one essentially different arrangement. Hence the answer is in this case only 60.

Definition.— Numbers are called *successive* when they proceed in order, each one differing from the preceding one by *unity*. The numbers are said to be *descending* when they commence with the greatest and continually decrease; they are said to be *ascending* when they commence with the least and continually increase; and such a series of numbers is said to ascend or descend (as the case may be) *from* the first number of the series.

Thus 17, 18, 19 are successive numbers ascending from 17.

So, 17, 16, 15, 14 are successive numbers descending from 17.

Again, if we speak of a series of five successive numbers descending from 100, we shall mean the numbers 100, 99, 98, 97, 96.

So if we speak of seven successive numbers ascending from 3, the numbers

3, 4, 5, 6, 7, 8, 9.

will be meant.

Thus, $\underline{|5}$ might be described as the continued product of five successive numbers, ascending from unity (or descending from 5); $\underline{|7}$, as the continued product of seven successive numbers ascending from unity (or descending from 7), and so on.

## RULE IV.

*Out of a given number of things,*

*the number of ways in which an arrangement of two things can be made is the product of the given number and the next lesser number;*

*the number of ways in which an arrangement of three things can be made is the continued product of three successive numbers descending from the given number;*

*the number of ways in which an arrangement of four things can be made is the continued product of four successive numbers descending from the given number;*

*and so on.*

The reason of this rule will be seen at once. For suppose we have seventeen given things; then, if we wish to make an arrangement of *two* things, we have the choice of seventeen things to place first, and then there are sixteen things left, out of which we have to choose one to place second, and complete our arrangement. Hence, by Rule I., the number of ways in which we can make an arrangement of two things, is 17.16.

So if we wish to make an arrangement of *three* things, we can place the first two in 17.16 ways, and we then have fifteen things left, out of which to choose one to come third, and complete our arrangement; therefore, by Rule I., the number of ways in which we can make an arrangement of three things is the product of 17.16 and 15, or 17.16.15: and so on.

Many questions which might be considered under Rule II. may be answered more directly by this rule. Thus —

*Question.*—How many three-lettered words could be made out of an alphabet of twenty-six letters, not using any letter more than once?

*Answer.* — 26.25.24 = 15600.

*Question.* — How many four-lettered words?

*Answer.* — 26.25.24.23 = 358800.

*Question.* — How many eight-lettered words?

*Answer.* — 26.25.24.23.22.21.20.19 = 62990928000.

*Question.*—Four flags are to be hoisted on one mast, and there are twenty different flags to choose from: what choice have we?

*Answer.* — By Rule IV. we have the choice of

20.19.18.17, or 116280

different ways.

The answer would evidently be the same if the flags were to be hoisted on different masts, for so long as there are four different positions to be occupied, the operation consists in the arrangement in these positions of four out of the twenty flags.

*Question.* — An eight-oared boat has to be manned out of a club consisting of fifty rowing members. In how many ways can the crew be arranged?

*Answer.*—We have simply to arrange eight men in order out of fifty men. Therefore Rule IV. applies, and the number of ways is

$$50.49.48.47.46.45.44.43, \text{ or } 2164694716800 0.$$

## RULE V.

*The number of ways in which twenty things can be divided into two classes of twelve and eight respectively, is*

$$\frac{\underline{|20}}{\underline{|12}\,.\,\underline{|8}}$$

*and similarly for any other numbers.*

Suppose that twenty persons have to take their places in twelve front seats and eight back seats. By Rule III. they can be arranged altogether in $\underline{|20}$ ways. But the operation of arranging them may be resolved into the following three operations:—

(1) The operation of dividing the twenty into two classes of twelve and eight.

(2) The operation of arranging the class of twelve in the twelve front seats.

(3) The operation of arranging the class of eight in the eight back seats.

Hence, by Rule II., $\underline{|20}$ is the product of the number of ways in which these three several operations can be performed. But by Rule III. the second can be performed in $\underline{|12}$ ways, and the third in $\underline{|8}$ ways;

therefore it follows that the first can be performed in

$$\frac{\lfloor 20}{\lfloor 12 \cdot \lfloor 8}$$

ways. This, therefore, expresses the number of ways in which twenty things can be divided into two classes, of which the first shall contain twelve things, and the second shall contain eight.

And it will be observed that our reasoning throughout is perfectly general, and would equally apply if, instead of the number twenty, divided into the parts twelve and eight, we had any other number, divided into any two assigned parts whatever.

Hence we can write down on the same plan the number of ways in which any given number of things can be divided into two classes, with a given number in each.

*Question.*—Eight men are to take their places in an eight-oared boat; but two of them can only row on stroke side, and one of them only on bow side; the others can row on either side. In how many ways can the men be arranged?

*Answer.*—The operation of arranging the men may be resolved into the following three simple and successive operations, viz.,

(1) To divide the five men who can row on either side into two parties of two and three, to complete stroke side and bow side respectively.

(2) To arrange stroke side when it is thus completed; and

(3) To arrange bow side.

The five men who can row on either side can be divided into two parties of two and three respectively, in

$$\frac{\underline{|5}}{\underline{|3}\,.\,\underline{|2}} \text{ or } 10$$

ways, by Rule V. And when this is done, stroke side, consisting of four men, can be arranged in $\underline{|4}$ or twenty-four different ways (Rule III.); and likewise bow side in twenty-four ways. Hence the whole arrangement can be made in 10 × 24 × 24, or 5760 ways.

## RULE VI.

*The number of ways in which twenty things can be divided into three classes of five, seven, and eight, respectively, is*

$$\frac{\underline{|20}}{\underline{|5}\,.\,\underline{|7}\,.\,\underline{|8}};$$

*and similarly for any other numbers.*

For, by Rule V., the twenty things can be divided into two classes of twelve and eight in

$$\frac{\underline{|20}}{\underline{|12}\,.\,\underline{|8}}$$

different ways, and, when this is done, the class of twelve can be divided into two classes of five and seven in

$$\frac{\underline{|12}}{\underline{|5}\,.\,\underline{|7}}$$

ways. Hence, by Rule I., both these can be done in

$$\frac{\underline{|20}}{\underline{|12}\,.\,\underline{|8}} \times \frac{\underline{|12}}{\underline{|5}\,.\,\underline{|7}} \quad \text{or} \quad \frac{\underline{|20}}{\underline{|5}\,.\,\underline{|7}\,.\,\underline{|8}}$$

different ways.

That is, twenty things can be divided into three classes of five, seven and eight severally, in

$$\frac{\underline{|20}}{\underline{|5}\,.\,\underline{|7}\,.\,\underline{|8}}$$

different ways; and since our reasoning is perfectly general, a similar result may be written down when the numbers are any other.

And it is easily seen that the reasoning may be extended, in the same manner, to the case of more than three classes.

*Question.*—In how many ways can three boys divide twelve oranges, each taking four?

*Answer.*—By Rule VI. the number of different ways in which twelve things can be divided into three classes of four each, is

$$\frac{\underline{|12}}{\underline{|4}\,.\,\underline{|4}\,.\,\underline{|4}} \text{ or } 34650.$$

*Question.*—In how many ways can they divide them, so that the eldest gets five, the next four, and the youngest three?

*Answer.*—By Rule VI. the number of different ways is

$$\frac{\underline{|12}}{\underline{|3}\,.\,\underline{|4}\,.\,\underline{|5}} \text{ or } 27720.$$

*Question.*—If there be fifteen apples all alike, twenty pears all alike, and twenty-five oranges all alike, in how many ways can sixty boys take one each?

*Answer.*—The boys have, in fact, to form themselves into a party of fifteen for the apples, a party of twenty for the pears, and a party of twenty-five for the oranges. They can therefore do it by Rule VI. in

$$\frac{\underline{|60}}{\underline{|15}\,.\,\underline{|20}\,.\,\underline{|25}}$$

different ways.

*Question.*—In how many ways can two sixes, three fives, and an ace be thrown with six dice?

*Answer.*—The six dice have to be divided into three sets, containing 2, 3, 1 severally, of which the first set are to be placed with *six* upwards; the second set with *five* upwards; and the third set with *ace* upwards. By Rule VI. it can be done in

$$\frac{\underline{|6}}{\underline{|3}\,.\,\underline{|2}\,.\,\underline{|1}} \quad \text{or } 60$$

different ways.

*Question.*— In how many ways may fifty-two cards be divided amongst four players, so that each may have thirteen?

*Answer.*— By Rule VI.,

$$\frac{\underline{|52}}{\underline{|13}\,.\,\underline{|13}\,.\,\underline{|13}\,.\,\underline{|13}}$$

A possible error must be guarded against in the application of Rules V. and VI.

Suppose we are given eight things, say the letters

$$A,\ B,\ C,\ D,\ E,\ F,\ G,\ H,$$

and are asked in how many ways it is possible to divide them into two parcels of four each. If the parcels are numbered *No.* 1 and *No.* 2, and are designed for different purposes, we may apply Rule V., and answer that the number of possible ways is

$$\frac{\underline{|8}}{\underline{|4}\,.\,\underline{|4}} \text{ or } 70.$$

In this case, to put

$$\left\{\begin{array}{l} A,\ B,\ C,\ D, \text{ into the \textit{first} parcel,} \\ E,\ F,\ G,\ H, \text{ into the \textit{second};} \end{array}\right.$$

and to put

$$\left\{\begin{array}{l} A,\ B,\ C,\ D, \text{ into the \textit{second} parcel,} \\ E,\ F,\ G,\ H, \text{ into the \textit{first},} \end{array}\right.$$

will be counted as *different ways* of disposing of the eight things. But if the parcels be perfectly indifferent

—if the eight things have simply to be disposed in two equal heaps, with no distinction between the heaps—then the two ways just indicated of disposing of the eight things will become identical; each being merged into the one way of putting

$$\begin{cases} A,\ B,\ C,\ D, \text{ into } \textit{one} \text{ parcel,} \\ E,\ F,\ G,\ H, \text{ into } \textit{another.} \end{cases}$$

In such a case as this, therefore, the Rule V. cannot be applied without some modification; we should, in fact, have to divide by 2 the result given by this rule.

So if there are twelve things to be divided into three different parcels—as, for instance, twelve oranges to be divided among three different boys—the Rule VI. may be applied. But if the parcels are indifferent, and we are simply asked in how many ways twelve things can be divided into three equal parts, the rule would want modification; we should, in fact, have to divide our result by $\underline{|3}$, the number of different orders in which the three parcels can be arranged.

When different numbers of things have to be put into the different parcels—as in the case when twenty things are to be divided into parcels of five, seven, and eight—no difficulty or doubt can arise, for the differences of number are sufficient to distinguish the different parcels, and to give an individuality to each; so that in such a case the Rule V. or VI. is always applicable.

It must be observed that there is some ambiguity in the manner in which the words sort and class are sometimes used, especially when we describe collections of articles as of different sorts or of the same sort, or of different classes or of the same class.

Thus, if letters have been spoken of as consonants and vowels, we may describe the alphabet as containing twenty letters of one sort, and six letters of the other sort; yet if we regard the individual character of each letter, we shall speak of a printer's fount as containing twenty-six different sorts of letters. Plainly, there are either two classes or twenty-six classes, according to the character adopted as the criterion of class.

For instance, we may describe the letters

$$a, a, a, x, x,$$

as *three of one sort and two of another sort.* But the letters

$$a, e, i, x, z,$$

regarded as vowels and consonants, might also be described as *three of one sort and two of another sort.*

Suppose now we are asked in how many different orders we can write down five different letters, of which three are of one sort and two of another sort, the answer will depend entirely on the sense in which "sort" is understood. If we suppose the letters to be such as

$$a, a, a, x, x,$$

where those of the same sort are absolutely identical with one another, having no personal individuality (so to speak), the answer will be

$$\frac{|\underline{5}}{|\underline{3} \cdot |\underline{2}}$$

(by Rule V.), since our only choice lies in dividing the five places into two sets of three and two, for the *a, a, a,* and *x, x.* But if the given letters be such as

$$a, e, i, x, z,$$

where the three, *a, e, i,* are of one sort as vowels, but each has an individual character of its own, and the two, *x, z,* are of one sort as consonants, but these also like the vowels distinct in their identity, then the answer becomes $|\underline{5}$, by Rule III., since the five letters are for the purposes of arrangement all different.

We shall avoid this ambiguity as much as possible, by speaking of things as of one *sort,* when there is no individual distinction amongst them, and of one *class* when they are united by a common characteristic, but capable, at the same time, of distinction one from another.

## RULE VII.

*The number of orders in which twenty letters can be arranged, of which four are of one sort (a, a, a, a, suppose), five of another sort (b, b, b, b, b, suppose),* *two of another sort (c, c, suppose), and the remaining nine all different, is*

$$\frac{\underline{|20}}{\underline{|4}\,.\,\underline{|5}\,.\,\underline{|2}}$$

*and similarly for any other numbers.*

For the operation of arranging the letters in order may be resolved into the following :—

(1) To divide the twenty places into four sets, of four, five, two, nine, respectively.

(2) To place the *a*, *a*, *a*, *a*, in the set of four places.

(3) To place the *b*, *b*, *b*, *b*, *b*, in the set of five places.

(4) To place the *c*, *c* in the set of two places.

(5) To *arrange* the nine remaining letters in the set of nine places.

Now by Rule VI., the operation (1) can be done in

$$\frac{\underline{|20}}{\underline{|4}\,.\,\underline{|5}\,.\,\underline{|2}\,.\,\underline{|9}}$$

different ways.

The operation (2) can be done in only one way, since the letters are all alike.

So the operations (3) (4) can be done in only one way each.

And the operation (5) can be performed in $\underline{|9}$ ways by Rule III.

Therefore by Rule II. the whole complex operation can be performed in

$$\frac{\underline{|20}}{\underline{|4}\,.\,\underline{|5}\,.\,\underline{|2}\,.\,\underline{|9}} \times \underline{|9}, \quad \text{or} \quad \frac{\underline{|20}}{\underline{|4}\,.\,\underline{|5}\,.\,\underline{|2}}$$

different ways.

And in the same way we can reason about any other case. Hence in any case, to find the number of orders in which a series of letters can be arranged which are not all alike, we have only to write down the fraction, having in the numerator the total number of the letters, and in the denominator the number of letters of the several sorts; each number being enclosed in the mark $\underline{|\;\;}$.

*Question.*—In how many orders can we arrange the letters of the word *indivisibility*?

*Answer.*— $\frac{\underline{|14}}{\underline{|6}} = 14.13.12.11.10.9.8.7 = 121080960.$

*Question.*—In how many orders can we arrange the letters of the word *parallelepiped*?

*Answer.*— $\frac{\underline{|14}}{\underline{|3}\,.\,\underline{|3}\,.\,\underline{|3}\,.\,\underline{|2}} = 201801600.$

*Question.*—In how many orders can we arrange the letters of the word *Llangollen*?

*Answer.*—75600.

## RULE VIII.

*Out of twenty things, a selection of twelve things can be made in the same number of ways as a selection of eight things (where* $12 + 8 = 20$*); and the number of ways is*

$$\frac{|\underline{20}}{|\underline{12}\,.\,|\underline{8}};$$

*and similarly for other numbers of things.*

For the selection of twelve (or eight) things out of twenty, consists of the operation of dividing the twenty things into two sets of twelve and eight, and rejecting one of the sets. Therefore (by the last rule), whichever set be rejected, the operation can be performed in

$$\frac{|\underline{20}}{|\underline{12}\,.\,|\underline{8}}$$

different ways.

*Question.*— Out of one hundred things, in how many ways can three things be selected ?

*Answer.*— By Rule VIII.,

$$\frac{|\underline{100}}{|\underline{97}\,.\,|\underline{3}};$$

or striking out from the numerator and the denominator all the successive factors from 1 to 97,

$$\frac{100 \cdot 99 \cdot 98}{\underline{|3}}$$

We observe that the numerator 100.99.98 expresses (Rule V.) the number of ways in which an *arrangement* of three things might be made out of one hundred things.

This suggests the following rule for the number of ways of selecting any number of things out of a larger number, which will often be found more convenient than Rule VIII., although both of course lead to the same result.

## RULE IX.

*Out of any given number of things,*

*the number of selections of two things may be obtained from the number of arrangements of two things, by dividing by* $\underline{|2}$;

*the number of selections of three things may be obtained from the number of arrangements of three things, by dividing by* $\underline{|3}$;

*the number of selections of four things may be obtained from the number of arrangements of four things, by dividing by* $\underline{|4}$;

*and so on.*

It will be sufficient to shew the reason of this rule in any particular case.

Suppose we have to make a selection of three things out of a given number of things; what is our choice in this case, compared with our choice in making an arrangement of three things.

The operation of making an *arrangement* of three things may be resolved into the two operations following, viz.:—

(1) To make a *selection* of three things out of the given things.

(2) To arrange in order the three selected things.

Therefore, by Rule I., the number of ways of making an *arrangement* of three things is equal to the number of ways of making a *selection* of three things, multiplied by the number of ways of arranging the three selected things.

But by Rule III., three things can be arranged in $|\underline{3}$ different ways.

Hence, the number of *arrangements* of three things, out of a greater number, is equal to the number of *selections* multiplied by $|\underline{3}$.

Or the number of *selections* of three things is equal to the number of *arrangements* divided by $|\underline{3}$.

And the same reasoning would apply if the number of things to be selected were any other instead of 3. Therefore the rule is true always.

The student being in possession of the two rules (VIII. and IX.) for writing down the number of ways in which any number of things can be selected out of a larger number, will, in any particular case, use the rule which may seem the more convenient. It will be observed that Rule VIII. gives the result in the more *concise* form when the number of things to be selected is a high number; but the fraction thus written down, though more concisely expressed, is not in such low terms as that which would be written down by Rule IX. Consequently, when the actual numerical value of the result is required, Rule IX. leaves the less work to be done, in cancelling out common factors from the numerator and the denominator. In many cases, it is simplest to take advantage of the principle of Rule VIII., that out of twenty things (suppose) the number of ways in which seventeen things can be selected is the same as the number of ways in which 20 − 17 or three things can be selected, and then to apply Rule IX. For, comparing the different forms of the result in this case, we observe that Rule VIII. gives

$$\frac{\underline{|20}}{\underline{|17}\,.\,\underline{|3}}$$

while Rule IX. gives

$$\frac{20.19.18.17.16.15.14.13.12.11.10.9.8.7.6.5.4}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16.17}$$

which might be simplified by dividing the numerator and denominator by the factors

$$4.5.6.7.8.9.10.11.12.13.14.15.16.17$$

But if we recognise the teaching of Rule VIII., that the number of ways of selecting seventeen things is the same as the number of ways of selecting three things, and then apply Rule IX. to find the number of ways of selecting three things, we can at once write down the result in the simple form

$$\frac{20.19.18}{1.2.3}.$$

*Question.*—Out of a basket of twenty pears at three a penny, how many ways are there of selecting six pennyworth?

*Answer.*—By Rule VIII., we can select eighteen out of twenty in as many ways as we can select two; and, by Rule IX., this can be done in

$$\frac{20.19}{1.2} \text{ or } 190$$

ways.

*Question.*—In how many ways can the same choice be exercised so as to include the largest pear?

*Answer.*—Taking the largest pear first, our only choice now lies in selecting seventeen out of the remaining nineteeen, which can be done (Rules VIII. and IX.) in

$$\frac{19.18}{1.2} \text{ or } 171$$

ways.

*Question.*—In how many ways can the same choice be exercised without taking the smallest pear?

*Answer.*—We have now to select eighteen pears out of nineteen. Therefore (Rules VIII. and IX.) our choice can be exercised in nineteen ways.

*Question.*—In how many ways can the same choice be exercised so as to include the largest, and not to include the smallest pear?

*Answer.*—Taking the largest pear first, we have then to choose seventeen more out of eighteen, which can be done (Rules VIII. and IX.) in eighteen ways.

*Question.*—Out of forty-two liberals and fifty conservatives, what choice is there in selecting a committee consisting of four liberals and four conservatives?

*Answer.*—The liberal committee-men can be chosen (by Rule VIII.) in

$$\frac{42.41.40.39}{1.2.3.4} \text{ or } 111930$$

different ways, and the conservative committee-men in

$$\frac{50.49.48.47}{1.2.3.4} \text{ or } 230300$$

different ways. Hence (by Rule I.) the whole choice can be exercised in

$$111930 \times 230300, \text{ or } 25777479000$$

different ways.

*Question.*—A company of volunteers consists of a captain, a lieutenant, an ensign, and eighty rank and file. In how many ways can ten men be selected so as to include the captain.

*Answer.*—Since the captain is to be one of the ten, the only choice lies in the selection of nine men out of the remaining eighty-two, which can be done (Rule VIII. or IX.) in

$$\frac{|\underline{82}}{|\underline{9}\,.\,|\underline{73}}$$

or

$$\frac{82.81.80.79.78.77.76.75.74}{|\underline{9}}$$

different ways.

*Question.*—In how many ways can ten men be selected so as to include *at least one* officer?

*Answer.*—By Rule VIII. ten men can be selected out of the whole company in

$$\frac{|\underline{83}}{|\underline{10}\,.\,|\underline{73}}$$

ways altogether. But the number of different ways that will include *no officer* will be the number of ways in which ten can be selected out of the eighty rank and file, that is, (by Rule VIII.)

$$\frac{\underline{|80}}{\underline{|10}\,.\,\underline{|70}}.$$

These must be subtracted from the whole number of ways in which ten men might be selected, and the remainder

$$\frac{\underline{|83}}{\underline{|10}\,.\,\underline{|73}} - \frac{\underline{|80}}{\underline{|10}\,.\,\underline{|70}}$$

will be the number of ways in which they may be selected so as to include at least one officer.

*Question.*—In how many ways can ten men be selected so as to include *exactly one* officer?

*Answer.*—The nine rank and file can be selected in

$$\frac{\underline{|80}}{\underline{|9}\,.\,\underline{|71}}$$

ways, and the one officer in three ways. Therefore the ten can be selected in

$$\frac{3 \times \underline{|80}}{\underline{|9}\,.\,\underline{|71}}$$

different ways.

*Question.*—There are fifteen candidates for admission into a society which has two vacancies. There are seven electors, and each can either vote for two candidates, or plump for one. In how many ways can the votes be given?

*Answer.* — Each voter can plump in fifteen ways, and can vote for two candidates in

$$\frac{15.14}{1.2} \text{ or } 105$$

ways (Rule IX). Therefore each elector can vote altogether in 120 ways. And there are seven electors; therefore all the votes can be given (by Rule II.) in

$$120 \times 120 \times 120 \times 120 \times 120 \times 120 \times 120$$
$$\text{or } 358318080000000$$

different ways.

It will be well to notice particularly the points which distinguish the next three examples.

In all of them we suppose twenty things of one class and six things of another class set before us, the individuals of each class being distinct; and in all of them a selection has to be made of three things out of each class. But while the first is a case of simple selection, in the second each set of three things has separately to be arranged in order, and in the third the whole six selected things have to be together arranged in order.

*Question.*— Out of twenty men and six women, what choice have we in selecting three men and three women?

*Answer.*— The men can be selected in

$$\frac{20.19.18}{1.2.3} \text{ or } 1140$$

different ways (Rule IX.), and the women in

$$\frac{6.5.4}{1.2.3} \text{ or } 20$$

different ways. Therefore, we have the choice of

$$1140 \times 20 \text{ or } 22800$$

different ways of making our selection.

*Question.*— Out of twenty men and six women, what choice have we in filling up six different offices, three of which must be filled by men, and the other three by women?

*Answer.*— We can allot the first three offices to three men in 20.19.18 or 6840 different ways (Rule IV.); and we can allot the other three offices to three women in 6.5.4 or 120 different ways. Therefore, we have the choice of

$$6840 \times 120 \text{ or } 820800$$

different ways of making our arrangement.

*Question.*— Out of twenty consonants and six vowels, in how many ways can we make a word, consisting of three different consonants, and three different vowels?

*Answer.*— We can select three consonants in

$$\frac{20.19.18}{1.2.3} \text{ or } 1140$$

different ways, and three vowels in

$$\frac{6.5.4}{1.2.3} \text{ or } 20$$

different ways. Therefore (by Rule I.), the six letters can be selected in

$$1140 \times 20, \text{ or } 22800$$

different ways, and when they are so selected, they can be arranged (by Rule III.), in $\underline{|6}$ or 720 different orders. Hence (by Rule I.), there are $22800 \times 720$ or 16416000 different ways of making the word.

*Question.*— Out of the twenty-six letters of the alphabet, in how many ways can we make a word consisting of four different letters, one of which must be always *a*?

*Answer.*— Since we are always to use *a*, we must choose three letters out of the remaining twenty-five. This can be done in

$$\frac{25.24.23}{1.2.3} \text{ or } 2300$$

ways. Then the whole set of four letters can be arranged in $\underline{|4}$ or 24 different orders. Hence we have the choice of

$$2300 \times 24, \text{ or } 55200$$

different ways of making the word.

The last answer might have been arrived at in another way, as follows:—

Without the limitation, we could make a word of four different letters in 26.25.24.23 or 358800 different ways. The question is how many of these will contain *a*. Now if all the 358800 words were written down on paper, since each is made of four letters, our paper would contain 358800 × 4, or 1435200 letters. And since no letter of the alphabet has been used with more favour than any other, it follows that each would occur 1435200 ÷ 26, or 55200 times. Therefore *a* must occur 55200 times; and since no word contains *a* more than once, 55200 words must contain *a*. That is, the number of words formed of four different letters of which *a* is one is 55200, as before.

*Question.*—Out of the twenty-six letters of the alphabet, in how many ways can we make a word consisting of four different letters, two of which must be *a* and *b*?

*Answer.*—We can choose the other two letters out of the remaining twenty-four in

$$\frac{24.23}{1.2} \text{ or } 276$$

ways, and then we can arrange the whole set of four letters in $\underline{|4}$ or 24 different orders. Hence we have the choice of

$$276 \times 24, \text{ or } 6624$$

different ways of making the word.

*Question.*—Out of twenty consonants and six vowels, in how many ways can we make a word consisting of three different vowels and two different consonants, one of the vowels being always *a*?

*Answer.*—We can choose the other two vowels in

$$\frac{5.4}{1.2} \text{ or } 10$$

ways, and the two consonants in

$$\frac{20.19}{1.2} \text{ or } 190$$

ways; hence our letters can be selected in 1900 ways, and when they are selected the set of five can be arranged in $\underline{|5}$ or 120 ways. Hence the whole number of ways of making the word is

$$1900 \times 120, \text{ or } 228000.$$

*Question.*—There are ten different situations vacant, of which four must be held by men, and three by women; the remaining three may be held by either men or women. If twenty male and six female candidates present themselves, in how many ways can we fill up the situations?

*Answer.*—The men's situations can be filled up in 20.19.18.17 or 116280 different ways, and the women's in 6.5.4 or 120 different ways. When this is done, there are nineteen persons left, all of whom are eligible for the other three situations. Hence these three can be filled up in 19.18.17 or 5814 different ways. Therefore, the whole election can be made in $116280 \times 120 \times 5814$, or 81126230400 different ways.

### ARRANGEMENTS OUT OF A NUMBER OF THINGS NOT ALL DIFFERENT.

We considered under Rule VII. the modifications of the case of Rule III., when the things out of which the arrangement is to be made are not all different. The corresponding modifications of Rule IV. are too intricate to be treated by a general rule in an elementary treatise on Arithmetic; but each case, as it arises, may be resolved into cases to which the preceding rules will apply. The manner of proceeding will be sufficiently illustrated by the following questions.

*Question.*—In how many ways can an arrangement of four letters be made out of the letters of the words *choice and chance?*

*Answer.*—There are fifteen letters altogether, of eight different sorts, viz., *c, c, c, c; h, h; a, a; n, n; e, e; o; i; d.* The different ways of selecting the four letters may, therefore, be classified as follows:

(1) all four alike,
(2) three alike and one different,
(3) two alike and two others alike,
(4) two alike and the other two different,
(5) all four different.

Now, the selection (1) can be made in only one way (viz. by selecting *c, c, c, c*), and when this selection of letters is made, they can be arranged in only one order;

therefore (1) gives rise to only one arrangement.

The selection (2) can be made in seven ways, for three letters alike can be selected in only one way (viz. *c, c, c*), and one different one in seven ways (*a, e, i, o, h, n, d*). And when this selection of letters is made, they can be arranged in four ways (Rule VII.);

therefore (2) gives rise to $7 \times 4$, or 28 arrangements.

The selection (3) can be made in $\frac{5 \cdot 4}{1 \cdot 2}$ or 10 ways (Rule IX.), since we have to select two out of the five pairs, *cc, hh, aa, nn, ee*. And when this selection of letters is made, they can be arranged in $\frac{\underline{|4}}{\underline{|2} \cdot \underline{|2}}$ or 6 ways (Rule VII.);

therefore (3) gives rise to $10 \times 6$, or 60 arrangements.

The selection (4) can be made in $5 \times 21$, or 105 ways, for we can select one of the five pairs, *cc, hh, aa, nn, ee* in five ways, and two out of the seven different sorts of letters that will then be left, in $\frac{7 \cdot 6}{1 \cdot 2}$ or 21 ways, (Rule IX.); and when this selection of letters is made, they can be arranged in $\frac{\underline{|4}}{\underline{|2}}$ or 12 ways (Rule VII.);

therefore (4) gives rise to $105 \times 12$, or 1260 arrangements.

The selection (5) of four different letters must be made out of the eight, *c*, *h*, *a*, *n*, *e*, *o*, *i*, *d*, therefore the number of arrangements which will come from such a selection must be 8 . 7 . 6 . 5 or 1680.

Hence, the whole number of arrangements of four letters out of the fifteen given letters is

$$1 + 28 + 60 + 1260 + 1680, \text{ or } 3029.$$

*Question.*—In how many ways can an arrangement of three things be made out of fifteen things, of which five are of one sort, four of another sort, three of another sort, and the remaining three of another sort?

*Answer.*—The three selected things may be either—

(1) all three alike,

or (2) two alike and one different,

or (3) all different.

Now, the selection of all three alike can be made in 4 ways, since we can take one of the four different sorts. And when this selection is made, the selected things can be arranged in only one order;

therefore (1) gives rise to only four arrangements.

The selection of two alike and one different can be made in 4 × 3, or 12 ways (Rule I.); for the two alike can be of any of the four sorts, and the one different of any one of the remaining three sorts. And when this selection is made, the things selected can be arranged in $\frac{\underline{|3}}{\underline{|2}}$, or three ways (Rule VII.);

therefore (2) gives rise to 12 × 3, or 36 arrangements.

And if all three selected things are to be different we shall have 4.3.2, or 24 arrangements (Rule IV.).

Hence, the whole number of arrangements of three things out of the fifteen given things is

$$4 + 36 + 24, \text{ or } 64.$$

*Question.*—In how many ways can an arrangement of five things be made out of the fifteen things given in the last question?

*Answer.*—The different ways of selecting five things may be classified as follows:—

(1) all five alike,
(2) four alike and one different,
(3) three alike and two others alike,
(4) three alike and two different,
(5) two alike, two others alike, and one different,
(6) two alike and three different.

Now by the application of Rules II., VII., IX., as in the preceding questions, it will be easily seen that

the selection (1) can be made in one way, and leads to one arrangement:

the selection (2) can be made in six ways, and leads to 6 × 5 or 30 arrangements:

the selection (3) can be made in twelve ways, and leads to 12 × 10 or 120 arrangements:

the selection (4) can be made in twelve ways, and leads to 12 × 20 or 240 arrangements:

the selection (5) can be made in twelve ways, and leads to 12 × 30 or 360 arrangements:

the selection (6) can be made in four ways, and leads to $4 \times 60$ or 240 arrangements.

Hence the whole number of different arrangements is

$$1 + 30 + 120 + 240 + 360 + 240, \text{ or } 991.$$

## RULE X.

*The whole number of ways in which a person can select some or all (as many as he pleases) of a given number of things, is one less than the continued product of* 2 *repeated the given number of times.*

For since he is at liberty to take none or all or as many as he pleases of the different things, he can dispose of each thing in two ways, for he can either take it or leave it. Now suppose there are five things, then he can act altogether in

$$2 \times 2 \times 2 \times 2 \times 2$$

different ways. But if he is not to reject *all* the things, the number of courses open to him will be one less than this, or

$$2 \times 2 \times 2 \times 2 \times 2 - 1.$$

And the same reasoning would apply if the number of things were any other number instead of five. Hence, the rule will be true always.

*Question.*—One of the stalls in a bazaar contains twenty-seven articles exposed for sale. What choice has a purchaser?

*Answer.*—He may buy either one thing or more, and there are twenty-seven things: therefore (by Rule X.), the number of courses open to him is one less than the continued product of twenty-seven *twos*, or 134217727.

*Question.*—What is the greatest number of different amounts that can be made up by selection from five given weights?

*Answer.*—By Rule X., $2 \times 2 \times 2 \times 2 \times 2 - 1$ or 31.

The different selections will not always produce different sums. Hence, we cannot always make thirty-one different sums. But under favourable circumstances, as, for instance, when the weights are 1lb., 2lbs., 4lbs., 8lbs., 16lbs., all the different selections will produce different sums, and then the number of different sums is thirty-one. Hence, thirty-one is the *greatest* number of different weights that can be made by a selection from five given weights.

In the case of the five weights, 1lb., 2lbs., 4lbs., 8lbs., 16lbs., the thirty-one different amounts that can be weighed consist of every integral number of pounds from one to thirty-one.

Thus, with single weights, we can weigh the following numbers of pounds, viz., 1, 2, 4, 8, 16; and then we have

| | | |
|---|---|---|
| $3=1+2$, | $7=1+2+4$, | $15=1+2+4+8$, |
| $5=1+4$, | $11=1+2+8$, | $23=1+2+4+16$, |
| $6=2+4$, | $13=1+4+8$, | $27=1+2+8+16$, |
| $9=1+8$, | $14=2+4+8$, | $29=1+4+8+16$, |
| $10=2+8$, | $19=1+2+16$, | $30=2+4+8+16$, |
| $12=4+8$, | $21=1+4+16$, | |
| $17=1+16$, | $22=2+4+16$, | $31=1+2+4+8+16$. |
| $18=2+16$, | $25=1+8+16$, | |
| $20=4+16$, | $26=2+8+16$, | |
| $24=8+16$, | $28=4+8+16$, | |

It may be observed that, if we had a 32lbs. weight, by adding it to each of the sets already obtained, we should get all the numbers from 33 to 63 inclusive; hence all the weights

1lb. 2lbs. 4lbs. 8lbs. 16lbs. 32lbs.

would enable us to weigh any number of pounds from 1 to 63.

Then the addition of a 64lbs. weight, would enable us to weigh any number up to 127, and so on.

*Question.*—What is the greatest number of different amounts that can be weighed with five weights, when each weight may be put into either scale?

*Answer.*—This is not a direct example of our rule, but it may be solved on a like principle to that by which the rule itself was established.

Each weight can be disposed of in three ways, that is, it can be placed either in the weight-pan, or in the pan with the substance to be weighed, or it can be left out altogether. Hence, all the weights can be disposed in $3 \times 3 \times 3 \times 3 \times 3$, or 243 ways (Rule II). But one of these ways would consist in rejecting *all* the weights; this must be cast out, and then there remain 242 ways. But in the most favourable case, half of these ways would consist in placing a less weight in the weight-pan than in the other, and these must be cast out. Hence there remain 121 different amounts that can be weighed under the most favourable circumstances with five weights, when it is permitted to place weights in the pan with the substance to be weighed.

The weights 1lb., 3lbs., 9lbs., 27lbs., 81lbs., will afford an instance of the most favourable case. In this instance, the 121 amounts that can be weighed consist of every integral number of pounds from 1 to 121. Thus—

| | |
|---|---|
| $2 = 3 - 1$, | $15 = 27 - 9 - 3$, |
| $4 = 3 + 1$, | $16 = 27 - 9 - 3 + 1$, |
| $5 = 9 - 3 - 1$, | $17 = 27 - 9 - 1$, |
| $6 = 9 - 3$, | $18 = 27 - 9$, |
| $7 = 9 - 3 + 1$, | $19 = 27 - 9 + 1$, |
| $8 = 9 - 1$, | $20 = 27 - 9 + 3 - 1$, |
| $10 = 9 + 1$, | $21 = 27 - 9 + 3$, |
| $11 = 9 + 3 - 1$, | $22 = 27 - 9 + 3 + 1$, |
| $12 = 9 + 3$, | $23 = 27 - 3 - 1$, |
| $13 = 9 + 3 + 1$, | $24 = 27 - 3$, |
| $14 = 27 - 9 - 3 - 1$, | &c. |

## RULE XI.

*The whole number of ways in which a person can select some or all (as many as he pleases) out of a number of things which are not all different, is one less than the continued product of the series of numbers formed by increasing by unity the several numbers of things of the several sorts.*

Thus, suppose we have the letters—

$$a, a, a, a, a,$$
$$b, b, b,$$
$$c, c, c, c,$$
$$d,$$
$$e,$$

viz., five of one sort, three of another, four of a third sort, one of a fourth sort, and one of a fifth.

The numbers of letters in the several classes are 5, 3, 4, 1, 1, and these, severally increased by unity, give the new series of numbers 6, 4, 5, 2, 2. The rule states that the whole number of ways in which a person may take some or all (as many as he pleases) of the given letters, is

$$6 \times 4 \times 5 \times 2 \times 2 - 1, \text{ or } 479.$$

The reason of the rule will be seen from the following considerations. Suppose the person were at liberty to take *none*, or all, or as many as he pleased of the letters. He could then dispose of the five *a*, *a*, *a*, *a*, *a* in *six* different ways, for he might take 5 or 4 or 3 or 2 or 1 or *none* of them. So he could dispose of the three *b*, *b*, *b*, in *four* ways, for he might take 3 or 2 or 1 or *none* of them. Similarly he could dispose of the *c*, *c*, *c*, *c* in *five* ways, and of the *d* in *two* ways, and of the *e* in *two* ways. Hence he might act altogether, in

$$6 \times 4 \times 5 \times 2 \times 2$$

different ways (Rule II). But if he is not to reject *all* the things, the number of courses open to him will be one less than this, or

$$6 \times 4 \times 5 \times 2 \times 2 - 1.$$

And the same reasoning would apply to any other case. Hence we may accept the rule as true always.

*Question.*— In how many ways can two booksellers divide between them 200 copies of one book, 250 of another, 150 of a third, and 100 of a fourth?

*Answer.*—Either man can take any number of books, but not either *none* or *all.* Therefore, the number of ways is one less than that given by the rule: *i. e.*, the division can be made in

$$201 \times 251 \times 151 \times 101 - 2$$

different ways; or in

$$769428201 - 2, \text{ or } 769428199$$

different ways.

## EXAMPLES ON CHOICE.

1.—Having four seals and five sorts of sealing wax, in how many ways can we seal a letter?

2.—There are five first-class carriages, eight second-class, seven third-class and three luggage-vans. In how many ways can a train be made consisting of one of each?

3.—How many changes can be rung upon eight bells? And in how many of these will an assigned bell be rung last?

4.—Out of a class of twelve boys, in how many ways can three boys be called up to say lessons?

5.—In how many ways can a set of twelve black and twelve white draught-men be placed on the black squares of a draught-board?

6.—In how many ways can a set of chess-men be placed on a chess-board?

7.—In how many ways can we arrange the letters of the word *possessions?*

8.—In how many ways can we arrange the letters of the words *choice and chance?*

9.—In how many ways can a triangle be formed, having its angular points at three of the angular points of a given hexagon?

10.—There are three teetotums, having respectively 6, 8, 10 sides. In how many ways can they fall? and in how many of these will two aces be turned up?

11.—A company of soldiers consists of three officers, four sergeants and sixty privates. In how many ways can a detachment be made consisting of an officer, two sergeants and twenty privates? In how many of these ways will the captain and the senior sergeant appear?

12.—In how many ways can four persons sit at a round table, so that all shall not have the same neighbours in any two arrangements?

13.— In how many ways can seven persons sit as in the last question? And in how many of these will two assigned persons be neighbours? And in how many will an assigned person have the same two neighbours?

14.—Out of a party of twelve ladies and fifteen gentlemen, in how many ways can four gentlemen and four ladies be selected for a dance?

15.—Out of twenty consonants and six vowels, in how many ways can a word be made, consisting of three different consonants and two different vowels, without placing all the consonants together?

16.—Out of twenty consonants and ten vowels, in how many ways can a word be formed consisting of three different vowels and three different consonants, the vowels and consonants being placed alternately?

17.—In how many ways can the foregoing questions be arranged, so that no question of *combination* shall come before any question of *permutation?*

18.—A plaything consists of eighteen cubical blocks; on each side of five of them a head is painted, on each side of seven a body, and on each side of six a pair of legs. How many different figures can be made by piecing them together?

19.—Having five pairs of gloves, in how many ways can a person select a right-hand and a left-hand glove which are not pairs?

20.—How many numbers less than 10,000 have a five in their arithmetical expression, and how many of them are divisible by five without remainder?

21.—In how many ways can a school of ninety boys divide themselves, so that twenty-four play football, twenty-two play cricket, thirty drill, four play racquets, and ten take a walk?

22.—From five apples, six pears, and three oranges, in how many ways can a person take fruit?

23.—A man has ten shares in the Great Western Railway Company, twelve in the North Western, seven in the Great Northern, two in the Great Eastern, five in the South Western. In how many ways can he sell shares?

24.—How many different signals can be made with a set of ten flags, using four at a time, (1) on a single mast, and (2) on a three-masted ship?

## CHAPTER II.

### CHANCE.

"There is very little chance of fine weather."

"Is there much chance of his recovery?"

"There is no chance of finding it."

"There is a great probability of war."

"This is a more probable result than the other."

"That is more likely to be mine than yours."

"There is less chance of her coming than of his."

—These are expressions in common use amongst us; the very commonness of their use shows that people in general have some idea of chance, and some conception of different degrees of probability in the occurrence of doubtful events. All understand what is meant by much chance and little chance; they distinguish events as very probable, probable, improbable, or very improbable; but no attempt is made in common conversation to measure with any accuracy the amount of probability attaching to any given event. If a Doctor is asked what chance there is of a patient's recovery, he may answer that there is much chance or little chance, but he cannot express with any precision the exact magnitude of his hope or of his fear. Yet his expectation of the event has a certain magnitude. He has a greater expectation of this patient's recovery than he has of the recovery of another, whose symptoms are more aggravated, and less expectation than in another case where the constitution is stronger. His expectation has a definite value, and if he were a sporting man, he would be prepared to offer or take certain definite odds on the event. But in common language, this definite amount of expectation or probability cannot be precisely expressed, because we have no recognised standard with which to compare it, no recognised amount of expectation or probability by which to measure it.

In fact, in describing the magnitude of any expectation which we entertain, we are in the same position as if we had to describe the length of a room, or the height of a tower, to a man who was not acquainted with a foot or a yard, or any of our standards of length. We could speak of the room as very long or very short, we could speak of the tower as very high or very low, but without some standard length recognised alike by ourselves and those whom we addressed, we could not give an accurate answer to either of the questions, How long is the room? or How high is the tower?

So when we are asked what chance we think there is of a fine afternoon, we may say that there is much chance or little chance, or we may even go further, and establish in our own minds a scale of expressions, distinguishing the different degrees of probability in some such way as follows:—

It is certain not to rain.
It is very unlikely to rain.
It is unlikely to rain.
It is as likely to rain as not.
It is likely to rain.
It is very likely to rain.
It is certain to rain.

but these expressions except the first, fourth, and last, are vague and indefinite, nor can we ever be sure that those with whom we are conversing attach exactly the same idea to each expression that we do.

This vagueness is of little consequence in common life, because in most cases it is impossible to make an accurate estimate of a chance, and the expressions are, perhaps, as accurate as the estimates themselves which we wish to express. But there are other classes of events concerning which it is possible to form accurate estimates of their degree of probability or likelihood of happening, and in these cases it is well to have some more precise method of expressing different degrees, than is afforded by the common expressions which we have quoted.

We must observe at the outset, that we use the words *chance* and *probability* as strictly synonymous. In common language, it is usual to prefer the former word when the expectation is small, and the latter when it is large. Thus we generally hear of "little chance," or of "great probability," but not so often of "great chance," or "little probability." This distinction, however, is not universal, and we shall entirely disregard it, using the two words chance and probability in the same sense.

It will be seen that probability always implies some ignorance on the part of the person entertaining the expectation, and the amount of probability attaching to any event will depend upon the degree of this ignorance. With omniscience, degrees of probability are incompatible; for omniscience implies certainty, and certainty precludes doubt, and degrees of probability are the measures of doubt.

Hence, there is no such thing as the *absolute* probability of an event, all probability being conditional on our ignorance, and varying when that condition varies. Thus the same event will be unequally probable to different persons, whose knowledge of the circumstances relating to the event is different. And to the same person, the expectation of any event will be affected by any accession of knowledge concerning the event.

For instance, suppose we see a friend set out with five other passengers in a ship whose crew number thirty men: and suppose we presently hear that a man fell overboard on the passage and was lost. So long as our knowledge is confined to the fact that one individual only has been lost out of the thirty-six on board, the probability that it is our friend is very small. The odds against it would be said to be thirty-five to one. But suppose our knowledge is augmented by the news that the man who has been lost is a passenger; though we still feel that it is equally likely to be any of the other five passengers, yet our apprehension that it is our friend becomes much greater than it was before. The odds against it are now described as five to one. Thus the probability that our friend is lost is seen to be entirely conditional on the respective degrees of our knowledge and ignorance; and so soon as our ignorance vanishes — so soon as we know all about the event, and become *as far as that event is concerned* omniscient,—then there no longer remains a question of probability; the probability is replaced by certainty.

This example will also illustrate the meaning of the ratio of probabilities. Since each of the passengers was equally likely to have been lost, it was evidently always six times as likely that the man lost was *some passenger*, as that it was our friend. So it was five times as likely that it was *a passenger, but not our friend*, as that it was our friend. Therefore, also, the probability that it was *a passenger, but not our friend*, was to the probability that it was a passenger in the *ratio* of 5 to 6.

Let us suppose another case. A number of articles are placed in a bag, and amongst them are three balls, alike in all respects, except that two of them are coloured white and the third black: all the other articles we will suppose to be coins, or anything distinguishable without difficulty from balls.

We present this bag to a stranger, and we give him leave to put in his hand in the dark, and to take out any one article he likes. But before he does this, we may consider what chance there is of his taking out a ball, or what chance there is of his taking out the black ball. Obviously we cannot form any accurate estimate of this chance, because it must depend upon the wants or the taste of the stranger influencing his will, whether he will prefer to take a ball or a coin, and being ignorant of his will in the matter, we cannot say whether it is likely or unlikely that he will select a ball.

But it is axiomatic, that if he draws a ball at all, it is twice as likely to be a white ball as to be a black one, or the respective chances of his drawing white or black are in the ratio of 2 to 1, and these chances are respectively two-thirds and one-third of the chance that he draws a ball at all.

We now proceed to show how the magnitude of a chance may be definitely expressed. We have already pointed out that the expressions used in common language are wanting in definiteness and precision, and we compared the expedients by which degrees of probability are usually indicated to the attempts which we should make to give an idea of the length of a room to a person unacquainted with the measures of a foot and a yard.

Now we observe, that the difficulty in this latter case ceases, so soon as the person with whom we are speaking agrees with us in his conception of any definite length whatever. If he can once recognise what we mean by the length of a hand, for instance, we can express to him with perfect accuracy the length of the room as so many hands; or, if he have an idea of what a mile is, we can precisely express the length of the room as some certain fraction of a mile. So, also, as soon as we have fixed upon any standard amount of probability that can be recognised and appreciated by all with whom we have to do, we shall be able to express any other amount of probability numerically by reference to that standard. The numbers 2, 3 would express probabilities twice or three times as great as the standard probability; and the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$ would express probabilities half, one-third, or two-thirds of the standard.

Now, it matters not how great or how small the standard be, provided it be a probability which all can recognise, and which all will alike appreciate. This is, indeed, the one essential which it has to fulfil; it must be such that all persons will make the same estimate of it. And that which best satisfies this condition, and, therefore, the most convenient standard with which to compare other probabilities, is that supreme amount of probability which attaches to an event which we know to be *certain* to happen. All understand what *certainty* is: it is a standard which all estimate alike. *Certainty*, therefore, shall be our unit of probability; and other degrees of probability shall be expressed as fractions of certainty.

But it may be asked, Is certainty a degree of probability at all, or can smaller degrees of probability be said to have any ratio to certainty? Yes. For if we refer to the instance already cited of the six passengers in the ship, we observe that the chance of the lost man being a passenger is six times as great as the chance of his being our friend. This is the case however great our ignorance of the circumstances of the event; and it will evidently remain true until we attain to some knowledge which affects our friend differently from his fellow-passengers. But the news that the lost man was a passenger does not affect one passenger more than another. Therefore, after receiving this news, it will still hold good that the chance of the lost man being a passenger is six times as great as the chance of its being our friend. But it is now certain that the lost man was a passenger; therefore the probability that it was our friend is one-sixth of certainty. Again in the instance of the balls and coins in the bag, we have already noticed that the chances of drawing white or black are respectively two-thirds and one-third of the chance of drawing a ball at all. And this is the case whatever this last chance may be. But suppose the man tells us that he is drawing a ball, not a coin, then this last chance becomes certainty; and therefore the chances of drawing white or black, become respectively two-thirds and one-third of certainty. Thus it is seen that certainty, while it is the supreme degree, is some degree of probability, or is such that another degree of probability can be compared to it and expressed as a fraction of it.

Of course, when we use unity to express certainty, the probability of the lost passenger being our friend will be expressed by the fraction $\frac{1}{6}$, and the chances of the ball drawn being white or black, will be expressed by the fractions $\frac{2}{3}$ and $\frac{1}{3}$.

After the explanations which we have already given, the reader will have no difficulty in accepting the following axiom.

## AXIOM.

*If an event can happen in a number of different ways (of which only one can occur), the probability of its happening at all is the sum of the several probabilities of its happening in the several ways.*

For instance, let the event be the falling of a coin. It can fall either *head* or *tail*, and only one of these ways can occur. The probability that it falls at all must be made up by addition of the probability that it falls *head* and the probability that it falls *tail*.

Again, let the event be that either *A*, *B*, or *C* should win a race in which there are any number of competitors. The event can happen in three ways, viz., by *A* winning, by *B* winning, or by *C* winning; and only one of these ways can occur. The probability that one of the three should win is equal to the sum of the probabilities that *A* should win, that *B* should win, and that *C* should win.

This is only saying that if a man would give £2 for *A's* chance of the prize, £3 for *B's* chance, and £4 for *C's* chance, he would give £2 + £3 + £4, or £9 for the promise that he should have the prize if any one of the three should win.

Again, if $\frac{1}{10}$ be the chance of a shot aimed at a target hitting the bull's eye, $\frac{1}{6}$ the chance of its hitting the first ring, and $\frac{1}{4}$ the chance of its hitting the outer ring, the chance that it hits one of these, *i. e*, the chance of its hitting the target at all, is $\frac{1}{10} + \frac{1}{6} + \frac{1}{4}$, or $\frac{31}{60}$.

## RULE I.

*The probability of an event not happening is obtained by subtracting from unity the probability that it will happen.*

For it is certain that it will either happen or not happen, or the probability that it will either happen or not happen is unity; and only one of these two (the happening and the not happening) can occur. Therefore, by the axiom, unity is the sum of the probabilities of the event happening and not happening; or the probability of its not happening is obtained by subtracting from unity the probability of its happening.

EXAMPLES.—If the chance of an event happening is $\frac{2}{5}$, the chance of its not happening is $1 - \frac{2}{5}$, or $\frac{3}{5}$.

If the chance of a plan succeeding is $\frac{7}{10}$, the chance of its failing is $1 - \frac{7}{10}$, or $\frac{3}{10}$.

If the chance of a shot hitting a target be $\frac{31}{60}$, the chance of its missing is $\frac{29}{60}$.

If the chance of $A$ winning a race be $\frac{1}{6}$, and the chance of $B$ winning it $\frac{1}{8}$, the chance that neither should win is $\frac{17}{24}$. For, by the axiom, the chance that one of them should win is $\frac{1}{8} + \frac{1}{6}$, or $\frac{7}{24}$; and therefore, by Rule I., the chance that this should not happen is $1 - \frac{7}{24}$, or $\frac{17}{24}$.

DEFINITION I.—Two probabilities which together make up unity, are called *complementary* probabilities.

DEFINITION II.—When it is said that the odds are three to two against an event, it is meant that the chance of the event failing is to the chance of its happening as three to two; and when it is said that the odds are three to two in favour of an event, it is meant that the chance of its happening is to the chance of its failing as three to two; and so for any other numbers.

## RULE II.

*If the odds be three to two against an event, the chance of the event not happening is*

$$\frac{3}{3+2},$$

*and the chance of its happening is*

$$\frac{2}{3+2};$$

*and so for any other numbers; the numerators of the two fractions being the two given numbers, and their common denominator the sum of the numbers.*

For the two fractions satisfy the condition required by Rule I., viz., that their sum should be unity, and that required by the definition, viz., that their ratio should be the same as the ratio expressing the given odds. Similarly,—

*If the odds be three to two in favour of an event, the chance of the event happening is*

$$\frac{3}{3+2},$$

*and the chance of its not happening is*

$$\frac{2}{3+2};$$

*and so for any other numbers; the numerators of the two fractions being the two given numbers, and their common denominator the sum of the numbers.*

EXAMPLES.—If the odds be ten to one against an event, the chance of its happening is $\frac{1}{11}$, and the chance of its failing is $\frac{10}{11}$.

If the odds be five to two in favour of the success of an experiment, the probability of success is $\frac{5}{7}$, and the probability of failure is $\frac{2}{7}$.

## RULE III.

*If an event can happen in five ways, and fail in seven ways, and if these twelve ways are all equally probable, and only one of them can occur, the odds against the event are seven to five, and the chances of its happening and failing are respectively*

$$\frac{5}{12} \text{ and } \frac{7}{12};$$

*and similarly for any other numbers.*

For since the event must either happen or fail, one of the twelve ways must occur; therefore the sum of their several probabilities is unity. But all the twelve ways are equally probable. Therefore the chance of the occurrence of any particular one is $\frac{1}{12}$, and the chance of the occurrence of one of the five which cause the event to happen is five times this, or $\frac{5}{12}$. So the chance of the occurrence of one of the seven which cause the event to fail is $\frac{7}{12}$.

Suppose, for example, that a die has twelve faces, of which five are coloured white and seven black. A person throws the die, and is to receive a prize if it fall white.

The odds are seven to five against his winning the prize. The chance that he wins is $\frac{5}{12}$, and the chance that he loses is $\frac{7}{12}$.

For all the twelve faces are equally likely to turn up, and one must turn up. Therefore the chance of any particular face turning up is $\frac{1}{12}$, and the chance of a white face turning up is five times this, or $\frac{5}{12}$.

*Or we might put it thus:*—Since there are five white and seven black faces, it is axiomatic that the chance of a white face is to the chance of a black face as five to seven. Now as soon as it is certain that the die is to be thrown, it is certain that either a white or a black face must turn up. The two chances must therefore now make up unity. But they still retain the ratio of five to seven, therefore they become respectively

$$\frac{5}{5+7} \text{ and } \frac{7}{5+7}.$$

And in the same way we might reason if the numbers were any other.

*Question.*—A party of twenty-three persons take their seats at a round table; shew that the odds are ten to one against two specified persons sitting together.

*Answer.*—Call the two specified persons $A$ and $B$. Then besides $A$'s place (wherever it may be) there are twenty-two places, of which two are adjacent to $A$'s place and the other twenty not adjacent. And $B$ is equally likely to be in any of these twenty-two places. Therefore (Rule III.), the odds are twenty to two, or ten to one, against his taking a place next to $A$.

The last rule may be expressed in a somewhat different form as follows:—

## RULE IV.

*If there be a number of events of which one must happen and all are equally likely, and if any one of a (smaller) number of these events will produce a certain result which cannot otherwise happen, the probability of this result is expressed by the ratio of of this smaller number to the whole number of events.*

For instance; if a man has purchased five tickets in a lottery, in which there are twelve tickets altogether and only one prize, his chance of the prize would be expressed by the ratio 5 : 12, or by the fraction $\frac{5}{12}$.

For convenience of reference we have given distinct numbers to the two Rules III. and IV., although they are only different statements of one and the same principle. This will be immediately seen, by considering the case of the lottery just instanced. We might at once have said that there were twelve ways of drawing a ticket, and five of these would cause the man to win, while the other seven would cause him to lose. Rule III. is therefore immediately applicable.

*Question.*—The four letters *a*, *e*, *m*, *n* are placed in a row at random: what is the chance of their standing in such order as to form an English word?

*Answer.*—The four letters can stand in $\underline{|4}$ or twenty-four different orders (*Choice*, Rule III.): all are equally likely and one must occur. And four of these will produce an English word—

*mane, mean, name, amen.*

Hence by the rule, the required chance is $\frac{4}{24}$ or $\frac{1}{6}$.

*Question.*—What is the chance of a year, which is not leap year, having fifty-three Sundays?

*Answer.*—Such a year consists of fifty-two complete weeks, and one day over. This odd day may be any of the seven days of the week, and there is nothing to render one more likely than another. Only one of them will produce the result that the year should have fifty-three Sundays. Hence (Rule IV.), the chance of the year having fifty-three Sundays is $\frac{1}{7}$.

*Question.* — What is the chance that a leap year, selected at random, will contain fifty-three Sundays?

*Answer.* — Such a year consists of fifty-two complete weeks, and two days over. These two days may be

Sunday and Monday,
Monday and Tuesday,
Tuesday and Wednesday,
Wednesday and Thursday,
Thursday and Friday,
Friday and Saturday,
Saturday and Sunday,

and all these seven are equally likely. Two of them (the first and last) will produce the required result. Hence (Rule IV.) the chance is $\frac{2}{7}$.

*Question.* — What is the chance that a year which is known not to be the last year in a century should be leap year?

*Answer.* — The year may be any of the remaining ninety-nine of any century, and all these are equally likely; but twenty-four of them are leap years. Therefore (Rule III.) the chance that the year in question is a leap year is $\frac{24}{99}$ or $\frac{8}{33}$.

*Question.*—Three balls are to be drawn from an urn which contains five black, three red, and two white balls. What is the chance of drawing two black balls and one red?

*Answer.*—Since there are ten balls altogether, three balls can be drawn in $\frac{10.9.8}{1.2.3}$, or 120 different ways, all equally likely. Now, two black balls can be selected in $\frac{5.4}{1.2}$, or ten ways, and one red in three ways, Hence, two black balls and one red can be drawn in $10 \times 3$, or 30 different ways. Thus we have 120 different ways of drawing three balls, whereof 30 ways will give two black and one red. Hence, when three balls are drawn the chance that they should be two black and one red is (by Rule IV.)

$$\frac{30}{120} \text{ or } \frac{1}{4}.$$

*Question.*—If from a lottery of thirty tickets, marked 1, 2, 3, &c., four tickets be drawn, what is the chance that those marked 1 and 2 are among them?

*Answer.*—Four tickets can be drawn out of thirty in $\frac{30.29.28.27}{1.2.3.4}$ ways. Four tickets can be drawn, so as to include those marked 1 and 2, in $\frac{28.27}{1.2}$ ways.

Hence, when four are drawn, the chance that these two are included is

$$\frac{28.27}{1.2} \div \frac{30.29.28.27}{1.2.3.4} = \frac{3.4}{29.30} = \frac{2}{145}.$$

The odds are, therefore, 143 to 2 against the event.

*Question.*—*A* has three shares in a lottery where there are three prizes and six blanks. *B* has one share in another, where there is but one prize and two blanks. Shew that *A* has a better chance of winning a prize than *B*, in the ratio of 16 to 7.

*Answer.*—*A* will get a prize unless his three tickets all prove blank. Now, three tickets can be selected in $\frac{9.8.7}{1.2.3}$, or 84 ways; and they can be selected so as to be all blank in $\frac{6.5.4}{1.2.3}$, or 20 ways. Hence the chance that they should be all blank is $\frac{20}{84}$ or $\frac{5}{21}$; and, therefore, the chance that this should not be so, or that *A* gets at least one prize, is $1 - \frac{5}{21}$, or $\frac{16}{21}$. But it is evident that the chance that *B* gets a prize is (Rule IV.) $\frac{1}{3}$ or $\frac{7}{21}$. Therefore, *A* has a better chance than *B* in the ratio of 16 to 7.

*Question.*—If four cards be drawn from a pack, what is the chance that there will be one of each suit?

*Answer.*—Four cards can be selected from the pack in $\frac{52.51.50.49}{1.2.3.4}$ or 270725 ways (*Choice*, Rule IX.); but four cards can be selected so as to be one of each suit in only $13 \times 13 \times 13 \times 13$ or 28561 ways (*Choice*, Rule II.). Hence the chance is

$$\frac{28561}{270725} \text{ or a little more than } \frac{1}{10}.$$

*Question.*—If four cards be drawn from a pack, what is the chance that they will be marked *one, two, three, four?*

*Answer.*—There are $4 \times 4 \times 4 \times 4$, or 256 ways of drawing four cards thus marked, and 270725 ways of drawing four cards altogether. Hence, the chance is

$$\frac{256}{270725},$$

or the odds are more than 1000 to 1 against it.

*Question.*—In a bag there are five white and four black balls. If they are drawn out one by one, what is the chance that the first will be white, the second black, and so on alternately?

*Answer.*—There are nine balls, five of one sort, four of another: they can, therefore, be arranged in

$$\frac{\underline{|9}}{\underline{|4}\,.\,\underline{|5}}, \text{ or } 126$$

different orders (*Choice*, Rule VII.). The balls are equally likely to be drawn in any of these orders; therefore, the chance that they should be drawn in the particular order, *white—black—white—&c.*, is $\frac{1}{126}$. That this order of colour corresponds to only one of the 126 arrangements is a direct consequence of our having disregarded all individuality among balls of the same colour when we calculated that number. (See *Choice*, page 33.).

*Question.*—In a bag are five red balls, seven white balls, four green balls, and three black balls. If they be drawn one by one, what is the chance that all the red balls should be drawn first, then all the white ones, then all the green ones, and then all the black ones?

*Answer.*—The nineteen balls can be arranged in

$$\frac{|\underline{19}}{|\underline{5}\,.\,|\underline{7}\,.\,|\underline{4}\,.\,|\underline{3}}$$

different orders (*Choice*, Rule VII.). All these are equally likely, and therefore the chance of any particular order is

$$\frac{|\underline{5}\,.\,|\underline{7}\,.\,|\underline{4}\,.\,|\underline{3}}{|\underline{19}}.$$

This will be the chance required, for all individuality among balls of the same colour has been disregarded; only one of the different arrangements will give the order of colours prescribed in the question.

*Question.*—Out of a bag containing 12 balls, 5 are drawn and replaced, and afterwards 6 are drawn. Find the chance that exactly 3 balls were common to the two drawings.

*Answer.*—The second drawing could be made altogether in

$$\frac{|\underline{12}}{|\underline{6}\,.\,|\underline{6}}, \text{ or } 924$$

ways. But it could be made so as to include exactly 3 of the balls contained in the first drawing, in

$$\frac{|\underline{5}}{|\underline{3}\,.\,|\underline{2}} \times \frac{|\underline{7}}{|\underline{3}\,.\,|\underline{4}}, \text{ or } 350$$

ways; for it must consist of a selection of 3 balls out of the first 5, and a selection of 3 balls out of the remaining 7 (*Choice*, Rules VIII. and II.). Hence, the chance that the second drawing should contain exactly 3 balls common to the first, is $\frac{350}{924}$ or $\frac{25}{66}$.

As the respective probabilities of various throws, with two common dice, are of practical interest, in their bearing upon such games as Backgammon, it may be well to discuss this case with some completeness.

It will be observed that as each die can fall in six ways, the whole number of ways in which the two dice can fall is $6 \times 6$ or 36. But these 36 different ways are not practically different throws, since, for example, it makes no difference in practice whether the first die falls *six* and the second *five*, or the first *five* and the second *six*. The number of practically different throws is, in fact, only 21, the 36 different ways of the dice falling being made up of six unique ways—

1 and 1, 2 and 2, 3 and 3, 4 and 4, 5 and 5, 6 and 6,

and 30 other ways, consisting of 15 essentially different throws, each repeated twice: thus—

1 and 2, 1 and 3, 1 and 4, 1 and 5, 1 and 6,
2 and 1, 3 and 1, 4 and 1, 5 and 1, 6 and 1,

2 and 3, 2 and 4, 2 and 5, 2 and 6,
3 and 2, 4 and 2, 5 and 2, 6 and 2,

3 and 4, 3 and 5, 3 and 6,
4 and 3, 5 and 3, 6 and 3,

4 and 5, 4 and 6,
5 and 4, 6 and 4,

5 and 6.
6 and 5.

Since each die is equally likely to fall in all different ways, the 36 different ways of the two dice falling are all equally likely; and, therefore, when the dice are thrown the probability of any particular way is $\frac{1}{36}$. But it cannot be said that all throws are equally probable, because *six-five* results practically in two ways out of the 36 ways of the dice falling, whereas *six-six* results in only one way. The correct statement is, that the probability of any assigned throw is $\frac{1}{36}$ if that assigned throw be *doublets*; but it is twice as much or $\frac{1}{18}$ if the assigned throw be *not doublets*. Thus the chance of throwing *six-three* is $\frac{1}{18}$, but the chance of throwing *three-three* is $\frac{1}{36}$.

*Question.*—When two dice are thrown, what is the chance that the throw will be greater than 8?

*Answer.*—Out of the 36 ways in which the dice can fall, there are six which give a result greater than 8, viz.:—

| | | |
|---|---|---|
| 5 and 4, | 5 and 6, | 5 and 5, |
| 4 and 5, | 6 and 5, | 6 and 6. |

Hence the required chance is $\frac{6}{36}$ or $\frac{1}{6}$.

*Question.*—What is the chance of throwing at least one ace?

*Answer.*—Of the thirty-six ways in which the dice can fall, eleven give an ace. Hence, the chance is $\frac{11}{36}$.

*Question.*—What is the chance of making a throw which shall contain neither an ace nor a six?

*Answer.*—Of the thirty-six ways, there are sixteen which involve neither one nor six. Hence, the chance is $\frac{16}{36}$ or $\frac{4}{9}$.

This question, as well as the preceding one, may be more conveniently solved by Rule VI.

*Question.*—What are the odds against throwing doublets?

*Answer.*—Of the thirty-six ways in which the dice can fall, six give doublets. Therefore, the chance for doublets is $\frac{6}{36}$ or $\frac{1}{6}$, and the chance against doublets $\frac{5}{6}$ (Rule III.). Therefore, the odds are five to one against doublets.

*Or we might reason thus:*—However the first die fall, the second die can fall in six ways, of which only one way will give the same number as on the first die. Hence, the odds are five to one against the second die falling the same way as the first, or the odds are five to one against doublets.

*Question.*—In one throw with a pair of dice, what is the chance that there is neither an ace nor doublets?

*Answer.*—The dice can fall in thirty-six ways, but in order that there may be neither an ace nor doublets, the first die must fall in one of five ways (viz. 2, 3, 4, 5, 6), and the second, since it may be neither an ace nor the same as the first, may fall in four ways. Hence, the number of ways which will produce the required result, is $5 \times 4$ or 20. And, therefore, the chance of this result is $\frac{20}{36}$ or $\frac{5}{9}$.

*Question.*—What is the chance of throwing exactly eleven?

*Answer.*—Out of the thirty-six ways, there are two ways which produce eleven; therefore, the chance is $\frac{2}{36}$ or $\frac{1}{18}$.

On the principle of the last answer, the reader will have no difficulty in verifying the following statements:

*In a single throw with two dice, the odds are*—

| | | | |
|---|---|---|---|
| 35 to 1 | against | throwing | 2, |
| 17 to 1 | ,, | ,, | 3, |
| 11 to 1 | ,, | ,, | 4, |
| 8 to 1 | ,, | ,, | 5, |
| $6\frac{1}{5}$ to 1 | ,, | ,, | 6, |
| 5 to 1 | ,, | ,, | 7, |
| $6\frac{1}{5}$ to 1 | ,, | ,, | 8, |
| 8 to 1 | ,, | ,, | 9, |
| 11 to 1 | ,, | ,, | 10, |
| 17 to 1 | ,, | ,, | 11, |
| 35 to 1 | ,, | ,, | 12. |

Thus the most frequent throw will be *seven*.

In some cases the purpose of a throw is equally answered, whether an assigned number appear on one of the dice, or whether it be the numbers of the two dice together make it. Let us consider, for example, the chance of throwing five in this way.

The chance of making a throw so that one die shall turn up *five* is $\frac{11}{36}$, and the chance of making a throw which shall amount to *five* is $\frac{4}{36}$. Therefore the chance of throwing *five* in one of these ways is $\frac{11}{36} + \frac{4}{36}$ or $\frac{15}{36}$.

On this principle the following statements may be easily verified.

*In a single throw with two dice, when the player is at liberty to count either the sum of the numbers on the two dice, or the number on either die alone, the odds are—*

| | | | | |
|---|---|---|---|---|
| 25 to 11, | | against | throwing | 1, |
| 24 to 12, | or 2 to 1 | ,, | ,, | 2, |
| 23 to 13, | | ,, | ,, | 3, |
| 22 to 14, | or 11 to 7 | ,, | ,, | 4, |
| 21 to 15, | or 7 to 5 | ,, | ,, | 5, |
| 20 to 16, | or 5 to 4 | ,, | ,, | 6, |
| | 5 to 1 | ,, | ,, | 7, |
| | $6\frac{1}{5}$ to 1 | ,, | ,, | 8, |
| | 8 to 1 | ,, | ,, | 9, |
| | 11 to 1 | ,, | ,, | 10, |
| | 17 to 1 | ,, | ,, | 11, |
| | 35 to 1 | ,, | ,, | 12. |

Thus the number which there is the greatest chance of making is *six*.

Definition.—If a person is to receive a prize on condition of some event happening, the sum of money for which his chance might equitably be sold beforehand is called his *expectation* from the event.

## RULE V.

*The expectation from any event is obtained by multiplying the sum to be realized on the event happening, by the chance that the event will happen.*

This rule may be illustrated as follows: Suppose a person holds five tickets in a lottery, where the whole number of tickets is twelve; and suppose there be only one prize, and let its value be one shilling.

The person in question gains the prize, if it happen that one of his tickets be drawn. The chance of this event is $\frac{5}{12}$; therefore, according to the rule, the person's expectation is $\frac{5}{12}$ of a shilling, or five-pence. And the correctness of this result may be immediately seen; for we observe, that if the person had bought all the twelve tickets he would have been certain of winning a shilling, and, therefore, he might, equitably, have given a shilling for the twelve tickets; but all the tickets are of equal value, and are equally valuable whether the same man hold one or more. Hence, each of them is worth a penny, and, therefore, the five in question are worth five-pence (as long as it is unknown which is drawn). Five-pence, therefore, is the sum that might equitably have been given for the assigned person's chance, and, therefore, by the definition this is his *expectation*.

*Question.*—A bag contains a £5 note, a £10 note, and six pieces of blank paper. What is the expectation of a man who is allowed to draw out one piece of paper?

*Answer.*—Since there are eight pieces of paper the probability of his drawing the £5 note is $\frac{1}{8}$; therefore, his expectation from the chance of drawing this note is $\frac{1}{8}$ of £5, or $\frac{5}{8}$ of a pound. Similarly, his expectation from the chance of drawing the £10 note is $\frac{1}{8}$ of £10, or $\frac{5}{4}$ of a pound. Therefore, his whole expectation is $\frac{15}{8}$ of a pound, or £1 17s. 6d.

*Question.*—What is the expectation of drawing a coin from a bag which contains one sovereign and seven shillings?

*Answer.*—The expectation from the chance of drawing the sovereign is $\frac{1}{8}$ of a sovereign, and the expectation from the chance of drawing a shilling is $\frac{7}{8}$ of a shilling. Hence, the whole expectation is 3s. $4\frac{1}{2}$d.

*Question.*—A person is allowed to draw two coins from a purse containing four sovereigns and four shillings. What is the value of his expectation?

*Answer.*—Two coins can be drawn in $\frac{8.7}{1.2}$ or 28 ways: of these $\frac{4.3}{1.2}$ or 6 ways will give two sovereigns, $4 \times 4$ or 16 ways will give a sovereign and a shilling, and the other 6 ways will give two shillings.

Therefore—

Chance of drawing 40 shillings $= \frac{6}{28}$,

Chance of drawing 21 shillings $= \frac{16}{28}$,

Chance of drawing 2 shillings $= \frac{6}{28}$.

Therefore the expectation is—

from the first chance, $\frac{6}{28} \times 40$, or $\frac{60}{7}$ shillings;

from the second chance, $\frac{16}{28} \times 21$, or 12 shillings;

from the third chance, $\frac{6}{28} \times 2$, or $\frac{3}{7}$ shillings.

Hence the whole expectation is $\frac{60}{7} + 12 + \frac{3}{7}$, or 21 shillings; or one-fourth of the whole sum in the bag.

This result might have been inferred at once from the consideration that, if all the eight coins had been drawn two and two, no drawing could be more likely to exceed in sovereigns than in shillings: (the number of sovereigns and shillings being the same). Hence the expectation from each of the four drawings must be the same; and therefore each must be one fourth of the whole sum to be drawn.

## RULE VI.

*The chance of two independent events both happening, is the product of the chances of their happening severally.*

That is, if the chance of one event happening be $\frac{5}{6}$, and the chance of another independent event happening be $\frac{7}{8}$, the chance that both events should happen is $\frac{5}{6} \times \frac{7}{8}$ or $\frac{35}{48}$.

This may be proved as follows:—

The chance of the first event is the same as the chance of drawing white from a bag containing six balls, of which five are white (Rule IV.)

The chance of the second event is the same as the chance of drawing white from a bag containing eight balls, of which seven are white.

Therefore the chance that both events should happen is the same as the chance that both balls drawn should be white.

But the first ball can be drawn in six ways, and the second in eight ways. Therefore (*Choice*, Rule I.), both can be drawn in $6 \times 8$, or 48 ways.

So the first can be *white* in five ways, and the second can be *white* in seven ways. Therefore both can be *white* in $5 \times 7$, or 35 ways.

That is, the two balls can be drawn in forty-eight ways (all equally likely), and thirty-five of these ways will give *double-white*. Hence (Rule IV.) the chance of double-white is $\frac{35}{48}$, and therefore the chance of the two given events both happening is $\frac{35}{48}$.

And the same reasoning would apply if the numbers were any others. Hence the rule is true always.

Example.—Suppose it is estimated that the chance that $A$ can solve a certain problem is $\frac{2}{3}$, and the chance that $B$ can solve it is $\frac{5}{12}$; let us consider what is the chance of the problem being solved when they both try.

The problem will be solved, unless they both fail.

Now the chance that $A$ fails is $\frac{1}{3}$: and the chance that $B$ fails is $\frac{7}{12}$.

Therefore the chance that both fail is

$$\frac{1}{3} \times \frac{7}{12}, \text{ or } \frac{7}{36}.$$

The chance that this should not be so, is

$$1 - \frac{7}{36}, \text{ or } \frac{29}{36}.$$

This is, therefore, the chance that the problem gets solved.

In the case just considered, four results were possible, viz. :—

(1) That $A$ and $B$ should both succeed:
(2) ,, $A$ should succeed and $B$ fail:
(3) ,, $A$ should fail and $B$ succeed:
(4) ,, $A$ and $B$ should both fail.

We may calculate the chance of these four events separately. Thus we have

Chance of $A$'s success $= \frac{2}{3}$, of $A$'s failure $= \frac{1}{3}$;

,, $B$'s success $= \frac{5}{12}$, of $B$'s failure $= \frac{7}{12}$.

Therefore, by the rule

(1) Chance that $A$ and $B$ both succeed

$$=\frac{2}{3}\times\frac{5}{12}=\frac{10}{36}:$$

(2) Chance that $A$ succeeds and $B$ fails

$$=\frac{2}{3}\times\frac{7}{12}=\frac{14}{36}:$$

(3) Chance that $A$ fails and $B$ succeeds

$$=\frac{1}{3}\times\frac{5}{12}=\frac{5}{36}:$$

(4) Chance that $A$ and $B$ both fail

$$=\frac{1}{3}\times\frac{7}{12}=\frac{7}{36}.$$

We observe that

$$\frac{10}{36}+\frac{14}{36}+\frac{5}{36}+\frac{7}{36}=\frac{36}{36}=1,$$

or the sum of the four probabilities is unity, as it ought to be, since it is certain that one of the four results must happen.

Further, we notice that the problem will be solved if any of the first three events out of (1), (2), (3) and (4) occur. Hence the chance of the problem being solved, might have been obtained by adding together the separate probabilities of these three events. Thus—

$$\frac{10}{36}+\frac{14}{36}+\frac{5}{36}=\frac{29}{36},$$

or the probability is $\frac{29}{36}$, as before.

It may be said that on an average

*Ten persons will die in the next ten years:*

| | | | | | |
|---|---|---|---|---|---|
| *out of every* | | 62 | *whose present age is* | | 30, |
| ,, | ,, | 45 | ,, | ,, | 40, |
| ,, | ,, | 35 | ,, | ,, | 50, |
| ,, | ,, | 25 | ,, | ,, | 60. |

We may apply such results as these to the solution of questions affecting Insurances and Life Annuities.

*Question.*—What are the odds against a person aged thirty living till he is sixty?

*Answer.*—The chance that he dies between thirty and forty is $\frac{10}{62}$; that he lives to forty and dies between forty and fifty is $\frac{52}{62} \times \frac{10}{45}$; that he lives to fifty and dies between fifty and sixty is $\frac{52}{62} \times \frac{35}{45} \times \frac{10}{35}$. Therefore the chance that he dies between thirty and sixty is

$$\frac{10}{62} + \frac{52}{62} \cdot \frac{10}{45} + \frac{52}{62} \cdot \frac{35}{45} \cdot \frac{10}{35}, \text{ or } \frac{149}{279}.$$

Hence the odds are 149 to 130, or about 8 to 7 against his living to be sixty.

*Question.*—What are the odds against a person at the age of forty living for thirty years?

*Answer.*—Proceeding as in the last question, we find the chance of his dying within thirty years to be

$$\frac{10}{45} + \frac{35}{45} \cdot \frac{10}{35} + \frac{35}{45} \cdot \frac{25}{35} \cdot \frac{10}{25}, \text{ or } \frac{2}{3}.$$

Therefore the odds are two to one against his living for thirty years.

*Question.*—What is the probability that two persons, *A* and *B*, aged respectively thirty and forty, will be alive ten years hence?

*Answer.*—The chance of *A* dying in the next ten years is $\frac{10}{62}$, and the chance of his living $\frac{52}{62}$. So the chance of *B* dying within ten years is $\frac{10}{45}$, and the chance of his living is $\frac{35}{45}$.

Therefore the chance that *A* and *B* will be both alive is

$$\frac{52}{62} \times \frac{35}{45}, \text{ or } \frac{182}{279}.$$

*Question.*—If it be eight to seven against a person who is now thirty years old living till he is sixty, and two to one against a person who is now forty living till he is seventy; find the probability that one at least of these persons will be alive thirty years hence.

*Answer.*—One at least will be living unless both be dead. The chance that the first be dead is $\frac{8}{15}$, and the chance that the second be dead is $\frac{2}{3}$: therefore the chance that both be dead is $\frac{8}{15} \times \frac{2}{3}$, or $\frac{16}{45}$, and the chance that this should not be so, or that *one at least* be alive is $1 - \frac{16}{45}$, or $\frac{29}{45}$.

## RULE VII.

*If there be two events which are not independent, the chance that they should both happen is the product of the chance that the first should happen, and the chance that when the first has happened the second should happen also.*

For instance, suppose we are asked what is the probability of drawing first a consonant and then a vowel, when two letters are drawn at random out of an alphabet of twenty consonants and six vowels.

The second event is dependent on the first; for if a consonant be drawn the first time, there are twenty-five letters left, of which six are vowels, and the chance that the second letter should be a vowel is $\frac{6}{25}$; but if a vowel be drawn the first time, there are twenty-five letters left, of which five are vowels, and the chance that the second letter should be a vowel is $\frac{5}{25}$.

According to the rule, however, we have to multiply the chance of the first event, which is $\frac{20}{26}$, by the chance of the second event happening *when the first has already happened*, which is therefore $\frac{6}{25}$, and thus we obtain the result—

$$\frac{20}{26} \times \frac{6}{25}, \text{ or } \frac{12}{65}.$$

The truth of this result may be seen in another way.

It is possible to select two letters *in order*, out of the alphabet, in $26 \times 25$ ways (*Choice*, Rule IV.), and all these are equally likely. But we can select two letters so that the first is a consonant and the second a vowel in only $20 \times 6$ ways (*Choice*, Rule I). Hence when two letters are drawn in order, the chance that the first is a consonant and the second a vowel is (as before)

$$\frac{20 \times 6}{26 \times 25}, \text{ or } \frac{12}{65}.$$

Indeed it will appear that this rule follows directly from the preceding one; for, since we have only to find the chance that *both* events should happen, we have not to do with the second event at all, except in the case when the first has happened. The probability of the double event must therefore be the same as if the chance of the second were always what it is when the first has happened (since we are not concerned with the case when the first has not happened). But if the chance of the second event can be treated as if it were always the same, without reference to the first event, it is to all intents and purposes independent of the first, and Rule VI. is therefore applicable.

*Question.*—One purse contains five sovereigns and four shillings; another contains five sovereigns and three shillings. One purse is taken at random and a coin drawn out. What is the chance that it be a sovereign?

*Answer.*—The chance that the first purse be selected is $\frac{1}{2}$, and if it be selected, the chance that the coin be a sovereign is $\frac{5}{9}$: hence the chance that the coin drawn out be one of the sovereigns out of the first purse is

$$\frac{1}{2} \times \frac{5}{9}, \text{ or } \frac{5}{18}.$$

Similarly the chance that it be one of the sovereigns out of the second purse is

$$\frac{1}{2} \times \frac{5}{8}, \text{ or } \frac{5}{16}.$$

Hence the whole chance of drawing a sovereign is

$$\frac{5}{18} + \frac{5}{16}, \text{ or } \frac{85}{144}.$$

*Question.*—What is the expectation from the drawing of the coin in the last question?

*Answer.*—The chance that it is a sovereign is $\frac{85}{144}$, and therefore the expectation from the chance of drawing a sovereign is $\frac{85}{144}$ of a pound, or $\frac{1700}{144}$ shillings.

If the coin drawn be not a sovereign, it must be a shilling, therefore the chance of drawing a shilling must be $1 - \frac{85}{144}$, or $\frac{59}{144}$ (Rule I.) Hence the expectation from the chance of drawing a shilling is $\frac{59}{144}$ of a shilling. Therefore the whole expectation from the drawing is

$$\frac{1700}{144} + \frac{59}{144}, \text{ or } \frac{1759}{144}$$

shillings, or 12s. $2\frac{7}{12}$d.

*Question.*—What would have been the chance of drawing a sovereign if all the coins in the last case had been in one bag, and what would have been the expectation?

*Answer.*—There would have been ten sovereigns and seven shillings in the bag; therefore, the chance of drawing a sovereign would have been $\frac{10}{17}$, and the chance of drawing a shilling $\frac{7}{17}$ (Rule I.) The expec-pectation would therefore have been

$$\frac{200}{17} + \frac{7}{17} \text{ or } \frac{207}{17}$$

shillings, or 12s. $2\frac{2}{17}$d.

The chance of drawing a sovereign is therefore in this case a little less, and the whole expectation very slightly less than in the former case.

*Question.*—There are three parcels of books in another room, and a particular book is in one of them. The odds that it is in one particular parcel are three to two; but if not in that parcel, it is equally likely to be in either of the others. If I send for this parcel, giving a description of it, and the odds that I get the one I describe are two to one, what is my chance of getting the book?

*Answer.*—The chance of getting the parcel described is $\frac{2}{3}$, and the chance that the book is in it is $\frac{3}{5}$;

therefore, the chance of getting the book in the described parcel is $\frac{2}{3} \times \frac{3}{5}$ or $\frac{6}{15}$.

The chance of getting a parcel not described is $\frac{1}{3}$, and the chance that the book is in it is $\frac{1}{5}$; therefore, the chance of getting the book in a parcel not described is $\frac{1}{3} \times \frac{1}{5}$, or $\frac{1}{15}$.

Therefore, the whole chance of getting the book at all is $\frac{6}{15} + \frac{1}{15}$, or $\frac{7}{15}$; or the odds are eight to seven against getting it.

*Question.*—In a purse are ten coins, of which nine are shillings and one is a sovereign; in another are ten coins, all of which are shillings. Nine coins are taken out of the former purse and put into the latter, and then nine coins are taken from the latter and put into the former. A person may now take whichever purse he pleases; which should he select?

*Answer.*— Since each purse contains the same number of coins, he ought to choose that which is the more likely to contain the sovereign. Now the sovereign can only be in the second bag, provided *both* the following events have taken place, viz.—

(1) That the sovereign was among the nine coins taken out of the first bag and put into the second.

(2) That it was *not* among the nine coins taken out of the second bag and put into the first.

Now the chance of (1) is $\frac{9}{10}$, and when (1) has happened the chance of (2) is $\frac{10}{19}$; therefore, the chance of both happening is $\frac{9}{10} \times \frac{10}{19}$, or $\frac{9}{19}$. This, therefore is the chance that the sovereign is in the second bag, and therefore (Rule I.) the chance that it is in the first is $1 - \frac{9}{19}$ or $\frac{10}{19}$. Hence, the first bag ought to be chosen in preference to the other.

## RULE VIII.

*The chance that a series of events should all happen is the continued product of the chance that the first should happen, the chance that (when it has happened) then the second should happen, the chance that then the third should happen, and so on.*

This is a simple extension of the last rule. For suppose there be four events, and let $\frac{1}{2}$ be the chance that the first should happen, and when the first has happened, let $\frac{3}{4}$ be the chance that the second should happen, and when these have happened, let $\frac{5}{8}$ be the chance that the third should happen, and when these have happened, let $\frac{1}{4}$ be the chance that the fourth should happen; by Rule VII., the chance that the first and second should both happen is $\frac{1}{2} \times \frac{3}{4}$, or $\frac{3}{8}$. We may now treat this as a single event, and then, again applying the same rule, we get $\frac{3}{8} \times \frac{5}{8}$, or $\frac{15}{64}$ as the chance that the first, second, and third should all happen. Treating this compound event as one event, we can again apply the same rule, and obtain $\frac{15}{64} \times \frac{1}{4}$, or $\frac{15}{256}$ as the chance that all the four events should happen. Thus the chance of all the events is

$$\frac{1}{2} \times \frac{3}{4} \times \frac{5}{8} \times \frac{1}{4},$$

the continued product of all the given chances.

*Question.*—There are three independent events whose several chances are $\frac{2}{3}$, $\frac{3}{5}$, $\frac{1}{2}$. What is the chance that one of them at least will happen?

*Answer.*—One at least will happen, unless all fail. The chance of all failing is $\frac{1}{3} \times \frac{2}{5} \times \frac{1}{2}$, or $\frac{1}{15}$. Hence the required chance is $1 - \frac{1}{15}$, or $\frac{14}{15}$.

*Question.*—There are three independent events whose several chances are $\frac{2}{3}$, $\frac{3}{5}$, $\frac{1}{2}$. What is the chance that exactly one of them should happen?

*Answer.*—The chance that the first should happen and the others fail is

$$\frac{2}{3} \times \frac{2}{5} \times \frac{1}{2}, \text{ or } \frac{4}{30}.$$

So the chance that the second should happen and the others fail is

$$\frac{3}{5} \times \frac{1}{3} \times \frac{1}{2}, \text{ or } \frac{3}{30}.$$

And the chance that the third should happen and the others fail is

$$\frac{1}{2} \times \frac{1}{3} \times \frac{2}{5}, \text{ or } \frac{2}{30}.$$

Hence, the chance that one of these should occur — that is, that *exactly one* of the three events should happen — is

$$\frac{4}{30} + \frac{3}{30} + \frac{2}{30}, \text{ or } \frac{9}{30}, \text{ or } \frac{3}{10}.$$

*Question.* — When six coins are tossed, what is the chance that one, and only one, will turn up head?

*Answer.* — The chance that the first should turn up head is $\frac{1}{2}$, and the chance that the others should turn up tail is $\frac{1}{2}$ for each of them. Therefore, the chance that the first should turn up head and the rest tail is

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}, \text{ or } \frac{1}{64}.$$

And there will be a similar chance that the second should alone turn up head, or that the third should alone turn up head, and so on.

Hence, the whole chance of some one, and only one, turning up head is

$$\frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64}, \text{ or } \frac{6}{64}.$$

*Question.*—When six coins are tossed, what is the chance that at least one will turn up head?

*Answer.*—The chance that all should turn up tail is

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}, \text{ or } \frac{1}{64}.$$

The chance that this should not be so, or that at least one head should turn up, is (Rule II.)

$$1 - \frac{1}{64}, \text{ or } \frac{63}{64}.$$

*Question.*—A person throws three dice, what are the respective chances that they should fall all alike, only two alike, or all different?

*Answer.*—The chance that the second should fall the same as the first is $\frac{1}{6}$, and the chance that the third should also fall the same is $\frac{1}{6}$. Hence, the chance that all three fall alike is

$$\frac{1}{6} \times \frac{1}{6}, \text{ or } \frac{1}{36}.$$

The chance that the second should fall as the first, and that the third should fall different, is

$$\frac{1}{6} \times \frac{5}{6}, \text{ or } \frac{5}{36};$$

and there is the same chance that the second and third should be alike, and the first different; or that the first and third should be alike, and the second different. Hence, the chance that some two should be alike, and the others different, is

$$\frac{5}{36} + \frac{5}{36} + \frac{5}{36}, \text{ or } \frac{15}{36}.$$

The chance that the second should be different from the first is $\frac{5}{6}$, and the chance that the third should be different from either is $\frac{4}{6}$. Hence, the chance that all three are different is

$$\frac{5}{6} \times \frac{4}{6}, \text{ or } \frac{20}{36}.$$

Therefore, the three chances required are $\frac{1}{36}$, $\frac{15}{36}$, $\frac{20}{36}$ respectively, their sum being unity, since the dice must certainly fall in some one of the three ways.

*Question.*—A person throws three dice, and is to receive six shillings if they all turn up alike, four shillings if two only turn up alike, and three shillings if all turn up different, what is his expectation?

*Answer.*—Referring to the last question, the chance of all turning up different alike is $\frac{1}{36}$; his expectation from this event is therefore $\frac{1}{36}$ of six shillings, or two pence. The chance of two only turning up alike is $\frac{15}{36}$ or $\frac{5}{12}$, and his expectation from this event is therefore $\frac{5}{12}$

of four shillings, or twenty pence. The chance of all turning up different is $\frac{20}{36}$ or $\frac{5}{9}$, and his expectation from this event is therefore $\frac{5}{9}$ of three shillings, or twenty pence. Therefore his whole expectation is 2 + 20 + 20, or 42 pence, or three shillings and sixpence.

We shall find the following notation very convenient:—

The symbol $3^2$ means $3 \times 3$, or 9:
,, $5^2$ means $5 \times 5$, or 25:
,, $5^3$ means $5 \times 5 \times 5$, or 125:
,, $2^4$ means $2 \times 2 \times 2 \times 2$, or 16:
,, $2^5$ means $2 \times 2 \times 2 \times 2 \times 2$, or 32:

and so on, whatever be the numbers; the small figure above the line denoting the number of times the other number is to be repeated, and the sign of multiplication being understood before every repetition.

So also $\left(\frac{2}{3}\right)^3 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$:

$$\left(\frac{3}{4}\right)^4 = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{81}{256}:$$

and so on.

*Question.*—A person goes on throwing a single die until it turns up ace. What is the chance (1) that he will have to make *at least* ten throws; (2) that he will have to make *exactly* ten throws?

*Answer.*—(1) The chance that he fails at any particular trial to throw an ace is $\frac{5}{6}$. The chance that he should fail the first nine times (by Rule VIII.) is $\left(\frac{5}{6}\right)^9$. This, therefore, is the probability that he will have to throw at least ten times.

(2) Since $\left(\frac{5}{6}\right)^9$ is the chance that he fails the first nine times, and $\frac{1}{6}$ the chance that he succeeds the next time, therefore by Rule VII., $\left(\frac{5}{6}\right)^9 \times \frac{1}{6}$ is the chance that he will have to throw exactly ten times.

*Question.*—A die is to be thrown once by each of four persons, *A*, *B*, *C*, *D*, in order, and the first of them who throws an ace is to receive a prize. Find their respective chances, and the chance that the prize will not be won at all.

*Answer.*—Since *A* has the first throw, he wins if he throws an ace; his chance is therefore $\frac{1}{6}$.

So *B* wins provided *A* fails and he succeeds. The chance of *A* failing is $\frac{5}{6}$, and of *B* succeeding is $\frac{1}{6}$. Therefore *B*'s chance of winning is

$$\frac{5}{6} \times \frac{1}{6}, \text{ or } \frac{5}{36}.$$

So $C$ wins provided $A$ and $B$ both fail, and he succeeds. The chance of $A$ and $B$ both failing is $\frac{5}{6} \times \frac{5}{6}$, or $\frac{25}{36}$; and then the chance of $C$ succeeding is $\frac{1}{6}$. Therefore $C$'s chance of winning is

$$\frac{25}{36} \times \frac{1}{6}, \text{ or } \frac{25}{216}.$$

So $D$ wins provided $A$, $B$, and $C$ all fail, and he succeeds. The chance of $A$, $B$, and $C$ all failing is $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$, or $\frac{125}{216}$; and then the chance of $D$ succeeding is $\frac{1}{6}$. Therefore $D$'s chance of winning is

$$\frac{125}{216} \times \frac{1}{6}, \text{ or } \frac{125}{1296}.$$

The prize is not won at all, provided all four fail to throw an ace. The chance that this should be the case is

$$\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}, \text{ or } \frac{625}{1296}.$$

*Question.*—Two persons, $A$ and $B$, throw alternately with a single die, and he who first throws an ace is to receive a prize of £1. What are their respective expectations?

*Answer.*—The chance that the prize should be won

at the first throw, is $\frac{1}{6}$:

at the second throw, is $\frac{5}{6} \times \frac{1}{6}$:

at the third throw, is $\left(\frac{5}{6}\right)^2 \times \frac{1}{6}$:

at the fourth throw, is $\left(\frac{5}{6}\right)^3 \times \frac{1}{6}$:

at the fifth throw, is $\left(\frac{5}{6}\right)^4 \times \frac{1}{6}$:

at the sixth throw, is $\left(\frac{5}{6}\right)^5 \times \frac{1}{6}$:

and so on.

But the first, third, and fifth, &c., throws belong to $A$, and the second, fourth, sixth, &c., belong to $B$. Hence $A$'s chance of winning is

$$\frac{1}{6} + \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6} + \&c.;$$

and $B$'s chance is

$$\frac{5}{6} \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^5 \cdot \frac{1}{6} + \&c.;$$

that is, $B$'s chance is equal to $A$'s multiplied by $\frac{5}{6}$. Hence $B$'s expectation is $\frac{5}{6}$ of $A$'s, or $B$'s is to $A$'s in the ratio of five to six. But their expectations must together amount to £1. Hence $A$'s expectation is $\frac{6}{11}$ of a pound, and $B$'s $\frac{5}{11}$ of a pound.

*Question.*—What is the chance that a person with two dice will throw aces exactly four times in six trials?

*Answer.*—The chance of throwing aces at any particular trial is $\frac{1}{36}$, and the chance of failing is $\frac{35}{36}$. Hence the chance of succeeding at four assigned trials, and failing at the other two is $\left(\frac{1}{36}\right)^4 \times \left(\frac{35}{36}\right)^2$. But aces will be thrown *exactly four times* if they be thrown at any set of four trials which might be asssigned out of the six trials, and if they fail at the remaining two. And (*Choice*, Rule IX.) it is possible to assign four out of six in $\frac{6.5.4.3}{1.2.3.4}$, or fifteen ways. Hence the chance required is fifteen times the chance of succeeding in four *assigned* trials, and failing at the other two. Therefore it is

$$\left(\frac{1}{36}\right)^4 \times \left(\frac{35}{36}\right)^2 \times 15, \text{ or } \frac{6125}{725594112}:$$

therefore the odds are more than 100,000 to 1 against the event.

*Question.*—If on an average nine ships out of ten return safe to port, what is the chance that out of five ships expected, at least three will arrive?

*Answer.*—The chance that any particular ship returns is $\frac{9}{10}$. The chance that any particular set of three ships should all arrive is $\left(\frac{9}{10}\right)^3$, and the chance that the other two should not arrive is $\left(\frac{1}{10}\right)^2$. Therefore the chance that a particular set of three should alone arrive is $\left(\frac{9}{10}\right)^3 \times \left(\frac{1}{10}\right)^2$, or $\frac{729}{100000}$. And out of five ships a set of three can be selected in $\frac{5.4.3}{1.2.3}$ or 10 ways. Hence the chance that some one of these sets of three should alone arrive is

$$\frac{729}{100000} \times 10, \text{ or } \frac{729}{10000}.$$

This is therefore the chance that *exactly three* ships should arrive.

Similarly the chance that any particular set of four should alone arrive is $\left(\frac{9}{10}\right)^4 \times \frac{1}{10}$, or $\frac{6561}{100000}$; and the chance that some one of the five possible sets of four should alone arrive is $\frac{6561}{100000} \times 5$, or $\frac{32805}{100000}$. This is therefore the chance that *exactly four* ships should arrive.

And the chance that all the five should arrive is $\left(\frac{9}{10}\right)^5$, or $\frac{59049}{100000}$.

But the chance that *at least three* should arrive is the chance that either *three exactly*, or *four exactly*, or *five exactly* should arrive: and is therefore the sum of the several chances of these exact numbers arriving: that is, the required chance is

$$\frac{7290}{100000} + \frac{32805}{100000} + \frac{59049}{100000}, \text{ or } \frac{99144}{100000}, \text{ or } \frac{12393}{12500}.$$

*Question.*—*A* and *B* play at a game which cannot be drawn, and on an average *A* wins three games out of five. What is the chance that *A* should win at least three games out of the first five?

*Answer.*—The chance that *A* wins three assigned games, and *B* the other two, is $\left(\frac{3}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^2$ or $\frac{108}{3125}$. But the three may be assigned in $\frac{5.4.3}{1.2.3}$, or 10 ways (*Choice* Rule IX.). Hence the chance that *A* should win some three games and *B* the other two is

$$\frac{108}{3125} \times 10, \text{ or } \frac{1080}{3125}.$$

Similarly the chance that *A* should win some four games and *B* the other one is

$$\frac{162}{3125} \times 5, \text{ or } \frac{810}{3125}.$$

And the chance that *A* should win all five games is

$$\left(\frac{3}{5}\right)^5, \text{ or } \frac{243}{3125}.$$

Therefore the chance that $A$ wins either three, or four, or all out of the first five games is

$$\frac{1080}{3125} + \frac{810}{3125} + \frac{243}{3125} = \frac{2133}{3125},$$

or the odds are rather more than two to one in $A$'s favour.

## RULE IX.

*If a doubtful event can happen in a number of different ways, any accession of knowledge concerning the event which changes the probability of its happening will change, in the same ratio, the probability of any particular way of its happening.*

It follows from the axiom that the probability of the event happening at all must be equal to the sum of the probabilities of its happening in the several ways.

*First,* suppose for simplicity that all the ways are equally likely. Let there be seven ways, and let the chance of each one severally occurring be $\frac{1}{10}$: then the chance of the event happening at all is seven times this, or $\frac{7}{10}$.

But suppose that our knowledge is increased by the information that the event happens nine times out of ten, or by such other information as brings our estimate of its probability up to $\frac{9}{10}$ instead of $\frac{7}{10}$, thus increasing the probability in the ratio of seven to nine.

It is still true that there are only seven ways of the event happening, all of which are equally likely: hence the probability of the event happening in any particular one of these ways is $\frac{1}{7}$ of $\frac{9}{10}$, or $\frac{9}{70}$, with our new information. Hence our information concerning the event, has increased the chance of its happening in an assigned way from $\frac{1}{10}$ or $\frac{7}{70}$ to $\frac{9}{70}$, that is, it has increased it in the ratio of seven to nine, the same ratio in which the probability of the event itself was increased.

And the same argument would hold if the numbers were any others, and therefore the rule is true, provided all the ways of the event happening are equally probable.

*Secondly,* suppose the ways are not equally probable. We may in this case regard them as groups of subsidiary ways, which would be equally probable. Then, as we have shewn, the chance of each one of these subsidiary ways would be increased (or decreased) in the same ratio as the chance of the event itself, and therefore the sum of the chances of any group of these subsidiary ways would be changed in the same ratio.

For instance, if the event could happen in any one of three ways, whose respective chances were $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{4}$, or $\frac{4}{12}$, $\frac{2}{12}$, $\frac{3}{12}$, we might divide the first of these ways into four subsidiary ways, the next into two ways, and the other into three ways, and the chance of each of these subsidiary ways would be $\frac{1}{12}$. If, therefore, by an accession of knowledge, the chance of the whole event were diminished in the ratio of three to two, each subsidiary way of the event's happening would have a diminished probability of $\frac{2}{3}$ of $\frac{1}{12}$, or $\frac{1}{18}$, and the probabilities of the three given *ways* would become respectively, $\frac{4}{18}, \frac{2}{18}, \frac{3}{18}$, or $\frac{2}{9}, \frac{1}{9}, \frac{1}{6}$: that is, they would be diminished in the same ratio as the chance of the event itself.

Thus we see that the rule is true always.

*Question.*—A bag contains five balls, which are known to be either all black or all white—and both these are equally probable. A white ball is dropped into the bag, and then a ball is drawn out at random and found to be white. What is now the chance that the original balls were all white?

*Answer.*—The probabilities are here affected by the observed event that a ball drawn out at random proved to be white.

We will first calculate the probabilities before this event was observed (which we will call *à priori* probabilities), and then consider how they are affected by the accession of knowledge produced by the observation of the event. (Probabilities modified by this knowledge may be distinguished as *à posteriori* probabilities.)

The event might happen in two ways; either by the balls having been all white, and any one of them being drawn, or by the five original balls having been black, the new one alone white, and this one drawn.

The *à priori* probability that all are white is $\frac{1}{2}$, and then the chance of drawing a white ball is 1 (or certainty). Hence the chance of the event happening in this way is $\frac{1}{2} \times 1$, or $\frac{1}{2}$.

So the *à priori* probability that the first five were black is $\frac{1}{2}$, and then the chance of drawing a white ball is $\frac{1}{6}$. Hence the chance of the event happening in this way is $\frac{1}{2} \times \frac{1}{6}$, or $\frac{1}{12}$.

Therefore the whole *à priori* chance of the event happening is $\frac{1}{2} + \frac{1}{12}$, or $\frac{7}{12}$.

But when the ball is drawn and observed to be white, this knowledge immediately increases the chance from $\frac{7}{12}$ to 1 (or certainty): that is, it increases the chance in the ratio of 7 to 12. Therefore, by Rule IX., the chances of the event happening in the several ways are increased in the same ratio.

Hence the *à posteriori* chance of the event having happened in the first way is $\frac{1}{2} \times \frac{12}{7}$, or $\frac{6}{7}$; and the *à posteriori* chance of its having happened in the second way is $\frac{1}{12} \times \frac{12}{7}$, or $\frac{1}{7}$. Or the chance of the original balls having been all white is now $\frac{6}{7}$, and the chance of their having been all black is $\frac{1}{7}$.

*Question.*—A penny is tossed ten times in succession, and always falls head. Supposing that one penny in every million that are coined has two heads, what is the chance that the penny in question has two heads?

*Answer.*—The penny has either two heads or a head and a tail: and the respective chances of these two cases *à priori* are $\frac{1}{1000000}$ and $\frac{999999}{1000000}$. In the first case the chance that head should fall ten times in succession is unity; in the second case it is $\left(\frac{1}{2}\right)^{10}$, or $\frac{1}{1024}$. Therefore we have *à priori*,—

(1) the chance that there should be two heads, and head should fall ten times is $\frac{1}{1000000}$:

(2) the chance that there should be head and tail, and head should fall ten times is $\frac{999999}{1024000000}$: and the chance that from one or other the same result should happen is

$$\frac{1}{1000000} + \frac{999999}{1024000000}, \text{ or } \frac{1001023}{1024000000}.$$

But after our knowledge is augmented by the observation of the fact that the penny falls head ten times in succession, this latter chance becomes unity, that is, it becomes multiplied by $\frac{1024000000}{1001023}$. Hence (Rule IX.) the chances (1) and (2) become multiplied in the same ratio. Therefore we have *à posteriori*,—

(1) the chance that there should be two heads, and head should fall ten times is

$$\frac{1}{1000000} \times \frac{1024000000}{1001023}, \text{ or } \frac{1024}{1001023}:$$

(2) The chance that there should be head and tail, and head should fall ten times is

$$\frac{999999}{1024000000} \times \frac{1024000000}{1001023}, \text{ or } \frac{999999}{1001023}.$$

The required chance (after the observed event) is consequently

$$\frac{1024}{1001023}, \text{ or rather more than } \frac{1}{1000}.$$

*Question.*—A purse contains ten coins, each of which is either a sovereign or a shilling: a coin is drawn and found to be a sovereign, what is the chance that this is the only sovereign?

*Answer.*—*A priori,* the coin drawn was equally likely to be a sovereign or a shilling, therefore the chance of its being a sovereign was $\frac{1}{2}$.

*A posteriori,* the chance of its being a sovereign is unity: or the chance is *doubled* by the observation of the event. Therefore (Rule IX.) the chance of any particular way in which a sovereign might be drawn is also doubled.

Now the chance that there was only one sovereign was *à priori*

$$\frac{10}{2^{10}} \text{ or } \frac{10}{1024};$$

and in this case the chance of drawing a sovereign would be $\frac{1}{10}$.

Hence the chance that there should be only one sovereign, and that it should be drawn was *à priori*

$$\frac{10}{1024} \times \frac{1}{10}, \text{ or } \frac{1}{1024}.$$

And the *à posteriori* chance that a sovereign should be drawn in this way is the double of this: *i. e.*, $\frac{2}{1024}$ or $\frac{1}{512}$; which is therefore the required chance.

*Question.*—A purse contains ten coins, which are either sovereigns or shillings, and all possible numbers of each are equally likely: a coin is drawn and found to be a sovereign, what is the chance that this is the only sovereign?

*Answer.*—*A priori*, the coin drawn was equally likely to be a sovereign or a shilling, therefore the chance of its being a sovereign was $\frac{1}{2}$.

*A posteriori*, the chance of its being a sovereign is unity: or the chance is doubled by the observation of the event.

Therefore (Rule IX.) the chance of any particular way in which a sovereign might be drawn is also doubled.

Now, *à priori*, eleven cases were equally probable, viz., that there should be

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 sovereigns.
10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0 shillings.

Therefore the chance of there being exactly one sovereign was $\frac{1}{11}$, and in this case the chance of drawing a sovereign was $\frac{1}{10}$.

Hence the chance that there should be only one sovereign, and that it should be drawn was, *à priori*,

$$\frac{1}{11} \times \frac{1}{10}, \text{ or } \frac{1}{110}.$$

And the *à posteriori* chance that a sovereign should be drawn in this way is the double of this, that is, $\frac{2}{110}$ or $\frac{1}{55}$, which is, therefore, the chance required.

*Question.*—Reference is made to a year which contained fifty-three Sundays, and was not the last year of a century. What are the odds against its being a leap-year?

*Answer.*—Of the ninety-nine years, excluding the last in any century, twenty-four are leap-years. Hence, before we consider the fact that the year in question contained fifty-three Sundays, the *à priori* chance that it was a leap-year is $\frac{8}{33}$, and that it was not a leap-year $\frac{25}{33}$.

The chance that a leap-year has fifty-three Sundays is $\frac{2}{7}$, and the chance that another year has fifty-three Sundays is $\frac{1}{7}$.

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Hence the chance that the year in question should be a leap-year, and have fifty-three Sundays, is

$$\frac{8}{33} \times \frac{2}{7}, \text{ or } \frac{16}{231}.$$

And the chance that it should not be a leap-year, and yet have fifty-three Sundays, is

$$\frac{25}{33} \times \frac{1}{7}, \text{ or } \frac{25}{231}.$$

Hence the whole *à priori* probability that the year should have fifty-three Sundays, is

$$\frac{16}{231} + \frac{25}{231}, \text{ or } \frac{41}{231}$$

But *à posteriori* this chance becomes certainty, or the probability gets multiplied by $\frac{231}{41}$. Hence also the probability that the fifty-three Sundays resulted from a leap-year is multiplied in the same ratio, and becomes

$$\frac{16}{231} \times \frac{231}{41}, \text{ or } \frac{16}{41},$$

and the chance that it is not a leap-year becomes

$$\frac{25}{231} \times \frac{231}{41}, \text{ or } \frac{25}{41}.$$

Thus the odds are now 25 to 16 against the year in question being a leap-year.

*Question.*—*A*, *B*, *C* were entered for a race, and their respective chances of winning were estimated at $\frac{2}{11}$, $\frac{4}{11}$, $\frac{5}{11}$. But circumstances come to our knowledge in favour of $A$, which raise his chance to $\frac{1}{2}$; what are now the chances in favour of $B$ and $C$ respectively?

*Answer.*—$A$ could lose in two ways, viz., either by $B$ winning or by $C$ winning, and the respective chances of his losing in these ways were *à priori* $\frac{4}{11}$ and $\frac{5}{11}$, and the chance of his losing at all was $\frac{9}{11}$. But after our accession of knowledge the chance of his losing at all becomes $\frac{1}{2}$, that is, it becomes diminished in the ratio of 18 : 11. Hence the chance of either way in which he might lose is diminished in the same ratio. Therefore the chance of $B$ winning is now

$$\frac{4}{11} \times \frac{11}{18}, \text{ or } \frac{4}{18};$$

and of $C$ winning

$$\frac{5}{11} \times \frac{11}{18}, \text{ or } \frac{5}{18}.$$

These are therefore the required chances.

*Question.*—One of a pack of fifty-two cards has been removed; from the remainder of the pack two cards are drawn and are found to be spades; find the chance that the missing card is a spade.

*Answer.*—*A priori*, the chance of the missing card being a spade is $\frac{1}{4}$, and the chance that then two cards drawn at random should be both spades is $\frac{12.11}{51.50}$, or $\frac{132}{2550}$.

Hence the chance that the missing card should be a spade, and two spades be drawn is

$$\frac{1}{4} \times \frac{132}{2550}, \text{ or } \frac{11}{850}.$$

The chance of the missing card being not a spade is $\frac{3}{4}$, and the chance that then two spades should be drawn is $\frac{13.12}{51.50}$, or $\frac{156}{2550}$. Hence the chance that the missing card should be not a spade, and two spades be drawn, is

$$\frac{3}{4} \times \frac{156}{2550}, \text{ or } \frac{39}{850}.$$

Therefore the chance that in one way or the other two spades should be drawn is

$$\frac{11}{850} + \frac{39}{850}, \text{ or } \frac{50}{850}, \text{ or } \frac{1}{17}.$$

But after the observation of the event this chance becomes certainty, or becomes multiplied by 17. Therefore the chance of either way from which the result might occur is increased in the same ratio.

So the chance that the given card was a spade becomes *à posteriori*,

$$\frac{11}{850} \times 17, \text{ or } \frac{11}{50}.$$

Questions as to the credibility of the testimony of witnesses will depend for their solution upon the last rule, and may be answered in a manner similar to that of the questions just considered. In most questions of this class, the testimony given, or the assertions made, constitute a phenomenon which might have occurred whether the event reported occurred or not, or in whatsoever manner it occurred. We may first investigate the *à priori* probabilities of such testimony being given, on the several hypotheses possible with respect to the occurrence of the event, and by summing them we may deduce the *à priori* probability of the testimony being given at all. If we then take into consideration the fact that the testimony has been given, this accession of knowledge raises the last probability into certainty, and therefore increases it in a definite ratio, which can be calculated. In the same ratio (by Rule IX.) must the probabilities be increased of the several ways in which the testimony may have been generated, or in which the event in question may have happened. Thus we obtain the *à posteriori* and final probability of any assigned manner in which the event could possibly have occurred. A few examples will fully illustrate this.

*Question.*—*A* speaks truth three times out of four, *B* four times out of five; they agree in asserting that from a bag containing nine balls, all of different colours, a white ball has been drawn; shew that the probability that this is true is $\frac{96}{97}$.

*Answer.*—We will consider those chances as *à priori*, which are independent of the knowledge that *A* and *B* make the report in question, and those as *à posteriori* which are subsequent to this knowledge.

The *à priori* chance that a white ball should be drawn is $\frac{1}{9}$, and in this case the chance that $A$ and $B$ should both assert it, is $\frac{3}{4} \times \frac{4}{5}$; hence the chance that $A$ and $B$ should both truly assert a white ball to be drawn, is *à priori*,

$$\frac{1}{9} \times \frac{3}{4} \times \frac{4}{5}, \text{ or } \frac{1}{15}.$$

The *à priori* chance that a white ball should not be drawn is $\frac{8}{9}$, the chance that $A$ should make a false report is $\frac{1}{4}$, and that he should select the white ball out of the eight which might be falsely asserted to have come up is $\frac{1}{8}$; hence in this case the chance that he asserts that the white is drawn is $\frac{1}{8} \times \frac{1}{4}$, and the chance that $B$ should make the same assertion is $\frac{1}{8} \times \frac{1}{5}$; therefore the chance that $A$ and $B$ should both falsely assert a white ball to be drawn is

$$\frac{8}{9} \times \frac{1}{8} \times \frac{1}{4} \times \frac{1}{8} \times \frac{1}{5}, \text{ or } \frac{1}{1440}.$$

Consequently the *à priori* chance that they both assert either truly or falsely that a white ball should be drawn is

$$\frac{1}{15} + \frac{1}{1440}, \text{ or } \frac{97}{1440}.$$

But *à posteriori* this chance becomes certainty, or it becomes multiplied by $\frac{1440}{97}$. Hence the chance of each way in which they may make the assertion, is multiplied in the same ratio. Therefore, *à posteriori*, the chance that they make the assertion truly is

$$\frac{1}{15} \times \frac{1440}{97}, \text{ or } \frac{96}{97};$$

and the chance that they make it falsely is

$$\frac{1}{1440} \times \frac{1440}{97}, \text{ or } \frac{1}{97}.$$

*Question.*—*A* gives a true report four times out of five, *B* three times out of five, and *C* five times out of seven. If *B* and *C* agree in reporting that an experiment failed which *A* reports to have succeeded, what is the chance that the experiment succeeded?

*Answer.*—The chance that the given reports should be made upon the experiment having succeeded is

$$\frac{1}{2} \times \frac{4}{5} \times \frac{2}{5} \times \frac{2}{7}, \text{ or } \frac{16}{350}.$$

The chance that the given reports should be made on the experiment having failed is

$$\frac{1}{2} \times \frac{1}{5} \times \frac{3}{5} \times \frac{5}{7}, \text{ or } \frac{15}{350}.$$

The *à priori* chance that in one way or other the given reports should be made is

$$\frac{16}{350} + \frac{15}{350}, \text{ or } \frac{31}{350}.$$

But, *à posteriori*, this is certain, or the chance is multiplied by $\frac{350}{31}$. Hence, also, the chance of each way in which the reports could be made is multiplied by $\frac{350}{31}$.

Therefore, *à posteriori*, the chance that the experiment succeeded is

$$\frac{16}{350} \times \frac{350}{31}, \text{ or } \frac{16}{31}.$$

We will conclude this chapter with some illustrations of the principles of probability, drawn from the game of whist.

This game is played with a pack of fifty-two cards, consisting of four suits of thirteen cards, marked differently. The cards are all dealt out to four players, of whom two and two are partners, so that each has thirteen cards. One of the dealer's cards is turned up, and the suit to which this card belongs is called trumps. Four particular cards in this suit—the ace, king, queen, and knave—are called honours.

It follows, from Rule IV., that the chance that the turned up card is an honour is $\frac{4}{13}$, and that it is not an honour is $\frac{9}{13}$.

*Question.*—What is the chance that each party in the game should have two honours?

*Answer.*—Besides the turned up card, there are fifty-one cards, of which twenty-five belong to the dealer and his partner, and twenty-six to their adversaries.

*First,*—Suppose the turned up card is an honour. The chance of this is $\frac{4}{13}$. Then the chance that one other honour should be among the twenty-five, and the remaining two among the twenty-six, is

$$3 \cdot \frac{25}{51} \cdot \frac{26}{50} \cdot \frac{25}{49};$$

therefore the chance that the turned up card should be an honour, and the honours equally divided, is

$$3 \cdot \frac{25}{51} \cdot \frac{26}{50} \cdot \frac{25}{49} \cdot \frac{4}{13}, \text{ or } \frac{100}{833}.$$

*Secondly,*—Suppose the turned up card is not an honour. The chance of this is $\frac{9}{13}$. Then the chance that two of the honours should be among the twenty-five, and the remaining two among the twenty-six, is

$$\frac{4 \cdot 3}{1 \cdot 2} \cdot \frac{25}{51} \cdot \frac{24}{50} \cdot \frac{26}{49} \cdot \frac{25}{48};$$

therefore the chance that the turned up card should not be an honour, and the honours be equally divided, is

$$6 \cdot \frac{25}{51} \cdot \frac{24}{50} \cdot \frac{26}{49} \cdot \frac{25}{48} \cdot \frac{9}{13}, \text{ or } \frac{225}{833}.$$

Hence the whole chance that the honours should be equally divided is

$$\frac{100}{833} + \frac{225}{833}, \text{ or } \frac{325}{833}.$$

In the same manner we may write down almost at sight the chances of the occurrence of other arrangements of the cards. We give a few examples :—

1.—If an honour turns up, the respective chances that the dealer and his partner have between them exactly *one*, *two*, *three*, or *four* honours, are respectively

$$\frac{26.25.24}{51.50.49}, \quad 3.\frac{25.26.25}{51.50.49}, \quad 3.\frac{25.24.26}{51.50.49}, \quad \frac{25.24.23}{51.50.49};$$

or $$\frac{312}{2499}, \frac{975}{2499}, \frac{936}{2499}, \frac{276}{2499}.$$

the sum of these fractions being unity, for one of the four cases must certainly occur.

2.—If an honour does not turn up, the respective chances that the dealer and his partner have *none*, or exactly *one*, *two*, *three*, or *four* honours, are

$$\frac{26.25.24.23}{51.50.49.48}, \quad 4.\frac{25.26.25.24}{51.50.49.48}, \quad 6.\frac{25.24.26.25}{51.50.49.48},$$

$$4.\frac{25.24.23.26}{51.50.49.48}, \quad \frac{25.24.23.22}{51.50.49.48};$$

or $$\frac{299}{4998}, \frac{1300}{4998}, \frac{1950}{4998}, \frac{1196}{4998}, \frac{253}{4998};$$

the sum of these fractions being unity, for one of the five cases must certainly occur.

3.—Before it is known whether an honour will turn up, the respective chances that the dealer and his partner have between them *none*, or exactly *one*, *two*, *three*, or *four* honours, are

$$\frac{207}{4998}, \frac{1092}{4998}, \frac{1950}{4998}, \frac{1404}{4998}, \frac{345}{4998};$$

the sum of the fractions being unity.

Hence, speaking approximately, we may expect that on the average, for every *one hundred* times the cards are dealt, the dealer and his partner will have four honours *seven* times, and the other players *four* times. The dealer and his partner will have three honours *twenty-eight* times, and the other players *twenty-two* times. And each party will have two honours the remaining *thirty-nine* times.

4.—The chance that each of the four players should have one honour is

$$\text{either } \frac{\underline{|4}.12.13^3}{51.50.49.48} \text{ or } \frac{\underline{|3}.13^3}{51.50.49},$$

(which happen to be equal) according as the turned up card is not, or is, an honour. Before the turned up card be seen, the chance is

$$\frac{9}{13} \cdot \frac{\underline{|4}.12.13^3}{51.50.49.48} + \frac{4}{13} \cdot \frac{\underline{|3}.13^3}{51.50.49}, \text{ or } \frac{9633}{249900}.$$

5.—If an honour turns up, the respective chances that the dealer should hold exactly *one, two, three,* or *four* honours are

$$\frac{39.38.37}{51.50.49}, \quad \frac{3}{1}\times\frac{12.39.38}{51.50.49}, \quad \frac{3}{1}\times\frac{12.11.39}{51.50.49}, \quad \frac{12.11.10}{51.50.49};$$

or

$$\frac{54834}{124950}, \quad \frac{53352}{124950}, \quad \frac{15444}{124950}, \quad \frac{1320}{124950}.$$

The sum of these fractions is unity, it being certain that the dealer has at least one honour.

6.—If an honour does not turn up, the respective chances that the dealer should hold *none,* or exactly *one, two, three,* or *four* honours, are

$$\frac{39.38.37.36}{51.50.49.48}, \quad 4.\frac{12.39.38.37}{51.50.49.48}, \quad 6.\frac{12.11.39.38}{51.50.49.48},$$

$$4.\frac{12.11.10.39}{51.50.49.48}, \quad \frac{12.11.10.9}{51.50.49.48};$$

or

$$\frac{82251}{249900}, \quad \frac{109668}{249900}, \quad \frac{48906}{249900}, \quad \frac{8580}{249900}, \quad \frac{465}{249900}.$$

The sum of these fractions is unity, since one of the five cases must certainly occur.

7.—Before it is known whether an honour will turn up, the respective chances that the dealer should hold *none*, or exactly *one*, or *two*, or *three*, or *four* honours are

$$\frac{56943}{249900}, \frac{109668}{249900}, \frac{66690}{249900}, \frac{15444}{249900}, \frac{1155}{249900};$$

the sum of these fractions being unity.

8.—If an honour turns up, the respective chances that a player, who is not the dealer, should hold no honour, or exactly *one*, or *two*, or *three* honours are

$$\frac{38.37.36}{51.50.49}, \quad 3.\frac{13.38.37}{51.50.49}, \quad 3.\frac{13.12.28}{51.50.49}, \quad \frac{13.12.11}{51.50.49};$$

or $$\frac{50616}{124950}, \frac{54834}{124950}, \frac{17784}{124950}, \frac{1716}{124950};$$

the sum of these fractions being unity; since the player must certainly hold *none*, or *one*, or *two*, or *three* honours.

9.—If an honour does not turn up, the respective chances that the player, who is not the dealer, should hold *none*, or *one*, or *two*, or *three*, or *four* honours are

$$\frac{38.37.36.35}{51.50.49.48}, \quad 4.\frac{13.38.37.36}{51.50.49.48}, \quad 6.\frac{13.12.38.37}{51.50.49.48},$$

$$4.\frac{13.12.11.38}{51.50.49.48}, \quad \frac{13.12.11.10}{51.50.49.48};$$

or $$\frac{73815}{249900}, \frac{109668}{249900}, \frac{54834}{249900}, \frac{10868}{249900}, \frac{715}{249900}.$$

10.—Before it is known whether an honour will turn up, the respective chances that a player, who is not the dealer, should hold *none*, or *one*, or *two*, or *three*, or *four* honours are

$$\frac{82251}{249900}, \frac{109668}{249900}, \frac{48906}{249900}, \frac{8580}{249900}, \frac{495}{249900};$$

the sum of these fractions being unity.

## EXAMPLES ON CHANCE.

1.—If ten persons form a ring, what is the chance that two assigned persons will be together?

2.—If ten persons stand in a line, what is the chance that two assigned persons will stand together?

3—If two letters are selected at random out of the alphabet, what is the chance that both are vowels?

4.—Compare the chances of throwing four with one die, eight with two dice, and twelve with three dice, having two trials in each case.

5.—A bag contains four red balls and two white. Three times in succession a ball is drawn and replaced. Find the chance that a red ball is drawn each time.

6.—What is the probability of throwing not more than eight in a single throw, with three dice?

7.—A bag contains six black balls and one red. A person is to draw them out in succession, and is to receive a shilling for every ball he draws until he draws the red one. What is his expectation?

8.—There are ten tickets, five of which are numbered 1, 2, 3, 4, 5, and the other five are blank. What is the probability of drawing a total of ten in three trials, one ticket being drawn out and replaced at each trial?

9.—What is the probability in the preceding question if the tickets are not replaced?

10.—A person has ten coins, which he throws down in succession. He is to receive one shilling if the first falls head, two shillings *more* if the second *also* falls head, four shillings *more* if the third *also* falls head, and so on, the amount doubling each time; but as soon as a coin falls tail, he ceases to receive anything. What is the value of his expectation?

11.—$A$ and $B$ play at chess, and $A$ wins on an average two games out of three. Find the chance of $A$ winning four games out of the first six that are not drawn.

12.—$A$ and $B$ play at chess, and $A$ wins on an average five games out of nine. Find $A$'s chance of winning a majority (1) out of three games, (2) out of nine games, (3) out of four games, drawn games not being counted.

13.—If the odds on every game between two players are two to one in favour of the winner of the preceding game, what is the chance that he who wins the first game, shall win at least two out of the next three?

14.—*A, B, C* play at a game in which each has a separate score, and the game is won by the player who first scores two points. If the chances are respectively $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, that any point is scored by *A, B, C,* find the respective chances of the three players winning the game.

15.—Assuming the results stated on page 99, what are the odds against a person aged thirty living to be seventy?

16.—What is the chance that three persons, aged respectively 30, 40, and 50, will be all alive twenty years hence, and what is the chance that at least two of them will be alive?

17.—Four flies come into a room in which there are four lumps of sugar, of different degrees of attractiveness, proportional to the numbers 8, 9, 10, 12; what is the chance that the flies will all select different lumps?

18.—Shew that the odds are eleven to three against a month selected at random, containing portions of *six* different weeks.

19.—Reference is made to a month which contains portions of six different weeks; what is the chance that it contains thirty-one days?

20.—A living man is known to be between thirty and fifty years old, and the odds are estimated at three to two that he is over forty. If he now die, what do these odds become? (See page 99.)

# APPENDIX.

## PERMUTATIONS AND COMBINATIONS TREATED ALGEBRAICALLY.

Definition I. A collection of $r$ things in a particular order is called a *permutation* or *arrangement* of $r$ things.

Definition II. A collection of $r$ things without regard to order is called a *combination* or *selection* of $r$ things.

## PROPOSITION I.

*If one operation can be performed in m ways, and (when it has been performed in any way) a second operation can then be performed in n ways, there will be mn ways of performing the two operations.*

For if we confine our attention to the case in which the former operation is performed in its *first* way, we can associate with this way any of the $n$ ways of performing the latter operation: and thus we shall have $n$ ways of performing the two operations, without recognising more than the *first* way of performing the former one.

Then, if we consider the *second* way of performing the former operation, we can associate with this way any of the $n$ ways of performing the latter operation: and thus we shall have $n$ ways of performing the two operations, using only the *second* way of performing the former one.

And so, corresponding to *each* of the $m$ ways of performing the former operation, we shall have $n$ ways of performing the two operations.

Hence, altogether we shall have $m$ times $n$, or $mn$ ways of performing the two operations. Q. E. D.

## PROPOSITION II.

*If one operation can be performed in m ways, and then a second can be performed in n ways, and then a third in r ways, and then a fourth in s ways (and so on), the number of ways of performing all the operations will be* $m \times n \times r \times s \times$ *&c.*

For by Prop. I., the first and second can be performed in $mn$ ways.

Then if we treat these two as forming one complete operation, and associate with it the third operation (which can be performed in $r$ ways), it follows again from Prop. I., that both these can be performed in $mn \times r$ different ways. That is, the first, second, and third of the original operations can be performed in $mnr$ ways.

Again if we treat these three as forming one complete operation, and associate with it the fourth operation (which can be performed in $s$ ways), it follows again from Prop. I. that both these can be performed in $mnr \times s$ different ways. That is, the first, second, third, and fourth operations can be performed in $mnrs$ ways, and so on. Q. E. D.

COROLLARY.—*If there be $x$ operations which can be performed successively in $m$ ways each, then all can be performed in $m^x$ ways.*

This follows from the proposition, by considering the particular case in which $m, n, r, s$, &c., are all equal.

EXAMPLES.—If there are $p$ candidates for the office of president, $s$ candidates for that of secretary, and $t$ candidates for that of treasurer, the election of the three officers can be made in $pst$ different ways.

If a telegraph has $m$ arms, and each arm is capable of $n$ different positions, including the position of rest, the number of signals that can be made is $n^m - 1$.

If there be $x$ things to be given to $n$ persons, $n^x$ will represent the whole number of different ways in which they may be given.

## PROPOSITION III.

*The number of different orders in which n different things can be arranged is*

$$n\,(n-1)\,(n-2)\,\ldots\ldots\,3.2.1.$$

For having to arrange the $n$ things, we may arrive at any possible arrangement, by taking them one by one, and placing them in the $n$ places in order.

The first place may be filled up by any of the $n$ things: that is, it may be filled up in $n$ different ways.

Then the second place may be filled up by any of the $n-1$ things that are left: that is, it may be filled up in $n-1$ different ways.

Then the third place may be filled up by any of the $n-2$ things that are now left: that is, it may be filled up in $n-2$ different ways.

Similarly the fourth place may be filled up in $n-3$ ways, the fifth in $n-4$ ways, and so on; and ultimately the last place may be filled up in only one way.

Hence (Prop. II.) the whole number of ways of filling up all the places, or making the whole arrangement, is the continued product of all these numbers, or

$$n\,(n-1)\,(n-2)\,\ldots\ldots\,3.2.1.$$

NOTE.—The continued product of all integers from 1 to $n$ is generally denoted by the symbol $\underline{|n}$.

Corollary.—*If n given things have to be devoted to n given objects, one to each, the distribution can be made in* $\underline{|n}$ *ways.*

Examples.—The number of ways in which $n$ persons can stand in a row is $\underline{|n}$. The number of ways in which they can form a ring is $\underline{|n-1}$. (See page 23.)

The number of ways in which $m$ ladies and $m$ gentlemen can form a ring, no two ladies being together, is $\underline{|m} \,.\, \underline{|m-1}$.

## PROPOSITION IV.

*Out of n different things, the number of ways in which an arrangement of r things can be made is*

$$n\ (n-1)\ (n-2)\ \ldots \textit{ to } r \textit{ factors},$$

or
$$n\ (n-1)\ (n-2)\ \ldots\ (n-r+1).$$

For we have to fill up $r$ different places in order with some of the $n$ given things. As in the last proposition, the first place can be filled up in $n$ ways, the second in $n-1$ ways, the third in $n-2$ ways; and so on for all the $r$ places.

Hence the whole number of ways of filling up all the $r$ places, or making the required arrangement, is

$$n\ (n-1)\ (n-2)\ \ldots \text{ to } r \text{ factors}.$$

Or, observing that the

| | | |
|---|---|---|
| 1st factor is | | $n$, |
| 2nd | ,, | $n-1$, |
| 3rd | ,, | $n-2$, |
| &c. | | &c. |
| $r$th | ,, | $n-(r-1)$ or $n-r+1$, |

we may write the result,—

$$n(n-1)(n-2)\ .\ .\ .\ (n-r+1).$$

Example.—The number of times a company of $mn$ men can form a rectangular column, having $m$ men in front, so as to present a different front each time, is $mn(mn-1)(mn-2)\ .\ .\ .\ .\ (mn-m+1)$.

## PROPOSITION V.

*Out of $n$ different things, when each may be repeated as often as we please, the number of ways in which an arrangement of $r$ things can be made is $n^r$.*

For the first place can be filled up (as before) in $n$ ways, and when it is filled up the second place can also be filled up in $n$ ways (since we are not now precluded from repeating the selection already made); and so the third can be filled up in $n$ ways, and so on, for all the $r$ places.

Hence (Prop. II., Cor.) all the $r$ places can be filled up, or the whole arrangement can be made, in $n^r$ different ways.

EXAMPLE.—In the ordinary scale of notation $10^r - 1$ different numbers can be made, each consisting of not more than $r$ figures.

## PROPOSITION VI.

*The number of ways in which $x + y$ things can be divided into two classes, so that one may contain $x$ and the other $y$ things, is*

$$\frac{(x+y)!}{x!\,y!}.$$

For suppose N represents the number of ways in which the division could be made; then the things in the first class can be arranged in $x!$ different orders (Prop. III.), and the things in the second class in $y!$ different orders, and therefore the whole set of $x + y$ things can be arranged in $x$ places of one class, and $y$ places of another class, in $N \,.\, x! \,.\, y!$ different ways (Prop. II.). But this must be the same as the number of ways in which the whole set of $x + y$ things can be arranged into *any* $x + y$ different places, which, by Prop. III., is $(x + y)!$. Hence we have the equation

$$N \,.\, x! \,.\, y! = (x+y)!,$$

or

$$N = \frac{(x+y)!}{x!\,y!}.$$

That is, the number of ways in which the required division can be made is

$$\frac{\lfloor x + y}{\lfloor x \lfloor y},$$

which was to be proved.

## PROPOSITION VII.

*The number of ways in which $x + y + z$ things can be divided into three classes, so that they may contain $x$, $y$, and $z$ things severally, is*

$$\frac{\lfloor x + y + z}{\lfloor x \lfloor y \lfloor z}.$$

For, by Prop. VI., the $x + y + z$ things can be divided into two classes, containing $x$ and $y + z$ things in

$$\frac{\lfloor x + y + z}{\lfloor x \lfloor y + z}$$

ways; and then the class of $y + z$ things can be subdivided into two classes, containing $y$ and $z$ things in

$$\frac{\lfloor y + z}{\lfloor y \lfloor z}.$$

Therefore the three classes of $x$, $y$, $z$ things can be made in

$$\frac{\lfloor x + y + z}{\lfloor x \lfloor y + z} \times \frac{\lfloor y + z}{\lfloor y \lfloor z}, \text{ or } \frac{\lfloor x + y + z}{\lfloor x \lfloor y \lfloor z}$$

ways (Prop. I.), which was to be proved.

Corollary.—We might similarly extend the reasoning if there were any more classes. *Thus, the number of ways in which $v+w+x+y+z$ things can be divided into five classes, containing respectively $v$, $w$, $x$, $y$, $z$ things, is*

$$\frac{\underline{|v+w+x+y+z}}{\underline{|v}\;\underline{|w}\;\underline{|x}\;\underline{|y}\;\underline{|z}}.$$

Examples.—The number of different ways in which $2n$ boys can divide themselves into two equal parties, to play a game, is

$$\frac{\underline{|2n}}{(\underline{|n})^2}.$$

The number of ways in which $mn$ things can be divided into $m$ parcels, of $n$ things each, is

$$\frac{\underline{|mn}}{(\underline{|n})^m}.$$

## PROPOSITION VIII.

*The number of different orders in which $n$ things can be arranged, whereof $p$ are all alike (of one sort), $q$ all alike (of another sort), $r$ all alike (of another sort), and the rest all different is*

$$\frac{\underline{|n}}{\underline{|p}\;\underline{|q}\;\underline{|r}}.$$

For the operation of making this arrangement may be resolved into the several operations following:—

(1) to divide the $n$ places which have to be filled up into sets of $p$ places, $q$ places, $r$ places, and $n-p-q-r$ places respectively:

(2) to place the $p$ things all alike in the set of $p$ places, the $q$ things all alike in the set of $q$ places, the $r$ things all alike in the set of $r$ places:

(3) to arrange the remaining $n-p-q-r$ things which are all different in the remaining set of $n-p-q-r$ places.

Now the operation (1) can be performed, by Prop. VII., in

$$\frac{\lfloor n}{\lfloor p \, \lfloor q \, \lfloor r \, \lfloor n-p-q-r}$$

different ways: the operation (2) can be performed in only one way: the operation (3), by Prop. III., in $\lfloor n-p-q-r$ ways.

Hence, (Prop. II.) the whole operation can be performed in

$$\frac{\lfloor n}{\lfloor p \, \lfloor q \, \lfloor r \, \lfloor n-p-q-r} \times \lfloor n-p-q-r, \text{ or } \frac{\lfloor n}{\lfloor p \, \lfloor q \, \lfloor r}$$

different ways. Q. E. D.

Corollary.—The same argument would apply if the number of sets of things alike were any other than three. Thus, for instance, *the number of orders in which n things can be arranged, whereof p are alike, q others alike, r others alike, s others alike, and t others alike, is*

$$\frac{n!}{p!\,q!\,r!\,s!\,t!}$$

Example.—If there be $m$ copies of each of $n$ different volumes, the number of different orders in which they can be arranged on one shelf is

$$\frac{(mn)!}{(m!)^n}.$$

## PROPOSITION IX.

*Out of n different things, the number of different ways in which a selection of r things can be made, is the same as the number of different ways in which a selection of n−r things can be made, and is*

$$\frac{n!}{r!\,(n-r)!}$$

For either operation simply requires the $n$ things to be divided into two sets of $r$ and $n-r$ things respectively, whereof one set is to be taken and the other left.

Therefore (by the last proposition), whichever set be rejected, the operation can be performed in

$$\frac{n!}{r!\,(n-r)!}$$

different ways.

The expression

$$\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$$

may be written

$$\frac{n\,(n-1)\,(n-2)\,.\,.\,.\,.\,.\,.\,.\,.\,.\,3.2.1}{\underline{|r}\,.\,(n-r)\,(n-r-1)\,\,.\,.\,.\,\,3.2.1}$$

or, dividing the numerator and denominator of the fraction by all the successive integers from 1 to $n-r$,

$$\frac{n\,(n-1)\,(n-2)\,.\,.\,.\,.\,.\,(n-r+1)}{\underline{|r}}$$

This result might have been obtained quite independently, as follows:—

Let $x$ represent the number of ways of making a selection of $r$ things out of $n$ things. The $r$ things thus selected might be arranged (Prop. III.) in $\underline{|r}$ different orders. Therefore (Prop. I.) $x \times \underline{|r}$ is the number of ways in which $r$ things can be selected out of $n$ things, and arranged in order. But by Prop. IV. this can be done in $n\,(n-1)\,(n-2)\,.\,.\,.\,.\,.\,(n-r+1)$ different ways. Therefore we have the equation

$$x \times \underline{|r} = n\,(n-1)\,(n-2)\,.\,.\,.\,.\,.\,(n-r+1),$$

which gives us

$$x = \frac{n\,(n-1)\,(n-2)\,.\,.\,.\,.\,.\,.\,(n-r+1)}{\underline{|r}}$$

the required expression.

EXAMPLES.—The number of ways in which a committee of $p$ liberals and $q$ conservatives can be selected out of $m$ liberals and $n$ conservatives, is

$$\frac{\underline{|m}}{\underline{|p}\;\underline{|m-p}} \times \frac{\underline{|n}}{\underline{|q}\;\underline{|n-q}}.$$

If there be $n-1$ sets containing $2a, 3a, 4a, \ldots (n-1)a$ things respectively, the number of ways in which a selection can be made, consisting of $a$ things out of each set, is

$$\frac{\underline{|2a}}{\underline{|a}\;\underline{|a}} \times \frac{\underline{|3a}}{\underline{|2a}\;\underline{|a}} \times \frac{\underline{|4a}}{\underline{|3a}\;\underline{|a}} \times \&c. \ldots \times \frac{\underline{|na}}{\underline{|(n-1)\,a}\;\underline{|a}}$$

or

$$\frac{\underline{|na}}{(\underline{|a})^n}.$$

## PROPOSITION X.

*The whole number of ways in which a selection can be made out of n different things is* $2^n-1$.

For each thing can be either taken or left; that is, it can disposed of in two ways. Therefore (Prop. II., Cor.) all the things can be disposed of in $2^n$ ways. This, however, includes the case in which *all* the things are rejected, which is inadmissible; therefore the whole number of admissible ways is $2^n-1$. Q. E. D.

## PROPOSITION XI.

*The whole number of ways in which a selection can be made out of $p + q + r +$ &c., things, whereof $p$ are all alike (of one sort), $q$ all alike (of another sort), $r$ all alike (of another sort), &c., is*

$$(p + 1)(q + 1)(r + 1) \ldots\ldots \text{\&c.} - 1.$$

For, of the set of $p$ things all alike, we may take either 0 or 1 or 2 or 3 or &c. or $p$, and reject all the rest; that is, the $p$ things can be disposed of in $p + 1$ ways. Similarly, the $q$ things can be disposed of in $q + 1$ ways, the $r$ things in $r + 1$ ways, and so on. Hence (Rule II.) all the things can be disposed of in $(p + 1)\ (q + 1)\ (r + 1) \ldots\ldots$ ways. This, however, includes the case in which *all* the things are rejected, which is inadmissible; therefore the whole number of admissible ways is

$$(p + 1)\ (q + 1)\ (r + 1) \ldots . \text{\&c.} - 1.$$

Q. E. D.

EXAMPLE.—If there be $m$ sorts of things and $n$ things of each sort, the number of ways in which a selection can be made from them is $(n + 1)^m - 1$.

If there be $m$ sorts of things, and one thing of the first sort, two of the second, three of the third, and so on, the number of ways in which a selection can be made from them is $\underline{|m + 1} - 1$.

## PROPOSITION XII.

### A NEW PROOF OF THE BINOMIAL THEOREM.

The Binomial Theorem was first published by Sir Isaac Newton, who was Lucasian Professor of Mathematics at Cambridge from 1669 to 1702. It furnishes a ready method of raising any given binomial expression to any required power.

We proceed to consider a question of combinations, and, from our results, to deduce a proof of the binomial theorem, applicable to all cases in which the exponent is a positive integer.

*Question.*—A painter has $x + y$ colours, of which $x$ are dark and $y$ are light colours. He has to paint $n$ croquet-balls (all of different sizes), each ball being one colour, but as many balls as he pleases the same colour. In how many ways can he paint the balls?

*Answer I.*—Since each ball can be painted with any one of the $x + y$ colours, and there are $n$ balls, the whole number of different ways in which the work can be done is (by Prop. II., Cor.)

$$(x + y)^n.$$

*Answer II.*—*If he paint all the balls dark,* each can be painted in $x$ different ways; therefore the work can be done in $x^n$ different ways.

*If he paint one light and the rest dark,* the selection of the one to be light can be made in $n$ ways; then the

$n-1$ can be painted dark in $x^{n-1}$ ways, and the *one* light in $y$ ways; therefore the work can be done in $n\,x^{n-1}y$, or, as we will write it for the sake of symmetry,

$$\frac{n}{1}\,x^{n-1}y$$

different ways.

*If he paint two light and the rest dark*, the selection of the two to be light can be made in

$$\frac{n\,(n-1)}{1.2}$$

ways (Prop. IX.), then the $n-2$ can be painted dark in $x^{n-2}$ ways, and the *two* light in $y^2$ ways: therefore the work can be done in

$$\frac{n\,(n-1)}{1.2}\,x^{n-2}y^2$$

ways.

*If he paint three light and the rest dark*, the selection of the three to be light can be made in

$$\frac{n\,(n-1)\,(n-2)}{1.2.3}$$

ways (Prop. IX.), then the $n-3$ can be painted dark in $x^{n-3}$ ways, and the *three* light in $y^3$ ways: therefore the work can be done in

$$\frac{n\,(n-1)\,(n-2)}{1.2.3}\,x^{n-3}y^3$$

ways.

And so on, until finally we consider the case in which *all are light*, in which case the work can be done in $y^n$ ways.

Hence the whole number of ways in which the work can be done is the sum of the series

$$x^n + \frac{n}{1} x^{n-1} y + \frac{n(n-1)}{1.2} x^{n-2} y^2$$
$$+ \frac{n(n-1)(n-2)}{1.2.3} x^{n-3} y^3 + \&c. \ldots. + y^n$$

which will have $n + 1$ terms altogether.

*The Binomial Theorem.*—The two answers to the question just investigated must give the same result numerically, or the two algebraical results must be equal: therefore we have

$$(x + y)^n = x^n + \frac{n}{1} x^{n-1} y + \frac{n(n-1)}{1.2} x^{n-2} y^2$$
$$+ \frac{n(n-1)(n-2)}{1.2.3} x^{n-3} y^3 + \&c. \ldots. + y^n.$$

We are thus furnished with a formula by which we can write down any power of a binomial expression, as a series of terms, consisting of powers of the two original terms. The statement of this formula is called the Binomial Theorem.

EXAMPLES.—

$$(x + y)^3 = x^3 + \frac{3}{1} x^2 y + \frac{3.2}{1.2} xy^2 + \frac{3.2.1}{1.2.3} y^3$$
$$= x^3 + 3x^2y + 3xy^2 + y^3.$$

So, $(a-2b)^4$

$$= a^4 + \frac{4}{1}a^3(-2b) + \frac{4.3}{1.2}a^2(-2b)^2 + \frac{4.3.2}{1.2.3}a(-2b)^3 + (-2b)^4$$

$$= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4.$$

The theorem will hold equally if we have any two fractions, $\frac{p}{q}$ and $\frac{r}{s}$ suppose, instead of the numbers $x$ and $y$. For

$$\left(\frac{p}{q}+\frac{r}{s}\right)^n = \left(\frac{ps+qr}{qs}\right)^n = \frac{(ps+qr)^n}{(qs)^n};$$

and since $ps$ and $qr$ are integers, we may apply the theorem, and write

$$(ps+qr)^n = (ps)^n + \frac{n}{1}(ps)^{n-1}qr + \frac{n(n-1)}{1.2}(ps)^{n-2}(qr)^2$$

$$+ \&c. \ldots + (qr)^n;$$

and therefore dividing by $(qs)^n$ we have

$$\left(\frac{p}{q}+\frac{r}{s}\right)^n = \left(\frac{p}{q}\right)^n + \frac{n}{1}\left(\frac{p}{q}\right)^{n-1}\frac{r}{s} + \frac{n(n-1)}{1.2}\left(\frac{p}{q}\right)^{n-2}\left(\frac{r}{s}\right)^2$$

$$+ \&c. \ldots + \left(\frac{r}{s}\right)^n,$$

which shews that the expansion follows the same law when any fractions are substituted for the terms $x$ and $y$.

EXAMPLES.—

$$\left(x+\frac{1}{x}\right)^4 = x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}.$$

$$\left(1-\frac{a}{b}\right)^4 = 1 - \frac{4}{1}\left(\frac{a}{b}\right) + \frac{4.3}{1.2}\left(\frac{a}{b}\right)^2 - \frac{4.3.2}{1.2.3}\left(\frac{a}{b}\right)^3 + \frac{4.3.2.1}{1.2.3.4}\left(\frac{a}{b}\right)^4$$

$$= 1 - \frac{4a}{b} + \frac{6a^2}{b^2} - \frac{4a^3}{b^3} + \frac{a^4}{b^4}.$$

$$\left(\frac{x}{2}+\frac{y}{3}\right)^3 = \left(\frac{x}{2}\right)^3 + \frac{3}{1}\left(\frac{x}{2}\right)^2\left(\frac{y}{3}\right) + \frac{3.2}{1.2}\left(\frac{x}{2}\right)\left(\frac{y}{3}\right)^2 + \frac{3.2.1}{1.2.3}\left(\frac{y}{3}\right)^3$$

$$= \frac{1}{8}x^3 + \frac{1}{4}x^2y + \frac{1}{6}xy^2 + \frac{1}{27}y^3.$$

It may here be observed that if $x$ be a small fraction, the powers $x^2$, $x^3$, &c., will be smaller still, and will rapidly become inconsiderable when the index is increased. Hence, a few terms of the expansion

$$1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \&c.$$

will give an approximately true value for $(1+x)^n$ when $x$ is small compared with unity.

It is proved in treatises on algebra that the formula of expansion

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \&c.$$

still holds when $n$ is a fractional or negative index. But in this case the series is interminable, and is only of practical use when $x$ is a proper fraction, when a finite number of terms will give an approximate value.

EXAMPLES.—

*To find the square root of* $1-x$ *in ascending powers of* $x$.

We have

$$\sqrt{1-x}=(1-x)^{\frac{1}{2}}$$

$$=1-\tfrac{1}{2}x-\frac{\frac{1}{2}\cdot\frac{1}{2}}{1.2}x^2-\frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{1.2.3}x^3-\frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3.4}x^4-\&c.$$

$$=1-\frac{x}{2}-\frac{1}{1.2}\left(\frac{x}{2}\right)^2-\frac{1.3}{1.2.3}\left(\frac{x}{2}\right)^3-\frac{1.3.5}{1.2.3.4}\left(\frac{x}{2}\right)^4-\&c.$$

*To find the square root of* 2.

We have

$$2=\frac{98}{49}=\frac{100}{49}\cdot\frac{98}{100}=\frac{100}{49}\left(1-\frac{1}{50}\right)$$

Therefore,

$$\sqrt{2}=\frac{10}{7}\sqrt{1-\frac{1}{50}}$$

But, as in the last example,

$$\sqrt{1-\frac{1}{50}}=1-\frac{1}{100}-\frac{1}{1.2}\left(\frac{1}{100}\right)^2-\frac{1.3}{1.2.3}\left(\frac{1}{100}\right)^3-\&c.$$

$$=1-\cdot01-\cdot00005-\cdot0000005-\cdot00000000625-\&c.$$

$$=\cdot9899494936\ldots\ldots$$

and therefore, multiplying by $\frac{10}{7}$

$$\sqrt{2}=1\cdot414213562\ldots\ldots$$

This is the readiest method of extracting the square root of 2, when a very high degree of accuracy is required. Very little labour would extend the result to 20 or 30 places of decimals. The binomial theorem may be similarly applied to find the square root of any other number, or to find cube roots or any other roots. Indeed when the fifth, seventh, or any higher root of a number is required, this method is the only practicable one, unless tables of logarithms are employed; and it has the advantage over the method by logarithms of bringing the result to any degree of accuracy required.

# APPENDIX II.

## DISTRIBUTIONS.

Most of the questions of Permutations and Combinations which we have considered have involved the division of a given series of things into two parts, one part to be chosen, and the other rejected. The theorem expressed arithmetically in Rule VI. (page 30), and algebraically in Proposition VII. (page 149), is the only one in which we have contemplated distribution into more than two classes. But as the number of things to be given to each class was in the terms of that theorem assigned, the problem was reduced to a case of successive selection, and was therefore classed with other questions of combinations. But when the number of elements to be distributed to each several class is unassigned, and left to the exercise of a further choice, the character of the problem is very much altered, and the problem ranks among a large variety which we class together as problems of *Distribution*.

Distribution is the separation of a series of elements into a series of classes. The great variety that exists among problems of distribution may be mostly traced to five principal elements of distinction, which it will be well to consider in detail before enunciating the propositions on which the solution of the problems will depend.

I.—*The things to be distributed may be different or indifferent.* The number of ways of distributing five gifts among three recipients, will greatly depend upon whether the gifts are all alike or various. If they are all alike (or, though unlike, yet indifferent as far as the purposes of the problem are concerned), the only questions will be (i.) whether we shall divide them into sets of 2, 2, 1 or 3, 1, 1, and (ii.) how we shall assign the three sets to the three individuals. If on the contrary the five gifts are essentially different, as *a*, *b*, *c*, *d*, *e*, then they may be divided into sets of 2, 2, 1 in 15 ways, and into sets of 3, 1, 1 in 10 ways, and then we shall have to assign the three sets which are in this case all essentially different (because their component elements are so), to the three individuals. In the first case, the sets could be formed in 2 ways, and when formed in either way they could be assigned in 3 ways, thus giving a complete choice of 6 distributions. In the second case, the sets could be formed in 25 ways, and when formed in any way they could be assigned in 6 ways, thus giving a complete choice of 150 distributions.

II.—*The classes into which the things are to be distributed may be themselves different or indifferent.* We here use the adjectives *different* or *indifferent* to qualify the abstract classes regarded as ends or objects to which the articles are to be devoted, without any reference to *à posteriori* differences existing merely in differences of distribution into the classes.

Where five gifts were to be distributed to three recipients, the distinct personality of the three recipients made the classes characteristically different, quite apart from the consideration of the differences of the elements which composed them. But if we had only to wrap up five books in three different parcels, and no difference of destination were assigned to the parcels, we should speak of the parcels as *indifferent*. The problem would be simply to divide the five things into three sets, without assigning to the sets any particular order. The distribution could be made in 2 ways if the things themselves were indifferent, and in 25 ways if they were different.

III.—*The order of the things in the classes may be different or indifferent*, that is, the classes may contain permutations or combinations. Of course this distinction can only arise when the things themselves are different, for we cannot recognise any order among indifferent elements. We shall avoid confusion by distinguishing arranged and unarranged classes respectively as *groups* and *parcels*. If three men are to divide a set of books amongst them, it is a case of division into *parcels*, for it does not matter in what order or arrangement any particular man gets his books. But if a series of flags are to be exhibited as a signal on three masts, it is a case of division into *groups*, for every different arrangement of the same flags on any particular mast would constitute a different signal.

IV.—*It may or may not be permissible to leave some of the possible classes empty.* It will entirely depend upon the circumstances out of which the problem arises, whether it shall be necessary to place at least one element in every class, or whether some of them may be left vacant; in fact, whether the number of classes named in the problem is named as a limit not to be transgressed, or as a condition to be exactly fulfilled. If we are to distribute five gifts to three recipients, it will probably be expected, and unless otherwise expressly stated it will be implied, that no one goes away empty. But if it be asked how many signals can be displayed by the aid of five flags on a three-masted ship, it will be necessary to include the signals which could be given by placing all the flags on one mast, or on two masts.

V.—*It may or may not be permissible to leave some of the distributable things undistributed.* This will be illustrated by a comparison of the Propositions XXV. and XXVI. below.

Propositions XIII., XIV., XV., and XVI. apply to the distribution of *indifferent* things.

The XVII. and following propositions embrace the different cases which arise in the distribution of *different* things.

The case in which the *parcels* are *indifferent* as well as the things to be distributed into them, is reserved to the last, as presenting peculiar difficulty. It will be found treated of in Prop. XXVIII.

## PROPOSITION XIII.

*The number of ways in which $n$* indifferent *things can be distributed into $r$* different *parcels* (*blank lots being inadmissible*) *is the number of combinations of $n-1$ things taken $r-1$ at a time.*

For we may perform the operation by placing the $n$ things in a row, then placing $r-1$ points of partition amongst them, and assigning the $r$ parts thus created, in order, to the $r$ parcels in order.

Hence the number of ways is the number of ways of placing $r-1$ points of partition in a selection out of $n-1$ intervals. Therefore it is the same as the number of combinations of $n-1$ things taken $r-1$ at a time. Q. E. D.

## PROPOSITION XIV.

*The number of ways in which $n$* indifferent *things* *can be distributed into $r$ different parcels (blank lots being admissible) is the number of combinations of $n+r-1$ things taken $r-1$ at a time.*

For the distribution of $n$ things, when blank lots are admissible, is the same as the distribution of $n+r$ things when they are not admissible, since in the latter case we have to place one thing in each of the $r$ parcels, and then to distribute the remainder as if blank lots were admissible. Hence, writing $n+r$ for $n$ in the result of Proposition XIII., we obtain the number required.

EXAMPLES.—Twenty shots are to be fired; the work can be distributed among four guns in $\frac{19 \,.\, 18 \,.\, 17}{1 \,.\, 2 \,.\, 3}$ or 969 ways, without leaving any gun unemployed. Or, neglecting this restriction, the work can be done in $\frac{23 \,.\, 22 \,.\, 21}{1 \,.\, 2 \,.\, 3}$ or 1771 ways.

Again, five partners in a game require to score 36 to win. The number of ways in which they may share this score (not all necessarily contributing), is

$$\frac{40 \,.\, 39 \,.\, 38 \,.\, 37}{1 \,.\, 2 \,.\, 3 \,.\, 4}$$

or 91390 different ways.

Again, in how many ways can five oranges be distributed amongst seven boys? Evidently two or more of them will get none. The answer is 462, viz.,

$$\frac{11.10.9.8.7.6}{1.2.3.4.5.6}.$$

## PROPOSITION XV.

*The number of ways in which n* indifferent *things can be distributed into r* different *parcels, no parcel to contain less than q things, is the number of combinations of* $n-1-r(q-1)$ *things taken r at a time.*

For if we first place $q$ things in each of the $r$ parcels we shall have $n-qr$ things left, and it will only remain to distribute them among the same $r$ parcels according to Proposition XIV., which shows that the number of ways of making the distribution is the number of combinations of $(n-qr)+r-1$ things taken $r$ at a time.

Q. E. D.

## PROPOSITION XVI.

*The number of ways in which n* indifferent *things can be distributed into r* different *parcels, no parcel to contain less than q things, nor more than* $q+z-1$ *things is the coefficient of* $x^{n-qr}$ *in the expansion of*

$$\left(\frac{1-x^z}{1-x}\right)^r$$

For if we multiply together $r$ factors, each represented by

$$x^q+x^{q+1}+\ldots\ldots+x^{q+z-1}$$

we shall have in our result a term $x^n$ for every way in which we can make up $n$ by the addition of one index $q$ or $q+1$ or $q+2$ or &c., or $q+z-1$ from each of the $r$ factors. Hence we shall have $x^n$ as many times as there are ways of distributing $n$ into $r$ parts, no part less than $q$ nor greater than $q+z-1$. Therefore the number of ways of so distributing $n$ is the coefficient of $x^n$ in the expansion of

$$(x^q+x^{q+1}+\ldots\ldots+x^{q+z-1})^r$$

or of

$$x^{qr}(1+x+x^2+\ldots x^{z-1})^r$$

which is the coefficient of $x^{n-qr}$ in the expansion of

$$(1+x+x^2+\ldots+x^{z-1})^r$$

or of

$$\left(\frac{1-x^z}{1-x}\right)^r \qquad \text{Q. E. D.}$$

Example.—The number of ways in which four persons, each throwing a single die once, can score 17 amongst them is the coefficient of $x^{17-4}$ in the expansion of

$$\left(\frac{x^6-1}{1-x}\right)^4$$

Now

$$(1-x^6)^4=1-4x^6+6x^{12}-\&\text{c}.$$

$$(1-x)^{-4}=\frac{1}{6}\left\{1.2.3+2.3.4x+3.4.5x^2+\ldots\right\}$$

And coefficient of $x^{13}$ in the product

$$=\frac{1}{6}\left\{14.15.16-4.8.9.10+6.2.3.4\right\}$$

$$=104.$$

## PROPOSITION XVII.

*The number of ways in which* $n$ different *things can be distributed into* $r$ different *parcels is* $r^n$, *when blank lots are admissible.*

For each of the $n$ different things can be assigned to any one of the $r$ parcels without thought of how the others are disposed of. Hence the $n$ things can be (severally and) successively disposed of in $r$ ways each, and therefore (*Choice*, Rule II.) all can be disposed of in $r^n$ different ways. Q. E. D.

## PROPOSITION XVIII.

*When blank lots are* not *admissible, the number of ways in which* $n$ different *things can be distributed into* $r$ *different parcels is* $\lfloor n$ *times the coefficient of* $x^n$ *in the expansion of* $(e^x-1)^r$,

Let $N_r$ denote the number of ways in which $n$ things can be distributed into $r$ different parcels, blank lots being inadmissible.

Then $rN_{r-1}$ will be the number of ways in which the distribution might be made if one parcel were left blank.

So $\frac{r(r-1)}{1.2} N_{r-2}$ will be the number of ways in which the distribution might be made if two parcels were left blank.

And so on,

But we know that if any number of blank lots were admissible, the distribution could be made in $r^n$ different ways. Therefore

$$r^n = N_r + \frac{r}{1} N_{r-1} + \frac{r(r-1)}{1.2} N_{r-2} + \&c. + rN_1,$$

or, if we establish the convention that $N^p$ is always to be replaced by $N_p$, we may write

$$r^n = (N+1)^r - 1.$$

Similarly, $$(r-1)^n = (N+1)^{r-1} - 1,$$
$$(r-2)^n = (N+1)^{r-2} - 1,$$

and so on.

Now multiply these equations in order by the coefficients in the expansion of $(1-x)^r$, and add (having regard to algebraical sign); then the first member of the resulting equation will be

$$r^n - \frac{r}{1}(r-1)^n + \frac{r(r-1)}{1.2}(r-2)^n - \&c. \ (\textit{till it stops})$$

and the second member, since the sum of the coefficients in the expansion of $(1-x)^r$ is zero, will be $\{(N+1)-1\}^r$ or $N^r$ or $N_r$. Therefore

$$N_r = r^n - \frac{r}{1}(r-1)^n + \frac{r(r-1)}{1.2}(r-2)^n - \&c. \text{ till it stops.}$$

that is, (Todhunter's *Algebra*, Art. 549.)

$N_r = \lfloor n$ times the coefficient of $x^n$ in the expansion of $(e^x - 1)^r$. Q. E. D.

EXAMPLES.—The number of ways in which five different commissions can be executed by three messengers is $3^5$ or 243. But if no one of the messengers is to be unemployed, the number of ways will be $\lfloor 5$ times the coefficient of $x^5$ in the expansion of $(e^x-1)^3$. But

$$(e^x-1)^3 = \left(x+\frac{x^2}{2}+\frac{x^3}{6}+\dots\right)^3$$
$$= x^3 + \frac{3}{2}x^4 + \frac{5}{4}x^5 + \dots$$

Hence the number of ways will be $\frac{5}{4}\times\lfloor 5$ or 150.

## PROPOSITION XIX.

*The number of r-partitions of n* different *things, i. e. the number of ways in which n* different *things can be distributed into r* indifferent *parcels, with no blank lots, is* $\lfloor n$ *times the coefficient of* $x^n$ *in the expansion of*

$$\frac{(e^x-1)^r}{\lfloor r}.$$

For every way of distributing the things into *r indifferent* parcels, must give rise to $\lfloor r$ ways of distributing them when the parcels are different. Hence, if $\mathfrak{N}_r$ denote the number of partitions, we have, by comparison with Prop. XVIII.,

$$\mathfrak{N}_r = \frac{N_r}{\lfloor r}$$

$= \lfloor n$ times the coefficient of $x^n$ in the expansion of

$$\frac{(e^x-1)^r}{\lfloor r} \qquad \text{Q. E. D.}$$

Example.—To divide the letters $a, b, c, d, e$ into three parcels. The number of ways will be $\underline{|5}$ times the coefficient of $x^5$ in the expansion of $(e^x - 1)^3 \div \underline{|3}$;

that is (as in the last example), $5.4 \times \frac{5}{4} = 25$.

The twenty-five divisions are easily seen to be ten such as $abc, d, e$, and fifteen such as $ab, cd, e$.

## PROPOSITION XX.

*The total number of ways in which* $n$ different *things can be distributed into* 1, 2, 3 ... *or* $n$ indifferent *parcels is* $\underline{|n}$ *times the coefficient of* $x^n$ *in the expansion of* $\frac{e^{e^x}}{e}$.

For with the notation of preceding theorems, we have

$$\mathfrak{N}_1 = \underline{|n} \text{ times the coefficient of } x^n \text{ in } \frac{e^x - 1}{\underline{|1}},$$

$$\mathfrak{N}_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{(e^x-1)^2}{\underline{|2}},$$

$$\mathfrak{N}_3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{(e^x-1)^3}{\underline{|3}},$$

and so on.

Therefore by addition

$$\mathfrak{N}_1 + \mathfrak{N}_2 + \mathfrak{N}_3 + \ldots (\textit{till it stops, i. e.} \ldots \mathfrak{N}_n)$$

is equal to $\underline{|n}$ times the coefficient of $x^n$ in the expansion of

$$\frac{e^x-1}{\underline{|1}} + \frac{(e^x-1)^2}{\underline{|2}} + \frac{(e^x-1)^3}{\underline{|3}} + \&c.,$$

this last series being carried to infinity if we please, since the terms beyond the $n^{\text{th}}$ do not involve $x^n$, and therefore the inclusion of them will not affect the coefficient of $x^n$.

But this series is the expansion of $e^{e-1}-1$, and the coefficient of $x^n$ therein is the same as in the expansion of $e^{e^x-1}$ or $e^{e^x} \div e$. Hence

$$\mathfrak{N}_1+\mathfrak{N}_2+\mathfrak{N}_3+\dots+\mathfrak{N}_n$$

is equal to $\underline{|n}$ times the coefficient of $x^n$ in the expansion of

$$\frac{e^{e^x}}{e}$$

Q. E. D.

## PROPOSITION XXI.

*The number of ways in which n* different *things can be arranged in r* different *groups* (*with no blank lots*) *is*

$$\frac{\underline{|n}\,\underline{|n-1}}{\underline{|n-r}\,\underline{|r-1}}.$$

For they can be arranged in one row in $\underline{|n}$ ways, and then the $r-1$ points of partition can be placed in a selection of the $n-1$ intervals in $\dfrac{\underline{|n-1}}{\underline{|n-r}\,\underline{|r-1}}$ ways.

But the number of ways in which $n$ things can be arranged in $r$ different groups must be the product of these two numbers (*Choice*, Rule I.), or

$$\frac{\underline{|n}\,\underline{|n-1}}{\underline{|n-r}\,\underline{|r-1}}.$$

Q. E. D.

N

## PROPOSITION XXII.

*The number of ways in which* $n$ different *things can be arranged in* $r$ indifferent *groups with no blank lots is*

$$\frac{\lfloor\underline{n}\,\lfloor\underline{n-1}}{\lfloor\underline{r}\,\lfloor\underline{n-r}\,\lfloor\underline{r-1}}.$$

For it is plain that for any one arrangement in this case we must have $\lfloor\underline{r}$ arrangements when the groups are not indifferent. Hence the result is $\frac{1}{\lfloor\underline{r}}$ of that in Proposition XXI., or

$$\frac{\lfloor\underline{n}\,\lfloor\underline{n-1}}{\lfloor\underline{r}\,\lfloor\underline{n-r}\,\lfloor\underline{r-1}}.$$

Q. E. D.

## PROPOSITION XXIII.

*The total number of ways in which* $r$ different *arranged groups can be made out of* $m$ *things all different, is the coefficient of* $x^{m-r}$ *in the expansion of*

$$\frac{\lfloor\underline{m}\; e^x}{(1-x)^r},$$

*blank groups being inadmissible.*

For if we use $n$ of the things at a time the groups can be made (by Theorem XI.) in

$$\frac{\lfloor\underline{m}\,\lfloor\underline{n-1}}{\lfloor\underline{m-n}\,\lfloor\underline{n-r}\,\lfloor\underline{r-1}}$$

ways. And $n$ may have any value from $r$ to $m$ inclusive. Hence the required number is

$$\underline{|m}\left\{\frac{1}{\underline{|m-r}}-\frac{1}{\underline{|m-r-1}}\cdot\frac{r}{\underline{|1}}+\frac{1}{\underline{|m-r-2}}\cdot\frac{r(r+1)}{\underline{|2}}+\&c. \text{ to } m-r+1 \text{ terms}\right\}.$$

which is equal to the coefficient of $x^{m-r}$ in the product of the two series

$$1+\frac{r}{\underline{|1}}x+\frac{r(r+1)}{\underline{|2}}x^2+\frac{r(r+1)(r+2)}{\underline{|3}}x^3+\&c.,$$

$$\text{and } \ldots+\frac{\underline{|m}\,x^{m-r}}{\underline{|m-r}}+\frac{\underline{|m}\,x^{m-r-1}}{\underline{|m-r-1}}+\frac{\underline{|m}\,x^{m-r-2}}{\underline{|m-r-2}}+\&c.,$$

which are respectively the expansions of $(1-x)^{-r}$ and $\underline{|m}\,e^x$. Hence the required number is the coefficient of $x^{m-r}$ in the expansion of

$$\frac{\underline{|m}\,e^x}{(1-x)^r}. \qquad \text{Q. E. D.}$$

## PROPOSITION XXIV.

*The total number of ways in which $r$* indifferent *arranged groups can be made out of $m$ things all different is the coefficient of $x^{m-r}$ in the expansion of*

$$\frac{\underline{|m}\,e^x}{\underline{|r}\,(1-x)^r},$$

*blank lots being inadmissible.*

This follows from the previous theorem, as Proposition XXII. from Proposition XXI.

## PROPOSITION XXV.

*The number of ways in which $n$* different *things can be arranged in $r$* different *groups (blank groups being admissible) is*

$$\frac{\lfloor\underline{n+r-1}}{\lfloor\underline{r-1}}.$$

For they can be arranged in one row in $\lfloor\underline{n}$ ways, and then the $r-1$ points of partition can be placed in the $n+1$ intervals (including the ends of the row) in

$$\frac{\lfloor\underline{n+r-1}}{\lfloor\underline{n}\,\lfloor\underline{r-1}}$$

ways by Proposition XIV.

The number of ways required is the product of these two numbers (*Choice*, Rule I.), or

$$\frac{\lfloor\underline{n+r-1}}{\lfloor\underline{r-1}}.$$

Q. E. D.

## PROPOSITION XXVI.

*The total number of signals that can be made by displaying arrangements out of $m$ flags on a set of $r$ masts, where each mast will hold any number of flags, is one less than the coefficient of $x^m$ in the expansion of*

$$\frac{\lfloor\underline{m}\; e^x}{(1-x)^r}.$$

If we use all the $r$ masts, the number of signals is (by Prop. XXIII.) the coefficient of $x^{m-r}$ in the expansion of

$$\frac{\lfloor m \, e^x}{(1-x)^r},$$

$$= \text{the coefficient of } x^m \text{ in expansion of } \frac{\lfloor m \, e^x x^r}{(1-x^r)}.$$

So, if we use $r-1$ given masts, the number of signals is the coefficient of $x^m$ in the expansion of $\frac{\lfloor m \, e^x \, x^{r-1}}{(1-x)^{r-1}}$, and the $r-1$ masts can be selected in $r$ ways, therefore the number of signals

$$= \text{the coefficient of } x^m \text{ in expansion of } \frac{r}{1} \cdot \frac{\lfloor m \, e^x \, x^{r-1}}{(1-x)^{r-1}},$$

and so on for $(r-2)$, $(r-3)$, &c. masts. Hence, the total number of signals, including the case when *no flag* is hoisted is the coefficient of $x^m$ in the expansion of

$$\lfloor m \, e^x \left\{ \left(\frac{x}{1-x}\right) + \frac{r}{1}\left(\frac{x}{1-x}\right)^{r-1} + \frac{r(r-1)}{1.2}\left(\frac{x}{1-x}\right)^{r-2} + \&c. \text{ to } r+1 \text{ terms} \right\},$$

or
$$\lfloor m \, e^x \left(1 + \frac{x}{1-x}\right)^r,$$

or
$$\frac{\lfloor m \, e^x}{(1-x)^r}.$$

Hence, excluding the case when no flag is hoisted, the number of signals is one less than the coefficient of $x^m$ in the expansion of

$$\frac{\lfloor m \, e^x}{(1-x)^r}, \qquad \text{Q. E. D.}$$

## PROPOSITION XXVII.

*To find the number of ways in which $n$* indifferent *things can be distributed into $r$* indifferent *parcels (no blank lots).*

Or

*To find the number of different $r$-partitions of $n$.*

Let $P_{n,r}$ denote the number of $r$-partitions of $n$, or the number of ways of distributing $n$ indifferent elements into $r$ indifferent parcels.

Suppose that in any distribution, $x$ is the smallest number found in any parcel. Then setting aside a parcel which contains $x$, all the other parcels contain not less than $x$, and therefore more than $x-1$. If we place $x-1$ in each of these $r-1$ parcels, the distribution can then be completed by distributing the remaining $n-x-(r-1)(x-1)$ or $n-1-r(x-1)$ things among the same $r-1$ parcels, and this can be done in $P_{n-1-r(x-1),\ r-1}$ ways. In this way we shall obtain all the distributions, by giving $x$ successively all its possible values. But since $x$ is the smallest number found in any parcel, $x$ cannot be greater than the greatest integer in $\frac{n}{r}$. Denote this integer by $\left\lfloor\frac{n}{r}\right\rfloor$. Then $x$ must have all values from 1 to this integer, and therefore

$$P_{n,r} = P_{n-1\ r-1} + P_{n-1-r,\ r-1} + P_{n-1-2r,\ r-1} + \ \ldots \text{ to } \left\lfloor\frac{n}{r}\right\rfloor \text{ terms.}$$

Now it is plain that $P_{n,1} = 1$ for all values of $n$.

Hence $P_{n,2} = P_{n-1,1} + P_{n-3,1} + P_{n-5,1} + \ldots$ to $\frac{|n}{2|}$ terms

$$= \frac{|n}{2|}$$

Again $P_{n,3} = P_{n-1,2} + P_{n-4,2} + P_{n-7,2} + \ldots$ to $\frac{|n}{3|}$ terms

$$= \frac{|n-1}{2\ |} + \frac{|n-4}{2\ |} + \frac{|n-7}{2\ |} + \ldots \text{ to } \frac{|n}{3|} \text{ terms.}$$

The summation will depend upon the form of $n$; thus,

| | |
|---|---|
| If $n = 6q$ | Then $P_{n,3} = \dfrac{n^2}{12}$ |
| $n = 6q \pm 1$ | $P_{n,3} = \dfrac{n^2 - 1}{12}$ |
| $n = 6q \pm 2$ | $P_{n,3} = \dfrac{n^2 - 4}{12}$ |
| $n = 6q + 3$ | $P_{n,3} = \dfrac{n^2 + 3}{12}$ |

Therefore $P_{n,3}$ is always *the integer nearest to* $\dfrac{n^2}{12}$ whether in excess or defect.

This integer is conveniently denoted by the symbol

$$\left|\frac{n^2}{12}\right|$$

Again, $P_{n,4} = P_{n-1,3} + P_{n-5,3} + P_{n-9,3} + \ldots$ to $\frac{|n}{4|}$ terms

$$= \left|\frac{(n-1)^2}{12}\right| + \left|\frac{(n-5)^2}{12}\right| + \left|\frac{(n-9)^2}{12}\right| + \ldots \text{ to } \frac{|n}{4|} \text{ terms.}$$

The summation will depend upon the form of $n$: thus,

If $n=12q$ then $P_{n,4}=\dfrac{n^3+3n^2}{144}$

$n=12q+1$ $P_{n,4}=\dfrac{n^3+3n^2-9n+5}{144}$

$n=12q+2$ $P_{n,4}=\dfrac{n^3+3n^2-20}{144}$

$n=12q+3$ $P_{n,4}=\dfrac{n^3+3n^2-9n-27}{144}$

$n=12q+4$ $P_{n,4}=\dfrac{n^3+3n^2+32}{144}$

$n=12q+5$ $P_{n,4}=\dfrac{n^3+3n^2-9n-11}{144}$

$n=12q+6$ $P_{n,4}=\dfrac{n^3+3n^2-36}{144}$

$n=12q-5$ $P_{n,4}=\dfrac{n^3+3n^2-9n+5}{144}$

$n=12q-4$ $P_{n,4}=\dfrac{n^3+3n^2+16}{144}$

$n=12q-3$ $P_{n,4}=\dfrac{n^3+3n^2-9n-27}{144}$

$n=12q-2$ $P_{n,4}=\dfrac{n^3+3n^2-4}{144}$

$n=12q-1$ $P_{n,4}=\dfrac{n^3+3n^2-9n-11}{144}$

Therefore $P_{n,4}$ is always the integer nearest to $\dfrac{n^3+3n^2}{144}$ when $n$ is even, and the integer nearest to $\dfrac{n^3+3n^2-9n}{144}$ when $n$ is odd: or, with the notation introduced above,

$$\begin{cases} P_{n,4} = \left|\dfrac{n^3+3n^2}{144}\right| \text{ when } n \text{ is even :} \\ P_{n,4} = \left|\dfrac{n^3+3n^2-9n}{144}\right| \text{ when } n \text{ is odd.} \end{cases}$$

By a like process we may deduce successively $P_{n,5}$, $P_{n,6}$, &c., and thus we may find $P_{n,r}$ for any values of $n$ and $r$, although we cannot write down a general expression for $P_{n,r}$ in any simple terms.

EXAMPLES.—There are twelve 3-partitions of 12, viz.

| | | | |
|---|---|---|---|
| 1 1 10 | 1 4 7 | 2 3 7 | 3 3 6 |
| 1 2 9 | 1 5 6 | 2 4 6 | 3 4 5 |
| 1 3 8 | 2 2 8 | 2 5 5 | 4 4 4 |

There are fifteen 4-partitions of 12, viz.

| | | |
|---|---|---|
| 1 1 1 9 | 1 2 2 7 | 2 2 2 6 |
| 1 1 2 8 | 1 2 3 6 | 2 2 3 5 |
| 1 1 3 7 | 1 2 4 5 | 2 2 4 4 |
| 1 1 4 6 | 1 3 3 5 | 2 3 3 4 |
| 1 1 5 5 | 1 3 4 4 | 3 3 3 3 |

The number of 4-partitions of 13 is the integer nearest to $(n^3+3n^2-9n)\div 144$ when $n=13$. Therefore there are 18 partitions, viz.

| | | |
|---|---|---|
| 1 1 1 10 | 1 2 3 7 | 2 2 2 7 |
| 1 1 2 9 | 1 2 4 6 | 2 2 3 6 |
| 1 1 3 8 | 1 2 5 5 | 2 2 4 5 |
| 1 1 4 7 | 1 3 3 6 | 2 3 3 5 |
| 1 1 5 6 | 1 3 4 5 | 2 3 4 4 |
| 1 2 2 8 | 1 4 4 4 | 3 3 3 4 |

*To find the number of 5-partitions of* 13.

We have $P_{13,5}=P_{12,4}+P_{7,4}$

$$=15+3$$

$$=18.$$

These eighteen partitions may be exhibited as follows:

| | | |
|---|---|---|
| 1 1 1 1 9 | 1 1 2 3 6 | 1 2 2 4 4 |
| 1 1 1 2 8 | 1 1 2 4 5 | 1 2 3 3 4 |
| 1 1 1 3 7 | 1 1 3 3 5 | 1 3 3 3 3 |
| 1 1 1 4 6 | 1 1 3 4 4 | 2 2 2 2 5 |
| 1 1 1 5 5 | 1 2 2 2 6 | 2 2 2 3 4 |
| 1 1 2 2 7 | 1 2 2 3 5 | 2 2 3 3 3 |

# APPENDIX III.

## DERANGEMENTS.

When we place a series of elements in a particular order we are said to *arrange* them. But if they have been already arranged, or if they have a proper order of their own, and we place them in other order, we are said to *derange* them. Thus derangement implies a previous arrangement in which each element had its own proper place, either naturally belonging to it or arbitrarily assigned to it.

The following Notation is useful:—

It is proved, in treatises on algebra, that if $e$ be the base of Napierian logarithms then, whatever be the value of $x$ positive or negative,

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \&c.$$

This series is continued *to infinity*, but by suffixing an integer to $e^x$ we obtain a symbol which conveniently expresses the sum of the same series continued only as far as the term in which that integer is the index of $x$. Thus—

$$e^x_n = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots + \frac{x^n}{\lfloor n}.$$

$$e^{-1}_n = 1 - 1 + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \ldots\ldots \pm \frac{1}{\lfloor n},$$

$$e^{-2}_3 = 1 - 2 + \frac{4}{\lfloor 2} - \frac{8}{\lfloor 3} = -\frac{1}{3}.$$

Let J denote the operation of changing any factorial into the next inferior factorial, as $\lfloor n$ into $\lfloor n-1$, or $\lfloor n-1$ into $\lfloor n-2$. And let JJ or $J^2$ indicate that the operation is to be performed a second time upon the result of the first, so that $J^2$ operating on $\lfloor n$ produces $\lfloor n-2$, and so on.

$$\begin{aligned} \text{Then}\quad J\lfloor n &= \lfloor n-1 \\ J^2\lfloor n &= \lfloor n-2 \\ J^3\lfloor n &= \lfloor n-3 \\ J^r\lfloor n &= \lfloor n-r \end{aligned}$$

Thus we may write—

$$\lfloor n . e^x_n = \lfloor n + \frac{n}{1}\lfloor n-1 . x + \frac{n(n-1)}{1 . 2}\lfloor n-2 . x^2 + \ldots\ldots$$

$$= (1 + xJ)^n . \lfloor n$$

where it is understood that $(1+xJ)^n$ is to be expanded by the law of the binomial theorem, and every term is then to operate upon $\lfloor n$.

As a particular case we may write

$$\lfloor n . e^{-1}_n = (1-J)^n . \lfloor n.$$

## PROPOSITION XXVIII.

*The number of derangements of a set of elements is one less than the number of permutations of the elements.*

For of all the permutations, one must give the proper order of the elements, and all the rest must be derangements.

## PROPOSITION XXIX.

*The number of ways in which a row of n elements may be so deranged that no element shall be in its proper place is* $\underline{|n}.e_n^{-1}$.

Let $\alpha, \beta, \gamma, \ldots \kappa$ denote the $n$ elements, and let $N$ represent the number of ways in which they can be permuted when unrestricted by any condition.

Also let $(A)$ express the condition that $\alpha$ is in its proper place and $(a)$ the condition that $\alpha$ is out of its proper place. Let $(B)$ and $(b)$ denote the same conditions with respect to $\beta$; and so on.

[With this notation $N\,(ACdk)$ will stand for the words, "*The number of permutations of the n things subject to the conditions that $\alpha$ and $\gamma$ are in their proper places and $\delta$ and $\kappa$ not in their proper places.*"]

Then we have

$$N = \underline{|n}$$

and $$N(A) = \underline{|n-1}$$

But since every permutation must satisfy one and only one of the conditions expressed by ($A$) and ($a$) it follows that

$$N(A) + N(a) = N$$

therefore $$N(a) = \lfloor\underline{n} - \lfloor\underline{n-1}$$

Now if we introduce the condition ($B$) our choice will be the same as if $\beta$ did not exist, and therefore the same as if we had $n-1$ elements to deal with instead of $n$ elements. Hence writing $n-1$ for $n$ we obtain from the last equation

$$N(aB) = \lfloor\underline{n-1} - \lfloor\underline{n-2}$$

And by subtraction—remembering that the conditions ($B$) and ($b$) are complementary—

$$N(ab) = \lfloor\underline{n} - 2\lfloor\underline{n-1} + \lfloor\underline{n-2}$$

Repeating our former operation, writing $n-1$ for $n$ on the introduction of the condition ($C$) we have

$$N(abC) = \lfloor\underline{n-1} - 2\lfloor\underline{n-2} + \lfloor\underline{n-3}$$

and by subtraction

$$N(abc). = \lfloor\underline{n} - 3\lfloor\underline{n-1} + 3\lfloor\underline{n-2} - \lfloor\underline{n-3}$$

Similarly

$$N\,(abcd) = \lfloor\underline{n} - 4\lfloor\underline{n-1} + 6\lfloor\underline{n-2} - 4\lfloor\underline{n-3} + \lfloor\underline{n-4}$$

and so on, the coefficients following the same law as in the Binomial theorem.

Hence finally,

$$N\,(abcd\ldots k) = \lfloor\underline{n} - \frac{n}{1}\lfloor\underline{n-1} + \frac{n\,(n-1)}{1.\,2}\lfloor\underline{n-2}$$
$$+ \frac{n\,(n-1)\,(n-2)}{1.\,2.\,3}\lfloor\underline{n-3} + \&c.$$

$$=\lfloor n\left\{1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\text{\&c. to } n+1 \text{ terms.}\right\}$$

$$=\lfloor n . e_n^{-1}. \qquad \text{Q. E. D.}$$

The student who is familiar with the use of symbols of operation will arrange the foregoing proof briefly as follows:

$$N = \lfloor n$$

$$N(A) = J \lfloor n$$

subtracting

$$N(a) = (1-J) \lfloor n$$

then

$$N(aB) = J\,(1-J) \lfloor n$$

and subtracting

$$N\,(ab) = (1-J)(1-J) \lfloor n = (1-J)^2 \lfloor n$$

And so every introduction of a condition such as $(A)$ produces a factor of operation $J$, and every introduction of a condition such as $(a)$ produces a factor of operation $(1-J)$

Hence $N(abc\ldots k) = (1-J)^n \lfloor n = \lfloor n . e_n^{-1}$.

## Another Proof of Proposition XXIX.

Let $f_n$ denote the number of ways of deranging a set of $n$ elements so that no element may be in its proper place.

Then the number of derangements so that exactly $r$ elements may be in their proper places will be

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} f_{n-r}$$

since such derangements are obtained by first selecting $r$ elements to retain their proper places and then deranging the remaining $n-r$.

But all the $\lfloor n$ permutations of the $n$ elements must be made up of those in which 0, 1, 2, &c., have their proper places.

Hence

$$\lfloor n = f_n + \frac{n}{1} f_{n-1} + \frac{n(n-1)}{1.2} f_{n-2} + \dots + n f_1 + 1$$

And, if we establish the convention that $f^r$ is always to be replaced by $f_r$ we may write

$$\lfloor n = (f+1)^n$$

Similarly $\lfloor n-1 = (f+1)^{n-1}$

$$\lfloor n-2 = (f+1)^{n-2}$$

and so on.

Now multiply these equations in order by the coefficients in the expansion of $(1-x)^n$ and add (having regard to algebraical signs) and we obtain an equation of which the first member is

$$\lfloor n + \frac{n}{1} \lfloor n-1 + \frac{n(n-1)}{1.2} \lfloor n-2 - \&c.$$

or $\lfloor n \left\{1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \&c. \text{ to } n+1 \text{ terms.}\right\}$

or $\lfloor n . e_n^{-1}$

while the second member is $\left\{(f+1) - 1\right\}^n$ or $f^n$ which by our convention represents $f_n$

Therefore $f_n = \lfloor n . e_n^{-1}$ Q. E. D.

Corollary I.—*The number of ways in which n elements can be deranged so that not any one of r assigned elements may be in its proper place (the rest being unrestricted) is*

$$\lfloor n - \frac{r}{1}\lfloor n-1 + \frac{r(r-1)}{1.2}\lfloor n-2 - \ldots \pm \lfloor n-r$$

$$\text{or } \lfloor n\left\{1 - \frac{r}{\lfloor n} + \frac{1}{\lfloor 2}\frac{r(r-1)}{n(n-1)} - \frac{1}{\lfloor 3}\frac{r(r-1)(r-2)}{n(n-1)(n-2)} + \&c. \text{ to } n+1 \text{ terms.}\right\}$$

This is established *passim* in the first proof of the Proposition.

Corollary II.—*The number of derangements of m+n elements so that m are displaced and n not displaced is*

$$\frac{\lfloor m+n}{\lfloor n} e_m^{-1}$$

Corollary III.—*If an arrangement of n elements be re-arranged at random, the chance that no element will be in its original position is* $e_n^{-1}$.

Corollary IV.—*If an arrangement of an infinite number of elements be re-arranged at random the chance that no element will be in its original position is* $e^{-1}$ *or* $\frac{1}{e}$.

Example.—Suppose we have the four elements *a b c d;* the number of derangements, so that all may be displaced, is by the proposition

$$\lfloor 4 \left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 9.$$

These nine derangements are as follows :

| | | |
|---|---|---|
| *b d a c* | *c a d b* | *d c a b* |
| *b a d c* | *c d b a* | *d c b a* |
| *b c d a* | *c d a b* | *d a b c* |

If it be required to derange the same terms so that two may remain *in situ* and two be displaced, the number of derangements is, by Corollary II.

$$12\left\{1 - 1 + \frac{1}{2}\right\} = 6.$$

These six derangements are as follows :

| | | |
|---|---|---|
| *a b d c* | *a d c b* | *a c b d* |
| *b a c d* | *c b a d* | *d b c a* |

## PROPOSITION XXX.

*The number of ways of deranging a series of n terms so that no term may be followed by the term which originally followed it is* $\lfloor n\, e_n + \lfloor n-1\, e^{-1}_{n-1}$

Let $\alpha, \beta, \gamma, \ldots \varkappa$ represent the $n$ terms. Then amongst the $\lfloor n$ arrangements of which the terms are capable there will be $\lfloor n-1$ in which any assigned sequence $\alpha\beta$ occurs : (for the arrangements will be obtained by regarding $\alpha\beta$ as one term and then arranging it with the remaining $n-2$ terms.) Similarly any two sequences which can consistently occur (as $\alpha\beta$, $\beta\gamma$, or $\alpha\beta$, $\gamma\delta$)* will be found in $\lfloor\underline{n-2}$ different arrangements. Any three consistent sequences will be found in $\lfloor\underline{n-3}$ different arrangements: and so on.

Hence there are $\lfloor\underline{n}$ arrangements altogether, among which we should find

$\alpha\beta$ in $\lfloor\underline{n-1}$ of them

$\beta\gamma$ in $\lfloor\underline{n-1} - \lfloor\underline{n-2}$ *more* of them

$\gamma\delta$ in $\lfloor\underline{n-1} - 2\lfloor\underline{n-2} + \lfloor\underline{n-3}$ more

$\delta\varepsilon$ in $\lfloor\underline{n-1} - 3\lfloor\underline{n-2} + \frac{3.2}{1.2}\lfloor\underline{n-3} - \lfloor\underline{n-4}$ more

and so on for all the $n-1$ sequences. Therefore the number of arrangements free from any of these sequences is

$$\lfloor\underline{n} - n\lfloor\underline{n-1} + \frac{n(n-1)}{1\,.\,2}\lfloor\underline{n-2} - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\lfloor\underline{n-3} + \&c.$$

$$+ \lfloor\underline{n-1} - (n-1)\lfloor\underline{n-2} + \frac{(n-1)(n-2)}{1\,.\,2}\lfloor\underline{n-3} + \&c.$$

$$= \lfloor\underline{n}\,.\,e^{n-1} + \lfloor\underline{n-1}\,.\,e^{-1}_{n-2}$$

or, adding and subtracting unity,

$$= \lfloor\underline{n}\,.\,e^{-1}_{n} + \lfloor\underline{n-1}\,.\,e^{-1}_{n-1}$$

Q. E. D.

The foregoing result may be written in very convenient form by the use of the notation explained on page 186. Thus

$$\lfloor\underline{n}\; e^{-1}_{n} + \lfloor\underline{n-1}\; e^{-1}_{n-1} = (1-J)^{n}\lfloor\underline{n} + (1-J)^{n-1}\lfloor\underline{n-1}$$

* Of course such sequences as $\alpha\beta$ $\alpha\gamma$ could not consistently occur, as $\alpha$ could not at the same time be followed by $\beta$ and $\gamma$.

or, since the operation $(1-J)^n$ is equivalent to the operations $(1-J)^{n-1}(1-J)$

$$= (1-J)^{n-1}\left\{\lfloor n - \lfloor n-1\right\} + (1-J)^{n-1}\lfloor n-1$$

$$= (1-J)^{n-1}\lfloor n$$

Or we may establish the result in this form independently as follows.

## PROPOSITION XXXI (otherwise.)

*The number of ways of deranging a series of n terms so that no term may be followed by the term which originally followed it, is* $(1-J)^{n-1}\lfloor n$

For there are $\lfloor n$ arrangements of the $n$ terms $\alpha, \beta, \gamma \ldots \varkappa$ and we should find

$\alpha\beta$ in $J\lfloor n$ of them

$\beta\gamma$ in $(J-J^2)\lfloor n$ or $(1-J)J\lfloor n$ more of them

$\gamma\delta$ in $(J-2J^2+J^3)\lfloor n$ or $(1-J)^2J\lfloor n$ more:

and so on.

Hence the whole number of arrangements containing at least one of the $n-1$ sequences $\alpha\beta, \beta\gamma, \gamma\delta, \ldots \iota\varkappa$ is

$$\left\{1+(1-J)+(1-J)^2+(1-J)^3+\ldots \text{ to } n-1 \text{ terms}\right\}J\lfloor n$$

$$= \left\{1-(1-J)^{n-1}\right\}\lfloor n$$

and therefore the number of admissible arrangements is

$$(1-J)^{n-1}\lfloor n$$

Q. E. D.

Corollary.—*The number of derangements of a series of n terms, free from any of r assigned sequences (which might occur simultaneously in one arrangement) is* $(1-J)^r \lfloor n$

Example.—Let us derange the series of four elements *abcd* so as to exclude the sequences *ab bc cd.*

By the proposition the number of derangements is $\lfloor 4 - 3\lfloor 3 + 3\lfloor 2 - \lfloor 1$, or 11.

And they are found on trial to be

| | | | |
|---|---|---|---|
| *acbd* | *bdca* | *cadb* | *dbac* |
| *adcb* | *badc* | *cbad* | *dcba* |
| | *bdac* | *cbda* | *dacb* |

## PROPOSITION XXXII.

*The number of ways of deranging the series of n terms* $\alpha, \beta, \gamma, \ldots \iota, \kappa$, *so that none of the n sequences* $\alpha\beta, \beta\gamma, \ldots \iota\kappa, \kappa\alpha$ *may occur is*

$$\lfloor n \, . \, e_{n-1}^{-1}$$

For, as before, there are $\lfloor n$ arrangements of the $n$ terms, and

$\alpha\beta$ occurs in $\lfloor n-1$ of them,

$\beta\gamma$ in $\lfloor n-1 - \lfloor n-2$ more of them,

$\gamma\delta$ in $\lfloor n-1 - 2\lfloor n-2 + \lfloor n-3$ more,

and so on for all the $n$ sequences, *except* that in the case of the last one the final term $\lfloor 0$ must be rejected since there cannot be any arrangements containing all the $n$ sequences.

Therefore the whole number of admissible arrangements is

$$n! - n\,(n-1)! + \frac{n(n-1)}{1.2}\,(n-2)! - \text{\&c. to } n \text{ terms.}$$

$$= n!\,.\,e^{-1}_{n-1}$$

a result which may also be written

$$n\,(1-J)^{n-1}\,(n-1)!$$

Example.—Let us derange the series of four elements *abcd* so as to exclude the sequences *ab bc cd da.*

By the proposition the number of derangements is $4!\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right\}$ or 8: and they are found to be

| | | | |
|---|---|---|---|
| *acbd* | *bdca* | *cadb* | *dbac* |
| *adcb* | *badc* | *cbad* | *dcba* |

## PROPOSITION XXXIII.

*If n terms be arranged in circular procession the number of ways in which they can be deranged so that no term may be followed by the term which originally followed it, is*

$$n!\left\{\frac{1}{n}-\frac{1}{n-1}+\frac{1}{2!\,(n-2)}-\frac{1}{3!\,(n-3)}+\ldots\ldots\pm\frac{1}{n!}\right\}$$

For the whole number of arrangements of $n$ things in circular procession is $(n-1)!$; and the sequence

$\alpha\beta$ occurs in $\lfloor n-2$ of them,

$\beta\gamma$ in $\lfloor n-2 - \lfloor n-3$ more,

$\gamma\delta$ in $\lfloor n-2 - \lfloor n-3 + \lfloor n-4$ more

and so on for all the $n$ sequences, provided we replace $\lfloor -1$ in the last term by unity. Hence the number of arrangements free from any of these sequences is

$$\lfloor n-1 - n\lfloor n-2 + \frac{n(n-1)}{1.2}\lfloor n-3 - \text{\&c. to } (n+1) \text{ terms.}$$

$$= \lfloor n\left\{\frac{1}{n} - \frac{1}{n-1} + \frac{1}{\lfloor 2(n-1)} - \frac{1}{\lfloor 3(n-3)} + \ldots \pm \frac{1}{\lfloor n}\right\}$$

Q. E. D.

If we establish the convention that $J^n\lfloor n-1 = \lfloor -1$ is to be replaced by unity, the above series is seen to be the algebraical expansion of $(1-J)^n\lfloor n-1$, in which form the result is easily remembered.

EXAMPLES.—

If $n=3$, the number of derangements may be written

$$\lfloor 2 - 3\lfloor 1 + 3\lfloor 0 - 1 = 1$$

or the only available derangement is the one in which the order of the terms is reversed.

If $n = 5$ we have

$$\lfloor 4 - 5\lfloor 3 + 10\lfloor 2 - 10\lfloor 1 + 5\lfloor 0 - 1 = 8$$

If *abcde* represent the original order the eight derangements may be exhibited as follows

*acbed* *aebdc* *acedb* *aecbd* *adceb*

*adbec* *acebd* *aedcb*

If $n = 6$ the number of derangements is 36: if $n = 7$ it is 229: and if $n = 11$ it is 1214673. And always if $n$ be a prime number the number of derangements, with 2 added, is a multiple of $n$.

In the foregoing propositions we have investigated the number of ways of deranging groups of elements subject to various laws. But as there can scarcely be a limit to the variety of laws which might be proposed to regulate the distribution in different cases, it would be an endless task to undertake a strictly complete discussion of the subject, or to make our treatise exhaustive. The cases which we have considered are those which most obviously arise, and the methods which we have applied to them will be easily adapted to a variety of other cases, or will suggest other methods of still wider applicability.

# APPENDIX IV.

## ON THE DISADVANTAGE OF GAMBLING.

"If (says Professor Rogers,) we are to understand the very elements of political economy, we must get rid of the impression, that if the contract be voluntary and the service be mutual, one man's gain is another's loss. . . . . The real truth is exactly the reverse; for one man's gain in all acts of free exchange is another man's gain.*" A fair bargain is a mutual benefit to the persons between whom it is made. If this were not so all commerce would be immoral, for no man could seek his own commercial profit without compassing the injury of his neighbour and so violating the law of civilised humanity.

But as a fair bargain is an advantage to both the contracting parties, so, speaking generally, a fair wager is a disadvantage to each party who enters into it. By a fair wager, we mean one in which each party's stake is equal to his mathematical expectation, calculated according to the principles laid down in the chapter on Chance: for instance, the wager is fair, if two men stake

* Manual of Political Economy, Oxford, 1868, p. 4.

equal sums, and either is to take the whole stakes according as a coin falls "head" or "tail." It is a common impression that such a wager as this, being obviously of the same import to each contracting party, can be neither an advantage nor a disadvantage to either, and that consequently, any the slightest odds must make it expedient for the party favoured by the odds to enter into the wager. For instance, according to this view, it must be decidedly expedient for any man to stake a pound against a pound and a penny, (if he can find any one foolish enough to give the odds of 241 to 240) that a coin will fall on an assigned side. Of course a wager may be made on such unequal terms that it may be decidedly expedient (from a selfish point of view) for one party to enter into it, but it must then be still more decidedly inexpedient for the other party. We do not deny that a man who does not scruple to take advantage of the ignorance or folly of another, or to exert against his neighbour the intellectual violence of superior knowledge or cunning,* may profitably enter into gambling speculations; but we combat the notion that there is a neutral advantage or disadvantage in a fair wager, and that the contract to play a game of pure chance for equal stakes, though it be not expedient, cannot be branded as inexpedient, and the delusion that, though no good be done, at least

* "Thou wouldst not take, by force or stealth,
What is not lawfully thy right;
But in the race for power and wealth
No wrong is done by mental might!"—*Monsell.*

no harm is done. It is this notion which palliates gambling. If it were only recognised that a fair wager were disadvantageous to each contracting party, it would be regarded as disreputable for one man to cast this disadvantage on another, even though he were accepting the like himself. But at present the evil motives which may lead men to gamble are covered by a reputable cloak, in the charitable hope that each party may be entering into a contract not disadvantageous to the other.

Every prospect of receiving anything of value, however doubtful the prospect be, we must regard as having itself some value. If a man have a chance, however small, of receiving £100, he will not relinquish his title to it without receiving something in return. He may take £75, or £50, or it may be £2 or £1 for his chance, according to his estimate of the probability of his getting the prize, but if he has any chance at all, his prospect must be worth something. When a man makes a wager, he is buying such a prospect as this. He pays down a certain sum of money and receives in return a doubtful prospect of a larger sum. Whether his bargain be advantageous or disadvantageous will depend upon whether the sum that he pays down is worth to him less or more than the prospect which he buys, and we can only decide this by considering whether he would gain or lose in the long run, if he repeated his operation on the same terms for a very great number of times. It is our object to prove that if the sum which he pay be the mathematical expectation of his prospect (and this is plainly the only sum which he can *fairly* pay in justice to the other contracting party), the bargain is disadvantageous to him: in the long run he will lose by the repetition of it.

For instance: if there be twenty tickets in a lottery for a prize which is worth £1, the value of the expectation of a man who holds one ticket is one shilling; and it is plain that the organiser of the lottery cannot without loss sell the tickets for less than one shilling each: the *fair* price of a ticket is one shilling. But according to our principle, it is inexpedient for a man to give so much as one shilling for a ticket; and though the twenty tickets together are undoubtedly worth twenty shillings, yet a single ticket is no more worth a shilling than a single glove is, by itself, worth half the price of a pair of gloves.

The twenty tickets in the hands of the original holder were worth a pound, but when distributed to twenty men (we say) they are not worth twenty shillings. The establishment of the lottery to effect the distribution was therefore inexpedient; the distribution itself was on the whole disadvantageous.

But if the distribution of the tickets be disadvantageous, their collection must be in the same degree advantageous. If therefore there exist in the nature of things such chances as those represented by the tickets of which we have spoken, it will be a beneficial and profitable act to collect those chances together. And this is precisely what an Insurance Company does, when it issues a policy undertaking to indemnify an owner against accidents which may befal his property. The man who insures his house against fire exchanges an uncertain position for a certain one. The man who buys a lottery ticket exchanges a certain position for an uncertain one. Insurance is the reverse of gambling, and is only wise in that gambling is foolish. The consent of the civilised world to the proposition that insurance is expedient is a tacit acknowledgment of the truth of the cognate proposition that gambling is inexpedient.

This will be seen more clearly by the consideration of an example.

Suppose that out of every twenty ships which make a particular voyage one is lost, and the remaining nineteen come safely to port. And suppose there is one ship making this voyage which with its cargo is worth £20,000. The value of the owner's expectation, according to our chapter on Chance, is just £19,000. But, according to the hypothesis which we are illustrating, it would be expedient for the owner to take a less sum than this for his expectation, say £19,000 $-x$. He may in consequence prudently pay £1,000 $+x$ to an Insurance Company in return for a guarantee that his £20,000 shall be secured him in full. And the Insurance Company, collecting together a great number of such risks, may profitably accept the bargain, their profit being entirely dependent on the fact that the shipowner is ready to accept for his contingent prospect an uncontingent sum, which is less than his mathematical expectation. For if the Insurance Company were to insure all the ships, securing to each owner his mathematical expectation, their own mathematical expectation of profit would be zero. They could only hope that the premiums received would in the long run balance the claims upon them, without leaving any profit to remunerate them for their trouble. Thus, the continued existence of Insurance Companies commercially successful is a standing witness to the fact, that a prudent man will commute a contingent prospect of value for less than the sum measured by his mathematical expectation.

But some one will object that mathematics must be utterly at fault, if an expectation is always worth less than the value which mathematics would assign to it. Not so. But if the mathematical result seems to contradict the conclusions of experience, or to violate the plain dicta of prudence, it is not that the mathematics have failed, but that two different problems have been confused. The mathematician solves one problem: the speculator seizes the result, and expects it to answer to another. It is not a true statement of the principle we have been enforcing to say that an expectation is always worth less than the value which mathematics would assign to it. The price which a speculator may prudently give for a contingent prospect of value depends always upon the amount of money he has to speculate with. But that which is commonly called the mathematical expectation of the contingency is the price which a man of infinite means might prudently pay for it. Therefore, the true statement of the case is rather that an expectation is worth less to a man of limited means, than the value which mathematics assign to the like expectation to a man whose means are unlimited.

The value of an expectation to any particular man, is the stake which he may prudently lay down for the sake of the expectation. That this depends upon the man's means is evident, as soon as we consider an extreme case. However foolish it may be for a man who is possessed of thousands of pounds to make a bet of £100, we are sensible that if the same bet were made by a man who possessed nothing in the world but the £100 which he risked the folly would be very much greater. Or, however advantageous it might be for the rich man to stake £100 in a speculative venture, it would scarcely be prudent for the poor man to do the same, and to run a risk of absolute ruin.

But the reader will expect that we appeal not to his sense of what is prudent, but that we rather shew him, by close mathematical reasoning, how the value of an expectation depends upon the means of the speculator.

The question whether a particular speculation is advantageous or disadvantageous to a particular man, can only be tested by considering whether, if he repeat the operation continually, he will in the end gain or lose. But what is to be understood by repeating the operation? If a boy, who has only a shilling, tosses for sixpence, he is staking half his property. Suppose he wins, and so becomes the possessor of eighteenpence. If we now speak of his repeating the operation, we may mean either of two things (1), that he is again to stake *sixpence;* or (2), that he is again to stake *half his property*, which will now be ninepence. So also, if he loses, there will be the like ambiguity as to whether he is to repeat the operation by staking his remaining sixpence, or only by staking threepence.

And so always, as soon as we take into consideration the funds at the disposal of a speculator, the stake which he lays down in any venture may be considered either as an absolute amount, or as a certain fraction of his entire fund. And by the repetition of his venture, we may either understand that he stakes the same sum again and again, or that he always stakes the same fraction of the fund which he holds at the time of making the venture.

But, whichever of these views we take, we are led mathematically to the same conclusion, that a fair wager is disadvantageous to a man whose means are limited, because, if it be repeated indefinitely, he will in the long run be the loser.

First,— Suppose that the same sum is repeatedly staked, and for the sake of simplicity suppose the sum to be gained the same as the sum staked, as in the case of an even bet. The speculator repeatedly stakes (say) a pound against a pound, so that every venture issues in making him either a pound richer or a pound poorer, each of these results being equally likely. If he continue his operation a great number of times, the balance of profit and loss will be continually varying. At one time he may have gained a considerable number of pounds, at another time he may find himself much poorer than when he began. And if his means were unlimited, he might go on gambling for ever; and there would be no reason to expect him to leave off a loser rather than a gainer. But his means being limited, this equilibrium of chances is disturbed. He has only (say) $n$ pounds to begin with. The balance of profit and loss may oscillate: now he may have gained $x$ pounds, and now he may have lost $y$ pounds, and the balance may again recoil; but as soon as ever a loss of $n$ pounds is reached, there is no more hope of restitution, for he has nothing more to venture. If his funds had been unlimited, there would have been an equal prospect of his leaving off richer or poorer, but the limitation of his resources, being apt to put a sudden termination on his operations at a time when the balance is most against him, interferes with the equality of chances, and occasions a presumption that he will leave off the loser. If, indeed, he were gambling with another speculator whose means were also limited, we should have to set against the prospect of the game abruptly terminating against him the counter prospect of its abruptly terminating in his favour by the exhaustion of the other man's funds; but, in fact, no one is restricted to gambling with one single opponent; the speculator deals with the public at large, with a world whose resources are practically unlimited. There is a prospect that his operations may terminate to his own disadvantage, through his having nothing more to stake; but there is no prospect that it will terminate to his advantage through the exhaustion of the resources of the world. Every one who gambles is carrying on an unequal warfare: he is ranged with a restricted capital against an adversary whose means are infinite.

We have said there is a chance of the man being ruined who with limited means continues to stake a pound against a pound in a fair wager. But if he prolong his play indefinitely, the chance of his being ruined is not distinguishable from certainty: it approaches nearer to certainty than by any assignable difference; and the chance that he should escape ruin is less than any assignable chance. Thus: —

## PROPOSITION XXXIII.

*If a man with limited funds repeatedly stake a pound against a pound, in a fair wager, the chance of his not being ultimately ruined is less than any assignable chance.*

Suppose his original funds are $n$ pounds: and let $R_x$ represent the chance of his being ultimately ruined when he has $x$ pounds in hand. At the same time $1-R_x$ will be the chance that he will escape ruin.

If at any time he have only one pound, the next venture must either ruin him or double his fund: the chance of either of these issues is one-half, and in the former case his chance of ruin is $R_0$ or unity, in the latter case it is $R_2$. Hence,

$$R_1 = \tfrac{1}{2} + \tfrac{1}{2}R_2$$

or

$$2R_1 = 1 + R_2$$

Similarly if at any time he have $x$ pounds a single venture must leave him with either $x-1$ or $x+1$ and so we have

$$2R_x = R_{x-1} + R_{x+1}$$

This being true for all values of $x$ it follows that $1, R_1, R_2, R_3 \ldots.$ are in arithmetical progression, and if we write

$$R_1 = 1-p$$

then

$$R_n = 1-np$$

and if we take the case of any one else whose funds are $z$ pounds his chance of ruin is $R_z = 1-zp$, where $p$ is a constant quantity whatever be the value of $z$. But however great a man's funds may be, his chance of ruin can never be negative. Hence, however great a value we assign to $z$, $R_z$ can never be less than zero, and therefore $zp$ (which is $1-R_z$) can never be greater than unity.

Therefore $p$ is not greater than $\frac{1}{z}$, and $np$ not greater than $\frac{n}{z}$ however great $z$ may be. But by choosing $z$ large enough while $n$ is finite we may make $\frac{n}{z}$ less than any assignable quantity, and therefore $np$ which is invariable and not greater than $\frac{n}{z}$ must be less than any assignable quantity. Therefore $1 - R_n$ is less than any assignable quantity, and therefore the chance that the man is ultimately ruined differs from certainty by less than any assignable chance, and the chance that he escapes ruin is less than any assignable chance.

Q. E. D.

The next proposition is not necessary to our present argument, but is added here as illustrating this part of the subject. It reduces to the case of the foregoing proposition when $\mu = 1$.

## PROPOSITION XXXIV.

*If a man with limited funds (£n) repeatedly stake a pound against a pound in a wager in which the odds on every venture are* $\mu : 1$ *in his favour, the chance that he is ultimately ruined is* $\frac{1}{\mu^n}$.

Arguing as in the last proposition, and observing that the chances of his losing or winning any venture are respectively $\frac{1}{1+\mu}$ and $\frac{\mu}{1+\mu}$, we have

$$R_1 = \frac{1}{1+\mu} + \frac{\mu}{1+\mu} R_2$$

or $$(1+\mu)\,R_1 = 1+\mu R_2$$

whence $$R_1 - R_2 = \frac{1}{\mu}\,(1-R_1)$$

So $$R_2 - R_3 = \frac{1}{\mu}\,(R_1-R_2)$$

and generally $$R_{x-1}-R_x = \frac{1}{\mu}\,(R_{x-2}-R_{x-1})$$

Therefore by continued multiplication

$$R_{x-1} - R_x = \frac{1}{\mu^{x-1}}\,(1-R_1)$$

$$= \frac{p}{\mu^{x-1}} \qquad \text{if } R_1 = 1-p.$$

Now give $x$ successively all the values from 1 to $n$ and add, and we get

$$1-R_n = p\left\{1 + \frac{1}{\mu} + \frac{1}{\mu^2} + \ldots + \frac{1}{\mu^{n-1}}\right\}$$

$$= p\,\frac{1-\dfrac{1}{\mu^n}}{1-\dfrac{1}{\mu}}$$

But $R_n$ can be zero only when $n$ is infinite, therefore

$$p = 1-\frac{1}{\mu}$$

and consequently

$$1-R_n = 1-\frac{1}{\mu^n}$$

or $$R_n = \frac{1}{\mu^n}$$

Q. E. D.

To resume our argument:—Proposition XXXIII. shews that, on the strictest mathematical principles, a man who continues to stake a constant sum in a fair wager must expect to be ruined in the end. Such a course of gambling—unregulated by any consideration of the gambler's disposable fund, until the absolute exhaustion of the fund places a peremptory limit on his play—is therefore manifestly inexpedient. And if it be inexpedient in the long run, it cannot be expedient even when carried to any limited extent.

For any limited extent of play which may be thought to be expedient, must leave the gambler either poorer or richer than he was at first. If it leave him poorer, it cannot be expedient. If it leave him richer, the same course of gambling which was expedient at first, must be, *à fortiori*, expedient now; and, therefore, it cannot be expedient for him to stop. Hence, no play, consisting of simple repetitions of the same venture, which is proved to be inexpedient in the long run, can be expedient when confined to any assignable extent.

But it may be said that a wise speculator will always regulate his ventures by his means. If, when he posesses a shilling, he stakes sixpence, the repetition of the act is not to be sought for in a constant staking of sixpence, but in a constant staking of half his fund. And if he always keeps as much as he stakes, it is plain that he can never be absolutely ruined. The reasoning of Proposition XXXIII. will not now apply. We must seek some other proof, if we would show that a course of gambling thus regulated is in the end disadvantageous.

The proof is easily found. Let us suppose that the man having $n$ pounds begins as before by staking one pound against one pound in an even wager. He stakes one $n$th part of his fund, and he continues the operation by staking every time not one pound but always one $n$th part of the fund which he then possesses. Every venture which issues in his favour increases his fund by one $n$th part, or multiplies it by $\left(1+\frac{1}{n}\right)$. Every venture which turns out against him decreases his fund by one $n$th part, or multiplies it by $\left(1-\frac{1}{n}\right)$. A favourable and an unfavourable issue will therefore together multiply his fund by

$$\left(1+\frac{1}{n}\right)\times\left(1-\frac{1}{n}\right) \text{ or } 1-\frac{1}{n^2}:$$

that is, the two operations will decrease his fund by $\left(\frac{1}{n^2}\right)$th part: and this will be the case whether the gain precede or follow the loss. Thus a gain and a loss do not balance one another, but they leave a net loss. And in any number of ventures in which there are the same number of profitable and unprofitable issues, there will be a resultant loss, greater as the number of ventures is greater. Now the man cannot expect in the long run to win oftener than he loses, and therefore if he repeat his operations indefinitely he must expect to lose in the end.

For example, if he begin with £100 and always stake one tenth of his fund, a single gain will raise his fund to £110; he will then stake £11, and if he lose he will only have £99 left. Or if he had lost the first venture he would have had £90 left: then he would have staked £9, and if he had won he would then have had £99 as before. In either order the gain and the loss reduce £100 to £99, and ten gains and ten losses in any order would reduce it to £90 8s. $11\frac{1}{2}$d.

## PROPOSITION XXXV.

*There are m tickets in a lottery for one prize of value A. To determine what price may be paid for a ticket by a man whose available fund is nA, so that by repeating his operation an average number of gains may balance an average number of losses.*

"Repeating his operation" will here mean that when the first venture is decided the man will purchase a ticket in another lottery in which the prize is the $n$th part of the amount of the fund which he *then* holds.

Suppose $\frac{xA}{m}$ the sum which he may pay for a ticket. Then every unsuccessful venture multiplies his fund by $1-\frac{x}{mn}$ and every successful venture multiplies it by $1+\frac{1}{n}-\frac{x}{mn}$. But in the long run he will have $m-1$ unsuccessful issues for one successful one. Therefore the average multiplier for $m$ ventures will be

$$\left(1+\frac{1}{n}-\frac{x}{mn}\right)\left(1-\frac{x}{mn}\right)^{m-1}$$

This must be unity since the gains, on the average, balance the losses. Therefore

$$1+\frac{1}{n}-\frac{x}{mn}=\left(1-\frac{x}{mn}\right)^{-(m-1)}$$

an equation to determine $x$.

If $n$ be a large number (as is usually the case) we may obtain an approximate solution. Thus

$$1+\frac{1}{n}-\frac{x}{mn}=1+\frac{(m-1)\,x}{mn}+\frac{(m-1)\,mx^2}{2m^2n^2}+\&c.$$

or $$0=-1+x+\frac{m-1}{2mn}x^2$$

whence $$x=1-\frac{m-1}{2mn}.$$

Consequently it is inexpedient for one to give more than

$$\left(1-\frac{m-1}{2mn}\right)\frac{A}{m}$$

for the chance $\left(\frac{1}{m}\right)$ of the prize $(A)$.

Corollary.—If he buy two tickets in the same lottery his chance of a prize is $\frac{2}{m}$. Hence writing $\frac{m}{2}$ for $m$ in the result of the proposition we obtain the price he may pay for two tickets, viz.

$$\left(1-\frac{m-2}{2mn}\right)\frac{2A}{m}$$

or he may pay for each of them

$$\left(1-\frac{m-2}{2mn}\right)\frac{A}{m}$$

So if he purchase $r$ tickets, he may pay for each of them

$$\left(1-\frac{m-r}{2mn}\right)\frac{A}{m}$$

or if he purchase all the $m$ tickets, he may pay for each its full representative value $\frac{A}{m}$.

## PROPOSITION XXXVI.

*There are $a+b+c+\ldots=m$ tickets in a lottery, and there are $a$ blanks, $b$ prizes each worth $\beta$, $c$ prizes each worth $\gamma$, and so on. To determine what price a man whose available fund is $n$ may prudently pay for a ticket.*

Let $\omega = b\beta + c\gamma + \ldots =$ the sum of the prizes, and let $\Omega = b\beta^2 + c\gamma^2 + \ldots$

Then the absolute value of a ticket is $\frac{\omega}{m}$.

Let $\frac{x\omega}{n}$ be the price which the man in question ought to pay for the ticket,

Then if we proceed as in the last question we find the average multiplier for $m$ ventures, viz.

$$\left(1-\frac{x\omega}{mn}\right)^a\left(1+\frac{\beta}{n}-\frac{x\omega}{mn}\right)^b\left(1+\frac{\gamma}{n}-\frac{x\omega}{mn}\right)^c\ldots$$

This must be equal to unity. Therefore

$$a \log\left(1-\frac{x\omega}{mn}\right) + b \log\left(1+\frac{\beta}{n}-\frac{x\omega}{mn}\right) + c \log\left(1+\frac{\gamma}{n}-\frac{x\omega}{mn}\right)$$
$$+ \dots = 0.$$

from which equation $x$ is to be found. If, as is usually the case, the amount at stake is small compared with the speculator's whole funds, $n$ must be a very large number; therefore we shall obtain an approximate value of $x$, by neglecting the terms involving high negative powers of $n$, in our equation.

Thus, expanding the logarithms, we have

$$0 = (1-x)\frac{\omega}{n} - \left(\frac{\Omega}{\omega^2} - \frac{2x - x^2}{m}\right)\frac{\omega^2}{2n^2} + \&c.$$

whence $$x = 1 - \frac{m\Omega - \omega^2}{2mn\omega}$$

Consequently, the price which the man may prudently pay for a ticket is

$$\left(1 - \frac{m\Omega - \omega^2}{2mn\omega}\right)\frac{\omega}{m}.$$

Corollary.—*The price which a man whose available fund is n pounds may prudently pay for a share in a speculation in which* $p_1$ *will be his chance of winning* $P_1$, $p_2$ *his chance of winning* $P_2$, *and so on (where* $p_1 + p_2 + \dots = 1$*) will be*

$$\Sigma(pP) - \frac{1}{2n}\left\{\Sigma(pP^2) - \{\Sigma(pP)\}^2\right\}.$$

This is derived immediately from the result of the proposition, by writing

$$\frac{\omega}{m} = \Sigma(pP) \quad \text{and} \quad \frac{\Omega}{m} = \Sigma(pP^2).$$

EXAMPLES.—Having $n$ shillings, what may one prudently pay to be entitled to a number of shillings equal to the number turned up at a single throw of a die?

$$\text{Here } \Sigma(pP)=\frac{1}{6}(1+2+3+4+5+6)=\frac{7}{2}$$

$$\Sigma(pP^2)=\frac{1}{6}(1+4+9+16+25+36)=\frac{91}{6}$$

And the required number of shillings is approximately

$$\frac{7}{2}\left(1-\frac{5}{12n}\right).$$

Again. If the throw be made with two dice, we shall have (see page 91)

$$\Sigma(pP)=7 \qquad \Sigma(pP^2)=2023\div36$$

And the required number of shillings will be approximately

$$7\left(1-\frac{37}{72n}\right).$$

It must be remembered that the results of the two foregoing propositions and the corollary are only obtained on the hypothesis that $n$ is very large compared with $\Sigma(pP)$ and with $\Sigma(pP^2)\div\Sigma(pP)$. This requires that the man's original fund should be very large compared not only with the amount which he stakes, but also with the amount which he has a chance (though it may be a very small chance) of winning.

In all practical cases the former conditions will be fulfilled, as no one would think of staking, in any single venture, his whole property, or a sum bearing any considerable ratio to his whole property. But cases will be likely to arise in which the latter condition is not satisfied, as when a man may purchase, for a sum comparatively small compared with his means, a ticket in a lottery in which there is a prize very many times larger than his whole property. In this case, the approximate results obtained above cannot be applied, and we must have recourse to the original equation in the form in which it was presented before we introduced the approximations which are now inadmissible. We may, however, still approximate, in virtue of the condition that the stake must be small compared with the speculator's whole funds. Thus :—

## PROPOSITION XXXVII.

*To find an approximate formula for the sum which a speculator may pay for any defined expectation, without assuming that his funds are necessarily large compared with the value of the prizes.*

Let $X$ (small compared with $n$, though not necessarily small compared with $P$) be the sum which a man whose available fund is $n$ may prudently pay for the chances $p_1$, $p_2$, $p_3$ ... of receiving prizes worth $P_1$, $P_2$, $P_3$, ... respectively, as in the last corollary.

Then the rigorous equation to determine $X$ is

$$\left(1+\frac{P_1}{n}-\frac{X}{n}\right)^{p_1}\left(1+\frac{P_2}{n}-\frac{X}{n}\right)^{p_2}\left(1+\frac{P_3}{n}-\frac{X}{n}\right)^{p_3}\ldots=1$$

which may be written

$$\left(1+\frac{P_1}{n}\right)^{p_1}\left(1+\frac{P_2}{n}\right)^{p_2}\left(1+\frac{P_3}{n}\right)^{p_3}\ldots$$

$$=\left(1-\frac{X}{n+P_1}\right)^{-p_1}\left(1-\frac{X}{n+P_2}\right)^{-p_2}\left(1-\frac{X}{n+P_3}\right)^{-p_3}\ldots$$

or, since $X$ is small compared with $n$,

$$=1+X\left(\frac{p_1}{n+P_1}+\frac{p_2}{n+P_2}+\ldots\right)$$

whence $$X=\frac{\left(1+\frac{P_1}{n}\right)^{p_1}\left(1+\frac{P_2}{n}\right)^{p_2}\left(1+\frac{P_3}{n}\right)^{p_3}\ldots-1}{\frac{p_1}{n+P_1}+\frac{p_2}{n+P_2}+\frac{p_3}{n+P_3}+\ldots}$$

This formula is equally applicable when there is a possibility of not receiving any prize, as the failure to receive a prize may be treated as the receiving of a prize of zero value: *i.e.* one of the quantities $P_1$, $P_2$, $P_3$ … will be in this case zero.

Corollary.—In the case when there is a single prize $P$, and the chance of gaining it is $p$, the formula becomes

$$X=\frac{\left(1+\frac{P}{n}\right)^{p}-1}{\frac{p}{n+P}+\frac{1-p}{n}}$$

$$=\frac{n(n+P)}{n+(1-p)P}\left\{\left(1+\frac{P}{n}\right)^{p}-1\right\}$$

EXAMPLES.—A man possessing a pound is offered a ticket in a lottery, in which there are 99 blanks, and one prize worth 100 pounds. What may he prudently pay for the ticket?

$$\text{Here} \qquad n=1 \qquad P=100 \qquad p=\frac{1}{100}.$$

$$X=\frac{101}{101}\left(\sqrt[100]{101}-1\right)$$

$$=\frac{101}{100}\times \cdot 0472=\cdot 0519$$

Hence he can only afford to pay about a shilling for the ticket.

Again. How much may a man with 10 pounds pay for the same ticket?

$$\text{Here} \quad n=10 \quad P=100 \quad p=\frac{1}{100}.$$

$$X=\frac{1100}{109}\left(\sqrt[100]{11}-1\right)$$

$$=\frac{1100}{109}\times \cdot 0242=\cdot 244.$$

Hence he can afford to pay about 4s. $10\frac{1}{2}$d. for the ticket.

Again. How much may a man with 100 pounds pay for the same ticket?

$$\text{Here} \quad n=100 \qquad P=100 \qquad p=\frac{1}{100}.$$

$$X=\frac{20000}{199}\left(\sqrt[100]{2}-1\right)$$

$$=\frac{20000}{199}\times \cdot 0069=\cdot 693.$$

Hence he may pay nearly 14 shillings for the ticket.

Again. Suppose the man has 1000 pounds. This sum is large compared with the value of the prize. Hence we may apply the simpler formula given in Proposition XXXVI. (Cor.), which leads to the result

$$1-\frac{1}{2000}\left(100-1\right)=1-\frac{99}{2000}=\cdot 9505.$$

Hence he may pay 19 shillings for the ticket.

If we had applied to this last case the formula of the present corollary we should have obtained the slightly more correct result ·9507, the difference between the two results being nearly one-fifth of a farthing.

The formula of Proposition XXXVII. is applicable in the case of the Petersburg problem, a problem of some intrinsic interest, but chiefly of importance on account of its having been repeatedly made the ground of objections to the mathematical theory of probability. This celebrated problem may be stated as follows:

## THE PETERSBURG PROBLEM.

*A coin is tossed again and again until a tail is turned up. If the first throw give a head one is to receive a florin, if the second also give a head one is to receive two florins more, if in addition the third throw give a head one is to receive four florins more, and so on, doubling the sum received every time; but as soon as a tail is turned up the play stops and one receives nothing further. The question is, what ought one to give for the expectation?*

The absolute value of the expectation is easily seen to be infinite. Thus—

The chance of winning the florin at the first throw is $\frac{1}{2}$: the value of the expectation is therefore one shilling.

The chance of winning the two florins at the second throw is $\frac{1}{4}$: the value of the expectation is therefore $\frac{1}{4}$ of two florins, or one shilling.

The chance of winning the four florins at the third throw is $\frac{1}{8}$: the value of the expectation is therefore $\frac{1}{8}$ of four florins, or one shilling, and so on.

Hence the value of the expectation attaching to each throw to which there is a possibility of the play extending, is one shilling. But the play may possibly extend to a number of throws larger than any assignable number. Therefore the whole expectation is worth a number of shillings larger than any assignable number: that is, it is infinitely great.

Or we may analyse the expectation by considering the various total sums which it is possible to receive, and the chance of each being received. Thus—

If a tail turn up the first time, nothing is received: and the chance of this is $\frac{1}{2}$.

If a tail turn up the second time and not before, one florin is received: and the chance of this is $\frac{1}{4}$.

If a tail turn up the third time and not before, $1 + 2$ florins are received: and the chance of this is $\frac{1}{8}$.

And so on. Hence

$\frac{1}{2}$ = chance of receiving 0

$\frac{1}{2^2}$ = ...................... 1 = 2 − 1 florins.

$\frac{1}{2^3}$ = .................. 1 + 2 = $2^2 - 1$ florins.

$\frac{1}{2^4}$ = ............... 1 + 2 + 4 = $2^3 - 1$ florins.

and so on, *ad infinitum.*

Therefore the value of the whole expectation is (in florins)

$$\left(\frac{1}{2} - \frac{1}{2^2}\right) + \left(\frac{1}{2} - \frac{1}{2^3}\right) + \left(\frac{1}{2} - \frac{1}{2^4}\right) + \text{\&c. } \textit{ad infinitum.}$$

Now the sum of this series to $r$ terms* is $\frac{r-1}{2} + \left(\frac{1}{2}\right)^{r+1}$, and to an infinite number of terms it is infinity.

Hence the value of the mathematical expectation is infinite, as we showed before.

But here a difficulty is raised. The mathematical expectation has been found to be of infinite value, and yet (it is objected) no one in his senses would give even such a moderate sum as £50 for the prospect defined in the problem.

The fallacy of this objection has been pointed out already. We have shown that the absolute value of a

* It is to be observed that the sum of this series to $r$ terms does not exactly correspond with the expectation when the play is limited to $r$ throws, because when the play is thus limited the chance of winning the whole number $(2^r - 1)$ is not $\frac{1}{2^{r+1}}$, but is $\frac{1}{2^r}$, the same as the chance of winning $2^{r-1} - 1$ florins.

mathematical expectation is not the price which a man of limited means ought to pay for the prospect. It expresses the value of the expectation to a man who is able to repeat the venture indefinitely without the risk of his operations being ever terminated by lack of means. The speculator's fund, to begin with, must be infinite in comparison with the stakes involved, before he may venture to give the absolute value of the mathematical expectation for any contingent prospect which he may desire to purchase. In the Petersburg problem the mathematical expectation is infinite: but if one is to give an infinite sum for the venture one must take care to hold funds *infinite in comparison with this infinity.* In other words, the speculation on these terms is only proper for one with respect to whose funds the infinite stake is inconsiderable. The stake which he lays down may be $\infty$, provided that his funds are $\infty \times \infty$.

But to find the sum which a man of limited means may pay for the expectation defined in our statement of the problem we may apply the result of Proposition XXXVII. Thus if $m + 1$ be the number of florins which the man possesses, the formula to determine the sum which he may pay may be written as follows:

$$X = \frac{\left(\frac{4}{m+1}\right)^{\frac{1}{2}}\left(1+\frac{m}{2}\right)^{\frac{1}{4}}\left(1+\frac{m}{4}\right)^{\frac{1}{8}}\left(1+\frac{m}{8}\right)^{\frac{1}{16}}\cdots - 1}{\frac{1}{2(m+1)} + \frac{1}{4(m+2)} + \frac{1}{8(m+4)} + \cdots}$$

For example. If the speculator possess nine florins we have $m=8$. The numerator is ·2137 ... and the denominator ·0966 .... Hence $X = 2·212$.

If he possess 33 florins we have $m=32$. The numerator is now ·0807 ... and the denominator ·0278 .... Hence he may pay nearly *three* florins.

If he possess 1025 florins we have $m=1024$. The numerator is now ·00488 ... and the denominator ·00097 .... Hence he may pay about *five* florins for the venture.

The result at which we have arrived is not to be classed with the arbitrary methods which have been again and again propounded to evade the difficulty of the Petersburg problem and other problems of a similar character. Formulæ have often been proposed, which have possessed the one virtue of presenting a finite result in the case of this famous problem, but they have often had no intelligible basis to rest upon, or, if they have been established on sound principles, sufficient care has not been taken to draw a distinguishing line between the significance of the result obtained, and the different result arrived at when the mathematical expectation is calculated.

We have not assigned any new value to the mathematical expectation; we have not substituted a new expression for the old; but we have deduced a separate result, which without disturbing the mathematical expectation has a definite meaning of its own. We have found not the fair price at which a contingent prospect may be transferred from one man to another, but the value which such a prospect has to a man in given circumstances. We have simply determined the terms at which a man may purchase a contingent prospect of advantage, so that by repeating the operation—*each time on a scale proportionate to his funds at that time*—he may be left neither richer nor poorer when each issue of the venture shall have occurred its own average number of times. By continuing the operation indefinitely, the recurrences of each issue will tend to be proportional to their respective probabilities, and, therefore, the condition we have taken is equivalent to the condition that *in the long run* the man may expect to be neither richer nor poorer.

It would be a great mistake to suppose that the price which one man may prudently give for a venture is the price which the man with whom he is dealing may prudently take for it, or that it is a fair price at which to make the compact. The price which the man may prudently give, is not even the price which the same man may prudently take if he change sides with his fellow gambler. *The sum in consideration of which a man possessed of n pounds may accept a position in which $p_1$ is the chance of his having to pay $P_1$, $p_2$ the chance of his having to pay $p_2$, and so on (where $\Sigma p = 1$)* must be obtained by changing the algebraical signs of $X$, $P_1$, $P_2$ ... in the formulæ of Propositions XXXVI. and XXXVII.

Thus on the hypothesis of Proposition XXXVI. (Cor.), we shall have

$$X = \Sigma(pP) + \frac{1}{2n}\left\{\Sigma(pP^2) - \{\Sigma(pP)\}^2\right\}$$

and in the more general case dealt with in Proposition XXXVII. we shall have

$$X = \frac{1 - \left(1 - \frac{P_1}{n}\right)^{p_1}\left(1 - \frac{P_2}{n}\right)^{p_2}\left(1 - \frac{P_3}{n}\right)^{p_3}\cdots}{\frac{p_1}{n - P_1} + \frac{p_2}{n - P_2} + \frac{p_3}{n - P_3} + \cdots}$$

The historical notes which follow are mainly derived from Mr. Todhunter's *History of the Mathematical Theory of Probability.*

The volume of the Commentarii of the Petersburg Academy for the years 1730 and 1731, was published in 1738. It contained a memoir by Daniel Bernoulli, entitled, *Specimen Theoriae novae de mensura sortis,* expounding a theory of *moral expectation* as distinguished from *mathematical expectation.* The author estimates that if a man's fund is increased by a small increment, the value of the increment to that man varies directly as the increment, and inversely as his original fund. But while he assumes this as a mathematical measure of what he regards as a moral value, Bernoulli does not attempt to give any proof of his assumption: and rightly, for it is beyond the province of mathematics to deal with such a subjective value as he speaks of. Mathematics can only be applied to the measure of such a quantity by some such arbitrary connection as that which he assumes. But that which he takes to represent his *moral expectation* is substantially identical with the quantity which we have been investigating, viz., the price to be paid for a venture, in order that repetitions indefinitely multiplied may tend to neutralise one another. Bernoulli draws from his theory the inference which we have established at the beginning of this Appendix, that even a fair chance is disadvantageous. The Petersburg problem, as he deals with it, is somewhat simpler than the modern variety of it, which we have enunciated above. $A$ is to receive a florin if head falls the first time; two florins if it falls the second time, and not before; four florins if it falls the third time, and not before, and so on. The mathematical expectation is infinite. For the moral expectation, Bernoulli gives an equation equivalent to that which we should write down in accordance with Proposition XXXVII.

Daniel Bernoulli's memoir contains a letter addressed to Nicolas Bernoulli, by Cramer, in which two methods are suggested of explaining the paradox of the Petersburg problem. One suggestion is, that all sums greater than $2^{24}$ are practically equal; the other (which is equally arbitrary), that the pleasure derivable from a sum of money varies as the square root of the sum. On one of these suppositions the expectation in the Petersburg problem, as enunciated by Bernoulli, would be 13; according to the other, it would be about 2·9.

D'Alembert (in the year 1754) maintained that a very small chance was to be regarded as absolutely zero. He does not suggest a limit to the smallness, but he gives an example in which the chance is $(\frac{1}{2})^{100}$. In another place he suggests that, in the Petersburg problem, we should take ($\beta$ being a constant)

$$\frac{1}{2^n(1+\beta n^2)}, \text{ instead of } \frac{1}{2^n},$$

as the chance that the head will not appear before the $n^{th}$ throw. From time to time he seems to have proposed a variety of arbitrary assumptions, for none of which any better reason can be assigned than that they lead to finite results.

Beguelin, in 1767, gave six different solutions of the Petersburg problem, with different results.

In 1777, Buffon, the Naturalist, published his *Essai d'Arithmétique Morale,* in which he speaks against gambling in language singularly resembling that which we have employed in the earlier pages of this Appendix. "Je dis qu'en général le jeu est un pacte mal-entendu, un contrat désavantageux aux deux parties, dont l'effet est de rendre la perte toujours plus grande que le gain; et d'ôter au bien pour ajouter au mal." But, among other arbitrary assumptions, this writer maintains that any chance less than $\frac{1}{10,000}$ is to be considered absolutely zero.

Laplace, whose great work, the *Théorie Analytique des Probabilités,* was published in 1812, has developed many of Daniel Bernoulli's ideas on this subject.

# MISCELLANEOUS EXAMPLES.

1.—There are five routes to the top of a mountain, in how many ways can a person go up and down?

2.—Out of 20 knives and 24 forks, in how many ways can a man choose a knife and fork? And then, in how many ways can another man take another knife and fork?

3.—In how many ways can the letters *a b c d* be arranged without letting *b* and *c* come together?

4.—*A* has 7 different books, *B* has 9 different books, in how many ways can one of *A*'s books be exchanged for one of *B*'s?

5.—In the case of the last question, in how many ways can two books be exchanged for two?

6.—Five men, *A*, *B*, *C*, *D*, *E*, are going to speak at a meeting, in how many ways can they take their turns without *B* speaking before *A*?

7.—In how many ways, so that *A* speaks *immediately* before *B*?

8.—Five ladies and three gentlemen are going to play at croquet, in how many ways can they divide themselves into sides of four each, so that the gentlemen may not be all on one side?

9.—The number of ways of selecting $n$ things out of $2n+2$ is to the number of ways of selecting $n$ things out of $2n-2$ as 99 to 7. Find $n$.

10.—One man has 4 books, another man has 6. In how many ways can they exchange books, each keeping the number he had at first?

11.—One man has 4 books, another has 6, and a third has 3. In how many ways can they exchange books, each keeping the number he had at first, but every one's set being altered?

12.—Four digits are arranged at random so as to form a number in the ordinary scale of notation. Two cyphers are then associated with them, and they are re-arranged at random so as again to form a number. Prove that the average value of the first number is to the average value of the second as 101 to 6734.

13.—A ferry-boat which can carry $n$ people has to convey $m$ people across a river. It takes a full load every time except the last: find the number of ways in which the work can be done.

14.—There are $2n$ guests at a dinner party; supposing that the host and hostess have fixed seats opposite to one another, and that there are two specified guests who must not be placed next to one another, find the number of ways in which the company can be placed.

15.—Out of three consonants and two vowels, how many words can be formed containing 2, 3, 4, or 5 letters, words being excluded in which two consonants or two vowels come together.

16.—How many five-lettered words can be made out of 26 letters, repetitions being allowed, but no consecutive repetitions (*i. e.*, no letter must follow itself in the same word).

17.—A boat's crew consists of eight men, of whom two can only row on the stroke side of the boat, and three only on the bow side. In how many ways can the crew be arranged?

18.—There are $m$ parcels, of which the first contains $n$ things; the second $2n$ things; the third $3n$ things; and so on. Shew that the number of ways of taking $n$ things out of each parcel is $\lfloor\underline{mn} \div \{\lfloor\underline{n}\}^m$

19.—How many different rectangular parallelepipeds can be constructed, the length of each edge being an integral number of inches not exceeding 10?

20.—The number of ways of dividing $2n$ different things into two equal parts, is to the number of ways of similarly dividing $4n$ different things, as the continued product of the first $n$ odd numbers to the continued product of the $n$ odd numbers succeeding.

21.—In how many ways can the letters of the word *f a c e t i o u s* be deranged without deranging the order of the vowels?

22.—In how many ways can the letters of the word *a b s t e m i o u s l y* be deranged without deranging the order of the vowels?

23.—In how many ways can the letters of the word *p a r a l l e l i s m* be deranged without deranging the order of the vowels?

24.—How many solutions can be given to the following problem? "Find two numbers whose greatest common measure shall be $G$ and their least common multiple $M = Ga^{\alpha} b^{\beta} c^{\gamma} d^{\delta}$; $a$, $b$, $c$, $d$ being prime numbers."

25.—How many solutions can be given to the following problem? "Find two numbers of which $G$ shall be a common measure, and $M$ (as in the last question) a common multiple."

26.—Prove that the number of ways in which $p$ positive signs and $n$ negative signs may be placed in a row, so that no two negative signs shall be together, is equal to the number of combinations of $p+1$ things taken $n$ together.

27.—In the expansion of $(a_1 + a_2 + \dots + a_p)^n$ where $n$ is an integer not greater than $p$, there are

$$\frac{\lfloor p}{\lfloor n \lfloor p-n}$$

terms, in none of which any one of the quantities $a_1\ a_2 \dots a_p$ occurs more than once as a factor; and the coefficient of each of these terms is $\lfloor n$.

28.—Out of 20 consecutive numbers, in how many ways can two be selected whose sum shall be odd?

29.—Out of 30 consecutive integers, in how many ways can three be selected whose sum shall be even?

30.—Out of $3n$ consecutive integers, in how many ways can three be selected whose sum shall be divisible by 3?

31.—If four straight lines be drawn in a plane and produced indefinitely, how many points of intersection will there *generally* be?

32.—If $n$ straight lines be drawn in a plane, no two being parallel and no three concurrent, how many points of intersection will there be?

33.—If $n$ straight lines be drawn in a plane, no two being parallel, and no three concurrent except $p$ which meet in one point, and $q$ which meet in another point, how many other points of intersection will there be?

34.—A square is divided into 16 equal squares by vertical and horizontal lines. In how many ways can 4 of these be painted white, 4 black, 4 red, and 4 blue, without repeating the same colour in the same vertical or horizontal row?

35.—Find the number of combinations that can be formed out of the letters of the following line (*Soph. Philoct.* 746):

ἀπαππαπαι παπαππαπαππαπαππαπαι,

taking them (1) 5 together, and (2) 25 together.

36.—In the case of the preceding question, if the number of combinations $r$ together is to the number $r-1$ together as $9:10$, find $r$, it being known that it lies between 17 and 24.

37.—The number of ways of selecting 4 things out of $n$ different things is one-sixth of the number of ways of selecting 4 things out of $2n$ things which are two and two alike of $n$ sorts: find $n$.

38.—If $pq+r$ different things are to be divided *as equally as possible* among $p$ persons, in how many ways can it be done? $(r<p)$

39.—In how many ways can a pack of cards be dealt to four players, subject to the condition that each player shall have three cards of each of three suits and four cards of the remaining suit?

40.—Into how many parts is an infinite plane divided by $n$ straight lines, of which no three are concurrent?

41.—Into how many parts is infinite space divided by $n$ planes, of which no four meet in a point?

42.—In how many ways can three numbers in arithmetical progression be selected from the series 1, 2, 3 ... $2n$, and in how many ways from the series 1, 2, 3 ... $(2n+1)$?

43.—If there be $n$ straight lines in one plane, no three of which meet in a point, the number of groups of $n$ of their points of intersection, in each of which no three points lie in one of the straight lines, is $\frac{1}{2}\,|\underline{n-1}$.

44.—120 men are to be formed at random into a solid rectangle of 12 men by 10; all sides are equally likely to be in front. What is the chance that an assigned man is in the front?

45.—If the letters of the alphabet are written down in a *ring* so that no two vowels come together, what is the chance that $a$ is next to $b$?

46.—If the letters of the alphabet are written down in a *row* so that no two vowels come together, what is the chance that $a$ is next to $b$?

47.—$A$, $B$, $C$ have equal claims for a prize. $A$ says to $B$, let us two draw lots, let the loser withdraw and the winner draw lots with $C$ for the prize. Is this fair?

48.—Five men, $A$, $B$, $C$, $D$, $E$, speak at a meeting, and it is known that $A$ speaks before $B$, what is the chance that $A$ speaks *immediately* before $B$?

49.—If $n$ things ($\alpha$, $\beta$, $\gamma$, &c.) be arranged in a row, subject to the condition that $\alpha$ comes before $\beta$, what is the chance that $\alpha$ comes *next* before $\beta$?

50.—Two numbers are chosen at random, find the chance that their sum is even.

51.—There are $n$ counters marked with odd numbers, and $n$ more marked with even numbers; if two are drawn at random shew that the odds are $n$ to $n-1$ against the sum of the numbers drawn being even.

52.—The figures 142857 are arranged at random as the period of a circulating decimal, which is then reduced to a vulgar fraction in lowest terms. Shew that the odds are 119 : 1 against the denominator being 7.

53.—There are ten counters in a bag marked with numbers. A person is allowed to draw two of them. If the sum of the numbers drawn is an odd number, he receives that number of shillings; if it is an even number, he pays that number of shillings. Is the value of his expectation greater when the counters are numbered from 0 to 9 or from 1 to 10?

54.—If a head counts for *one* and a tail for *two*, shew that $3n$ is the most likely number to throw when $2n$ coins are tossed. Also shew that the chance of throwing $3(n+1)$ with $2(n+1)$ coins is less than the chance of throwing $3n$ with $2n$ coins in the ratio $2n+1 : 2n+2$.

55.—A bag contains $2n$ counters, of which half are marked with odd numbers and half with even numbers, the sum of all the numbers being S. A man is to draw two counters. If the sum of the numbers drawn be an odd number, he is to receive that number of shillings; if an even number, he is to pay that number of shillings. Shew that his expectation is worth (in shillings)

$$\frac{S}{n(2n-1)}$$

56.—If in the case of the last question there be $m+n$ counters, of which $m$ are marked with odd numbers, amounting to $M$, and $n$ with even numbers amounting to $N$, the man's expectation is worth

$$\frac{M+N-(m-n)(M-N)}{\frac{1}{2}(m+n)(m+n-1)}$$

57.—What are the odds against throwing 7 twice at least in three throws with two dice?

58.—Two persons play for a stake, each throwing two dice. They throw in turn, $A$ commencing. $A$ wins if he throws 6, $B$ if he throws 7: the game ceasing as soon as either event happens. Shew that $A$'s chance is to $B$'s as 30 to 31.

59.—Four persons draw each a card from an ordinary pack. Find the chance (i) that one card is of each suit: (ii) that no two cards are of equal value: (iii) that one card is of each suit and no two of equal value.

60.—Each of four persons draws a card from an ordinary pack. Find the chance that one card is of each suit, and that in addition, on a second drawing, each person shall draw a card of the same suit as before.

61.—A bag contains $\frac{1}{2}n(n+1)$ counters, one marked 1, two marked 4, three marked 9, &c. A person draws out a counter at random, and is to receive as many shillings as the number marked on it. Prove that the value of his expectation varies as the square of the number of counters in the bag.

62.—$A$ and $B$ throw for a certain stake, $A$ having a die whose faces are numbered 10, 13, 16, 20, 21, 25; and $B$ a die whose faces are numbered 5, 10, 15, 20, 25, 30. The highest throw to win, and equal throws to go for nothing. Prove that the odds are 17 to 16 in favour of $A$.

63.—A pack of cards consists of $p$ suits of $q$ cards each, numbered from 1 up to $q$. A card is drawn and turned up: and $r$ other cards are drawn at random. Find the chance that the card first drawn is the highest of its suit among all the cards drawn.

64.—$A$ and $B$ play for a stake which is to be won by him who makes the highest score in 4 throws of a die. After two throws, $A$ has scored 12, and $B$ 9. What is $A$'s chance of winning?

65.—A bag contains 6 shillings and 2 sovereigns. What is the value of one's expectation if one is allowed to draw till one draws a sovereign?

66.—There are $m$ white balls and $m$ black ones: $m$ balls are placed in one bag, and the remaining $m$ in a second bag, the number of white and black in each being unknown. If one ball be drawn from each bag, find the chance that they are of the same colour.

67.—In the last question, if $m = 4$, and a ball of the same colour has been drawn from each, find the chance that a second drawing will give balls of the same colour: (i) if the balls drawn at first have been replaced, and (ii) if they have not been replaced.

68.—A player has reckoned his chance of success in a game to be $e$, but he considers that there is an even chance that he has made an error in his calculation affecting the result by $e'$ (either in excess or defect). Shew that this consideration does not affect his chance of success in a single game, but increases his chance of winning a series of games.

69.—Shew that in taking a handful of shot from a bag it is more probable that an odd number will be drawn than an even number.

70.—A bag contains $m$ white and $n$ black balls, and from it balls are drawn one by one till a white ball is drawn. $A$ bets $B$ at each drawing, $x$ to $y$, that a black ball is drawn. Prove that the value of $A$'s expectation at the beginning of the drawing is $\frac{ny}{m+1} - x$

71.—Counters ($n$) marked with consecutive numbers are placed in a bag, from which a number of counters ($m$) are to be drawn at random. Shew that the expectation of the sum of the numbers drawn is the arithmetic mean between the greatest and least sums which can be indicated by the number of counters ($m$) to be drawn.

72.—$A$ and $B$ play a set of games, in which $A$'s chance of winning a single game is $p$, and $B$'s chance $q$. Find

(i), the chance that $A$ wins $m$ out of the first $m+n$.

(ii), the chance that when $A$ has won $m$ games, $m+n$ have been played.

(iii), the chance that $A$ wins $m$ games before $B$ wins $n$ games.

73.—The face of a die, which should have been marked ace, has been accidentally marked with one of the other five numbers. A six is thrown twice in two throws. What is the chance that the third throw will give a six?

74.—One of two bags contains 10 sovereigns, and the other 10 shillings. One coin is taken out of each and placed in the other. This is repeated 10 times. What is now the expectation of each bag?

75.—$A$, $B$, $C$ are candidates for an office, the election to which is in the hands of $8m+1$ electors. $3m$ votes, together with the casting vote if necessary, are promised to $A$, and $2m$ votes to $B$. In how many ways can the remaining votes be given so that $A$ may be successful?

76.—$A$ writes to $B$ requiring an answer within $n$ days. It is known that $B$ will be at the address on some one of these days, any one equally likely. It is a $p$-days' post between $A$ and $B$. If one in every $q$ letters is lost in transit, find the chance that $A$ receives an answer in time. ($n > 2p$.)

77.—What is the probability that a number, consisting of

7 digits, the sum of which is 59, will be exactly divisible by 11?

78.—There are $n$ vessels containing wine, and $n$ vessels containing water. Each vessel is known to hold $a$, $a+1$, $a+2$, ... or $a+m-1$ gallons. Find the chance that the mixture formed from them all will contain just as much wine as water.

79.—A man has left his umbrella in one of three shops which he visited in succession. He is in the habit of leaving it, on an average, once in every four times that he goes to a shop. Find the chance of his having left it in the first, second, and third shops respectively.

80.—If $mn$ balls have been distributed into $m$ bags, $n$ into each, what is the chance that two specified balls will be found in the same bag? And what does the chance become when $r$ bags have been examined and found not to contain either ball?

81.—One card out of a pack has been lost. From the remainder of the pack, thirteen cards are drawn at random, and are found to consist of two spades, three clubs, four hearts, and four diamonds. What are the respective chances that the missing card is a spade, a club, a heart, or a diamond?

82.—A number of persons $A$, $B$, $C$, $D$ ... play at a game, their chances of winning any particular game being $\alpha$, $\beta$, $\gamma$, $\delta$ ... respectively. The match is won by $A$ if he gains $a$ games in succession; by $B$ if he gains $b$ games in succession; and so on. The play continues till one of these events happens. Shew that their chances of winning the match are proportional to

$$\frac{(1-\alpha)\alpha^a}{1-\alpha^a},\ \frac{(1-\beta)\beta^b}{1-\beta^b},\ \&c.$$

83.—$A$ goes to hall $p$ times in $q$ consecutive days and sees $B$ there $r$ times. What is the most probable number of times that $B$ was in hall in the $q$ days?

Ex.—Suppose $p=4 \quad q=7 \quad r=3$.

84.—If $M_r$ be the number of permutations of $m$ things taken $r$ together, and $N_r$ the number of permutations of $n$ things taken $r$ together, prove that the number of permutations of $m+n$ things $r$ together will be obtained by expanding $(M+N)^r$, and in the result replacing the indices by suffixes.

85.—Find the number of positive integral solutions of the equation $x+y+z+ \ldots$ ($p$ variables) $= m$, the variables being restricted to lie between $l$ and $n$, both inclusive.

86.—In how many ways can 26 different letters be made into six words, each letter being used once and only once?

87.—A body of $n$ members has to elect one member as a representative of the body. If every member gives a vote, in how many ways can the votes be given?

88.—In the case of the last question, how many different forms may the result of the poll assume, regarding only the number of votes given to each member and not the names of his supporters ?

89.—In a company of sixty members, each member votes for one of the members to fill an office. If the votes be regarded as given at random, what is the chance that some member shall get a majority of the whole number of votes ? Also determine the chance of the same event on the hypothesis that every different result of the poll (considered as in the last question) is equally likely to occur.

90.—In how many ways can 3 sovereigns and 10 shillings be put into 4 pockets ? (One or more may be left empty.)

91.—In how many ways can 12 sovereigns be distributed into five pockets, none being left empty ?

92.—In how many ways can 20 books be arranged in a bookcase containing five shelves, each shelf long enough to contain all the books ?

93.—In how many ways can a person wear five rings on the fingers (not the thumb) of one hand ?

94.—A debating society has to select one out of five subjects proposed. If thirty members vote, each for one subject, in how many ways can the votes fall ?

95.—In the last question, what is the chance that upwards of twenty votes fall to some one subject ?

96.—A bag contains $m$ counters marked with odd numbers, and $n$ counters marked with even numbers. If $r$ counters be drawn at random the chance that the sum of the numbers drawn be odd is $\frac{1}{2}(1+\mu)$, and that it be even $\frac{1}{2}(1-\mu)$, where $\mu$ is the coefficient of $x^r$ in the expansion of

$$\frac{\lfloor m+n-r \lfloor r}{\lfloor m+n}(1-x)^m(1+x)^n$$

97.—The number of ways in which $r$ things may be distributed among $n+p$ persons so that certain $n$ of those persons may have one at least is

$$(n+p)^r - n(n+p-1)^r + \frac{n(n-1)}{1.\ 2}(n+p-2)^r - \&c.$$

98.—Show that for $n$ different things $1-$(number of partitions into 2 parts) $+ \lfloor 2$ (number of partitions into 3 parts) $- \ldots \pm \lfloor n-1$ (number of partitions into $n$ parts)$=0$.

99.—Find the number of 5-partitions of 21.

100.—Two examiners working simultaneously examine a class of 12 boys, the one in classics the other in mathematics. The boys are examined individually for five minutes each in each subject. In how many ways can a suitable arrangement be made so that no boy may be wanted by both examiners at once?

101.—If $f_n$ denote the number of derangements of $n$ terms in circular procession so that no term may follow the term which it followed originally,

$$f_n + f_{n+1} = \lfloor n e_n^{-1}$$

102.—A pack of $n$ different cards is laid face downwards on a table. A person names a certain card. That and all the cards above it are shewn to him, and removed. He names another; and the process is repeated until there are no cards left. Find the chance that, in the course of the operation, a card was named which was (at the time) at the top of the pack.

103.—Three different persons have each to name an integer not greater than $n$. Find the chance that the integers named will be such that every two are together greater than the third.

104.—A person names a group of three integers (not necessarily different, but each one not greater than $n$). Find the chance that the integers named will be such that every two are together greater than the third.

105.—If three numbers be named at random, they are just as likely as not to be proportional to the sides of a possible triangle.

106.—A list is to be published in three classes. The odds are $m$ to 1 that the examiners will decide to arrange each class in order of merit, but if they are not so arranged, the names in each will be arranged in alphabetical order. The list appears, and the names in each class are observed to be in alphabetical order, the numbers in the several classes being $a$, $b$, and $c$. What is the chance that the order in each class is also the order of merit?

107.—How many different throws can be made with $n$

dice, those throws being considered the same in which the same set of numbers is turned up?

108.—Prove that the most likely throw in the last question is one in which the numbers turned up are all different, if $n$ is not greater than 6. And find the most likely throw when $n$ is greater than 6.

109.—There are $2n$ letters, two and two alike of $n$ different sorts. Shew that the number of orders in which they may be arranged, so that no two letters which are alike may come together, is

$$\frac{1}{2^n}\left\{\lfloor\underline{2n} - \frac{n}{1}2\lfloor\underline{2n-1} + \frac{n(n-1)}{1.2}2^2\lfloor\underline{2n-2} - \text{\&c. to } 2n \text{ terms}\right\}$$

110.—From a bag containing $m$ gold and $n$ silver coins, a coin is drawn at random, and then replaced; and this operation is performed $p$ times. Find the chance that all the gold coins will be included in the coins thus drawn.

111.—A train, consisting of $p$ carriages, each of which will hold $q$ men, contains $pq-m$ men. What is the chance that another man getting in, and being equally likely to take any vacant place, will travel in the same carriage with a given passenger.

112.—If the chance of a trial succeeding is to its chance of failing as $m:n$, the most likely event in $(m+n)r$ trials is $mr$ successes and $nr$ failures.

113.—If $n$ witnesses concur in reporting an event of which they received information from another person, the chance that the report is true will be $(p^{n+1}+q^{n+1}) \div (p^n+q^n)$ where $p=1-q$ is the chance of the correctness of a report made by any single person.

114.—The reserved seats in a concert room are numbered consecutively from 1 to $m+n+r$. I send for $m$ consecutive tickets for one concert and $n$ consecutive tickets for another concert. What is the chance that I shall find no number common to the two sets of tickets?

115.—Two persons are known to have passed over the same route in opposite directions within a period of time $m+n+r$, the one occupying time $m$, and the other time $n$; find the chance that they will have met.

116.—If $p=1-q$ be the chance of success at any trial, what is the chance that in $r+n$ trials there should be at least $r$ consecutive successes, (i) when $n<r$ and (ii) when $m$ is the greatest integer in $n \div r$.

117.—If $n$ numbers be selected at random, what are the respective chances that their continued product in the common scale of notation will end with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9?

118.—There are $n$ tickets in a bag numbered 1, 2, 3, ... $n$. A man draws two tickets at once, and is to receive a number of sovereigns equal to the product of the numbers drawn. What is his expectation?

119.—What would be the expectation in the last question if three tickets were drawn and their continued product taken?

120.—If a set of dominoes be made from double blank up to double $n$, prove that the number of them whose pips are $n-r$ is the same as the number whose pips are $n+r$, and the number is the coefficient of $x^{n-r}$ in the expansion of $(1-x-x^2+x^3)^{-1}$; and the total number of dominoes is $\frac{1}{2}(n+1)(n+2)$.

121.—If from the dominoes in the last question a man is to draw one at random, and to receive as many pounds as there are pips on the domino drawn, what is his expectation worth ?

122.—If $p$ be a prime number whose reciprocal in the decimal scale of notation is expressed by a recurring period of $p-1$ digits, and if the digits of this period be rearranged at random, the chance that the new period thus formed will belong to a fraction whose denominator is $p$, will be $(q+1)^{r-1}(\lfloor q)^{10} \div \lfloor p-2$, where $q$ is the quotient and $r$ the remainder when $p$ is divided by 10.

123.—A vessel is filled with three liquids whose specific gravities in descending order of magnitude are $S_1$, $S_2$, $S_3$. All volumes of the several liquids being equally likely, prove that the chance of the specific gravity of the mixture being greater than $S$ is

$$\frac{(S_1-S)^2}{(S_1-S_2)\ (S_1-S_3)}, \text{ or } 1-\frac{(S-S_3)^2}{(S_2-S_3)\ (S_1-S_3)},$$

according as $S$ lies between $S_1$ and $S_2$, or between $S_2$ and $S_3$.

# ANSWERS TO THE EXAMPLES.

### Examples on Choice (pages 62–64).

**1.** 20. **2.** 840 ways; or, considering the arrangement, 20160 ways. **3.** 40320, 5040. **4.** 1320.

**5.** $\dfrac{\underline{|32}}{\underline{|12}\,\underline{|12}\,\underline{|8}}$. **6.** $\dfrac{\underline{|64}}{\underline{|32}\,(\underline{|8})^2\,(\underline{|2})^6}$

**7.** 166320. **8.** $\underline{|15} \div 192$. **9.** 120. **10.** 480, 22.

**11.** $\dfrac{18.\underline{|60}}{\underline{|20}\,\underline{|40}}$, $\dfrac{3.\underline{|60}}{\underline{|20}\,\underline{|40}}$

**12.** 3. **13.** 360, 120, 24. **14.** 675675. **15.** 1436400. **16.** 9849600. **17.**—If there be $m$ of one sort, and $n$ of the other, the number of ways is $\underline{|m}\,\underline{|n}$. **18.** $6^6 - 6^4$. **19.** 20. **20.** 3439, 1271. **21.** $\underline{|90} \div \underline{|24}\,\underline{|22}\,\underline{|30}\,\underline{|4}\,\underline{|10}$ **22.** 167. **23.** 20591. **24.** 5040, 75600.

### Examples on Chance (pages 138–141).

**1.** $\frac{2}{9}$. **2.** $\frac{1}{5}$. **3.**—The alphabet containing 20 consonants and 6 vowels, $\frac{3}{65}$. **4.**—The chances are proportional to 14256 : 12060 : 10175. **5.** $\frac{8}{27}$. **6.** $\frac{7}{27}$

**7.** 3 shillings. **8.** $\frac{33}{1000}$. **9.** $\frac{1}{60}$. **10.** 10 shillings.

**11.** $\frac{112}{243}$. **12.** $\frac{425}{729}$, $\frac{245753125}{387419489}$, $\frac{625}{2187}$. **13.** $\frac{16}{27}$

**14.** $\frac{7}{12}$, $\frac{17}{54}$, $\frac{11}{108}$. **15.** 257 to 208. **16.** $\frac{130}{837}$, $\frac{16025}{17577}$

**17.** $\frac{2560}{28561}$. **19.** $\frac{7}{9}$. **20.** 31 to 15.

## Miscellaneous Examples (pages 231–250).

**1.** 25. **2.** 480, 437. **3.** 12. **4.** 63. **5.** 756.
**6.** 60. **7.** 24. **8.** 30. **9.** 5. **10.** 209. **11.** 59733.
**13.** If $m=qn+r$, $|\underline{m} \div (|\underline{n})^q |\underline{r}$. **14.** $(4n^2-6n+4)|\underline{2n-2}$
**15.** 66. **16.** 10156250. **17.** 1728. **19.** 220.
**21.** 3023. **22.** 332629. **23.** 277199. **24.** 16.
**25.** $\frac{1}{2}n\ (n-1)$ where $n=(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)$.
**28.** 100. **29.** 2030. **30.** $\frac{1}{2}n(3n^2-3n+2)$. **31.** 6.
**32.** $\frac{1}{2}n(n-1)$. **33.** $\frac{1}{2}n(n-1)-\frac{1}{2}p(p-1)-\frac{1}{2}q(q-1)$.
**34.** 576. **35.** 15, 12. **36.** 19. **37.** 6.

**38.** $\frac{|\underline{pq+r}}{(|\underline{q})^p(q+1)^r}$. **39.** $\frac{(|\underline{13})^4}{(|\underline{3})^{12}(|\underline{4})^3}$. **40.** $\frac{1}{2}(n^2+n+2)$.

**41.** $\frac{1}{6}(n+1)(n^2-n+6)$. **42.** $n(n-1)$, $n^2$. **44.** $\frac{11}{120}$.

**45.** $\frac{1}{10}$. **46.** $\frac{2}{21}$. **47.**—Their respective chances are $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{2}$. **48.** $\frac{2}{5}$. **49.** $\frac{2}{n}$. **50.** $\frac{1}{2}$.

**53.**—The latter is better as 11 to 9. **57.** 25 : 2.

**59.** $\frac{2197}{20825}$, $\frac{704}{4165}$, $\frac{264}{4165}$. **60.** $\frac{52728}{112559125}$.

**63.** $\frac{p}{r+1}\left\{1-\frac{\lfloor\underline{pq-q}\;\lfloor\underline{pq-r-1}}{\lfloor\underline{pq}\;\lfloor\underline{pq-q-r-1}}\right\}$ **65.** 22 shillings.

**66.**—$\frac{m-1}{2m-1}$. **67.**—$\frac{9}{98}$, $\frac{1}{35}$. **73.**—$\frac{13}{54}$.

**74.** $21\pm19(\cdot8)^{10}$ crowns. **76.** $\left(1-\frac{2p}{n}\right)\left(1-\frac{1}{q}\right)^2$. **77.** $\frac{4}{21}$.

**78.** Middle coefficient of $\left(\frac{1+z+z^2+\ldots+z^{m-1}}{m}\right)^{2n}$

**79.** $\frac{16}{37}$, $\frac{12}{37}$, $\frac{9}{37}$. **80.** $\frac{n-1}{mn-1}$, $\frac{n-1}{mn-rn-1}$.

**81.** $\frac{11}{39}$, $\frac{10}{39}$, $\frac{3}{13}$, $\frac{3}{13}$. **83.** If $(q+1)\,r\div p$ be an integer, this and the next lower integer are equally likely. If this be not an integer, the next lower integer. (Example: 5 or 6.)

**86.** $\frac{1771}{24}\times\lfloor\underline{26}$. **87.** $n^n$. **88.** $\lfloor\underline{2n-1}\div\lfloor\underline{n}\;\lfloor\underline{n-1}$.

**89.**—In the first case, the odds against the event are as the sum of the first thirty terms to the sum of the remaining thirty terms, in the expansion of the binomial $(59+1)^{59}$. In the second case, the chance is $60\;\lfloor\underline{60}\;\lfloor\underline{88}\div\lfloor\underline{29}\;\lfloor\underline{119}$.

**90.**—5720. **91.** 330. **92.** $\lfloor\underline{24}\div\lfloor\underline{4}$. **93.** 6720.

**94.** 46376. **99.** 105. **100.** $(\lfloor\underline{12})^2\,e_{12}^{-1}$. **102.** $1-e_n^{-1}$

**103.** $\frac{1}{2}\left(1+\frac{1}{n^2}\right)$. **104.** $\frac{1}{2}\left\{1+\frac{3(n+1)}{2n^2+4n+1+(-1)^n}\right\}$

**106.** $\frac{m+1}{m+\lfloor\underline{a}\,\lfloor\underline{b}\,\lfloor\underline{c}}$ **107.** $\frac{\lfloor\underline{n+5}}{\lfloor\underline{n}\,\lfloor\underline{5}}$.

**108.**—If $n$ lie between $6r$ and $6(r+1)$, the most likely throw will be one in which each number appears either $r$ or $r+1$ times.

**114.** $\frac{(r+1)(r+2)}{(m+r+1)(n+r+1)}$. **115.** $\frac{mn+mr+nr}{(m+r)(n+r)}$.

**117.**—Chance of the final digit being 0 is $(10^n-8^n-5^n+4^n) \div 10^n$. Chance of 5 is $(5^n-4^n)\div 10^n$. Chances of 1, 3, 7, 9, are equal, each being $4^{n-1}\div 10^n$. Chances of 2, 4, 6, 8, are equal, each being $4^{n-1}(2^n-1)\div 10^n$. **118.** $(n+1)(3n+2)\div 12$. **119.** $\frac{1}{8}n(n+1)^2$. **121.** $n$ pounds.

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