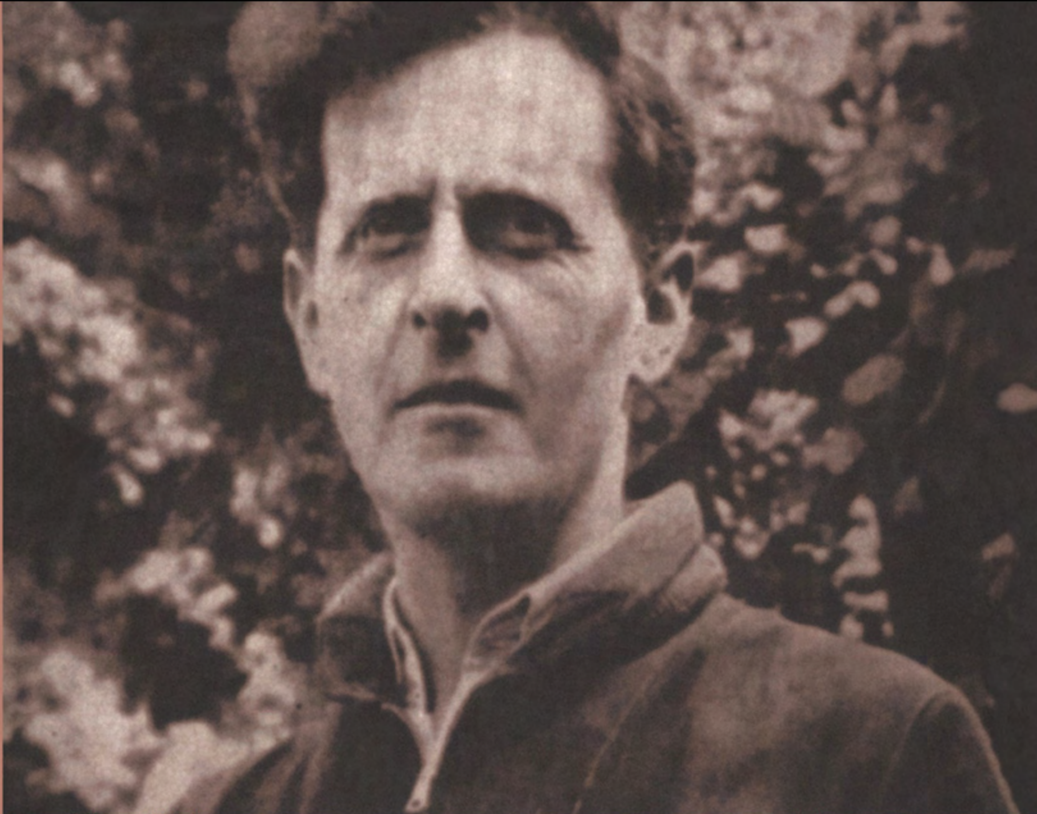

Wittgenstein's philosophy of mathematics

Pasquale Frascolla

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PREFACE

This book is intended to be an exposition, as unbiased as possible, of the conceptions on mathematics which were progressively elaborated by Wittgenstein during the entire course of his reflections on the subject (from the *Tractatus* up to 1944). The fact that I have based my work exclusively on already published material (including notes on lectures and on conversations) does not mean that I believe a philologically credible reconstruction of the evolution of Wittgenstein's thought – obtained by accurately analysing the manuscripts and typescripts he left – to be of little importance. Rather, I have set myself to do something which could contribute to a critical reading of the texts which have not yet been published: that is, to advance general interpretative conjectures, which may later find further confirmation in those texts (in addition to that furnished, in my opinion, by the published writings), or else refutation. Although this strategy properly applies to the writings after 1929 and not to the *Tractatus,* a systematic exposition of Wittgenstein's view of mathematics has appeared to me indispensable also in relation to his first work. The results that I have reached on this subject are reported in Chapter 1. As far as the writings of his so-called intermediate phase (1929–33) are concerned, and those of the decade 1934–44, I have tried to extract the general lines of Wittgenstein's approach to mathematics in these two periods. I have singled out the watershed between them to be the radical development of his considerations on rule-following and his consequent abandoning of a certain strong version of verificationism, which he endorsed in the intermediate phase. Even here there is a justification for the direction my work has taken: in my opinion, in spite of how he conceived, in those years, his own philosophical work, and hence of the form that his reflections assumed, Wittgenstein developed – with increasing coherence – an overall conception of mathematics, which I have called "quasi-formalism" (a conception – to some extent – already present in the *Tractatus,* and whose mature core is a full-blooded nominalistic view of necessity). Certainly, this is not a novelty at all. But I believe that even today it is useful to seek to give a less rhapsodic and more systematic formulation of Wittgenstein's views on mathematics. Indeed, often those who are not struck by the fascination of his writings "throw the baby away with the bath water".

On the other hand, it happens equally frequently that those who are struck by that fascination remain, afterwards, prisoners of Wittgenstein's jargon and reformulate his ideas in such a way that there is no increase at all in our capacity of critical evaluation. This book intends to offer a contribution towards the overcoming of the one-mindedness of both these attitudes.

P.F.
Scuola Normale Superiore, Pisa

ABBREVIATIONS

AWL Ambrose, A. (ed.) *Wittgenstein's Lectures, Cambridge 1932–1935,* Blackwell, Oxford, 1979.

BB *The Blue and Brown Books: Preliminary Studies for the 'Philosophical Investigations',* ed. R. Rhees, Blackwell, Oxford, 1958.

LFM Diamond, C. (ed.) *Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge 1939,* Cornell University Press, Ithaca, N.Y., 1976.

NB *Notebooks 1914–1916,* ed. G. H. von Wright an G. E. M. Anscombe, trans. G. E. M. Anscombe, Blackwell, Oxford, 1961.

PG *Philosophical Grammar,* ed. R. Rhees, trans. A. J. P. Kenny, Blackwell, Oxford, 1974.

PI *Philosophical Investigations,* ed. G. E. M. Anscombe and R. Rhees, trans. G. E. M. Anscombe, 2nd edn, Blackwell, Oxford, 1958.

PR *Philosophical Remarks,* ed. R. Rhees, 2nd edn, trans. R. Hargreaves and R. White, Blackwell, Oxford, 1975.

RFM *Remarks on the Foundations of Mathematics,* ed. G. H. von Wright, R. Rhees and G. E. M. Anscombe, trans. G. E. M. Anscombe, 3rd edn, Blackwell, Oxford, 1978.

T *Tractatus Logico-Philosophicus,* trans. D. F. Pears and B. F. McGuinness, Routledge & Kegan Paul, London, 1961.

WVC McGuinness, B. F. (ed.) *Ludwig Wittgenstein and the Vienna Circle: Conversations recorded by Friedrich Waismann,* trans. J. Schulte and B. F. McGuinness, Blackwell, Oxford, 1979.

Z *Zettel,* ed. G. E. M. Anscombe and G. H von Wright, trans. G. E. M. Anscombe, 2nd edn, Blackwell, Oxford, 1981.

1

THE PHILOSOPHY OF ARITHMETIC OF THE *TRACTATUS*

PRELIMINARIES

In considering the state of the exegesis of the two groups of propositions of the *Tractatus* devoted by Wittgenstein to mathematics (propositions 6.02–6.031 and 6.2–6.241), one realizes immediately that the critical assessment of this topic is still at an embryonic stage. There are no overall expositions that can be compared by a close examination of their respective textual support and weighing of their arguments. Moreover, it is of primary importance to avoid the pitfalls of certain exegetical suggestions which, despite the authority of the proponents, are irremediably misleading. Only after having re-established a correct reading of the text, even in its "technical details", can one tackle some general theses concerning the philosophy of mathematics which are explicitly stated in the *Tractatus* and which, until these misinterpretations are clarified, are almost bound to remain obscure. Two such significant mistakes can be found, respectively, in Black's and in Anscombe's interpretation and will be examined in this section.

One of the ideas underlying the reconstruction presented in this chapter regards the real import of the inductive definition given by Wittgenstein in 6.02. In my opinion, it must be read as a definition – by induction on the number of occurrences of the sign "*+1*" in a term of the form *0+1+1+…+1* – of the endless series of expressions

"Ω^{0}'x", "Ω^{0+1}'x", "Ω^{0+1+1}'x", "$\Omega^{0+1+1+1}$'x", and so on,

which belong to the language of the general theory of logical operations.[1] The subsequent informal characterization of the concept of a natural number as the exponent of an operation, as well as a large part of the peculiar way in which Wittgenstein construes arithmetical identities, are closely connected with the inductive definition of the above series of expressions.[2] Obviously, we will dwell upon this definition at length in the course of the present chapter. But a preliminary question arises: how should the symbol "Ω", which occurs in the preceding expressions (and in their respective *definientia*), be interpreted? Let us consider Black's answer to this question.[3] He writes that natural numbers are introduced by Wittgenstein "in connexion with the operation, Ω'x, that has just been *defined* in 6.01"; and he then continues that "Wittgenstein's basic idea can be illustrated in connexion with some *other* operation, say the one expressed in ordinary language by the expression 'parent of x'".[4] Now, it is undoubtedly true that the symbol "Ω" occurs for the first time in 6.01 of the *Tractatus,* where Wittgenstein introduces a special notation to represent the general form of a logical operation, i.e. the operational scheme to which every procedure of generation of a truth-function of any given set of propositions can be reduced. The propositions concerned run as follows: "If we are given the general form according to which propositions are constructed, then with it we are also given the general form according to which one proposition can be generated out of another by means of an operation" (*T* 6.002); "Therefore the general form of an operation Ω'(η) is

[ξ, N(ξ)]'(η) (=[η, ξ, N(ξ)]).

This is the most general form of transition from one proposition to another" (*T* 6.01). Notice that, if we put the letter "p" with a bar, closed between parentheses (which Wittgenstein uses to denote the set of elementary propositions), in the base position of the operation-sign "[ξ, N(ξ)]", we get:

"[ξ, N(ξ)]'(p) (=[p, ξ, N(ξ)])"

where the expression of the general form of a truth-function, given by the Austrian philosopher in proposition 6, occurs on the right of "=".[5] This proves that by the complex symbol "[ξ, N(ξ)]'(η)" Wittgenstein means the procedure of successive application of the operation of joint negation, starting from any set of propositions, clearly explained by Russell in his 1922 'Introduction'.[6] It satisfactorily represents the general form of a logical operation since such an operation is conceived as a procedure to generate, from one or more given propositions, a proposition which is a truth-function of these; and *whatever the procedure may be,* the appropriate iteration of Sheffer stroke-operation will produce exactly that truth-function. Contrary to Black's explicit opinion quoted above, Wittgenstein, in 6.01, does not propose a *definition* of the symbol "Ω", whereby "Ω" would denote this procedure of successive application of the operation N of joint negation. Plainly, it is used there to make reference to a logical operation *in general,* namely, as an operation variable, and it continues in this role both in 6.02 and in all the other propositions of the *Tractatus* in which it occurs.[7]

But it is not only the textual evidence that leads to this conclusion. In favour of the thesis that Wittgenstein uses the symbol "Ω" as an operation variable, there is also another strong reason: that it can be adopted as a base for a coherent reconstruction of the propositions of the *Tractatus* devoted to mathematics. I shall present this systematic account in the next two sections. For the time being, let us scrutinize the absurd consequences entailed by Black's reading of 6.01, which Black himself draws in his comment on 6.02.[8] Let us suppose that the variable "x" stands for a set of propositions, which has a non-tautological and non-contradictory proposition *H* as its only element. Then, given the meaning Black ascribes to "Ω", "Ω'x" would stand for the set of the truth-functions of *H*, namely the set whose only elements are *H*, ~*H*, the tautology and the contradiction (strangely enough, he does not mention these two truth-functions of *H*).[9] Applying the operation once more, nothing new would be obtained, but the set Ω'x would always be reproduced; thus:

$\Omega\text{'}x = \Omega\text{'}\Omega\text{'}x = \Omega\text{'}\Omega\text{'}\Omega\text{'}x = \Omega\text{'}\Omega\text{'}\Omega\text{'}\Omega\text{'}x$, and so on,

for any other finite sequence of occurrences of "Ω'". Using the definitions found in 6.02, the above result can be restated as follows:

$\Omega^{1}\text{'}x = \Omega^{2}\text{'}x = \Omega^{3}\text{'}x = \Omega^{4}\text{'}x$, and so on.

But, as we shall see in detail in the next section, according to Wittgenstein, a numerical identity **"t = s"** is an arithmetical theorem if and only if the corresponding equation "$\Omega^{t}\text{'}x = \Omega^{s}\text{'}x$", which is framed in the language of the general theory of logical operations, can be proven.[10] Unsurprisingly, Black's comment on 6.02 which begins with the statement: "Wittgenstein's idea is hard to grasp", ends with the question: "does it follow that m = n for all positive integers?".[11] Although Black poses it as a question, the absurd consequence that follows from his interpretation of the symbol "Ω" is actually unavoidable. Whatever the set of propositions for which it is assumed that the variable "x" stands, one will always get – with only one application of the operation meant by Black by "Ω" – a set of propositions which will be left unchanged by each of its other iterations. And this will bring about the catastrophic identification of all positive numbers.

Now, it seems to me truly ungenerous to hold that Wittgenstein had a view of arithmetic which leads to such a conclusion, even if the criticism is made cautiously and in question-form. It is rather incongruous that, faced with the macroscopic absurdities which he is compelled to ascribe to Wittgenstein, owing to his interpretation of the text, Black does not consider it more reasonable to appeal to a minimal principle of interpretative charity and to revise and rectify his own position. Black's commentary on 6.241, the closing proposition of the second group of propositions of the *Tractatus* concerning mathematics, is another example which reveals the same attitude. In considering the case of the identity "2 × 2 = 4", Wittgenstein gives here a valuable example of how a proof of an equation of operation theory, corresponding to a true arithmetical identity, appears in the light of his conceptions. The importance of this proposition has been, in general, enormously underestimated by the exegetes of Wittgenstein's writings, although a detailed analysis could have led to a fruitful insight into the philosophy of arithmetic of the *Tractatus*. The following comment from Black crudely expresses this common stance, passing off, moreover, faults of his own interpretation as Wittgenstein's shortcomings: "Wittgenstein's proposed 'proof' is eccentric and would not satisfy contemporary standards of mathematical rigour".[12]

In Black's *Companion* we also find a typical and significant case testifying how the lack of understanding of the literal meaning of Wittgenstein's text, and of some apparently minor "technical" details (such as the role of the symbol "Ω" at first sight), produces inevitably the unaccountability of some of the philosophically most telling theses of the *Tractatus*. Take proposition 6.22: "The logic of the world, which is shown in tautologies by the propositions of logic, is shown in equations by mathematics". If someone proposed an interpretation of the philosophy of logic of the *Tractatus* which cannot account for the fundamental thesis that tautologies show the logic of the world, its formal features (namely, the traits it shares with every possible world), this circumstance would be rightly considered a very good reason to raise doubts about the soundness of that interpretation. And yet, inasmuch as it concerns the philosophy of mathematics of the *Tractatus,* a quite different standard of judgement has been adopted. Nobody will deny the philosophical import of the thesis, expressed in 6.22, that arithmetical identities (or, more properly, the equations of the theory of logical operations onto which they are mapped in Wittgenstein's reconstruction of arithmetic), like tautologies, show the logic of the world. Nonetheless, Black refers to it with the following laconic comment: "It is hard to see how what is shown in equations can be assimilated in this way to what is shown in tautologies".[13] The situation does not improve if one considers other influential commentators of the *Tractatus*. Anscombe quite overlooks proposition 6.22; Fogelin quotes it in full, but only in order to comment that Wittgenstein's treatment of equations is modelled on his treatment of tautologies, without adding a minimal explanation of this claim; similarly, Block quotes it without further remarks, whereas Ayer, enumerating the aspects of similarity between mathematical equations and tautologies, ignores paradoxically the fundamental element underlined in 6.22. Savitt, on the other hand, frankly admits that the proposition in question continues to be "very mysterious".[14] This bibliographical sample, though obviously very exiguous, seems to me sufficiently significant to be a valid support to the above statement that an inadequate analysis of the symbolism used by Wittgenstein in his definitions has impeded the understanding of his general positions.

But now let us return to the exegetical issues by examining some aspects of Anscombe's explanation of the group of propositions 6.02–6.031. First of all, she wonders why Wittgenstein did not take the simplest and most direct route in his definition of the formal concept of number: (i) explain the meaning of the symbol "0" and of the symbol "+1"; (ii) then introduce the definition given at 6.03 ("The general form of an integer is [0, ξ, ξ+1]"), the meaning of which would have been perfectly clear, if the definitions of "0" 4 and "+1" had already been on hand. But, as we shall see in detail later, the inductive definition in 6.02 aims precisely at determining the meaning of these two symbols, at least in the sense that, according to the Austrian philosopher, the meaning of every arithmetical term of the form 0+1+1+...+1, for any finite number of occurrences of "+1", will be shown by a certain corresponding expression of the language of the operation theory, as defined by the inductive definition in question. This is like saying, simply, that the definition ought to determine the nature of the elements of the series, the general term of which is expressed by the variable "[0, ξ, ξ+1]". In short, Wittgenstein has actually done what Anscombe denies he did. I shall not dwell upon a thorough exposition of Anscombe's interpretation of this part of the *Tractatus,* since I wish to draw attention only to the source from which, in my opinion, her misleading reading issues. With reference to the inductive definition in 6.02, she says that it "explains the meaning of a zero exponent of the operator 'Ω' and also the meaning of an exponent of the form 'n+1' given the meaning of the exponent 'n'". Then, she adds in a footnote that Wittgenstein uses "Ω" instead of the capital Latin letter "O" in order to avoid the unperspicuous expression resulting from the addition of "0" as exponent of "O"; the use of the lower case Greek letter "ν" instead of the usual numerical variable "n", would follow, consequently, by assimilation to the use of "Ω".[15] On the basis of this assumption on the role of the variable "ν", we are led to believe that Wittgenstein, laying down the inductive definition that opens 6.02, is merely introducing new expressions employing the *already available* numerical symbols. In the *Tractatus,* natural numbers would be considered by him, somewhat generically and without further explanation, as the results of the successive addition of a unit, starting from zero. In other words, Anscombe maintains that, in the first part of 6.02, Wittgenstein is proceeding by a usual induction on n. Hence, the purpose of the inductive definition which opens Wittgenstein's treatment of mathematical concepts would be to explain the meaning of the particular kind of arithmetical context in which numerals occur as exponents of "Ω", taking the standard definitions of numerals in terms of "0" and "+1" and, obviously, the meaning of the two primitives as given (notice that the definitions of "1", "2" and "3" in these terms are actually supplied by Wittgenstein only in the last part of 6.02). So, at this point, the question Anscombe puts at the beginning of her comment on these propositions inevitably arises: why did Wittgenstein not give an advance explanation of the meaning of "0" and "+1" (which is required to determine both that of standard numerals and that of the variable for the formal concept of number) but preferred to put the explanation of the meaning of the expressions in which a numeral occurs as exponent of the symbol "Ω" first?

This is a problem which will disappear with the different interpretation that I will develop systematically in the next section and which I wish to outline briefly now. Contrarily to Anscombe's opinion, it seems quite unreasonable to suppose that, by the inductive definition at the beginning of 6.02, which *opens* Wittgenstein's treatment of numbers, he is defining a new function, assuming numerals, their definitions in terms of the primitive notation and the meaning of the primitives "0" and "+1" as *already* given. I think that Wittgenstein's true aim in 6.02 can be described as follows: he intends to define the endless expressions of the form $\Omega^{0+1+...+1}$'x, where "0+*1*+*1*+...+*1*" is the term of the language of the general theory of logical operations which corresponds to the usual arithmetical term "0+1+1+...+1". This he does with a reductionist purpose in mind: the meaning of "0+1+1+...+1" should be derived from the meaning of the expression of the language of operation theory in which the corresponding numerical term occurs as exponents of the variable "Ω". In other words, the *definiens* of "$\Omega^{0+1+...+1}$'x", with n ≥ 0 occurrences of "+1", will *show* the meaning of the arithmetical term "0+1+1+...+1", with the same number of occurrences of "+1".[16] The ambiguity of the text of the *Tractatus,* which also misled Anscombe, originates from Wittgenstein's erroneous use of the numerical variable "ν", which conceals his true intentions in supplying the inductive definition at the beginning of 6.02. To fulfil them correctly, it has to be construed as a definition framed in a metalanguage which has the language of the theory of logical operations as its object-language. If my interpretation is right, the variable "ν" should be regarded as a schematic letter for an expression of the form *0+1+1+...+1*. Expressions of this form belong to the language of the theory of logical operations and each of them is used in it as an exponent attached to "Ω" to represent a specific formal property common to the elements of a definite, wide class of linguistic (non-mathematical) constructs. Although the inductive definition is formulated by Wittgenstein in a substantially misleading way, it is nothing but a definition of the infinite set of expressions of the form $\Omega^{0+1+1+...+1}$'x, by induction on the number of occurrences of "*+1*" in the exponent "*0+1+1+...+1*". The variables "ν" and "μ", employed by Wittgenstein in defining the arithmetical function of product in 6.241, also need to be re-interpreted as schematic letters for arbitrary arithmetical terms of the language of the theory of logical operations (where the class of these terms has been previously introduced by a syntactic definition which perfectly parallels the definition usually given for the terms of arithmetic). I believe this is the only way in which the reduction of arithmetic to the general theory of logical operations, outlined by Wittgenstein in the *Tractatus,* can be effected. Of course, in trying to perform the reduction satisfactorily, the non-Wittgensteinian distinction between language and metalanguage is needed, and reference to numbers (in the metalanguage in which the definitions must be correctly restated) has to be made. As in all similar cases, this circumstance weakens the reductionist claim, but it is reasonable to think that Wittgenstein would have considered it a straightforward confirmation of his overall idea on the substantial ineffability of our "knowledge" of forms.

Recapitulating: according to Wittgenstein, the meaning of each of the usual arithmetical terms of the form 0+1+1+...+1 can be derived from the meaning of the corresponding expression of the general theory of logical operations "$\Omega^{0+1+1+...+1}$". This is the true import of the pivotal thesis according to which a number is the exponent of an operation (*T* 6.021). In the language of the theory of operations, a symbol "*n*" will be taken as an abbreviation of "*0+1+1+...+1*" with n occurrences of "*+1*", in perfect analogy with standard numerals of arithmetic. By means of the explicit definition of "*n*" and of the inductive definition in 6.02, the symbolic context "Ω^{n}'x" can then be translated into an expression in which neither "*n*" nor "*0+1+1+...+1*" (with n occurrences of "*+1*") further appear, i.e. into a string "Ω'Ω' ... Ω'x", with n occurrences of "Ω'". Similarly, the expression "Ω^{t}'x" (from the operation theory language) will correspond to the arithmetical term **"t"** and the relevant definitions will make it possible to translate the former into a specific, complex arrangement of strings of "Ω'". As a result, the equation "Ω^{t}'x = Ω^{s}'x" will be univocally correlated to the arithmetical identity **"t = s"**. Elementary processes of numerical calculation will be reduced to manipulatory procedures of "Ω'" strings – the addition of a further "Ω'" to a given string; the subdivision and grouping of the string elements into sub-strings; the creation of longer strings by the combination of shorter ones, etc. – based, to a large though not exclusive extent, on the properties of the application of a logical operation.[17] The relations between the configurations thus obtained within the theory of operations will *show* the possibilities of symbolic transformation corresponding to the arithmetical identities usually classified, in non-*Tractarian* jargon, as "true arithmetical identities".

The systematic and detailed exposition of this theory, including the treatment of complex arithmetical terms of the form (**t × s**) and (**t+s**), is the somewhat laborious task of the next section. We shall see that important specifications need to be added to what has been said so far concerning Wittgenstein's use of arithmetical terms as exponents of an operation variable. There are, it is true, other examples of obvious misinterpretations, like that implied, I believe, by Block's totally unjustified statement that Wittgenstein's definition of numbers presupposes the laws of addition[18] (actually, according to Wittgenstein, both successor and addition can be reduced, in a peculiar but certainly not incongruous way, to the abstract notions of application of a logical operation and of composition of two such operations). But at this point, instead of continuing this inevitably fragmentary review of the bibliography on Wittgenstein's early work, it seems to me more helpful to present what constitutes, in my opinion, a thorough and as far as possible faithful exposition of the *Tractatus* view of arithmetic.

SYSTEMATIC EXPOSITION

It is expedient, of course, to start from the beginning, and the beginning is the first part of proposition 6.02. Here we find the following inductive definition:

$x = \Omega^{0}{}'x$ Def.

$\Omega'\Omega^{v}{}'x = \Omega^{v+1}{}'x$ Def.[19]

Let us consider one by one the symbols included therein. First of all, the variable "x" is usually employed by Wittgenstein as an object variable, in the peculiar sense that the word "object" has in the *Tractatus* (and, consequently, as a symbol that shows the form of a name).[20] However, in this context, it is used by him for a slightly different purpose: that of showing the form of an expression which has not been generated by the application of a logical operation. As seen, the symbol "Ω" is an operation variable, the symbol of the formal concept of operation. The essential aspects of the notion of logical operation in the *Tractatus* are as follows: (i) an operation is a uniform procedure, by the application of which certain expressions (in particular, propositions) can be generated from given expressions (again, propositions), whenever the base and the result of the operation are linked by a certain formal relation ("The concept of the operation is quite generally that according to which signs can be constructed according to a rule" (*NB*, entry 22/11/16)); "The operation is what has to be done to the one proposition in order to make the other out of it. And that will, of course, depend on their formal properties, on the internal similarity of their forms" (*T* 5.23, 5.231)[21] ; (ii) there is no *object* that corresponds to an operation sign as its fixed and distinguishable semantical value ("The occurrence of an operation does not characterize the sense of a proposition. Indeed, no statement is made by an operation, but only by its result, and this depends on the bases of the operation" (*T* 5.25)); (iii) an operation can always be applied to the result of its own application: the result of one of its applications is again a possible base for a new application of the operation ("an operation can take one of its own results as its base" (*T* 5.251)). Wittgenstein calls "successive application of an operation" any finite sequence of applications of the operation to the result of its own application, starting from the result of its application to an expression which is not generated by means of an application of the same operation (*T* 5.2521).

We will now go on to Wittgenstein's use of the sign "'", which occurs both in the *definiendum* and in the *definiens* (of the inductive step) of the inductive definition. The single inverted comma is part of the notation which represents *the form* of a result of the application of an operation to a given base; it is not attached to constants such as "N" or "~", but is used by Wittgenstein whenever he wants to speak of the result of an operation in general, or of the general form of an operation (as in 6.01). Probably, the use of this sign is borrowed, with a slight modification, from *Principia Mathematica,* where, if “R” is a dyadic relational predicate, “R‘x” is defined as the only object y having the relation R to x.[22]

Now we have to make the crucial step: the interpretation of the symbols “0” and “ν+1”. As already mentioned in the first section, I believe that, in order to give a correct presentation of Wittgenstein’s intentions, one must assume the availability of the syntactical category of the terms of the form *0+1+1+...+1* in the language of the general theory of logical operations. We can assert that “*0*” is the sign of that form with no occurrences of “*+1*”, and “ν” is a schematic letter for a generic expression of the form, namely, for an expression of such a form with any number n ≥ 0 of occurrences of “*+1*”.[23]

Before continuing our exposition, it would be advisable to make another change in the original notation of the *Tractatus*. We shall use the symbol “*S*” in place of the symbol “*+1*” since the symbol “+” will later designate the operation of addition, which is not treated explicitly by Wittgenstein in the *Tractatus*. Let us reconsider, at this point, the inductive definition from which Wittgenstein’s treatment of numbers starts. We have:

“Ω^{0}’x” meaning the same as “x”,
“Ω^{S0}’x” meaning the same as “Ω’x”,
“Ω^{SS0}’x” meaning the same as “Ω’Ω’x”,
“Ω^{SSS0}’x” meaning the same as “Ω’Ω’Ω’x”,

and so on, for each number n ≥ 0 of occurrences of “*S*”.

Despite the misleading appearance that the use of the variable “ν” creates, the definition has to be construed as a definition by induction on the number of occurrences of the sign “*S*” in the exponent of an expression of the form $\Omega^{SS...S0}$’x. Its base lays down the meaning of such an expression where there is no occurrence of “*S*”; the inductive step establishes the meaning of an expression of this form with n+1 occurrences of “*S*” (for each n ≥ 0), given the meaning of an expression of the same form with n occurrences of “*S*”. We shall shortly see how, according to Wittgenstein, the meaning of each term of the arithmetical language “SS ... S0” (of each original arithmetical term “0+1+1+...+1”) can be grasped by means of the above definition: it is *shown* by the *definiens* of the corresponding expression of the operation theory language in which “*SS ... S0*” occurs as exponent of “Ω”. The inductive definition in 6.02 is conceived as a sort of reduction of the notions of zero and successor to the abstract notion of application of a logical operation, a reduction which the Austrian philosopher proposes as an alternative to the logicist explication of these two arithmetical primitives in terms of the notion of class.[24]

Obviously, the necessary condition for the understanding of Wittgenstein’s definition is the explanation of the nature of the elements of the series x, Ω’x, Ω’Ω’x, Ω’Ω’Ω’x, and so on (i.e. the explanation of the meanings of those terms which provide the *definientia* for the corresponding *definienda* “Ω^{0}’x”, “Ω^{S0}’x”, “Ω^{SS0}’x”, “Ω^{SSS0}’x”, and so on).

Being entirely constructed with variables, every member of the series of expressions “x”, “Ω’x”, “Ω’Ω’x”, “Ω’Ω’Ω’x”, ..., exhibits that which is called by Wittgenstein “a form”. Each configuration “Ω’Ω’ ... Ω’x” shows the common form of all those linguistic expressions that are “concrete” instances of a certain possibility of symbolic construction, namely of a specific *logisch Urbild.* Thus, we can say that each numerical configuration of the form *SS ... S0,* belonging to the operation theory language, is employed by Wittgenstein as part of an abbreviated notation for an expression that shows the formal structure common to the elements of a certain class of linguistic expressions. We can describe explicitly the elements of the series of forms x, Ω’x, Ω’Ω’x etc. as follows: “x” shows the form of any expression which is an initial expression of a series generated by the successive application of an operation (namely, an expression which has not been generated by any application of the operation); “Ω’x” shows the form of any expression which is the result of the successive application of an operation, consisting of a single application of the operation to an initial expression; “Ω’Ω’x” shows the form of any expression which is the result of the successive application of an operation, constituted by the application of the operation to the result of its own application to an initial expression; and so on. What Wittgenstein had in mind can be easily explained by taking the example of the operation of negation (this will be denoted by the tilde “~”). The symbol “x” shows the form of both the propositions “it’s raining” and “it’s cold”, inasmuch as neither is generated as a result of the negation of another proposition (and, of course, the form of any proposition which is not generated by the application of the logical operation of negation); “Ω’x” shows the form of the propositions “~it’s raining’ and “~it’s cold”; “Ω’Ω’x” that of the propositions “~~it’s raining” and “~~it’s cold”, and so on.[25] Although, in the *Tractatus,* Wittgenstein takes into consideration only those operations that have propositions both as bases and as results of their application, it may be useful, for illustrative purposes, to introduce a further type of generative procedure of linguistic expressions from given expressions. Let us consider the operation that, given any dyadic predicate “R”, generates the expression “the R of **a**” when applied to an appropriate singular term “**a**”. For example, taking the dyadic predicate “father of”, the series of expressions “Paul”, “the father of Paul”, “the father of the father of Paul”, “the father of the father of the father of Paul” etc. can be considered as constituted by the following: an expression which is not the result of any application of the operation; an expression that is the result of the application of the operation to the proper noun “Paul”; an expression that is the result of the application of the operation to the definite description “the father of Paul”, i.e., to the expression which is the result of the immediately preceding application of the operation; an expression that is the result of the application of the operation to the definite description “the father of the father of Paul”, i.e., to the expression which is again the result of the immediately preceding application of the operation; and so on. Making these assumptions, the variable “x” can be used to show the form of the proper noun “Paul”, “Ω’x” the form of the definite description “the father of Paul”, “Ω’Ω’x” the form of “the father of the father of Paul”, “Ω’Ω’Ω’x” the form of “the father of the father of the father of Paul” and so on.

If we allow ourselves to violate Wittgenstein’s prohibition of speaking of what is shown by language and, in particular, of speaking of formal properties of linguistic expressions, the content of 6.02 becomes more explicit and the sense in which one can speak of a reduction of the arithmetical primitives to the notion of application of a logical operation is clarified. The meaning of the symbolic configuration “Ω^{0}’x” is identified with the meaning of the variable “x”. Since, in this context, “x” shows, at the utmost level of generality, the form of an expression which is the initial expression of a series generated by the successive application of an operation, we can say that “*0*” stands for the formal property constituted by the number of times the operation is applied to produce such an expression (and the meaning of the numeral “0”, belonging to the usual arithmetical language, is reduced to this property). Similarly, as the meaning of “Ω^{S0}’x” is identified with the meaning of “Ω’x”, and the latter shows, at the same utmost level of generality, the form of an expression which is the result of the successive application of an operation consisting of one application to an initial symbol, we can say that “*S0*” stands for the formal property constituted by the number of times the operation is applied to produce such an expression (and again, the meaning of the usual numeral “S0” is reduced to this property). In the same way, “*SS0*” stands for the formal property common to all the expressions having the form shown by “Ω’Ω’x”: this is, of course, the number of times an operation is applied to generate any one of them (so the meaning of the usual numeral “SS0” is reduced to this property); and so on, for all the other terms “*SS ... S0*”. In this peculiar Wittgensteinian sense, we can say that the first part of 6.02 effects a reduction of the meaning of each of the infinite arithmetical terms “SS ... S0” to the general notion of application of a logical operation. The introduction of the corresponding terms of the operation theory language as exponents of “Ω” signals the recognition of a certain formal aspect of any expression generated by the successive application of a logical operation: the number of applications of the operation.[26]

In the last part of 6.02, Wittgenstein defines the first three standard numerals using the terms “*SS ... S0*” in a way which perfectly parallels the usual procedure followed in arithmetic (by definition, a numeral “*n*” of the operation theory language will be an abbreviation for the term “*SS ... S0*”, with n occurrences of “*S*”, and will occur exclusively as exponent of “Ω” or as part of a more complex term attached to “Ω” as exponent). Finally, propositions 6.021, 6.022 and 6.03 solve the problem of defining the general notion of a natural number (or, in the original terminology of the *Tractatus,* the notion of an integer). Here again we have a typical example of Wittgenstein's way of going about this kind of thing. Given the class of the expressions "*0*", "*S0*", "*SS0*", "*SSS0*" etc., their common formal structure can be recognized (this new aspect of symbolism can be seen). Consequently, an appropriate variable expressing this formal concept has to be introduced in the notation. As explained above, the "concrete" material forming the base for the reconstruction of arithmetic in the theory of operations is provided by the series of linguistic expressions generated by the successive application of an operation. Each term of the form *SS ... S0* is introduced in the operation theory language as a notational device to express the formal property of all the expressions generated by the same number of applications of an operation to the result of its own application. So the real import of the reductionist thesis put forward by Wittgenstein in 6.021 ("A number is the exponent of an operation") is clear: it asserts the reducibility of the meaning of every numeral "**n**" to the meaning of "*n*" in its occurrence as exponent of "Ω". To reach the general concept of a natural number, a further step is needed: according to 6.022, "the concept of number is simply what is common to all numbers, the general form of a number". In representing this form Wittgenstein follows the method adopted by him in all cases in which the entities falling under a formal concept constitute a series. The relation of each term to its immediate successor in the series is a constant formal relation. In this case, a formal concept must be expressed by a variable for an arbitrary term of the series, i.e. a variable which indicates the first term of the series and the form of the uniform procedure to generate the immediate successor of any given term. Obviously, even the schematic expression "*SS ... S0*" (or, in the original notation of the *Tractatus,* the schematic expression "0+1+1+...+1") holds good. In accordance with the previous directions, we find the more perspicuous variable "[0, ξ, ξ+1]" (*T* 6.03): its meaning can be immediately derived from the primitive context of occurrence of the corresponding (italicized) signs "*0*" and "*+1*" in the theory of logical operations (the inductive definition in 6.02), inasmuch as it is construed along the lines suggested above. In these limits, the promised reduction of the primitive arithmetical notions and of the concept of a natural number to the general notion of successive application of an operation has been effected.

The introduction of the variable for the formal concept of a natural number completes the first part of our systematic exposition of the interpretation of arithmetic in the *Tractatus*. But the true import of Wittgenstein's conception is far from finished with the reduction of the meaning of a numeral "**n**" to the formal property shown by the corresponding string of the operation theory language "Ω'Ω' ... Ω'x", with n occurrences of "Ω'". Two fundamental aspects of his construction should be taken into account to realize this import effectively: (i) the way in which the arithmetical functions of sum and product are reduced to the general notion of application of a logical operation, with the consequent possibility of correlating, to any given arithmetical identity, an appropriate equation of the theory of operations (in such a manner that the first is a theorem of numerical arithmetic if and only if the second is a theorem of the latter theory); (ii) the relationship between arithmetic and the world that is implied by the mapping of true arithmetical identities onto theorems of the general theory of logical operations. In the remainder of the section we will examine these two points, starting from point (i).

Before we begin the analysis of how Wittgenstein extends his interpretation of arithmetic to the arithmetical functions of product and sum of two numbers, it is expedient to give a warning. The textual evidence on which the further reconstruction is grounded is extremely meagre: it amounts, in substance, to proposition 6.241 of the *Tractatus*. Nonetheless, the content of this proposition, together with the results already achieved so far, is sufficient, in my opinion, to give a firm support to the following exposition.

We have seen that a correct understanding of the inductive definition in 6.02 requires that numerals in primitive notation (i.e. in terms of the italicized symbols "*0*" and "*+1*" or "*S*") be on hand in the language of the theory of operations. Furthermore, every numeral "*n*" is introduced in this language by a definition which strictly parallels the analogous definition of standard numerals in the language of arithmetic. Similarly, the inclusion of complex arithmetical terms in Wittgenstein's system requires that the more extensive category of arithmetical terms is previously introduced in the language of operation theory by means of a syntactic definition which is assumed to parallel the definition of the same class of expressions of the arithmetical language. To this end, we lay down that every numeral "*n*" is an arithmetical term (of the operation theory language), and that, if "*r*" and "*s*" are arithmetical terms (of that language), then "(*r* × *s*)" and "(*r*+*s*)" are too. Let "**t**" be an arithmetical term (belonging to the language of arithmetic): in the case that it is a numeral "**n**", we know from the inductive definition in 6.02 the meaning of the expression "Ω^{t}'x". Now, let us suppose that "**t**" is an arithmetical term of the form (**r × s**) or (**r+s**): then we must cope with the problem of defining the corresponding expressions "$\Omega^{(r\times s)}$'x" and "$\Omega^{(r+s)}$'x". In fact, the meaning of "**t**" will be shown by the specific grouping of the elements of a sequence of "Ω'" (without numerical exponents) into which, by means of these definitions, the expression "Ω^{t}'x" can be transformed. This grouping will show *the arithmetical structure of a certain complex operational model of sign construction*. Thus, for instance, the configuration "(Ω'Ω)'(Ω'Ω)'(Ω'Ω)'x" will correspond, in the sense explained above, to the complex arithmetical term "(2 × 3)", and the configuration "((Ω'Ω)'(Ω'(Ω'Ω)))'x" will correspond to the complex arithmetical term "(2+3)". In Wittgenstein's reconstruction of arithmetic, numerical calculations will consist in manipulations of strings of "Ω'", governed by the general properties of the notion of application of an operation (in the broad sense of "application" that we are going to examine). For example, the proof of the operation theory equation "$\Omega^{(2\times3)}$'x = Ω^{6}'x" will be constituted by the transformation of the configuration "(Ω'Ω)'(Ω'Ω)'(Ω'Ω)'x" into the sequence without internal groupings "Ω'Ω'Ω'Ω'Ω'Ω'x". Analogously, the proof of the equation of operation theory "$\Omega^{(2\times3)}$'x = $\Omega^{(3\times2)}$'x", which replaces the usual arithmetical proof of the identity "2 × 3 = 3 × 2", will be constituted by the transformation of the configuration "(Ω'Ω)'(Ω'Ω)'(Ω'Ω)'x" into the configuration "(Ω'(Ω'Ω))'(Ω'(Ω'Ω))'x".

Let us begin our exposition with the analysis of Wittgenstein's treatment of the function of the product of two numbers (which is the only one he explicitly deals with). As soon as we approach the definition given by him in 6.241, we encounter a problem quite similar to that which we faced in connection with the definition contained in 6.02, i.e. the problem of the interpretation of the variables "μ" and "ν". If Wittgenstein's aim is, in fact, that which I have attributed to him, then the use of numerical variables here is totally misleading. The end-result of his reconstruction of arithmetic should be the assignment of a meaning to every expression of the operation theory language "Ω^{t}'x", for every arithmetical term "**t**". This process of determination of the meaning must match the inductive construction of the syntactic notion of arithmetical term (which is paralleled by the definition of the syntactic category of arithmetical terms of the operation theory language). Suppose, for the moment, that the definitions given in 6.02 are sufficient to fulfil precisely this function, in so far as the inductive base – numerals – is concerned (a provisional supposition that will be discarded later on). In place of the original definition of product

$\Omega^{\nu\times\mu}$'x = $(\Omega^{\nu})^{\mu}$'x,

we should state the following definition:

$\Omega^{(r\times s)}$'x = $(\Omega^{r})^{s}$'x.

But here a problem immediately arises: that of the occurrence of "Ω^{ν}" inside a pair of curved brackets in the *definiens* of Wittgenstein's original definition and, correspondingly, of the occurrence of "Ω^{r}" inside a similar pair of brackets in the *definiens* of the definition that, according to my plan, should replace the original one. That here we are facing a real problem can be easily seen as follows. It is obvious that the pair of brackets indicates that the exponent "*r*" applies to "Ω", whereas the exponent "*s*" applies to the whole expression "Ω^{r}". However, given the results obtained so far (the infinite set of equations of the form Ω^{n}'x = Ω'Ω' ... Ω'x, with n occurrences of "Ω'", supplied by the inductive definition of "$\Omega^{SS...S0}$'x" together with the definition of "*n*"), we can assume, by inductive hypothesis, the knowledge of the meanings of the contexts "Ω^{s}'x" and "Ω^{r}'x", and of those obtained from them by replacing "Ω" with an operation sign whose meaning is already known. But we cannot take for granted the knowledge of the meaning of a schematic expression such as "Ω^{r}" not followed by "'" and by "x", nor, thereby, that of the *definiens* of the above definition.[27] To get closer to the solution of the problem, it is helpful to make a step by step analysis of Wittgenstein's proof of the operation theory equation corresponding to the identity "2 × 2 = 4", as presented in proposition 6.241 of the *Tractatus*. In fact, despite certain ambiguities of the Austrian philosopher's notation, his proof contains valuable suggestions to overcome our difficulty. Starting with the expression "$\Omega^{(2\times2)}$'x", Wittgenstein, in a few *apparently* easily justifiable steps, arrives at the expression "Ω^{SS0}'Ω^{SS0}'x". One first applies the definition of "$\Omega^{(r\times s)}$'x", with the numeral "2" in place of both "*r*" and "*s*"; then the inductive definition contained in 6.02, with "Ω^{2}" in place of "Ω"; and finally, the definitions of numerals contained in the last part of 6.02. Thus we have[28]

[1]	$\Omega^{(2\times2)}$'x = $(\Omega^{2})^{2}$'x	[by definition of "$\Omega^{(r\times s)}$'x"]
[2]	= $(\Omega^{2})^{SS0}$'x	[by definition of "2"]
[3]	= Ω^{2}'Ω^{2}'x	[by inductive definition in 6.02 with "Ω^{2}" in place of "Ω"]
[4]	= Ω^{SS0}'Ω^{SS0}'x	[by definition of "2"].

However, in this short series of transformations there is something wrong: in the step from line [2] to line [3] the inductive definition in 6.02 is used with "Ω^{2}" in place of "Ω", and nowhere in the construction of arithmetic outlined in the *Tractatus* up to proposition 6.241 is the meaning established of the symbol "Ω^{n}" alone; only the meaning of every context such as "Ω^{n}'x" is provided by the definitions in 6.02. At the same stage of the proof, the pair of curved brackets enclosing "Ω^{2}" in line [2] has obviously disappeared, as it is no longer necessary to separate the scopes of the two exponents. As a consequence, the expression so obtained in line [3] seems to be directly interpretable by means of the definitions given in 6.02. If these apparently insignificant details are not noticed, the way in which Wittgenstein's proof proceeds may be found very surprising. Overlooking the kind of substitution on which the application of the inductive definition in 6.02, passing from line [2] to line [3] of the proof, is based, it would be expected that from line [4] Wittgenstein goes directly on to the expression "Ω^{SS0}'Ω'Ω'x" by another application of the inductive definition. On the contrary, the proof continues as follows:

[5] Ω^{SS0}'Ω^{SS0}'x = (Ω'Ω)'(Ω'Ω)'x.

The complex symbol "(Ω'Ω)", which first appears in line [5], explicitly expresses the meaning of "Ω^{2}" in lines [1] and [2], and also the meaning of its two occurrences in line [3] (and of the two occurrences of "Ω^{SS0}" in line [4]). But what does "(Ω'Ω)" mean? There is only one reasonable answer to this question, given the role of the symbol "Ω": "(Ω'Ω)" shows *the form of the operation resulting from the composition of a given operation with itself*, the form of the operation commonly known as "the second iteration of the (given) operation". In Wittgenstein's translation of the language of arithmetic into the language of the general theory of logical operations, the expression which corresponds to the term "2 × 2" is, then, "(Ω'Ω)'(Ω'Ω)'x" that appears in line [5]. This symbolic configuration displays the form of any expression obtained by applying the second iteration of an operation to the result of its own application to an initial symbol, or, more briefly and perspicuously, the form of the result of the *double* successive application of the *second* iteration of an operation. Thus, in the notation of Wittgenstein's theory, we have two different expressions with two distinct meanings: "Ω'Ωx" shows the form of the result of the application of an operation to the result of its own application to an initial symbol; "(Ω'Ω)'x" shows the form of the result of the application of the operation resulting from the composition of an operation with itself (the second iteration of the operation) to an initial symbol. As usual, the relation between "(Ω'Ω)'x" and "Ω'Ω'x" is expressed by the following equation:

[A] (Ω'Ω)'ξ = Ω'Ω'ξ.[29]

By the definitions in 6.02, "Ω'Ωx" can be abbreviated to "Ω^{2}'x"; but in 6.241 "(Ω'Ω)'x" too is abbreviated to "Ω^{2}'x", without the slightest explanation. In Wittgenstein's proof of the operation theory equation corresponding to the identity "2 × 2 = 4", the occurrence of "Ω^{2}'x" in line [3], as also the occurrence of "Ω^{SS0}'x" in line [4], is ambiguous. The step from line [4] to line [5], however, solves this ambiguity, giving *the only interpretation* of "Ω^{SS0}'x" which permits the conclusion of the proof. In fact, if the alternative interpretation were adopted and, on the ground of the inductive definition in 6.02, the step from line [4] to the line

[5*] Ω^{SS0}'Ω^{SS0}'x = Ω^{SS0}'Ω'Ω'x

were made, the proof could not be continued further (because the inductive definition allows only the transformation of "Ω^{SS0}'x" into "Ω'Ω'x", and not the transformation of "Ω^{SS0}'ξ" into "Ω'Ω'ξ", which would be needed to go on with the proof). At this point, nothing obstructs the justification of the two steps of the proof that immediately follow line [5] (that Wittgenstein combines into a single step). The inductive definition in 6.02 cannot take a part in the justification for the simple reason that it does not concern the expression "(Ω'Ω)'x" occurring in line [5]; rather, it is necessary to make use of the rule [A] which, however, is not stated anywhere in the *Tractatus*.[30] The reconstruction of the proof can be concluded without further difficulties, as follows:

[6] (Ω'Ω)'(Ω'Ω)'x	= (Ω'Ω)'Ω'Ω'x	[by definition [A]]
[7]	= Ω'Ω'Ω'Ω'x	[by definition [A]]
[8]	= Ω^{SSSS0}'x	[by inductive definition in 6.02] T

[9] $= \Omega^{4}$'x [by definition of "*4*"].

Now let us return to our main task: that of giving the definition of "$\Omega^{(r\times s)}$'x". It will be recalled that we were proceeding by induction on the syntactic complexity of an arithmetical term, and our difficulties lay in the fact that, if all we have done so far has been to fix the meaning of "Ω^{n}'x", then we cannot assume, by inductive hypothesis, the knowledge of the meaning of "Ω^{r}" when it occurs in a context such as "(Ω^{r})"; therefore, we cannot understand the *definiens* of the definition. It is necessary, then, to start from the beginning of the inductive path again, in order to complete its base. We know that the definitions in 6.02 establish the meaning of an expression "Ω^{n}'x", for every n. Following Wittgenstein faithfully, we would have to add the definition of an expression "Ω^{n}", for every n, so as to determine the meaning of the occurrence of "Ω^{n}" in the context "(Ω^{n})". But this course has the considerable inconvenience of rendering the contexts "Ω^{n}'x" and "$\Omega^{SS...S0}$'x" ambiguous, as has been seen scrutinizing lines [3] and [4] of Wittgenstein's proof of the equation "$\Omega^{(2\times 2)}$'x = Ω^{4}'x". Diverging from the original notation, we introduce in the language of operation theory an infinite set of new expressions, in which (italicized) numerals occur as exponents on the left of "Ω":

[B] ${}^{0}\Omega = I$, where I'ξ = ξ,
${}^{1}\Omega = \Omega$
${}^{2}\Omega = (\Omega'\Omega)$,
${}^{3}\Omega = (\Omega'(\Omega'\Omega))$,
${}^{4}\Omega = (\Omega'(\Omega'(\Omega'\Omega)))$,

and so on.[31]

So "${}^{n}\Omega$" is an abbreviated notation for the expression which shows the form of the nth iteration of an operation. Finally, in order to handle these definitions, we need a general rule (belonging to the theory of operations), of which the equation [A] is a particular instance:

[C] $(\Omega'\Psi)'\xi = \Omega'\Psi'\xi$,

where "Ψ", like "Ω", is an operation variable (differing from the line which Wittgenstein would certainly have taken, I will not adopt the exclusive interpretation of operation variables). A general proposition easily follows from what has been established so far:

[D] for every n ≥ 0, ${}^{n}\Omega$'x = Ω' ...'Ω'x, where "Ω'" occurs n times.

For our purposes, this completes the presentation of the base of the inductive definition of the double series of expressions "Ω^{t}'x" and "${}^{t}\Omega$". Now, the definition of product can be rewritten in this manner:

[I] $\Omega^{(r\times s)}$'x = ${}^{r}\Omega^{s}$'x.

The correlative definition is to be added:

[II] ${}^{(r\times s)}\Omega = {}^{s\,r}\Omega$

This time, we know by inductive hypothesis the meaning of the expression "${}^{r}\Omega$" and therefore we can understand the meaning of the *definiens* of the new definition of "$\Omega^{(r\times s)}{}'x$". Similarly, as by inductive hypothesis we know both the meaning of "${}^{r}\Omega$" and that of "${}^{s}\Omega$", we have no difficulty in understanding the meaning of the *definiens* of the definition of the expression "${}^{(r\times s)}\Omega$". Returning for a moment to Wittgenstein's proof of the operation theory equation corresponding to the identity "$2 \times 2 = 4$", it will be noticed that the "arithmetical core" of the proof is constituted by the series of steps from line [5] to line [7]. Here the possibility of transforming a string composed of two groups of two symbols "Ω" each into a four "Ω" string without internal groupings is *shown* (the first string perspicuously *showing* what the arithmetical process of multiplying two by two consists in).

In order to complete our systematic reconstruction, the function of sum, which is overlooked by Wittgenstein, has to be introduced. In the light of the explanations made so far, this task is not too difficult. The expressions "$\Omega^{(r+s)}{}'x$" and "${}^{(r+s)}\Omega$" must be defined, assuming, by inductive hypothesis, that the meanings of "$\Omega^{r}{}'x$", "$\Omega^{s}{}'x$", "${}^{r}\Omega$" and "${}^{s}\Omega$" are already known (as well as that of the expression "$(\Omega'\Psi)$", which denotes the composition of two arbitrary operations). Then we can state the following pair of definitions:

[III] $\Omega^{(r+s)}{}'x = ({}^{r}\Omega'{}^{s}\Omega)'x$
[IV] ${}^{(r+s)}\Omega = ({}^{r}\Omega'{}^{s}\Omega)$.

It might be interesting to test our tentative reconstruction of Wittgenstein's system, proving the equation corresponding to an arithmetical identity somewhat more complex than "$2 \times 2 = 4$". Let us consider, for example, the identity "$(2+3) \times (2+1) = 15$". According to Wittgenstein's reinterpretation of arithmetic, the equation "$\Omega^{((2+3)\times(2+1))}{}'x = \Omega^{15}{}'x$" must be proven. The proof goes as follows:

[1] $\Omega^{((2+3)\times(2+1))}{}'x$
$= ({}^{2+3)}\Omega^{(2+1)}{}'x$ [definition [I]]
[2] $= ({}^{2(2+3)}\Omega'{}^{1(2+3)}\Omega)'x$ [from definition [III] by substitution of "Ω" with "${}^{(2+3)}\Omega$"]
[3] $= (({}^{(2+3)}\Omega'{}^{(2+3)}\Omega)'{}^{(2+3)}\Omega)'x$ [from [B] by substitution of "Ω" with "${}^{(2+3)}\Omega$"]
[4] $= ((({}^{2}\Omega'{}^{3}\Omega)'({}^{2}\Omega'{}^{3}\Omega))'({}^{2}\Omega'{}^{3}\Omega))'x$ [definition [IV]]
[5] $= ((((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega))))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega))))'x$ [[B]]
[6] $= (((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega'))))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'x$ [[C]]
[7] $= (((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega))))'(\Omega'\Omega)'(\Omega'(\Omega'\Omega))'x$ [[C]]
[8] $= (((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega))))'(\Omega'\Omega)'\Omega'\Omega'\Omega'x$ [[C and A]]
[9] $= (((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega))))'\Omega'\Omega'\Omega'\Omega'\Omega'x$ [[A]]

[10] = ((Ω’Ω)’(Ω’(Ω’Ω)))’((Ω’Ω)’(Ω’(Ω’Ω)))’Ω’Ω’Ω’Ω’Ω’x [[C]]
[11] = ((Ω’Ω)’(Ω’(Ω’Ω)))’(Ω’Ω)’(Ω’(Ω’Ω))’Ω’Ω’Ω’Ω’Ω’x [[C]]
[12] = ((Ω’Ω)’(Ω’(Ω’Ω)))’(Ω’Ω)’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’x [[C and A]]
[13] = ((Ω’Ω)’(Ω’(Ω’Ω)))’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’x [[A]]
[14] = (Ω’Ω)’(Ω’(Ω’Ω))’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’x [[C]]
[15] = (Ω’Ω)’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’x [[C and A]]
[16] = Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’Ω’x [[A]]
[17] = Ω^{15}’ x [inductive definition in 6.02 and definition of “*15*”].

Once again, the “arithmetical core” of the proof is in the possibility *shown* by the comparison of line [5] with line [16], namely, the divisibility of a fifteen “Ω’” string into two major subgroups: the first comprising two equal sub-subgroups, each made up of a sub-sub-subgroup of two symbols “Ω’” and a sub-sub-subgroup made up of a sub-sub-sub-subgroup of one symbol “Ω’” and of a sub-sub-sub-subgroup made up of two symbols “Ω’”; the second comprising two sub-subgroups, again one of two symbols “Ω’” and the other made up of a sub-sub-subgroup of one symbol “Ω’” and of a sub-sub-subgroup made up of two symbols “Ω’”. In the same way, if the proof of the equation corresponding to the identity “((1+1)+(1+1)) = (1+(1+(1+1)))” were carried out, one could easily verify how, by a process of successive substitutions, the recognition of a fundamental arithmetical “fact” is made possible: that “((Ω’Ω)’(Ω’Ω))’x” and “(Ω’(Ω’(Ω’Ω)))’x” are two different possible groupings of the elements of one and the same string: “Ω’Ω’Ω’Ω’x”.[32] The proof carried out above makes use of the method of substitution of identicals that Wittgenstein indicates, in 6.24, as “the method by which mathematics arrives at its equations”. Nevertheless, the proof goes on directly from the initial definitions of the theory, without exploiting numerical identities already proven. Actually, the substitutional method could be applied, restricting it, however, only to arithmetical terms occurring as exponents on the right-hand side of the variable “Ω”. For instance, if the equation “$\Omega^{(2+1)}$’x = Ω^{3}’x” has already been proven, it may be used in the proof of the equation “$\Omega^{((2+3)\times(2+1))}$’x = Ω^{15}’x”, simplifying it considerably.[33]

There is a further problem, not of a strictly exegetical nature, which is left open by the systematic account of Wittgenstein’s theory of arithmetic proposed here. It concerns the meaning of the inverted comma “’”. At the beginning of the section it was seen that “’” is part of the notation whereby Wittgenstein represents the form of the result of the application of an operation to a given base (a notation also used to make reference to an operation in general). In outlining the formal concept of operation, we have assumed as a suitable base for the application of an operation a proposition or, more generally, a meaningful expression (either simple, or generated by a successive application of the same or of another operation). [34] This assumption is suggested mainly by Wittgenstein’s use of the variable “ξ” in the base position of an operation sign. In proposition 6.241 of the *Tractatus,* however, we encountered the expression "(Ω'Ω)", in which the operation variable itself appears in the position where the operation base sign should be. One can say that Wittgenstein's introduction of this new expression supplies the textual evidence on which the treatment of complex arithmetical terms in the frame of operation theory, proposed here as a faithful realization of the Austrian philosopher's intentions, is grounded. Since, in my interpretation, the context "(Ω'Ω)" represents the form of the second iteration of an operation, a first conjecture which can be put forward is that Wittgenstein conceived the composition of two operations (and, in particular, the process of composition of an operation with itself) as nothing but a particular case of application of an operation to a given base. Alternatively, one must conclude that, in the *Tractatus,* the sign "'" is ambiguous, since it is used to express both the application of an operation and the composition of two operations. But in this case, two different primitives of the theory of operations would correspond to the primitive arithmetical notion of successor.[35]

At this point, we can consider our exposition of the *Tractatus* system of arithmetic to be concluded. The results we have achieved so far are sufficient to undertake the second block of propositions of the *Tractatus* concerning mathematics (propositions 6.2–6.241). As we have seen, the group of propositions 6.02–6.031 (together with the fundamental 6.241) gives the co-ordinates of Wittgenstein's interpretation of arithmetic. Propositions 6.2–6.24, on the other hand, state the central philosophical theses on the status of arithmetical identities, on the nature of the process of numerical calculation and on the role played by vision in this process. It must be stressed that Wittgenstein's reflections on these topics entail no reference to general arithmetical theorems. They concern exclusively elementary numerical arithmetic, namely the sort of identities typically exemplified by "2 × 2 = 4" and by "(2+3) × (2+1) = 15", the treatment of which ought to have been thoroughly clarified by the above exposition.

The next section of this chapter will be devoted entirely to an examination of the block of propositions 6.2–6.24. However, in my opinion, the conclusions we have reached so far provide ample grounds for a conjecture concerning the crucial proposition 6.22 (which in the first section I proposed as a striking example of how some of the main theses of the *Tractatus* philosophy of mathematics are still obscure): "The logic of the world, which is shown in tautologies by the propositions of logic, is shown in equations [*in den Gleichungen*] by mathematics". Let us consider, for example, the numerical identity "2 × 2 = 4" and ask ourselves which feature of the logic of the world it shows. Of course, in order to give an answer to the question, the first thing to do is to focus our attention on the translation of the arithmetical identity into the language of the general theory of logical operations. Our question has to be reformulated as follows: which feature of the logic of the world is shown by the correctness of the equation "$\Omega^{(2\times2)}$'x

= Ω^{4}’x”? We know that “$\Omega^{(2\times2)}$’x” is an abbreviated notation for the expression “(Ω’Ω)’(Ω’Ω)’x” and that the latter shows the form of the result of the double application of the second iteration of an operation. Moreover, “Ω^{4}’x” is an abbreviated notation for the expression “Ω’Ω’Ω’Ω’x” which shows the form of the result of the successive application of an operation, consisting in three applications to the result of its own application, starting from the result of its application to an initial symbol. The following general conclusion can be drawn from the proof of the correctness of the aforementioned equation. Given any logical operation and any initial symbol (relative to the operation), the expression generated by the double application of the second iteration of the operation to the given initial symbol will have the same meaning as the expression generated by three applications of the operation to the result of its own application, starting with the result of its application to that initial symbol.[36] Consider the example of the operation of negation and the proposition “it’s raining”: denoting the second iteration of negation by “(~’~)”, and taking as proven the correctness of the equation corresponding to the arithmetical identity “2 × 2 = 4”, we can state that “(~’~)’(~’~)’it’s raining” is tautologically equivalent to the proposition “~’~’~’~’it’s raining” (here I have kept the sign of application of an operation, in its double meaning – a sign which is not attached by Wittgenstein to the tilde nor to any other operation constant – to conform to the notation of the theory of operations in which arithmetic is interpreted). Similarly, we can take the operation that, for any given dyadic predicate “R”, generates the expression “the R of **a**” when applied to an appropriate singular term “**a**”. Then, the result of the application of the second iteration of this operation to “**a**” is represented by the definite description “(the R of the R) of **a**”. Thus, starting from the proper noun “Paul” and proceeding as previously explained with reference to the predicate “father of”, we can state, given the correctness of the equation corresponding to the arithmetical identity “2 × 2 = 4”, that the two definite descriptions “the paternal grandfather of the paternal grandfather of Paul” and “the father of the father of the father of the father of Paul” have the same meaning. We can get a closer understanding of 6.22 by comparing an example of a tautology like “it’s raining ≡ ~’~’ it’s raining” with a biconditional such as “(~’~)’(~’~)’it’s raining ≡ ~’~’~’~’it’s raining”. The truth of “it’s raining ≡ ~’~” it’s raining” depends exclusively on the circumstance that this *sinnlos* proposition is an instance of the form exhibited by “*p* ≡ ~’~*p*”, a formula which is true for every truth-possibility of “*p*”. Likewise, the truth of the biconditional “(~’~)’(~’~)’it’s raining ≡ ~’~’~’~’it’s raining” depends exclusively on the circumstance that “(~’~)’(~’~)’it’s raining” and “~’~’~’~’it’s raining” have the forms shown by “$\Omega^{(2\times2)}$’x” and by “Ω^{4}’x”, respectively, and that the operation theory equation “$\Omega^{(2\times2)}$’x = Ω^{4}’x” is correct.[37] Thus, not even the truth of the latter biconditional is a contingent matter, because it is not determined by the actual configuration of the world.

If we adopt a non-Wittgensteinian jargon and say that the tautological equivalency of "it's raining" and "~'~'it's raining" is formally grounded because it depends exclusively on the meaning of the logical particle "~", then we can also say that the tautological equivalence of "(~'~)'(~'~)'it's raining" and "~'~'~'~'it's raining" rests *on even more formal grounds,* because it depends only on the properties of the abstract notion of logical operation (and in no way on the meaning of "~"). The parallel between the tautologousness of a logical formula and the correctness of the equation corresponding to a true numerical identity can be brought to a conclusion. The tautologousness of "$p \equiv$ ~'~'p" shows the following formal feature of the world (namely, a feature that is shared with every possible world): the state of affairs, pictured by a meaningful proposition, is the same state of affairs pictured by its double negation. The correctness of "$\Omega^{(2\times2)}$'x = Ω^{4}'x" shows the following formal trait of the world: let "**A**" be an expression generated by the double application of the second iteration of a given operation to a given initial symbol, and let "**B**" be the expression resulting from three applications of this operation to the result of its own application, starting from the result of its application to the same initial symbol; then, the state of affairs pictured by any proposition which contains some occurrences of "**A**" is the same state of affairs pictured by the proposition obtained by replacement of one or more occurrences of "**A**" with the expression "**B**" in the first proposition. Generally speaking, if: (i) "**t**" and "**s**" are arithmetical terms; (ii) "**A**" is an expression generated by means of an operation from a given initial symbol, and the form of "**A**" is shown by "Ω^{t}'x"; (iii) "**B**" is an expression generated by means of the same operation from the same initial symbol, and the form of "**B**" is shown by "Ω^{s}'x"; (iv) the equation "Ω^{t}'x =Ω^{s}'x" is a theorem of operation theory; then the state of affairs pictured by a proposition which contains some occurrences of "**A**" is the same state of affairs pictured by any proposition obtained from the former by replacing one or more occurrences of "**A**" with "**B**" (whatever the operation, and the corresponding initial symbol from which "**A**" and "**B**" are generated, may be).[38] In conclusion, if to show an aspect of the logic of the world amounts to showing a formal feature of it, then the tautologousness of a logical formula and the correctness of the equation corresponding to an arithmetical identity are, from this point of view, perfectly at par: both properties accomplish this task. Mathematics and the world come in contact only in virtue of the peculiar relation between mathematical language and the logical forms of the picturing-facts language. As we shall see, despite the radical changes that the notion of logical form will undergo in the further development of Wittgenstein's philosophy, this fundamental thesis will remain substantially unaltered.

The above reformulation of the content of 6.22 as a general metalogical theorem, though unfaithful to Wittgenstein's own style, seems to me both sufficiently corroborated by the textual evidence and consistent with the

Austrian philosopher's conception of forms in the *Tractatus*. Of course, his brief outline, without further elaboration, is not enough to account for numerical specifications in ordinary language asserting the number of the elements of the extension of a material concept, and for the formal relations between these statements. In this case, as in those previously seen, a non-mathematical conclusion can be inferred from a set of non-mathematical premises by means of a mathematical proposition, in conformance with Wittgenstein's remark on the use of mathematical propositions "in life" (6.211). Moreover, 6.2331 contains the statement "Calculation is not an experiment", which many years later will become a sort of slogan condensing his view on the normative role of the outcomes of calculations. In the next section it will be worthwhile to say something concerning the theme touched on in 6.2331.

THE "KNOWLEDGE" OF FORMS: VISION AND CALCULATION

In his masterful essay on the relation between the *Tractatus* picture theory and the theory of judgement provisionally outlined by Russell in his *Theory of Knowledge,* David Pears construes the opposition between the two conceptions as a conflict between an Aristotelian and a Platonic view of forms.[39] According to Pears, the mirror-like relationship between language and the world, upon which the picture theory of the *Tractatus* rests, comes into being through the absorption of the form of states of affairs by propositions. This absorption produces a perfect match between possibilities of combination of a name with other names in a proposition and possibilities of combination of its *Bedeutung* with the *Bedeutungen* of the other names in a state of affairs. The Aristotelian spirit of Wittgenstein's conception is revealed by the thesis that forms of objects (to which forms of states of affairs are reducible) are features essentially inherent to objects and not self-subsistent ideal entities. Wittgenstein opposes this point of view to Russell's 1913 conception of forms which sees forms as completely general facts, placed in an ideal world and accessible only by means of a vaguely specified "logical experience".

I think that my exposition of Wittgenstein's interpretation of arithmetic corroborates Pears's thesis on the Aristotelian character of the *Tractatus* theory of forms. We have seen that, according to this exposition, the path leading to arithmetical notions has its "concrete" base in the series of linguistic expressions generated by means of the successive application of a logical operation. The introduction of numerals into the language of operation theory is a notational device to express the formal property constituted by the number of applications of an operation to the result of its own application. More complex terms are introduced in that language to represent *the arithmetical structure of the forms* of the expressions generated by processes (of growing complexity) of iteration, composition of iterations, iteration of iterations, of a given operation. The idea that forms of linguistic expressions have an arithmetical structure is really the cornerstone of Wittgenstein's treatment of arithmetic in the *Tractatus*.

Besides the essential connection between arithmetical terms and the forms of certain linguistic expressions, there is a second, fundamental assumption on which the whole group of propositions 6.2–6.241 of the *Tractatus* is based. In order to clarify this assumption, we can once again take a remark made by Pears as our starting point. We have said that Pears maintains that Wittgenstein's solution to the problem of the nature of forms is a solution that is "Aristotelian in spirit": forms, which Russell had placed in a Platonic world, are conceived by the Austrian philosopher as "essential features of objects".[40] But he immediately observes that Wittgenstein's version of Aristotelianism is actually rather strange. This "strangeness" derives directly from Wittgenstein's thesis on the impossibility of using language to speak meaningfully of forms. For this reason, we have the "odd" situation of a metaphysics that is more empirical than that of Russell's, but at the same time condemned to ineffability.[41] Certainly, Pears is right in saying that Wittgenstein's solution is a "strange" version of Aristotelianism. Nonetheless, it needs to be stressed with equal emphasis that it is the only version which is consistent with the semantical conceptions of the *Tractatus*. In fact, given Wittgenstein's theory of the sense of propositions, the concession that one could speak meaningfully of forms would have amounted to reintroducing those ideal objects and ideal states of affairs invoked by Russell and rejected "in Aristotelian spirit" by the Austrian philosopher. In other words, if the picture theory of the *Tractatus* had not been accompanied by the prohibition of speaking of forms, it would have entailed the abandonment of Aristotelianism in favour of Platonism. Then, however strange Wittgenstein's version of Aristotelianism may be, it is undoubtedly the only version which is admissible in a semantical frame dominated by the picture theory. Let us explain this point by briefly attempting to construe the equation "$\Omega^{(2\times2)}$'x" = Ω^{4}'x" as a meaningful proposition. This equation would picture the contingent ideal state of affairs constituted by the mutual reducibility of the two operational schemes of sign construction exhibited by "(Ω'Ω)'(Ω'Ω)'x" and by "Ω'Ω'Ω'Ω'x" respectively. To speak of a contingent state of affairs pictured by "$\Omega^{(2\times2)}$'x" = Ω^{4}'x" is like admitting the conceivability of a situation in which this equation is incorrect; but this would imply that the whole logical space in which real facts are placed would no longer be the same. The possibility that two different facts are pictured, respectively, by a proposition in which an expression generated by a double application of the second iteration of a given operation (when applied to an initial symbol) occurs, and by the proposition obtained from the former by replacing that same expression with the expression generated by three applications of the same operation to its own result – starting from the result of its application to the same initial symbol – would be conceded. But, in virtue of the general properties of the concept of operation, this is an impossible world, exactly in the same way in which, according to the view of propositions held in the *Tractatus,* no possible world can contain two different facts corresponding to a proposition and to its double negative. In conclusion, the attempt to attribute a contingent status to numerical identities is bound to fail because we cannot conceive formal properties of the world alternative to those which constitute our logical space and, at the same time, remain comfortably immersed in the latter. To attribute to our world formal properties different from those which we actually acknowledge would mean simply to abandon our logical space for a new one; and, according to Wittgenstein, this cannot be done.

The fundamental assumptions of picture theory entail that one cannot speak meaningfully, in language, of the forms of language; this restriction of the sayable is tantamount to ruling out the possibility that forms be conceived as a peculiar sort of objects (as simple constituents of contingent ideal states of affairs), and thus to ruling out the possibility that they provide "the substance" of a second world. This circumstance explains the radical, extreme nature of the consequences deriving from the anti-Platonic conception of forms maintained by Wittgenstein in the *Tractatus.* Given the three following premises:

1 logic and mathematics deal (in a manner we are shortly going to examine in detail) with formal properties and relations of linguistic expressions;
2 the role of Wittgensteinian objects of an ideal world cannot be assigned to forms;
3 a thought, a meaningful proposition, is the logical picture of a contingent configuration of objects and is true if this configuration exists, false otherwise;

only one conclusion can be drawn, if the coherence of the whole has to be saved: the results achieved in logic and mathematics cannot be formulated in meaningful propositions expressing a thought (and, *a fortiori,* the predicates "true" and "false" cannot be appropriately applied). This is exactly the drastic and somewhat disconcerting inference that Wittgenstein makes, as regards mathematics, in 6.2 and 6.21: "The propositions of mathematics are equations (*Gleichungen*), and therefore pseudo-propositions (*Scheinsätze*). A proposition of mathematics does not express a thought."

Now the reasons become clear why it is quite correct to speak of the existence of a logicist point of view in the philosophy of mathematics of the *Tractatus*. Both in logic and mathematics, suitable notations are constructed in order to render perspicuous those formal properties of linguistic expressions which, for the reasons mentioned above, cannot be described meaningfully. In logic, the notation of propositional variables, of symbols of propositional functions, of sentential connectives, of quantifiers, etc., is needed, according to Wittgenstein, to construct formulae which clearly exhibit forms of propositions. These formulae can be used to check whether a given proposition has a certain metalogical property, or whether certain metalogical relations hold between two or more given propositions. The method of checking for a metalogical property or relation is either a mechanical procedure of decision, as occurs with the truth-table method and with any other equivalent method; or a semi-mechanical procedure of generation, as occurs with the derivation of tautological formulae in an axiomatized logical calculus. In exactly the same way, arithmetical notation (numerals and complex arithmetical terms) is introduced as part of a symbolism devoted to exhibiting perspicuously the forms of the results of the successive application of all sorts of iterations and compositions of logical operations. Arithmetical calculation has a perfectly analogous role to that of logical calculation: to ascertain that the relation of identity of meaning – which, according to Wittgenstein, cannot be meaningfully spoken of – holds between any two given expressions having certain specified forms. Thus, the first part of 6.2 states: "Mathematics is a logical method". However, for a thorough understanding of the content of the group of propositions 6.23–6.241, a further development of the comparison between logic and mathematics is required. To this purpose it is expedient to verify whether, and to what extent, Wittgenstein's view on logical *sinnlos* propositions applies also to the equations into which numerical identities are translated. As it concerns logic, the pivotal thesis is what Wittgenstein himself calls the "fact" which "contains in itself the whole philosophy of logic" (*T* 6.113). The "fact" in question concerns the process of recognition of the truth of a tautology (and of the falsity of a contradiction) and, more generally, the process of recognition of the formal properties of a proposition and of the formal relations between propositions. Suppose that a certain proposition is given. After its form has been perspicuously exhibited by means of logical notation, we are able to decide, using the method of truth-tables or some other equivalent procedure, whether the proposition under consideration is true for all the truth-possibilities of its component propositions, or whether it is false for all the truth-possibilities or whether it is true for some and false for some others. At this point, if only values **T** or only values **F** have been obtained, and if this procedure has been carried out in order to settle the truth-value of the given proposition, then our work can be considered concluded. In these two limiting cases (tautology and contradiction) the truth-value can be settled by applying suitable procedures of sign manipulation without "going out of language". In contrast, if the tested proposition is true for some of the truth-possibilities of its component propositions (true in some possible worlds) and false for others of them (false in some other possible worlds), then the only method to decide its truth-value is "to go out and see" which of the possible worlds has actually come true, or, in other words, what is the effective configuration of the world. Although in 6.113 Wittgenstein speaks only of the possibility of recognizing that logical propositions are true "from the symbol alone" (*am Symbol allein*), it is obvious that he considers the peculiar mark of all formal properties and relations precisely the possibility of being recognized in this way.[42] This is a well known aspect of Wittgenstein's philosophy of logic and corresponds to the traditional opposition between *a priori* and *a posteriori* knowledge. But, besides the theme of the verifiability of the formal properties of a proposition (or of a set of propositions) "without going out of language", there is an interesting difference, which Wittgenstein points to, between two kinds of "knowledge" of the formal domain.[43] It is clearly stated in 6.1221 (b): "For example, we see (*ersehen*) from the two propositions themselves that '*q*' follows from '*p* ⊃ *q* . *p*', but it is also possible to show it in *this* way: we combine them to form '*p* ⊃ *q* . *p* : ⊃ : *q*', and then show that this is a tautology". Many aspects of the philosophy of mathematics of the *Tractatus* can be clarified only when this distinction is taken into account. In so doing, we shall see that not even Wittgenstein's conception of arithmetical identities is completely covered by what we have said so far. A crucial difference between tautological formulae of logic and correct equations of operation theory appears and a new problem arises. It is the problem of contending with what, in an entry of the *Notebooks,* Wittgenstein described as "the old old objection against identity in mathematics. Namely the objection that if 2 × 2 were really the *same* as 4, then this proposition ["2 × 2 = 4"] would say no more than a = a" (*NB,* entry 6/9/14). Thus, we are led to Frege's solution of the problem of informativeness of identities, based on the distinction between *Sinn* and *Bedeutung* of a sign, and to Wittgenstein's attempt to put a radical alternative to it, exploiting the notion of plurality of ways of seeing a symbol.[44]

To illustrate the crucial difference between tautologies and equations briefly mentioned above, consider, for example, the logical formula "*p* ⊃ *q* . ≡ . ~*q* ⊃ ~*p*". Its being a tautology shows that any two propositions of the form *p* ⊃ *q* and ~*q* ⊃ ~*p* are tautologically equivalent, or, by virtue of Wittgenstein's extensional criterion of synonymy between propositions, that they have the same sense. It is obvious that the symbol "≡" is the sentential connective known as "biconditional"; therefore, its occurrence in a proposition which is an instance of the formula "*p* ⊃ *q* . ≡ . ~*q* ⊃ ~*p*" does not express the metalogical relation of tautological equivalency, namely, the semantic relation of propositional synonymy. Now, even the correctness of the equation corresponding to an arithmetical identity such as "2 × 2 = 4" shows the identity of meaning of any two expressions – generated by one and the same operation starting from the same initial symbol – which have, respectively, the form $\Omega^{(2\times2)}$'x and Ω^{4}'x. But, the identity sign "=" which occurs in the equation "$\Omega^{(2\times2)}$'x = Ω^{4}'x" – different from the propositional connective "≡" occurring in the tautological formula "*p* ⊃ *q* . ≡ . ~*q* ⊃ ~*p*" – does not belong, according to Wittgenstein, to language, at least when language has been regimented by the precise rules of logical syntax. To get a better clarification of the question, it may be useful to resort here to the non-Wittgensteinian distinction between object-language and metalanguage, and, at the same time, to Wittgenstein's distinction between *sinnlos* propositions, on one hand, and pseudo-propositions (*Scheinsätze*), on the other. Whereas an instance of a tautological formula is a limiting case of a proposition constructed in object-language, and is thus a *sinnlos* proposition, an instance of an operation theory equation is an attempt to express a metalinguistic relation – in particular, a semantic relation – and is thus a *Scheinsatz*. Such an equation should closely approximate a general metalogical pseudo-proposition of the type "a proposition of the form **A** logically implies a proposition of the form **B**" or "a proposition of the form **A** is logically equivalent to a proposition of the form **B**", but not a tautological formula. In fact, a formula such as "$p \supset q \ . \equiv . \sim q \supset \sim p$" does not attempt to assert the metalogical theorem that is shown by its tautological nature. On the other hand, an equation such as "$\Omega^{(2\times 2)}$'x = Ω^{4}'x", because of the presence of the symbol "=", seems to affirm that which is shown by its correctness.

To further complicate matters, the Austrian philosopher actually speaks of the "*Identität der Bedeutung*" (6.2322) and of the "*Bedeutungsgleichheit*" (6.2323) of the two expressions occurring on the right and on the left of the sign "=" in a correct equation (for any suitable pair of arithmetical terms "**t**" and "**s**", of the two schemes "Ω^{t}'x" and "Ω^{s}'x"), and not merely of the identity of meaning of any two related instances of these schemes. It is beyond doubt that in this context he does not use the word "*Bedeutung*" in the way Frege does speaking of the "*Bedeutung*" of a singular arithmetical term. More precisely, in only one case does Wittgenstein's *Bedeutung* coincide with the object denoted by a singular term and thus with the Fregeian *Bedeutung*: the case of the names of simple objects (which according to Wittgenstein are, on the other hand, devoid of the Fregeian *Sinn*). In all other cases in which Wittgenstein speaks of the "*Bedeutung*" of an expression, as happens, for example, with arithmetical terms (and consequently with the expressions of the language of the theory of operations in which they occur as exponents of "Ω"), he has no intention of introducing some other kind of entity of which the term or the expression would be proxy in language. Only those signs whose *Bedeutungen* are objects can be combined with other signs of the same kind to form logical pictures of possible states of affairs (meaningful propositions). But there are also meaningful signs which cannot be employed to construct propositions, since their *Bedeutung* is not an object for which they stand. An arithmetical term or, better, an expression in which an arithmetical term occurs as exponent of the operation variable "Ω" is a sign of the latter sort. Such a sign is meaningful because it is used to exhibit a formal property which is common to the linguistic expressions of a certain class and it is evident that Wittgenstein is referring to this property as the "*Bedeutung*" of the sign.

This is the background which one needs to take into account if propositions 6.231 and 6.232 (first part) are to be understood. They run as follows: “It is a property of affirmation that it can be construed (*daß man sie ... auffassen kann*) as double negation. It is a property of ‘1+1+1+1’ that it can be construed as ‘(1+1)+(1+1)’. Frege says that the two expressions have the same meaning (*Bedeutung*) but different senses (*Sinn*).” In fact, Frege would have said that the two arithmetical terms occurring on the left and on the right of “=” in the identity “(1+(1+(1+1))) = ((1+1)+(1+1))” have different senses in so far as the one identifies a number in one way and the other in another way; and would have maintained that they have the same *Bedeutung,* the number 4.[45] According to Wittgenstein, there is no ideal object (one and the same) which is identified in different ways by the two arithmetical terms in question. When he speaks of the “*Identität der Bedeutung*” of two arithmetical terms “**t**” and “**s**”, he means the reciprocal transformability of the forms shown by the correlated terms “Ω^{t}’x” and ”Ω^{s}’x”. And, since a form is not an object named or described by a term of the latter sort, but is that which is shown by it, the possibility of mutual reduction of two forms coincides with the possibility of mutual transformation of the two involved expressions “Ω^{t}’x” and ”Ω^{s}’x”.[46] For this reason, when Wittgenstein speaks of construing an expression as another one, he means precisely the recognition of this possibility of *symbolic* process. To construe an arithmetical term “**t**” as the term “**s**” means to see (*ersehen* (6.232)) the possibility of transforming into one and the same form the two forms which “Ω^{t}’x” and ”Ω^{s}’x” are intended to exhibit. And, to see this amounts, for the reason explained above, to recognizing that the two expressions can be transformed into one and the same expression (e.g. to recognizing that “((Ω’Ω)’(Ω’Ω))’x” and “(Ω’(Ω’(Ω’Ω)))’x” are two possible groupings of the elements of the same string “Ω’Ω’Ω’Ω’x”). The “informative content”[47] of a proven equation, in which two different terms “Ω^{t}’x” and ”Ω^{s}’x” occur on the sides of “=”, could only be communicated by a general metalinguistic statement on the substitutability *salva significatione* of any two expressions constructed in compliance with the operational models shown by “Ω^{t}’x” and ”Ω^{s}’x” (and so, on the intersubstitutability of the latter inside the calculation). But, owing to the assumptions made in the *Tractatus,* this “content” cannot be stated meaningfully. Any propositional interpretation of an equation having been precluded, only Wittgenstein’s vague characterization in 6.2323 remains: “An equation merely marks the point of view from which I consider the two expressions: it marks their equivalence in meaning”. A

t this point, let us summarize the results achieved so far:

1 the equation into which a true identity “**t = s**” is translated draws attention to the synonymy of any two linguistic expressions having the form exhibited by the schemes “Ω^{t}’x” and “Ω^{s}’x”, and, in the sense explained above, to

the *Bedeutungsgleicheit* of the two schemes themselves; its content, though, as all the other aspects that language shows, cannot be meaningfully asserted;

2 2 the correctness of the equation amounts to the mutual reducibility of the two schemes "Ω^{t}'x" and "Ω^{s}'x", namely, to the formal possibility of obtaining both by grouping the elements of one and the same string of "Ω'" in two different ways.

But, in the light of this interpretation, the *Tractatus* philosophy of mathematics appears in a somewhat odd situation. On one hand, arithmetic consists of equations; on the other, these equations are, so to speak, vanishing entities, because they are not genuine propositions nor limiting cases of genuine propositions, such as tautologies, and, if language were rigorously regimented, they would disappear, as would all the other pseudo-propositions. What really matters in arithmetic is not the equation itself, but its unsayable content, namely, the *Bedeutungsgleichheit* of the two terms involved. The equality sign plays no assertive role, and once the identity of meaning has been recognized, the mathematical result has been achieved. As Wittgenstein puts it: "But the essential point (*das Wesentliche*) about an equation is that it is not necessary in order to show that the two expressions connected by the sign of equality have the same meaning, since this can be seen (*sich dies ... ersehen läßt*) from the two expressions themselves" (*T* 6.232). At this point, the following questions arise spontaneously: if this is how things stand, why does the equality sign have to be introduced in the arithmetical notation? and why do mathematical pseudo-propositions (equations) have to be formulated? In other words, if our "cognitive" relationship to forms is one of immediate and intuitive grasping – of perceiving what is shown by symbolism – why should we fix in *Scheinsätze* the results gained in this non-discursive activity? Moreover, if what is shown by symbolism coincides with what can be immediately seen – if the *Identität der Bedeutung,* and the mutual substitutability, of two arithmetical terms "must be manifest in the two expressions themselves" (*T* 6.23) – why are procedures of numerical calculation needed? To give a plausible answer to these questions, let us start with the analogous questions which can be raised with reference to logic. On what grounds are a logical notation and a mechanical or semi-mechanical procedure for checking metalogical properties and relations introduced? For example, that the metalogical relation of entailment holds between two propositions can be seen, according to Wittgenstein, as soon as one is able to grasp their forms, and hence recognize that the inclusion relation holds between the respective sets of their truth-grounds (i.e. between the respective sets of verifying possible worlds). But this does not mean that an empirical speaker would be able, in all cases, to recognize the existence of this relation. Only an omniscient God – or the metaphysical subject, *die Grenze der Welt* (*T* 5.632) – endowed with an extensionally 30 complete mastery of the formal domain would never need a logical notation, a tabular method, or a formalized logical calculus in order to ascertain whether a certain metalogical property or relation holds true. This God is omniscient in the sense that, for him, what is shown by symbolism actually coincides with what he sees. Properly speaking, seeing a formal connection could not be characterized as an experience since, according to Wittgenstein, only contingent configurations of objects can be really experienced. In my opinion, when he speaks of the vision of such a connection, he uses the verb "to see" (or "to perceive" etc.) metaphorically, in order to stress the logical inconceivability of the existence of a different connection. Apart from this, there is a notable difference between God and the empirical users of language, concerning the "knowledge" of the formal domain: contrarily to God, we are not endowed with an *extensionally complete* mastery of the formal relations holding between linguistic expressions. The immediate visibility of these relations is for us an ideal, and the introduction of a highly artificial notation is nothing but a tool for approximating this ideal. This is clearly true of logical notation, since it serves to exhibit perspicuously the forms of expressions, with the aim of a correct application of those techniques of checking metalogical properties of propositions which formal logic sets up. The need for logical notation and logical methods of checking metalogical properties arises only in view of our extensionally limited ability in perceiving formal relations, i.e. of our empirical limitations in the "knowledge" of the formal domain. Only divine "knowledge" of this domain is extensionally complete, whereas human "knowledge" may not be so. Here the adverb "extensionally" is decisive: God can know something we do not know only in the sense that He knows something that now we may not know but, in principle, we are able to know. Wittgenstein's implicit assumption of the effective decidability of the fundamental metalogical properties and relations determines precisely these very narrow limits in which we can talk about a gap between God and us in the "knowledge" of forms. If there is something that God knows and that we do not, we can fill this gap in our "knowledge" by applying the appropriate effective method of calculation. Then, if it is true that, contrarily to God, we do not see everything which shows itself, it is also true that, in virtue of the assumption of decidability, this difference has a purely empirical nature: in every specific case, it can be annulled, in principle, by the application of a suitable algorithm. According to the *Tractatus,* formal connections which, *de facto,* are not perceived by us can exist; but only in the sense that, in any case, their existence can be ascertained through the mediation of sign transformations carried out in accordance with a known effective method.

Now, to return to mathematics, or rather, to the equations correlated to arithmetical identities. Arithmetical procedures of calculation, too, exist only because of the gap between what is shown by symbolism and what we see *de facto*. For a God who is able to completely master the relations of mutual reducibility between forms, for a God who actually perceives everything which the symbolism shows, numerical arithmetic would be superfluous. Such a God would have no reason to formulate those pseudo-propositions which equations are, and less so, to use them in a step by step procedure of calculation. Indeed, only the empirical limitations of our skill in grasping the relations holding between forms make it indispensable to resort to equations and to arithmetical calculus. The very existence of mathematics is due only to these limitations. But the effective decidability of arithmetical identities (of the corresponding equations onto which they are mapped) – which, not by chance, are the only mathematical "propositions" dealt with by Wittgenstein in the *Tractatus* – implies that, in the field of mathematics too, the difference between God's "knowledge" and ours is merely empirical, extensional. In any specific case, the gap between God and us can be bridged by the suitable application of numerical calculation techniques (of the corresponding procedures in the theory of operations). Even if the domain of formal connections does not coincide with the connections *de facto* perceived by us, it does coincide with that of connections which, in principle, can be recognized by the application of an appropriate available algorithm.

By applying to mathematics the content of 6.1262, originally about logic, we can assert that arithmetical calculation is merely a mechanical expedient to facilitate the recognition of the correctness of equations in complicated cases.[48] Equations supply the material for the application of that process of symbolic manipulation in which arithmetical calculation consists. Indeed, the recognition of the identity of meaning of two expressions "Ω^{t}'x" and "Ω^{s}'x" is tantamount to the recognition of their mutual intersubstitutability. Then, starting from an expression in which an arithmetical term occurs as exponent of "Ω" and applying the relevant definitions and the equations whose correctness has already been perceived or proven, a process of substitution of identicals (in the sense of symbols which can be transformed one into the other) can be carried out. When this procedure comes to an end, the identity of meaning between the last expression thus reached and the starting expression will be proven.[49] The existence of a formal relation is recognized through the mediation of a step by step procedure of substitution of terms, the *Bedeutungsgleichheit* of which has been already established. Arithmetical calculation, like any other logical method of checking formal properties and relations, is a process of symbolic manipulation. The possibility of ascertaining the correctness of an equation by means of a procedure which "does not go outside language" makes it clear that this correctness has nothing to do with the actual configuration of the world: "And the possibility of proving the propositions of mathematics means simply that their correctness can be perceived without its being necessary that what they express should itself be compared with the facts in order to determine its correctness" (*T* 6.2321).[50]

There is a final subject which Wittgenstein touches on briefly in the second block of propositions of the *Tractatus* dedicated to mathematics. It is the question of the role of intuition for the solution of mathematical problems: "The question whether intuition [*Anschauung*] is needed for the solution of mathematical problems must be given the answer that in this case language itself provides the necessary intuition. The process of *calculating* serves to bring about this intuition" (*T* 6.233, 6.2331). Let us ask ourselves what shape a mathematical problem assumes in the frame of Wittgenstein's general conception of mathematics expounded above. A mathematical problem can be nothing but a problem concerning the correctness or incorrectness of a particular equation. It poses an interrogative about the possibility or the impossibility of transforming the form exhibited by an expression, in which an arithmetical term occurs as exponent of "Ω", into the form exhibited by another expression of the same kind. According to Wittgenstein, to answer the question of the role of intuition in the mathematical practice of solving problems, one needs simply to recall a certain part of our semantical competence concerning the meaning of the term "calculation". The execution of a calculation is the construction of a sign figure that shows the mutual reducibility of two forms, transforming step by step two different configurations of one string of "Ω'" into each other. But the possibility shown by a numerical calculation – by a proof in the theory of operations – as any other feature of the formal structure of language, is an "object" of intuition or, in the terms of Wittgenstein's preferred metaphor, of vision. Thus the second part of 6.2331 runs as follows: "Calculation is not an experiment": calculation fixes what logically *must* be the case and therefore its result is seen *in* signs, whereas the result of an experiment – what *is* the case – is described *by means of* signs. Here we are at one of the central themes of Wittgenstein's conception of mathematics (and, in virtue of the relationship between mathematics and forms of non-mathematical language, of his whole philosophy). The gradual evolution of the notion of intuition firstly into that of decision, and then into that of inclination to react in a certain way to symbols, indicates that it was a nucleus destined to remain unchanged in the further devolpment of Wittgenstein's thought.

FOUNDATIONS OF MATHEMATICS (I)

Unlike Wittgenstein's writings after 1929, the *Tractatus* contains no analysis or discussion of the philosophical conceptions underlying the different foundational approaches (with the obvious exception of Russell's version of logicism, the theory of types). However, it may be useful to briefly evaluate Wittgenstein's own position in the *Tractatus* with the aid of those categories – platonism, constructivism, intuitionism and formalism – that are usually employed to label the points of view of the rival foundational schools.

Let us begin with the opposition platonism\constructivism. If platonism is understood to be a conception which assumes the existence of a realm of mind-independent mathematical entities, identified by means of our definitory procedures, and whose pre-existent properties and relations are discovered by applying the classical methods of proof; and if constructivism is understood to be a conception of mathematical entities that construes definitions as means to constitute these entities, and proofs as processes whereby properties and relations between these entities are "concretely" exhibited; then we can safely say that, in the *Tractatus,* Wittgenstein places himself beyond this conflict since he rejects the basic assumption common to both rival conceptions.[51] The main point on which platonists and constructivists disagree concerns the way in which the existence of mathematical entities has to be conceived. But, from both points of view, it is perfectly legitimate to speak of mathematical objects and to make assertions, even hypothetically, about their properties and relations. On the other hand, Wittgenstein conceives arithmetic as dealing with numerical properties of forms of linguistic expressions. He regards these forms as mere possibilities of symbolic construction, which cannot be treated as objects. This is like depriving arithmetical equations of any descriptive content: if one tried to formulate in a "proposition" the content of such an equation, the outcome would be the expression of a metalinguistic rule about the intersubstitutability *salva significatione* of the linguistic constructs of certain forms (or, within arithmetic, of certain expressions of the language of operation theory).

Nonetheless, the way in which Wittgenstein deals with the infinite in the *Tractatus* can be plausibly interpreted as expressing a view somewhat in agreement with a constructivist attitude. Also in assessing this topic, one must not forget the essential connection between arithmetic and language, which is the characteristic hallmark of Wittgenstein's conception (the former, after all, is really parasitic on the latter). Constructivist restrictions on the treatment of the mathematical infinite, which certainly can be found in the *Tractatus,* derive directly from more general assumptions on the meaning of "infinite", when referred to a set of linguistic expressions. This point may be elucidated by examining an example which has its own interest, apart from the question under scrutiny: the theses maintained by Wittgenstein in proposition 4.1273. Here Wittgenstein tackles the problem of expressing in logical notation (*in der Begriffsschrift*) a proposition of the form "*b* is a successor of *a*" (with respect to a given relation **R**). He argues that, in order for this task to be accomplished, an expression for the general term of the series of forms exhibited respectively by "*a* **R** *b*", "(∃x): *a* **R** x . x **R** *b*", "(∃x, y): *a* **R** x . x **R** y . y **R** *b*", and so on, is required. For, to say that *b* is an **R**-successor of *a* is asserting the logical sum of the infinite set of propositions, the forms of which are shown by the above series of expressions (*T* 4.1252 and 4.1273). The notation for an arbitrary term of such a series of forms would consist, as usual, in a variable containing an expression for the first term of the series and exhibiting perspicuously the form of the operation that generates every other term from its immediate predecessor.[52] What we are interested in is Wittgenstein's criticism of the Frege–Russell theory of the proper ancestral relation of a given relation.[53] As is well known, this theory had been put forward in view of a reduction of the notion of order of the elements of a series to purely logical concepts. Wittgenstein writes that Frege and Russell, laying down their definitions, overlooked the formal nature of the concept of general term of a formal series. What does this rather obscure charge mean? To give an answer to the question, let us first remark that the first part of Wittgenstein's analysis of the proposition "*b* is a successor of *a*" is in complete agreement with Frege's. Of course, also according to the German logician, if one says that *b* is an **R**-successor of *a*, one means that either *a* stands in the relation **R** with *b* or *a* stands in the relation **R** with an object which stands in the relation **R** with *b*, or *a* stands in the relation **R** with an object which stands in the relation **R** with an object which stands in the relation **R** with *b*, and so on. But, in Frege's view, this is a somewhat unsatisfactory explanation simply because of the occurrence of the phrase "and so on". The latter seems to refer to the process of moving one's attention along the series, for any finite number of steps, from *a* onwards, thus introducing a psychological element into mathematics. An explication coherent with Frege's anti-psychologistic and objectivistic attitude can be achieved only on condition that this "and so on" be analysed away. The "intuitive" reference to the order of the elements of an infinite series must be ruled out, and the proposition "*b* is a successor of *a*" must be translated into a finite and completely explicit proposition built up exclusively from logical vocabulary. Otherwise, in Frege's opinion, a non-analytical residue would undermine the judgements concerning the order of the elements of an infinite series. And this, for the German logician, would be tantamount to the final defeat of logicism.[54] In perfect agreement with Frege, Bertrand Russell clearly states the aim of the logicist definition of the proper ancestral of a given relation: "It is this 'and so on' that we wish to replace by something less vague and indefinite".[55]

In Wittgenstein's opinion, this aim is ill conceived. If the concept of general term of the series of forms $a\ \mathbf{R}\ b$, $(\exists x): a\ \mathbf{R}\ x\ .\ x\ \mathbf{R}\ b$, $(\exists x, y): a\ \mathbf{R}\ x\ .\ x\ \mathbf{R}\ y\ .\ y\ \mathbf{R}\ b$, and so on, is a formal concept, then the variable expressing it is able to exhibit, for any given *a*, *b* and **R**, the constant formal relation linking a proposition of the series, apart from the first, to its immediate predecessor; and thus to exhibit the uniform operation by means of which any proposition can be generated from the proposition that precedes it in the series. The general term of this series of forms directly shows the content of the notion of serial order that Frege and Russell tried to reduce to logic by their definition. For, to stress the formal nature of the concept of general term of the above series of forms, as Wittgenstein does, is the same as claiming that such a series cannot be identified independently from the order of its elements, or, in *Tractarian* jargon, that the order is not an "external" relation between the elements. Of course, what is true for the understanding of the serial order of the propositions is also true for the understanding of the order of the series to which *a* and *b* belong (inasmuch as, in grasping the first, one grasps the second). This is why the general term expresses nothing more than that informally expressed by "and so on", and constitutes the irreducible core of the notion of serial order: "The concept of successive applications of an operation is equivalent to the concept 'and so on'" (*T* 5.2523). Plainly, Wittgenstein thinks that this notion is not reducible to more basic ones, and the alleged circularity of the Frege–Russell definition is, in his opinion, a strong proof of this. At this point, the problem of the presence of a constructivist orientation in the philosophy of mathematics of the *Tractatus* can be dealt with. We have seen that a proposition of the form "*b* is a successor of *a*" is the logical sum of an infinite set of propositions. As is known, one of the cornerstones of the theory of meaning in Wittgenstein's first work is the principle of functionality of sense: the sense of a truth-function of a given set of propositions is a function of the sense of these propositions.[56] Therefore, in order to understand the proposition "*b* is a successor of *a*", the sense of each one of the propositions belonging to the relevant infinite series has to be understood (the knowledge of the meanings of "*b*", "*a*" and "**R**" being assumed). Now, in Wittgenstein's view, to accomplish this "infinite" task amounts, precisely, to knowing the effective law of generation of the propositions belonging to the set. Indeed, the variable that represents the general term of the series should be a recursive definition by which the infinite totality can be grasped at one stroke. In this way, the strong constructivist thesis that any reference to an infinite set is nothing but a reference to the logically unlimited possibility of application of an effective rule of generation of the set elements, enters in Wittgenstein's conception of the infinite. It is to be noticed, however, that it is not a question of a rule for producing mathematical objects unlimitedly, since objects of this kind are not included in the ontology of the *Tractatus*. Rather, it is simply the theoretically unlimited possibility of generating meaningful linguistic expressions (propositions, in the case of the infinite set involved in a statement of the form "*b* is a successor of *a*") from one another, ordered in a series by a constant formal relation. Analogously, the infinity of the set of natural numbers is construed as the possibility, unlimited in principle, of constructing symbols "SS ... S0" ("0+1+1+...+1" in the original notation of the *Tractatus*), and, via the reduction of arithmetic to the general theory of operations, it corresponds to the logical possibility of endless iteration of an operation; and an operation is not a process of construction of ideal entities from given ideal entities, but a procedure to generate a meaningful linguistic expression from a given meaningful linguistic expression (or from a given set of such expressions), grounded on the existence of a formal relation between base and result.

Now to return to the analysis of 4.1273. Here Wittgenstein does not restrict himself to charging Frege–Russell's definition of the proper ancestral relation with a misunderstanding of the formal nature of the concept of general term of a formal series. As mentioned above, he adds another severe attack: "the way in which they [Frege and Russell] want to express general propositions like the one above [like the proposition "*b* is a successor of *a*"] is incorrect; it contains a vicious circle" (*T* 4.1273). At first sight, this part of 4.1273 merely echoes the usual objection raised, from a constructivist point of view, against the logicist definition of the ancestral relation of a relation: namely, the objection concerning the impredicative nature of the logicist definition of the predicate "**R**-successor of *a*". Moreover, it is quite reasonable to presume that this was known to Wittgenstein, who was familiar with Russell's work. However, given his rejection of the notion of a mathematical object, it is difficult to think that he agreed with the typical constructivist justification for the claim of unsoundness of impredicative definitions: the conviction that mathematical definitions are constitutive of the defined entities. But it is easy to show that the vicious circularity of Frege–Russell's explication of the proposition "*b* is a successor of *a*" is equally found in the conditions for sense of the proposition into which the former has to be translated according to such an explication, i.e. of the universal generalization "*b* has all the hereditary properties of *a*". Among the propositions forming the set whose logical product is asserted by this generalization, there are propositions in which "*b* is a successor of *a*" occurs as one of their truth-functional components, and, in such a situation, the principle of functionality requires that the meaning of "*b* is a successor of *a*" should be already known if the logicist explication is to be understood. Impredicative definitions are rejected by Wittgenstein not because they contradict a constructivist view of the existence of mathematical objects, but rather because they violate certain requirements on the composition of the sense of complex propositions.

At the end of the first group of remarks devoted to mathematics, Wittgenstein places the following interesting proposition: "The theory of classes is completely superfluous in mathematics. This is connected with the fact that the generality required in mathematics is not *accidental* generality" (*T* 6.031). Once having outlined his interpretation of numerals as exponents of an operation variable, he wishes to contrast his own tentative reduction of arithmetic to operation theory with the logicist programme of reduction of arithmetic to logic. We know that, for the reasons expounded in the last section, the philosophy of arithmetic of the *Tractatus* can be aptly described as a kind of logicism. But the Austrian philosopher resolutely rejects the translation of number theory into the theory of classes, namely the reduction of the former to logic, in the broad sense of the term "logic" where logic includes set theory. Proposition 6.031 explains the grounds for this rejection. Apparently, this explanation has a typical constructivist flavour because it seems to raise the question of the correct interpretation of a generalized statement concerning the elements of an infinite domain (e.g. a statement about all numbers or about some number), and to rule out any extensional interpretation. I do not believe, however, that this is at issue in 6.031. What Wittgenstein is really denying here is that the general validity of *any* mathematical statement can be accounted for by the theory of classes. And, as he identifies *tout court* mathematical pseudo-propositions with numerical identities (with the operation theory equations onto which they are mapped), the question of the nature of general validity arises in connection with this sort of expression, and not with universally or existentially quantified propositions.

A double clarification is needed in order to achieve a correct understanding of 6.031. First, the sense in which a numerical identity is *generally valid in an essential way* must be explained. Second, one must justify the objection to the theory of classes: that its theorems – even if generally valid – are such only accidentally. We can expound the objection more explicitly. The reductionist programme of logicism turns out to be a translation of arithmetical axioms into propositions of the language of the theory of classes (in the type-theoretical version), and to be a proof of these translations from the axioms of the latter theory. This translation is considered by Wittgenstein not only superfluous, but also harmful since it changes propositions whose general validity is essential into propositions that, if valid, are endowed merely with an accidental general validity. As known, the opposition between essential and accidental general validity of logical propositions is elaborated in the *Tractatus* in overt polemic with Russell's conception of logic. In effect, the above mentioned inadequacy of the logicist reduction of arithmetic follows as a simple corollary from what Wittgenstein considered one of the main flaws of Russell's theory of types: among the axioms of the theory, there are propositions that, despite their complete generality, do not satisfy the basic requirement of logical validity, i.e. truth in all possible worlds.[57] To clarify in what the attribution to mathematical propositions of an essential general validity consists, it is sufficient to recall what we said in the preceding sections. Let us refer to an example we used previously, the numerical identity "$2 \times 2 = 4$", and see in what sense its validity is both general and essential. We must translate this identity into the corresponding equation of operation theory: "$\Omega^{(2\times2)}$'x = Ω^{4}'x". The general validity of the equation can be explained along already known lines: it shows that, given any operation and any initial symbol, the expression obtained by the application of the second iteration of the operation to the result of its application to the initial symbol has the same meaning as the expression obtained by three successive applications of the operation to the result of its own application, starting from the result of its application to the same initial symbol. The generality of the numerical identity is due to its purely formal character: to the circumstance that both schematic expressions "$\Omega^{(2\times2)}$'x" and "Ω^{4}'x" exhibit forms of linguistic expressions. The essential character of this general validity issues from the same source. According to Wittgenstein, an essential general validity is one that does not depend whatsoever on the real configuration of the world, on the particular arrangement of objects which actually occurs. But an equation such as "$\Omega^{(2\times2)}$'x = Ω^{4}'x" concerns only the mutual reducibility of two operational models of linguistic construction. The independence of its general validity from the effective configuration of the world is implied by the fact that such a formal relation exists exclusively in virtue of the properties of the concept of application of an operation (in the broad sense including composition with another operation). These properties have nothing to do with the contingent configuration of the domain of objects: in other words, the basic assumptions of the general theory of operations, unlike the critical axioms of the theory of types, are not existential statements on the universe of objects in its totality. Now we are in a better position to evaluate Wittgenstein's objection to the theory of classes. The axioms of the theory of types, despite their complete generality, are not tautological; if they are true, they are only accidentally true – true because of the fortuitous circumstance that a certain possible configuration of the world is its real configuration. Even if the type-theoretical translations of arithmetical propositions were generally valid, in view of the realization of this configuration of the world, it would be only a fortunate chance. And this is like saying that the logicist translation of arithmetical propositions cancels that fundamental feature of mathematics which is the essential general validity of its propositions. According to Wittgenstein, the formal purity of mathematics can be saved only on condition that no existential assumption, stating properties of the domain of objects in its totality, is introduced. This very condition should be fulfilled by his reconstruction of arithmetic in the frame of the general theory of operations. Since arithmetic deals only with formal properties of linguistic expressions generated by processes of iteration and composition of logical operations, and since the infinity of the set of natural numbers is exclusively connected with the logically unlimited possibility of applying an operation to the result of its own application, no explicit existential axiom about the universe of objects has to be adopted. According to Wittgenstein, what holds true for logic also holds true for mathematics (another aspect of his logicism): existential assumptions about the universe of objects – being fully absorbed by the vocabulary of our language – are presupposed, but never asserted by mathematical propositions.

Our discussion of Wittgenstein's criticism of the logicist (type-theoretical) foundation of arithmetic can be finished here. I wish to conclude this sketchy comparison of the *Tractatus* philosophy of mathematics with the other foundational points of view by a brief discussion of his relations with formalism and with intuitionism.[58] Through our exposition it should be clear that, since his first masterpiece, Wittgenstein's reflections on mathematics follow a threefold plan: (i) to rule out that the validity of mathematical pseudo-propositions depends in some way on the contingent configuration of objects in the world; (ii) to rule out, with equal force, that mathematical results state properties and relations of ideal objects; but, at the same time, (iii) to avoid that mathematics be construed as a mere activity of manipulation of meaningless signs (after a purely formalistic manner). An arithmetical term "**t**" has a meaning (*Bedeutung*), which it derives from the meaning of the schematic formal expression "Ω^{t}'x". What really matters to Wittgenstein is not the denial that an arithmetical term has a meaning, but rather that this meaning is an ideal object, named or described by the term. In his view, even the distinction between a numeral and a number is quite legitimate, in perfect analogy with the distinction, explicitly made by him, between a propositional sign (*Satzzeichen*) and a proposition (*Satz*). That between numerals and numbers is not an ontological distinction between two kinds of entities (material entities versus ideal entities), but a distinction between two ways of considering the one linguistic reality: the way in which one considers a sign as a mere physical entity and the way in which one takes into account its role of notational device to represent a certain formal property.[59] Nonetheless, a point has to be stressed, which to a large extent shortens Wittgenstein's distance from formalism and justifies, in my opinion, the description of his early conception of mathematics as quasi-formalistic (a label that, as we shall see, is fitting for every stage of the whole development of his philosophy of mathematics). The recognition of the mutual reducibility of two models of linguistic construction – when calculation is required – is the outcome of a rule-governed process of transformation of a certain grouping of the elements of a string of "Ω'", *exhibiting* the first model, into a different one, *exhibiting* the second model. Thus, although arithmetical signs within calculations are not considered as mere physical structures, doing mathematics is appropriately described as a sign manipulation activity.

The last remark leads us to the comparison with intuitionism. We have said that the reference to vision, intuition or immediate recognition plays a decisive role in Wittgenstein's conception of our "knowledge" of the formal domain. And there is also a certain similarity between Wittgenstein's theses on the purely instrumental role assigned to mathematical notation and even to the formulation of theorems (equations), on the one hand, and certain typical ideas of Brouwer concerning logical and mathematical language, on the other. However, a decisive element of disagreement between the two conceptions can be easily identified. When Wittgenstein, dealing with mathematics and logic, speaks of vision, he does not intend to supply a psychologistic foundation to mathematical activity. Rather, he resorts to the notion of intuition or to the metaphor of vision to describe the relationship between speakers and what is shown by language (the domain of necessity) and to contrast it with the meaningful expression of a thought (the picture of a contingent state of affairs). It goes without saying that all this has nothing to do with the psychologization of mathematics of intuitionists. [60]

I have not yet analysed a subject which perhaps the reader might judge of some importance in an overall evaluation of the philosophy of mathematics of the *Tractatus*. In the frame of his general theory of operations, Wittgenstein covers a very small portion of mathematics, namely, that limited and not very interesting part of arithmetic consisting in numerical identities. Frank P. Ramsey, reflecting on this situation, remarked that Wittgenstein's view, "as it stands, ... is obviously a ridiculously narrow view of mathematics". [61] The entire remaining part of mathematics, even of the theory of numbers itself, is passed over in silence in the *Tractatus*. Bearing in mind Wittgenstein's well known attitude of not considering it his job to make a "detailed" investigation to prove the effective applicability of his general conceptions, one can conjecture that he took for granted the possibility of extending his interpretation to all of mathematics (no further philosophical achievement would have been needed, in his opinion, for the accomplishment of this task). On the other hand, Wittgenstein, restricting himself to the most elementary part of arithmetic, could easily admit a gap between what language shows and what we see *de facto*; and, in this way, he could account for the existence of numerical calculations, without this admission meaning the acknowledgement – quite absurd for him – of formal properties and relations with which we have no means to become acquainted. However, having pointed this out, it is legitimate to ask ourselves how Wittgenstein would have resolved the problem if the interpretation of the rest of mathematics in the framework of his theory of forms had proven unfeasible. Would he have rectified or improved his interpretation, or would he have condemned the resistant part of mathematics, leaving it to its fate (as the *Tractatus* suggests doing with the theory of classes)? This is not an idle question since it is one of the main problems which is raised by Wittgenstein's later philosophy of mathematics. Despite his later abandonment of any kind of reconstructive programme of arithmetic, the question of the eventual revisionary import of Wittgenstein's approach on current mathematical practice will occur again, persistently demanding a credible answer.

2

VERIFICATIONISM AND ITS LIMITS

The intermediate phase (1929–33)

INTRODUCTION

When one approaches Wittgenstein's reflections on mathematics during the years 1929–33, one is struck immediately by the proportion of space that they occupy in his total philosophical production of this period.[1] In the *Tractatus,* the treatment of the fundamental notions of arithmetic, together with the analysis of numerical identities and the procedure of numerical calculation, aims at showing that an interpretation perfectly consistent with the assumptions of the general conception of logical forms maintained there could be provided – at least of a small portion of mathematics. This interpretation is seen as an alternative to the project of set-theoretical foundation of mathematics proposed by the logicism of Frege and Russell and, more specifically, to its type-theoretical realization. In light of the exposition made in the first chapter, one can affirm that, even though within the narrow limits of a rough outline, the objective put forward by Wittgenstein in the parts of the *Tractatus* dedicated to mathematics is achieved.

It is quite natural that, when returning to his philosophical work and beginning gradually to dismantle several of the general premises of the linguistic theory of the *Tractatus,* Wittgenstein is forced inevitably to re-examine his initial ideas concerning mathematics too: ideas certainly tied in a significant measure to those very assumptions which he now begins to question. This obvious remark, however, does not justify the broad extent of his reflections on mathematics in this period: in fact, there is an evident disproportion between the marginal role they play in the *Tractatus* and the centrality that they assume in the new phase. In order to account for this discrepancy, one could give, with good reasons, an "external" explanation. The 1920s had seen the consolidation and the affirmation of two main theoretical tendencies in the debate over the foundations of mathematics: Brouwer's and Weyl's intuitionism and Hilbert's formalism (as well as the presentation by Ramsey of an updated and, under certain essential aspects, amended version of logicism). Now, whereas the background of the philosophy of mathematics of the *Tractatus* is constituted exclusively by the works of Frege and Russell (whose idea that mathematics and logic are substantially the same thing is accepted, even though in the peculiar sense explained in Chapter 1), Wittgenstein's writings between 1929 and 1933 testify that the range of his interests in this field increases notably. His frequent references to theses and texts by Brouwer, Weyl, Hilbert, Skolem, and his discussion of themes typical of foundational inquiry – from the interpretation of the generalized propositions to the problem of the consistency of an axiomatic system; from the investigation on the status of the proof by complete induction to the clarification of the notion of a real number – provide evidence of his close attention to the developments of the debate over foundations. To this, one can add his acquaintanceship with the philosophical positions expressed by G. H. Hardy, a mathematician who, although not directly involved in foundational research, was one of the most eminent mathematicians of Cambridge.[2] There can be no doubt that this material supplies more than one cue for Wittgenstein's inquiry into mathematics. Nonetheless, in order to understand how his contact with specific topics of foundational studies is such a rich source of inspiration, one must introduce an "internal" consideration regarding the relationship which, according to him, connects the domains of mathematics and grammar. For, in my opinion, the ultimate reasons for Wittgenstein's interest in the philosophy of mathematics are to be found in this linkage.[3] It is true that his position on this subject will come to maturity only in the writings of the decade 1934–44. However, his reflections on mathematics in the intermediate phase are worthy of close scrutiny for several important reasons. First, certain themes, which are dealt with extensively in the writings of that period, will not be re-examined, in later writings, to comparable depth. Secondly, the core of his more mature ideas, i.e. the view of necessity derived from his rule-following considerations, has its clear antecedent in the intermediate phase conception of the relation between the general and the particular within mathematics (within grammar). Last but not least, these writings testify to the difficulty met by Wittgenstein in saving the sharp opposition between the notion of linguistic rule (and, therefore, of necessity) and that of meaningful proposition, established in the *Tractatus*. As, according to the *Tractatus,* arithmetic shows relations between forms of picturing-facts language, so, according to the 1929–33 conception, it produces rules of grammar of the factual language which describes the outcomes of manipulations of arithmetical symbols, as well as the outcomes of certain operations – union, partition etc. – on classes of empirical objects (in the case of cardinal arithmetic). But, in the intermediate phase, a new idea, quite extraneous to the *Tractatus* universe, emerges: that, in certain cases, the content of a rule of grammar can be expressed in a genuine meaningful *proposition*.

There are several questions which Wittgenstein must answer in his new attempt to attain a satisfying view of the relationship between mathematics and the rest of language. Here we state some of them: (i) How can the result of a particular manipulation of physical signs (e.g. ink or chalk strokes) acquire the universal normative value typical of a mathematical result? (ii) Is it legitimate to say that the application of a general procedure of calculation displays grammatical properties and connections which, *in some sense,* pre-exist our effective acknowledgement of them? And to what extent can a mathematical proposition – that is decidable by the application of such an algorithm – be comparable to a genuine meaningful proposition? (iii) What is the relationship between sense and proof of a mathematical proposition like, in all those cases in which proof does not consist in the mere application of a set of already known rules of calculation? (iv) How can something regarding the internal possibilities of a calculus, considered as a whole – for instance, whether certain expressions can or cannot be obtained by manipulating the signs of the calculus according to its rules – be proved mathematically? (v) How should one construe universally and existentially generalized statements, which are obviously found in mathematics, once the latter is looked upon as a sign manipulation activity governed by rules, and producing new rules?

In the course of this chapter, we shall see how certain typical stances of Wittgenstein's conception of this period are reflected in his replies to these questions, above all, his verificationist approach in the theory of meaning of propositions. The notion of rule will stand out as the pivotal notion of Wittgenstein's philosophy of mathematics even during the intermediate phase. His critical considerations on the theme of rule-following, clearly stated for the first time only in the following biennium 1934–5, will widen and radicalize the cues already present in the transitional phase. Then, the generalized application of the thesis that there are no unacknowledged necessary connections will lead inevitably to the abandonment of verificationism and to the consequent decline of the notion of mathematical proposition. So one can conclude that the presence of this notion hallmarks just a brief interlude in the overall development of Wittgenstein's reflections on mathematics.[4]

FINITE CARDINAL NUMBERS: THE ARITHMETIC OF STROKES

Returning to his philosophical work, Wittgenstein at once realized the necessity to distance himself from the project of reducing mathematics to logic, outlined in the *Tractatus*. A foundation of arithmetic on the abstract notion of logical operation now appears superfluous to him; in reality, "a nebulous introduction of the concept of number by means of the general form of operation – such as I gave – can't be what's needed" (*PR* §109). Even though in the *Tractatus* this reduction does not take the form of a true definition of numerals and of the predicate "natural number" (framed in the language of the basic theory), Wittgenstein sees the rejection of the reductionist programme as a denial of *any attempt* to give such a definition. Furthermore, in his writings of the intermediate phase, this rejection accompanies the acknowledgement, absent in the *Tractatus,* of the central role of the concept of cardinal number of a class. In *Philosophical Remarks,* Wittgenstein argues that one should assume numerals as known, simply identifying numbers with what these terms represent (*darstellen*); in a note subsequently added in the margins of this passage, he writes that "instead of a question of the definition of number, it's only a question of the grammar of numerals" (*PR* §107). This is the same point of view that Wittgenstein adopts, even more explicitly, in *Philosophical Grammar,* where, to the question "What are numbers?", he answers "What numerals signify", and where the analysis of the meaning of the general term "number" – analysis here understood as "an exposition of the grammar of the word 'number' and of the numerals" – is contrasted with the search for its definition (*PG,* II, Ch. IV, §18, p. 321). Plainly, this different approach is in perfect agreement with the new general orientations of Wittgenstein's analysis of language, upon which we will not dwell. However, the fact that he speaks overtly of the meanings of numerical terms is not without importance. He had done so in the *Tractatus* too, differentiating his position from a purely formalistic approach. At the same time, in speaking of the *Bedeutungen* of numerical terms, one runs, in his opinion, a serious risk: that numbers will be conceived as objects, or in other words, that numerals will be construed as proper nouns, as expressions that have a meaning in so far as they denote ethereal objects (or even worse, for those who rashly identify the meaning of a noun with the object denoted by it, as expressions whose meaning is a *sui generis* kind of object, i.e. a number). In one of the conversations recorded by Waismann, Wittgenstein manifests, even if rather cryptically, his concern for what he judges to be a total misunderstanding of the grammar of numerals and of arithmetical terms in general: numbers, contrary to objects, are not represented (*vertreten*) by signs which are proxy for them in language, but simply "are there" (*WVC,* p. 34). Wittgenstein's problem is once again that of avoiding the dilemma between the conception which identifies numbers with signs, i.e. with certain types of physical structures (of chalk or ink) devoid of meaning – a conception which, getting rid of ideal or mental entities, might have attracted him strongly for the same reasons as behaviourism in philosophy of psychology – and the conception, attributed to Frege, which rejects formalism, reintroduces meaning and treats arithmetical signs as signs standing for something, as names or descriptions of ideal objects.[5] In effect, the view, maintained in the *Tractatus,* of how formal properties of language can be represented in language already points to a way out from this "unpleasant" alternative. It is the view according to which an arithmetical term does not have a meaning in so far as it denotes or describes a formal property, but in so far as it is part of an abbreviated notation for an expression which directly shows that property (e.g. "**n**" has the meaning shown by the *definiens* of "Ω^{n}'x"). This original solution has what to Wittgenstein's eyes appears a great virtue: calculation no longer needs to be conceived, in Frege's fashion, as the instrument whereby the oneness of the object denoted by two expressions with different senses is ascertained. Suppose, in fact, that the meaning of an arithmetical term "t" is the arithmetical structure of the operational scheme of sign construction exhibited by "Ω^{t}'x". Then, one gets a double result: that calculation can be satisfactorily described as a mere procedure of sign manipulation, and that the outcomes of calculations – the operation theory equations into which numerical identities are translated – can be construed as disguised statements of metalinguistic theorems (about the intersubstitutability *salva significatione* of the expressions constructed in compliance with the operational models shown by "Ω^{t}'x" and "Ω^{s}'x", occurring on the two sides of "=" in a proven equation). It is true that an arithmetical *Scheinsatz* happens to be in the odd situation that it must play a role expressly prohibited by the basic assumptions of the theory of language of the *Tractatus* (one among a large family of similar oddities). But this, after all, is of little importance for the early Wittgenstein, since an ideal user of language – the metaphysical subject – would need neither to construct equations, nor to transform them by calculations.

In his intermediate phase, Wittgenstein does not betray the fundamental insight of the *Tractatus* about the meaning of numerals: a sequence of strokes like " | | |" carries out the function of a numeral inasmuch as it is used as a paradigmatic representation of the class property of having three elements. Processes to determine whether a given set of empirical objects has three members can vary (and, according to Wittgenstein, the setting of a 1–1 correlation with the paradigm is only one procedure among many). The point is that, when used as a numeral, the sequence " | | |" acquires the property in question *by definition*, namely, in *Tractarian* jargon, it *shows* the property. In the sign construction of the arithmetic of strokes, such sequence serves to set up what can, and what cannot, be meaningfully said of any class to which the number of elements represented paradigmatically by the sequence has been attributed (in the same way in which, by comparison of patches of colours employed as paradigms, rules on what can and cannot be meaningfully said about coloured surfaces are drawn). Once numerical signs are construed as paradigms, elementary arithmetical processes can be thoroughly described in terms of operations on symbolism, without this entailing an unconditional adhesion to formalism. Manipulations of stroke sequences do not aim at ascertaining physical properties of the specific sign structures used in the calculation, nor, even less, at providing inductive evidence for a universal empirical hypothesis on the outcomes of analogous transformations of classes of objects having the corresponding cardinal numbers. As mentioned above, the application of elementary processes of numerical calculation leads, according to

Wittgenstein, to the adoption of grammar rules of non-mathematical language (as we shall see later on, also of that portion of non-mathematical language whereby the effectively performed operations on signs are described). In the 1929–33 writings, this topic begins to receive a more extensive treatment with respect to the scanty handling in the *Tractatus,* even if we are still far from the developments of the following decade.

Wittgenstein's reflections are prompted by a classical Fregeian theme: the interpretation of numerical specifications as attributions of a property to a concept. In *Philosophical Remarks* Wittgenstein notes that the possibility of interpreting *all* numerical specifications as assignments of a property to a concept depends on the exclusive acceptance of the model of analysis of propositions in terms of argument and function. In *Philosophical Grammar* he observes that several propositional forms, differing from one another with respect to the grammatical systems of the concept-words occurring in them, are forcibly compressed under this canon of representation of the logical structure of propositions. [6] Nonetheless, in §119 of *Philosophical Remarks,* there is the admission that "you can ascribe a number to the concept that collects the extension", and, in a passage of *Philosophical Grammar* which precedes the text summarized above by only a few pages, one finds the affirmation that "as Frege said, a statement of number is a statement about a concept (a predicate)" (*PG,* II, Ch. IV, §19, p. 332). Actually, no contradiction exists between Wittgenstein's two positions, if certain strict constraints that he places on the use of the terms "concept" and "predicate" are respected. In the passages under examination, Wittgenstein uses the term "concept" in the same way as "*eigentliche Begriff*" is employed in the *Tractatus,* namely meaning what he now also calls "material concept" or "material function" (*PR* §113). It is a limitation which has very considerable consequences in Wittgenstein's philosophy of mathematics too, since Frege's model of analysis will not be applicable to numerical specifications *internal* to mathematics – like the statement that there are six permutations of a set of three elements, or that there are two roots of a second degree equation.[7]

Having made this point, let us look at what Wittgenstein refers to when discussing the Fregeian analysis of numerical specifications. He has in mind the explanation, advanced by the German logician in *Grundlagen der Arithmetik,* of the propositions of the form "the number 0+1+1+...+1 belongs to the concept F" (an explanation that, as is known, precedes the definition, by means of the equinumerosity relation, of the notion of the number belonging to the concept F). Following his usual convention on the exclusive interpretation of variables, Wittgenstein rewrites these propositions as follows: "~(∃x) Fx" for the proposition "the number 0 belongs to the concept F"; "(∃x) Fx . ~(∃x,y) Fx . Fy" for the proposition "the number 0+1 belongs to the concept F"; "(∃x,y) Fx . Fy: ~(∃x,y,z) Fx . Fy . Fz" for the proposition "the number 0+1+1 belongs to the concept F"; and so on.[8] Referring to the number attributed to a concept, Wittgenstein states that it is "an external property of the concept" (*PG,* II, Ch. IV, §19, p. 332). What he means by this is that the relation between a concept and the number of elements of its extension is a contingent relation, in the sense that the concept does not change as this number varies. This may appear evident *per se*: the concept expressed by the predicate "man", for instance, is defined and identified by its distinguishing marks (e.g. those expressed by the predicates "two-footed" and "unfledged"), and of course this concept remains the same, be there one thousand men or ten billion. Restating this thesis in a style more adherent to Wittgenstein's linguistic approach, one can say that a numerical specification of this sort does not contribute to the determination of the meaning of the predicate but, taking such meaning as given, asserts, though in general terms, the existence of a certain state of affairs. With exclusive reference to material concepts, Wittgenstein lays down a strict opposition between number as an external property of a concept, on one hand, and number as an internal property of the extension of a concept, on the other. This opposition is presented as follows:

> What is meant by saying number is a property of a class? Is it a property of ABC (the class), or of the adjective characterizing the class? There is no sense in saying ABC is three: this is a tautology and says nothing at all when the class is given in extension. But there is sense in saying that there are three people in the room. Number is an attribute of a function defining a class: it is not a property of the extension. A function and a list are to be distinguished. ... We say something different when we talk about a class given in extension and when we talk about a class given by a defining property. Intension and extension are not interchangeable.
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> (*AWL,* pp. 205–6)

The notion of cardinal number as an internal property of an extension or of a class is introduced in *Philosophical Remarks,* where we find the assertions that "the number is an *internal* property of the extension" and that "numbers are pictures [*Bilder*] of the extensions of concepts" (*PR* §§119, 100). Correspondingly, in *Grammar,* it is argued that "a cardinal number is an internal property of a list" and that "a number is a schema for the extension of a concept" (*PG,* II, Ch. IV, §19, p. 332). A class considered from a purely extensional point of view is, according to Wittgenstein, a class that is given by means of the list of its elements. By definition, such a list is the only notation by which reference can be made to a class in extension, and this brings about the restriction of the use of the term "class" to finite classes.[9] Wittgenstein's statement that the number of elements of a class is one of its internal properties can be explained as follows: no proposition, in which reference is made to a class not by means of a description such as "the class of the F's", but by means of an explicit list of its members, can keep its sense unaltered if the cardinal number of the class changes. In fact the class, mentioned extensionally, is no longer the same as this number varies. This, I think, is what Wittgenstein means in the following passage: “And now – I believe – the relation between the extensional conception of classes and the concept of a number as a feature of a logical structure is clear: an extension is a characteristic of the sense of a proposition” (*PR* §105). His further thesis that a number is the picture or schema of the extension of a concept, in my opinion, simply reformulates the idea that a numeral is used to represent paradigmatically the class property constituted by the number of its elements. For example, arithmetical properties established through the appropriate manipulations of the numerical sign “ | | | |” – like the property of being able to be constructed by joining two sequences of two strokes each – concern, in the terminology sometimes adopted by Wittgenstein in this period, *the symbol* “ | | | |”. This means that the recognition of this property corresponds to the acceptance of a grammar rule regarding the factual statements whereby the results of certain manipulations of four-element classes are described. Acknowledging the sign construction under discussion as an arithmetical proof, one rules out, for instance, that it *makes sense* to describe, as a correct performance of the empirical process of union of two sets of two objects each, any operation which does not yield a four-element set. It is no wonder that, according to Wittgenstein, one cannot assert meaningfully that “ | | | |”, considered not as a physical structure but as the sample four, contains four elements (as, in the *Tractatus,* one could not *say* that two is the number of times an operation has been iterated to generate an expression of the form Ω’Ω’x). The reasons for this prohibition are the same as those which impede the meaningful attribution of the length of one foot to a ruler used as a sample foot, i.e. that such a statement would be true by definition and thus would not be properly a statement. [10] As for numbers, we find ourselves in the situation described by Wittgenstein with these words: “And if we say numbers are structures we mean that they must always be of a kind with what we use to represent them [*sie darstellen*]” (*PR* §107).[11]

The idea that the sequences of strokes in an arithmetical construction, like the figures of a geometrical proof, take the part of *paradigms, symbols,* or, in *Tractarian* terms, *variables,* will develop, in later writings, into the conception of mathematical proof as the picture of an experiment. For the present we will deal with the thesis that the normative force of an arithmetical result lies in the fact that a grammar rule is obtained from it, a rule which excludes whole classes of non-mathematical descriptions as senseless. This thesis applies not only to the language whereby empirical processes of union, partition etc. of classes of objects are described, but also to the language whereby sign manipulations themselves – calculations worked out by an individual – are spoken of. The latter application of the thesis leads to what Wittgenstein calls “the geometrical interpretation” of arithmetical results; taking into account this interpretation, the reasons for the distance of the Austrian philosopher’s view from formalism can be better clarified. In a description of the purely "phenomenal" aspect of mathematical activity, such as that to which Wittgenstein deliberately restricts himself, constructions of primitive arithmetic appear as the outcomes of processes of manipulation of suitable physical structures. In one of the conversations recorded by Waismann, he affirms that "mathematics is always a machine, a calculus.... A calculus is an abacus, a calculator, a calculating machine; it works by means of strokes, numerals, etc." (*WVC,* p. 106); and again: "What we find in books of mathematics is not a *description of something* but the thing itself. We *make* mathematics. Just as one speaks of 'writing history' and 'making history', mathematics can in a certain sense only be made" (*WVC,* p. 34). In *Philosophical Remarks* one finds the following similar affirmation:

> Let's remember that in mathematics, the signs themselves *do* mathematics, they don't describe it. The mathematical signs *are* like the beads of an abacus. And the beads are in space, and an investigation of the abacus is an investigation of space.... You can't write mathematics, you can only do it. (And for that very reason, you can't 'fiddle' the signs in mathematics).
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> (*PR* §157)

Finally, in *Grammar,* after having explained the notion of cardinal number as an internal property of the extension of a concept, and after having introduced the sign " | | | |" for the number 4, he writes: "What arithmetic is concerned with is the schema | | | |. – But does arithmetic talk about the lines I draw with pencil on paper? – Arithmetic doesn't talk about the lines, it *operates* with them" (*PG*, II, Ch. IV, §19, p. 333). In view of what will be said in the following section, it is expedient to stress that, in this context, Wittgenstein does not consider arithmetic as a complex of general procedures of calculation, uniformly applicable to every case of a certain kind. Each single addition, multiplication etc. is seen as a much more primitive sign construction, endowed with its own specificity. In speaking of the type of processes carried out in this free activity of sign construction, one can stick to vague terms: the operations of elementary arithmetic are operations of juxtaposition of sequences of strokes; of grouping into sub-sequences the strokes of a given sequence (e.g. to demonstrate that 11 divided by 3 is equal 3, with a remainder of 2); of grouping in two different ways the strokes of a single sequence (e.g. to demonstrate that 5+(4+3) = (5+4)+3)); of correlation of the strokes belonging to certain sequences with the strokes of some other sequences; etc. This picture strongly suggests a comparison with Hilbert's conception of the intuitive theory of numbers, namely, of the most elementary part of arithmetic, which consists only of processes of manipulation of concrete numerical signs or figures.[12] As is known, finitary (content-endowed) statements of the intuitive theory of numbers can be validated, according to Hilbert, by means of absolutely reliable processes, which do not require use of axioms nor of rules of inference, being grounded on the direct vision of the combinatorial properties of sequences of concrete objects: strokes or signs "1". In his writings of the intermediate phase Wittgenstein, too, maintains that arithmetical properties and relations are recognized not by applying logical principles and rules but by intuition (*Einsicht*) or vision of sign structures.[13] But there is a crucial difference between Hilbert's and Wittgenstein's orientation. According to the German mathematician, the space–temporal reality of signs supplies a domain of totally surveyable entities, whose combinatorial properties can be verified with absolute certainty. On the other hand, not only does Wittgenstein disregard the exigency of finding an area of mathematical practice which would be immune to the dangers that threaten its most abstract parts; but, what is more interesting to us now, his way of conceiving the relationship between arithmetic and the sign reality is very different from that of Hilbert's. In Hilbert's view, finitary statements of the intuitive theory of numbers *are about* properties of sequences of discrete objects, given in the immediate experience. On the contrary, according to Wittgenstein, the result of a construction within the arithmetic of strokes is not *a statement* about the signs (neither as tokens, nor as types), understood as meaningless physical structures. Calculation, i.e. the process of manipulation of signs, "is only a study of logical forms, of structures" (*PR* §111). In order to grasp the content of this thesis and to evaluate its true import against Hilbert's stance, Wittgenstein's geometrical interpretation of arithmetical results, mentioned above, has to be introduced. The relation between arithmetic and geometry is presented as follows: "You could say arithmetic is a kind of geometry; i.e. what in geometry are constructions on paper, in arithmetic are calculations (on paper). – You could say it is a more general kind of geometry" (*PR* §109). Obviously, to get the point of the comparison, one needs to start with Wittgenstein's interpretation of geometrical theorems, a matter on which he is extremely clear:

> "the sum of the angles of a triangle is 180 degrees" means that if it doesn't appear to be 180 degrees when they are measured, I will assume there has been a mistake in the measurement. So the proposition is a postulate about the method of describing facts, and therefore a proposition of syntax.
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> (*PG,* II, Ch. III, §17, p. 320)

No doubt, a geometrical theorem, proven with the aid of a drawing on the blackboard, does not describe the properties of this figure, but has an universal normative value. It establishes what *must* be the case whenever certain determinate operations are *correctly* performed on any given figure which has been *correctly* identified as a triangle. The crucial point is Wittgenstein's interpretation of *must*. The proposition "Necessarily, if the inner angles of a figure correctly identified as a triangle are correctly measured, then their sum equals to 180°" does not express an empirical law on the outcomes of the measurements effectively carried out, nor a forecast on the outcomes of future measurements of such kind. Necessity has a purely linguistic nature because it corresponds to the adoption of a rule of syntax, which rules out as senseless every empirical description of the form "by the correct measurement of the inner angles of the figure T, correctly identified as a triangle, the sum s has been obtained", where "s" is other than "180" or, more generally, is a term such that "$s \neq 180$" is a theorem of arithmetic (here the further problem of the admitted margins of error in the process of measurement is deliberately overlooked). In short, it cannot be that, by a correct measurement of the inner angle of a given figure, correctly identified as a triangle, a different result from that established by the theorem is obtained, simply because the alleged description of such a situation would be senseless. *By definition,* if the inner angles of a figure correctly identified as a triangle are correctly measured, their sum equals to 180°. This is the reason why the geometrical theorem is able to supply a standard to evaluate the measurements actually carried out: an outcome different from 180° implies either an error in the process of measurement, or a mistake in the identification of the given figure. Of course, the result of the manipulations performed on the particular figure employed in the proof of the theorem would be too weak a ground for *the truth of a universal statement* about all the figures being the same shape as the proof-figure. According to Wittgenstein, this circumstance shows that the theorem is not at all a statement of this kind but is, indeed, the disguised expression of a rule of grammar:

> It wouldn't be possible for a doctor to examine *one* man and then conclude that what he had found in his case must also be true of every other. And if I now measure the angles of a triangle and add them, I can't in fact conclude that the sum of the angles in every other triangle will be the same. It is clear that the Euclidean proof can say nothing about a totality of triangles. A proof can't go beyond itself.
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> (*PR* §131)

When Wittgenstein says that a construction within the arithmetic of strokes plays the same role as a geometrical construction, he wishes to stress that a grammar rule of the language whereby concrete manipulations of signs are described is drawn, similarly, from the stroke-construction. An arithmetical result has a universal normative value in so far as it establishes what *must* be obtained by the correct performance of transformations – of the same type as those carried out in the proof – of groups of signs whose shape has been correctly identified as the same shape as the sequence of strokes used in the proof. As in geometry, this necessity arises from a purely linguistic ground. A rule is adopted that excludes as senseless every empirical statement whereby an outcome different from that obtained in the proof is described as the result of the correct performance of such-and-such transformations on such-and-such sequences of strokes. In other words, to have *that* result is assumed to be an essential property of the correct performance of the sign manipulation in question, in the sense that such a correct performance yields it *by definition*: "In the calculus process and result are equivalent to each other" (*PG,* II, Ch. VII, §39, p. 457). Thus, an arithmetical theorem is not a universal statement about the properties of certain types of sets of concrete objects, rashly founded on the evidence provided by the single process of sign transformation which constitutes the proof. According to Wittgenstein, it is not a statement at all, but the disguised expression of a grammar rule of the language whereby empirically given processes of sign manipulation are described. An arithmetical theorem, like a geometrical one, supplies a standard to evaluate the calculations actually carried out: a result different from that established by the theorem implies either that the sign transformations have been wrongly performed or that a mistake in the identification of the signs has been made. If, abandoning the purely geometrical interpretation, sequences of strokes are considered as pictures of the extension of concepts, i.e. as meaning cardinal numbers of classes of empirical objects, then an arithmetical construction serves to fix a property of the grammatical space in which all finite sets of such a sort are placed.

In contrast to Hilbert's opinion, the universal validity and the normativeness of the results of arithmetic of strokes is not founded on a privileged epistemic relation between the knowing subject and the realm of concrete signs and of their combinatorial properties (a relationship which would imply that uncontroversial truths could be established about that domain). As, in Wittgenstein's view, it is not a question of the ascertainment of the truth of statements about concrete objects such as signs, but of the acceptance of grammar rules of the language whereby empirical manipulations of these signs are described,

> it isn't even necessary for the construction actually to be carried out with pencil and paper, but a description of the construction must be sufficient to show all that is essential. (The description of an experiment isn't enough to give us the result of the experiment: it must actually be performed.) The construction in a Euclidean proof is precisely analogous to the proof that 2+2 = 4 by means of the Russian abacus.
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> (*PR* §131 and *PG,* II, Ch. III, §15, p. 307)

Consequently, the appeal to intuition-vision of signs has, in Wittgenstein's writings, a role completely different from that which it plays in Hilbert's. Whereas the latter, considering signs as the concrete objects which finitary statements of the intuitive theory of numbers deal with, uses the word "vision" in its literal meaning, Wittgenstein gives it an eminently metaphorical meaning: as in the *Tractatus,* visual perception is the model of the relationship between the speaker and the domain of the acknowledged necessary connections in language. And, since a construction within the arithmetic of strokes enlarges or modifies this "grammatical knowledge", the acknowledgement of the rule expressed, though not explicitly, by its result is characterized in terms of vision of certain aspects of symbolism.

Hitherto, only the role attributed by the intermediate Wittgenstein to a proven mathematical "proposition", together with the elements of continuity of the conception of this period with that of the *Tractatus,* has been stressed. It goes without saying that this preliminary short account gives rise to more problems than it solves. First of all, Wittgenstein must face the problem of the relation between a mathematical theorem, on one hand, and its proof, on the other. Even conceding that a theorem is a disguised metalinguistic expression of a grammar rule, can one say that its proof compels us to adopt it? Of course, in answering this question, the relation between the sense of the sentence which expresses the theorem and the meanings which the words occurring in it have *before* the proof is given (in their occurrences in the premises from which the theorem is deduced – for instance, definitions) has to be taken into account. The analysis of this typically Wittgensteinian theme will show that, in the intermediate phase, the continuity with the *Tractatus* is actually weakened by the irruption of that notion of *mathematical proposition,* which had been expressly banished in the early work.

MATHEMATICAL PROPOSITIONS

The conception of primitive numerical signs as paradigms, expounded in the previous section, actually expresses only one aspect of Wittgenstein's anti-formalism in the intermediate period. A further aspect of this stance is contained in the following thesis, which is usually referred to in the presentation of his positions of those years: a sign belonging to the language of a branch of mathematics, far from being a mere physical entity, has a meaning inasmuch as it is used according to certain definite rules of calculation. Then, inevitably, the case of chess pieces is put forward: their meaning – the only relevant thing to the purpose of the game – is completely contained in the rules determining their correct use. It is beyond dispute that Wittgenstein maintained this point, and moreover, it is undeniable that he viewed it as a valid alternative to formalism (and, at the same time, to the contentuistic conception of mathematics, in the generic sense of an interpretation that assigns mathematical statements a descriptive function). The textual evidence in favour of this is impressive. In one of the conversations recorded by Waismann, Wittgenstein says:

> In Cambridge I have been asked whether I believe that mathematics is about strokes of ink on paper. To this I reply that it is so just the sense in which chess is about wooden figures. For chess does not consist in pushing wooden figures on wood. . . . It does not matter what a pawn

> looks like. It is rather the totality of rules of a game that yields the logical position of a pawn. A pawn is a variable, just like "x" in logic.... For Frege the alternative was this: either we deal with strokes of ink on paper or these strokes of ink are signs of *something* and their meaning is what they go proxy for. The game of chess itself shows that these alternatives are wrongly conceived – although it is not the wooden chessmen we are dealing with, these figures do not go proxy for anything, they have no meaning in Frege's sense. There is still a third possibility, the signs can be used the way they are in the game.... If we construct a figure in geometry, once again we are not dealing with lines on paper. The pencil-strokes are the same thing as the signs in arithmetic and the chessmen in chess. The essential thing is the rules that hold of those structures.
>
> (*WVC,* pp. 103–5)

The same ideas are also expressed in *Philosophical Remarks*: "The system of rules determining a calculus determines the 'meaning' [*Bedeutung*] of its signs too. Put more strictly: The form and the rules of syntax are equivalent" (*PR* §152); "only the group of rules defines the sense [*Sinn*] of our signs, and any alteration (e.g. supplementation) of the rules means an alteration of the sense" (*PR* §154). The first thing to be remarked about these claims is that it is rather queer that Wittgenstein sees them as a radical alternative to formalism. The central thesis – that the meaning of the signs of mathematical language, just like that of chess pieces, is determined by the rules governing their use in calculations – can be found almost word for word in the outline of formalism given by J. Thomae and criticized by Frege. In a passage by Thomae, quoted by the German logician and very probably known to Wittgenstein, one can read:

> For the formalist, arithmetic is a game with signs, which are called empty. That means they have no other content (in the calculating game) than they are assigned by their behaviour with respect to certain rules of combination (rules of the game). The chess player makes similar use of his pieces; he assigns them certain properties determining their behaviour in the game, and the pieces are only the external signs of this behaviour.[14]

In his attempt to distance himself from formalism, Wittgenstein seems not to be able to do anything better than propose an out-dated version of formalism itself. Nonetheless, in my opinion, the truly interesting point, brought out by his proposal, is a different one: it shows that, in his intermediate phase, Wittgenstein exploits a not completely problematic notion of sign transformation rule, or, more precisely, a notion still *partially* safe from the attacks he launches a short while afterwards. In confirmation of this, it can be observed that Wittgenstein uses it as the basis of one of the distinguishing hallmarks of his philosophy of mathematics during this period: i.e. his aforementioned attempt to accommodate the notion of meaningful mathematical *proposition*.

Consider a concrete instance of mathematical system, or calculus, often discussed by Wittgenstein for illustrative purpose: the method of multiplication of two natural numbers in decimal notation. It can be seen as being constituted by a series of particular numerical identities (those which one learns through multiplication tables), and by a certain set of general rules for the calculation of the product of any two given numbers, written in decimal notation. According to Wittgenstein, the meaning of the variable "$\xi \times \eta$ (where "ξ" and "η" are schematic letters for numerical expressions in decimal notation) is laid down by the introduction of this procedure of calculation. The meaning of every arithmetical term belonging to the whole system of expressions having the form shown by the variable "$\xi \times \eta$" is established with reference to that one general procedure. Any definite description obtained by replacement of "ξ" and "η" in the schema "the product of ξ times η" with two numerical expressions means, indeed, the outcome of the correct application of *that* method to the two numbers, where the method is referred to by means of a general formulation of its operational rules. In Wittgenstein's words: "it is this method that settles the meaning [*Bedeutung*] of '$\xi \times \eta$' and so settles *what* is proved [proving, for example, a numerical identity like '25 × 25 = 625']" (*PG,* II, Ch. VI, §36, p. 345).[15] In a 1932–3 lecture, he explains thoroughly in what the determination of the meaning of the variable "$\xi \times \eta$" by the formulation of the general rules for calculating the product of two numbers consists in:

> Compare this with being taught to multiply. Were we taught all the results or weren't we? We may not have been taught to do 61 × 175, but we do it according to the rule which we have been taught. Once the rule is known, a new instance is worked out easily. We are not given all the multiplications in the enumerative sense, but we are given *all* in one sense: any multiplication can be carried out according to rule. Given the law for multiplying, any multiplication can be done.
>
> (*AWL,* p. 8)

The meaning of a general mathematical term of such a kind (schematic descriptions and, as we shall see, certain predicates too) *transcends* the set of the sign figures acknowledged – in any given moment – as resulting from correct applications of the method. What the schema "the product of ξ times η" means can be explained in general terms and the concept of multiplication so introduced carries out a normative function when one calculates what number is identified by the definite description obtained by replacement of the schematic letters with two specific numerals. Indeed, the judgement that, in a given sign construction, exactly *those* rules are followed, and that they are *correctly* applied, is grounded on the availability of the concept (though framed in general terms, its definition establishes what operations *must* be performed and what counts as a correct step). A strictly analogous consideration holds for an arithmetical predicate such as "prime". According to Wittgenstein, its meaning is established by supplying the general procedure of calculation, by whose application the presence of the property expressed by the predicate can be checked in any given case: "The concept 'prime number' is the general form of investigation of a number for the relevant property; the concept 'composite' is the general form of investigation for divisibility etc." (*PR* §161); "You could call '=5', 'divisible by 5', 'not divisible by 5', 'prime', arithmetical predicates and say: arithmetical predicates always correspond to the application of a definite, generally defined, method" (*PR* §204). The assumption that the meaning of an arithmetical predicate can be given by means of a definition which employs more general predicates whose meanings are known – though apparently obvious – is, within the overall framework of Wittgenstein's conception, extremely telling. Take a statement such as "11,003 is prime". Once this is proven, a grammar rule, excluding as senseless certain empirical descriptions of the form "such-and-such outcome has been obtained by a correct application of the decision procedure for the property of being prime" – and thus providing a new criterion for correctness of the operations of dividing 11,003 –, is adopted. Then, the attribution to the predicate "prime" of a meaning transcending the extension acknowledged up to the moment of the proof amounts to the assumption that the *general rules* of the method of checking the property are able normatively to condition the process leading to the adoption of such a *particular rule*. In other words, the meaning that the predicate has, *before* any given application of the procedure of calculation definitionally associated with it, would be able to impose rigid constraints on the decision by which a given sign construction is ratified as the sort of construction that *must* be obtained whenever the method is correctly applied (and so, on the decision of adopting a definite grammar rule and not a different one). For this reason, when a certain result is obtained, the acceptance of the rule corresponding to it seems to be forced by the recognition of *that* necessary connection whose existence could be conjectured, in the general terms in which the decision procedure is framed, before the latter was applied. In the case of a mathematical concept associated with a procedure of calculation, the relation between the general and the particular – though involving only the meanings of certain linguistic expressions – is like that which holds for material concepts (of course, the understanding of the meaning of a predicate expressing a concept of the latter sort does not coincide with the knowledge of the ratified cases of its application, and thus the question whether a given object falls under the concept is genuine).[16] It must be noticed that general mathematical terms of this kind are really in an exceptional situation compared with any other general formal term, namely with any other general term such that the assertion that it applies to a given object is nothing but a disguised expression of a grammar rule. In fact, that certain signs express, to some extent, more than we actually mean by them, or, in other words, the existence of unacknowledged necessary connections, is thus admitted: we understand the meaning of "11,003" but it may be that this number is prime without this being known by us, whereas, for example, we cannot understand the meaning of "yellow" without knowing that yellow is lighter than blue. In the very restricted sense in which the "direct perception" of a necessary connection has not *yet* taken place because a procedure of calculation on hand has not *yet* been applied, one can say that a certain number is the result of a given operation, or that it has the property expressed by an arithmetical predicate such as "prime", even if one does not know how things really stand: "We may only put a question in mathematics (or make a conjecture), where the answer runs: 'I must work it out'" (*PR* §151).

With the conception of the meaning of mathematical terms defined by reference to a general method of calculation, we have arrived on the threshold of Wittgenstein's verificationism. A preliminary remark should be made at this point. In his intermediate phase, the Austrian philosopher is fully aware of the fact that the word "proposition", even considering only its use within mathematics, denotes a myriad of grammatically distinct structures, perhaps related one to the other by some loose analogy, and not different kinds of one and the same omnicomprehensive grammatical category. Numerical identities such as "2+3 = 5", algebraic laws such as "a+b = b+a", number-theoretical conjectures such as Golbach's conjecture, etc., are all called "propositions" (and the same holds true for terms such as "proof", "existence", etc.).[17] In effect, a conspicuous portion of Wittgenstein's analyses in the writings of the intermediate phase seeks to throw light on these grammatical distinctions. But, at the same time, there is an undeniable attempt to give a general characterization of the notion of proposition, i.e. a sharp delimitation of the use of the word "proposition", clearly revealing a verificationist theory of meaning. These two attititudes are not contradictory: by the acknowledgement of the grammatical variety of uses of that word and by the simultaneous proposal of a definite limitation of its employment, Wittgenstein wishes to remove from the scope of what can be meaningfully stated – of what can be subject of genuine knowledge – *a large part* of the domain of necessary connections. It is once again the old goal achieved in the *Tractatus* through the distinction between saying and showing: in the intermediate phase too, the majority of the results of that activity of stipulation of grammar rules which, in Wittgenstein's view, is carried out in mathematics is banished from the domain of what can be legitimately expressed in propositions. But at the same time, and rather unexpectedly, a weakening of the clear-cut separation between expression of a sense and "vision" of an internal connection springs from the blending of verificationism and his theory of concept-formation by reference to general calculating procedures: the notion of mathematical proposition bridges the (Wittgensteinian) gap between these two very distant worlds.

All the data of the problem are to be found in §148 of *Philosophical Remarks*. Here Wittgenstein states his verification principle of the sense of mathematical propositions:

> We might also ask: what is it that goes on when, while we've as yet no idea how a certain proposition is to be proved, we still ask "Can it be proved or not?" and proceed to look for a proof? If we "try to prove it", what do we do? Is this a search which is essentially unsystematic, and therefore strictly speaking *not a search at all,* or can there be some plan involved? How we answer this question is a pointer as to whether the as yet unproved – or as yet unprovable – proposition is senseless or not. For, in a very important sense, every significant proposition must teach us through its sense how [*wie*] we are to convince ourselves whether it is true or false. "Every proposition says what is the case if it is true."

The quasi-quotation from *Tractatus* 4.022 puts the seal on the formulation of a strong version of the verification principle: the sense of a proposition is the way in which its truth-value can be settled. Namely, whoever understands a proposition does not merely know, in general terms, the form that any proof of the proposition should have (a form depending on the logical structure of the proposition). A much more specific knowledge is needed: one which manifests itself overtly in the ability of giving a description, *again in general terms,* of the sign operations by whose application the proposition can be decided. However, some clarifications about Wittgenstein's employment of the verification principle within mathematics are called for. In saying that a mathematical proposition shows, to everyone who understands it, how things stand if it is true, one runs the risk of suggesting that the understanding of such a proposition is bound up with the knowledge of a possible ideal state of affairs, whose existence would verify the proposition. Nor, not to fall in this trap, is it enough to maintain, as Wittgenstein does in the passage quoted, that the understanding of the sense amounts to the knowledge of a decision procedure. This thesis has to be accompanied by a crucial specification about the nature of truth and falsity in mathematics: as it concerns mathematical propositions, the predicate "true" is synonymous of the predicate "provable", and the predicate "false" of "refutable". This is what Wittgenstein observes in the passage of §148 of *Philosophical Remarks,* immediately following that quoted above: "And with a mathematical proposition this 'what is the case' must refer to the way [*die Art und Weise*] in which it is to be proved". Using a well proven terminology, one could say that, according to Wittgenstein, the understanding of the sense of a proposition consists in the knowledge of its assertibility conditions, interpreted as knowledge of the method by whose application a proof or a refutation can be obtained. But, in presenting things in this usual way, one has not to forget that, within the framework of Wittgenstein's quasi-formalistic approach, a proof is not the instrument whereby properties either of mind-independent or of mind-constructed objects are discovered. Proof is a sign figure which produces the adoption of grammar rules about what, by definition, is to count as the result of the correct performance of the operations of a given calculus in a particular case. Nonetheless, the extension of the verificationist notion of proposition to the domain of mathematics has to preserve the following basic principle: a gap must exist between the understanding of the sense of a proposition and the knowledge of its truth-value; a gap that can be filled only by an (eventual) application of the decision procedure. As in mathematics the ascertainment of the truth of a proposition is the construction of a proof, and the ascertainment of its falsity is the construction of a refutation, to preserve this pivotal principle means to make a sharp distinction between understanding the sense of a mathematical proposition and knowing its proof (or its refutation). Wittgenstein himself states the problem explicitly: "My explanation mustn't wipe out the existence of mathematical problems. That is to say, it isn't as if it were only certain that a mathematical proposition made sense when it (or its opposite) had been proved" (*PR* §148); "We come back to the question: In what sense can we *assert* a mathematical proposition? That is: what *would* mean nothing would be to say that I can only assert it if it's correct. – No, to be able to make an assertion, I must do so with reference to its sense, not its truth" (*PR* §150). Taking into account that a meaningful proposition is what can be given in answer to a meaningful question, one can recognize the same request in the following passage from *Grammar*: "Where there is no method of looking for an answer, there the question too cannot have any sense. – Only where there is a method of solution is there a question (*of course that doesn't mean: 'only where the solution has been found is there a question'*)" (*PG*, II, Ch. V, §25, p. 377, my italics). The application of a method of calculation to decide a mathematical proposition will have to solve a genuine doubt, i.e. will have to determine which horn of the dilemma holds: "We can't say 'I will work out *that* it is so', we have to say '*whether* it is so', i.e., whether it is *so* or otherwise" (*PG,* II, Ch. V, §23, p. 366).

But where, within mathematics, do we find the verificationist conditions for the appropriate use of the word "proposition" to be satisfied? The answer to this question is to be found in Wittgenstein's notion of a mathematical concept definitionally associated to a general procedure of calculation. The sense of any one proposition, belonging to the system of the instances of the schematic expression "ξ is a prime number", can be fully explained in terms of the meaning of decimal notation and of the meaning of the formulation of the check-method for the property of being a prime number. Thus, it is not the actual application of the method, namely, the actual construction of a proof, that gives sense to any particular proposition of the system; the sense of any such proposition can be understood without its truth-value being known. In the course of a comparison between the status of a simple arithmetical identity and that of a true mathematical theorem such as the fundamental law of algebra, Wittgenstein says:

> the proposition 26 × 13 is essentially one of a system of propositions (the system given in the formula $a \times b = c$), and the corresponding question one of a system of questions. The question whether 26 × 13 equals 419 is bound up with one particular *general method* by means of which it is answered.... [The fundamental law of algebra] seems to get its sense from the proof, while the propositions stating what the product in a multiplication is do not.... In the case of the question about the product of 26 and 13, there is something about it which makes it look like an empirical question. Suppose I ask whether there is a man in the garden. I could describe beforehand a complicated way of finding out whether there is or not. There is a resemblance of the multiplication question to this one, in that before you find out I could tell you how to find out.
>
> (*AWL,* pp. 197–8)[18]

Conversely, only when the sense of a proposition is not generated by its proof but by its being a member of a whole system of propositions, each one decidable by applying one and the same general calculating procedure, is it permissible to say that, by an application of the procedure, *that proposition is proven* (the proof does not determine what has been proven, but settles the truth-value of a proposition which has its own sense before the proof is given). This happens, for example, with numerical identities stating the outcome of the multiplication of two numbers in decimal notation. In Wittgenstein's words: "Hence I can only say '25 × 25 = 625 is proved' if the method of proof is fixed independently of the specific proof. For it is this method that settles the meaning of '$\xi \times \eta$', and so settles *what* is proved" (*PG,* II, Ch. VI, §36, p. 435). Then, the *Art und Weise* of the proof – with which knowledge the understanding of the sense of a given mathematical proposition is identified by Wittgenstein – is nothing but the general method for checking every proposition of the system to which the proposition in question belongs:

> So if I want to raise a question which won't depend on the truth of the proposition, I have to speak of *checking* its truth, not of proving or disproving it. The method of checking the truth corresponds to what one may call the sense of the mathematical proposition. The description of this method is a general one and brings in a system of propositions, for instance of propositions of the form a × b = c. ... If it's impossible to speak of such a check, then the analogy between "mathematical proposition" and the other things we call propositions collapses.
>
> (*PG,* II, Ch. V, §23, p. 366)

The verification principle imposes two other crucial requisites on the notion Wof proposition: (i) the negation of a meaningful proposition must also be a meaningful proposition; (ii) a proposition must be an element of the classical calculus of truth-functions, and, in particular, the Law of Excluded Middle must hold for every expression of mathematical language to which the status of proposition is assigned. Let us begin with point (i), which, within the overall frame of Wittgenstein's conception of mathematics, raises the greatest difficulty. In order to satisfy this, the application of the decision procedure must solve a meaningful doubt (that 11,003 is prime or composed; that 25 × 85 is equal to 1,825 or different from it; etc.); i.e., the alternative to the fact that a given object falls under the concept must not be ruled out by the *primitive definitions* of the calculus (if it is to be ruled out, this will be *a derived rule*). For this reason – assuming that one operates within the arithmetic of naturals – the phrase "11,003 is a natural number" is not a mathematical proposition, whereas "11,003 is a prime number" is. But Wittgenstein strongly feels the most urgent difficulty with regard to the negation of a mathematical proposition and expresses it overtly as follows: "Negation in arithmetic cannot be the same as the negation of a proposition, since otherwise, in 2 × 2 ≠ 5 I should have to make myself a picture of how it would be for 2 × 2 to be 5" (*PR* §203). This difficulty becomes fatal if the understanding of the sense of a proposition requires the acquaintance with a proof of the proposition. In fact, a grammar rule that excludes calling "a correct performance of the multiplication of 2 times 2" any given sign construction having "5" as its end result, is drawn from the proof of the inequality "2 × 2 ≠ 5". By this proof, the conceivability of the sort of sign figure in which the proof of the false proposition "2 × 2 = 5" should consist in is indeed ruled out. If the existence of a proof were the condition of the meaningfulness of a proposition, and if the understanding of its sense were identified with the acquaintance with such a proof, then one of either a proposition or its contradictory would be condemned to senselessness. On the contrary, if the understanding of a proposition is bound not to the knowledge of the sequence of formulae in which a proof would consist, but only to the knowledge of the general decision procedure for all the propositions of the same system, then such a proposition can be both asserted and negated meaningfully. In discussing how one could have sought for the trisection of the angle by ruler and compass, this construction being logically impossible, Wittgenstein says:

> But the same paradox would arise if we asked "is 25 × 25 = 620?"; for after all it's *logically* impossible that that equation should be correct; I certainly can't describe what it would be like if ... – Well, a doubt whether 25 × 25 = 620 (or whether it = 625) has no more no less sense than the method of checking gives it. It is quite correct that we don't here imagine, or describe, what it is like for 25 × 25 to be 620.
>
> (*PG*, II, Ch. V, §27, p. 392)

It is the conceivability of an error in the application of the calculating method by which a new rule is derived from the primitive rules of the calculus that gives sense also to a false proposition and, in particular, to the negation of a proven mathematical proposition: "Whether an expression has sense depends upon the calculus. I can imagine the kind of mistake which could lead one to say 26 × 13 = 1560, or that 4 is the first digit of π, and thus I could say that the corresponding questions about them are genuine" (*AWL,* p. 200); "What is the contradictory of what is proved? – For that you must look at the proof. We can say that the contradictory of a proved proposition is what would have been proved instead of it if a particular miscalculation had been made in the proof.... the negation is an exclusion within a predetermined system. *I can't negate a definition as I can negate an equation derived by rules*" (*PG,* II, Ch. V, §24, p. 373, my italics). Once this argument has been accepted, the claim in point (ii) is easily justified. By one and the same application of the decision procedure, a proposition can be proven and, at the same time, its negation can be refuted (or, alternatively, the proposition can be refuted and, simultaneously, its negation can be proven). The validity of all the laws of classical logic is a simple immediate consequence of the restriction of the range of mathematical propositions to the domain of the expressions that can be used to make assertions decidable, in principle, by applying an available general method of calculation.[19]

Within the boundaries of strong verificationism, the relation between sense and proof of a mathematical proposition and the relation between sense and verification of an empirical proposition are so like that the *qualified* use of the word "proposition" is justified. The situation changes radically as soon as one crosses these boundaries, i.e. as soon as one considers the part of mathematical activity that is not reducible to the application of known general methods of calculation. In the new setting, the notion of the mathematical proof as the means for deciding a proposition, which, however, has a sense of its own, vanishes, and the notion of the proof as the only source of the sense of the theorem, apparently so implausible, replaces the former. The reason for this change is Wittgenstein's peculiar conception of the relation between the general and the particular in mathematics (in grammar). In the case of a predicate such as "prime", and in all similar cases, the meaning of the formulation of the method of calculation, definitionally associated with the predicate, imposes strict constraints on the acknowledgement of a given sign figure as the result of the correct application of the general method in a particular case (and thus on the adoption of the grammar rule drawn from the proof, which lays down what kind of sign construction has to be considered, by definition, as the result of a correct application of the operational rules of the procedure). For, basing oneself upon the meaning attributed to the formulation of these general rules, one knows what rules have to be followed, what conditions have to be satisfied in order that the predicate can be applied and what steps have to be considered as in compliance with the rules. Beyond the verificationist boundaries, things stand otherwise: no definition of a mathematical predicate by means of more general predicates, no formulation – however precise it may be – of rules of calculation or inference, is able to carry out a normative function on the decisions on what is to count, in a new case, as a correct application of the defining predicates and the operational rules (on the decisions of adopting certain grammar rules of the language whereby the relevant sign manipulations are described). The content of the rule of the geometry of signs, expressed covertly by the theorem, is determined by the proof: this directly shows the sort of sign transformations which are considered, by definition, as correct performances of the operations involved, *whatever* the starting definitions of the proof-process are.[20] In my opinion, this is the true antecedent of Wittgenstein's later considerations on rule-following: these will extend to the whole field of mathematics, *with no exceptions whatsoever,* the intermediate Wittgenstein's view of the relation between the general and the particular in grammar. For Wittgenstein's theses will entail that the meaning assigned, in any given moment, to the formulation of a *general* rule has no normative power on the successive linguistic decisions on what is to count, by definition, as the result of the correct application of the rule in each new *particular* case.[21]

An example often dealt with by Wittgenstein, which vividly illustrates the effects of the jump beyond the boundaries of verificationism, is provided by numerical specifications within mathematics. In order to be meaningful regardless of the existence of a proof, such a specification should belong to a whole system of statements of the same kind, which are decidable through the application of a general calculating procedure. This is the verificationist condition to be satisfied in order that the meaning of an instance of a description schema such as "the number of the Γ's" can be understood, where this number is not known. A portion of the grammar of an expression which can appropriately substitute the schematic letter "Γ" may be unknown, in the restricted sense that the available method of calculation – by whose application the gap in knowledge can, in principle, be filled – has not yet been applied. Let us see what happens when the verificationist conditions are not satisfied. Of the variable "pure colour", Wittgenstein says: "if [someone] says that for him there are 5 pure colours, in that case we don't understand him, or must suppose we completely misunderstand one another. This number is demarcated in dictionaries and grammars and not within language" (*PR* §114). Obviously, by this he is denying that the number of pure colours can be stated meaningfully. In *Philosophical Remarks,* the distinction between numerical specifications concerning material concepts and numerical specifications concerning variables is framed in terms of the *Tractarian* opposition between what can be said by means of a proposition and what shows itself: "What distinguishes a statement of number about the extension of a concept from one about the range of a variable? The first is a proposition, the second not. For the statement of number about a variable can be derived from the variable itself. (It must show itself.)" (*PR* §113). A few years later, the same distinction is restated more explicitly by Wittgenstein: "What distinguishes a statement of number about a concept from one about a variable? The first is a proposition about the concept, the second a grammatical rule concerning the variable" (*PG,* II, Ch. IV, §20, p. 348). It is expedient to make a detailed analysis of an instance of numerical specification within mathematics, dealt with by Wittgenstein: namely, the attribution of the number six to the formal concept expressed by the variable "permutation of a three-element set" (the statement "there are six permutations of a three-element set").[22] Firstly, Wittgenstein stresses that, contrarily to numerical specifications such as "there are six men in the room", i.e. to the attribution of a number to a material concept, numerical specifications within mathematics are not *general* statements at all. The justification of this claim runs as follows: the concept expressed by the predicate "man in the room" is defined by its distinguishing marks, not by the list of the men who are actually in the room; therefore, the proposition "there are six men in the room" is not logically equivalent to any conjunction of singular propositions in which six specified individuals are mentioned, and is correctly expressed by a generalized proposition in which a string of six different existentially quantified variables occurs (according to the modified Fregeian model presented in the previous section). On the other hand, if the variable "permutation of a three-element set" (a set generically denoted by "{A, B, *C*}") means the same as the disjunctive variable "$= ABC \vee = ACB \vee = BAC \vee = BCA \vee = CAB \vee = CBA$", then the proposition "there are six permutations of a three-element set" is logically equivalent, in virtue of this grammar rule, to the conjunction "*ABC* is a permutation of the set {*A*, *B, C*} and *ACB* is a permutation of the set {*A, B, C*} and ... and *CBA* is a permutation of the set {*A, B, C*}" (where the dots are simply dots of laziness). The difference between the two situations is shown by the circumstance that only by the addition of the further (factual) premise that **a, b, c, d, e** and **f** are all the men in the room can the generalized proposition "there are six men in the room" be inferred by the conjunction "**a** is in the room and **b** is in the room and ... and f is in the room" (and vice versa). For this reason Wittgenstein is able to affirm: "To say that 6 permutations of 3 elements are possible cannot say less, i.e. anything more general, than is shown by the schema:

A B C
A C B
B A C
B C A
C A B
C B A

. . . . The proposition that there are 6 permutations of 3 elements is identical with the permutation schema ..." (*PG,* II, Ch. IV, §20, pp. 348–9). But the construction of this schema can be considered as nothing but the construction of a proof of the statement that there are six permutations of a three-element set; thus, we have here a case in which the proof determines the sense of the proven proposition. The implicit presupposition of Wittgenstein's claim is that the definition of a formal predicate such as "permutation of a three-element set" by means of more general formal predicates, whose meanings are assumed known, cannot play the role usually attributed to it. According to Wittgenstein, except in the case of mathematical concepts definitionally associated with general procedures of calculation, a formal concept can be identified only through the list of the objects which, by definition, fall under it. But this appears to be really an odd idea, which is plainly contrary to ordinary mathematical practice. Nothing seems to be more natural, for example, than defining the formal predicate "permutation of a three-element set" by employing as its distinguishing marks such more general formal concepts as that of a three-element set, of an injective function from a set to a set, of a surjective function from a set to a set, etc. Following this routine, a numerical specification such as "there are six permutations of a three-element set" would have the same logical form as the generalized empirical proposition "there are six men in the room" (ininfluent details apart). Were the definition understood, and were a list of triplets of elements *A, B* and *C* given, anyone would be able to count them, in the same way in which anyone could count the apples on a table. Of course, the conclusion that there are six permutations of the set {*A, B, C*} could be eventually drawn only if the listed triplets are acknowledged as *all and only all* the permutations of the set. It is at this step of the argument that Wittgenstein's "odd" idea enters. Whether the observed triplets exhaust the set of all such permutations depends, obviously, on what triplets are acknowledged as permutations of the generic set {*A, B, C*}, namely, on the meaning of the predicate "permutation of the set {*A, B, C*}". But, in considering the simpler case of the permutations of a two-element set, Wittgenstein says: "What I see in *AB, BA* is an internal relation which therefore cannot be described. That is, *what* cannot be described is that which makes this class of permutations complete" (*PG,* II, Ch. IV, §20, p. 349). Plainly, he presupposes that a definition of the formal predicate "permutation of a two-element set", given *before* the counting is worked out, has no normative power in the determination of the units to be counted (whereas what makes counting the elements of a set an empirical process is precisely the fact that it rests on a previous, stable identification, in grammar, of what counts as a unit to be counted). For this reason, according to Wittgenstein, the determination of the number of the elements of the extension of a formal predicate and the determination of its meaning are one and the same grammatical process: by the statement that there are six permutations of a three-element set, such permutations are *distinguished,* as are the cases in grammar, *not counted* (and the same holds true for the algebraic theorem that there are two roots of a second degree equation, etc.).

The general combinatorial theorem "$P(n) = n!$" having been proven, the logical situation of the statements about the number of permutations of finite sets changes radically. Before this proof is on hand, the content of the rule "there are six permutations of a three-element set" is exhibited by the sign construction which demonstrates it. Once the theorem "$P(n)= n!$" has been established, a general method of calculation is correlated to the schematic description "the number of permutations of an **n**-element set". Every proposition of the form "the number of permutations of an **n**-element set is equal to **m**" can be decided by an application of the calculating procedure. Then, the conditions that strong verificationism imposes on the legitimate use of the pair of expressions "meaningful proposition"/"meaningful problem" are satisfied and Wittgenstein is able to conclude: "It is clear that there is a mathematical question: 'How many permutations of – say – 4 elements are there?, a question of precisely the same kind as 'What is 25 × 18?'. For in both cases there is a general method of solution. But still it is only with respect to this method that this question exists" (*PG,* II, Ch. IV, §20, p. 349).

Of course, the area of mathematics in which, in compliance with the verification principle, it is legitimate to speak of meaningful unproven mathematical propositions, meaningful mathematical hypotheses and unsolved problems is very narrow. In a passage of *Philosophical Remarks,* already partially quoted, Wittgenstein observes: "We may only put a question in mathematics (or make a conjecture) where the answer runs 'I must work it out'. ... What 'mathematical questions' share with genuine questions is simply that they can be answered" (*PR* §151). Also in *Grammar,* Wittgenstein points to the limitations induced by his adhesion to the strong form of verificationism:

> Tell me *how* you seek and I will tell you *what* you are seeking. ... Where you can ask you can look for an answer, and where you cannot look for an answer you cannot ask either. Nor can you find an answer. ... "the equation yields S" means: if I transform the equation in accordance with certain rules, I get S. Just as the equation 25 × 25 = 625 says that I get 625 if I apply the rules for multiplication to 25 × 25. But in this case these rules must already be given to me before the word "yields" has a meaning, and before the question whether the equation yields S has a sense.
>
> (*PG,* II, Ch. V, §24, p. 370; §25, pp. 377–8)

Only within the narrow part of mathematical practice that consists in solving problems by applications of known general methods of calculation does the relation between sense and proof of a mathematical proposition appear in the usual guise. But Wittgenstein himself signals that mathematical activity cannot be reduced to this sort of school practice of doing exercises and that, on the contrary, the really interesting part of doing mathematics – the activity to which professional mathematicians are engaged – has little in common with the performance of the operations of calculation needed for checking mathematical propositions, in the verificationist meaning of the word "proposition":

> One could lay down: "whatever one can tackle [*anfassen*] is a problem. – Only where there can be a problem, can something be asserted". Wouldn't all this lead to the paradox that there are no difficult problems in mathematics, since if anything is difficult it isn't a problem? What follows is, that the "difficult matematical problems", i.e. the problems for mathematical research, aren't in the same relationship to the problem "25 × 25 = ?" as a feat of acrobatics is to a simple somersault. They aren't related, that is, just as very easy to very difficult; they are "problems" in different meanings of the word.
>
> (*PG*, II, Ch. V, §25, pp. 379–80; *PR* §151)

In this connection, Wittgenstein's preliminary clarification involves the grammar of the verbs "to look for" and "to discover". In general, it is correct to describe the activity of a certain individual *X* as an activity of looking for the object which he identifies as **A** or as a **B** (where "**A**" and "**B**" are, respectively, a schematic letter for singular terms and a schematic letter for predicates), only if *X* knows what he is looking for and the way the search has to be conducted. The object looked for must be a logically possible object, i.e. the singular term "**A**" or the general term "**B**" has to be a meaningful term of *X*'s language. With this, obviously, the object looked for is not assumed to exist, but to occupy a point of the logical space in which *X* is immersed. But this minimal condition is not sufficient: according to Wittgenstein, even if *X* knows what **A** is or what a **B** is, and therefore is logically able to recognize **A** or to recognize an instance of **B**, whenever he meets it, we would not say that he is looking for **A** or for a **B**, unless he is acting systematically with a view to finding **A** or a **B**. The availability of a search-method is a grammatical requisite to use "looking for" correctly in language; and when the knowledge of the meaning of the singular term denoting the object looked for (and, correspondingly, the knowledge of the meaning of the general term an instance of whose extension is looked for) coincides with the knowledge of the method to find it (or, respectively, to find one instance), the two mentioned conditions merge into a verificationist conception of looking for. A similar point is true of the use of the verb "to look for" in a linguistic context such as that provided by any instance of the schematic expression "to look for the solution of the problem whether it is the case that **p**" (where "**p**" is a schematic letter for declarative sentences). The activity of *X* can be characterized as a looking for the solution of a problem of such a kind, i.e. as an activity directed to the aim of ascertaining whether the state of affairs described by a given sentence exists, only when the sentence has a meaning in the language of X and X is acting systematically in order to determine whether the sentence is true. And here, too, the two conditions merge into a single one, if, in compliance with the verification principle, the understanding of the sense of the sentence coincides with the knowledge of a method to decide it.[23] In mathematics too it is correct to speak of looking for an object or for the solution of the problem whether an object has, or has not, a certain property, only within the boundaries established by the verification principle. Whoever masters a decision procedure, definitionally correlated to an arithmetical predicate "**Q**" (or to a description schema "**D**"), knows what is a **Q** and, whenever a finite interval is fixed, the calculations by which a **Q** (and, correspondingly, the object identified by an instance of the schematic description "**D**") can be found. Of such an individual one can legitimately say that he is looking for a **Q** (or for the object identified by the description), or for the solution of the problem of whether a given object has the property expressed by "**Q**". As it concerns the verb "to discover", symmetrical restrictions hold:

> What is hidden must be capable of being found.... Also, what is hidden must be completely describable before it is found, no less than if it had already been found. It makes good sense to say that an object is so well hidden that it is impossible to find it; but of course the impossibility here is not a logical one; i.e. it makes *sense* to speak of finding an object, to describe the finding; we are merely denying that it will happen.
>
> (*PG,* II, Ch. V, §22, p. 363)

"There is no meaning to saying you can describe beforehand what a solution will be like in mathematics *except in the cases where there is a known method of solution*" (*AWL,* p. 7, my italics).[24] In conclusion, also in mathematics one can say that an object with a certain property, or the solution of the problem of whether a given object has a certain property, can be *found out,* only when this object – or this solution – can be *looked for,* in the narrow sense in which, from a verificationist point of view, one can speak of looking for.

Wittgenstein's thesis that one should not call "mathematical searching" and "mathematical discovery" the activity and its eventual outcome, which are usually referred to by these expressions, seems to be as odd as the idea that, where the verificationist requisites are not satisfied, it is the proof which gives sense to a theorem. And yet this is Wittgenstein's stance: "And 'search' must always mean: search systematically. Meandering about in infinite space on the look-out for a gold ring is no kind of research. You can only *search* within a system: And so there is necessarily something you *can't* search for" (*PR* §150); "In mathematics, we cannot talk of systems in general, but only *within* systems. They are just what we can't talk about. And so, too, what we can't search for" (*PR* §152); and moreover:

> Does it count as looking for something, if I am unaware of Sheffer's system and say I would like to construct a system with only *one* logical constant? No! Systems are certainly not all in one space, so that I could say: there are systems with 3 and with 2 logical constants and now I am trying to reduce the number of constants *in the same way.* There is no "*same way*" here.
>
> (*PG,* II, Ch. V, §22, p. 361)

Wittgenstein's attitude has the same source as his ideas about the relation between sense and proof beyond the boundaries of verificationism. For example, take Sheffer's case: before the "discovery" of his logical constant, could he have meaningfully described *what* he was engaged in looking for? It seems that the answer to this question should be straightforwardly affirmative. Sheffer could have said that he was looking for a logical operation by means of which all the truth-functions of any given finite set of atomic propositions could be generated. The ability of giving beforehand a meaningful description of what one is looking for is founded on a crucial condition: that *precisely what is looked for can be described in general terms*. Wittgenstein's objection to this seemingly natural way of looking at things is that the meaning of the generic description of the object given *before* it is "directly perceived", or the meaning that the expression "in the same way" has *before* the "discovery", is not able to normatively condition the adoption of the grammar rule which makes a certain sign procedure what was looked for. Things stand inversely: it is *the free decision* of counting, by definition, a certain operational procedure as the procedure identified, though in general terms, by the description provided before the "discovery" (or of considering it, by definition, as obtained in the same way as other known procedures), which establishes *ex novo* how the description – or the expression "in the same way" – have to be understood after the "discovery". Wittgenstein's insistence on the completeness of mathematical systems and on the inapplicability to mathematics of the distinction between knowledge by description and knowledge by acquaintance are only other variations on this theme.[25]

Mathematical verificationism of the intermediate phase amounts to the admission that we are not all-seeing in grammar: as in the *Tractatus,* unacknowledged internal connections are accepted, though only in the narrow, purely extensional sense in which the lack of knowledge can be made good, in principle, by the application of a general method of calculation. But, as soon as one leaves the area of mathematics in which genuine propositions can be stated, the principle of *esse est percipi* holds true: nothing shows itself, and thus nothing exists, beyond what we effectively see. Then, we are all-seeing simply because there are no unacknowledged necessary connections: "What I mean could also be expressed in the words: one cannot discover any connections between parts of mathematics or logic that was already there without one knowing" (*PG*, II, Ch. VII, §42, p. 481). In describing the process that, according to him, cannot be properly called "mathematical discovery", Wittgenstein often resorts to the metaphor of seeing. The old contrast between what can be said and can be verified or falsified, on one hand, and the domain of what shows itself and can be only seen, on the other, returns here. Genuine novelties in mathematics cannot be expected as results of a rational activity of solving problems, since this is carried out, by definition, *within* a given system. When a new system is recognized, what really happens is a sort of revelation:

> where we can only expect the solution from some sort of revelation [*Offenbarung*], there isn't even a problem. A revelation doesn't correspond to any question. It would be like wanting to ask about experiences belonging to a sense organ we don't yet possess. Our being given a new sense, I would call revelation. Neither can we *look for a* new sense (sense-perception).
>
> (*PR* §149 and *PG,* II, Ch. V, §25, p. 377)

Every proposition not belonging to a system of propositions which are decidable by the application of known general rules of calculation, as it concerns its sense, hangs in the void. In order to account for the gap between what such a "proposition" says before and what it says after its proof, Wittgenstein resorts to the distinction between calculus and prosa. Before the proof, of the sense of a sentence such as that expressing the fundamental law of algebra or the Euclidean theorem that there are infinite prime numbers, there existed "only a rough pattern ... in the word-language" (*PG,* II, Ch. V, §24, p. 374). The point is that only after a proof has been supplied does one get a piece of mathematics, i.e. the disguised expression of a rule of the geometry of signs, whose content is directly shown by the proof. As the meanings of the words occurring in the sentence to be proven do not impose any constraint on what sign figure will eventually be acknowledged as a proof of the theorem, the demonstration will have to exhibit the sense of the geometrical statement that the correct performance of the involved sign operations must yield a certain outcome (the proof is the paradigm of their correct performance and, therefore, the operations carried out in it are correct by definition):

> We might also put it like this: the completely analysed mathematical proposition is its own proof. Or like this: a mathematical proposition is only the immediately visible surface of a whole body of proof and this surface is the boundary facing us. A mathematical proposition – unlike a genuine proposition – is essentially the last link in a demonstration that renders it visibly right or wrong.... Only on the assumption that there's a body behind the surface, has the proposition any significance for us.
>
> (*PR* §162)

Contrary to what happens with the experimental confirmation of a proposition of physics, or with the application of the decision procedure for a genuine mathematical proposition, the proof "is part of the grammar of the proposition, ... belongs to the *sense* of the proved proposition, i.e. determines that sense. It isn't something that brings it about that we believe a particular proposition, but something that shows us *what* we believe" (*PG,* II, Ch. V, §24, pp. 370, 375). A mathematical conjecture such as Fermat's or Goldbach's conjecture challenges mathematicians not to establish its truth, but to give it a sense. Until a proof is available, a mathematical conjecture is merely an empirical structure (*empirisches Gebilde*) (*PG,* II, Ch. V, §22, p. 362). The inductive evidence in favour of Goldbach's conjecture is completely irrelevant from the mathematical point of view, since it contributes in no way to creating that necessary connection between the concept of even number greater than 2 and the concept of prime number which only a proof will be able to produce.[26] On the other hand, the mathematical construction which demonstrates the universal conjecture certainly does not prove the empirical hypothesis put forward on the basis of the outcomes of the checks carried out hitherto (the hypothesis that, for any given even number greater than 2, the application of the calculating procedure will result in the finding out of two primes whose sum equals the number). In fact, once proven, the theorem will express a rule of the language whereby the performances of the pertaining sign operations are described, not the empirical statement that, in working out the calculations, an even number greater than 2 and not equal to the sum of two primes will never be found. The empirical-extensional process of enumeration of the favourable cases and mathematical proof are usually considered as two different procedures of validation of one and the same universal proposition. But, according to Wittgenstein: "nothing is more fatal to philosophical understanding than the notion of proof and experience as two different but comparable methods of verification" (*PG,* II, Ch. V, §22, p. 361). How generality in mathematics and, specifically, in number theory has to be interpreted in order that this gross error is avoided will be the subject of the next section.

THE MATHEMATICAL INFINITE

Quantifiers in mathematics

In the light of what has been seen in the previous section, we can safely state that, according to Wittgenstein, even in the case of "**P**" being an effectively decidable predicate, the knowledge of its associated check-method, in the absence of a proof, is not sufficient for the understanding of the sense of the universal generalization "$(x)\mathbf{P}(x)$". Goldbach's conjecture, obtained by universal quantification of a predicate of such a sort, is a typical example of an expression which has yet to receive mathematical sense from a proof and which serves only to incite mathematicians towards its construction.[27] One of the premises of Wittgenstein's claim is the rejection of the idea that the meanings of the propositional forms "$(x)\mathbf{F}(x)$" and "$(\exists x)\mathbf{F}(x)$" are supplied by the interpretation of arithmetical universal and existential generalizations in terms of logical product and logical sum of the infinite set of the instances of the schematic expression "$\mathbf{F}(x)$" (where "**F**" is confined to expressions of decidable arithmetical properties). The verificationist orientation lies behind this refusal.[28] Whoever construes "$(\exists x)\mathbf{F}(x)$" as the infinite disjunction "$\mathbf{F}(0) \vee \mathbf{F}(1) \vee \mathbf{F}(2) \vee$ ... and so on *ad infinitum",* and "$(x)\mathbf{F}(x)$" as the infinite conjunction "$\mathbf{F}(0) \geq \mathbf{F}(1) \geq \mathbf{F}(2) \geq$ ... and so on *ad infinitum",* supposes that the meaning of quantifiers can be derived simply by transferring to the new contexts the meaning they have when they range over a finite domain. But, according to Wittgenstein, the differences between the two cases are much more important than their analogies. On the basis of the verification principle, the clarification of the grammar of a proposition involves the answers to questions such as "How is the proposition used? What is regarded as the criterion of its truth? What is its verification?", with the very restrictive clause that "if there is no method provided for deciding whether the proposition is true or false, then it is pointless, and that means senseless" (*PG,* II, Ch. VII, §39, p. 452). If a proposition has sense in so far as it communicates, to everyone who understands it, a method to settle its truth-value, then no generalization on an infinite domain can be dignified as a meaningful proposition and, at the same time, be construed as an infinite conjunction or disjunction. In fact, the only check-procedure which could be associated to such a truth-function would be a step by step examination of any one of the singular decidable propositions forming the infinite set of propositions to which the operation of logical sum or product is applied. But this process, of course, would not be a decision procedure. In conceiving arithmetical generality in terms of logical sum and product, it is typically assumed that the application of the extensional check-procedure is *logically* in order and that the method "cannot be employed, but only because of human weakness" (*PG,* Part II, p. 452; *PR* §124). In this confusion between a logical impossibility and an empirical, biological one – which in Russell's and Ramsey's writings had its most raw expression –, the grammar of the word "infinite" is also the object of a misunderstanding by no means innocuous. This word is treated as if it were the designation of a huge number, with the consequence that the impossibility of completing an infinite series of sign operations is strictly paralleled to the impossibility, perhaps due to lack of time, of carrying out a very large number of such operations. According to Wittgenstein, the difference between the two cases is brought out by one crucial circumstance: whereas, in the case of an empirical impossibility of doing something, a description of an attempt at doing it is meaningful, in the case of a logical impossibility it is also impossible to try to carry it out. In conclusion, it makes no sense to speak of an infinite process of verification of a proposition because, by definition, an infinite process is one that can never be completed: "A proposition that deals with all numbers cannot be thought of as verified by an endless striding, for, if the striding is endless, it does not lead to any goal.... That is to say, the endless path does not have an end 'infinitely far away', it has no end" (*PG,* p. 455; *PR* §123). "**P**" being a decidable predicate, any given finite set of propositions "**P**(0)", "**P**(1)", "**P**(2)", ..., "**P**(**n**)" can, in principle, be checked. But it makes no sense to speak of the checking of their totality, since this process cannot (logically) be completed by proceeding step by step; and this simply amounts to saying that the expression "**P**(0) ≥ **P**(1) ≥ **P**(2) ≥ ... and so on *ad infinitum*", usually considered as equivalent to "(x)**P**(x)", is not a logical product at all.[29]

Wittgenstein's considerations, summarized hitherto, establish only a negative thesis: if quantifiers are extensionally construed, then, on the basis of the verification principle, unproven generalizations on an infinite domain – such as Goldbach's conjecture – are condemned to senselessness. With bounded quantifications of decidable arithmetical predicates, the situation changes. Here a general specification of the meaning of quantifiers can be supplied, which provides generalized propositions with sense even before a proof is given. In fact, the extensional method of checking case by case is an appropriate decision procedure for such a sort of generalization. If the quantifier ranges over the set of natural numbers included in the interval from m to n, then the existential generalization of a decidable predicate "**P**" is synonymous to a finite disjunction, and the universal generalization to a finite conjunction. Whoever understands the meaning of "**P**" – i.e. whoever knows its definitionally associated method of calculation –, and knows under what conditions (whose existence is ascertainable in principle) one is authorized to assert a disjunction or a conjunction, will be able to provide a logically unobjectionable description of the series of operations that must be carried out to settle the truth-value of the (bounded) existential or universal generalization. Thus, the requirements imposed by Wittgenstein's strong verificationism on the use of the expression "mathematical proposition" are satisfied: the sense of generalizations which are constructed by bounded quantification of a decidable predicate is not generated by their proof. In presenting his verificationist thesis that the decision procedure corresponds to the sense of a mathematical proposition, Wittgenstein mentions precisely the "propositions of the form '(∃k)nm ...', and '~(∃k)nm ...', which bring in intervals" (*PG,* Part II, Ch. V, §23, p. 366).

Leaving the field of the finite, where verificationism and extensionalism are easily combined, the problem of the interpretation of generalizations over an infinite domain is still open. Wittgenstein starts with some considerations on the relation between contingency and universality within mathematics. Can one speak meaningfully of a situation in which, by chance, all natural numbers have the property expressed by a decidable predicate

"**P**"? According to Wittgenstein, a positive answer to this question amounts to the admission of the conceivability of a situation of such a sort: the statement that, for every *n,* the singular proposition "**P**(**n**)" is provable would not be inferrable from the knowledge of a certain general formal connection, but would present itself as a mathematically irreducible given, concerning the totality of the outcomes of the application, to every *n*, of the decision procedure for the predicate "**P**". Once it is put in these terms, the description of a supposedly possible situation in which all natural numbers happen by chance to have a certain property is immediately revealed to be senseless. This description entails precisely the conceivability of those infinite applications as forming a completed totality: "The expression 'by chance' indicates a verification by successive tests, and that is contradicted by the fact that we are not speaking of a finite series of numbers" (*PG,* II, Ch. VII, §39, p. 457). The sharp opposition of universality and contingency in arithmetic is founded on the distinction between the existence of a general mathematical result showing the rule according to which, for any given *n*, a proof of "**P**(**n**)" can be constructed, and the mere verification, case by case, of the truth of single propositions "**P**(**n**)". As, by definition, no universal conclusion can be arrived at by means of the latter process, nothing can hold by chance for all numbers: "the proposition 'It's possible – though not necessary – that *p* should hold for all numbers' is nonsense. For in mathematics '*necessary*' and '*all*' go together" (*PR* §154 and *PG,* II, Ch. VI, §35, pp. 428–9). Wittgenstein's criticism of extensionalism develops, therefore, into a conception of arithmetical universality which connects it essentially to the notion of a general rule of sign construction. The attribution of a certain property to all natural numbers has meaning only in reference to a sign process which allows us to survey (*übersehen*) all of them, to check them "at *one* stroke" (*mit einem Schritt*) (*PG,* II, Ch. VII, §39, p. 455; *PR* §123). From Wittgenstein's quasi-formalistic point of view, such a process must exhibit *a form,* namely, an unlimited possibility of sign construction in compliance with a general rule. In the case of "(x)**P**(x)", the demonstration would provide a proof-schema by means of which, for any given *n,* the proof of the singular proposition "**P**(**n**)" can be obtained. Wittgenstein distinguishes two types of such proof-schemas. One is exemplified by the proof of the algebraic identity "$2x = x+x$"; here a uniform method to generate, for any given *n,* a proof of the numerical identity "**2n** = **n**+**n**" is supplied: it will be enough to substitute, in the algebraic proof, the variable "x" with the numeral "**n**". As Wittgenstein explicitly says: "an algebraic proof is the general form of a proof which can be *applied* to any number" (*PR* §122, note [1]). The second type of proof-schema is the proof by complete induction: this includes the proof of the singular proposition "**P**(1)" (the inductive base) and shows the form of the passage from the proof of any proposition "**P**(**n**)" to the proof of "**P**(**n**+1)", i.e. the general rule by which the proof of any proposition "**P**(**n**+1)" can be generated from the proof of the proposition "**P**(**n**)" (inductive step). Inductive proof is the general term for the infinite series of proofs of the singular propositions "**P**(1)", "**P**(2)" etc., in the same way as "[1, ξ, ξ+1]" is the general term for the series of natural numbers. For any given *n*, the proof of "**P**(**n**)" can be obtained starting from the proof of "**P**(1)" and proceeding for *n* – 1 steps in the series of proofs, following the general rule of passage exhibited by the inductive step of the proof.[30]

As has been rightly observed, there is a remarkable affinity between Wittgenstein's interpretation of arithmetical universality and Skolem's finitistic construction of quantifier-free arithmetic.[31] Wittgenstein often suggests that in mathematics it would be correct to give up the use of "all", and its translation into the logical notation "(x)", reserving the use of this expression for contexts which can be extensionally interpreted, i.e. where the domain of the bound variable is finite. However, the truly idiosyncratic aspect of the Austrian philosopher's stance is found in the thesis that the proof of "(x)**P**(x)" is the very source of its sense. The relation of a universal proposition of number theory to its proof is at times likened by Wittgenstein to the naming relation: the proposition is like a Russellian logical proper name for its proof, and the existence of the latter is thus the necessary condition for the former to be meaningful. Now, it might seem strange that the above general characterization of what a proof of a universally quantified proposition should consist in does not suffice, given the knowledge of the meaning of "**P**", for the understanding of "(x)**P**(x)", independently of its proof. To find a plausible explanation for this renunciatory attitude (so different from that of intuitionists), one has to bear in mind that Wittgenstein's purpose in providing that characterization is totally negative: he wishes to banish the extensional interpretation of the universal quantifier, *without committing himself to an alternative, intensional one*. This alternative is impeded, in fact, by his assumption that no definition of the expression "proof of a proposition obtained by universal quantification of a decidable arithmetical predicate" – framed in general terms – would be able normatively to condition our subsequent decisions on what sign constructions are to count, by definition, as proofs of such a kind of proposition. Wittgenstein's overall position on the matter can be appreciated only by taking into account the claim that the general import of a proof of "(x)**P**(x)" rests completely in its applicability to constructing proofs of singular numerical propositions "**P**(**n**)". Infinity is contained in the proof-schema simply as unlimitedness of this logical possibility of application. But the acknowledgement of the unlimited applicability of the proof-schema corresponds to the adoption of the grammar rule which rules out as senseless any empirical proposition stating that the outcome "**P**(**n**)" (for some given *n*) has not been obtained by a correct performance of the sign operations schematically exhibited by the proof. In virtue of the rule, whenever this has not been obtained, it must be concluded that an error in the effectively carried out sign transformations has been made. Both the extensional and the intensional interpretation (in terms of a generic description of the relevant type of proof) of "$(x)\mathbf{P}(x)$" having been excluded, such a universal statement has to be considered as nothing but a disguised, abbreviated expression of the above grammar rule. As the rule makes reference to the very operations shown in the proof-schema, the existence of the latter is required for "$(x)\mathbf{P}(x)$" to have a *mathematical content* (as opposed to the vague, empirical statement that can be made with it before the proof is given): it is "as if the proposition formed a sign only in a purely external way and you still needed to give the sign a sense from within" (*PG,* II, Ch. VII, §39, p. 455; *PR* §122). But, in effect, even more renunciatory consequences derive from Wittgenstein's analysis of universality in arithmetic. The determination of the application-scope of a given proof-schema is not at all *the conclusion* to which it leads, since, obviously, it *says* nothing about its own applicability (*we* are the ones who establish, by definition, what this scope should be): "We ought not to confuse the infinite possibility of its application with what is actually proved. The infinite possibility of application is *not* proved!" (*PR* §163). The acknowledgement of the unlimited applicability of a schema for the construction of numerical proofs amounts to the adoption of a linguistic rule, which is not justifiable by any deductive argument: "That one can run the number series through the rule is a form that is given; nothing is affirmed about it and nothing can be denied about it. Running the stream of numbers through is not something which I can say I can prove" (*PG,* II, Ch. VI, §36, p. 434). As happens whenever it is not a matter of simply checking the outcome of one application of an available calculating procedure (namely, whenever it is not a matter of deciding a proposition which is meaningful, on the basis of the verification principle), Wittgenstein resorts to the metaphor of seeing. As a consequence, it is the very process of construction of the proof-schema corresponding to a universal arithmetical generalization that, substantially, loses its importance. On one hand, the mere circumstance that, in the schematic construction, variables are employed is not sufficient to guarantee that this sign figure would be seen as the expression of a general rule to generate particular numerical proofs. For instance, even the algebraic proof of the identity "$2x = x+x$" might be meant as a sign transformation from which a grammar rule concerning exclusively the letter "x" is derived. On the other hand, a proof-figure in which variables occur is not really needed in order that the universal validity of an arithmetical condition "$\mathbf{F}(x)$", i.e. the unlimited possibility of generating, according to a general rule, proofs of numerical instances of "$\mathbf{F}(x)$", is acknowledged. Even in the calculation of the expansion of the numerical term "$(5+2)^2$" one might see the general rule for expanding the binomial, and, in this case, nothing would be added by the proof of the algebraic identity "$(a+b)^2 = a^2+2ab+b^2$". Similarly, even in a finite initial segment of the series of proofs of "$\mathbf{P}(1)$", "$\mathbf{P}(2)$", "$\mathbf{P}(3)$" etc., one might see the general rule according to which every step in the series has to be performed, and, at this point, the addition of a proof by induction would be quite superfluous (*PR* §164 and *PG,* II, Ch. VI, §§36–7). In this way, the rejection of the extensional interpretation of universal generalizations in terms of infinite logical product, based on typical verificationist grounds, does not merely develop into the giving up of the use of the universal quantifier within mathematics, but into a far more austere depreciation of the role of general proofs.

Wittgenstein's treatment of existential generalizations is certainly more sketchy and less systematic than that of universal generalizations. As usual, he starts from the criticism of the extensional conception from the verificationist standpoint. As the step by step checking of singular propositions "$\mathbf{P}(1)$", "$\mathbf{P}(2)$", "$\mathbf{P}(3)$" etc. (where "$\mathbf{P}$" stands for an arithmetical decidable predicate) is not a decision procedure for the generalization "$(\exists x)\mathbf{P}(x)$", the extensional interpretation of the existential quantifier implies the meaningfulness of the generalization, prior to a proof, only when the logical operator ranges over a finite numerical interval. Extensional interpretation of the quantifier, finiteness of the domain of the bound variable and the constructive nature of existence proofs are three ingredients which meld into a (strong) verificationist conception of existential generalizations as genuine propositions. Of course, things become complicated as soon as one leaves the range of finiteness. In some cases, even an existential generalization on an infinite domain can be construed in such a way that the verificationist requirement is fulfilled. For example, consider *the system* of statements of the form "there is a real solution of the equation $x^2+ax+b = 0$", namely "$(\exists x)(x^2+ax+b = 0 \geq \text{x is real})$". The sense of every statement of such a form is not determined by the corresponding proof: this holds true only when the statement is not understood as referring to the extensional procedure of calculating the outcomes of successive trials of substitution of "x" with numerical symbols, but to the outcomes of the application of the resolutive formula for second degree equations. For any given pair (a, b), the existential statement is equivalent to the assertion that at least one of the two solutions of the equation, calculated by means of the known algorithm, is real, and any statement of this form can be effectively decided, as the strong verification principle demands: "Now consider the question 'does the equation $x^2+ax+b = 0$ have a solution in the real numbers?'. Here again there is a check and the check decides between (∃ ...), etc. and ~(∃ ...), etc." (*PG,* II, Ch. V, §23, p. 366). Exceptions apart, an existential statement will get sense only through the construction of a proof. However, a significant difference between Wittgenstein's treatment of the existential quantifier and of the universal one can be easily pointed out. In the latter case, the criticism of extensionalism prompts Wittgenstein to give a remarkably expounded characterization of what, *in general,* a proof of the universal validity of an arithmetical condition should consist in. For this reason, it may appear that he is looking for a general notion of proof of a generalization constructed by universal quantification of a decidable arithmetical predicate. But everything he says on the relation between sense and proof of such generalizations testifies that no general notion of their proof could have normative force on any subsequent decision on whether a given sign construction is to count as a proof. As it concerns mathematical existence, even this misleading impression vanishes: Wittgenstein sticks strictly to his precept that, beyond the verificationist boundaries, a general mathematical notion, like a grammar category such as that of pure colours, does not transcend the list of its acknowledged particular instances. There is no way of explaining the meaning of the expression "mathematical existence" other than by saying that mathematical existence is the result proven by any sign construction that we decide must be called "existence proof". And, of course, this is a void characterization as long as no restriction is imposed to the adoption of such grammar rules:

> The mistake lies in pretending to possess a clear *concept* of existence. We think we can prove a something, existence, in such a way that we are then convinced of it *independently of the proof*.... Really, existence is what is proved by the procedures we call 'existence proofs'.... We have no concept of existence independent of our concept of an existence proof.
>
> (*PG*, II, Ch. V, §24, p. 374)

What is pointed out in this passage is, according to Wittgenstein, the mistake which intuitionists make when they affirm that every existence proof must be constructive. Obviously, there is nothing wrong in proposing to circumscribe in a certain way the use of the expression "existence proof"; nonetheless, in this situation, one needs to be aware of the fact that, in making such a proposal, one has not discovered anything, but has merely suggested a linguistic decision:

> "Every existence proof must contain a construction of what it proves the existence of". You can only say "I won't call anything an 'existence proof' unless it contains such a construction".... When the intuitionists and others talk about this they say: "This state of affairs, existence, can be proved only thus and not thus". And they don't see that by saying that they have simply defined what *they* call existence.
>
> (*PG*, II, Ch. V, §24 p. 374)

Recursive arithmetic and algebra

The cue for Wittgenstein's reflections on universality within arithmetic came, without doubt, also from his reading of the paper in which Thoralf Skolem presented systematically the so-called primitive recursive arithmetic.[32] However, the verificationist background of Wittgenstein's position continues to be central in this context. An induction is not obtained as the result of the application of a general method, which could supply, for any given member of a certain class of universal generalizations, either an inductive proof or a refutation. Thus, the verification principle does not allow one, in reference to a proof of such a kind, to speak of looking for it and of discovering the solution of a genuine mathematical problem:

> So he has seen an *induction*! But was he *looking for* an induction? He didn't have any method for looking for one. And if he hadn't discovered one, would he *ipso facto* have found a number which does not satisfy the condition? – The rule for checking can't be: let's see whether there is an induction or a case for which the law does not hold.... Prior to the proof asking about the general proposition made no sense at all, and so wasn't even a question, because the question would only have made sense if a general method of decision had been known *before* the particular proof was discovered.
>
> (*PG,* II, Ch. VI, §31, pp. 400–2)

Despite such affirmations, the relation between sense of a theorem of this form and inductive proof has certain peculiar features which distinguish it from all the other cases in which – a general decision procedure being not on hand – the proof is given the task of creating the mathematical meaning of a certain verbal expression. At least apparently, there are good reasons to think that inductive proofs do not belong properly to the category of meaning-producing proofs. In effect, when Wittgenstein observes "We are not saying that when f(1) holds and when $f(c+1)$ follows from $f(c)$, the proposition $f(x)$ is *therefore* true of all cardinal numbers; but: 'the proposition $f(x)$ holds for all cardinal numbers' *means* 'it holds for $x = 1$, and $f(c+1)$ follows from $f(c)$'" (*PG*, II, Ch. VI, §32, p. 406), he seems to be setting up a false opposition. Indeed, if Wittgenstein's general grammatical stipulation is adopted, it follows that the proof of the base and of the inductive step is in fact the proof of the universal generalization. In other words, Wittgenstein's point of view would be rather akin to the usual approach to the problem. According to the latter, the introduction, among the axioms of arithmetic, of some general form of the Principle of Complete Induction, or, alternatively, the employment of the Principle as a primitive rule of inference (as in Skolem's work) testifies, on one hand, that logical rules alone are not sufficient, as obvious, for the transition from the inductive premises to the universal conclusion; on the other hand, the decision to adopt such an axiom or such a rule of inference may be seen precisely as the decision to make a stipulation which contributes to the determination of the sense of any universal generalization over the domain of natural numbers. It is quite clear that, if this stipulation is explicitly expressed in an axiom or in a rule of inference, then the related universal conclusion can be *inferred* from the (proven) base and inductive step of any given inductive proof. In this situation, proof by induction really appears to be the means of answering the question whether an arithmetical condition holds for all natural numbers. Of great interest is the fact that Wittgenstein himself seems to be making the same point, which, of course, could produce much difficulty for his overall conception. Consider his discussion of Skolem's inductive proof of the associative law of addition "$a+(b+c) = (a+b)+c$" (which he refers to by "**A**") (*PG,* II, Ch. VI, §30). Firstly, he restates Skolem's proof as follows:

$$\left.\begin{array}{l} a+(b+1) = (a+b)+1 \\ a+(b+(c+1)) = a+((b+c)+1) = (a+(b+c))+1 \\ (a+b)+(c+1) = ((a+b)+c)+1 \end{array}\right\} \quad \mathbf{B}$$

where the first line is the inductive step of the recursive definition of the sum, and the steps in the second and third lines are all performed by applying, with suitable substitutions, that very definition. The step from the group of equations **B** to the associative law could be legitimized by the following general stipulation (*allgemeine Bestimmung*):

$$\left.\begin{array}{ll} \alpha & \varphi(1) = \psi(1) \\ \beta & \varphi(c+1) = F(\varphi(c)) \\ \gamma & \psi(c+1) = F(\psi(c)). \end{array}\right\} \phi(c) \underline{\underline{\Delta}};\ \psi(c) \quad \mathbf{R}$$

Expressed in words, this grammatical rule establishes that every proposition of the form "the equation Δ holds for all natural numbers" means that the corresponding equations α, β, γ hold.[33] By adopting this general linguistic convention, any problem of whether a given arithmetical equation holds true for all natural numbers appears as possessing a definite sense, independently from the circumstance that an inductive proof – solving it in the affirmative – has been found. What is really surprising is that Wittgenstein seems to endorse this conclusion: "All I can do is to explain: the question whether A holds for all cardinal numbers is to mean: 'for the functions

$$\varphi(\xi) = a+(b+\xi),\ \psi(\xi) = (a+b)+\xi$$

are the equations α, β, γ valid?' And then that question is answered by the recursive proof of **A**, if what that means is the proofs of α, β, γ (or the laying down of α and the use of it to prove β and γ)" (*PG,* II, Ch. VI, §30, p. 398). A similar conclusion, though more comprehensive, can be drawn if, instead of the stipulation **R**, the other general grammatical convention, already mentioned above, is laid down, namely: "the proposition $f(x)$ holds for all cardinal numbers" means: it holds for $x = 1$, and $f(c+1)$ follows from $f(c)$. In fact, for any given decidable predicate "**P**", the question of whether all natural numbers are **P** becomes equivalent, by definition, to the question of whether the inductive base and the inductive step hold; and, if by "inductive proof" of the universal generalization the proof of the base and of the inductive step is meant, then it becomes quite reasonable to say that the construction of this proof gives a positive answer to the starting question.

These considerations, which Wittgenstein himself outlines, threaten to upset the whole framework of his intermediate phase view of mathematics. It seems that the absence of a decision procedure can be counterbalanced, as it concerns the normative conditioning of the acknowledgement of internal relations, by a grammatical stipulation such as that stated above. Assuming as known the meaning of the predicate "**P**", the understanding of the universal statement of number theory "$(x)\mathbf{P}(x)$" would be grounded on the mastery of such a general notion of proof by induction, prior to the (eventual) construction of the specific corresponding induction. A large part of Wittgenstein's remarks on Skolem's arithmetic aims at avoiding this undesired conclusion. His strategy entails an attack on three fronts: (i) to lay down that the word "proof" cannot be correctly applied to any group of formulae which is an instance of the schema **R**; (ii) as a corollary of (i), to deny that, in proving by induction the algebraic properties (associative, commutative, etc.) of the operations of sum and product, a reduction in the number of the fundamental laws of arithmetic is effected; (iii) to show that the general stipulation expressed by the schema **R** is superfluous, since it is not able to provide the variable "inductive proof" with a meaning which transcends the set of the acknowledged particular instances of inductive proofs (namely, a meaning that, as with the genuine mathematical predicates in the verificationist sense, carries out a normative function on the decision of counting a given sign process as an inductive proof of a theorem). Let us examine one at a time points (i) – (iii). Concerning (i), Wittgenstein maintains that if the general convention **R** is stated by saying that an equation Δ is *proven* as valid for all natural numbers whenever the corresponding equations α, β, γ are *proven,* then an extremely misleading form of expression is used. The ground for his claim is that the meaning of the verb "to prove", in its occurrence in a statement such as "the equation α has been proven", is totally different from the meaning it has when, on the strength of that stipulation, the statement "the equation Δ has been proven" is made. Namely, the stipulation extends the use of the substantive "proof" and of the verb "to prove", moving from the meaning they have in linguistic contexts of the first kind and laying down their applicability to a class of cases which have nothing to do with the former. According to Wittgenstein, here we are facing a manifest and arbitrary twisting of the grammar of these words. The norm of conduct to be kept to is that "it is inadvisable to call something a 'proof' ... when the ordinary grammar of the word 'proof' doesn't accord with the grammar of the object under consideration" (*PG,* II, Ch. VI, §33, p. 414). But now Wittgenstein must reply to the question: what is *the ordinary grammar* of the word "proof"? According to him, only a finite chain of equations, the left member of the first equation being A and the right member of the last being B, and every link of the chain being obtained either by application of some primitive law or of previously proven equations, can be correctly called a "proof of an equation A = B". Once this proposal of linguistic regimentation is accepted, the consequence that no reduction of the fundamental laws of arithmetic can be effected by the inductive proofs of the algebraic properties of the operations of sum and product inevitably follows. Consider, for instance, Skolem's proof of the associative law of sum **A**. As already seen, in the proof of the base and of the inductive step of the induction which "proves" **A**, only the recursive definition of sum is used as paradigm. But this does not mean that a chain of equations leading from "$a+(b+c)$" to "$(a+b)+c$", each step of which having been performed only on the basis of the definition "$a+(b+1) = (a+b)+1$", has been constructed. If only in this circumstance one is legitimized, by definition, to speak of a reduction of the associative law of the sum to the recursive definition of this arithmetical function, then, obviously, no reduction in the number of fundamental laws of arithmetic is effected by the inductive proof. For the same reason, the fact that, in the proof of the inductive step of the proof of the commutative law of sum, the associative law is used as paradigm to justify a certain step does not entail the reduction of this law to the associative law. Thus Wittgenstein observes: "If I am right that the independence remains intact after the recursive proof, that sums up everything I have to say against the concept of recursive 'proof'" (*PG,* II, Ch. VI, §34, p. 425). It goes without saying that Wittgenstein's arguments rest on the presupposition that the notion of an arithmetical proof cannot be enlarged by the adoption, either as an axiom or as a rule of inference, of some form of the Principle of Complete Induction. What is really of significance, in my opinion, is the fact that he seeks to justify this presupposition by resorting to the ordinary grammar of the word "proof". In fact, this attempt shows that he is not able to resist the temptation to adopt a certain meaning of this word – by him dignified as "the ordinary" one – as a standard to evaluate the correctness of other uses. And this is a doubly disconcerting move: first, because, according to his own view, no constraint can be imposed by a general notion on the decision to count sign structures of a certain kind as proofs; second, because the uses of "proof" ruled out by his grammatical restrictive choice are inserted in a widely shared mathematical usage. In spite of this, as we shall see, Wittgenstein resorts to this strategy, both in his writings of the intermediate phase and in his later writings, whenever he wishes to critically evaluate some aspect of current mathematical practice.[34] The core of Wittgenstein's view is expressed in thesis (iii), i.e. his firm belief about the absolute mathematical irrelevance of any formulation of the Principle of Complete Induction, either in axiomatic form, or as a rule of inference or even in the form of a general stipulation such as **R**. We know that, according to the Austrian philosopher, the acknowledgement of a given sign construction as an inductive proof is nothing more than the acknowledgement of the unlimited applicability, in principle, of a certain type of schema of construction of numerical proofs. It corresponds to the adoption of a rule of the geometry of signs which condemns certain descriptions of the empirically given sign manipulations as senseless. In claiming the *immediate* nature of this acknowledgement, namely, in denying that it is like a process which passes through the verification of the accordance of the given construction to an abstract schema (common to all the constructions of a certain sort), Wittgenstein keeps a hold on his fundamental point that such a grammatical invention cannot be disciplined by the meaning previously attributed to the formal representation of an inductive proof: "the individual proof needs our acceptance of it as such (if 'proof' is to mean what it means); and if it doesn't have it, no discovery of an analogy with other such constructions can give it to it.... the fact that the connection between **B** and **A** is in accordance with a rule can't show that **B** is a *proof* of **A**" (*PG*, II, Ch. VI, §33, pp. 415, 418). There is no doubt that one may notice the analogies between the proofs of the fundamental laws of arithmetic, single out the formal structure common to all of them and represent it by a schema such as **R**. But, as it concerns the determination of the meaning of the variable "inductive proof", this process is not at all assimilable to the intensional characterization of the meaning of a genuine predicate such as "prime" through the correlation of a decision procedure. The verificationist requirement for the introduction of mathematical concepts remaining unfulfilled, the meaning of the variable "inductive proof" comes to coincide *sic et simpliciter* with its acknowledged extension, i.e., with the class of the sign constructions actually identified as inductive proofs:

> Isn't our principle: not to use a *concept-word* where one isn't necessary? – That means, in cases where the concept-word really stands for an enumeration, to say so.... What I mean is: in Skolem's calculus we don't *need* any such concept, the list *is sufficient*.... The concept of generality (and of recursion) used in these proofs has no greater generality than can be read immediately from the proofs.
>
> (*PG*, II, Ch. VI, §33, pp. 417, 418)

Rejecting the enrichment of the notion of proof through the adoption of the Principle of Complete Induction as a rule of inference, the aspect of Skolem's system of arithmetic changes. Recursive definitions of the arithmetical operations of sum and product are schematic rules for generating infinite series of numerical definitions: as a recursive proof of a fundamental law of arithmetic is the general term of infinite series of numerical proofs – exhibiting the form of the first term of any series and the form of the step from any given term to the immediately successive term in the series – so a recursive definition of an arithmetical operation is the general term of infinite series of numerical definitions, showing the form of the first term of any series and the general instruction that must be followed to generate the immediate successor of any given term of the series. What has been said about inductive proofs holds for recursive definitions too: their general import lies completely in the unlimited possibility of applying them in Skolem's calculus to generate particular numerical definitions, and this is not stated but shown by signs. Wittgenstein's last problem is that of explaining what the relationship is between the fundamental laws of arithmetic and the respective inductions, improperly called "proofs". According to him, a law such as the associative law of addition "$a+(b+c) = (a+b)+c$", in so far as it is considered as a purely algebraic formula, is a primitive rule of substitution of the calculus with letters, adopted by free convention. If, in the selection of the paradigms of sign transformation constituting this calculus, the properties of the sum of natural numbers were not taken into account, the inductive proof of the associative law could not have any relevance to the justification of the choice of these paradigms, and, in particular, could not give any support to the choice of "$a+(b+c) = (a+b)+c$". Things stand otherwise if, in laying down the algebraic rules, one decides to proceed in accordance with the properties that the operations of sum and product possess within numerical arithmetic, in virtue of their recursive definitions. Inductive proofs cannot fill the gulf between arithmetic and algebra because, according to Wittgenstein, they do not prove the fundamental laws. Nonetheless, they can provide justification for the choice of these laws in the calculus of letters, showing what we mean by "agreement" between algebraic laws and numerical calculations (e.g. when the stipulation suggested by the inductive proof of the associative law is laid down, a new criterion for having correctly carried out the operation of addition in certain cases is supplied): "An induction doesn't prove the algebraic proposition since only an equation can prove an equation. But it justifies the setting up of algebraic equations from the standpoint of their application to arithmetic" (*PR* §167).

In conclusion, Skolem's achievements are not threatened by Wittgenstein's reinterpretation, even though the rejection of a general notion of inductive proof looks unpromising from the mathematical point of view. But with the theory of real numbers, although for exactly opposite reasons, the approach of the Austrian philosopher is not so innocuous.

Real numbers

What we have seen up to now in this chapter proves that two basic tendencies underlie Wittgenstein's writings on mathematics during the intermediate phase, and that they are constantly on the point of clashing. On one hand, the idea that a mathematical concept (individual concepts included), to which no decision procedure is associated, coincides with its acknowledged extension (bearing in mind that, in mathematics, by stating that a certain object belongs to the extension of a predicate – or is the extension of a definite description – a linguistic rule, though disguised, is set up). On the other hand, his criticism of the extensionalist conception of the mathematical infinite prompts Wittgenstein to give a characterization of the meaning of certain general terms, even though a correlated decision procedure is obviously not available. Thus, he appears to be imposing severe grammatical constraints on the applicability of these terms and, for this very reason, to reveal a strongly revisionary attitude towards classical mathematics. In Wittgenstein's remarks on the nature of real numbers and of the continuum, the second of the two tendencies definitely prevails. As, by definition, an infinite structure is a structure that can never be completed, there is no intermediate reality – such as the infinite sequence of all the rational approximations of an irrational number – between the finite expansions effectively carried out (by calculations acknowledged as correct) and the potentially endless process of their generation. The latter, for example, is a procedure for yielding decimal digits according to a general effective law (*Gesetz*) or prescription (*Vorschrift*). And it is with such a law that a real number has to be directly identified: "We could also say: '√2' means the method whereby x^2 approximates to 2. Only a path approaches a goal, positions do not. And only a law approaches a value.... The letter 'π' stands for a law.... A real number *yields* extensions, it is not an extension. A real number is: an arithmetical law which endlessly yields the places of a decimal fraction" (*PR* §§185–6). That effective prescriptions for generating convergent sequences of decimal fractions are called "numbers" is because of the existence of a calculus, whose rules are significantly analogous to the rules of the calculus with rational numbers (*PG*, II, Ch. VII, §43, pp. 484–5). The identification of a real number with an arithmetical law for calculating approximated rational values has the immediate effect of depriving the notion of an irregular infinite decimal of any ground (and, with it, the notation "0, *abcd ... ad inf.*", where the expression "*ad inf.*" does not refer implicitly to a law of production of digits). The idea that the concept of an infinite sequence can be detached from that of a general rule – by reference to which such a sequence can be grasped as a whole – is a sub-product of the view of laws as merely convenient devices for representing infinite extensions. Once it is assumed that a completed infinite totality corresponds to an endlessly applicable rule of generation, the step which leads to giving up rules is brief. In classical mathematics, the notion of an infinite decimal expansion does not necessarily require reference to a law because an infinite process of arbitrary selection, for every $n \geq 1$, of a fraction $m/10^n$ (with m included between 0 and 9) – i.e. a selection, every step of which is made without compliance to some law formulated beforehand – is conceived as a completed process: "And if there is an infinite reality, then there is also contingency in the infinite. And so, for instance, also an infinite decimal that isn't given by a law" (*PR* §143). Wittgenstein also exploits his verification principle in attacking the claim that the question of whether the totality of laws needs to be supplemented by irregular infinite decimals in order to get the continuum is a meaningful one. In order for the assertion of the existence of such irregular infinite decimals besides infinite decimals generable by means of laws to have sense, it must be logically possible to tell the difference between the situation in which this statement would be true and that in which it would be false. But as, in each conceivable circumstance, an irregular infinite decimal would be represented only by a certain finite sequence of rational approximations, we would never be able to point out a feature whereby we could distinguish a situation where such an irregular infinite decimal exists from a situation in which only some approximating decimal fractions exist:

> And so we cannot say that the decimal fractions developed in accordance with a law still need supplementing by an infinite set of irregular infinite decimal fractions that would be "brushed under the carpet" if we were to *restrict* ourselves to those *generated by a law.* Where is there such an infinite decimal which is generated by no law? And how would we notice that it was missing? Where is the gap it is needed to fill?
>
> (*PR* §181)[35]

In my opinion, the most interesting aspects of Wittgenstein's reflections on real numbers do not concern, however, his criticism of the extensional conception, but his radicalization of the constructivist standpoint. Firstly, he refuses an intensional interpretation of the notion of an infinite sequence not generable by applying an effective rule, such as that proposed by intuitionists through the introduction of free-choice sequences. The grounds for this rejection emerge clearly from his remarks on the geometrical process of construction of an endless monotonous succession of nested intervals with rational extremes on the axis of the numbers, such that the length of the nth interval tends to 0 as *n* increases (the length of the *n*th interval being equal to 10^{-n} if decimal intervals are dealt with, and being equal to 2^{-n} if, instead, intervals obtained by successive bisections are dealt with). Wittgenstein considers the following prescription for the construction of a succession of intervals of the second type. (A) Divide in two equal parts a given unitary segment *AB* and toss a coin: if it is heads, take the left half; if it is tails, take the right half; then proceed successively to a new bisection according to the same rule. An arithmetical translation of this prescription runs as follows: (A') place "0" at the *n*th place of the expansion of an infinite binary fraction if on the *n*th toss of a coin it is heads, and place "1" if it is tails (this process corresponds to an endless sequence of choices between "0" and "1"). Wittgenstein's first objection is that one cannot reasonably say that such an unlimitedly applicable procedure *generates* a point. According to him, the impression that, by restricting step by step the nested intervals, a point is generated, arises from an illicit transposition of what holds for segments of the visual field to segments of Euclidean geometry. One can say of a visual segment that, by shrinking, it is approximating more and more to a visual point, in the sense that it becomes more and more similar to it. On the contrary, by shrinking an Euclidean segment, it always remains in equal measure different from a point

> since its length, so to say, never gets anywhere near a point. If we say of a Euclidean line that it is approximating to a point by shrinking, that only makes sense if there is an already designated point which its ends are approaching; it cannot mean that by shrinking it *produces* a point.
>
> (*PG*, II, Ch. VII, §42, pp. 477–8)

Only if the existence of a point enclosed in all the nested intervals of the series is postulated is one authorized to speak of a process of approximation to something (indeed, this is the right guaranteed by the so-called Postulate of Continuity of classical mathematics). Wittgenstein's second objection states that whoever is unwilling to assume such an existence as given independently from the process of construction of the series (e.g. intuitionists and Wittgenstein himself) cannot claim that, through the simple unlimited possibility of shrinking the intervals, *one* point is produced. The constructivist conception of infinite sequences must admit only lawlike sequences because only the univocal predetermination, by means of a law, of each step in the process of production of the approximated rational values is able to identify *one* generative procedure and hence *one* specific real number or point on the line: "What I mean might be put like this: for a real number, a construction and not merely a process of approximation must be conceivable. – The construction corresponds to the unity of the law" (*PR* §186).

From the simultaneous rejection both of the extensional standpoint and of the intuitionist attempt to free the constructivist conception of an endless arithmetical process from the requirement of the presence of an effective law of computation, it seems reasonable to expect a restriction of the notion of infinite sequence to the sequences that can be produced through the successive application of a calculating method. This limitation would be perfectly in harmony with Wittgenstein's quasi-formalistic orientation, i.e. with his way of seeing mathematics essentially as a rule-governed activity of sign manipulation. And there is no doubt that he constantly uses recursive real numbers, such π as and √2, as models of what, in his conception, should be meant by "real number": in fact, only "via the recursion each stage becomes arithmetically comprehensible" (*PR* §194). To this, Wittgenstein adds a further requirement for genuine real numbers, that of the effective comparability with the rationals: for any given rational number, the question whether it is less than, equal to or greater than the structure under consideration must be effectively decidable (*PR* §191). Thus, one may feel rather amazed at discovering that certain prescriptions, which are *not* considered by Wittgenstein as genuine generators of real numbers, actually satisfy both the requisite of effectiveness and that of the effective comparability with the rationals, no more nor less than the recipes for the calculation of the decimal expansion of π and √2. After Wittgenstein, we adopt the following abbreviations: "$\overset{5\to 3}{\sqrt{2}}$" stands for the definite description "the real number generated by the prescription, write out the digits of √2 and, whenever a '5' occurs, substitute a '3' for it"; "π'" stands for the definite description "the real number generated by the prescription, write out the decimal expansion of π and, whenever the group of digits '777' occurs, substitute the group '000' for it"; "*P*" stands for the description "the infinite binary fraction such that at the *n*th place of the expansion there occurs a '1' or a '0' according to whether *n* is prime or not". According to Wittgenstein, none of these three definite descriptions identifies a real number. Since, in each of the three cases, one deals with an effective rule for producing an infinite sequence of decimal or binary fractions, and since, for any given rational number, one can decide whether $\overset{5\to 3}{\sqrt{2}}$ (or π', or *P*) is less than, equal to or greater than the rational under consideration, the problem of explaining the reasons for such a severe attitude has to be solved. As seen, the understanding of the meaning of a symbol such as "√2" coincides with the knowledge of an arithmetical procedure by means of which rational numbers whose squares differ as little as one likes from 2 can be generated. Of course, the results of the application of the arithmetical method for producing approximated rational values are expressed in one definite numerical notation; decimal notation, for example. But the arithmetical meaning of the process of generating rational numbers whose squares approximate more and more to 2 does not depend in any essential way on the particular notation used to represent its results. It is like the inside of the trunk, where "the tree's vital energy is", namely, "the living essence" of the real number, whereas the digits produced in a certain notation are like the bark of the tree, or the dead chalk excretions by which the snail builds its shell (*PG*, II, Ch. VII, §42, p. 475; *PR* §182). Let us return, now, to examining the pseudo-real numbers which Wittgenstein denotes by "$\overset{5\to 3}{\sqrt{2}}$" and by "π'". It is doubtlessly true that the operation consisting in producing digits of the decimal expansion of √2 and, whenever a "5" occurs, substituting it by a "3", is an effectively performable operation, just as the operation of producing digits of the decimal expansion of π and, whenever a "777" group occurs, substituting it by a "000" group. However, in order for such an operation of substitution to have a mathematical meaning, it is not enough that, for any given place of the decimal expansion of √2, and for any given three-place sequence of the decimal expansion of π, one can effectively decide whether the condition for applying the operation of substitution is satisfied (this would be tantamount to seeking to interfere with the vital energy of a tree by acting superficially on its bark). According to Wittgenstein, the introduction of the operation of substitution of digits can contribute to defining a genuine real number only on condition that *the series of places of the expansion where the operation must be applied* is itself understood intensionally. This means that this contribution can be given only if the occurrences of "5" in the decimal expansion of √2, and of "777" in the decimal expansion of π, are not like "chance" results of calculations. Only if the law which governs these occurrences is known and, consequently, only if we are able to acknowledge *the necessity* of each of them, can the operation of substitution be inserted in a substratum of rules and thus can determine the mathematical essence of a new real number. In fact, only when such a law is known does it become *senseless* to say, for any given *n* prescribed by the law, that a digit different from "5" has been obtained at the *n*th place by a *correct* application of the method of calculation of the decimal expansion of √2. The law would supply a new criterion for the correctness of the empirically effected applications of this method (if a digit different from "5" has been obtained at one of the places of the expansion of √2 where, according to the law, a "5" occurs, then an error must have been made in carrying out the calculations): thus, the operation of substitution could be inserted in what Wittgenstein considers a truly mathematical, and not a merely empirical, setting (and the same holds for π'). While the law remains unknown, i.e. while the corresponding grammatical rule is not yet adopted, it is the recurrent temptation to identify a real number with an infinite extension which confers on $\overset{5\to 3}{\sqrt{2}}$ and π' the appearance of real numbers.[36] Wittgenstein's stance can be further clarified by examining his observations on the pseudo-real number *P*. The recipe for yielding the digits of the expansion of *P* satisfies both the requirement of effectiveness, given the decidability of the predicate "prime", and that of the effective comparability with the rationals. But the operation of placing "1" or "0" at the *n*th place of a binary expansion, according to whether *n* is or is not prime, raises again the same question raised by the operation of substitution in the cases of $\overset{5\to 3}{\sqrt{2}}$ and π'. The former operation can contribute to the creation of a real number only if its application is not based on the outcomes of those "arithmetical experiments" which are the successive checks for the property of being prime in the series of natural numbers. The operation would be admissible only if a law were known which establishes *the necessity* of those outcomes, namely, if a law of generation (or, more weakly, of distribution) of the primes were known. In consequence of the acceptance of such a law, it would be *senseless* to say that by correctly applying the decision procedure for the property of being prime to an *n* such that "1" occurs at the nth place of the expansion of *P*, a negative answer is obtained. Thus, the law would supply a new criterion for the correctness of the applications of this decision procedure (whenever, for an *n* such that at the *n*th place of *P* a "1" occurs, the application of the decision procedure has yielded a negative answer, then a mistake must have been made in carrying out the pertaining calculations). If such a law were known, any given finite sequence of "1" and "0" constructed in accordance with the prescription and which, in the absence of such a law, appears to be completely accidental would manifest a mathematical regularity:

> I must be able to write down a part of the series, in such a way that you can *recognize* the law. That is to say, no *description* is to occur in what is written down, everything must be represented. The approximations must themselves form what is *manifestly* a series. That is, the approximations themselves must obey a law.
>
> (*PR* §190)

Wittgenstein's resorting, in this connection, to his classical opposition between seeing and describing aims at undermining a possible defence of the pseudo-real numbers at issue. At least apparently, one could advance a justification for the claim to their legitimacy which acknowledges the need for grasping "at one stroke" the totality of the outcomes of those "arithmetical experiments" by which it is verified whether a certain arithmetical condition holds. One could maintain, in fact, that the law which prescribes the places of the decimal expansion of √2 in which "5" occurs and the law on the occurrences of the group "777" in the decimal expansion of π – just as the law of prime number distribution –, though unknown to us, do exist. Therefore, the prescriptions to generate $\overset{5\rightarrow 3}{\sqrt{2}}$, π' and *P* could be intensionally understood to the extent that they would contain an allusion, an indirect reference, to these unknown, yet existent, laws. But, according to Wittgenstein, this way out is closed. A mathematical regularity is not like a state of affairs describable in general terms and whose existence can be assumed hypothetically. In fact, a grammar rule which presupposes the existence of a proof corresponds to a mathematical law: only the proof *shows* directly what is meant by saying that such-and-such results must be obtained if such-and-such operations are correctly performed. The principle of *esse est percipi* holds for mathematical regularities: otherwise, we ought to suppose that the signs of our language could have a meaning that is still unknown to us:

> The connection that I think I do not see does not exist.... If you give us a law for this [that of the primes] distribution, you give us a *new* number series, *new* numbers. (A law of the calculus that I do not know is not a law.) (Only what I *see* is a law; not what I *describe*. That is the only thing standing in the way of my expressing more in my signs than I can understand.
>
> (*PG*, II, Ch. VII, §42, p. 480)[37]

However, Wittgenstein's conception, as we have expounded it hitherto, seems to meet an insurmountable difficulty. Let us again compare *P* with √2: one could maintain that, just as, for instance, the occurrence of "1" at the fifth place of the binary expansion of *P* cannot be mathematically understood unless the distribution of the primes is known, so neither can the occurrence of "4" at the third place of the decimal expansion of √2 be mathematically understood, since there is no general law prescribing it, which is known beforehand. But, according to Wittgenstein, there is a sharp difference between the two situations. The absence of a law by which one could infer the result that must be obtained at a certain given place of the decimal expansion of √2 before the calculation is effectively made is comparable to the absence of a law by which one could infer the result that must be obtained by multiplying two given numbers before the calculation is effectively carried out (or, more generally speaking, to the absence of a rule by which the outcomes of the applications of a decision procedure in certain particular cases could be inferred, prior to their effective computation). The point is that, whenever the product of two given numbers is worked out, the mathematical concept of product is not modified, because its intensional nature is independent of the process of partial determination of its extension. Similarly, the calculation which yields, for a given n, the digit occurring at the nth place of the decimal expansion of √2 does not change *the concept of the square root of* 2, which is identified *tout court* with the effective process for generating rational numbers whose squares approximate more and more to 2. Rather, it simply expands the knowledge about the decimal representation of the rational approximations of √2: "We don't understand why there is a 4 at the third decimal place of √2, but we don't *need* to understand it.... In fact, in the end the decimal system as a whole withdraws into the background, and then only what is essential to √2 remains in the calculation" (*PR* §193). In contrast, in the case of *P*, the unlimited possibility of writing "1" or "0" according to the outcomes of the arithmetical experiments is all that we have. In making these experiments, things go in a certain way but we do not see why they *must* go that way (a rule of the geometry of signs is lacking): in Wittgenstein's words, we have a hollow tree trunk or a shell without the living snail.

Set theory

One is able to have the right appreciation of the distance of Wittgenstein's approach to the problem of mathematical infinite from the set-theoretical framework (both of platonistic and constructivist orientation) only by taking into account the quasi-formalistic background of the former. If the only mathematical reality is the concrete reality constituted by the languages of calculi, then infinity must be involved in the rules of the systems of symbolic manipulation. The problem of the infinite regards essentially the notation (not only the mathematical one, since it arises in connection with that portion of common language whereby lived experience of space and time is described, and also in connection with the thing-language). Consider the primitive numerical notation: a positive integer is a sign sequence obtained from "1" by adding the sign "+1" to the right a finite number of times, a sequence that is used as a paradigm in those processes of sign transformation (proofs) which lead us to the adoption of certain definitions, i.e. of certain rules of the geometry of signs (the circularity of the first part of this informal explanation did not worry Wittgenstein, given his rooted conviction about the substantial irreducibility of the pair of notions finite/infinite). When one speaks of the endlessness of the series of the integers, one refers simply to the unlimited logical possibility of reiterating the operation of inserting (on the right) the sign "+1", starting from the sign "1":

> The infinite number series is itself only such a possibility – as emerges clearly from the single symbol for it "(1, *x*, *x*+1)". This symbol is itself an arrow with the first "1" as the tail of the arrow and "*x*+1" as its tip and what is characteristic is that – just as length is inessential in an arrow – the variable x shows here that it is immaterial how far the tip is from the tail.
>
> (*PR* §142)

Similarly with the decimal numerical notation: whoever knows this method of construction of numerical expressions, based on the positional meaning of the digits, understands the infinity of natural numbers. The grammar of the system sets no limits on the application of the operation of placing a digit on the left or on the right of any sequence of digits (in the first case, the inserted digit must be other than "0"): "The rules for a number-system – say, decimal system – contain everything that is infinite about the numbers. That, e.g. *these rules* set no limits on the left or right hand to the numerals; *this* is what contains the expression of infinity" (*PR* §141). The classical opposition between reality (actual infinite) and possibility (potential infinite) is inevitably resolved in favour of the latter as soon as it is referred to the language. If reality is that of the actually constructed signs, then, in any given moment, it can only be finite. Moreover, there cannot be an "ethereal" actualization of infinity, since it is logically impossible to conceive a process of generation of endless numerical symbols as completed: "The infinite number series is only the infinite possibility of finite series of numbers. It is senseless to speak of the *whole* infinite number series, as if it, too, were an extension. Infinite possibility is represented by infinite possibility. The signs themselves only contain the possibility and not the reality of their repetition" (*PR* §144). However, in admitting that mathematical reality – in so far as it is identified with certain actually given sign constructions – is confined to finiteness, Wittgenstein is not maintaining that the meaning of the expression "positive whole numbers" has to be explained with reference to the feasibility of the sign operations involved. According to the intermediate Wittgenstein, the strict finitistic positions are undermined by a misunderstanding which is perfectly symmetrical to the error made by those who, in seeking to justify the extensional view of the infinite, mistake the logical impossibility of completing an endless process for a mere biological impossibility. The feasibility of a certain sign operation has nothing to do with that possibility which is meant when the infinity of the series of the naturals is spoken of: this refers only to the acceptance of a grammar rule which rules out as senseless an expression such as "the greatest natural number" (in the given notation). Only within the strict finitistic interpretation would it be reasonable to assume that the possibility at issue is inevitably conditioned by factors such as the availability of sufficient space and time, of writing materials, and the existence of certain human capabilities, etc. But, according to Wittgenstein, a clear-cut distinction has to be made between a rule and its acknowledged formal properties, on one hand, and the results empirically obtained, or that can be empirically obtained, on the other (as shown clearly by the example of those effective laws for yielding rational approximations with which real numbers are identified). Thus, infinity has to be construed as unlimited applicability *in principle* of a rule of sign construction: "Someone might perhaps say: True, but the numerals are still limited by their use and by writing materials and other factors. That is so, but that isn't expressed in the *rules* for their use, and it is only in these that their real essence is expressed" (*PR* §141).[38]

But, as soon as the notion of what is permitted, in principle, by the rules of a calculus is accepted, one runs the risk of construing in a misleading way this possibility of constructing numerical symbols which have not been actually written down yet. There is the strong temptation, in fact, to conceive the material process of their construction as if it were a sort of perceptible manifestation of structures which pre-exist, in a more rarefied, ethereal, sense of "existence". This, for Wittgenstein, is one of the sources of mathematical platonism:

> Of course, the natural numbers have only been written down up to a certain highest point, let's say 10^{10}. Now what constitutes the *possibility* of writing down numbers that have not yet been written down? How odd is this feeling that they are all somewhere already in existence! (Frege said that before it was drawn a construction line was in a certain sense already there.) The difficulty here is to fight off the thought that possibility is a kind of shadowy reality.
>
> (*PG*, II, Ch. II, §10, p. 281)

Nonetheless, the intermediate Wittgenstein's strong verificationism consents to provide the statement that a term of a series exists prior to its effective construction with an acceptable, non-platonistic sense: it has to be understood as asserting simply that a general rule of sign construction, by whose application it can, in principle, be generated, is known. As seen on pages 79–85, the claim that a certain given number can be prime without our knowing about this, i.e. without the truth-value of the statement that it is prime being known to us, must be interpreted as affirming that an effective method of checking for the property of being prime is on hand. The existence of an element of a series is on a par with a number's possessing a decidable property. The availability of an effective general rule of sign construction and the availability of a decision procedure for a predicate supply plausible sense to the reference to a domain which goes beyond the mathematical entities generated hitherto and to a number's having an unproven mathematical property. The description of a not yet constructed entity, *framed in the general terms of the rule of construction,* has enough logical force to impose rigid constraints on the future acknowledgement of a given sign structure as the structure which, by definition, must be identified by the description (on the future adoption of this grammar rule). Thus, the formation of the concept of result of the such-and-such application of the such-and-such rule of construction – namely, the adoption of the norm which establishes what must be counted as the result of the correct application of the general rule in that case – appears at least to be conditioned by the concept of the general rule:

> It is clear that we can follow a rule like |*a,* ξ, ξ+1 |. I mean by really following the rule for constructing it without previously being able to write down the series. In that case it's the same as if I were to begin a series with a number like 1 and then say "now add 7, multiply by 5, take the square root of the result, and always apply this complex operation once again to the result".
>
> (*PG*, II, Ch. 2, §10, p. 282)

> Supposing ... in order to get to my results I had written down what you may call "the rule of squaring", say algebraically. In this case this rule was involved in a sense in which no other rule was. We shall say that the rule is *involved* in the understanding, obeying, etc., if, as I should like to express, the symbol of the rule forms part of the calculation.
>
> (*BB*, p. 13)

Of course, it makes sense to speak of constructing signs in compliance with a certain specified rule only on condition that the rule is something which is univocally identifiable, independently from the results of its own application. For instance, from the standpoint of Wittgenstein's strong verificationism, the doubt over the occurrence of a certain digit at a certain place of the decimal expansion of √2 has a perfectly definite sense before the calculation is carried out; and it is the knowledge of the general law for yielding rational approximations that provides the doubt with a sense. Similarly, the rule of construction shown perspicuously by the symbol " |1, ξ, ξ+1 |", and the associated formal concept of positive whole number, are intensionally identifiable. Indeed, according to Wittgenstein, the most appropriate use of this symbol is within a system of symbols of the same kind (i.e. inside a system of recursive definitions of infinite series of expressions), such as, for example, " |5, ξ, √ξ |" and " |√[(ξ+7) × 5] |". But this obviously entails that of each of these laws we have a definite concept, which is completely independent from the results of their application.[39]

Infinity is a characteristic of the possibilities of sign construction within a notational system, shown by the rules governing the use of the expression "and so on". It is not further reducible, by means of definitions, to other, more primitive, notions. In formulating explicitly the rules for using a variable such as "1, 1+1, 1+1+1, and so on", one could not avoid, according to Wittgenstein, resorting to another variable of the same kind. Thus, with infinity, one is in the typical situation where the process of giving definitions comes to an end and makes way for the vision/intuition of the grammatical properties of the law of sign generation. A large part of Wittgenstein's criticism of set theory derives precisely from the thesis that our understanding of the infinite has to do with the "ultimate grammatical given". One aspect of his critique covers the Cantorian notion of an infinite class, grounded on the separation between infinity and law, or rather, on the idea that laws are mere expedients for representing infinite extensions. According to the Austrian philosopher, the term "class" has meaning only when it refers to a collection whose elements can be listed. As the logical possibility of enumeration is involved here, his conception makes the word "class" synonymous with the expression "finite class", disregarding the empirical limitations which the material process of forming a list undergoes. On the other hand, as we know, infinite lists are logically impossible and the only legitimate sense in which, from Wittgenstein's anti-extensionalist point of view, one can speak of infinity is with reference to a law of generation. The original sin of set theory is the idea of a "merely possible" symbolism or calculus, i.e. a symbolism which would be logically possible, but whose realization would be impeded by the empirical limitations which human activity of sign manipulation inevitably undergoes (*PG*, II, Ch. VII, §40, p. 469). Wittgenstein suggests comparing the foundations of set theory to the notion of a calculus with signs of infinite length – for example lists with infinite items – which, if they could be written down, would constitute the only suitable representations of the actual infinite.[40] Cantorians admit that the actual infinite cannot be represented (*dargestellt*) in the usual symbolism of mathematics, but maintain that one can resort to a new notation – that of set theory – by means of which the actual infinite, if not represented, can be described (*beschrieben*):

> Set theory attempts to grasp the infinite at a more general level than the investigation of the laws of the real numbers. It says that you can't grasp the actual infinite by means of mathematical symbolism at all and therefore it can only be described and not represented. The description would encompass it in something like the way in which you carry a number of things that you can't hold in your hand by packing them in a box. They are then invisible but we still know we are carrying them (so to speak, indirectly). One might say of this theory that it buys a pig in a poke. Let the infinite accommodate itself in this box as best it can.

(*PG*, II, Ch. VII, §40, p. 468)

This suggestion of an "underlying imaginary symbolism", which would be able to directly represent the infinite, undermines set theory from the beginning: in fact, if, as Wittgensteins thinks, the impossibility of such a symbolism has a logical nature, then there is nothing at all that the set-theoretical description should replace. But what he judges as the most pernicious effect of the set-theoretical approach is that it enormously strengthens the tendency towards the adoption of *a descriptivist model* in the conception of internal relations. In my opinion, the primary source of Wittgenstein's resolute opposition to set theory is his awareness of this risk. As known, highly general notions, such as that of a well-ordered set or the Dedekindian notion of an infinite set, are defined in the theory. Once concepts of such a sort are introduced, the question of whether a given mathematical structure falls under one of them can be raised: the questions of whether the relation of *smaller than* is a well-ordering of the set of the naturals, and of whether it is a well-ordering of the rationals too, and, again, of whether the set of even numbers is infinite in the Dedekindian sense are, though very simple, all perfectly admissible. Wittgenstein's objection to the set-theoretical approach is that it treats the relation between the general and the particular in a way that fits the domain of physical objects, not that of mathematical structures:

> This is always a case of the mistake that sees general concepts and particular cases in mathematics. In set theory we meet this suspect generality at every step.... The distinction between the general truth that one can know, and the particular that one doesn't know, or between the known description of the object, and the object itself that one hasn't seen, is another example of something that has been taken over into logic from the physical description of the world.
>
> (*PG*, II, Ch. VII, §40, p. 467)

In trying to explain the grounds of Wittgenstein's critique one has to start from his conception of mathematical theorems as disguised rules of grammar. It holds true also of a theorem affirming that a certain given structure has a certain property defined in general terms (as, for example, that **N**, in its natural order, is a well ordered set). In acknowledging a given sign structure as a proof of the theorem and, therefore, in seeing it as something which induces us to adopt the corresponding rule, the definition of the general concept is not able to play any normative role. According to Wittgenstein's view, a general concept has a purely extensional nature, in the sense that it does not transcend the class of the mathematical structures which, with or without the mediation of a proof, are counted, by definition, as particular cases falling under the concept (which are *seen* as such). This is true of all mathematical concepts, with the notable exception of those concepts correlated to effective methods for deciding whether any given object falls under them. In fact, to speak of a mathematical structure is nothing but a slightly overemphatic way of referring to the meaning of certain signs; and, for this very reason, to attribute a not yet seen formal property to such a structure amounts to assuming that the corresponding signs have a meaning which we are not able to understand. This lies at the foundation of Wittgenstein's pivotal principle that in mathematics (as in the whole of grammar) *esse est percipi,* and of its corollary about the completeness of "mathematical knowledge":

> In mathematics there isn't any such thing as a generalization whose application to particular cases is still unforeseeable. That's why the general discussions of set theory (if they aren't viewed as calculi) always sound like empty chatter, and why we are always astounded when we are shown application for them. We feel that what is going on isn't properly connected with real things.
>
> (*PG*, II, Ch. VII, §40, p. 467)

Set theory, not only the Cantorian one, has, in Wittgenstein's opinion, a typical "perverse" way of managing things: the essential properties of mathematical structures are detached from the structures themselves (abstraction), and these are presented in a guise that creates the appearance that they might be lacking the properties in question. Instead of the single structures, whose properties coincide with their acknowledged properties, one finds, on one hand, general notions, and amorphous sets on the other. The task assigned to proofs is that of discovering which abstract concepts apply to the sets previously rendered amorphous: "With this there goes too the idea that we can use language to *describe* logical forms. In a description of this sort the structures are presented in a package and so it does look as if one could speak of a structure without reproducing it in the proposition itself" (*PG*, II, Ch. VII, §40, p. 468). Wittgenstein's first objection seems to be that any definition of a given structure, inasmuch as it identifies that structure, is condemned to make reference, though covertly, to its essential properties and thus to be viciously circular. It is of some importance that in this connection he gives as an example Russell's definition of the ancestral relation of a relation, which already in the *Tractatus,* for the same reason, he had charged with vicious circularity: "Concepts which are packed up like this may, to be sure, be used, but our signs derive their meaning from definitions which package the concepts in this way; and if we follow up these definitions, the structures are uncovered again. (Cf. Russell's definition of 'R*')" (*PG*, II, Ch. VII, §40, p. 468). A second objection to the descriptivist style of set theory concerns its consequences on how the relation between sense and proof should be construed. Using the abstract notions of the theory, it becomes quite natural to think that a mathematical quandary can be meaningfully formulated and that a proof decides which alternative is the true one. But, if – as Wittgenstein believes – the proof of one, differing from the verification of a genuine proposition, rules out as senseless the other, then set theory ends up by assuming the conceivability of the inconceivable. Thus, as regards the notion of well-ordering, Wittgenstein observes:

> For instance, when we say that we can arrange the cardinal numbers, but not the rational numbers, in a series according to their size, we are unconsciously presupposing that the concept of an ordering by size does have a sense *for rational numbers,* and that it turned out on investigation that the ordering was impossible (which presupposes that the *attempt* is thinkable).
>
> (*PG*, II, Ch. VII, §40, p. 461)

Similarly, as regards Dedekind's definition of the notion of an infinite set, Wittgenstein says:

> the definition pretends that whether a class is finite or infinite follows from the success or failure of the attempt to correlate a proper subclass with the whole class; whereas there just isn't any such decision procedure. – 'Infinite class' and 'finite class' are different logical categories; what can be significantly asserted of the one category cannot be significantly asserted of the other.
>
> (*PG*, II, Ch. VII, §40, pp. 464–5)

Thus, the set-theoretical "chatter" is not at all innocuous since, through its descriptivist approach to internal relations, it leads to the concealment of some fundamental features of the relation between sense and proof in mathematics. However, the question whether, according to Wittgenstein, something in set theory has to be saved, and, if so, what, still remains open.

FOUNDATIONS OF MATHEMATICS (II)

As we have said in the first section of this chapter and have already partially seen, critical references to the major schools of foundational research are not lacking in Wittgenstein's writings of the intermediate phase. Let us begin by expounding his point of view on the metamathematical investigations inspired by Hilbert's programme.[41] In discussing the problem of the consistency of the axioms of a formalized mathematical theory, Wittgenstein sets himself two main objectives: first, to supply a correct interpretation of the claim that "a contradiction that nobody has seen might be hidden in the axioms from the very beginning, like tuberculosis" (*WVC*, p. 120); second, to prove how unfounded the fear is that the eventual discovery of a contradiction in the system of arithmetic can jeopardize all the results obtained in mathematics up until then; namely, to show that this is nothing but a "superstitious fear" (*WVC*, p. 196). It is undeniable that, when faced with the peremptory assertion that "it does not make sense to talk of *hidden contradictions*" (*WVC*, p. 174), which Wittgenstein contrasts to the thesis that an axiomatic system might be inconsistent without our knowing, one feels a certain sensation of discomfort. Indeed, at least at first sight, the challenged claim appears not only to be quite meaningful, but even factually true: is not the system of Frege's *Grundgesetze* a conspicuous example of a logical system which contained a hidden contradiction, subsequently brought to light by Russell's discovery of the celebrated antinomy of the class of all the classes not belonging to themselves? Should we perhaps say, following Wittgenstein's indications, that until Russell had discovered it, the contradiction of Frege's system did not exist ("a contradiction is a contradiction only *if it is there*" (*WVC*, p. 120)[42])? An answer to these rather disconcerting questions can be found only by placing Wittgenstein's theses in the context of his overall conception of mathematics in the intermediate phase. First of all, on the basis of the verification principle, the question as to whether, by carrying out derivations within a certain formal system, a formula of the form "*p*. ~ *p*" might ever be obtained, has a meaning only on condition that a method for deciding it is available: "as long as no procedure for finding a contradiction is given, there is no sense in wondering if our inferences might not eventually lead to a contradiction" (*WVC*, p. 120); "Have we a method for finding the contradiction? If not, then there is no question here. For you cannot look for anything *ad infinitum*" (*WVC*, p. 143). If such a method were at our disposal, the statement that there is a contradiction in the system would have a well defined sense prior to its eventual proof (and, likewise, the question as to whether a contradiction is or is not derivable from the axioms would have meaning before the answer has been given). In this hypothetical situation, we could even say that *we do not know yet* whether the system is coherent, and the application of the decision procedure would solve this genuine doubt. The conditions laid down by the verification principle are the only ones under which it would make sense to speak of the presence or lack of a formal property – such as the derivability of a contradiction from the given set of axioms –, without the presence (or absence) of the property having been previously established. Then, we would be faced with one of those situations where Wittgenstein's postulate that in mathematics there are no unacknowledged internal relations and properties loses its validity. A system could be inconsistent without our knowing, in the restricted sense in which this gap in our grammatical "knowledge" could always be filled by the application of an available calculating procedure. A contradiction could be hidden only in the same sense in which, for example, the property of 11,003 being prime, or the property of √2 having "4" as the eighteenth digit of its decimal expansion, could be. In these contexts, one can talk about the possible presence of a formal characteristic we do not yet *see,* because what we mean can be fully explained with reference to a definite calculation method.

As regards the problem of the consistency of arithmetic, trouble arises, obviously, from the unsatisfied verification conditions which are required to speak meaningfully about the possible presence of a contradiction hidden within its axioms:

> What would a hidden contradiction be, after all? I can say, for example: The divisibility of the number 357567 by 7 is hidden for just as long as I have not applied a certain criterion – the rule for division, I suppose. To turn the hidden divisibility into an open one I need only apply the criterion. Is it the same with contradiction? Obviously not. I cannot bring a contradiction to light by applying a criterion, can I? So I say that all this talk about a hidden contradiction does not make sense, and the danger mathematicians talk about – as if contradiction could be hidden in present-day mathematics like a disease – this danger is a mere figment of the imagination.
>
> (*WVC*, p. 174)

This was exactly the case with the *Grundgesetze's* system and Russell's discovery. Then there is a negative answer to the question we stated previously, i.e. if, following Wittgenstein's indications, we have to conclude that in Frege's system the contradiction was not there before Russell discovered it. As the inconsistency of the calculus was not a property the presence or absence of which could be established by the application of a suitable algorithm, its acknowledgement did not leave the calculus unchanged. In this and in all similar cases (among which Wittgenstein mentions the discovery of the non-independence of one of the axioms of a system), to see a new formal property of a calculus means to see a new calculus. This thesis is an immediate consequence of Wittgenstein's postulate that, beyond the verificationist boundaries, in mathematics there are no unacknowledged connections and properties. A decision procedure for the syntactic properties of a formalized system not being on hand, such a system comes to coincide, in this aspect, with the system identified by its actually acknowledged properties. Thus, according to Wittgenstein, one should not say that Russell, deriving the contradiction from the axioms of *Grundgesetze,* has established the inconsistency of Frege's logical calculus, but that he has invented a new calculus, *this one* with the property of being inconsistent. This is the reason why the Austrian philosopher maintains that, so long as a contradiction is not seen, a calculus is perfectly in order. Correspondingly, from the verificationist point of view, the attempt to prove the consistency of arithmetic cannot be legitimately described as true mathematical research. According to Wittgenstein, one can search only within a system, namely, within a space of acknowledged possibilities, and with the knowledge of a method for finding the object looked for (as happens whenever a general procedure of calculation is at disposal and it is only a matter of applying it to a particular case). The situation of the proof of consistency of arithmetic is quite similar to that of the search for a proof of Golbach's conjecture. No description, *in general terms,* of what one is seeking to prove is able to impose any constraint on our future grammatical behaviour, i.e. on our eventual seeing, in a given sign construction, a compelling reason to adopt a new grammar rule and criterion for the correctness of the derivations carried out within the formal system. Were it adopted, the rule would exclude empirical description of the kind "by a correct derivation, a formula of the form '*p* . ~ *p*' has been obtained" as senseless; and whenever a contradiction was effectively derived, the conclusion that a mistake in performing the derivation *must* have been made, would be inferred. According to Wittgenstein, both in the case of the proof of consistency and in the case of the proof of Goldbach's conjecture, we are faced with an asystematical (*planlose*) attempt to construct a new calculus, namely, a sign construction which would induce us to adopt a new rule of the geometry of signs: "It is a stroke of luck, as it were, that I come to see the new system. To be sure, I can go over to the new system; but I cannot look for it, I cannot reach it by means of transformation, and I cannot come to see its possibility by means of a proof" (*WVC,* p. 146). What has been observed regarding Sheffer's "discovery" holds true also for the "discovery" of a proof of consistency of arithmetic: the passage from the old to the new system – from the system whose consistency has not yet been proven to the system whose consistency has been proven –, is not a codifiable process, since, as long as we are confined within the former, we cannot be led by its rules to the latter, and the new system comes into being only when we actually see it.

Taken literally, the claim that the metamathematical investigations of the 1920s were nothing more than asystematical attempts to invent new calculi from nothing seems as absurd as the thesis that an axiomatic system cannot contain a hidden contradiction, except for an as yet unfound contradiction, which can however be found by applying a known method for searching. But Wittgenstein knew about, at least in broad outline, Hilbert's and his pupils' efforts to make the consistency problem mathematically manageable, and he certainly did not wish to deny, paradoxically, this state of affairs. To clarify the real import of his claims, it is of great importance to link them to what he says about Hilbert's so-called simple model of the proof of consistency.[43] According to this model, the proof of consistency is a proof by induction on the length of a derivation in the formal system. As seen earlier (pp. 79–85), Wittgenstein denies that the knowledge of the general form of the inductive process is sufficient for understanding the sense of a universal generalization, in the absence of an inductive proof (hence, also for understanding the sense of the statement that no contradiction can be derived in the object-system). The search for a proof of consistency is asystematical to the extent that no description of the form which such a proof should have is able to condition the "direct perception" of the mathematical property at issue. In other words, the process whereby a certain sign construction induces us to adopt the grammar rule excluding as senseless any empirical statement of the kind "by a correct derivation, a formula of the form '*p*. ~ *p*' has been obtained" – i.e. our *seeing* a given construction in this way – is indisciplinable by any general characterization of the required sign transformations, supplied beforehand. But this amounts to admitting that the knowledge of Hilbert's simple model of the consistency proof does not provide the "logical hypothesis" that no contradiction can be derived in arithmetic with a definite sense.

Also Wittgenstein's re-interpretation of the relation between the metamathematical enquiry and the object-system is based on the peculiar nature that he attributes to inductive proofs. An inductive proof of the consistency of an axiomatic system plays, with respect to the object-system, the same role that an inductive proof of a fundamental law of arithmetic plays with respect to numerical arithmetic. Consider, for instance, the proof of the associative law of the sum. It induces us to adopt the grammar rule which excludes as senseless empirical descriptions of the kind "by a correct application of the recursive definition of the sum, for three given numbers *a*, *b* and *c* two different results have been obtained by adding a to (*b*+*c*) and by adding (*a*+*b*) to *c*", and, consequently, to adopt a new criterion for the correctness of the sign transformations performed in numerical arithmetic. Similarly, the proof of consistency induces us to adopt the grammar rule which rules out as senseless empirical descriptions of the kind "by a correct application of the rules of tranformation of the system, a formula of the form '*p* . ~ *p*' has been obtained in a finite number of steps", and, thereby, to adopt a new criterion for the correctness of the derivations performed in the object-system. By matching the results that can be achieved in metamathematical investigations to the results obtained in Skolem's primitive recursive arithmetic, Wittgenstein sets himself a negative objective, which he expresses as follows:

> The system of calculating with letters is a new calculus; but it does not relate to ordinary calculation with numbers as a metacalculus does to a calculus. *Calculation with letters is not a theory.* This is the essential point. In so far as the "theory" of chess studies the impossibility of certain positions it resembles algebra in its relation to calculation with numbers. Similarly, Hilbert's "metamathematics" must turn out to be mathematics in disguise.
>
> (*WVC,* p. 136)

Mathematics, thus, is contrasted to theories, and by maintaining that metamathematics is nothing but disguised mathematics, Wittgenstein is suggesting precisely that it is not a theory at all. In fact, a theory is a system of statements which have sense independently from their proof. But mathematical theorems are expressions of rules of grammar and only where the verificationist requirements are satisfied can they be considered to be close to genuine statements. Metamathematical theorems, proven by induction, have a completely different status; a theorem of this sort "is related to the proof as a sign to the thing signified. The proposition is a name for the induction. The former goes proxy for the latter; it does not follow from it" (*WVC*, p. 135). However, the passage to a new calculus, produced by an inductive proof of consistency, has no power of legitimization over the object-system. It is true that such a proof can actually modify our attitude, prompting us to look for a mistake in the sign transformations whenever a supposedly correct derivation of a contradiction is supplied. But the mathematical practice within a calculus – the production of rules of the geometry of signs – cannot depend in any essential way on the invention of a new piece of mathematics, even though the latter regards the internal possibilities of the former system:

> No calculus can decide a philosophical problem. A calculus cannot give us information about the foundations of mathematics. So there can't be any "leading problems" of mathematical logic, if those are supposed to be problems whose solution would at long last give us the right to do arithmetic as we do. We can't wait for the lucky chance of the solution of a mathematical problem.
>
> (*PG*, II, Ch. III, §12, p. 296)

In fact, the construction of a portion of mathematics which deals with the internal possibilities of a calculus is still only a process of sign manipulation, exposed, of course, to all the empirical "uncertainties" that can threaten the calculating practice which takes place within the object-calculus (possibilities of unnoticed errors etc.). On the other hand, there are no mathematical procedures which decide the correctness of the application of the term "calculus" in any given case. Thus, the meaning of "calculus" does not transcend its actually acknowledged extension:

> If we are asked: but is it now really certain that it isn't a different calculus being used, we can only say: if that means "don't we use other calculi too in our language?" I can only answer "I don't know any others at present".... But the question cannot mean "can no other calculus be used?". For how is the answer to that question to be discovered? A calculus exists when one describes it.
>
> (*PG*, II, Ch. I, §1, p. 245)

"I said earlier 'calculus is not a mathematical concept'; in other words, the word 'calculus' is not a chesspiece that belongs to mathematics" (*PG*, II, Ch. III, §12, p. 296). By this Wittgenstein is not denying the feasibility or the legitimacy of a mathematical treatment of the notions of formal system and of proof (in a given formal system), but, rather, is rejecting the idea that, through a treatment of such a kind, something like the essence of a calculus or of a proof can be grasped. The terms "calculus" and "proof" are two typical instances of general terms the meaning of which is given only through the enumeration of the systems of rules of sign transformation – and of the sign structures – that, by definition, are called, respectively, "calculi" and "proofs". Moreover, the introduction of a new grammar rule concerning any one of these two terms is characterized by the absolute freedom of our linguistic decisions with respect to the meaning attributed to the term by some preceding general definition. Wittgenstein's evaluation of the effects of the eventual "discovery" of a contradiction in a calculus is a simple corollary of the above thesis. Consistency is usually conceived as a necessary property of every axiomatic system worthy of this name, since from a contradiction anything would follow. Wittgenstein briefly sketches various arguments to show that the adoption of grammar rules regarding the term "calculus" are not subject to any logical constraint – not even consistency – and that, therefore, our calculating practice is not in need of any legitimization through mathematical proofs that these supposedly necessary requirements are really satisfied. He stresses: (i) the innocuousness of a contradiction in a system, as long as this is considered in a purely formalistic manner; (ii) the possibility of finding applications for inconsistent calculi; (iii) the irrelevancy to physics of the eventual discovery of a contradiction in some branch of mathematics; (iv) the limited, localized import of the occurrence of a contradiction in a calculus, in the sense that it would leave untouched the parts of the system not directly involved in the derivation of the contradiction. A real difficulty may arise, with the occurrence of a contradiction, if the axioms are construed as primitive rules of sign manipulation and the theorems as derived rules (e.g. in an axiomatization of numerical arithmetic, as rules of substitution). If there were two derived rules, one of which lays down the substitutability of a certain arithmetical term with another, while the second rule forbids this very substitution, we would not know how to proceed when, at a given step of a process, of calculation, we were faced with this term. But the fact that we would not know how to react in this situation signals only the existence of an empirical characteristic of our linguistic practice, not the existence of a logical reason at the foundations of this practice (if a justification which transcends the anthropological given of our linguistic usage is meant by the latter):

> We don't have any reaction to a contradiction. We can only say: if it's really meant like that (if the contradiction is *supposed* to be there) I don't understand it. Or: it isn't something I've learnt. I don't understand the sign. I haven't learnt what I am to do with it, whether it is a command, etc.... "The rules may not contradict each other" is like "negation, when doubled, may not yield a negation". That is, it is part of the grammar of the word "rule" that if "p" is a rule, "$p \,.\, \sim p$" is not a rule.... Here too we cannot give any foundation (except a biological or historical one or something of the kind); ... Once again we have a grammatical structure that cannot be given a logical foundation.

(*PG,* II, Ch. III, §14, pp. 303–4)

However, if we were faced with such a conflict between two rules of sign manipulation, we could easily re-establish order by adopting a new rule; and within this new calculus, we could go on confidently.

In discussing the use of quantifiers in mathematics and the notion of a real number, we have already had the occasion of touching the problem of the relations between the intermediate Wittgenstein and the intuitionism of Brouwer and Weyl. Another interesting aspect of this issue regards the question of the validity of the Law of Excluded Middle. Preliminarily, it must be noted that Wittgenstein sees Brouwer's rejection of the Law of Excluded Middle as equivalent to the introduction in logic of a third value – undecidability – next to provability and refutability of a proposition. As known, such an interpretation does not reflect faithfully the meaning that intuitionists assign to sentential connectives (as proven by the intuitionistic validity of the negation of the negation of the Law of Excluded Middle).[44] But this does not remove the task of explaining the reasons for Wittgenstein's sharp opposition to Brouwer's standpoint that there can exist unsolvable mathematical problems, namely, problems which we will never be able to solve. According to Wittgenstein, this idea is a mere sub-product of the extensional conception of the infinite. For example, if π is understood extensionally, the question of whether the group of digits "777" occurs in its decimal expansion can be condemned to remain necessarily without an answer (by calculating one after the other the digits of the expansion, it might be the case that we will never be in the position to be able to affirm that the figure "777" occurs, nor in the position to negate it): "Of course, if mathematics were the natural science of infinite extensions of which we can never have exhaustive knowledge, then a question that was in principle undecidable would certainly be conceivable" (*PR* §174). Brouwer's use – in order to define entities so that the statement that they possess a certain mathematical property is neither provable nor refutable – of undecided mathematical questions, such as those regarding the occurrence of a certain sequence of digits in the decimal expansion of π, is based, according to Wittgenstein, on an extensionalist residue. The ground for this charge seems to be contained in the following argument: to use an undecided mathematical alternative in the definition of a certain mathematical entity, its sense must obviously be understood; but, in so far as it is not an alternative which can be decided by applying a known calculating procedure, it can appear to be meaningful only because of the underlying illusory extensional interpretation:

> If someone says (as Brouwer does) that for $(x)\ .\ {}^{f}{}_1x = f_2x$, there is, as well as yes and no, also the case of undecidability, this implies that "(x) ..." is meant extensionally and we may talk of the case in which all x happen to have a property. In truth, however, it's impossible to talk of

> such a case at all and the "(*x*) ..." in arithmetic cannot be taken extensionally.
>
> (*PR* §174)

The tacit presupposition of this attack is the rejection, by Wittgenstein, of the idea that the meaningfulness of a mathematical proposition (of a mathematical alternative) can be based not on the knowledge of a decision procedure nor on that of a proof, but rather on a much weaker general knowledge: that of the form which the proof of any statement having the same logical structure of the statement under consideration should have. If the rejection of extensionalism is not accompanied by the acceptance of some general notion of proof, only an actually constructed proof is able to provide the proposition (the alternative) with a sense (except in the case of a proposition belonging to a whole system of propositions, each one of these being decidable by the application of a general method of calculation). This crucial assumption explains why, according to the Austrian philosopher, the supposition of the existence of undecidable mathematical questions is untenable. In order to understand a given mathematical problem, in fact, we should have some sort of knowledge of the *type* of solution it would have; and, if the problem were unsolvable, we would never become acquainted with the same (we would for ever be unable *to see* the solution directly). But if a description of *the form* of the solution cannot play any normative role on the linguistic decisions by which certain mathematical connections will be eventually ratified as constituting the solution of the problem, then it will have a definite sense only when these internal relations are directly seen. An equivalent formulation of Wittgenstein's point of view runs as follows: there is no internal connection that, in principle, cannot be acknowledged by us, because admitting the existence of such a sort of connection would be tantamount to supposing that our signs have a meaning which we do not yet understand:

> The supposition of undecidability presupposes that there is, so to speak, an underground connection between the two sides of an equation; that though the bridge cannot be built in symbols, it does exist, because otherwise the equation would lack sense. – But the connection only exists if *we* have made it by symbols; the transition isn't produced by some dark speculation different in kind from what it connects (like a dark passage between two sunlit places).
>
> (*PG,* II, Ch. V, §25, p. 377)

"A connection between symbols which exists but cannot be represented by symbolic transformations is a thought that cannot be thought. If the connection is there, then it must be possible to see it" (*PR* §174). Wittgenstein's criticism of the notion of unsolvable problems preludes, as is to be expected, a defence of the Law of Excluded Middle. If the range of meaningful mathematical propositions is determined by the requirements of the strong verificationist view expounded in the third section (Mathematical propositions), then the universal validity of the Law of Excluded Middle is guaranteed by definition. The failure of this logical principle is the sure sign that the verificationist boundaries have been crossed. In this case it is not a matter of the non-applicability of one or more logical principles to a certain class of mathematical propositions, but, rather, it is the very grammatical decision of speaking of propositions that is at stake: "I need hardly say that where the law of the excluded middle doesn't apply, no other law of logic applies either, because in that case we aren't dealing with propositions of mathematics. (Against Weyl and Brouwer)" (*PR* §151); "if the question of the truth or falsity of a proposition is *a priori* undecidable, the consequence is that the proposition loses its sense and the consequence of this is precisely that the propositions of logic lose their validity for it" (*PR* §173); "The whole approach that if a proposition is valid for one region of mathematics it need not necessarily be valid for a second region as well, is quite out of place in mathematics, completely contrary to its essence. Although many authors hold just this approach to be particularly subtle and to combat prejudice" (*PG,* II, Ch. VII, §39, p. 458).

Lastly, we have to tackle the problem of the eventual revisionary import of the intermediate Wittgenstein philosophy of mathematics on current mathematical practice. In his 1929–33 writings, many declarations of drastic exclusion of any influence of his discussions and analyses on the activity of mathematicians may be found. These declarations agree with Wittgenstein's overall conception, which he was developing in this period, of the task of philosophy. Philosophers have to repress the strong inclination to intervene within mathematics, limiting themselves to considering what mathematicians say of their own activity (their "ideology"), and to denouncing the grammatical misunderstandings underlying the interpretations put forward by mathematicians:

> The philosopher only marks what the mathematician casually throws off about his activities. The philosopher easily gets into the position of a ham-fisted director, who, instead of doing his own work and merely supervising his employees to see they do their work well, takes over their jobs until one day he finds himself overburdened with other people's work while his employees watch and criticize him. He is particularly inclined to saddle himself with the work of the mathematician.
>
> (*PG,* II, Ch. V, §24, p. 369)

But, in the light of the exposition made hitherto, it seems undeniable that Wittgenstein did not follow his own precept. The most interesting aspect of the issue is that his manifest violation has two completely opposing sources, which bears witness to the unresolved tensions which run through his reflections during the intermediate phase. On one hand, his critique of the extensionalist conception of the infinite prompts him towards a characterization of certain general mathematical notions which goes abundantly beyond the verificationist limits and causes drastic and overt exclusions, with respect to both classical and constructivist mathematics (as clearly exemplified by his treatment of the notion of real number). On the other, it is the very conception of general notions in mathematics (in grammar) – to which, as the example of real numbers shows, Wittgenstein does not remain completely faithful – that seems inevitably destined to produce effects on mathematical practice. According to such a view, the meaning of a mathematical term, with the notable exception of the terms definitionally associated to a decision procedure, coincides with its acknowledged extension. Were the identification of mathematical concepts – both general and individual – with their extensions taken seriously, there would be disruptive consequences on mathematical practice. In fact, this would place in doubt one of the most fecund tendencies of mathematics, namely, to construct ever more abstract theories. Wittgenstein's conviction about the mathematical irrelevancy of the general notion of an inductive process of definition and of proof (and, given his assumptions on the nature of mathematical induction, his conviction about the substantial irrelevancy of the general notion of an effective recursive process), and his overt criticism of set theory, testify this potential "dangerous" effect of his stance. Of course, it is with respect to mathematics that the Austrian philosopher's conception of the relation between the general and the particular in grammar fully reveals its revisionary import because, here, it does not clash merely with some theory of language but with a well-proven practice. Wittgenstein's own explicit evaluation of the possible consequences that his approach would have on current mathematical practice clearly suffers from the ambiguity of his position. He often resorts to the distinction between calculus and prosa to support his claim of the practical ineffectualness of his critical remarks. For example, referring to set theory, he says:

> When set theory appeals to the human impossibility of a direct symbolization of the infinite it brings in the crudest imaginable misinterpretation of its own calculus. It is of course this very misinterpretation that is responsible for the invention of the calculus. But of course that doesn't show the calculus in itself to be something incorrect (it would be at worst uninteresting) and it is odd to believe that this part of mathematics is imperilled by any kind of philosophical (or mathematical) investigations.... What set theory has to lose is rather the atmosphere of clouds of thought surrounding the bare calculus, the suggestion of an underlying imaginary symbolism, a symbolism which isn't employed in its calculus, the apparent description of which is really nonsense.

(*PG*, II, Ch. VII, §40, pp. 469–70)

From Wittgenstein's quasi-formalistic point of view, even axiomatic set theory is a calculus like any other, having its own rules of sign transformation, and only the eventual application of its results as grammar rules – inside or outside mathematics – could confer a greater or lesser interest on it. But, at the same time, the Austrian philosopher is perfectly aware of the fact that, if his conception of mathematics were largely shared among mathematicians, it would have significant consequences on the style of their work, impeding or limiting the further growth of mathematics:

> A philosopher feels changes in the style of a derivation which a contemporary mathematician passes over calmly with a blank face. What will distinguish the mathematicians of the future from those of today will really be a greater sensitivity, and *that* will – as it were – prune mathematics; since people will then be more intent on absolute clarity than on the discovery of new games. Philosophical clarity will have the same effect on the growth of mathematics as sunlight has on the growth of potato shoots. (In a dark cellar they grow yards long).
>
> (*PG*, II, Ch. V, §25, p. 381)

Here Wittgenstein actually ventures a prophecy on the future of mathematics, foreseeing that mathematicians will change their sensitivities and, consequently, will modify the style of their work: then, constructive exigencies will be sacrificed to "philosophical clarity". Since one of the main results of philosophical clarification regards the correct understanding of the relation between the general and the particular in mathematics, one can rightfully affirm that mathematicians will give up constructing more and more abstract theories which have general notions as their object. But nothing was more dear to Wittgenstein than "absolute clarity"; and it would have been of no importance to him if, to obtain it, the "onwards movement in building ever larger and complicated structures", typical of our civilization, would have been interrupted.[45]

3

FROM FACTS TO CONCEPTS
The later writings on mathematics (1934–44)

THE CRISIS OF VERIFICATIONISM: RULE-FOLLOWING

In the previous chapter we saw how the existence of general calculating procedures led Wittgenstein, in the years 1929–33, to introduce, albeit with a certain amount of caution, the notion of mathematical proposition. With this notion, the possibility of distinguishing between the understanding of the sense and the knowledge of the proof is admitted for every proposition belonging to a whole system of propositions which can be decided by the application of one and the same effective method. As seen, the propositional interpretation of a mathematical expression of this kind, e.g. an arithmetical identity, is an interpretation in terms of what the Austrian philosopher calls "the geometry of signs". From his quasi-formalistic point of view, to say, for instance, that the identity "25 × 25 = 625" is true is the same as asserting that the correct application of certain rules of sign transformation – set up in advance – to the pair of numerals ("25", "25") yields a sequence of expressions, a figure, which ends with the numeral "625".[1] But this statement has not to be construed as an empirical statement about physical signs (considered neither as tokens nor as types): it does not describe results usually obtained by people trained in a certain routine of application of the rules at issue, nor expresses a prediction on the results they will probably obtain. A mathematical theorem is always the disguised expression of a rule, which establishes what *must* be obtained whenever the pertaining calculating processes are carried out correctly. Its normative value is clearly revealed by its "geometrical" formulation: that the correct application of the general calculating procedure to expressions of such-and-such a sort *necessarily* yields such-and-such result (and, therefore, a mistake must have been made every time a different result is worked out). The decisive point, of course, is Wittgenstein's view of the purely linguistic nature of necessity. Suppose that, by transforming certain expressions – written in decimal notation – in accordance with the rules of a given general procedure of calculation, the sequence of digits "$a_1a_2... a_n$" is obtained and that the entire sign construction is ratified as correct. This amounts to the adoption of a grammar rule which rules out as senseless any empirical statement that a certain figure – whose shape is other than that of "$a_1a_2 \ldots a_n$" – has been obtained by applying correctly the method of calculation to numerical expressions of such-and-such a sort. Obtaining "$a_1a_2 \ldots a_n$" is assumed as a criterion to affirm that, starting with expressions correctly identified as being the same shape as the starting expression of the proof, the rules have been followed, and thus, as a criterion to call any given series of sign operations that have been carried out on such expressions "a correct application of the general procedure". That, from certain numerical expressions, one *must* obtain a sequence of digits which is the same shape as "$a_1a_2 \ldots a_n$" if the rules have been correctly followed is, indeed, only another way of saying that it *would be senseless* to identify any result differing in shape from "$a_1a_2 \ldots a_n$" as something that has been obtained by having correctly applied the general procedure to those expressions. By the acknowledgement of a certain sign figure as the construction yielded by a correct application of the calculation rules, the occurrence – as end-result of any given similar process of sign transformation – of an expression having the same shape as the end-result of that figure is elevated to the dignity of a grammatical criterion for the empirical predicate "being yielded by a correct calculation" (obviously not the sole criterion). As noted previously, this thesis regarding the essential, internal nature of the relation that links a correct calculation to its result is enunciated by Wittgenstein, in the writings of his intermediate phase, when he says that, in mathematics, "process and result are equivalent to each other".[2] The theme of the relation between calculation and experiment, which, as is well known, is one of the subjects that Wittgenstein most reflected upon in the ten years from 1934 to 1944, is nothing more than a development and a refinement of this old idea. The following section of this chapter will be devoted exclusively to its examination. For the time being, it is opportune to go a bit more deeply into some implications of the verificationist approach. Consider the arithmetical proposition "11,003 is prime", assuming that, until now, it has been neither proven nor refuted, and try to formulate, as explicitly as possible, the statement that can be made with it. Suppose that the method of calculation associated with the predicate "prime" consists in dividing the checked number by each one of the numbers that are less than it is and greater than or equal to 2, and in ascertaining whether the remainder of each of these divisions is equal to or different from 0. Then, we have the following hypothetical statement: by correctly applying the checking procedure to 11,003, only remainders different from 0 are obtained. When a proof of this statement is effectively constructed, i.e. when a certain sequence of signs is acknowledged to be the proof that, from the pertinent divisions, only remainders different from 0 are obtained, then the statement is transformed into the expression of a grammar rule. With this acknowledgement, the normative dimension would be introduced: that "must" which is the true hallmark of mathematics. But, according to Wittgenstein's strong verificationism, a statement such as

"11,003 is prime" can be understood without knowing its eventual proof. In fact, the knowledge that is required for this understanding is simply that of the existence of an internal relation of logical equivalence (by definition) between "11,003 is prime" and "for every *x* greater or equal to 2 and less than 11,003, if 11,003 is correctly divided by *x*, a remainder different from 0 is obtained". Knowing the existence of this relation means knowing the assertibility-conditions for the proposition "11,003 is prime": i.e. which algorithm (defined in general terms) must be applied, and what results (again described in general terms such as "remainder of a division", "different from 0" etc.) such eventual application should have, in order that the conclusion that 11,003 is prime can be inferred. The construction of the proof, if it works, shows that the assertibility-conditions are satisfied. As we know from Chapter 2, the grammatical "knowledge" that Wittgenstein's verificationism demands with reference to a predicate such as "prime" is the knowledge of the existence of a relation of logical equivalence (by definition) between *any* instance of the schematic expression "*y* is prime" and the corresponding instance of the schema "for every *x* greater than or equal to 2 and less than *y*, if *y* is correctly divided by *x*, a remainder that is different from 0 is obtained". It supplies, at one stroke, the knowledge of the conditions for asserting each single proposition that belongs to the infinite system of instances of the schema "*y* is prime". Obviously, if in order to understand the statement that 11,003 is prime one must know its assertibility-conditions, one must also understand the proposition that expresses such conditions, namely the proposition "for every *x* greater than or equal to 2 and less than 11,003, if 11,003 is correctly divided by *x* a remainder that is different from 0 is obtained". This means that one should be able to understand the singular propositions (finite in number): "if 11,003 is correctly divided by 2, a remainder different from 0 is obtained", "if 11,003 is correctly divided by 3, a remainder different from 0 is obtained", and so on. As this understanding cannot coincide with the knowledge of the truth-value of each of these singular propositions (namely, with that of the remainders obtained by correctly applying the algorithm of division to each pertaining pair of numbers), then, in learning the rules of the dividing procedure, a *general notion* must have been acquired about what the correct application of the rules themselves consists in. The content of this notion should be independent from the acknowledged instances of correct application of the rules, i.e. it should transcend the class of the particular sign processes ratified as correct applications of the rules. If this intensional notion were available, judgements on the correctness of the application of the calculating rules in particular cases would be genuine judgements, in the verificationist sense: only on this condition could they be understood without their truth-value being known, through the knowledge of their assertibility-conditions, framed in general terms. The conformity of a step in a calculation to a general rule for manipulating signs could itself be checked by means of a calculating process. In a situation of such a sort, a statement on the correctness of a certain step could be understood without knowing whether it is true or false, because only an empirical circumstance (namely, the fact that the algorithm for deciding whether an operation on signs has been correctly performed has not yet been applied) would separate us from that knowledge. At this point, it would appear that the verificationist interpretation of the sense of a mathematical proposition as knowledge of its assertibility-conditions primes a regress *ad infinitum*. Can this be avoided? In one way it certainly can: by refusing to consider as genuine statements, from the verificationist point of view, the statements asserting what is to count as the result of the correct application – in any given case – of the rules of a general calculating procedure. This is the path that Wittgenstein followed from 1934–5 onwards in his reflections on following a rule. The point is that, instead of saving verificationism, his considerations end up by destroying it, by undermining at its very roots that possibility of an intensional characterization of some mathematical concepts on which the verificationist conception was based.

In reality, since his writings of the intermediate phase, which, however, rested to a large extent on a non-problematic notion of rule, Wittgenstein had already perceived the disquieting existence of an "unbridgeable gulf between rule and application, or law and special case" (*PR,* I, §164). In a passage from *Philosophical Grammar,* to be found only a few lines after the explicit assertion that one can effectively write a sequence by following a general rule given beforehand, he expresses all his newly born perplexity with these words:

> But we might ask: how does it happen that someone who now applies the general rule to a further number is still following *this* rule? How does it happen that no further rule was necessary to allow him to apply the general rule to this case in spite of the fact that this case was not mentioned in the general rule? And so we are puzzled that we can't bridge over this abyss between the individual numbers and the general proposition.
>
> (*PG,* II, Ch. II, §10, p. 282)

One of the conclusions that Wittgenstein very quickly reached regards the second of the two questions contained in the preceding quotation, i.e. the question that we have come upon whilst re-expounding his verificationist conception. If there were a general rule R_2 that was able to establish how, in any given case, a certain rule of sign transformation R_1 should be applied, then the statement that a certain number is yielded by the correct application of R_1 could be justified by resorting to such a higher order rule, and the bridge over the abyss between R_1 and its applications would be built. Taking into consideration the rule for generating the numerical series that begins with 0 and every other term of which is obtained by adding 10 to the number that immediately precedes it in the series, Wittgenstein wonders: "Why *must* one write 110 after 100? Is there an answer to this question? There is, namely that this is what one usually does after instruction. But isn't there another answer? Couldn't we answer the question 'Why did you add 10 when given the rule?' by giving another rule for following the rule 'Add 10'?" (*AWL*, p. 132). In §§84–6 of *Philosophical Investigations,* this very problem recurs, even though it is placed in a context where it is superimposed on the separate question of the possibility of following rules that are vague or of employing concepts that violate Frege's requisite of clear-cut boundaries.[3] Wittgenstein stresses two things: (i) as far as the higher order rule R_2 is concerned, just the same problem can be posed that it is called upon to resolve in relation to rule R1; in other words, in the face of a new element of its domain of application, it should be established *how* R_2 must be applied to it ("Can't we imagine a rule determining the application of a rule, and a doubt which *it* removes – and so on?" (*PI*, I, §84)); (ii) the ability to imagine doubts about how a general rule should be applied to a particular case does not mean that such doubts actually arise. Thus, not only does the idea of resorting to a new rule, with whose help what is to count as the result of the correct application of a given rule can be determined, entail a regress *ad infinitum*. The point is that this journey backwards from the formulation of one rule to the formulation of another that specifies how the first one should be applied is taken for a mistaken need to find reasons, which clashes with the well-known non-problematic nature of the common practice of applying rules: "So I can say, the sign-post does after all leave no room for doubt. Or rather: it sometimes leaves room for doubt and sometimes not. And now this is no longer a philosophical proposition, but an empirical one" (*PI*, I, §85).

That the problem of following a rule can find its solution only on the condition that one comes out of the spiral of formulating rules to interpret rules and returns to the communal practice of applying them, is the final conclusion of Wittgenstein's reflections. But this conclusion is reached by means of a series of intermediate steps that have to be clarified, one by one. The first of these regards the nature of a statement of the type "the result of the correct application to *m* of the rule R is *n*", e.g. of the aforementioned statement "the result of the correct application to 100 of the rule 'Add 10' is 110" (or, equivalently, "the twelfth term of the series generated by correctly applying the general rule 'Let 0 be the first term and every other term be obtained by adding 10 to the term that immediately precedes it in the series' is 110"). This is without doubt a mathematical statement, where the copula "is" is used atemporally; thus, in Wittgenstein's conception, it is itself the disguised expression of a rule that establishes the existence of an internal relation between *the concept* of following the general rule "Add 10" in the particular case under consideration, and a certain number. In other words, it expresses the grammar rule which excludes as senseless any empirical, temporal statement that affims that a number other than 110 has been obtained by applying correctly the rule "Add 10" to 100. The textual evidence on this pivotal point is impressive. In a lecture held by Wittgenstein in 1935, he observes: "To say that if one did anything other than write 110 after 100 one would not be following the rule is itself a rule. It is to say 'This rule demands that one write 110'. And this is a rule for the application of the general rule in the particular case" (*AWL*, p. 133). Many years later, referring to the rule of adding 1 ("+1") and to the statement that it yields 5 if applied correctly to 4, Wittgenstein writes:

> If it is not supposed to be an empirical proposition that the rule leads from 4 to 5, then *this,* the result, must be taken as the criterion for one's having gone by the rule. Thus the truth of the proposition that 4 + 1 makes 5 is, so to speak, *overdetermined.* Overdetermined by this, that the result of the operation is defined to be the criterion that this operation has been carried out. The proposition rests on one too many feet to be an empirical proposition. It will be used as a determination of the concept 'applying the operation +1 to 4'. For we now have a new way of judging whether someone has followed the rule. Hence 4 + 1 = 5 is now itself a rule, by which we judge proceedings.
>
> (*RFM,* VI, §16)

Things do not change if, instead of taking some general formulation of a rule, one starts from particular examples of its application (for instance, from a finite initial segment of the series such as 0, 10, 20, 30, 40): to say that 110 is obtained from 100 by proceeding *in the same way* as when 10 is obtained from 0, 20 from 10, and so on, is the same as establishing a rule that determines the meaning of the expression "proceeding in the same way" in the context in question (the negation of that statement will not be false, but senseless). In conclusion, the mathematical statement that a certain number is the result of following a rule R in a given particular case has a normative force on the practice of sign manipulation inasmuch as it enunciates a grammar rule about the expression "applying correctly R". This is why, when dealing with the problem of the nature of the disguised definitions which establish what is to count as the result of the correct application of a general rule in a particular case, the theme of intuition crops up, although in a cautious and, at this stage of Wittgenstein's development, somewhat critical way. As seen in the preceding chapters, Wittgenstein often describes as a kind of intuition or vision the direct relationship between the speaker and the domain of necessary truths (of grammar rules). But we have already observed that his use of the terms "intuition" and "vision", in this context, is highly metaphoric and is not meant to allude to *a psychological phenomenon* of acquaintance with properties and internal relations. As early as 1935 (even if it had been anticipated previously), he rejects the use of the term "intuition" as inappropriate, not only because it is overly subject to mentalistic implications but, above all, because it suggests a mistaken assimilation of the process for determining the meaning of linguistic expressions to a sort of cognitive process. Wittgenstein oscillates between different terms to use in the place of “intuition”: “decision” (*Entscheidung*) is the one that comes up most often in critical passages, even if it is always accompanied by certain warnings (“It is no act of insight, intuition, which makes us use the rule as we do at the particular point of the series. It would be less confusing to call it an act of decision, though this too is misleading, for nothing like an act of decision must take place, but possibly an act of writing or speaking” (*BB*, II, §5, p. 143)); or used with a certain caution (“It would almost be more correct to say not that an intuition was needed at every stage, but that a new decision was needed at every stage” (*PI*, I, §186)); or with the addition of some qualification capable of eliminating undesired connotations (“‘I have a particular concept of the rule. If in this sense one follows it, then from that number one can only arrive at this one’. That is a spontaneous decision” (*RFM*, VI, §24)). But whether it is a matter of decisions or, as Wittgenstein sometimes says, of inclinations, or even a matter of purely and simply acting in a certain way, the connections thus produced belong to the conceptual network, not to the description of facts.

Until now, we have simply tried to clarify one essential element in the scene where the problem of rule-following is set.[4] The point is that neither *empirical* judgements such as “Smith has correctly applied to 100 the rule expressed by ‘Add 10’” nor *empirical* judgements of the type “Smith masters the law for the generation of the sequence 0, 10, 20, 30, and so on” are at issue. The former presuppose that the standard of correctness of the application to 100 of the rule “Add 10” has already been established, namely, that the proposition “the result of the correct application to 100 of the rule ‘Add 10’ is 110” has already been filed away in the archives and adopted as a grammar rule. Similarly, when the usual behavioural tests are carried out in order to attribute to someone the mastery of a general rule, the answers that he *must* give – in order that such an attribution be considered reasonably founded from the empirical point of view – are taken for granted (whereas one aspect of the problem of rule-following is precisely that of what answer one *must* give in each single correct application of the rule; i.e. whether there is a justification, and if so what, for the fact that it is precisely the obtainment of a particular result that is assumed as the criterion for the rule having been followed). Sections 143–55 of *Philosophical Investigations* deal, indeed, with criteria that guide the formulation of the empirical judgements whereby the mastery of a rule, and, more generally, the understanding of a system of signs, is ascribed to someone. And it is only after the long parenthesis dedicated to reading that Wittgenstein, in §185, begins to examine the real problem of rule-following (what I will call “the first component” of the problem finds a precise formulation in §186). One of the less felicitous aspects of Kripke’s essay on rules and private language is, in my view, precisely the lack of a clear-cut distinction between empirical statements on the conformity of a given behaviour to a rule (or on the mastery of a general rule), on the one hand, and grammatical statements that *define,* case by case, the concept of correct application of a rule, on the other. Considering, quite rightly, the former as genuine statements, but, at the same time, projecting inadvertently onto them what holds only for the latter – i.e. that there are no facts of any type that render them true, facts which in some way they are responsible to – Kripke is forced to introduce in this context the distinction between truth-conditions and assertibility-conditions. In my opinion, instead, as far as empirical statements on rule-governed behaviours are concerned, the criteria that play a role are the ones that Wittgenstein himself calls "the current criteria"; and it is quite immaterial whether they are formulated in terms of truth-conditions or of assertibility-conditions (in any case, according to Wittgenstein, it is a question of taking public linguistic phenomena into consideration, of making the usual tests for the possession of a certain ability in the use of signs etc.). When Colin McGinn, in his critical assessment of Kripke's essay, affirms that, given Wittgenstein's adhesion to a redundancy theory of truth, "it cannot be that [he] really wishes to deny that semantic sentences have truth conditions – on pain of denying that they express proposition", he seems to perpetuate the aforementioned mistake.[5] Indeed, if by "semantic sentences" he means the sentences used for attributing to someone the understanding of the meaning of the linguistic formulation of a rule or, more generally, of the meaning of a phrase, then McGinn's is only a slightly contorted way of saying that, according to Wittgenstein, such attributions are empirical assertions. But, so as to avoid misunderstandings, it is necessary to add that one of the aims of Wittgenstein's rule-following considerations is precisely that of *denying* that the real "semantic sentences" relevant to this context, namely, the definitions of what is to count as the result of the correct application of a rule in a given case, express genuine propositions.

Having cleared up this misunderstanding, let us examine the first component of Wittgenstein's rule-following considerations, which is to be found in §§185–96 of the *Investigations*. We have seen that each proposition of the form "the result of the correct application to *m* of the rule R is *n*", once "petrified", supplies a criterion for the application of the *concept* of following the rule R in the case mentioned. Now, one could maintain that, in adopting one of these specific rules, one does nothing more than extract what was already implicitly contained in some general formulation of the rule R. One would be dealing, then, with a typical situation where Wittgenstein's verificationism would apply: once the general rule R has been understood, what separates us from the adoption of a particular rule, concerning the result of the application of R in a given case, would be only the empirical circumstance that a certain operation, such as an inferential transition, has not yet been carried out. Referring to the rule "Add 2" Wittgenstein's interlocutor states: "what I meant was, that he should write the next but one number after *every* number that he wrote; and from this all those propositions follow in turn" (*PI*, I, §186). It is certainly not fortuitous that, in the 1935 lecture we have already cited, the problem as to whether the individual steps conforming to a given rule are, in some mysterious way, already contained in it, was presented by Wittgenstein as an exemplification of the more general problem of what one means when one affirms that a logical consequence of a proposition is already contained within it. When, in the *Investigations,* the Austrian philosopher, in reply to his interlocutor, observes: "that is just what is in question: what, at any stage, does follow from that sentence. Or, again, what, at any stage, we are to call 'being in accord' with that sentence", he is denying that the logical connection between a rule and the result of its application in a particular case is already there, though implicitly, and that only an empirical "distance" from it remains, which, in principle, can always be filled. The internal relations are *created* by the decisions that, case by case, we are inclined to take regarding what follows from a general rule, i.e. by the decisions that, case by case, determine the meaning of the description "the result of the correct application of the rule". The mentalist view of meaning as a process capable of performing all the steps before they have been made and the platonist view of an ideal world of necessary connections that are pre-existent to our effective acknowledgement are, respectively, the variant "towards the inner" and "towards the outer" of one and the same misunderstanding on the nature of grammar rules: that an independent reality corresponds to these norms, to these conceptual connections, and that our acknowledgement of their existence is justified inasmuch as it is able to mirror that reality.[6] Constructivism and radical conventionalism both merge in the Wittgensteinian conception of following a rule. The relation between the concept of result of the correct application of a calculating rule in a specific case and a certain number comes into being only in the moment in which it is ratified by us (according to the principle that, in grammar, *esse est percipi*). But this acknowledgement cannot even be bound to the fulfilment of general logical conditions because there is no stable conceptual background for the creation of new internal connections: rather, this creation involves the meaning of every expression actually used to formulate those conditions and also the criteria that establish when the conditions are to be considered satisfied. This is how, in Wittgenstein's words, the supposedly stable background very quickly crumbles:

> "But am I not compelled, then, to go the way I do in a chain of inferences?" – Compelled? After all I can presumably go as I choose! – "But if you want to remain in accord with the rules you *must* go this way." – Not at all, I call *this* "accord". – "Then you have changed the meaning of the word 'accord', or the meaning of the rule." – No; – who says what "change" and "remaining the same" mean here?
>
> (*RFM*, I, §113)

Consider, for instance, the sequence 2, 4, 6, 8,... and the attempt, apparently quite reasonable, to justify the statement (rule) that the successive term in the series must be 10 because only by writing "10" is one proceeding in the same way that one has proceeded when constructing the initial segment. This attempt may be effective on condition that one is provided with a concept for proceeding in the same way, which is independent of the acknowledgement, in each particular case, of a given action as the one falling, by definition, under the concept. But assume, instead, that, in affirming that one proceeds in the same way if he writes "11", one is laying down a grammatical stipulation that determines the meaning of the expression "proceeding in the same way" in the step under examination; then no formulation of the condition that should be satisfied in the continuation of the initial segment of the series, given beforehand, can exercise any normative function. As it is not a question of sticking to a pre-existent meaning, but of creating freely a new one, then the empirical fact that everybody who has received a certain training acknowledges that, by writing "10", one is proceeding in the same way, simply shows the existence of an agreement in conferring meaning to the expressions of our language ("We do as a matter of fact all make the same decision" (*AWL*, p. 134)). Of course, what we have said holds as well if, in the place of an initial segment, we have a general formulation of the law for generating the series: from §186 of the *Investigations,* what comes out is that we are not provided with a general notion of logical consequence which is independent of the cases effectively ratified as such. If it is a logically free decision that establishes the meaning of "following from the law" in each new case – namely, if it is a freely stipulated convention that makes a certain proposition, by definition, a logical consequence of the law – then the advance formulation of some general condition to which the deriving process ought to conform is emptied of its true point. It is only an empirical regularity that all those people who have received a certain type of training, or, more generally, all those who share a certain form of life, agree in the practice of conferring meaning to expressions of their language. This picture, in which a total uniformity of behaviour from the anthropological point of view is a *pendant* of an absolute freedom from the logical point of view, is, in my opinion, the first component of Wittgenstein's conception of rules.[7]

The second component is to be found in the much discussed and celebrated §§198–202 of the *Investigations*. Here Wittgenstein faces two problems, one after the other, which are closely interconnected: (i) what is it that confers meaning to the formulation of a rule? (ii) how can one give an account – inside the framework delineated by the reply to the first question – of the uneliminable distinction between thinking oneself to be following a rule and actually following it (a distinction which seems to be made dubious by precisely the acceptance of that framework)? It is the solution to problem (ii) that requires the introduction of the so-called community view of rule-following.[8]

Section 198 opens with a question that reproposes the problem already faced in §186: "But how can a rule shew me what I have to do at *this* point?"; nonetheless, at this stage of the analysis, one can no longer take the path attempted by the interlocutor in §186, i.e. of calling into question the way in which the rule was meant at the moment it was formulated (with its eventual store of application examples), because §§187–97 have dismantled the "primitive" conception of meaning on which that attempted reply was based. From this comes the discouraged admission by the interlocutor: by means of an appropriate interpretation of the formulation of the rule, whatever one does can be brought into accord with the rule, and can be considered to conform to it. The point is this: if the formulation of a rule has meaning, then, certainly, whoever understands it must be able to extract from it the criteria to discriminate between those actions that are compatible with the rule and those that are not. But if such a meaning were determined by *an interpretation* of the rule formulation, i.e. according to what is suggested by Wittgenstein at the end of §201, by replacing the given expression of the rule with a new formulation, then "the paradox" presented at the beginning of this section would follow: that "every course of action can be made out to accord with the rule" (once again, the discouraged admission that the interlocutor is forced into at the beginning of §198). That the presentation of this paradox points out the existence of a misunderstanding can *already* be inferred, according to Wittgenstein, by the fact that, in trying to establish a meaning for the rule formulation which is able to provide the rule with its normative role, one passes from one interpretation to another, and then to yet another, etc., in the illusive hope that, in the end, one will arrive at an expression of the rule that has the required effectiveness. Here the occurrence of the adverb "already" is important: if it is true that one can already see from this regress that a misundertsanding is present, it can be seen even more clearly by what Wittgenstein himself calls "the answer" to the paradox: "If everything can be made out to accord with the rule, then it can also be made out to conflict with it. And so there would be neither accord nor conflict here" (*PI*, I, §201). Be careful: this is not a reformulation of the paradox, it is the answer to it. And it constitutes the answer because it shows that, by taking into account only the verbal expression of a rule, the same action can be considered to be as much in agreement as in disagreement with it: but this entails that, *remaining inside the sphere of interpretations,* the concepts of accordance and of conflict with a rule collapse, and, with them, the very concept of rule vanishes into nothing. This is the reason why, at the first mention of the paradox at the beginning of §198, Wittgenstein raises the objection: "That is not what we ought to say, but rather: any interpretation still hangs in the air along with what it interprets, and cannot give it any support". This means that what is interpreted and its interpretation, the initial formulation of a rule and its reformulation, are all in the same boat because the latter reproposes, in exactly the same way, all the problems raised by the former. From this one must come to the fundamental conclusion that "interpretations by themselves do not determine meaning" (*PI*, I, §201). In order that the formulation of a rule acquires a meaning, what is needed is the existence of a stable usage, a custom, an institution, i.e. a practice of rule-following. Indeed, only on this condition can the way of grasping (*Auffassung*) the rule, supplied by an interpretation, be contrasted to the way of grasping it that is exhibited in the actions by us qualified, case by case, as conforming or not conforming to the rule (not in the empirical judgements on the behaviour of somebody, but in the atemporal statements, the definitions, on what *one must* call "following the rule" or "violating a rule", or also in the former, but only inasmuch as they reveal the latter).[9] Wittgenstein's solution to the above mentioned problem (i) is therefore that there is no intensional understanding of the formulation of a rule that is able to supply the criteria for distinguishing between actions that are in accord and actions that conflict with it. Thus, the verbal expression of a rule does not have any content which transcends *the class of actions* acknowledged as atemporally conforming to it, which transcends the ratified practice of following it:

> And the *like this* (in "go on like this") is signified by a number, a value. For at *this* level the expression of the rule is explained by the value, not the value by the rule. For just where one says "But don't you see...?" the rule is no use, it is what is explained, not what does the explaining. (*Z* §§301–2)

In the terminology of the intermediate phase, we can say that the concept of following a rule R, as all the other formal concepts, is identified with its ratified extension, with the class of actions which are acknowledged as being with R in the internal relation expressed by the formal dyadic predicate "being a correct application of" or "being in accord with".

Problem (ii) still remains open. The possibility of distinguishing between thinking oneself to be following a rule and really following it, i.e. the possibility that one believes oneself, mistakenly, to be following a rule, is, for Wittgenstein, a grammatical property of the term "rule"; thus his conclusions on rule-following cannot cancel it. But let us see what becomes of this distinction if one considers a rule R and an individual who does not belong to a community of people who agree to call certain actions, carried out by them habitually, "following R". In this hypothesis it is not necessary, in my opinion, that the individual in question be a Robinson Crusoe: the same conclusion can be reached even if one imagines that he lives in a society where there is no agreement in deciding which actions conform (atemporally) to R. To me, it seems quite natural to consider those customs, habits, institutions – whose existence is necessary, according to Wittgenstein, in order that the formulation of a rule acquires meaning – as *social* customs, habits and institutions. But, as is well known, the adversaries to the community view have maintained that it is perfectly legitimate to apply those concepts also to an individual who is not an effective or even virtual member of a community of rule-followers.[10] For the sake of argument, let us assume that our isolated individual has a practice, a habit, which could be called "following the rule R". This supposition entails that he has carried out, and carries out more or less constantly, some actions that he qualifies as conforming or not conforming to R; i.e. actions which he decides whether they are or are not with R in the *grammatical relation* of being in accord with R. Now, as seen before, the verbal expression "R" does not have a meaning from which our individual can derive criteria for discriminating between actions that, by definition, are in accord with R and actions that, by definition, go against R. On the contrary, it is those single actions which he decides must be called "conforming to R" (or "not conforming to R") that show the way in which he conceives R. Suppose, then, that he sincerely affirms that, by carrying out a particular action, one follows R: plainly, this action seems to him to conform to R, and he believes that one follows R if one carries it out.[11] How could he introduce, under these conditions, the crucial difference between what seems to him, on the one hand, and what really is, on the other hand (and therefore, between believing to follow and really following the rule)? As far as the concept of following R is concerned, he has only the extension hitherto acknowledged by him, namely, the set of actions ratified by him as instances, by definition, of following R (and, similarly, he has only the extension hitherto acknowledged by him, as far as the concept of behaving in the same way that he has behaved on a certain number of previous occasions is concerned). Hence, there is no court to which he can submit his atemporal judgements about the conformity to R of a certain type of behaviour. If being in accord with the rule R coincides with being acknowledged as such by our individual, then his beliefs – or rather his decisions – on what is to count as being in accord with R are the law. The possibility of making a distinction between believing to follow a rule and following a rule is preserved if one introduces the concept of a community, i.e. if the practice of following a rule is constituted by the actions that the members of a community unanimously call, by definition, "conforming to the rule". In this case, in fact, one is able to distinguish between an action that one individual believes to be in accord with a rule R and an action that is unanimously acknowledged as such by the members of the community, and to base on this very distinction the difference between individual appearance and objective reality of correctness. There are some specifications to be made regarding the community view. The first is that it is not sufficient that the members of the community agree on obtaining the same result when they apply a rule in a given case: what is needed is that among them agreement reigns in *acknowledging* that what is obtained is the same result. In other words, the results must be the same when they are considered from *inside* and not outside the community (one could find that what *we* judge as being the same result may be judged as being a different one by the members of the community, and vice versa). In the second place, since agreement is required in the judgements on what is to count, by definition, as following the rule in a given case, an individual is to be considered isolated even if he physically lives in a community but such a sort of agreement does not reign in the community (or, rather, one should say that in this case one would not be dealing with a community, but with a group of individuals that are close only in a physical way). And here, once again, following a rule would collapse in believing to follow a rule. In the third place, it should be stressed again that, in this context, what are under discussion are the standards on which empirical judgements about a person's conformity to a rule are based (i.e. disguised definitions such as the conditional: "If one correctly applies the rule 'Add 2' to 1,000 one obtains 1,002"). It is clear that this rule does not forge a grammatical relation between the expression "the result of the correct application of 'Add 2' to 1,000" and the expression "the number which nearly all the members of the community agree upon in considering as the result of the correct application of the rule 'Add 2' to 1,000", but rather between the first expression and the numeral "1,002". In fact, the identity "the number which nearly all the members of the community agree upon in considering as the result of the correct application of the rule 'Add 2' to 1,000 = 1,002" is a contingent identity, whose truth strongly suggests the adoption of the rule that makes obtaining 1,002 a criterion for the correct application of the rule in question to 1,000. Therefore, this identity states the empirical state of affairs whose existence is the very presupposition for the distinction between seeming and being, as it concerns applying correctly the rule "Add 2" to 1,000, to make sense.

If we reconsider all that has been said up to this point, I believe that the community view's ultimate motivations can be grasped. Wittgenstein's analysis is primarily concerned with the disguised definitions that establish the meaning of the expression "following the rule" in each given case (for a certain rule R), or, equivalently, with the atemporal statements on what is to count as the result of the correct application of the rule in each case. The pivotal point is that, for Wittgenstein, one is not dealing with factual statements: if they were to have this status, the problem of the distinction between believing oneself to be following the rule and really following it would by no means take on the dramatic proportions that it has in the *Investigations*. Using the phrase in a non-philosophical way, one could say that – were these statements empirical – it would be for the facts to decide whether an action that to one person seems to conform to a given rule (in an atemporal sense) really does (in the same sense). But for Wittgenstein there are no facts, neither of a mental nor of an ideal nature, which atemporal statements – statements about the existence of internal relations – must be responsible to. With what, therefore, can one compare the statement, made by an individual, that the expression "applying correctly the rule 'Add 2' to 1,000" *means* obtaining 1,003? The term of comparison cannot be other than the communal rules, because the rules adopted unanimously, and in a completely natural way, by its members supply the only "objective reality" against which the grammatical decisions of an individual can be measured. As far as the internal relations of the community language are concerned, reality and appearance obviously coincide, because, if something seems to be to all the members of the community, then it is (in the jargon of the *Tractatus,* it is shown in language). In conclusion, the notion of a community is essentially, grammatically interwoven with that of following a rule: "Could there be only one human being that calculated? Could there be only one that followed a rule? Are these questions like, say, this one: 'Can one man alone engage in commerce?'" (*RFM*, VI, §45).

We can finally return to the question that was our starting point, that of mathematical verificationism. There is no doubt that Wittgenstein's considerations on rule-following destroy the very premises of that conception, i.e. the possibility of distinguishing between the understanding of the sense and the knowledge of the proof of a mathematical proposition. In fact, no logical constraint on the acknowledgement, in a given sign figure, of the result of the correct application of the decision method can be imposed by means of a general definition of the operations constituting the method, supplied beforehand. The devastating effect that the Wittgensteinian reflections on following a rule have on verificationism is not, obviously, that of excluding the possibility of giving a description, in general terms, of the procedure to follow and of the results to obtain in order that one can conclude, for example, that 11,003 is a prime number. Rather, such a description has no normative force on the future decisions whereby any given sequence of arithmetical expressions will be – or will not be – filed away in the archives as a proof of the statement. Only the construction of a sign figure ratified as a proof will show what counts, for us, as a correct application of the general procedure in the particular case under examination. Once again, the presumable agreement on the matter of all those people that have had a particular type of training and that share a certain form of life is an anthropological given that cannot be further reduced to a common intensional understanding of the general calculating method. In Wright's words, as a product of Wittgenstein's rule-following considerations, "there is in our understanding of a concept no rigid, advance determination of what is to count as its correct application".[12] Thus, the formulation of a general calculating technique, like that of multiplying two numbers in decimal notation, has meaning only in so much as a practice of sign construction, unanimously called "multiplying correctly" or "calculating according to the technique", exist. But if the concept of that technique is reduced to its ratified extension, then only the effective construction of the sign figure unanimously acknowledged as the correct multiplication of two given numbers will be able to show what is meant when one says that *their product* is equal to a certain other number.

Of course, Wittgenstein's new conception leads to a loss of weight of that distinction between genuine mathematical propositions, which belong to a whole system of propositions decidable by the application of a general algorithm, and isolated mathematical propositions, only improperly called "propositions", which had marked his brief verificationist interlude. For the expressions in both categories, only their proof can determine how they must be understood *as mathematical propositions* (in fact, they are rules that have been adopted because of the agreed upon acknowledgement of certain sign figures as proofs). In effect, the distinction does not disappear from the horizon but, for the aforesaid reason, it loses its importance, becoming a simple illustration of the internal heterogeneity of mathematics. This development has not been an instantaneous one, however.[13] Only from the years 1937–8 does the situation clearly change. It is witnessed by those remarks in Part I of *Remarks on the Foundations of Mathematics* where Wittgenstein faces the problem of believing that a mathematical operation yields a certain result (*RFM*, I, §§106–12). The problem can be stated in this way: is it legitimate to speak, for instance, of the belief in the fact that the result of the multiplication of 13 by 13 is 169? (admitting its legitimacy is the same as admitting the characterization of the arithmetical identity "13 × 13 = 169" as being a proposition). The reply is: it depends on what one calls "the multiplication of 13 by 13". If one intends to call as so only the sign figure obtained by *correctly* applying the rules of the product, i.e. a certain sequence of numerical expressions that has "169" as its final term, then the object of the belief – the proposition that the result of the multiplication of 13 by 13 is equal to 169 – is true by definition; but then, to speak of a belief, and, correlatively, of a proposition, ends up with minimizing the sharp difference with linguistic contexts such as "I believe it will rain" (in other words, to wipe out the *fundamental difference* between the function of rule carried out by a mathematical proposition and that of an empirical proposition). Alternatively, if by the description "the multiplication of 13 by 13" one means even a mistaken multiplication, then the object of the belief that the result of the multiplication of 13 by 13 is 169 is an empirical proposition regarding the expression that usually appears – or that will appear – at the end of a sign manipulation process which is described as the application (be it right or wrong) of certain general rules of calculation (and here it would be appropriate to speak of a belief). There is, then, a third case that the alternative outlined by Wittgenstein excludes: that the proposition which is the object of the belief is the *unproven mathematical proposition* that 169 is the result of the multiplication of 13 by 13, namely, that it is the very proposition that the proof induces us to accept as being true by definition. Obviously, one is dealing with the very possibility admitted by Wittgenstein during his verificationist phase. But the *mathematical hypothesis* that 13 × 13 is equal to 169 would make sense only if the normative content of the expression "the multiplication of 13 by 13" could be drawn from the general rules for calculating the product of two numbers: and this is precisely what Wittgenstein's rule-following considerations rule out as impossible. As the pure and simple reference to calculating rules is not able to establish "which pattern is the multiplication 13 × 13", the agreement between the results that an individual usually gets by calculating and those results that are filed away in the archives, entrusted to the handbooks, is just a lucky empirical circumstance; and in case of conflict, the latter would count, thus excluding the individual in question from the practice of calculating.[14]

The radical change in Wittgenstein's position with respect to the intermediate phase is clearly shown by his different attitude to the notion of mathematical proposition. In his 1929–33 writings, his concern to "save" this notion, in spite of the pressure to the contrary put upon him by other tendencies in his approach to mathematics, was evident, and directed much of his analyses. In his writings on mathematics during the last phase, for the reasons indicated, what one witnesses is a veritable decline of the notion of mathematical proposition. It no longer has any attraction for Wittgenstein. Connected to the notion of mathematical proposition remain only the misleading suggestions that lead to matching mathematics and empirical science (and thus, almost inevitably, to platonism). Therefore, there is a repeated distancing from the idea that such a notion is really indispensable: "Of course, we teach children the multiplication tables in the form of little *sentences,* but is that essential? Why shouldn't they simply: *learn to calculate*? And when they can do so haven't they learnt arithmetic?" (*RFM*, I, §143);

> Might we not do arithmetic without having the idea of uttering arithmetical propositions, and without ever having been struck by the similarity between a multiplication and a proposition? Should we not shake our heads, though, when someone shewed us a multiplication done wrong, as we do when someone tells us it is raining, if it is not raining? – Yes; and here is a point of connexion. But we also make gestures to stop our dog, e.g. when he behaves as we do not wish. We are used to saying "2 times 2 is 4", and the verb "is" makes this into a proposition, and apparently establishes a close kinship with everything that we call a "proposition". Whereas it is a matter only of a very superficial relationship.
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> (*RFM*, Appendix III, §4)

A mathematics without propositions is a mathematics without statements and without truth-values: thus, so many years later and in a so greatly changed overall theoretical context, the radical claim of the *Tractatus* that a mathematical proposition does not express a thought definitively prevails.

MATHEMATICAL PROOFS AS PARADIGMS

According to Wittgenstein, a conditional such as "If one correctly applies the rule 'Add 2' to 1,000, one obtains 1,002" – inasmuch as it belongs to the geometry of signs – is a necessary conditional.[15] The peculiarity of the position that he is led to by his rule-following considerations concerns the nature of necessity. The source of necessity is a linguistic decision that rules out as *senseless* any empirical statement which identifies a number other than 1,002 as the result yielded, on a certain occasion, by a correct application of the rule "Add 2" to 1,000. In employing the term "decision", Wittgenstein has no intention of referring to a speaker's conscious deliberation, because, on the contrary, the adoption of that rule has proven to be spontaneous, natural and inevitable for all those who have had a certain linguistic training and share certain inclinations, whether they be innate or acquired (those who share a particular form of life). His resorting to the word "decision" reflects, rather, Wittgenstein's desire to destroy the idea that the necessary relations between a rule and the results of its correct application in particular cases exist before our acknowledgement; and, moreover, to undermine the strong temptation of seeing this acknowledgement as no more than the knowledge of an empirical state of affairs, which is inevitably subject to mistakes (if one were dealing with knowledge, then obviously one could be mistaken in believing that a certain result is obtained by correctly applying the rule). To Wittgenstein, the acknowledgement of necessary connections is a sort of creation, subject to conditioning of a pragmatic, or even biological, nature, but without constraints of a logical nature. In other words: the process by which it is established what is to count as the correct application of a rule in a given case is a process that involves the will, not the intellect, and the former is free from the logical point of view. The principal consequence of this approach is a distinction between two levels in the problem of the correctness of the application of a rule in a given case. On the level of the will, one establishes what is to count, *by definition,* as the result of the correct application of a rule. Here what is at stake is not whether a statement has strong or weak grounds, but the convenience or, simply, the naturalness of a practical decision that concerns the formation of a concept. By adopting the definition, a unit of measurement is fixed, whereby empirical rule-governed sign transformations can be evaluated; and thus substance is given to the very notion of mistake in applying the rule. Only then is the level of the intellect reached, i.e. the level of the empirical judgements on the correctness of individual performances of the sign operations: judgements that can be formulated in terms of deviation from the previously chosen standard (for instance, the statement that, on a certain occasion, X has correctly applied the rule "Add 2" to 1,000 becomes grammatically equivalent to the statement that he has written "1,002", and this is an empirical statement, subject to confirmation or disproof).[16]

Wittgenstein's solution to the problem of rule-following entails, a rejection of both the platonistic conception and of the empiristic conception of necessity. Platonists have a "mythological" view of conceptual connections, whereas empiricists (or formalists) are not able to distinguish between conceptual connections and observable regularities, thus depriving themselves of the possibility of fully realizing the normative function that the former have. According to the platonist's view, it is necessary for the result of the application of the rule "Add 2" to 1,000 to be equal to 1,002, and the relationship between speakers and the domain of necessary facts is of a cognitive nature (actually, it is the privileged model of this kind of relationship). With a manoeuvre that, in Wittgenstein's opinion, is typical of the sublimation of logic to a sort of ultra-physics – namely, by postulating an ontological "ethereal" counterpart of conceptual connections – "norm" comes to mean "ideal", and the acknowledgement of necessity is seen as a *sui generis* kind of knowledge. [17] One could wonder whether it is not a prejudicial conventionalistic–nominalistic inclination regarding the nature of necessity which makes Wittgenstein discard, as totally inappropriate, this picture of a world of necessary connections between ideal objects. The answer is not an easy one. It is certain that there is, in the background, his criticism of the general conception of the meaning of a linguistic expression which invariably connects meaning to the function of denoting objects and that, consequently, construes the understanding of the meaning of any expression in terms of knowledge of the criteria for identifying its reference. [18] But the rule-following considerations literally take the ground from under the feet of the assumption that a mathematical object, a definite calculating procedure, is denoted, for instance, by the expression "Add 2", and that such an object is in a pre-existing relation with the number that is the result of the correct application of the procedure to a given number. If that assumption were correct, the understanding of the meaning of the expression "Add 2" should, obviously, have a normative role in the process by which one establishes that a certain number is the result of the correct application of the arithmetical procedure, denoted by it, to a given number. And this presupposes that the understanding of the meaning of "Add 2" at least brings about the univocal identifiability of the procedure in question. But Wittgenstein's conclusions can be summarized in the thesis that the meaning attributed, in any given moment, to "Add 2" is not able to impose any logical constraint on what must be considered to be the result of the correct application of the rule and thus, *a fortiori,* is not able to intensionally identify any specific entity. In fact, the existence of such an intensional characterization would make it possible to base upon it the judgements on the correctness of the applications of the rule: whereas, in Wittgenstein's opinion, even this "knowledge" would be in itself ineffectual, because it would be necessary to decide, each time, how to apply it. In total coherence with his position in the *Tractatus,* he sharply separates facts from concepts, knowledge from rules.

The deeply rooted conviction that the world of eternal, ideal connections be an ontological projection of the rigidity of the grammatical "must" and of the relentlessness with which we learn to apply grammar rules goes hand in hand with the exclusion of mathematics from the domain of knowledge: "If you know a mathematical proposition, that's not to say you yet know *anything*. I.e. the mathematical proposition is only supposed to supply a framework for a description" (*RFM*, VII, §2). An alternative to platonism which at first sight is feasible is the empiristic-formalistic conception: that one gets a certain result by applying a rule R to a given number is simply a regularity observed in the sign behaviour of the vast majority of people who have been trained in the use of R. Here, necessity disappears and one is left with what actually happens: thus, apparently, the only "realistic" attitude is assumed, "the moonshine" about impalpable objects is avoided, and one stays anchored in the concrete or, as Wittgenstein ironically adds, one is shown to be free from that sort of obscurantism which seems to nest in the statement "that mathematics does not *treat* of signs, or that pain is not a form of behaviour" (*RFM*, III, §76). This view, however, does not hold water for two reasons. In the first place, if the connection between a given number and the concept of correctly applying a rule R in a certain case were an empirical connection, the latter should be able to be identified independently from the former (proceeding correctly in that case should not entail, by definition, that the number in question be obtained). But Wittgenstein's rule-following consideratons entail, precisely, that such an identification is impossible. In the second place, if the connection were empirical, one could not account for the normative dimension of the mathematical statement that a certain number is the result of the correct application of the rule. A universal connection is empirical to the extent that it is exposed to experimental outcomes, hence – in the case in point – to the extent that the results obtained by applying the rule to cases of the relevant type are permitted to confirm or disconfirm the generalization. From an apparently correct sign transformation that terminates with a number different to the one that is usually obtained, the conclusion should be inferred that, at times, the correct application of the rule produces the deviant outcome in question: that sometimes, for instance, the correct application of the rule "Add 2" to 1,000 yields 1,003. In effect, only the necessitation of the conditional "If one correctly applies the rule 'Add 2' to 1,000, one obtains 1,002" – namely, having placed it among the things that are not under discussion, having sheltered it from external events – allows the conclusion to be drawn that, in the above-mentioned circumstances, the rule *must not have been* correctly applied. Conceptual nature of the connection and its normative value are one. Thus the empiristic conception, in its "unscrupulous" attempt to do without internal relations, impedes the understanding of the role of the proposition "I must have miscalculated". And this deficiency is not a trifle, taking into account Wittgenstein's well-known claim that the clarification of the role of this proposition "is really the key to an understanding of the 'foundations' of mathematics" (*RFM*, III, §90).

The anti-platonistic and anti-empiristic orientation of his conception of rule-following fixes the co-ordinates of Wittgenstein's approach to the problem of the nature of mathematical theorems and of the relation between a theorem and its proof. Against empiricism, he maintains that mathematics concerns the essence of things, the network of concepts, their internal, atemporal relations, that is to say, that proven mathematical propositions do not describe facts, but determine the form facts take; against platonism, he formulates the general principle that "*essence* is expressed by grammar" (*PI*, I, §371), that "it is not the property of an object that is ever 'essential', but rather the mark of a concept" (*RFM*, I, §73), and that, for this very reason, essence is never discovered, but created. Thus, for instance, the geometrical theorem that the sum of the internal angles of a triangle is equal to 180° concerns the essence of a triangle or, equally, the concept of triangle; but this is only a more emphatic way of saying that it expresses in disguise a grammar rule regarding the word "triangle". The idea that a theorem is a disguised grammatical proposition, the expression of an accepted linguistic rule, reproposes the standpoint that Wittgenstein had already set down in his writings of the intermediate phase. In effect, on this theme, there is a striking continuity with the positions taken in the *Tractatus* and, in particular, with the true significance of his early adhesion to the logicist programme. Mathematical propositions are not concerned either with ideal objects or physical ones (nor with signs) and enter in contact with the empirical world (the only world) in so much as they set up rules of the language whereby empirical states of affairs are described (they determine the shape of the network but do not describe the property of the world that the network covers): "*So* much is true when it's said that mathematics is logic: its moves are from rules of our language to other rules of our language. And this gives it its peculiar solidity, its unassailable position, set apart. (Mathematics deposited among the standard measures)" (*RFM*, I, §165). [19]

The thesis that mathematical theorems are used to infer "material" propositions from other propositions of the same kind, and, within mathematics, to derive other theorems, seems to be out of the question, and even trivial. Of course, what makes Wittgenstein's position problematic (and philosophically interesting) is his attempt to conceive theorems as though they were *nothing but* linguistic rules, adopted in consequence of corresponding proofs. One of the nerve-centres of Wittgenstein's philosophy of mathematics is precisely this "adoption of a grammar rule in consequence of a proof". In fact, not only does the acceptance of a grammar rule seem to be as far away as possible from the ascertainment of the truth of a proposition by means of a demonstration; but the rule-following considerations render extremely elusive the notion of derivation – through the application of general rules of sign transformation – of grammar rules from given grammar rules, with which it would seem plausible to identify the notion of proof from Wittgenstein's standpoint. To solve this problem, one crucial point needs to be clarified, to which much space is dedicated in *Remarks on the Foundations of Mathematics* and in the 1939 *Lectures*: what makes a certain sign construction – a geometrical drawing, or a figure made of groups of numerals arranged in a certain way, or even a long sequence of written sentences – a mathematical proof? Do not forget that, whether one is demonstrating a geometry theorem or whether one is proving an elementary arithmetical identity, or an advanced result of the theory of transfinite cardinals, the phenomenal aspect of mathematics – the only one which philosophical investigation should, in Wittgenstein's opinion, consider – is always that of a sign manipulation activity (on paper, on the black-board, aloud or, simply, in the mind). The quasi-formalistic leaning of Wittgenstein's approach is the source of his very great interest for the distinction between experiment and proof. Indeed, once one decides to stick to this approach, the line separating the two sorts of procedure – proof and experiment – seems dangerously to fade away (just as the request for external criteria for internal processes seems to annul the distinction between pain and behaviour typical of pain). Through the mere observation of the sign operations that have been performed by a certain individual, one cannot distinguish, for instance, if he has worked out the sum of 200 plus 200, or whether he has carried out an experiment to see what comes out when the rules of addition are applied by him to these two numbers. The "observable reality" is the same in both cases, and it does not at all help to resort to the eventual internal accompaniments of sign transformations. According to Wittgenstein, the difference exists, but it is not, so to speak, in the nature of things, in the physical or mental setting of the sign manipulation process. Rather, it lies in the use that is made of the sign construction. Often this difference in use is conceived, in a totally misleading way, as a difference between the ontological domain of calculations and that of experiments (the ideal reality of numbers as opposed to the psychological or physical reality of experiments). And this cannot surprise us, in the light of what has been said at the beginning of this section regarding the platonistic interpretation of necessity: in fact, it is precisely the normative function attributed to a sign construction that characterizes our practice of calculating, and marks out the grammatical difference between it and making experiments. A certain manipulation of signs or of other concrete objects shows us, initially, what has been obtained by performing the operations in question (experiment): the passage to the proof is *the passage from what is the case to what must be the case,* and *we* do it when we commit ourselves to a specific routine in the use of the sign construction. Wittgenstein has used up a lot of energy in his attempt to clarify, in detail, what this jump from mere being the case to necessity, from facts to concepts, is like, and I wish to summarize his conclusions on the matter. [20] First of all, this passage can be appropriately described as a passage from the ascertainment of empirical facts to the adoption of language rules. Following Wittgenstein, let us suppose that we have on paper a sequence of five strokes and a star-shaped figure, and that we want to establish whether the number of strokes is the same as the number of vertices of the star, by setting up a 1–1 correlation between the two sets under examination. Hence, we perform certain sign operations, like tracing lines etc., and, when considering their end-result, namely a certain figure, we say that the two sets have the same number of elements. It is a situation that is perfectly analogous to the following: given the mastery of scales as a comparison system for two objects' weights and, in particular, given the knowledge of the rule that two objects are the same weight if the scale-pans, with each of the two objects placed on one of the pans, are level, one establishes whether two particular objects A and B are the same weight or not. What makes this procedure an experiment is our attitude towards its result: *we are willing to accept it, whatever it might be.* The *concept* of correctly comparing the number of elements of the two sets under examination poses no restriction on the result of its application to the two concrete figures considered (or, which is after all the same thing, the question regarding the correctness of the application of the process can have a reply without taking its result into account). The first step along the path that leads from experiment to proof, is the ascription of a paradigmatic function to the sequence of strokes and the star-shaped figure that have been manipulated in the course of the experiment. This is realized by means of a sort of initial baptism, i.e. through the adoption of the two following grammar rules: call every figure that is the same shape as *this* (namely, the sequence of strokes occurring in the experiment construction) "hand", and every figure that is the same shape as *this* (namely, the star employed in the experiment construction) "pentacle". Thus the sequence of strokes in the construction becomes a hand *by definition,* or a hand-paradigm (and the star in the construction, a pentacle-paradigm). Let us go on, then, with the adoption of another grammar rule, which is: call every sign construction that is the same shape as *this* (again, the construction obtained by means of the experiment) "figure yielded by the correct comparison of the number of elements of a hand with the number of vertices of a pentacle". This is the same as rendering the construction a paradigm of the correct comparison of these numbers, a picture of what, *by definition,* such a correct procedure consists in. Since the picture construction contains as a part a picture of the *result* of the experiment construction, the yielding *that* result, namely, the 1–1 correspondence between the elements in a hand and the vertices in a pentacle, becomes *a distinguishing mark of the concept of correct comparison of the number of elements in a hand with the number of vertices in a pentacle.* As a consequence of the passage from experiment to proof, the connection between process and result becomes atemporal and supplies a new conceptual tool for the formulation of empirical judgements: if the 1–1 correlation between the elements of a sign figure identified as a hand and the vertices of a sign figure identified as a pentacle fails, we should conclude either that the correlation has not been carried out correctly or that the figures in question were not, actually, a hand and a pentacle. [21] Now consider a second example of the passage from experiment to proof which, because of its very diversity from the one examined previously, efficaciously illustrates the general features of the process. Suppose that the usual algorithm for the addition of two numbers in decimal notation has been given (it does not matter whether in general form or by a sufficient number of examples), and suppose that the addition of 200 and 200 has never been carried out. If somebody puts pen to paper and applies the algorithm to this pair of numbers, he is doing no more than producing a certain sign figure that, as a matter of fact, terminates with the numeral "400". This figure is not, as such, a proof of the arithmetical identity "200 + 200 = 400". It could have been constructed, for instance, to satisfy a curiosity about which sign the person in question would have written down as final sign, had he been given the task of applying the algorithm of addition to the numerals "200" and "200", written on a sheet of paper. The construction would verify, then, the empirical hypothesis that, in this situation, the last numerical sign written will be "400" and would therefore be an integral part of an experiment. In this case, too, the passage from experiment to proof is effected by means of the adoption of a grammar rule. [22] One decides to call "figure yielded by the correct application of the algorithm of addition to 200 and 200" every sign construction that is the same shape as the figure obtained in the experiment. As a result, the latter assumes the role of paradigm of the correct addition of 200 find 200, i.e. becomes the picture of what, *by definition,* the correct calculation of the sum of 200 plus 200 consists in. Obviously, the picture of the outcome of the experiment – of the last line in the experiment construction – is part of the picture construction; and hence, that the result of the correct application of the algorithm in question be 400 becomes true by definition. At this point, a new tassel is added to the conceptual apparatus, a new necessary connection has been invented, and upon it the empirical judgements on the correctness of the applications of the algorithm of addition can be based (whoever, adding 200 and 200, obtains any number other than 400 *must* have miscalculated):

> Proof, one might say, must originally be a kind of experiment – but is then taken as a picture.... The process of adding *did* indeed yield 400, but now we take this result as the criterion for the correct addition – or simply: for the addition – of these numbers. The proof must be our model, our picture, of how these operations have *a result*.... The proof defines "correctly counting together". The proof is our model for a particular *result's being yielded,* which serves as an object of comparison (yardstick) for real changes.
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> (*RFM*, III, §§23–4)

It seems to me that Wittgenstein's conception of the relation between experiment and mathematical proof is utterly coherent with his solution to the problem of rule-following, expounded in the last section. It constitutes, in an obvious sense, a natural generalization of this solution. As seen, the problem of rule-following may be formulated thus: what makes 1,002 the result of the correct application of the rule "Add 2" to 1,000? Wittgenstein's reply is that it is only our spontaneous decision, which is agreed upon by all those who have had a certain training and possess certain linguistic inclinations, that establishes it. What one is dealing with, in fact, is the creation *ex novo* of a conceptual connection, which could not be derived in any way from the concept of the rule and from the concept of the number 1,000 such as we know them before the decision is taken. A completely analogous reply is supplied by Wittgenstein to the question: what makes a certain sign construction a mathematical proof of the equality of the number of elements in a hand and the number of vertices in a pentacle? The answer is: the decision to adopt a certain definition, relative to the expression "figure yielded by the correct comparison of the number of elements in a hand with the number of vertices in a pentacle", a decision that is equivalent to attributing to the sign construction in question the role of paradigm of that correct procedure. Here, too, what is behind the decision is the constancy in the outcomes of the experiments, i.e. the almost complete agreement in the results that men trained like us obtain by manipulating figures like hands and pentacles. Once again, the passage to the sphere of necessity or conceptual connections does not have a cognitive basis but an exclusively practical one (which falls within the domain of the will). This conventionalistic component, wedged by Wittgenstein into the heart of the notion of proof, has effects that are both devastating and pervasive on many of the most accredited ideas on mathematics. Later on we will see that the decision to adopt the definition that establishes the necessary and sufficient conditions for the use of a predicate like "figure yielded by the correct comparison of the number of elements in a hand with the number of vertices in a pentacle", or like "figure yielded by the correct addition of 200 and 200", is but the last step in a whole succession of decisions of the same type, each one being logically independent from the decisions that precede it in the sequence. For the time being, let us stop to examine two objections that may be raised to Wittgenstein's view, as it has been outlined up to now. Both express a clear rejection of the most radical consequence of that conception: that the creative activity of mathematicians has to do with the production of the meaning of certain linguistic expressions – with the formation of concepts – and not with establishing the truth of certain propositions. The first runs as follows: let us assume that a general definition has been given of the *lesser than, equal to* and *greater than* relations between the cardinal numbers of two sets (in terms of the notion of 1–1 correspondence), or that these mathematical concepts have been explained by examples. The proof of equinumerosity of the set of the elements in a hand and of the set of the vertices in a pentacle does not show what the statement that these two sets have the same number of elements means, because we already know what, in general, two sets being equinumerous means. The proof simply shows us that the two geometrical entities in question are related exactly in this way, i.e. it leads up to the recognition of *the truth* of this proposition. Similarly, the proof of the identity "200 + 200 = 400" does not show what the statement "400 is the result obtained by correctly adding 200 and 200" means; if the algorithm for the addition of two numbers in decimal notation has been explained, the sense of every instance of the schema "*x* is the result of the (correct) addition of *y* and *z*" can be understood. By calculation we do nothing more than establish that 400, 200 and 200 are related exactly in that way, namely, once again, the truth of a proposition. The intermediate Wittgenstein's conception of the relation between the general and the particular in grammar already entailed the rejoinder to this apparently reasonable objection. The rule-following considerations, from this point of view, have only had the effect of extending the pivotal argument – on which that conception was based – to the whole field of mathematics, cancelling the position of exceptionality assigned, in the years 1929–33, to systems of propositions which are decidable by means of a uniform calculating procedure. In short: the definition of a formal concept (general or individual), given at a certain moment, is not able to impose constraints on the subsequent decisions which lay down that an object, by definition, falls under the formal concept. But if, for example, one does not presuppose the meaning of the schematic expression "*x* is the result of the (correct) addition of *y* and *z*" as given, one cannot even say that any of its instances is a meaningful proposition, the proof of which would eventually prove its truth. It is the task of mathematics to create, case by case, the meaning of the single propositions having that form. We start, so to speak, with nothing, and only the proof, for instance, of the proposition "400 is the result of the correct addition of 200 and 200" is able to determine its sense: in fact, it directly shows what, by definition, the correct addition of 200 and 200 is like, and how it yields 400 as its result. This conception of the relation between mathematics and meaning, according to Wittgenstein, should both clear up the misunderstanding about the nature of meaning at the roots of platonism and, at the same time, "save the truth" of formalism:

> Is it already mathematical alchemy, that mathematical propositions are regarded as statements about mathematical objects, – and mathematics as the exploration of these objects? In a certain sense it is not possible to appeal to the meaning of the signs in mathematics, just because it is only mathematics that gives their meaning.
>
> (*RFM*, V, §16)

The fact that the ultimate root at the abyssal distance between Wittgenstein's position and the "received view" of mathematical activity lies in the diversity of the respective underlying theories of meaning is vividly illustrated by the clash between the Austrian philosopher and Alan M. Turing during the sixth Cambridge lecture held in 1939. The mathematical problem from which the discussion originates is that of finding, for the heptadecagon, a construction with ruler and compasses which is analogous to the construction, already available, of the pentagon. More precisely, the investigation regards the nature of a positive judgement of analogy that is given in reference to a certain geometrical construction, obtained for the first time by a creative mathematician. For the very general relative term "analogous to", the claim made above concerning the expressions "correct comparison of the number of elements of two sets" and "*x* is the result of the correct addition of *y* and *z*" holds true. The meaning that "analogous to" has before the construction does not impose restrictions upon the meaning that it should have in the proposition of grammar: "this construction (of the heptadecagon) is analogous to the construction of the pentagon with ruler and compasses". According to Wittgenstein, accepting the construction in question as the solution to the geometrical problem means adopting a grammar rule that determines, by means of a direct reference to the new figure, the meaning of "analogous to" in that proposition (the rule would run as follows: call a sign construction "a construction of the heptadecagon analogous to that of the pentagon with ruler and compasses" if and only if it is the same shape as that proposed by the mathematician). The positive judgement of analogy is a disguised formulation of this rule; and only on the basis of the standard supplied by the rule can empirical judgements be formulated on geometrical sign figures effectively drawn. Turing's reaction to Wittgenstein's stance reproposes the very idea of meaning overthrown by the analysis of rule-following: "*Turing*: It certainly isn't a question of inventing what the word 'analogous'means; for we all know what 'analogous' means" (*LFM*, p. 66). [23] But Wittgenstein replies that the problem is that of inventing a *new* meaning for "analogous", and clearly points out that the only constraints this creative activity is subjected to are constraints of a pragmatic nature (i.e. they can be characterized in terms of utility, naturalness and so on, for those who have received a certain type of training and share a certain form of life): "*Wittgenstein*: The point is indeed to give a new meaning to the word 'analogous'. But it is not merely that; for one is responsible to certain things. The new meaning must be such that we who have had a certain training will find it useful in certain ways" (*LFM*, p. 66). At this point the real subject of the conflict has emerged in its entirety. To the further explanations furnished by Wittgenstein, Turing does not oppose any argument in favour of his own position, but limits himself to restating his disagreement with the Austrian philosopher's point of view: "*Turing* [asked whether he understood]: I understand but I don't agree that it is simply a question of giving new meanings to words" (*LFM*, p. 67). All that is left to Wittgenstein is to take note of the fact that Turing does not formulate other objections. Turing's restated (but no longer argued) disagreement is then led back by Wittgenstein to the "ideological" sphere, to the fear of the revisionary effects that his (Wittgenstein's) conception would have on mathematical practice and to the feeling that it entails an undeserved depreciation of the importance of the work of mathematicians (from discoverers of eternal truths to mere creators of meanings of words). If this is the point, the Austrian philosopher has no difficulty in tranquilizing his interlocutor:

> *Wittgenstein*: Turing doesn't object to anything I say. He agrees with every word. He objects to the idea he thinks underlies it. He thinks we're undermining mathematics, introducing Bolshevism into mathematics. But not at all. We are not despising the mathematicians; we are only drawing a most important distinction – between discovering something and inventing something.
>
> (*LFM*, p. 67)

Before going on to examine the second of the two above-mentioned objections, let us pause to consider a fundamental consequence of the conception of mathematical proof as a paradigmatic picture of the correct performance of certain sign transformations. It concerns the relation between the sense of the theorem and its proof. Return to the proposition "400 is the result of the correct addition of 200 and 200". It expresses the rule that rules out as senseless any empirical statement which identifies a number other than 400 as the result yielded, on a certain occasion, by a correct addition of 200 and 200. This rule, in reality, is already expressly contained in the linguistic stipulation that determines the meaning of the predicate "figure yielded by the correct addition of 200 and 200" by means of a direct reference to the experiment construction which terminates with the numeral "400" (and which, due to the adoption of this definition, is elevated to the rank of paradigm of correctly adding 200 and 200, i.e. to the rank of proof of the identity "200 + 200 = 400"). As seen, the picture construction contains, as its final part, a picture of the outcome of the experiment construction; thus, the definition of the predicate "figure yielded by the correct addition of 200 and 200" entails that it would not be false, but senseless, to describe a number other than 400 as the result of the correct addition of 200 and 200, carried out on a certain occasion: in fact, such a description would contradict a definition. If we consider a case that is only a little more complex than the addition of 200 and 200, such as the product of 25 times 25, we can realize the fact that the proof already contains in itself the contents of the identity "25 × 25 = 625" and, moreover, a whole series of other grammar rules. In effect, the paradigm of correctly multiplying 25 and 25, namely, the proof of the above identity, excludes as senseless any empirical statement that identifies a number, other than that occurring in the appropriate place of the picture construction, as a partial result of correctly multiplying 25 times 25 on a certain occasion (for instance, the identification of any number other than 125 as the result obtained in the first line of a correct multiplication of 25 times 25). This complete subordination of the sense of the theorem to its proof does not exhaust what Wittgenstein says about the relation between these two terms. Rather, it is its ever-present background, in so far as it is entailed by his pivotal thesis on the role of a mathematical proof as paradigm of the correct performance of certain sign operations, as picture of an experiment.

The second objection to Wittgenstein's ideas involves the notion of possibility. One could maintain that the agreement in the way that certain operations with signs (or with other empirical objects) are carried out, and in the results thus obtained, is not an irreducible given, but has behind it the mathematical impossibility of a different situation. By this, of course, one does not mean that nobody, for instance, when tracing straight joining lines between the elements of a drawn hand and the vertices of a drawn pentacle, finds himself, at the end, with a vertex of the pentacle that is not correlated to any of the elements in the hand. One means, rather, that it is mathematically impossible to carry out the correlation correctly and then find oneself in that situation, because it is a mathematical property of the set of elements in a hand and of the set of vertices in a pentacle to have the same number of elements. It is this property and, with it, this mathematical impossibility, that are brought to light by the proof. Wittgenstein's reply to this objection is that it is our decisions on which sign constructions to adopt as models of the correct performance of certain sign operations (logical inferences included) to establish what is mathematically possible and what is not. We do not discover some pre-existent possibilities or impossibilities, but we determine what the field of possible and of necessary is. To understand Wittgenstein's point, one must take into consideration the situation where a certain sign construction has not yet been elevated to the rank of paradigm, i.e. where the grammar rule that establishes what is to be called "figure yielded by the correct application of such-and-such procedure" has not yet been adopted. If proofs only took account of ideal pre-existent possibilities and impossibilities, it would be impossible, already at this stage, for a *correct* comparison of the number of elements in a hand and the number of vertices in a pentacle to lead to the result that the former is less than the latter. But the meanings that the expressions "1–1 correlation", "subset" and so on have at the moment the figure is constructed do not impose logical constraints on how it should be called. We could unanimously find it totally natural to proceed in a way which for us, today, is deviant and to elevate the resulting construction to being the paradigm of the correct 1–1 correlation of the set of elements in a hand *with a proper subset* of the set of vertices in a pentacle. At this point, it would become mathematically impossible for the set of elements in a hand to be equinumerous with the set of vertices in a pentacle. In short, it is of no effect to appeal to an ideal world of possibilities, which the mathematical results, if correct, would reflect, because no world of that type can be identified independently from the decisions made, from time to time, on the "state of affairs" that constitute that world:

> We multiply 25 × 25 and get 625. But in the mathematical realm 25 × 25 is *already* 625. – The immediate [objection] is: then it's also 624, or 623, or any damn thing – for any mathematical system you like.... There would be an infinity of shadowy worlds. Then the whole utility of this breaks down because we don't know which of them we're talking about.... You never get beyond what you've decided yourself; you can always go on in innumerable different ways.... You want to make an investigation, but no investigation will do, because there is always freedom to go into another world.
>
> (*LFM*, p. 145)

The conventionalistic component introduced in such a way into the core of the notion of proof; the absolute logical freedom in the choice of a sign construction as the model of the correct application of a certain symbolic technique in a particular case; the fact that mathematics is the realm of necessity, but only by virtue of a totally free act of the will, has however, its price. In the "received view", the agreement in accepting a mathematical proof has its basis in a common cognitive capacity: it is a matter of exercising it in an appropriate way, and the rest, namely the recognition of the truth of a proposition by means of the step by step process of its proof, is the "happy end" which – under epistemically ideal conditions – one cannot escape from, given the compelling character of the succession of these steps. In Wittgenstein's conception, the agreement to adopt a certain paradigm of the correct execution of certain operations (that is, after all, the agreement to adopt a certain grammar rule), is an irreducible primary given, a given of our natural history. It is grounded only on the empirical regularity of sign behaviour, and of our physical setting, which manifests itself in the fact that the vast majority of those who have had a certain training obtain the same result whenever asked to apply certain transformation rules to certain signs. Without the existence of this double harmony in the symbolic behaviour of the members of the community, the activity that we call "calculating" (in the general sense which includes also inferring) would not exist. The norms produced by this activity have, in fact, the pretence of universal validity for all the members of the community and, what is more, belonging to that community is defined through one's acceptance of those norms:

> We say that a proof is a picture. But this picture stands in need of ratification, and that we give it when we work over it. True enough; but if it got ratification from one person, but not from another, and they could not come to any understanding – would what we have here be calculation? So it is not the ratification by itself that makes it calculation but the agreements of ratifications.... The agreement of

> ratifications is the pre-condition of our language-game, it is not affirmed in it.
>
> (*RFM*, VII, §9)

I believe that what has been expounded so far is sufficient to face the following problem posed by Georg Kreisel: why, in Wittgenstein's conception of mathematics, are proofs needed, given that a theorem is considered to be a linguistic rule and a rule of such a sort, "as ordinarily understood, is a matter of simple decision"? [24] It seems to me that, according to the Austrian philosopher, there are no theoretical obstacles in imagining a linguistic practice in which what is to count as the result of the correct application of a calculating technique in each single case is established without going through a proof. The point is, of course, that *we* would not call it "calculating": and this means that no theoretical problem arises regarding why we resort to proofs, but a problem could arise regarding an appropriate description of our practice. A fundamental characteristic of our form of life is that a linguistic rule regarding the meaning of a description like "the result of the correct multiplication of 12 by 12" is adopted only as a part of another accepted rule that sets up the meaning of the predicate "figure yielded by the correct multiplication of 12 by 12". In so far as adopting the latter definition means making a certain sign construction the paradigm of the correct multiplication of 12 by 12, which contains the result of the operation, the passage through the proof reveals itself to be an anthropological condition for the ratification of the rule-theorem. Wittgenstein, in effect, pushes his analysis even further. The subordination of the acceptance of a certain linguistic convention to the presentation of a proof responds to our need to *be persuaded* to extend the conceptual apparatus in a certain direction. For instance, when setting up the rule that it does not make sense to speak of the greatest prime number, what is needed is that every single inferential step in a sequence of propositions (say the Euclidean proof that, among the series of natural numbers, the occurrence of prime numbers is endless) is acknowledged by us as being correct, and that the final decision regarding the true import of the whole sequence of propositions appears to us as being totally natural. In other words, what is needed is the spontaneous ratification – by all those who have received a certain training – of the construction under examination as a sequence of correct applications of logical rules, a sequence that leads to the exclusion, by definition, of the possibility of a greatest prime number:

> Do not look at the proof as a procedure that *compels* you, but as one that *guides* you. – And what it guides is your conception of a (particular) situation.... *In the course* of this proof we formed our way of looking at the trisection of the angle, which excludes a construction with ruler and compass.... In the course of the proof our way of seeing is changed.... Our way of seeing is remodelled.
>
> (*RFM*, IV, §30)

From powerful and sophisticated means for establishing truths, mathematical proof is transformed into a tool of persuasion for producing conceptual changes that are not justifiable in any way except than in terms of the shared inclination to accept them, generated by the proof. The skill of the creative mathematician lies, therefore, in his ability to bring about a common inclination, among the members of the community of specialists, to adopt a certain definition. His research problems

> are like the problem set by the king in a fairy tale who told the princess to come neither naked nor dressed, and she came wearing fish net. That might have been called not naked and yet not dressed either. He did not really know what he wanted her to do, but when she came thus he was forced to accept it. The problem was of the form, Do something which I shall be inclined to call neither naked nor dressed. It is the same with a mathematical problem. Do something which I will be inclined to accept as a solution, though I do not know now what it will be like.
>
> (*AWL*, pp. 185–6)

THE PROBLEM OF STRICT FINITISM

It is well known that the notion of mathematical proof as the paradigm of how the correct performance of certain sign transformations (the correct application of certain rules) yields a certain result does not exhaust the contents of the later Wittgenstein's reflections on this theme in the slightest. In the course of a thorough, and sometimes cryptic, discussion about the fundamental claims of the logicist translation of arithmetic – conducted in Part III of *Remarks on the Foundations of Mathematics* – the requirements are clearly formulated which any sign construction should satisfy in order that the status of proof can be conferred: a proof must be perspicuous (*übersichtlich*), surveyable (*übersehbar*); it must supply a memorable picture, it must be easily reproducible and recognizable again. [25] In the studies on Wittgenstein's philosophy of mathematics, the question on how to assess these requirements – which, amongst other things, seem to be reconciled with difficulty to other tendencies present in it – has been posed time and time again. In this section I will point out, first of all, how such requirements, in my opinion, should *not* be interpreted; then I will suggest a tentative solution to the problem, based on the role that the relation of *being the same shape as* plays in the ascription of a paradigmatic function to an experiment construction.

Since the publication of the celebrated review of the *Remarks* by Kreisel, Wittgenstein's observations on the perspicuity, surveyability etc. of a mathematical proof have often been considered as witness to the Austrian philosopher's strong inclination towards that conception of mathematics called, by Bernays, "strict finitism", and by Wang "anthropologism" (or, in the rather more prudent words of Kreisel, as bearing witness to the fact that

"[Wittgenstein] concentrates on the strictly finitist aspects of mathematics" [26]). To be able to decide whether – and if so, to what extent – Wittgenstein's positions can be really considered akin to this conception, it is expedient to re-examine the question of their relation with finitism in general (a theme already touched on in Chapter 2). According to the short characterization supplied by Kreisel, finitist mathematics can be so qualified by the nature of the objects and by the sort of mathematical facts to which its attention is confined: the objects are "*concrete* objects which are thought of as reproducible, are to be recognisable, and surveyable, i.e. thought of as built up of discrete parts whose structure can be surveyed"; [27] the facts are constituted by the purely combinatorial properties of finite sets of such objects. Now, when one attributes to the later Wittgenstein the endorsement of an even more extreme form of constructivism, which admits only a small portion of finitist mathematics to be legitimate, one assumes that he has reached this position by eliminating the residues of idealization of mathematical practice still present in the finitist conception. As Kreisel explains, the latter does not take into account the difference "between constructions which consist of a finite number of steps and those which can actually be carried out, or between configurations which consist of a finite number of discrete parts and those which can actually be kept in mind (or surveyed)". [28] Leaving to one side, for the time being, the question of strict finitism, it seems to me that even the previous characterization of finitist mathematics does not adapt itself at all to Wittgenstein's view of mathematics. Actually, the latter is linked to a (generic) finitist conception under two principal aspects: for each of them, one can point out not only an important element of convergence with finitism but also a no less important element of divergence. First of all, there is the problem of the mathematical infinite, which we have dealt with at great length when commenting upon Wittgenstein's writings of the intermediate phase. The rejection of the extensional notion of the infinite – and, in particular, of the extensional interpretation of quantifiers when the domain of the bound variable is infinite – is obviously in tune with analogous rejections and analogous proposals of reinterpretations of generalized statements that, from positions that varied greatly, were put forward during the 1920s. Wittgenstein's famous matching of finitism and behaviourism, [29] united by their denial of the existence of something (infinite sets and inner states, respectively), in the correct but badly executed attempt to avoid confusion (that between the infinite and a very large quantity, and that between an inner state and a private entity), shows, on this point, the agreement and, at the same time, the distance between the Austrian philosopher's position and finitism. The denial of the existence of infinite sets is a mistaken way to draw a grammatical distinction which, though it may be opportune, should be done differently: by showing that the grammar of the word "infinite" cannot in the slightest be clarified by taking into account only the picture of something huge, a picture which usually accompanies the use of the word. As Wittgenstien affirms in one of his lectures in 1939: "If one were to justify a finitist position in mathematics, one should say just that in mathematics 'infinite' does not mean anything huge. To say 'There's nothing infinite' is in a sense nonsensical and ridiculous. But it *does* make sense to say we are not talking of anything huge here" (*LFM*, p. 255). The second aspect of finitism, that is relevant for Wittgenstein's conception of mathematics, concerns the formalistic connotations that it assumes in the Hilbertian version, and which are also expressed in the presentation of finitist mathematics given by Kreisel. Certainly, the supremacy of sign over meaning in mathematics sets up, in Wittgenstein's view, a barrier against that misunderstanding of the nature of meaning that is, in his opinion, at the roots of platonism (namely, of the conception that construes necessary relations, created *ex novo* by the agreed adoption of definitions, into pre-existent relations between ideal objects, which would be gradually brought to light by mathematical discoveries). But the idea that "genuine" mathematics *is about* the combinatorial properties of certain sets of concrete objects corresponds to another, equally pervasive, misunderstanding of the nature of mathematics. Mathematics, to Wittgenstein, is about nothing, not even concrete objects, and, as a consequence, does not discover general facts regarding finite sets of such objects. Properties of sets of concrete objects (signs) can be ascertained by experiments; but the effect of sign manipulations, when the yielded construction is unanimously ratified as a mathematical proof and not as the outcome of an experiment, is the adoption of a definition, the formation of a concept. In Wittgenstein's opinion, there is not a region of mathematics, such as finitist mathematics, that deals with concrete objects, and which is distinct from the other branches of mathematics that study properties and relations of more abstract entities. In doing mathematics, *one always operates* with signs, whether they be physical or mental: digits are manipulated, sequences of formulae of a formalized language are generated, long chains of propositions of the informal mathematical language are written down. What a numerical calculation and the derivation of a formula in a formalized system and the construction of the proof of a theorem in an intuitive mathematical theory have in common is the paradigmatic function assigned to the figure obtained in each one of the three cases: it is used as a model of how the involved operations yield the related result (it does not matter if one is dealing with the instructions of an algorithm of numerical computation, with the transformation rules of a formal system or with the usual non-explicit rules of inference). The various branches of mathematics are not distinguished by the nature of the "objects" and the sort of "facts" they are concerned with; rather, what vary are the demonstrative procedures, which form what Wittgenstein calls "the MOTLEY of techniques of proof" (*RFM*, III, §46). Obviously, the background to this view of mathematics is supplied by the rule-following considerations. The meaning given, at any given moment, to the formulation of a set of sign transformation rules is not able to determine what is to count, in each single case, as the result of their correct application. A proof that makes use of these rules shows what, by definition, the yielding of such-and-such result through a sequence of correct applications of the rules means. It consolidates the sign construction which all those who have been trained in the use of rules habitually agree in obtaining – when they are asked to apply them to certain expressions – into a model, into a norm. It is on the basis of these premises that Wittgenstein's requirements of perspicuity, surveyability etc. of a mathematical proof should be evaluated. How little they have to do with finitism, or with that severe variant of it called "strict finitism", can be seen by comparing the comments by Wittgenstein on Cantor's proof of the non-denumerability of the set of real numbers with those dedicated to the possibility of proving, within the arithmetic of strokes, any arithmetical identity with high enough numbers. For the finitist, the first of the two proofs certainly does not come within the boundaries of the part of mathematics which studies the combinatorial properties of finite sets of concrete objects, whereas the second comes within these very boundaries (as regards the latter, a strict finitist could have a different opinion, but here that is irrelevant). Now, Wittgenstein moves some criticisms against the platonistic interpretation of the true import of Cantor's proof; nevertheless they do not originate in any way from a presupposed identification of legitimate mathematics with finitist mathematics and, even less so, from the violation, by Cantor's proof, of the requirements imposed by strict finitism. Once the appropriate clarifications have been made about what, in his opinion, it really demonstrates, Cantor's proof is more than good enough for Wittgenstein, in spite of the certainly non-finite nature of the "objects" it deals with. [30] Conversely, notwithstanding the concrete nature of the sequences of strokes and notwithstanding the finite character of the operations that can be carried out on them, Wittgenstein rejects the idea that a sign construction within the arithmetic of strokes can be elevated to the rank of proof of an identity with quite high numbers. As we will see later on, it is true that this rejection is grounded on the requirements of perspicuity, surveyability, and so on, of a proof; but, in order that the intention of radicalizing finitism in the direction indicated by Kreisel be plausibly attributed to Wittgenstein, it would be necessary to forget that proofs which are finitistically (not only strict finitistically) unacceptable are actually accepted by him or are not questioned on the basis of the restriction of admissible mathematical procedures to the finitary ones.

Yet, there is another aspect of the later Wittgenstein's philosophy of mathematics that has played no small part in rendering plausible the identification of his positions with strict finitism. A finitist admits into his mathematics concrete structures composed of any finite number of discrete parts and effective procedures for the manipulation of these structures, which can be carried out in any finite number of steps. A strict finitist contests the legitimacy of this reference to an arbitrary finite number (of parts of a concrete structure or of steps in an effective procedure). It contains, in fact, an element of idealization with respect to the practical limitations to which our activity of sign manipulation is subject. By refusing this level of abstraction, considerations on the complexity of objects and on the number of applications of the operations become decisive for determining what can be proven and what cannot. As a consequence, in the introduction of a mathematical concept, the reference to the possibility in principle of carrying out the operations needed to verify whether a given object falls under the concept is replaced by the reference to the practical possibility of finishing them. In the words of Wright: "the philosophical and mathematical exegesis of the strict finitist attitude will involve extensive commerce with a range of concepts of human intellectual capability: the actual cogency – or, following the terminology of the later Wittgenstein, *surveyability* – of proofs, the actual intelligibility of symbols, and so on". [31] Wittgenstein's apparent closeness to strict finitism springs from the very fact that, when laying down his conditions on the notion of mathematical proof, he resorts to concepts that involve the notion of human intellectual capability, mentioned by Wright. The difference, as Dummett was the first to notice, is that, in Wittgenstein, these concepts are not introduced with the same purpose as strict finitism, i.e. *not* with the purpose of distinguishing between effective operations performable in principle and operations that are humanly feasible. [32] What we have said above concerning the relation between Wittgenstein's philosophy of mathematics and finitism should, at least, already make Dummett's claim plausible. Later on, we will see how the requirement not to exceed certain upper bounds of length and complexity – in a sign construction that must be elevated to the rank of mathematical proof – directly follows from the role as paradigm of the correct performance of certain sign operations that Wittgenstein assigns to proofs. Yet, things are not so simple because there is another element that contributes to further muddling the question of the relation between the Austrian philosopher's conception and strict finitism. It has to do with the fact that – even though on the basis of completely independent arguments – also the later Wittgenstein denies that the notion of possibility, in principle, of performing certain operations can be used for determining the meaning of a mathematical term or the sense of a mathematical proposition. It is the rule-following considerations that have this effect, and we can explain the reasons for this by returning briefly to the verificationism of the intermediate phase. As we know, for the Wittgenstein of this phase, a predicate such as "prime" has a definite meaning inasmuch as it is associated with an effective decision procedure. Correspondingly, *every* proposition in the system of the instances of the schematic expression "ξ is prime" makes sense, even if it has not yet been proven. An unproven proposition such as "11,003 is prime" has sense because of the normative force exercised by the meaning of the predicate on the acknowledgement of a given sign construction as a proof of the proposition. Between the decision procedure and the result of its application in each particular case there is no logical abyss to be filled with a decision but only an empirical distance that, in principle, can always be crossed. This reference to the possibility in principle is appropriate because, according to the intermediate Wittgenstein, *every* proposition belonging to the system of instances of the scheme "ξ is prime" gets its sense in relation to the general checking method associated with the predicate: there are no distinctions founded on the practical limitations to which the application of the algorithm is inevitably subject. The abandonment of verificationism, as a consequence of the reflections on rule-following, implies an innovation on even this last point, and the reason is easily given. The sense conferred upon any formulation of the general decision procedure associated with a mathematical predicate has become logically uninfluential in determining what is to count, in each single case, as the result of the correct application of the procedure. Now a definition is needed that creates *ex novo* the meaning of the predicate "figure yielded by the correct application of the algorithm" (to such-and-such expressions); this is effected by assigning a paradigmatic role to the sign construction which, as a matter of fact, is almost invariably obtained by all those who have been trained to manage the algorithm, when they are asked to apply it to the expressions in question (namely, passing from what simply is the case, to what must be the case; from fact to concept). At this point, the appeal to what can be obtained in principle with a sign transformation procedure (numerical algorithm, rules of inference etc.) loses meaning for mathematics. If only the effective exhibition of the proof is able to determine the sense of each one of the instances of a schematic expression such as "ξ is prime", then the only limitations that bear weight in mathematics are those that are met during the concrete sign activity of constructing proofs. So, to an interlocutor who insists on the possibility of imagining, for each step taken in a shortened calculation, also a corresponding step in the primitive calculation (obviously, a whole series of primitive steps), Wittgenstein laconically objects: "That is just it: *we can imagine that it could be done* – without doing it" (*RFM*, III, §53). Vice versa, if the practice of constructing proofs within a certain symbolism is blocked – beyond a certain degree of complexity of its sign structures – by the existence of limitations in our manipulatory or recognitional capabilities, it is going off track to speak of the mathematical connections that could be ascertained by operating with the symbolism in question, if these practical limitations did not exist: indeed, without our effective recognition/ratification, such connections simply do not exist. As Wright says, with telling effect: "if we cannot agree whether some tortuous calculation shows a particular number to be prime, there can be no fact of the matter, furnished by the concepts themselves, of which we are merely incapable of practical recognition. Without our ratification of them, there are no such facts." [33]

The preceding considerations allow us, at least, to place Wittgenstein's requirements of perspicuity, surveyability etc. of a mathematical proof in the appropriate setting. They should be connected to the anthropological matrix of the notion of proof, i.e. to that particular routine of use that, according to Wittgenstein, turns an experiment construction into a proof. It is strange that Dummett and Wright, both of whom have clearly indicated this direction for research, have both suggested interpretative developments that appear to me to be utterly unconvincing. Consider how Dummett explains the reason why Wittgenstein introduces the requirement of surveyability:

> A mathematical proof, of which computations are a special case, is a proof in virtue of our using it to serve a certain purpose; namely, we put the conclusion or result in the archives, that is, treat it as unassailable and use it as a standard whereby to judge other results. Now something cannot serve this purpose, and hence is not a mathematical proof, *unless we are able to exclude the possibility of a mistake's having occurred in it*. [34]

As for Wright, he affirms:

> it is essential that we feel able to be absolutely confident that there is no error or oversight in the proof; and this, of course, is impossible if the proof is unsurveyably long or complex. We have to feel that the proof has taught us *how* to get a certain result; and this confidence requires us to be sure that the result did not depend upon some unnoticed element in the procedure, for example an "error". [35]

I believe that there can be little doubt as to the fact that both Dummett and Wright reason roughly as follows: a limit in the length and complexity of a sign construction is necessary for it to be able to carry out the role of mathematical proof (which, according to Dummett, is the role of making us file away the result among the standards by which calculations are judged, whereas, according to Wright, more appropriately, it is that of providing us with the model of how the correct performance of certain operations yields a certain result). In fact – they continue – if a sign construction is not surveyable, then we cannot be sure of the fact that it does not contain errors and hence we do not feel we are authorized to confer that role on it. This interpretation contains, in my opinion, a complete overturning of Wittgenstein's position. In the first place, if, before elevating a given sign construction to the rank of proof, we should be able to exclude the possibility that it contains errors, *we should already have a notion of what is to count as the correct performance of the involved operations*. But Wittgenstein's rule-following considerations show that it is only when a sign construction is given the status of proof that we form the new concept of the correct performance of those operations in the case in question. Thus, to speak of a search for eventual errors in the sign construction that is candidate for the role of mathematical proof means forgetting the true cornerstone of Wittgenstein's whole approach to the question (and this, I believe, is the reason for the inverted commas with which Wright encloses "error", in the passage quoted). From this initial misunderstanding, another – equally dangerous – immediately follows. In order to admit a sign construction into the atemporal realm of proofs, should we really have reached an epistemically privileged position, from which the presence of mistakes could be excluded with certainty? To answer this question, it is expedient to distinguish between two situations which are very different one from the other. Suppose that, in relation to a given sign construction, the passage from what simply is the case to what must be the case has already been made; then, we can check the correctness of a certain calculation, carried out by somebody, simply by verifying that it is the same shape as the picture construction (namely, we can check the correctness of every step in the calculation by using the standard steps already put in the archives). Here, of course, our judgement on whether the given calculation is correct is based on the "usual rules for comparing and for copying" and it is open to error, like all the other empirical judgements. The situation that Dummett and Wright refer to, instead, is that where one is dealing with the decision whether to entrust a given construction with the role of paradigm of the correct performance of a sign process (of a numerical calculation, of a formal derivation, of an informal deduction). In this case, however, the certainty of the absence of errors is not so much an indispensable epistemic condition for making this passage into the domain of necessity as, on the contrary, *a consequence* of the fact that such a passage has taken place: "*What* is unshakably certain about what is proved? To accept a proposition as unshakably certain – I want to say – means to use it as a grammatical rule: this removes uncertainty from it" (*RFM*, III, §39); "But... I say again: 'Calculating is right – as it is done'. There *can* be no mistake of calculation in '12 × 12 = 144'. Why? This proposition has assumed a place among the rules" (*RFM*, III, §73). Consistent with his usual approach to the problem of the relation between scepticism and certainty, Wittgenstein does not look for a cognitive foundation for the latter but leads it back to a decision on the use of language. This decision, on the other hand, can always be given up: this would happen the moment that someone were able to change our previous inclination to ratify all the steps in the proof as correct, to weaken its persuasive strength, i.e. the moment we were willing to admit that, actually, there was an error in the proof (even if we cannot *now* say how such a thing could happen, because *now* the proof is our standard of correctness). The accusation – moved by Dummett – that Wittgenstein would confuse the *a priori* and necessary character of mathematics with the certainty of their results, that he wrongly assumes "that we never accept a proof unless we are not merely assured but *certain* of its validity" [36] and, hence, that he is not able to account for the possibility of the presence of some mistake in a proof, seems to me to be ungrounded. Obviously Wittgenstein does not deny the circumstance that, before filing a certain construction along with the proofs, we scrutinize it once again, or more. But, for the above-mentioned reasons, the description that Dummett and Wright give of the way that Wittgenstein interprets these typical features of doing mathematics does not appear to me to be appropriate: according to their presentation, it is as though one were concerned with reaching an absolutely certain cognitive basis, whereas, in reality, one is simply concerned with *letting oneself be persuaded* to model the concept apparatus in a certain way, to orient the will in a certain direction (*to be firmly convinced that it must be so does not mean knowing for certain that it is so*).

Wittgenstein's requirement of a limit for both the length and the complexity of a sign construction that is candidate for the role of proof cannot be justified, therefore, by saying that, if it were not satisfied, we could not be absolutely certain of the absence of errors. We must look for a justification elsewhere, and the place to look for it, in my opinion, is in the characteristics of the process whereby, according to Wittgenstein, a given experiment construction is elevated to the rank of picture construction or paradigm. In the second section of this chapter we saw that this passage is realized by means of a sort of ostensive definition of the predicate "figure yielded by correctly performing such-and-such operations on such-and-such a basis". In virtue of this definition, the statement that a figure has the property expressed by the predicate means that it is the same shape as the construction referred to indexically in the formulation of the grammar rule. In this way, the latter construction becomes the paradigm of the correct performance of the operations involved and the passage from experiment to mathematical proof takes place. The proof is representative of a visual shape, used for distinguishing between sign figures generated by correctly applying general rules (instances of proofs which have been correctly carried out) and sign figures that are generated by some incorrect application of the rules, namely, that contain some mistake. Inasmuch as it is representative of a visual shape, a proof furnishes *the model of what a sign construction is like*, when it is obtained with a correct performance of the operations involved. Following Wittgenstein, one may say that the symbols contained in the proof – just as the whole sign structure – do not have the identity of physical objects but that of shapes. [37] What in a proof must be beyond discussion, what must be beyond every doubt, is its shape. Indeed, if there were doubts regarding what the shape of a given construction C is, how could one use the property of being the same shape as C to define the expression "figure yielded by correctly performing such-and-such operations on such-and-such a basis", or else to fix, in terms of sameness of shape with C, the criteria for the correct performance of these operations (and, hence, of the correct reproduction of the proof)? The requirements of perspicuity, surveyability, reproducibility etc. of a mathematical proof are justified to the extent that a sign construction which does not satisfy them cannot be used in setting up the grammar rule that transforms it into a paradigm; and this because such a rule exploits the notion of sameness of shape. In this way, the above-mentioned properties become constitutive of the notion of mathematical proof: "Where a doubt can make its appearance whether *this* is really the pattern of *this* proof, where we are prepared to doubt the identity of the proof, the derivation has lost its proving power. For the proof serves as a measure" (*RFM*, III, §21); "I should like to say that where surveyability is not present, i.e. where there is room for a doubt whether what we have really is the result of this substitution, the *proof* is destroyed. And not in some silly and unimportant way that has nothing to do with the *nature* of proof" (*RFM*, III, §43). The search for the motivations behind the apparently strict finitist requirements imposed by Wittgenstein onto the notion of proof leads us, once again, to the conclusions of his reflections on rules. If one starts with the idea that the concept of result of the correct application of a general rule, in each single case, must be created by a decision that is not logically responsible to the meanings previously attributed to the formulation of the rule (nor responsible to some instances of its application, previously acknowledged as correct, and to the meaning of terms such as "identical", "analogous" etc.), then the role that the traditional conception assigns to the understanding of rules must be assigned to other factors. Linguistic decision takes the place of what, in that conception, was the discovery (errors always being possible) of necessary relations between rules and results of their correct application. Yet, such a decision would not be made if the intrinsic compelling logical strength of rules, now vanished, were not substituted by the capability of persuasion of the sign figure which we decide to entrust with the role of paradigmatic picture of how the correct performance of certain operations leads to a certain result. In conclusion, in posing the requirements of perspicuity, surveyability etc. of a proof, Wittgenstein is simply stressing that hypothetical conclusions are not drawn concerning what occurs when certain manipulations of sign structures of any length and complexity are carried out (hence, a proof, as opposed to an experiment, is reproduced by simply copying it). Rather, the proof is the figure that we are inclined to see as the model of how a certain sign process yields a certain result (we could say that this request updates Wittgenstein's old maxim according to which what a proof proves cannot be described but only shown by it). And a sign construction is perspicuous, surveyable etc., precisely when it induces us to use it as a picture, as a measure: "'A proof must be capable of being taken in' means: we must be prepared to use it as our guide-line in judging" (*RFM*, III, §22); "'Proof must be capable of being taken in' really means nothing but: a proof is not an experiment. We do not accept the result because it results once, or because it often results. But we see in the proof the reason for saying that this *must* be the result" (*RFM*, III, §39); "'Proof must be surveyable': this aims at drawing our attention to the difference between the concepts of 'repeating a proof', and 'repeating an experiment'" (*RFM*, III, §55).

It should be clear, now, that Wittgenstein's observations on the perspicuity, surveyability etc. of proofs are essentially concerned with the capability of a notational system of furnishing standards for the correctness of application of its procedures of sign transformation. This capability varies from one system to another: the question of the unsuitability of sign constructions for this use, when certain limits of length and complexity are exceeded, soon appears in some notations, whereas it tends to fall into the background in some others (for instance, in the languages of many branches of informal mathematics). In effect, Wittgenstein's interest is prevalently focused on the problem of the relation between two mathematical systems, one of which is – or contains as a part – a translation of the other. In the words of Kreisel, the Austrian philosopher's analysis "applies generally to *proof theoretical reductions* or *translations,* either mapping of proofs of one kind into proofs of another, or theorems (provable formulae) of one system into theorems of another". [38] Wittgenstein's gaze is not turned to the mathematical aspects of the problem but to its philosophical implication: namely, to the claim that only the *possibility in principle of proving,* within a system S, a theorem which corresponds – according to certain translation rules – to a theorem proven within a system T, supplies a definitive justification of the result obtained in T. The two typical examples taken into consideration by Wittgenstein concern: (1) the relation between the Russellian logical calculus and numerical arithmetic; and (2) the relation between the arithmetic of strokes (or the arithmetic that uses the primitives "0" and "+1" and which introduces the operations of sum and product by means of the usual recursive definitions) and the arithmetic in the decimal system. Since the contrary has often been maintained, perhaps it would be worthwhile to note that Wittgenstein's preponderant attention towards the most elementary parts of logic and of arithmetic is not due to a deprecative tendency to underestimate the more advanced parts of mathematics and logic; neither, as we know, is it due to an inclination towards strict finitism. The true reason why his enquiries take this direction lies, instead, in his intention to critically evaluate the foundational claims of logic on arithmetic (and on all mathematics). More generally, he wishes radically to revise the very idea of a hierarchy between two mathematical systems, one of which, even though it cannot be used in practice for carrying out the relevant proofs, is still the ultimate foundation of the other (for instance, of a hierarchy between one system in a certain notation and another system which appears as a "mere", albeit practically indispensable, reformulation of the former in an abbreviated notation). The attitude that tends to relegate to a merely practical sphere – which is mathematically insignificant – the restrictions regarding the feasibility of proof construction in a calculus, when certain upper bounds in the complexity of their formulae and the length of their derivations are exceeded, originates from a conception of sign transformation processes that do not take into account Wittgenstein's rule-following considerations. This circumstance is shown particularly clearly in the case of logical calculus. As Wittgenstein says: “We incline to the belief that *logical* proof has a peculiar, absolute cogency, deriving from the unconditional certainty in logic of the fundamental laws and the laws of inference. Whereas propositions proved in this way can after all not be more certain than is the correctness of the way those laws of inference are *applied*” (*RFM*, III, §43). As soon as one bears in mind that “the proof is not its foundations plus the rules of inference, but a *new* building” (*RFM*, III, §41), or as soon as one realizes that axioms and rules of inference do not contain in themselves, already pre-determined, the results of their correct application, then the cogency of a logical derivation comes to depend upon its capacity to convince us to see it as the model of the correct performance of a certain sequence of operations on formulae. It must induce us to consider it to be representative of the shape possessed by every correct performance of this sequence of operations, i.e. it must persuade us to adopt *the definition* that identifies a correct performance of these operations with a sign construction which is that shape. Wittgenstein calls this capability “the geometrical cogency of a proof”: such a cogency depends upon parameters such as the perspicuity of the formulae that occur within it and the length of the derivation; and it depends upon them in an essential way, to the extent that symbols which are not perspicuous and derivations which are too long would put into question our inclination to assign it the role of paradigm. Indeed, the shape of a construction that is to be used as a model of how a certain result is yielded by the correct performance of a certain series of operations must be clearly identifiable (to acknowledge as proof a derivation which is not surveyable would be tantamount, I believe, to giving an ostensive definition of the predicate “red” by pointing to a sample immersed in a half-light). It is not that, beyond certain upper bounds of length and complexity, there are mathematical proofs that cannot be carried out because of empirical limitations in our physical or intellectual capabilities; simply, there are no such proofs (in that calculating technique): “The consideration of *long* unsurveyable logical proofs is only a means of shewing how this technique – which is based on the geometry of proving – may collapse, and new techniques become necessary” (*RFM*, III, §46). A quite similar point can be made, according to Wittgenstein, as regards arithmetic in primitive notation. If the signs used exceed a certain length, the manipulation processes generate constructions which do not have any “characteristic visual shape” (*charakteristiche visuelle Gestalt*) (*RFM*, III, §11). Hence, it becomes impossible to establish whether any other figure is a correct reproduction of the shape of such a construction: and this means that it cannot be used as paradigm, it cannot be a mathematical proof. It is not that, in primitive arithmetic, there are proofs of arithmetical identities with arbitrarily high numbers which cannot in practice be carried out because our powers of visual discrimination etc. are limited. These proofs do not exist and, therefore, in that calculus, not even the corresponding arithmetical identities exist: "Now I ask: could we also find out the truth of the proposition 7034174 + 6594321 = 13628495 by means of a proof carried out in the first notation? – Is there a proof of this proposition? – The answer is: no" (*RFM*, III, §3). This radical thesis has immediate consequences on the problem of the foundational role that logic would play with regard to numerical arithmetic, or that arithmetic in primitive notation would play with regard to decimal arithmetic. Consider, first of all, the problem of the construction of the logical formula which, according to the usual translation method, corresponds to a true arithmetical identity, and of the proof that it is a tautological formula. This construction and this proof can be carried out without any difficulty as it concerns the logical translation of arithmetical identities with low numbers; but the claim that, for every true arithmetical identity, a definite logical formula *could* be constructed and its being tautological *could* be proven rests on the idea that what matters is the performability, in principle, of certain operations. In fact, with high enough numbers, the logical formulae to construct and check would be completely unmanageable in practice (without the intervention of arithmetic there would be great obstacles even in deciding which, of two such formulae, is the correct translation of a given arithmetical identity). That claim is therefore groundless, given the vacuity, for Wittgenstein, of the notion of performability in principle of the operations in question (the same goes for the claim that, for every arithmetical identity provable in the decimal system, a proof in the primitive arithmetic *could,* in principle, be constructed). There is also a second argument, of equal weight, against the idea of a hierarchy among calculi. Suppose that the complex logical formula corresponding to an arithmetical identity unanimously ratified as being correct reveals itself not to be tautological when subjected to checks, or that the addition of two high numbers in primitive arithmetic gives a different result from the one that is acknowledged to be correct when the two numbers are added together with the usual addition technique in the decimal system. In both cases, one would use the results of the "secondary" systems (numerical arithmetic and the usual decimal arithmetic, respectively) to correct those obtained in the "primary" systems (logical calculus and primitive arithmetic, respectively): "A shortened procedure tells me what *ought* to come out with the unshortened one. (Instead of the other way round.)" (*RFM*, III, §18). In such cases, when one affirms that, if miscalculations had not been made, one would have obtained even in the "primary" calculus the result ratified as being correct in the "secondary" calculus, then one uses the latter to introduce a new criterion of correctness for sign transformation processes carried out in the former (one extends, so to speak, the field where the result of the "secondary" calculus exercises its normative strength). And this is not because there is a privileged calculus, "which lies at the source of mathematics", that supplies us with the "right" results, but because we are firmly determined not to put those results into doubt which have been reached by means of the most familiar calculating techniques. What is more, where the lack of perspicuity and surveyability drives us to resort to the notion of what can, in principle, be proven, it is the effectively performable calculations which, in reality, determine what *should* be found, if the practical limitations were able to be overcome. A third argument which Wittgenstein elaborates against logicism regards the possibility of genetically developing the various branches of mathematics, starting with the Russellian calculus, without introducing new concepts and new methods into it, such as new rules of comparison for its sign structures. One is not concerned, here, with knowing whether one can construct, for every proof carried out in a certain mathematical system (arithmetic, for instance), a corresponding logical proof, but with clarifying another point: that the introduction of new symbols by means of definitions, the use of notational devices such as the numeration of variables and the like – all of which are indispensable for *being able to perform in the logical calculus* the operations of the mathematical system to be translated – constitute, in reality, a passage into a new calculus (which Wittgenstein describes also as the discovery of a new aspect of the old calculus, giving the example of the technique, which he outlined in the *Tractatus,* for introducing numbers as exponents of a logical operation). By this enrichment of the logical calculus, its signs are rendered perspicuous and its proofs become surveyable (with the aim, for instance, of being able to effectively carry out the symbolic processes which correspond to arithmetical calculations): as Wittgenstein himself says, its expressions are transformed into shapes. Since it is only on this condition that its sign constructions can be used as paradigms, an important consequence follows, concerning the status of definitions. The conception according to which definitions are mere abbreviations, eliminable in principle, reproposes a notion which for Wittgenstein is discredited: if, by replacing the defined symbols with the complex expressions in primitive notation which define them, a sign construction loses its perspicuity, then it follows that it is wrong to affirm that mathematical processes, which can be carried out using the abbreviated notation, could also be realized within the primitive one, and that only empirical limitations impede this. The construction with defined symbols is not the simple, manageable substitute for an unmanageable, but existent, mathematical proof, which in practice cannot be carried out; and this simply because the latter does not exist as a proof:

> I want to say: if you have a proof-pattern that cannot be taken in, and by a change in notation you turn it into one that can, then you are producing a proof, where there was none before.... The assumption is that the definitions serve merely to abbreviate the expression for the convenience of the calculator; whereas they are part of the calculation.

> By their aid expressions are produced which could not have been produced without it.
>
> (*RFM*, III, §2)

One last group of Wittgenstein's observations concerns that which, apparently, is the strongest argument in support of the reductionist claim: it is founded upon the existence of a theorem that shows, for every provable identity in numerical arithmetic, how to construct a proof of the corresponding formula in logical calculus (or upon the existence of the theorem that shows, for every true identity in decimal arithmetic, how to construct a proof of its translation into primitive arithmetic). The objection that Wittgenstein moves to the idea that the existence of such theorems resolves, once and for all, the question in favour of the foundationalist thesis, is based on his conception of the nature of necessity. With the proof of the theorem on the translatability of numerical arithmetic into logic, it is acknowledged that, if F is a true arithmetical identity, then by correctly carrying out certain specified derivations in the logical calculus, the formula Ψ(F) can be obtained (where Ψ(F) corresponds to F according to the general translation method chosen). As in the case of any other mathematical theorem, it does not state an empirical regularity which regards what actually occurs when one follows the instructions for the translation and carries out the sign manipulations shown by the proof of the theorem (even if such a regularity is observed in all those cases where the length of the logical proofs is sufficiently small). Rather, the proof of the theorem convinces us to attribute a normative strength to the involved notion of correspondence. In fact, the theorem expresses in disguise the grammar rule that it is senseless to affirm that a correct logical derivation has yielded a formula that is different from Ψ(F) (when faced with a derivation which terminates with a formula other than Ψ(F), we must say either that it contains some error or that the translation of F is mistaken). According to Wittgenstein, the crucial point, however, is that the ascription of normative strength to this notion of correspondence has, at its very foundation, only a natural, spontaneous decision on the way in which a conceptual connection between the two calculi should be set up:

> If someone tries to shew that mathematics is not logic, what is he trying to shew?... He is not trying to shew that it is impossible that, for every mathematical proof, a Russellian proof can be constructed which (somehow) "corresponds" to it, but rather that the acceptance of such a correspondence does not lean on logic.... "But can't one prove logically that both transformations must lead to the same result?" – But what is in question here is surely the result of transformations of signs! How can logic decide this?
>
> (*RFM*, III, §53).[39]

WITTGENSTEIN'S QUASI-REVISIONISM

Cantor's diagonal proof and transfinite cardinals

Wittgenstein's claim that the statement "the set of all sequences of natural numbers is not enumerable" acquires its sense only through Cantor's proof is commented upon by Kreisel with the following ironic exclamation: "One could only wish that all one's assertions had as much sense as the assertion of the non-enumerability of the set of all sequences before its proof!". [40] On this point, according to Kreisel, there is no reasonable doubt, because the notions involved in the assertion in question are impeccably definable and, hence, fully understandable, *before* the proof is presented. Plainly, this is enough to make the statement meaningful, independently from any eventual proof; the proof is no more than the tool for establishing its truth. As we know, Kreisel's criticism rests upon certain assumptions regarding the meaning of mathematical expressions which Wittgenstein definitely rejects. And, in effect, the Austrian philosopher's starting observations on Cantor's proof aim precisely at overturning the "received view" of the relation between sense and proof of a mathematical proposition. First, he gives a general warning regarding the relation between the expression of a mathematical result in ordinary language and its proof:

> The result of a calculation expressed verbally is to be regarded with suspicion. The calculation illuminates the meaning of the expression in words. It is the finer instrument for determining the meaning. If you want to know what the verbal expression means, look at the calculation; not the other way about. The verbal expression casts only a dim general glow over the calculation: but the calculation a brilliant light on the verbal expression.
>
> (*RFM*, II, §7)

That the real import of a theorem can be appreciated only by taking its proof into consideration (think of the case of existence theorems) is a pacifically acceptable claim even for those who do not in any way share Wittgenstein's ideas on meaning in mathematics. These ideas, instead, permeate the further development of his analysis. The original "mistake" in the usual interpretation of the relation between sense and proof of Cantor's theorem lies, according to Wittgenstein, in maintaining that the understanding (before the proof) of the statement that the set of real numbers cannot be ordered in a denumerable series can be based upon a previous intensional characterization of the general notion of series (of the order type of natural numbers, that is the only order type of denumerable series considered by Wittgenstein). As seen in Chapter 2, this claim is already shown to be groundless in the light of the intermediate Wittgenstein conception of the relation between the general and the particular in mathematics (for that part of mathematics which is beyond the verificationist boundaries); and, *a fortiori*, it is groundless on the basis of the generalization of that conception, yielded by Wittgenstein's rule-following considerations. Suppose that a sound definition of the notion of series has been formulated. The grammar rule expressed in disguise by a theorem of the kind "**S** is a series" is stipulated freely, in the sense that in no way can its adoption be justified by resorting to the meanings *previously* assigned to the words occurring in the theorem and, in particular, to the word "series". In this same sense, the acknowledgement of a sign construction as a proof of the theorem is also free: a figure is a proof in so far as it persuades us to adopt that phraseology, namely, in so far as it produces the inclination to accept the rule that, by definition, **S** is called "a series" (in the jargon of the intermediate phase, the proof induces us to "see" a series in **S**). In such a process, all that is left to us is to let ourselves be guided by the "perception" of analogies with the structures that have already been normatively identified as series (and whenever an analogy of such a sort is acknowledged, the predicate "analogous" is provided with a new meaning). The situation of somebody who, for the first time, poses the problem of whether the set of real numbers can be ordered in a denumerable series is, from Wittgenstein's point of view, very different from that which Kreisel describes as obvious. What is lacking is the stable semantic background with which the definition of the notion of denumerable series is supposed to supply the search for the solution of the problem. Such a sort of background exists only when one is verifying *the empirical hypothesis* that a certain object **a** falls under a given concept **F**; but it vanishes when, as always in mathematics, one is creating a conceptual connection, is laying down *the grammar rule* that **a** is an **F**. Thus, the resources of one who faces the question of the denumerability of the set of the reals are, in effect, very modest:

> The mistake begins when one says that the cardinal numbers can be ordered in a series. For what concept has one of this ordering? One has of course a concept of an infinite series, but here that gives us at most a vague idea, a guiding line for the formation of a concept... the expression ["infinite series"] stands for a certain analogy between cases, and it can e.g. be used to define provisionally a domain that one wants to talk about.
>
> (*RFM*, II, §16)

Now, in order that the question regarding the possibility of ordering the set **R** of the reals in a denumerable series has what Wittgenstein calls "a clear sense", the extension to **R** of the analogy between the structures previously acknowledged as denumerable series would need to be conceivable; but that could only happen if either the analogy in question had also been acknowledged (and then the problem would already be solved) or, at least, if there were a method for deciding, given any arbitrary set, whether it can be ordered in a series or not. Since, in the situation in question, neither of the two above-mentioned alternatives holds, the following "pessimistic" conclusion must be drawn: "Asked: 'Can the real numbers be ordered in a series?' the conscientious answer might be: 'For the time being I can't form any precise idea of that'" (*RFM*, II, §16). [41] By posing the question on the denumerability of the set **R**, we show our determination to extend our system of grammar rules (in particular, the rules for using the phrase "order which is analogous to the order of the naturals"). Thus, we are dealing with the expression of a practical orientation, not with a genuine, theoretical doubt: as a consequence, we can escape the untenable assumption according to which Cantor's proof has shown that a possibility hypothetically conceived by us (that of the denumerability of **R**) was, actually, a logical absurdity.

What has been outlined above is the appropriate description, in Wittgenstein's opinion, of the situation that precedes Cantor's proof. Let us now see how, in his view, the effect of this demonstration should be described. Like every other mathematical proof, even the diagonal proof plays the role of paradigm: to be precise, it is the model of the construction of a denumerable sequence of naturals, which is different from all the denumerable sequences of naturals that belong to a given enumerable set of such sequences. The meaning of the expression "denumerable sequence of naturals which is different from all the denumerable sequences of naturals that belong to a given enumeration" is determined in reference to Cantor's construction: the grammar rule is adopted that, by this expression, one must mean the diagonally generated sequence. The task of a creative mathematician like Cantor is not to establish an indisputable truth, but to induce us to enrich in a certain way our linguistic apparatus: "Cantor gives a sense to the expression 'expansion which is different from all the expansions in a system', by proposing that an expansion should be so called when it can be proved that it is diagonally different from the expansions in a system" (*RFM*, II, §31). The ability of the creative mathematician is shown by his proposing a sign construction (proof) which produces, in all those who have received a certain training, the willingness to introduce a certain new conceptual tool: "'I want to shew you a method by which you can serially *avoid* all these developments'. The diagonal procedure is such a method. – 'So it produces a series that is different from all of these'. Is that right? – Yes; if, that is, you want to apply these words to the described case" (*RFM*, II, §8). Let us now come back to the question of the non-denumerability of **R**. Once that, in reference to Cantor's construction, the meaning of the expression "real number which is different from all the elements in a given denumerable sequence of real numbers" has been determined, it is utterly natural to adopt the further rule that excludes as senseless the expression "denumerable sequence of all the real numbers": indeed, for any given denumerable sequence of reals, a real number which does not belong to the sequence can be constructed diagonally. If one limits oneself to the consideration of these aspects of Wittgenstein's position, one can appreciate the core of truth contained in his reiterated declarations of non-interference in mathematical practice. In fact, it seems clear to me that, in the observations examined hitherto, Wittgenstein simply undertakes to supply a reinterpretation of the result reached by Cantor, without attacking its mathematical validity. Certainly, unlike Wittgenstein, a mathematician would say that the meaning of the expression "real number different from all the elements in a denumerable sequence of real numbers" was well known before Cantor's proof, and that the latter, far from conferring meaning to that expression, has simply shown how to construct an object to which it applies. Furthermore, he would deny that the proof simply persuades us to adopt the rule that excludes as senseless the expression "denumerable sequence of all the real numbers", and would maintain, on the contrary, that it compels us, on the strength of the definitions, the axioms and the rules of inference accepted *before* carrying it out, to recognize the truth of the statement that, for any given denumerable sequence of reals, there is one real number that does not belong to it. The clear-cut contrast between the two interpretations brings us back to the radical divergences between the theories of meaning underlying them: but the calculating method devised by Cantor is safe from this clash.

Things radically change, however, when one begins to consider *the numerical consequences* that the classical theory of the transfinite deduces from Cantor's proof. Here, Wittgenstein's opposition is not limited to his proposal to use – in the description of the results obtained in set theory – a terminology that accurately avoids the presentation of the mathematician's activity as a sort of enquiry into the physics of ideal entities (a misleading picture that springs from mistaken assumptions on the nature of meaning and necessity). He outlines various arguments which, altogether, end up by questioning the legitimacy of classical set theory. It is true that Wittgenstein has no intention of elaborating a programme of alternative reconstruction of certain parts of mathematics. Nevertheless, there is, in his writings, the idea that a "correct" interpretation of certain notions can contribute to lessening their interest to mathematicians, to the point of bringing about a sort of "natural perishing" of the branches of mathematics where they are treated. In this sense, it seems totally appropriate to me to qualify Wittgenstein's position as being "quasi-revisionary". [42] The main objection that he moves to the classical interpretation of Cantor's proof concerns the thesis that the result of the proof has to do with the comparison between the number of elements of the set of natural numbers (called by Wittgenstein "cardinal numbers") and the number of elements of the set of real numbers. If this thesis is subscribed to, then, according to Wittgenstein, the proof becomes a "puffed-up proof", in the sense that it is attributed with the capability to do something which, actually, it does not do:

> If it were said: "Consideration of the diagonal procedure shews you that the concept 'real number' has much less analogy with the concept

> 'cardinal number' than we, being misled by certain analogies, are inclined to believe" that would have a good and honest sense. But just the opposite happens: one pretends to compare the "set" of real numbers in magnitude with that of cardinal numbers. The difference in kind between the two conceptions is represented, by a skew form of expression, as difference of extension.
>
> (*RFM*, II, §22)

The first reason for this charge is the rejection of the extensional conception of the infinite, a rejection that accompanies Wittgenstein's reflections on the subject from the *Tractatus* right up to his last writings. To this, a second, essential reason should be added: even if it is accepted that, using Frege's and Russell's method of definition by abstraction, one can introduce the symbol "$\aleph_0$", the grammatical conditions in order that it can be treated as a numeral are not satisfied. As far as the first point is concerned, an interesting clue as to Wittgenstein's position are those inverted commas enclosing the term "set" where, in the passage quoted above, the set of real numbers is mentioned. That one must not speak of real numbers (or irrational numbers) as forming a totality is clearly spelt out by the Austrian philosopher: "It might be said: Besides the rational points there are *diverse systems* of irrational points to be found in the number line. There is no system of irrational numbers – but also no super-system, no 'set of irrational numbers' of higher-order infinity" (*RFM*, II, §33). By saying that no *system* of irrational numbers exists, he means that, by virtue of the diagonal proof, given any enumeration of irrationals, there will always be some irrational number not included in it. When he denies the existence of the *set* of irrational numbers, instead, Wittgenstein is making a statement that is not justified by a theorem, but simply expresses the endorsement of a certain conception of the mathematical infinite: i.e. the conception according to which the reference to an infinite totality is to be understood as the reference to our decision not to pose grammatical limits on the application of a generation law of linguistic expressions. To speak of the number of elements of the set of real numbers is therefore already a symptom of some confusion. The most interesting aspects of Wittgenstein's position emerge when he calls into question even the legitimacy of introducing the notion of the cardinal number of the series of natural numbers. Apparently, we come across a certain tolerance: indeed, liberalizing his intermediate phase position, Wittgenstein acknowledges as acceptable the grammatical inclination to extend the use of the expression "1–1 correlation" from the case of finite classes to that of infinite sequences. Once this concession has been made, even the symbol "$\aleph_0$" may be used to indicate the infinity of a sequence (namely, the grammatical property of a technique for producing expressions which is constituted by the unlimited applicability of the technique):

> Thus we have a grammatical class "infinite sequence", and equivalent with this expression a word whose grammar has (a certain) similarity

> with that of a numeral: "infinity" or "$\aleph_0$". This is connected with the fact that among the calculi of mathematics we have a technique which there is a certain justice in calling "1–1 correlation of the members of two infinite series", since it has a similarity to such a mutual correlation of the members of what are called "finite" classes.
>
> (*RFM*, II, §38)

But, according to Wittgenstein, to concede that the symbol "$\aleph_0$" can be used to make reference to the property which a technique for generating a series of symbols possesses whenever *we* are not inclined to call any term "the last term in the series" is far from admitting that it can be interpreted as *the number of elements* of an infinite sequence. This inflexible preclusion appears to be very surprising, especially if, as is reasonable, one tries to place it in the general framework of his conception of mathematics. What is at stake is the legitimacy of a definition: if there exists any justification for adopting the grammar rule whereby the statement that a sequence has $\aleph_0$ elements comes to mean that it can be 1–1 correlated with the set of naturals. One may also concede that the appeal to the analogy with the notion of cardinal number of a finite class does not exploit any general notion of analogy, but determines the meaning that one decides to give the predicate "analogous" in this context; and that the only justification for this linguistic decision is that it is agreed upon by practitioners of mathematics. But one could maintain that the practice of the vast majority of mathematicians (and not only the classical mathematicians, as far as "$\aleph_0$" is concerned) constitutes, in effect, the anthropological state of affairs which tips the balance definitely in favour of the notion of (the least) transfinite cardinal number. If one calls attention to the existence, amongst the mathematicians themselves, of hostile attitudes to the Cantorian theory of the transfinite (finitism, intuitionism, and so on), then it seems that, from Wittgenstein's point of view, the only appropriate conduct would consist in taking note of the different grammatical inclinations existent in the community of mathematicians, and in evaluating, with the greatest philosophical tolerance, the different practices that draw inspiration from these inclinations. In reality, things are the exact opposite: not only does Wittgenstein reject the notoriously disputable axioms of set theory ("We might say: if you did not understand *any* mathematical proposition better than you understand the Multiplicative Axiom, then you would *not* understand mathematics" (*RFM*, VII, §33)), but he qualifies the expression "number of elements in an infinite series" as being senseless. The analogies that lead to the use of "$\aleph_0$" as a numeral are judged by him to be inappropriate, in spite of the wide spread that this "perverse" grammatical inclination has among professional mathematicians. It is only by enriching with a new element the framework outlined so far of Wittgenstein's positions that this apparent incongruity can be accounted for. In expounding the Austrian philosopher's considerations on rule-following, his conception of mathematical proof as paradigm of the correct performance of certain sign transformations, and the requirements of perspicuity, surveyability etc. of a mathematical proof, we have concentrated our attention exclusively on what Wittgenstein calls "the geometrical application" of proofs and theorems:

> But now can I say that the conception of a proof as "proof of constructability" of the proved proposition is in some sense a simpler, more primary, one than any other conception?... This would point to a geometrical application. For the proposition whose truth, as I say, is proved here, is a geometrical proposition – a proposition of grammar concerning the transformations of signs. It might for example be said: it is proved that it makes *sense* to say that someone has got the sign... according to these rules from... and...; but no sense etc. etc.
>
> (*RFM*, III, §38)

This conception, which I have called "quasi-formalistic", offers, in my opinion, a unifying key for the interpretation of Wittgenstein's philosophy of mathematics, in all the phases of its development. However, it does not provide an understanding of the reasons for the quasi-revisionary attitude that he has to certain parts of mathematics, above all set theory. Nor it is sufficient, to this purpose, to recall Wittgenstein's extreme anti-extensionalism. There is a further principle which underlies his attitude to the theory of transfinite cardinals and which is again linked to the question of the meaning of mathematical expressions. In a first, generic formulation, the principle has no revisionary implications, but simply seems to repropose a typical theme of the logicist approach to the foundations of mathematics (a theme which was explicitly put forward by Frank P. Ramsey in his criticism of Hilbert's formalism):

> I want to say: it is essential to mathematics that its signs are also employed in *mufti*. It is the use outside mathematics, and so the *meaning* of the signs, that makes the sign-game into mathematics. Just as it is not logical inference either, for me to make a change from one formation to another (say from one arrangement of chairs to another) if these arrangements have not a linguistic function apart from this transformation.
>
> (*RFM*, V, §2)

For instance, in order that the meaning of the symbols for finite cardinals can be understood, it is not enough to limit oneself to taking into consideration their occurrence in those disguised rules of the geometry of signs that arithmetical identities are; rather, one must consider their use in the empirical statements describing the results of processes of counting objects. This very plausible exigency, lacking in any revisionary import, is transformed, however, into something very different when Wittgenstein states, loud and clear, the following general thesis: "Concepts which occur in 'necessary' propositions must also occur and have a meaning in non-necessary ones" (*RFM*, V, §41). Here, it is not claimed that mathematical terms *can,* but that they *must* (*müssen*) occur meaningfully even in non-mathematical propositions: one is not dealing, therefore, with a mere advice to adopt – when one wishes to clarify the meaning of mathematical expressions – a holistic perspective, but with the institution of a rigid hierarchy among their uses. The extra-mathematical use of mathematical expressions, indeed, so peremptorily required by Wittgenstein, supplies the term of comparison for evaluating the legitimacy of their employment *within* mathematics. The mathematicians' grammatical inclinations should be compared with the meanings that the expressions of the mathematical language have in their non-mathematical applications. The criterion of the pure and simple agreement in the practice of mathematicians finds a limitation in this principle; and it is from this that the quasi-revisionary tendencies present in Wittgenstein's philosophy of mathematics originate. The case of the mathematical infinite and of the theory of transfinite cardinals illustrates the situation in an exemplary way. The attack on the extensionalist conception of the infinite rests, fundamentally, on the priveleged role that is attributed to the uses of the adjective "infinite" in the common language: they serve to distinguish between correct and incorrect inclinations in its use within mathematics. In the common language, in short, it does not denote a quantity greater than every finite quantity, but expresses the acknowledgement of the grammatical possibility (meaningfulness) of referring to a finite quantity which is as great as one likes:

> The point is that Frege hasn't told us what has the number endless. You were led to think that probably if it were used at all it would be used for an immense collection of things. "The number of cardinal numbers" looks like "the number of a row of trees" – whereas we use it in sentences like "Jackie already knows endless (or $\aleph_0$) multiplications...." I can say, "Ask any sum you like: I give you an $\aleph_0$ choice." But you can't then say, "Give me $\aleph_0$ shillings"; this would not mean anything.
>
> (*LFM*, pp. 169–70)

This is the first step along the path that leads Wittgenstein to the rejection of the interpretation of "$\aleph_0$", and, *a fortiori,* of the other symbols for transfinite cardinals, as numerals. In contrast, consider a numeral for a finite cardinal number – "10", for instance. According to Wittgenstein, its classification as "a numeral" can be traced directly back to the fact that it is used, outside mathematics, to indicate the number of elements of finite classes of objects. Indeed, if it were used *only* in the formulation of mathematical propositions of the kind "the number of the points of intersection in a pentacle = 10", we would have no right to interpret it in the usual (numerical) way. Such a proposition does not state the result of the process of counting the elements of the extension of the geometrical predicate "point of intersection in a pentacle" but, in Wittgenstein's view, is the disguised expression of a grammatical stipulation, which contributes to the determination of the meaning of the predicate. By correlating *atemporally* the numerals from "1" to "10" with the points of intersection of a particular drawn pentacle – i.e. by constructing a geometrical proof – we do not count the elements of an empirically given set, but establish how the intersection points – in any figure correctly identified as a pentacle – *must* be distinguished, and therefore what is to count, by definition, as an element of the set in question (given the purely extensional nature of every mathematical concept, we determine the meaning of the predicate itself). In short, according to Wittgenstein, an expression cannot be appropriately classified as a numeral if it enters *only* in the formulation of linguistic stipulations on what must be called, by definition, "a such-and-such". In laying down these conventions, we do not, indeed, establish "quasi-experimentally" the number of objects to which a mathematical predicate applies, but only determine its meaning. In discussing a quite similar example, Wittgenstein says: "The number 3 can be the number of apples or the number of roots of a certain equation. If we didn't know how to use 3 *outside* mathematics at all, we should get no idea of its use if we said it is the number of roots of this equation. For in mathematics 3 is the number of roots either by definition or by proof" (*LFM*, p. 140). To return, now, to the problem of the meaning of "$\aleph_0$" (or of the equivalent expression "infinite sequence"). As far as what has been seen above, this symbol is not used outside mathematics as a numeral: it is another part of speech. Moreover, the fact that, within set theory, one makes assertions such as "the set of algebraic numbers has $\aleph_0$ elements" or "the set of rational numbers has $\aleph_0$ elements" is not sufficient to conclude that the predicate "having $\aleph_0$ elements" expresses a numerical property or – which is the same – that the symbol "$\aleph_0$" is legitimately used as a numeral (since these statements are disguised expressions of grammar rules). In conclusion, the adoption of a certain grammatical criterion in order to call a mathematical structure "an infinite sequence", founded upon a natural extension from the finite to the infinite of the notion of 1–1 correspondence, has nothing to do, in Wittgenstein's opinion, with the introduction of a method for determining a number: "'Having the number so-and-so' is used differently *in* mathematics and *outside* mathematics. So with the expression 'having the number $\aleph_0$'; it will be wrong to think we know how it is used if we say there are $\aleph_0$ cardinal numbers – which have it either by definition or by proof" (*LFM*, pp. 140–1); "From the fact, however, that we have an employment for a *kind* of numeral which, as it were, gives the number of the members of an infinite series, it does not follow that... we have *here* some kind of employment for something like a numeral. For there is no grammatical technique suggesting employment of such an expression" (*RFM*, II, §38).

At this point, two last questions remain open. First of all, Wittgenstein has to explain where the mathematicians' inclination to treat "$\aleph_0$" as a numeral arises from; moreover, where the classical mathematician's inclination to see, in Cantor's proof of the non-denumerability of **R**, a compelling reason for asserting the inequality "$2^{\aleph_0} > \aleph_0$" originates from, in spite of the fact that this inequality is not anchored in any practice (where, plainly, the practice in question is not the practice within mathematics, because this exists, but that of applying the symbols – occurring in the inequality – in non-necessary propositions). The Austrian philosopher's explanation is quite meagre: habitually, one lets oneself be seduced by the picture evoked by such descriptions as "the number of natural numbers" or by such propositions as "the number of real numbers is greater than the number of natural numbers", namely, by the picture of a huge quantity and by that of a quantity that is greater than another quantity, which is itself infinite: as though it were these pictures, and not the use of the related expressions outside and within mathematics, that provide these expressions with a meaning. The seductive power of these pictures flourishes on the ground of mistaken theories of meaning and necessity, where mathematics is conceived as a sort of physics of ideal entities: the theory of transfinite cardinal appears to be the exploration of a region where one comes across entities with the most surprising properties (dense sets which, nevertheless, let themselves be enumerated; numbers which do not change if 1 is added to them; and so on). It is in this sense that, in Wittgenstein's opinion, set theory is a branch of mathematics of whose applications mathematicians have a "quite fantastic" conception. But what is destined to remain of set theory, if mathematicians subject themselves to the sort of mental cleansing that Wittgenstein urges? On this point, the Austrian philosopher manifests some uncertainty, wondering repeatedly if mistaken ideas concerning the application of a calculus must necessarily prejudice its status as part of mathematics (and gives the example of the difficulties met in the past with the interpretation of the symbol "$\sqrt{-1}$"). Yet, the whole of his observations on set theory authorizes us to think that his purpose is that of attenuating *interest* in this part of mathematics. According to him, such an interest depends entirely upon a mistaken conception of the relations among mathematics, meaning and the pictures invariably associated with the use of certain expressions; a conception that is rooted in the current style of thought and life. Thus, a by-product of his battle for the change of this style will be that mathematicians will turn their backs on set theory. With reference to Hilbert's famous words on the resolute will of the vast majority of mathematicians not to let themselves be expelled from Cantor's paradise, Wittgenstein comments: "I would say 'I wouldn't dream of trying to drive anyone out of this paradise'. I would try to do something quite different: I would try to show that this is not a paradise – so that you leave of your own accord" (*LFM*, p. 103).

The Law of Excluded Middle

As seen in Chapter 2, the question concerning the applicability of the Law of Excluded Middle in mathematics is inextricably entwined with the destiny of the notion of mathematical proposition. Plainly, as soon as restrictions are imposed on this notion, namely, on the mathematical expressions whereby meaningful assertions can be made, the field of tautologies obtainable by substituting the propositional variable "*p*" in the formula "$p \vee \sim p$" is circumscribed. The strong verificationist identification of the condition of propositional meaningfulness with the availability of a general decision method yielded an equivalence, for every given mathematical expression, between the validity of the Law of Excluded Middle and the attribution of the status of mathematical proposition. By not admitting meaningful mathematical alternatives for which a decision procedure is not on hand, the intermediate Wittgenstein not only advanced a severe limitation – with respect to the practice of classical mathematicians – to the use of the logical Law, but also distanced himself from the intuitionist criticism to the generalized application of the Law. Brouwer's exploitation of undecided mathematical alternatives for constructing instances of non-applicability of the Law of Excluded Middle was led back by Wittgenstein to the persistence of extensionalist residues (the "appearance" of meaningfulness of undecided mathematical alternatives would derive precisely from these residues). In one of the sections in *Remarks on the Foundations of Mathematics,* there is a passage which seems to repropose Wittgenstein's old point of view: "In an arithmetic in which one does not count further than 5 the question what 4 + 3 makes doesn't yet make sense. On the other hand the problem may very well exist of giving this question a sense. This is to say: the question makes *no more* sense than does the law of excluded middle in application to it" (*RFM*, V, §11). [43] It is quite reasonable to suppose that the principle applies to the alternative between "4 + 3 = 7" and "~ (4 + 3 = 7)", inasmuch as the series of numbers and the domain of the addition have been suitably extended, and not as a consequence of the fact that, by means of the new algorithm, the identity has been proven. Hence, Wittgenstein's remark must be evaluated, in my opinion, as expressing an unreflected-upon return to the old verificationist suggestions. On the whole, with the decline of the notion of mathematical proposition, the Law of Excluded Middle – in the sphere of mathematics – ends up occupying a completely marginal position, where it is transformed into something very different from a logical law. To clarify this point, it would be better to proceed by distinguishing the case of an identity which has already been proven, such as "4 + 3 = 7", from the case of an undecided alternative, such as that of the occurrence of a certain figure ϕ (for instance, the sequence of digits "777") in the decimal expansion of π. In its geometrical meaning, the identity "4 + 3 = 7" can be restated as the following grammar rule: it makes no sense to describe any empirically given sign behaviour by saying that, by means of a correct application of the operation of addition to the pair (4, 3), a number other than 7 has been obtained. Its negation "~ (4 + 3 = 7)" can be paraphrased as follows: it makes no sense to describe any empirically given sign behaviour by saying that, by means of a correct application of the operation of addition to the pair (4, 3), the number 7 has been obtained. What has the appearance of an innocuous application of the Law of Excluded Middle, namely, the apparent tautology "4 + 3 = 7 ∨ ~ (4 + 3 = 7)", is, in reality, the "disjunction" of the two aforementioned grammar rules: "The opposite of 'there exists a law that *p*' is not: 'there exists a law that ~*p*'. But if one expresses the first by means of P, and the second by means of ~*P*, one will get into difficulties" (*RFM*, V, §13); "If 'you do it' means: you must do it, and 'you do not do it' means: you must not do it – then 'Either you do it, or you do not' is not the law of excluded middle" (*RFM*, V, §17). The difficulties referred to by Wittgenstein arise, of course, in the case of propositions that have been neither proven nor refuted: here, the suppression of the expression "there exists a law" creates apparent exceptions to the validity of the logical principle ("apparent", because the alternative between "there exists a law that *p*" and "there exists a law that ~*p*" is not a disjunction between a proposition and its negation). Before facing this theme, let us examine the consequences of Wittgenstein's conception, to the extent that it applies to an identity such as "4 + 3 = 7". We spoke before of "4 + 3 = 7 ∨ ~ (4 + 3 = 7)" as a disjunction, enclosing "disjunction" in inverted commas. I believe that Wittgenstein actually used sentential connectives only to combine two declarative sentences, not the expressions of two linguistic rules. Now, when he introduces the proposition "there exists a law that *p*", he certainly does not employ it for asserting the empirical existence of a certain grammar rule in the language of the community; rather, he has in mind the use one makes of the proposition when giving expression to the rule in question (when stating the geometrical interpretation of "*p*"). However, even if the empirical interpretation of "there exists a law" were adopted, the alternative between "4 + 3 = 7" and "~(4 + 3 = 7)" would continue not to be really an alternative, and their disjunction, though factually true, would have an anthropological, not a mathematical, content. Wittgenstein's implicit answer to the question: "For how about all the other mathematical propositions, say '$25^2 = 625$?'; isn't the law of excluded middle valid for these *inside* mathematics?" (*RFM*, V, §18), is, then, in my opinion, that the law does not apply (by "all the other mathematical propositions", he means those propositions which, as opposed to "ϕ occurs in the expansion of π", have already been decided or can be decided by means of a known algorithm).

Most of Wittgenstein's observations are addressed to the question of the application of the Law of Excluded Middle to undecided mathematical propositions, where a method is not known either for proving or for refuting them. When one asserts an instance of the logical principle, such as "ϕ occurs in π ∨ ϕ does not occur in π", one maintains, obviously, that each of the two disjuncts has a perfectly clear sense, notwithstanding the fact that we do not know how to prove either. The assumption of such an understanding as being possible may have, in Wittgenstein's view, two different sources. The first is the conception of the infinite decimal expansion of π as a completed totality. Letting oneself be guided by the misleading picture of an infinite sequence as a very long finite sequence, one forms an idea of the occurrence and of the non-occurrence of ϕ in the decimal expansion of π on the model of the meaning that the expressions "occurrence of such-and-such a figure in a given sequence" and "non-occurrence of such-and-such a figure in a given sequence" have when the sequence is finite. With this, one disregards the circumstance that only in the latter case is there a method which – problems of surveyability, perspicuity and so on, aside – can be applied for deciding the question of the occurrence of the figure. It is true that, as a consequence of the rule-following considerations, being able to examine step by step the members of a finite series does not imply the meaningfulness of both the unproven disjuncts; or, equally, it does not imply the existence of a predetermined reply to the question of the occurrence of a certain figure in a finite series, before our acknowledging what this reply must be (as Wittgenstein admitted, to a certain extent, in his verificationist interlude). Nonetheless, the two situations cannot be matched because, for the Austrian philosopher, the grammatical inclination to consider the infinite in analogy to the finite is anyway "perverse". A second source of the illusion that at least one of the two disjuncts can be understood goes back to the knowledge of other cases of occurrence of a figure in an infinite series (for instance, of the occurrence of the figure "41" in the expansion of √2). That it is an illusion follows from the fact that there is no meaning of the expression "the same way" on which the understanding of the "hypotheses" can be founded. On the contrary, it is only the acknowledgement that the figure in question appears in the series in the same way as a figure occurs in another series (or in the same way as a different figure occurs in the same series) that establishes what we mean, in the context in question, by "the same way": "Suppose I were to ask: what is meant by saying 'the pattern... occurs in this expansion'? The reply would be: 'you surely *know* what it means. It occurs as the pattern... in fact occurs in the expansion.' – So *that* is the way it occurs? – But *what way* is that?" (*RFM*, V, §12). Thus, we are led relentlessly back to the conclusion that only a proof will be able to give mathematical substance to one of the two disjuncts concerning the occurrence of ϕ in the decimal expansion of π. The proof transforms one of the them into the expression of *a grammar rule which concerns the results of the correct application of the law for generating digits of the expansion.* A proof of the non-occurrence of ϕ, for instance, would exclude as senseless any empirical description of the type: "by correctly applying the law for the expansion of π, the figure ϕ has been generated"; and a constructive proof of the existence of ϕ, which shows us that it occurs in the interval between the mth and the nth decimal place, would rule out as senseless any empirical description of the type: "by correctly applying the law for the expansion of π up to the nth decimal place, the figure ϕ) has not been obtained" (a slightly different rule would be obtained from a non-constructive proof of the existence of ϕ). Once one of the two disjuncts is proven, one again finds oneself, as far as the application of the Law of Excluded Middle is concerned, in a situation quite similar to that previously described with reference to the identity "4 + 3 = 7" and its negation. On the other hand, to appeal to the logical law *before* one of the two disjuncts has been proven is tantamount, in Wittgenstein's opinion, to formulating a sort of *second order rule*; namely, a non-mathematical rule which prescribes the development, either in one direction or the other, of the system of mathematical rules regarding the law for the expansion of π:

> But what are you saying if you say that one thing is clear: either one will come on ϕ in the infinite expansion, or one will not? It seems to me that in saying this you are yourself setting up a rule or postulate.
>
> (*RFM*, V, §9)

> But does this mean that there is no such problem as: "Does the pattern ϕ occur in this expansion?"? – To ask this is to ask for a rule regarding the occurrence of ϕ. And the alternative of the existence or non-existence of such a rule is at any rate not a mathematical one. Only within a mathematical structure which has yet to be erected does the question allow of a *mathematical* decision, and at the same time become a demand for such a decision.
>
> (*RFM*, V, §20) [44]

Wittgenstein's position, if taken to be a rejection of the employment of the Law of Excluded Middle in inferential processes, seems to inevitably bear a potential revisionary import on classical mathematical practice. I do not believe, however, that it is so, for one fairly paradoxical reason, which I should like to illustrate by returning briefly to Wittgenstein's attitude to set theory. Notwithstanding his explicit and recurrent suggestion not to distance oneself from the concrete practice of mathematicians, to be always aware of the existence of a "solid core" beyond the "glistening concept-formations", he, when dealing with the motivations of set-theoreticians, limits himself to denouncing the fascination of certain misleading pictures, thus completely ignoring the specific mathematical reasons which may lead to the acceptance of set-theoretical axioms (for instance, what one is able to prove by using the Axiom of Choice). [45] An analogous consideration holds, in my opinion, also for Wittgenstein's attitude to the question of the validity of the Law of Excluded Middle. In criticizing the "false pictures" and the inadequate theories of meaning which, in his view, form the basis for the usual justifications of the employment of this logical principle inside mathematics, Wittgenstein seems to take no account of the concrete ground of its use: indeed, he completely disregards the differences between an inferential practice which accepts the law as valid and a practice that does not acknowledge it. Hence, every eventual revisionary implication of his substantial rejection of the applicability of the Law of Excluded Middle in mathematics is passed over in silence.

Consistency

The famous (or ill-famed) observations of the later Wittgenstein concerning the problem of the consistency of an axiomatic system confirm two previously stressed points. In the first place, there is the quasi-revisionary character of his approach: "My aim is to alter the *attitude* to contradiction and to consisency proofs. (*Not* to shew that this proof shews something unimportant. How *could* that be so?)" (*RFM*, III, §82). In the second place, there is his tendency to underestimate the inferential aspect of mathematical practice. In the context at issue, such a tendency reveals itself clearly in several forms: in the rather hasty way he deals with antinomies (both set-theoretical and semantic) and in his insistence on the absolutely marginal role that they have in ordinary linguistic practice; in his linking the importance usually attributed to the contradiction to mistaken ideas concerning the foundational role, with respect to the rest of mathematics, assigned to Frege's logical calculus (to set theory); lastly, in minimizing the relevance of *mathematical* remedies to the antinomies – in particular, of Russell's theory of types – and in his total disregard for the enormous stimulus function that the analysis of antinomies had in the development of mathematical logic (Gödel's theorem included). I believe that this tendency has the same source as Wittgenstein's critical attitude to the Law of Excluded Middle, i.e. his openly declared diffidence with regard to the notion of mathematical proposition. The propositional expression of the result of a calculation – or, more generally, of the conclusion of a proof – is a disguised rule of the geometry of signs, namely, is the expression of a grammar rule that sets by definition which types of empirical descriptions of sign behaviour make sense and which do not. Moreover, a proof is merely a sign construction which is able to induce us to adopt such a geometrical rule and shows the real content of the theorem. But, when the mere force of persuasion replaces the compulsion exercised by logical rules, and when the acceptance of a definition replaces the establishment of the truth of a proposition, there is no more room for the usual notion of inference and hence of the derivation of a contradiction: rather, they lose much of their importance.

Let us now have a quick look at the aspects of the habitual attitude to contradictions, which the Austrian philosopher judges should be corrected. For the time being, we shall assume that it is rational, in some unspecified sense of the term, to accept the Principle of Non-Contradiction, or, equally, to restrict the notion of a calculus to those sign manipulation procedures that do not generate contradictory results. As we know, already in his writings of the intermediate phase, Wittgenstein denies that, beyond the verificationist boundaries, it makes sense to speak of the existence of a hidden contradiction. Obviously, his considerations on rule-following reinforce this claim. For instance, the statement that a finite sequence of correct applications of the derivational rules of a formal system generates an expression of the form $p.\sim p$ as its end formula does not make sense as *hypothesis* on the existence of necessary connections. It acquires a mathematical meaning only in relation to a given sequence of formulae which has been effectively ratified as a sequence of such a sort. Once the existence of unacknowledged grammatical connections, and hence the meaningfulness of logical hypotheses, have been excluded, the fear for the eventual presence of a hidden contradiction in a calculus is shown to be simply a symptom of a mistaken conception of the nature of necessity: "Can we be certain that there are not abysses now that we do not see? But suppose I were to say: The absysses in a calculus are not there if I don't see them!" (*RFM*, III, §78). Wittgenstein's rule-following considerations also serve to show how groundless the shared evaluation is, regarding the supposedly catastrophic effects that the derivation of a contradiction would have for a calculus. Usually, an appeal is made to the tautology "$p \supset (\sim p \supset q)$" to conclude that, if a formula and its negation are derivable, then every formula is derivable in the system. Of course, this is a statement that can be proven within the metatheory of the object-system. According to Wittgenstein, we may be inclined to accept it or not, just as for any other theorem, but there are no logical constraints that compel us to: there are still other paths to try, and – of each one of them – it can legitimately be affirmed that, by following it, one is working in conformity with the (formation and transformation) rules and the definitions given at the time the system was constructed. It is this conception – an immediate corollary of his rule-following considerations – which allows the Austrian philosopher to de-dramatize the question of the banalization of the calculus, which would inevitably be brought about by the derivation of a contradiction. Instead of accepting the aforesaid metatheorem, one can always adopt a rule that rules out as incorrect the derivations which make use of Scoto's law or any of its equivalents: "One may say, 'From a contradiction everything would follow'. The reply to that is: Well then, don't draw any conclusions from a contradiction: make that a rule" (*LFM*, p. 209). So far we have taken it for granted that it would be mistaken not to take steps in the case of a derivation of a contradiction in a system. One of Wittgenstein's central concerns is that of giving the "right" interpretation to that assumption. The point is not that of maintaining the convenience of making use of incoherent calculi but that of clarifying to what extent our inclination to judge the acknowledgement of the validity of the Principle of Non-contradiction as being the only rational conduct can be justified. I shall not tackle this subject, because it goes beyond the limits of an exposition of Wittgenstein's philosophy of mathematics, and involves the most general features of his philosophy. I simply note that, at the roots of his position, there is the rejection of every neutral and objective foundation for this justification, and that this rejection is closely connected to the role assigned by him to mathematics: that of supplying useful or useless forms of description and not true or false descriptions, "units of measure" and not "results of measurements". [46]

We now come to the question of the status of the consistency proof. The field being cleared of all the equivocations regarding the presumed indispensability of such a proof for manipulating confidently the signs of the object-system, all that is left is to evaluate its effective import (these equivocations can be considered to be a little dated, especially in the light of the results of the research done by mathematical logicians on the relations between the object-system and the relative metatheoretical apparatus). As in his writings of the intermediate phase, Wittgenstein has an inductive proof of consistency in mind. From a proof of that kind, one can derive a new criterion for calling a derivation in the object-system "a correct derivation": one adopts the grammar rule according to which it has no sense to affirm that, by a finite number of correct applications of the derivational rules, a sequence of formulae which terminates with a formula of the form *p*. ~*p* has been generated. If we are confident in the fact that we will apply the derivation rules correctly, we may extract from the inductive proof a valid reason for foreseeing that, by manipulating the formulae of the system, we will not run across a contradiction: "that prediction is the application that first suggests itself to us, and the one for whose sake we have this proof at heart. The prediction is not: 'No disorder will arise *in this way*' (for that would not be a prediction: it is the mathematical proposition) but: 'no disorder will arise'" (*RFM*, III, §86). This is the only reasonable confidence that a consistency proof can offer us. Whoever is in search of an "absolute" confidence is hopelessly off track: indeed, nothing can prevent the eventuality that a sequence of formulae which today we would not call "correct derivation of a contradiction" will be unanimously ratified as such in the future. And, at that moment, no objection could be raised to whoever maintains that the new use of the expression "correct derivation of a contradiction" is quite compatible with the meaning we have *always* given it (once again, by virtue of the rule-following considerations). If *this* is what worries us, then we can only hope for "a good angel".

NOTES

THE PHILOSOPHY OF ARITHMETIC OF THE *TRACTATUS*

1 I believe that the systematic presentation of the language of this theory and the explicit formulation of its basic principles are needed for a clear understanding of Wittgenstein's ideas on arithmetic. Italicized numerical exponents of "Ω" occurring in the above series of expressions belong to this language: their use will be explained thoroughly in note 10.

2 Wittgenstein speaks of the "general form of an integer" ("*allgemeine Form der ganzen Zahl*") (6.03), clearly meaning "non-negative integer". As he does not deal with negative integers, we speak more simply here and afterwards of natural numbers. Our quotations from the text of the *Tractatus* are taken from the Routledge & Kegan Paul 1961 edition, with the Pears and McGuinness English translation.

3 See Black (1964), pp. 313–14.

4 *Ibid.,* p. 313 (my italics).

5 The signs "(η)", "(p)", etc. must be interpreted in accordance with the rule established by Wittgenstein in 5.501: in the notation of the *Tractatus,* a sign of this kind denotes the set of propositions which are the values of the variable between parentheses.

6 Cf. Russell (1922).

7 Thus, "Ω" is the variable expressing the formal concept of operation, in compliance with Wittgenstein's general treatment of formal concepts and the related notation (cf. propositions 4.126–4.1274).

8 Cf. Black (1964), p. 314.

9 Cf. Black (1964), p. 314. Each of *H*, ~*H*, *H* ∨ ~*H*, *H*. ~*H* is representative of a set of tautologically equivalent propositions.

10 I use *t* and *s* (in italics) as schematic letters for arithmetical terms inasmuch as the latter are parts of the language of the general theory of logical operations (namely, the language to which the variable "Ω", with or without an arithmetical term as exponent, belongs). On the other hand, **t** and **s** (in bold) will be used as schematic letters for arithmetical terms inasmuch as they belong to the language of usual arithmetic (this being included in the language in which we speak about the operation theory language). Similarly, italics will be used for numerical constants introduced in the language of operation theory, while the non-italicized numerical constants and variables will be employed as usual in the language of arithmetic. The reasons for this duplication of the arithmetical terms will be explained later in the text. Now I wish to stress that the use of schematic letters for arithmetical terms of the operation theory language, in place of the numerical

variables used improperly by Wittgenstein both in the definition in 6.02 and in his definition of product in 6.241, constitutes, in my opinion, the other indispensable step (together with the acknowledgement of the role of "Ω" as a variable) to achieve exactly his intended objective. In order to avoid confusion and interference with other texts, in presenting Black's and Anscombe's positions I have confined myself, and will continue to do so, to their notation, which contains the original defect of Wittgenstein's. However, in the last part of this section, I will briefly outline the alternative approach that I intend to develop, and from then on italics will be used for arithmetical constants belonging to the language of operation theory and for schematic letters standing for arbitrary arithmetical terms of this language (in contrast, the variable "Ω" will appear in normal typeface). Italics were used for the exponents of the expressions of the series mentioned on pages 1–2 because there I was speaking about a definition by induction on the length of a term of the form *0 + 1 + 1 +... + 1* (i.e. on the number of occurrences of "*+ 1*"), belonging to the operation theory language.

11 Black (1964), p. 314.

12 Cf. Black (1964), p. 343.

13 Black (1964), p. 341.

14 Cf., respectively, Anscombe (1959), Fogelin (1976), Block (1975), Ayer (1985), Savitt (1979).

15 Cf. Anscombe (1959), p. 125.

16 As forewarned in note 10, italics are used in order to differentiate arithmetical symbols belonging to the language of the general theory of operations from similar symbols of ordinary arithmetic, appearing in normal typeface (that I also use when discussing the former); schematic letters required for general statements about the former also appear in italics. In order to understand Wittgenstein's intentions it is useful to see what he wrote many years later about the introduction of numerals as exponents of the negation sign "~" in the logical formula "*~p*" (cf. *RFM,* Part III, §46).

17 As we shall see, in order that the reduction suggested by Wittgenstein is effected, the notion of composition of two operations is also required.

18 In his paper already quoted in note 16.

19 Wittgenstein places the *definiendum* on the right-hand side of the sign "=". In the definitions stated in the sequel, even when quoting Wittgenstein's own definitions, the usual convention of placing the *definiendum* on the left and the *definiens* on the right of "=" will be followed. Furthermore, the abbreviation "Def." will be omitted.

20 Cf. Wittgenstein (1922), 4.1272.

21 What Wittgenstein says here about propositions holds generally for every other kind of expression, even if the only operations he considers in the *Tractatus* are operations on propositions.

22 Cf. Whitehead and Russell (1910–13), Introduction to the first edition; (1957), p. 31. Cf. also Anscombe (1959), p. 124, fn 1. However, other important clarifications will have to be added regarding the meaning given by Wittgenstein to the sign "'".

23 I replace "0" and "+1" with the italicized symbols "*0*" and "*+1*" for the reasons already explained in the first section.

24 A somewhat fuller treatment of the conflict between the two rival conceptions will be made in the last section.

25 The fact that, because of their tautological equivalence, the propositions "it is raining" and "~ ~ it is raining" are synonymous according to the extensional criterion adopted by Wittgenstein in the *Tractatus* does not erase the difference between their forms, which is what concerns us here.

26 Apart from the obvious differences, I think it is undeniable that Wittgenstein's treatment of natural numbers as exponents of an operation already contains the basic insight on which the representation of natural numbers in Church's λ-calculus is founded.

27 Obviously, this difficulty does not arise in any way from the fact that Wittgenstein's original numerical variables have been replaced by schematic letters for arithmetical terms of the operation theory language.

28 In the following reconstruction of Wittgenstein's proof I have replaced, consistent with my earlier notation, each term "*0 + 1 + 1 +... + 1*" with the corresponding term "*SS... S0*".

29 I use "ξ" and not "x" for the base of the iteration of an operation, since such iteration can be applied to any appropriate expression, not only to an expression which has not been generated by an application of the operation.

30 Actually, the general theory of operations, to which arithmetic has to be reduced, does not receive a systematic treatment in the *Tractatus*.

31 Strictly, we should have proceeded by induction, as in 6.02. However, I have decided not to go deeply into details because my objective is only that of clarifying, along general lines, Wittgenstein's intentions (or, cautiously speaking, the intentions that I attribute to him). In the attempt to fulfil this purpose, I have adopted the symbolism and principles of the general theory of operations that I consider to be closer to those that Wittgenstein presumably had in mind. Generally, I prefer not to depart too much from the text of the *Tractatus,* even though this choice may involve some unpleasant complications. But it must be admitted that the above interpretation of "${}^{0}\Omega$" as the operation which, when applied to a given base, reproduces it identically, diverges slightly from Wittgenstein's position. As Anscombe suggests, commenting on proposition 5.23, "an operation is what has to happen to a proposition in order to turn it into a *different* one" (Anscombe 1959, p. 117, author's italics); and the fact that Wittgenstein, in 6.02, defines "Ω^{0}'x" as the form of the result of the non-application of an operation to x, and not as the result of the application of the identical operation to x, confirms Anscombe's hypothesis. The reasons why the Austrian philosopher proceeded as he did should have been made clear in our explanation of the inductive definition in 6.02; on the other hand, now we cannot be faithful to Wittgenstein's approach since ${}^{0}\Omega$ needs to be identified with a specific operation (obviously, it cannot be identified with "the absence of an operation"). In note 33 I will briefly outline the technical problems which Wittgenstein's treatment of the number 0 in 6.02 raises within his interpretation of arithmetic in the theory of logical operations.

32 Of course, this is the possibility which Wittgenstein mentions in 6.231: "It is a property of '1 + 1 + 1 + 1' that it can be construed as '(1 + 1) + (1 + 1)'". The next section will be largely devoted to the examination of this central theme of the philosophy of mathematics of the *Tractatus*.

33 A non-restricted application of the substitution method encounters some difficulties. It is true that the following general theorem can be proven: for every pair of arithmetical terms **t** and **s**, if Ω^{t}'x = Ω^{s}'x, then ${}^{t}\Omega$'x = ${}^{s}\Omega$'x; but, in order to carry out the substitutions freely, a more general result is required: for every pair of arithmetical terms **t** and **s**, if Ω^{t}'ξ = Ω^{s}'ξ, then ${}^{t}\Omega$'ξ = ${}^{s}\Omega$'ξ. I think that the

validity of the latter theorem is obstructed by Wittgenstein's treatment of 0, and particularly by "*0*" having been introduced by him to represent the formal property of being an initial expression, i.e. the property of not being generated by any application of an operation (see the previous remarks in note 31).

34 What Wittgenstein himself calls "an expression" ("*ein Ausdruck*") (*T* 3.31).

35 It goes without saying that the plausibility of the thesis attributed to Wittgenstein by the first of the alternative interpretations mentioned in the text is highly debatable.

36 If the operation in question is an operation on propositions and if, correspondingly, the initial symbol is a proposition, it is more appropriate to speak of sameness of sense, where the criterion of synonymy is the tautological equivalence. If we take into account, as we already have and will again in the text, operations that yield definite descriptions as their results, we can assume – without departing, I believe, from Wittgenstein's positions – the following criterion of sameness of meaning: two definite descriptions ξ_1 and ξ_2 have the same meaning if any two propositions constructed one from the other by replacement of some occurrence of ξ_1 with ξ_2 are tautologically equivalent. Of course I do not claim to discuss here the countless well-known problems raised by these criteria of synonymy. What matters here is only their role in explaining proposition 6.22.

37 In the next section we will dwell on the difference between tautological formulae of logic and equations of the general theory of operations, corresponding to true arithmetical identities. Obviously, it arises from the difference of status between sentential connectives occurring in a tautology and the sign of identity of meaning "=", which occurs in such an equation.

38 Recalling that the only linguistic contexts acknowledged as meaningful in the *Tractatus* are extensional contexts.

39 Cf. Pears (1977). See also Pears (1979, 1987).

40 Cf. Pears (1979), p. 202.

41 *Ibid.,* p. 202.

42 See 6.122.

43 I put "knowledge" in inverted commas because, according to Wittgenstein's conception, it is legitimate to speak of knowledge only in connection with that which a meaningful proposition expresses. The verb "*kennen*", sometimes used by Wittgenstein in referring to forms (for example, to the forms of objects, in 2.0123 and 2.01231), corresponds, *mutatis mutandis* (as Malcolm has rightly underlined in Malcolm (1986), ch. I), to the Russellian "to be acquainted with". To be more precise, it shares with the latter the fundamental property of not expressing (alleged) knowledge of truths, but a relation which is presupposed by the possibility of formulating true and false propositions. Of course, it is the same relation that Wittgenstein calls "*erkennen*" in 6.122 (we have used the verb "to recognize", in accordance with the English translation of Pears and McGuinness). The use of inverted commas wherever we speak of cognitive processes, knowledge etc. with reference to the recognition of formal properties and relations signals our deviation from the *Tractatus* positions. "To see (*ersehen*)", "to recognize (*erkennen*)", "to perceive (*einzusehen*)" are the verbs which Wittgenstein usually employs when attempting to describe, even though in a substantially metaphorical way, the relation between a speaker and the formal structure of language.

44 An analysis of the general views concerning identity put forward in the *Tractatus* is beyond the scope of this work. I shall tackle only the question of the role of the symbol "=" when it occurs in an equation of the theory of operations.

45 I have rewritten the numerical expressions mentioned by Wittgenstein in accordance with the syntactic definition of arithmetical term previously employed in this book.
46 The thesis that any term of mathematical language does not name nor describe but shows (*zeigt*) or represents (*dargestellt, darzustellt*) a form will continue to play a decisive part in Wittgenstein's reflections on mathematics after 1929.
47 Inverted commas are needed because, according to Wittgenstein, an equation actually says nothing.
48 "Proof in logic is merely a mechanical expedient to facilitate the recognition of tautologies in complicated cases" (*T* 6.1262).
49 See *T* 6.24. However, it was stressed in the last section that, in Wittgenstein's translation of arithmetic into the general theory of operations, the substitution method cannot be so widely applied as Wittgenstein seems to have believed (at least, according to my reconstruction of his translation).
50 See also *T* 6.234 and *T* 6.2341.
51 The above description of the platonism\constructivism conflict is obviously a rough, schematic outline, but it is sufficient for our present purposes.
52 Wittgenstein does not give this variable but see Anscombe (1959), pp. 128–9.
53 A criticism which can plainly be extended also to Dedekind's theory of the chain of a set.
54 See Frege (1879), Part III; (1884), Part 5, section 80.
55 Russell (1919), p. 21.
56 See *T* 3.318 and 5.2341 ("The sense of a truth-function of *p* is a function of the sense of *p*").
57 See, for example, *T* 6.1231, 6.1232, 6.1233.
58 The Austrian philosopher will devote broader discussions and analyses both to set theory and to the interpretation of arithmetic into logic in his post-1929 writings. We shall deal with them in the following chapters. As regards formalism, in the brief analysis which follows in the text, I shall use the term "formalism" in a somewhat vague sense which is closer to the view of Heine and Thomae (formerly criticized by Frege) than to Hilbert's much more sophisticated one. We shall speak of the latter only when later Wittgenstein references to it are examined.
59 In order to give a more accurate formulation of this distinction, the further distinction between token and type should be used in the way suggested by Frank P. Ramsey with reference to the pair "propositional sign"\"proposition" (see Ramsey 1923, 1925).
60 As Waismann says in Appendix A of *WVC,* p. 219: "To a certain extent it is true that mathematics is based on intuition, namely the intuition of symbols".
61 Ramsey (1925), p. 17.

VERIFICATIONISM AND ITS LIMITS: THE INTERMEDIATE PHASE (1929–33)

1 I am referring in particular to *Philosophical Remarks* and to Chapters 9, 10, 15, 16, 17, 18 and 19 of *Big Typescript,* which, as is known, were published almost unaltered as Chapters I–VII of Part II of *Philosophical Grammar.* Keeping in mind that we are dealing with "second hand" material, I will also use Waismann's notes of the conversations which Wittgenstein held, from 1929 to 1932, with several members of the Vienna Circle, and Ambrose-Macdonald's notes of Wittgenstein's lectures from the period 1932–35.

2 The texts of Weyl, Hilbert, Skolem and Hardy, discussed by Wittgenstein, will be cited later in the chapter.
3 As already remarked in Chapter 1, the relationship between mathematics and rules of non-mathematical language is, in my opinion, a feature of Wittgenstein's philosophy of mathematics which remains constant in its overall development.
4 As far as I know, the first organic expositions of Wittgenstein's problem known today as "the problem of following a rule" are contained in two lectures of the Lent Term of the year 1935 (see *AWL,* pp. 130–5), and in section 5 of Part II of the *Brown Book*.
5 See *WVC,* pp. 103–5.
6 See *PR* §§109 and 115; *PG,* II, Ch. IV, §19, p. 345.
7 We shall deal with Wittgenstein's treatment of numerical specifications internal to mathematics in the next section and, with reference to his later writings, in Chapter 3, pages 157–66.
8 See *PG,* Part II, Ch. IV, §19, p. 333 (I have changed the original notation in a completely non-essential manner).
9 See *PR* §118; and *PG,* Part II, Ch. VII, §40, p. 461. We shall dwell upon the reasons and the consequences of this radical restriction on pages 72–99.
10 See *PG,* II, Ch. IV, §19, p. 346.
11 The question of the representative function of numerical symbols belonging to notations different from the primitive notation – e.g. to decimal notation – and of its eventual dependence on their translatability, in principle, into expressions written in the primitive symbolism, is only skimmed in the intermediate phase writings. We shall come back to this question in Chapter 3, when the later developments of Wittgenstein's reflections on this will be examined.
12 Cf., for example, Hilbert (1922, 1925). Wittgenstein's knowledge of these texts seems certain. A passage from the first of these two works is also cited and commented on in one of the conversations with the members of the Vienna Circle (cf. *WVC,* p. 137); moreover, in discussing Hilbert's programme of consistency, Wittgenstein often mentions the formula "$0 \neq 0$" as the formula whose non-derivability from the axioms of the formalized theory of numbers has to be proven; and this is the formula mentioned for the same purpose by Hilbert (1925, 1927) (cf. *WVC,* p. 119). On this point see note 88 by the editor in *WVC*. We shall come back to the connection between Wittgenstein's view and finitism in Chapter 3, pages 142–56.
13 See *PR* §§104, 107; *PG,* II, Ch. IV, §19, p. 347.
14 As quoted by Geach and Black in (1980), pp. 163–4.
15 See also *AWL,* pp. 195–201.
16 That the criteria of application of an empirical general term are rigidly determined by means of an appropriate convention or are bound to keep a certain margin of indeterminacy is quite a different matter.
17 See, for example, *PR* §161. In the next two sections of this chapter, we shall elaborate upon some consequences of this statement on Wittgenstein's view.
18 Though this passage is from the notes of a lecture given by Wittgenstein in 1935, and therefore does not belong to the so-called intermediate phase, I thought it worthwhile quoting because, in my opinion, it clearly expresses the theses we are dealing with. This time difference testifies, however, to the difficulties that Wittgenstein encountered in overcoming the verificationist conception of the relation between sense and proof of a mathematical proposition. As we shall have the opportunity of noting later on, even when his rule-following

considerations will destroy the very premises of verificationism, Wittgenstein will still be tempted to save the propositions decidable by a general method of calculation from the consequences of these considerations.

19 In the last section of this chapter we shall consider Wittgenstein's interpretation (in the intermediate phase) of the intuitionist refusal of the Law of Excluded Middle. His later reflections on this subject will be examined in Chapter 3, pages 167–71.

20 The point of Wittgenstein's frequent references – in discussing mathematics – to Kant's *a priori* synthetic judgements is to be found in this thesis (see, for example, *PG,* II, Ch. VI, §31, p. 404).

21 Consequently, the more usual presentation of the problem of rule-following (what sets up the way in which a finite segment of an infinite series should be continued, in order that the continuation would be in compliance with a given general rule of formation of the series?), to which the later Wittgenstein often resorts, appears to me not essential to the Austrian philosopher's argument. My interpretation of the argument is contained in the first section of Chapter 3.

22 Here we are assuming that the general formula for calculating the number of permutations of a set of *n* elements is not on hand. As we shall see, once the formula is known, every statement on the number of permutations of any finite set becomes a meaningful statement, according to the verificationist criterion.

23 See *PR* §28 and *PG,* II, Ch. V, §22.

24 These passages give grounds for referring Wittgenstein's affirmations such as "In mathematics description and object are equivalent" *(PG,* II, Ch. VII, §39, p. 419) exclusively to the part of mathematics where the verification principle does not apply.

25 I draw attention again to the fact that this theme is the true antecedent of later Wittgenstein's rule-following considerations.

26 Wittgenstein's position on this point cannot be fully understood unless his refusal of the extensionalist conception of mathematical infinity is taken into account. The next section is devoted to this subject.

27 See *PG,* II, Ch. V, §22, p. 361; §24, p. 371.

28 An overall exposition of the views on generality, elaborated by Wittgenstein in his intermediate phase, is beyond the scope of this book.

29 See *PG,* II, Ch. VII, §39, p. 456.

30 See *PR* §164 and *PG,* II, Ch. VI, §36, p. 430. I will not go into the details of Wittgenstein's analysis of proof by complete induction here, as this will be dealt with in the next section of this chapter.

31 See Marion, Wittgenstein and finitism, forthcoming.

32 Cf. Skolem (1923). As known, this work is discussed at length in *Philosophical Remarks* and *Philosophical Grammar.* Wittgenstein speaks indifferently of inductive proof and recursive proof; actually, he always conceives an inductive proof as a schematic representation of an effective rule of construction of an infinite series of numerical proofs. The idea that reference to an infinite totality has to be understood in terms of reference to an effective law of generation of linguistic expressions, defined inductively, goes back to the *Tractatus,* and is constantly present in Wittgenstein's thought.

33 On the relation between Wittgenstein's formulation of this rule, stating the uniqueness of the function defined by recursion, and Goodstein's equational calculus, see Marion, forthcoming. In referring to the rule, or to some slight notational variant of it, Wittgenstein uses from time to time the symbols "**R**",

"(ρ)", "V" (see *PG,* II, Ch. VI, §33, pp. 409, 414; §37, p. 441). I will confine myself to the symbol "**R**", except in quoting passages where one of the other symbols occurs.

34 I am referring, in particular, to his criticism of set theory (see pages 92–9 of this chapter, concerning the intermediate phase, and pages 157–66 of Chapter 3, concerning his writings of the decade 1934–44).

35 It is clear that Wittgenstein disregards any consideration about cardinality, which might lead to admitting the incompleteness of the set of lawlike sequences.

36 In *Philosophical Grammar,* Wittgenstein's position appears more flexible: he states the general thesis that "the laws corresponding to the irrational numbers all belong to the same type to the extent that they must all ultimately be recipes for the successive construction of decimal fractions" (*PG,* II, Ch. VII, §41, p. 474), but also stresses the existence of different kinds of irrational numbers and maintains that it does not matter whether π' is called "a real number" or not; what really matters, however, is that it "is not a number in the same sense as π" (*PG,* II, Ch. VII, §42, p. 476).

37 This is the very reason for Wittgenstein's hostility towards Brouwer's pendulum number: when the relations of greater than, equal to or less than are not effectively decidable, according to Wittgenstein's second requirement, it does not make sense to speak of a real number. This is not simply a matter of not being able to know the result of the comparison, but of the non-existence of a result, since no effective method for yielding it is on hand (see *WVC,* p. 73).

38 The problem of whether Wittgenstein, in his later writings on mathematics, endorses a strict finitist conception is one of the crucial points of the interpretation of his mature position. The third section of Chapter 3 will be devoted exclusively to its examination.

39 See *PG,* II, Ch. II, §10, pp. 282, 286.

40 See *PG,* II, Ch. VII, §40, p. 469. The possibility of a symbolic system containing expressions of infinite length, which could be manipulated by a being not subject to the biological limitations of humans, was assumed by Ramsey as the basis for his simplification of the theory of types (see Ramsey (1925), p. 42). It can be conjectured that Wittgenstein, in his remarks, had in mind Ramsey's work.

41 It is expedient to point out that the major part of the relevant material is contained in Waismann's notes of conversations Wittgenstein had between 1929 and 1932 with several members of the Vienna Circle. Thus, to the usual and well known difficulty of extracting from Wittgenstein's writings certain basic argumentative lines, is to be added the further complications owing to the peculiar nature of this text. It seems to me, however, that some of Wittgenstein's orientations can be reconstructed with sufficient reliability.

42 In this connection, Wittgenstein's belief that the set-theoretical antinomies would not be genuine *mathematical* contradictions, rather, by-products of grammatical ambiguities of the ordinary language (see *WVC,* pp. 120–1), has no relevance. No matter what he thought with regard to this point, it is obvious that Russell's antinomy can be derived within Frege's system by pure calculation, i.e. without any interpretation of the signs being involved.

43 See Wittgenstein's reference to Hilbert (1922) in *WVC,* p. 137.

44 For Wittgenstein's position see, for example, *PR* §174; for the true intuitionist position see Dummett (1977), pp. 17–19.

45 See *PR,* Foreword. The three themes touched on in this section – the consistency problem, the question of the validity of the Law of Excluded Middle and that of

the eventual revisionary import of Wittgenstein's approach to mathematics – will be thoroughly re-examined, with reference to the writings of the years 1934–44, in the last section of Chapter 3.

FROM FACTS TO CONCEPTS: THE LATER WRITINGS ON MATHEMATICS (1934–44)

1 In the propositional formulation of a mathematical result, mathematical symbols should always be considered as occurring autonomously.

2 *PG*, II, Ch. VII, §39, p. 457. This is a slight modification of proposition 6.1261 of the *Tractatus*: "In logic process and result are equivalent. (Hence the absence of surprise)".

3 On this point see Kripke (1982), pp. 81–3.

4 Baker and Hacker have rightly insisted on the internal character of the relation between the concept of rule and the concept of what agrees with it in a particular case, that is to say, of what constitutes the result of its correct application in that case, matching it to the relation that ties the concept of expectation to the concept of what satisfies the expectation, or the concept of proposition to the concept of the fact which renders it true (Baker and Hacker 1984, p. 72). The same point is made by Shanker (1987), Ch. I. I consider, however, their rejection of the community view to be wrong. In my opinion, in fact, the later Wittgenstein conception can be aptly described as *a community view of internal relations*.

5 McGinn (1984), p. 71, fn 17, and p. 77.

6 It is for *such* linguistic norms that what Kripke says about empirical judgements of conformity of a given behaviour to a rule holds: i.e. that there are no truth-conditions for them, facts that make them true. But whereas it seems to me to be fully in the spirit of Wittgenstein to speak of assertibility-conditions for empirical judgements, it is not the same for grammar rules, in so far as the latter are not asserted but stipulated.

7 Regarding the full-blooded conventionalism, to which Wittgenstein's rule-following considerations lead, Dummett (1959) is still a major reference-point; amongst the many who have developed this theme, see Wright (1980). In the course of this chapter we will return to this theme many times.

8 One of the clearest presentations of the community view is to be found, in my opinion, in Chapter 9 of Malcolm (1986).

9 Cf. *PI*, I, §201.

10 Cf. for instance, McGinn (1984) and Baker and Hacker (1984).

11 I use the impersonal form "one follows R if one carries it out" to point out the grammatical, atemporal, nature of this statement. As, according to Wittgenstein himself, it is not appropriate to speak of believing or thinking with reference to norms, the opposition between *individual belief* and *reality* would be more rightly presented as an opposition between *individual appearance* and *reality.* In effect, in the famous §258 of *Philosophical Investigations,* at the core of his private language argument, it is precisely in the latter terms that Wittgenstein frames the same question: "But in the present case I have no criterion of correctness. One would like to say: whatever is going to seem right to me is right. And that only means that here we can't talk about 'right'."

12 Wright (1980), p. 21.

13 See, for example, the passage from the notes of the lecture held by Wittgenstein in 1935, in *AWL,* p. 197.

14 See *RFM,* I, §112. The thesis that emerges from §§106–12 of *RFM* is, in substance, re-stated in one of Wittgenstein's 1939 lectures, where he contrasts the learning of multiplication rules with learning the result of single multiplications, and discusses invention and discovery in relation to the outcome of the multiplication of 136 by 51 (cf. *LFM,* Lecture 10).
15 See Wright (1980, p. 452; 1990, p. 96).
16 If I am not very much mistaken, the idea that judgements on correctly following a rule presuppose the availability of a concept, the existence of a linguistic norm that lays down what, in each single case, must be considered, by definition, to be the result of the correct application of the rule, is what is suggested by Kreisel's affirmation that, according to Wittgenstein "there is a non-empirical residue in the notion of a rule of language" (cf. Kreisel 1958, p. 139). The distinction between definitions and empirical judgements is clearly formulated by Wittgenstein with reference to the use of the dyadic predicate "analogous to" in the sixth of his Cambridge lectures in 1939 (cf. *LFM,* Lecture VI); Canfield (1981, pp. 82–5) has rightly insisted on this. We will deal with the definitory use of "analogous to" later in this section.
17 Cf. *RFM*, VII §61.
18 Cf., for instance, *PI,* I, §10.
19 This thesis is clearly enunciated also in the 1939 *Lectures,* for instance in Lecture 28.
20 The analysis that follows is an attempt to systematically reformulate the contents of §§25–41 of Part I of *RFM*.
21 The next section in this chapter will be devoted to an analysis of the fundamental relation of *being the same shape as,* used in the ostensive definitions of "hand", "pentacle" and "figure yielded by the correct comparison of the number of elements of a hand with the number of vertices of a pentacle". Note that the experiment construction and its relevant parts are referred to indexically in the above definitions.
22 For simplicity, now we are assuming that the tokens "200" and "200", which occur in the experiment construction, already play a paradigmatic role, namely, are representative of their own shape (cf. *RFM,* IV, §11).
23 It is useful to read Canfield (1981) on the contents of this lecture.
24 Cf. Kreisel (1958), p. 140.
25 See, for instance, *RFM,* III, §§1, 2, 9, 11, 55.
26 Kreisel (1958), p. 147.
27 *Ibid*., pp. 147–8 (Kreisel's italics).
28 *Ibid*., p. 149.
29 See *RFM*, II, §61.
30 Kielkopf's attempt to show how the observations on Cantor's proof are inspired by Wittgenstein's presumed strict finitist orientation does not convince me at all (cf. Kielkopf 1970, pp. 135–8). We will come back to the contents of these observations, in detail, in the last section of this chapter.
31 Wright (1982), p. 204 (Wright's italics).
32 Cf. Dummett (1959).
33 Wright (1980), p. 129.
34 Dummett (1959, 1978a), p. 180 (my italics).
35 Wright (1980), p. 130 (Wright's italics). The inverted commas that enclose the last occurrence of "error" in the passage quoted betray, in my opinion, the author's perception of the inadequacy of his own presentation of Wittgenstein's position.

36 Dummett (1978b), in Shanker and Shanker (1986), p. 119.
37 Cf. *RFM*, III, §§10–11 and 39.
38 Kreisel (1958), p. 149 (Kreisel's italics).
39 On Wittgenstein's evaluation of the effective import of the general theorem of translatability of arithmetic into logic, see Wright (1980), p. 135.
40 Kreisel (1958), p. 153.
41 It is interesting to note how the second alternative suggested by Wittgenstein, NOTEregarding the possibility of conferring sense to the question of the denumerability of **R**, seems to repropose the old verificationist conception of the sense of a question or of a mathematical conjecture. This bears witness to how strong and recurrent was Wittgenstein's temptation to subtract at least one part of mathematics from the "strange" destiny it is condemned to by the thesis that the proof determines the sense of a question or a conjecture.
42 I have taken the expression "quasi-revisionism" from Wrigley (1980), keeping its original meaning: to indicate a programme that does not seek to modify certain parts of mathematics, but *our attitude* towards them.
43 The sections of *RFM* devoted to the Law of Excluded Middle are 9–20 of Part
44 On this point see Fogelin (1968).
45 This theme is discussed by Wrigley (1980).
46 For an exhaustive discussion on this theme, see Wright (1980), Ch. XVI.

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