Consciousness: A Hyperspace View by Saul-Paul Sirag

Introduction

Ordinary reality, objectified by the methods of measuring space, time and matter, is a subrealm of a larger reality. This is an ancient idea — at least as ancient as Plato's cave story in which prisoners are chained in such a way that they identify themselves with their own shadows on the cave wall. It seems clear that Plato meant to imply that the larger reality is hyperdimensional — i.e., although we tend to identify ourselves with our 3-d bodies, there is a higher dimensional realm in which we are higher dimensional beings of which our 3-d bodies are mere shadows. This interpretation of the cave parable is augmented by Plato's motto for his Academy: "Let no one enter here without geometry" (cf. Hinton, 1904, 1980).

The idea that reality is hyperdimensional is entertained today by physicists attempting to unify all the physical forces in a unified field theory. It is reasonable to suppose that the detailed description of this hyperdimensional reality will yield a theory of *consciousness.*

Consider the following aspects:

1. The unification of electricity and magnetism worked out by James Clerk Maxwell (1831–1879) entailed the first satisfactory theory of light: i.e., light is an electromagnetic wave. This theory introduced the unexpected idea that visible light is only a tiny part of the electromagnetic spectrum. Subsequent discovery of radio waves and x-rays confirmed this theory. Analogously, we can expect something new to come out of the unification of all the forces. We should not be too surprised to see that this something is consciousness and that the unified theory provides a basis for a "spectrum" of states of consciousness.

2. In philosophy, consciousness is usually discussed in the context of the mind-body problem: Are the basic entities of the world mind-like or body-like, or some mixture of mind-like and body-like entities? The defining characteristic of body-like is assumed to be extension in space. The defining characteristic of mind-like is assumed to be sensation. So the question becomes: Can the world be constructed from sensations alone, extensions alone, or some mixture of both? There are other possible views; for example: both sensations and extensions are derived from different combinations of some neutral entity (cf. Russell, 1954).

The extension-alone school (materialism) maintains that consciousness is an epiphenomenon of the complex structures of the brain. The sensation-alone school (idealism) maintains that consciousness (or a universal mind) is the ultimate reality and that the material world, as well as the existence of individual minds, is a construct within the universal mind. The sensation-and-extension school (dualism) maintains that consciousness is an aspect of reality separate from, but somehow interacting with, the material world. The chief criticism against dualism is that an entity which is not extended in space cannot interact with matter (which is extended in space). The view that consciousness and matter arise from different combinations of some neutral set of entities is called neutral monism.

It is easy to imagine that a hyperspace view of reality will entail a reevaluation of the traditional categories of the mind-body problem.

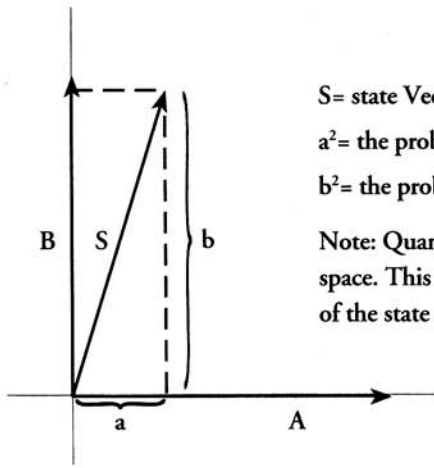
3. The unification of all the forces is possible only if a theory can be constructed unifying general relativity (Einstein's theory of gravity) with quantum mechanics. The main problem is that these two theories appear to be incompatible because general relativity is a deterministic theory, whereas quantum mechanics employs a fundamentally nondeterministic description of measurement. A possible means of reconciliation is suggested by the fact that quantum mechanics itself has a deterministic side as well as the well-known nondeterministic side. The reconciliation of these two opposed aspects of quantum theory is called the quantum measurement problem. To state this problem clearly, a brief description of quantum theory is necessary.

A quantum system is represented by a vector (called the state vector) which rotates in an abstract space (which may very well be infinite-dimensional). Note: A *vector* is an arrow-like entity with both length and direction. The rotation of the state vector is deterministic in the sense that if its position is known at one time, its position at another time can be calculated. (It is just like a clock hand except that a clock hand is rotating in a 2-d space, whereas the quantum state vector might be rotating in a hyperspace.) As long as no *measurement* is made on the system, the state vector keeps rotating smoothly in the state space according to a deterministic equation called Schroedinger's equation; but, as soon as a measurement is made, the state vector immediately jumps to a vector (an *eigenvector*) corresponding to an allowed value (an *eigenvalue*) of the particular measurement that is being made. This jump is called the collapse of the *wave function* (another name for the state vector), but it is more appropriately described as the projection of the state vector onto an eigenvector belonging to a measurement. Most important: It is completely undetermined which eigenvector is projected out of the state vector by the measurement; however, we can calculate the probability of this projection, and we can verify this probability by repeating the measurement over and over or by making simultaneous measurements on a multitude of similarly prepared systems. Each type of measurement (e.g., a position measurement or a momentum measurement) has its own set of allowed states (eigenvectors) which belong to allowed values (eigenvalues). The eigenvectors corresponding to a type of measurement provide a *coordinate system* for the space in which the state vector rotates (state space). Each type of measurement provides such a coordinate system.

Note: Since a coordinate system is imposed on a vector space arbitrarily (by choosing a measurement), we will consider a coordinate system as equivalent to a point of view (cf. Weinberg, 1987).

A crucial question is: Under what conditions do different types of measurement provide identical coordinate systems for the state space? And what is the consequence of this coincidence?

FIG. A1 STATE SPACE FOR 2–EIGENSTATE QUANTUM SYSTEM



S= state Vector A= eigenstate A B= eigenstate B

a^2= the probability that S will be found in state A

b^2= the probability that S will be found in state B

Note: Quantum state vectors exist in a complex state space. This drawing shows only the real part of the state vector and eigenstates.

The answer is somewhat abstract: To each type of *measurement* corresponds an *operator* which acts on the state vector and on the state space in which the state vector lives. Of all the vectors of the state space, let us pick out only those which do not change direction, but change only their length under the action of the operator. These vectors are the eigenvectors belonging to the operator. The factor by which an eigenvector changes length is the eigenvalue belonging to the eigenvector (e.g., if the operator doubles the length of the eigenvector, the eigenvalue is 2). Now it is a theorem of pure mathematics that *commuting* operators have the same eigenvectors. Note: If $AB = BA$, where A and B are operators, then we say that A and B *commute*. Thus, in this case, A and B provide the same coordinate system (the same set of eigenvectors for the state space on which they act).

The consequence for quantum theory is that any two types of measurement represented by commuting operators can be made in any order. Thus, in unified field theory, we are looking for a complete set of commuting operators whose eigenvalues are the complete set of "simultaneous" eigenvalues of the world. As we shall see, these operators are expected to be the basis elements of *a maximal commutative subalgebra of a Lie algebra* (and these terms will be defined in due time).

If two types of measurement are represented by *noncommuting* operators ($AB \neq BA$), the order in which the measurements are made affects the outcome of the measurements.

This is the basis of *Heisenberg's Uncertainty Principle*. For example, *position* and *momentum* are measurements corresponding to *noncommuting* operators, so that $PQ - QP = h/(i2\pi)$, where P is the momentum operator (in the x direction), Q is the position operator (in the x direction), i is the square root of minus one, and h is *Planck's constant*, which is the very small quantity $\approx 10^{-27}$ erg seconds.

This implies that the uncertainty in position times the uncertainty in momentum is always greater than or equal to Planck's constant — which is the usual way the uncertainty principle is stated.

(Note: The smallness of Planck's constant accounts for the fact that we didn't notice it until we investigated atomic phenomena. If we measure ordinary-sized objects, the ordinary measurement errors are much larger than the fundamental uncertainty due to Planck's constant.)

The most peculiar thing in quantum theory is not, however, the Uncertainty Principle. Rather it is the Measurement Problem (cf. Wheeler and Zurek, 1983), which we state as follows: How can we reconcile the deterministic evolution of the state vector with the random projection of the state vector onto a measurement eigenvector?

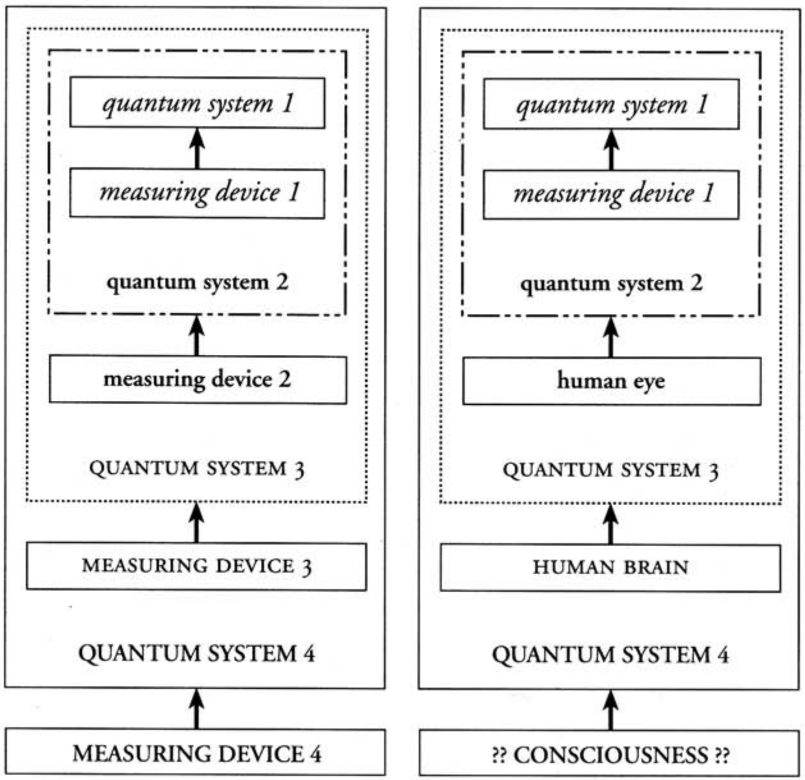
The problem becomes more acute when we realize that measuring devices themselves are ultimately composed of quantum entities so that, in principle, there is a single state vector **V** which corresponds to the combination of the measured system and the measurement device. This state vector **V** must rotate deterministically like any other quantum state vector. The projection of **V** onto an eigenvector is induced by a measurement of the measuring device by another measuring device (which, for the time being, is considered outside the quantum system). Of course, this second measuring device can be combined with the above system into an even larger quantum system and treated similarly. In fact, an indefinitely long chain of such devices can be hooked together. The most interesting question is: What is the ultimate observer? Is it the apparatus on the physicist's laboratory table? Is it the physicist's eyes? His optic nerves? His brain? His consciousness? (see Fig. A2)

According to the mathematician John von Neumann and the physicist Eugene Wigner (cf. Wigner, 1979), the best solution to the measurement problem is that ultimately consciousness projects the state vector onto the eigenvector. However, if one adopts the Wigner interpretation of quantum theory, the *nonlocality* of the state vector as implied by Bell's Theorem (cf. d'Espagnat, 1976, 1979, 1983; Herbert, 1985) suggests that this consciousness which projects the state vector must be a *universal consciousness*.

Of course, many other solutions to the measurement problem have been proposed. None of them, however, has been universally accepted by physicists. And all of them are as strange, in their own ways, as the Wigner proposal. For example, Heisenberg (1958) proposed that an eigenvector is essentially a

FIG. A2

THE VON NEUMANN CHAIN — WIGNER'S INTERPRETATION



mental entity in the sense that it describes not the state of the physical world but rather the state of our knowledge of the physical world.

We should expect that a view of unified field physics, in which hyperspace is considered real, would throw new light on the quantum measurement problem. And we should not be surprised if consciousness is involved in this picture.

4. *Cosmology,* the study of the large-scale structure of spacetime, presents a special challenge to unified field theory. On the large scale, gravity is the dominant force, so that the unification of gravity with the other forces should have implications for cosmology. Moreover, it is believed that the forces become unified only in extremely high-energy interactions, such as occurred shortly after the big bang explosion of the universe. Thus it is hoped that unified field theory can provide clues to the origin of the universe. It should be clear from the discussion above that quantum cosmology will entail a state vector for the universe as a whole. In this case, the problem of having an observer outside the universe to project the state vector onto eigenvectors becomes acute. The keyword here is *outside,* and it should not be surprising to find that a hyperspace view of reality provides a solution to this problem.

5. Among physicists working on *unified field theory,* it is widely believed that the deepest aspect of the world is the *symmetry principle,* which we state as follows: The rule describing how the state vector evolves does not change even though the direction of the state vector can be altered by the changes in point of view (i.e., coordinate system) (cf. Weinberg, 1987).

These changes in point of view which leave the evolution rule unchanged are called *symmetries,* and together they make up a mathematical structure called the *symmetry group* of the world. The symmetry principle severely restricts the form of the rule of state-vector evolution. In fact, it is believed that is possible to specify the rule uniquely by specifying the symmetry group. Thus, in order to unify all the forces, we must answer the question: What is the symmetry group of the world? Since each element of this symmetry group corresponds to a change in point of view, it is clear that the symmetry principle requires the basic rule of evolution to be independent of point of view (i.e., coordinate-system independent).

Some of these symmetries are spacetime symmetries, such as the fact that the rules of physics are independent of the time and place and orientation of the measuring equipment. It is rather amazing that from these quite plausible requirements we can derive the following laws: conservation of energy, conservation of linear momentum and conservation of angular momentum. Moreover, the rules of Einstein's theory of special relativity can be derived from the postulate that the laws of physics must remain invariant under any coordinate system change which does not stretch the vectors in spacetime. The constancy of the speed of light c is built in here by the assumption that c is the intrinsic conversion factor necessary to put space and time into the same coordinate system — ct is a distance, if t is time. In the last several decades it has become increasingly clear that similar symmetry statements can be made about vectors in the state spaces of quantum mechanics. These are called "internal" symmetries in order to distinguish them from the symmetries of spacetime.

Symmetry and Groups

Because of the central role that symmetry plays in the unified field theory, it is necessary to present a more detailed description of symmetry and symmetry groups.

We define a *symmetry* as a change in an object which leaves some property of the object unchanged (or *invariant*). We use (rather loosely) the name of the change to identify the symmetry. For example, we say that a sphere is rotationally symmetric because its shape does not change under any rotation. We say that a cylinder is axially symmetric, because it does not change its appearance under a rotation about one particular axis (cf. Weyl, 1949, 1952; Elliott and Dawber, 1979).

The set of all symmetry transformations of an object forms a mathematical structure called a *group*, the most important feature of which is that one transformation followed by another transformation is equivalent to a third transformation. This property of the group is called *closure*. There are three other necessary properties:

1. *Associativity:* $a(bc) = (ab)c$ where a, b, c are group elements

2. *Identity element:* $ea = ae = a$ where a is any group element and e is the identity element of the group

3. *Inverse elements:* $aa^{-1} = a^{-1}a = e$ where a is any group element and a^{-1} is the inverse of a; i.e., every element has an inverse.

There are many extra properties which a group might possess, the most important of which is

4. *Commutativity:* $ab = ba$ where a, b are group elements.

In other words, the order in which we perform all the transformations is irrelevant. We call such a group commutative. A noncommutative group may have a commutative subgroup. If this commutative subgroup commutes with all the group elements, it is called the *center* of the group.

Groups come in two types: *continuous* groups and *discrete* groups. A continuous group is also a space. Thus a continuous group possesses not only the above algebraic properties which make it a group but also the geometric properties which make it a space. This means that the elements of the group are also the points of a space.

A discrete group is not a space (or is a 0-dimensional space if one insists in calling it a space). However, it may be a subgroup of a continuous group, i.e., a set of points set at intervals in a space. The simplest example is the group of integers, which can be viewed as a set of discrete points on the real number line. The identity element is 0; the inverse elements are the negative integers. The integers are closed and associative under the operation of addition. Moreover, the integers form a subgroup of the real numbers (viewed as a group under addition). The integers form an infinite but countable set. In fact, we ordinarily use the positive integers to do our counting.

There are also discrete groups which have a finite number of elements. These groups are called *finite groups.* The most important finite groups are called *symmetric* groups. A symmetric group S_n consists of all the *permutations* (reorderings) of n objects. The group property of closure arises out of the fact that a permutation of a permutation yields another permutation. For each n, there exists a permutation group of $n!$ (pronounced *n factorial*) elements, because there are *n!* ways to permute (rearrange) *n* objects. For example, there are 4! = 4 x 3 x 2 x 1 = 24 ways to permute four objects. It should be noted that S_n is a *noncommutative* group, except in the two trivial cases $n = 1$ or 2.

If the space of a continuous group is *smooth,* the group is called a *Lie group* (named for the Norwegian mathematician Sophus Lie, 1842–1899). The best examples of Lie groups are groups consisting of rotations of vectors in vector spaces (which could, for example, be the state spaces of quantum theory).

One must be very careful to distinguish between two spaces here: (1) the space which the group is and (2) the space on which the group acts. One must also keep in mind that the group can act on itself as a transformation group. In this special case of self-action, (1) and (2) are the same space. This is hardly an unimportant technical detail. For as we will see, the action of the Lie symmetry group on itself corresponds to *force fields,* whereas the action of the group on an "outside" space corresponds to *matter fields.* And this is the fundamental basis of the distinction between these two types of field.

The set of all *rotations* of a sphere is a very useful example of a Lie group. There is a continuous infinity of rotations of an ordinary sphere (called a *2-sphere,* with the label S^2, because it is a 2-d space, coordinatized by latitude and longitude, let us say). These rotations themselves form a 3-d space, called *the 3-sphere mod plus or minus 1,* with the label $S^3/\{\pm 1\}$. The group name for this space is *SO(3),* which means the set of all *special-orthogonal* 3-by-3 *matrices* (cf. Schutz, 1980; Poor, 1981).

Note: A *matrix* is a rectangular array of numbers. The word *orthogonal* refers to the fact that the matrices rotate without stretching the vectors on which they act; the word *special* implies that volume remains invariant under the action of the matrices. Thus the 3 in the label *SO(3)* refers not to the 3-d space of the group itself but to the dimensionality of the space on which the group acts, i.e., the 3-d vector space in which the sphere S^2 is embedded as the set of all unit-length 3-d vectors.

In general, the set of *n*-by-*n* special orthogonal matrices is a group called *SO(n)* which acts on an *n-dimensional* vector space and in so doing rotates an $n - 1$ dimensional sphere S^{n-1}.

The importance of *SO(3)* may be realized by considering its finite subgroups, which are the symmetry groups of various *polyhedra* inscribed in the sphere S^2 on which *SO(3)* acts. (cf. Weyl, 1952; Du Val, 1964; Slodowy, 1983). Each element of the group leaves the polyhedron invariant — i.e., looking the same (see Fig. A3).

The finite subgroups of *SO(3)* are:

1. All of the *cyclic* groups c_n (n = any integer) are commutative groups which rotate *n*-sided pyramids. We can imagine these pyramids as inscribed in S^2 with the base inscribed in the "equator" and the apex at the "north" pole.

2. All the *dihedral groups* d_n (n = any integer) are (except for $n = 1$ or $n = 2$) noncommutative groups which rotate *n*-sided *oranges.* We can imagine such an orange as S^2 partitioned into *n* sectors with the two vertices at the two poles. The noncommutativity of the symmetry group arises from the fact that a flip-over of the orange, exchanging the "north" and "south" poles, followed by a clockwise rotation through angle *x,* is different from a clockwise rotation through angle *x* followed by a flip. (To make this concrete, try it on a beach ball with different colored segments.)

3. The three *regular polyhedral* groups, which are symmetry groups of the *five platonic solids:* the *tetrahedron,* the *octahedron,* the *cube,* the *icosahedron,* and the *dodecahedron.* (Note: The octahedron and cube have the same symmetry group; likewise for the icosahedron and the dodecahedron.) All three groups are noncommutative.

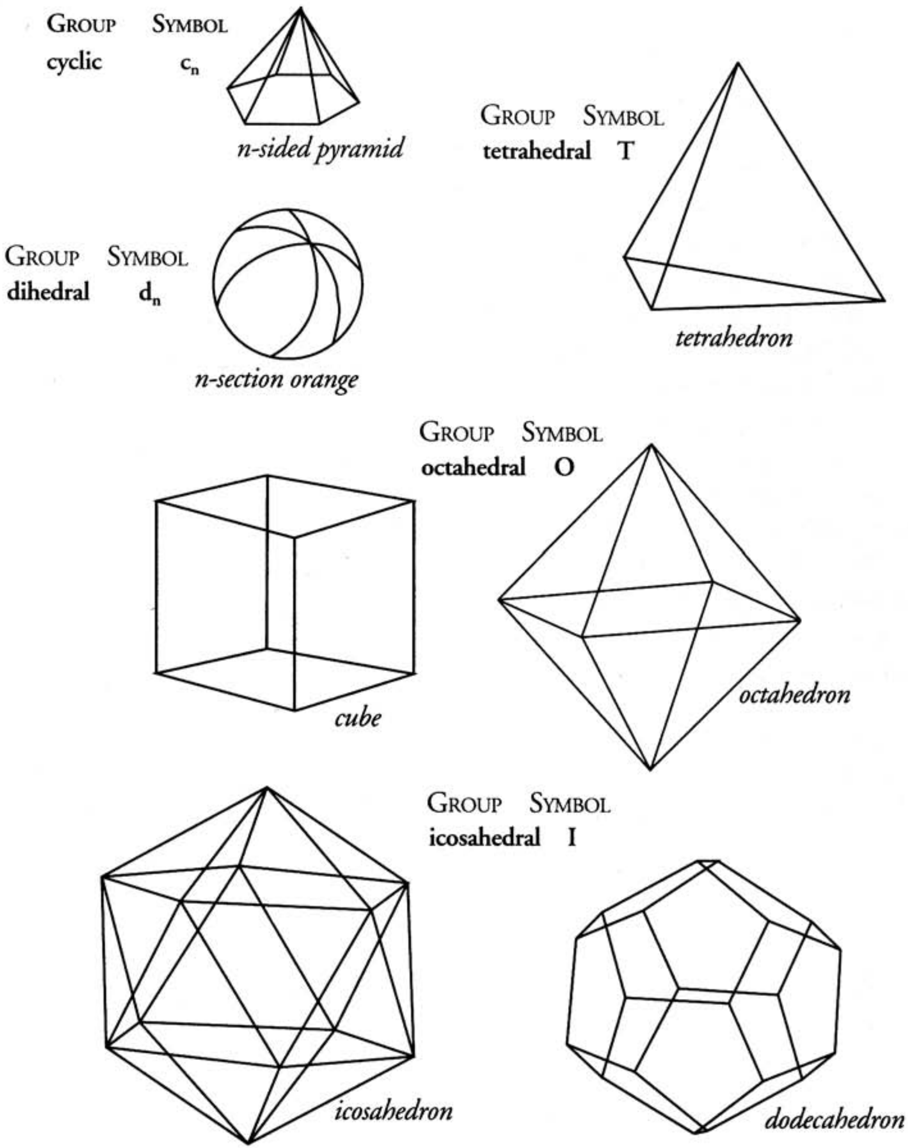
T is the 12-element *tetrahedral* group, which we depict as the set of symmetries of the tetrahedron inscribed in S^2.

O is the 24-element *octahedral* group, which we depict as the set of symmetries of the octahedron inscribed in S^2. This is also the symmetry group of the cube, since the six faces of the cube correspond to the six vertices of the octahedron, and the eight faces of the octahedron correspond to the eight vertices of the cube.

(The noncommutativity of *O* is easily seen by rotating a cube with six different colored faces. Or simply visualize the following: The

Fig. A3

Symmetry figures embedded in S^2 correspond to finite subgroups of SO(3). The regular figures (all sides equal) are the five Platonic solids.



three axes about which a cube rotates could be labeled — according to aeronautical terminology — yaw, pitch, and roll. Then imagine you are the pilot of an airplane which is flying upright and level due west, and consider a pitch downward through 90° followed by a yaw to the left through 90°; you are now flying due south and with your right wing pointed to the ground. Now go back to the original starting position and reverse the order of the two maneuvers. You will "end up" with your nose headed straight to the ground!)

I is the 60-element *icosahedral* group, which we depict as the set of symmetries of the icosahedron inscribed in S^2, as well as the symmetry group of the dodecahedron, since the 12 faces of the dodecahedron correspond to the 12 vertices of the icosahedron, and the 20 faces of the icosahedron correspond to the 20 vertices of the *dodecahedron.*

The finite groups c_n, d_n, and the regular polyhedral groups *T*, *O*, and *I* form the complete set of finite subgroups of *SO(3)* (cf. Weyl, 1952). There is a corresponding set of finite subgroups of the Lie Group *SU(2)*. This group is the set of all *special-unitary* 2-by-2 matrices: the elements of *SU(2)* rotate the vectors in the *complex* 2-d vector space on which the matrices act. In other words, *SU(2)* acts on the 2-d complex space (4-d real space) just like *SO(3)* acts on 3-d real space. In fact, *SU(2)* as a space is called the *double cover* of *SO(3)*. This is because *SU(2)* as a space is the sphere S^3, whereas *SO(3)* is $S^3/\{\pm 1\}$. This means that *SO(3)* can be derived from *SU(2)* by pairing up positive and negative elements of *SU(2)* to form elements of *SO(3)*. Geometrically, this would be like considering the "north" and "south" poles of a sphere as the same element (cf. Penrose, 1978).

Thus each *x*-element finite subgroup of *SO(3)* has a 2*x*-element double cover in *SU(2)*. These finite subgroups of *SU(2)* can be called *polyhedral double groups*. For example, the 24-element octahedral group *O* has a 48-element double cover in *SU(2)* called the *octahedral double group OD*, and *O* can be derived from *OD* by pairing up positive and negative elements in *OD*. We can write: $O = OD/\{\pm 1\}$.

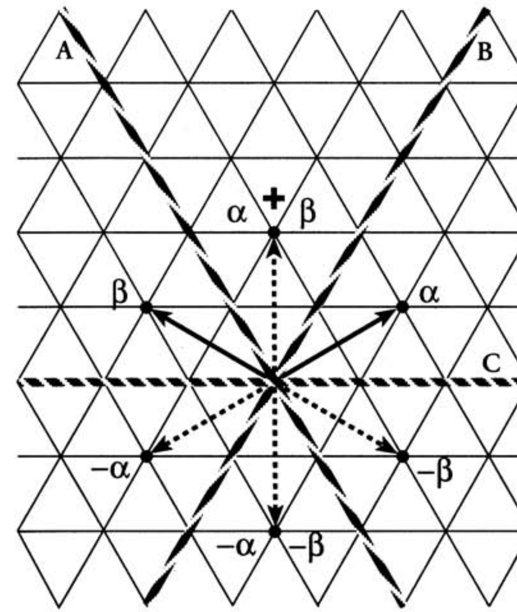
Another important class of finite groups is *reflection groups* (Coxeter, 1973), the most useful examples of which are the *Weyl groups* of Lie groups. A Weyl group acts on the *reflection space* h^* of the Lie algebra of the Lie group. The concept of a reflection space plays an important role in our theory, so it is necessary to describe this idea, charming in itself, in some detail. It will perhaps interest the reader to know ahead of time that the observable quantities (eigenvalues) of a quantum system, mentioned previously, exist in a reflection space, so that we could also call this space *eigenvalue space.*

An ordinary *mirror* seems to transfer objects from one side of the mirror to a similar space on the other side, seemingly changing the direction of objects as if the space in front of the mirror had simply flipped through the mirror. Imagine a two-sided mirror that actually could transfer objects, without changing their size, in both directions — the space in front of the mirror would be flipped, without stretching, behind the mirror, and vice versa, while the 2-d space of the mirror itself is left unmoved. This action of a 2-d mirror plane on a 3-d vector space is called a *reflection* in mathematics. And the idea generalizes to any number of dimensions, provided the mirror of an *n*-dimensional vector space is a plane of one dimension less, called a *hyperplane.* (For example, a line is the mirror hyperplane of 2-d space, and a 3-d plane is the mirror hyperplane in a 4-d space.) Only in this way can the mirror be two-sided. Just as a line has two sides in an ordinary plane but an infinity of "sides" in a 3-d space, so a 2-d plane is 2-sided in 3-d space but infinite-sided in 4-d space. Notice how these considerations provide some intuition about hyperspace.

Since we are by now accustomed to the idea of group elements acting on a vector space, we should not be surprised to learn that a reflection, as defined above, is a group element, and that a reflection is its own inverse element. That is, if *r* is a reflection, then *r* following *r* has the same end effect as doing nothing to the vector space. So we can write: $r^2 = 1$, where 1 is our symbol for the identity element (which means "do nothing").

Suppose we have two or more mirror planes intersecting each other. At the point where all the mirrors intersect, one vector (a *mirror vector*) is defined for each mirror, so that the vector corresponding to a mirror points away from the mirror *orthogonally* (at a right

FIG. A4 THE A_2 REFLECTION SPACE



A and B are basic mirrors (set at 60° to one another).

C is a mirror generated by reflection in either A or B.

α and β are basis roots (set at 120° to one another).

The other four roots are generated by reflections in the mirrors. The nodes (crossing points) are the weight lattice. Each node is a different weight.

Note: The root lattice is a sub-lattice of the weight lattice. The large dots are part of the root lattice.

Fig. A5 A–D–E Coxeter Graphs

Lie algebra label	Coxeter graph	Total number of mirrors
A_n	0–0–0–. . . –0 (n nodes)	$(n^2 + n)/2$

D_n	0–0–0–. . . –0 (n nodes)
	    |
	    0
		$n^2 - n$ (n greater than 3)

E_6	0–0–0–0–0
	       |
	       0
		36

E_7	0–0–0–0–0–0
	       |
	       0
		63

E_8	0–0–0–0–0–0–0–0
	       |
	       0
		120

angle). There would, of course, be a negative mirror vector attached to the back of each mirror at the same point. We have pictured our mirrors as living in a vector space, and we usually imagine a vector space as having some set of *basis vectors* to provide a *coordinate system* for the vector space. Moreover, we customarily make this coordinate system rectangular by having the basis vectors orthogonal to each other, such as in the familiar x–y–z axis system for a 3-d vector space. For any vector space, however, there is an infinity of coordinate systems, not all of them rectangular. To set up a coordinate system, we need one basis vector for each dimension of the space, but these basis vectors need not be orthogonal (or even all of the same length) (see Fig. A4).

We can now define a *reflection space* as a vector space whose basis vectors are mirror vectors.

This implies that all the reflection activity possible in such a space is generated by the basic mirrors attached to the basis vectors. All the other mirrors are derived by reflection in these basic mirrors. Each basic mirror defines a reflection in these basic mirrors. Each basic mirror defines a reflection group element and all the elements of the reflection group are generated by (i.e., are combinations of) the reflections defined by the basic mirrors.

It is known (cf. Coxeter, 1973) that in setting up reflection spaces, the angles between the basic intersecting mirrors can only be: 60°, 45°, or 36° (except in the 2-d case where the angles can be 180°/p, where p is any integer larger than 2). Thus all possible reflection spaces can be defined by a graph (called a *Coxeter graph* or *Dynkin diagram*) for each reflection space. The nodes in each graph stand for the basic mirrors.

The most important reflection spaces have basic mirrors which are all set at 60° to each other. These are called the *A-D-E* Coxeter graphs, which are displayed in Figure A5.

Thus, we have two infinite series: A_n and D_n (corresponding to two types of n-dimensional reflection spaces, and the "exceptional series" consisting of the three graphs E_6, E_7 and E_8 (corresponding to three exceptional reflection spaces of dimensions 6, 7, and 8).

Not only do these graphs classify reflection spaces, but they define a *nesting* of lower dimensional reflection spaces within the higher dimensional reflection spaces. This is because a lower-rank graph can always be derived by removing a node from a higher-rank graph. (The *rank* of a graph is the number of nodes, which is equal to the dimensionality of the reflection space defined by the graph.) For example, starting from E_8, we can remove appropriate nodes to create the following two hierarchies of nestings:

$$E_8,\ E_7,\ E_6,\ D_5,\ D_4,\ A_3,\ A_2,\ A_1;$$
$$E_8,\ E_7,\ D_6,\ D_5,\ D_4,\ A_3,\ A_2,\ A_1;$$

and many other hierarchies as well.

The nesting hierarchies imply that the lower dimensional reflection spaces are contained in higher reflection spaces. In fact, since a mirror in an n-dimensional reflection space is an $(n-1)$-dimensional plane, we can regard this mirror plane itself as a reflection space defined by a graph of $n-1$ nodes; and, in turn, each $(n-2)$-dimensional mirror can be considered a reflection space defined by a graph of rank $n-2$; and so on, all the way down through the hierarchy of reflection spaces. In effect, each node in an n-rank graph (of *A-D-E* type) is a graph of rank $n-1$.

As one might guess, this hierarchy of reflection spaces is also a hierarchy of algebraic and topological structures, which are of great beauty and utility (cf. Gilmore, 1981).

As mentioned above, the reflection spaces are Lie algebra reflection spaces h^*. In fact, the *A-D-E* labels name various Lie algebras. Thus we can expect an intimate relationship between Lie groups, Lie algebras and Weyl reflection groups, since all three structures can be derived from the Coxeter graphs.

For example, given the A_2 Coxeter graph, we specify a 2-d reflection space (Fig. A4) with two basic mirrors (i.e., lines) intersecting at 60°. Call the mirrors *A* and *B*. The two mirror vectors, α and ß, attached to these mirrors are angled at 120° to each other. These are the basis vectors of this reflection space. By reflection in the basic mirrors (and a third mirror generated from either of the basic mirrors), we generate four new vectors: –α, –ß, α + ß, and –α – ß. Thus we have six vectors attached in pairs to each side of three mirrors. These six mirror vectors, usually called *roots,* are *eigenvalue vectors* in the sense that the coordinates of each root are eigenvalues of an eigenvector belonging to two commuting operators h_1 and h_2, which form the basis of h, the *maximal commutative subalgebra* (the *Cartan subalgebra)* of A_2. The *noncommuting* part of the Lie algebra requires a basis made of one eigenvector corresponding to each root. Thus the A_2 algebra is 8-d, with eight basis vectors: h_1, h_2, and six noncommuting eigenvectors.

Note: An element of an *algebra* is both a *vector* and an *operator,* because an algebra has both additive and multiplicative properties. Imagine each element represented by a matrix. The addition of two matrices corresponds to the addition of two vectors. We represent the algebra acting on its own vector space by the multiplication of one matrix by another. In a *Lie algebra,* multiplication of elements X and Y is defined as the combination $XY - YX$, which is written via Lie brackets: $[X, Y]$. Thus, if we consider X an operator and Y a vector, we can write the *eigenvector equation* as:

$$[X, Y] = aY$$

where Y is an eigenvector with eigenvalue a belonging to the operator X. In the case of A_2, for example, we have:

$$[h_1, h_2] = 0;$$

$$[h_2, h_1] = 0;$$
$$[h_1, e] = 2e;$$
$$[h_2, e] = -e.$$

Thus h_1 and h_2 are commutative and are eigenvectors of each other with eigenvalue 0. And e is an eigenvector of both h_1 and h_2, with eigenvalues 2 and –1, respectively. The root associated to e is a vector in the reflection space h^* with coordinates 2 and –1. There are five other eigenvectors following the same pattern as e but with different eigenvalues, and thus corresponding to the other five roots.

In general, a Lie algebra of rank n is defined by an n-node Coxeter graph, such that there is an n-dimensional Cartan subalgebra with n commuting operators as basis, and a set of noncommuting operators corresponding 1:1 to the roots in the reflection space defined by the Coxeter graph. In fact, the reflection space is called the *dual space* h^* of the Cartan subalgebra h. Moreover, h^* and h can be considered two different views of the same space, which we now call $\mathbf{C}^n$, because we regard the Lie algebras as complex Lie algebras (cf. Humphreys, 1972).

Since quantum theory restricts observable quantities to eigenvalues of measurement operators, these quantities are to be found in some Lie algebra reflection spaces. For example, the first *grand unified field theory* proposed the A_4 reflection space as the unified eigenvalue space. This 4-d reflection space has: one dimension for electric charge, one for weak charge, and two for strong (color) charge (cf. Georgi, 1981,1982).

Note: A_4 is the label for a complex Lie algebra whose compact Lie group is *SU(5),* with Cartan subgroup T^4, so that the A_4 reflection space is also the reflection space of the group *SU(5).* In general, the Lie algebra A_n with reflection space $\mathbf{C}^n$ has a compact Lie group $SU(n+1)$. Thus the first grand unified field theory is usually referred to as the *SU(5)* theory.

The term *grand unified theory* is really a misnomer because gravity is missing, and it is mainly the attempt to include gravity in unified field theory that has forced us to look to much larger Lie algebra reflection spaces. For example, the most celebrated version of *superstring theory* proposes to unify all the forces via the 16-dimensional reflection space $E_8 \times E_8$. This is possible only because the hierarchy of reflection space embeddings provides for an embedding of A_4 in E_8 (cf. Duff, 1986).

Since reflection spaces seem to be the key unification structures, we would do well to study them in more detail. One of the most striking facts of recent mathematics is a 1:1 correspondence between the finite subgroups of *SU(2)* mentioned above and the *A-D-E* series of Lie algebras (cf. McKay, 1980; Slodowy, 1983; Arnold, 1986). We list the most important aspects of this correspondence in Figure A6 (the terms will be explained in due course).

The *extended Coxeter graph* of a particular n-rank Lie algebra defines an infinite number of mirror planes in the n-dimensional reflection space of the Lie algebra. This is accomplished by starting from the n basis vectors defined by the ordinary Coxeter graph (cf. p. 335), and then constructing a new vector corresponding to the node marked with an asterisk. We lengthen all the other vectors by the factors indicated in the extended graph. These $n+1$ vectors are now in *balance:* if they were force vectors, the total force would be zero.

These vectors-in-balance define a mirror plane which forms a closed *alcove* (called a *fundamental alcove*) with the basic mirror planes. An observer in the closed alcove would see the space of the alcove reflected in all the mirror walls of the alcove so that a tessellation of mirror-walled alcoves (called *Coxeter alcoves*) would be generated, thus filling the entire reflection space (cf. Coxeter, 1973; Bröcker and Dieck, 1985) (see Fig. A5).

For example, the A_3 fundamental alcove is a tetrahedron. Put a candle in its center; then an infinity of candles in the centers of tetrahedral alcoves will be reflected in the four mirrored walls of the fundamental tetrahedron.

There are many uses for the *balance numbers* in the extended graphs. The sum of the balance numbers for a particular graph is called the *Coxeter number c.* We can calculate the *mirror number m* of intersecting mirrors in an n-dimensional reflection space as $m = nc/2$. We can check that for E_7 the number of mirrors is 63 = 7 x 18/2, because the Coxeter number is 18 = 1 + 2 + 3 + 4 + 3 + 2 + 1 + 2. Since there are two roots for each intersecting mirror, the number of roots is simply 126 = 7 x 18. And thus the *dimensionality* of the E_7 Lie algebra is the rank plus the number of roots: 133 = 7 + 126.

This is all on the Lie algebra side. What

FIG. A6 THE MCKAY CORRESPONDENCE

LIE ALGEBRA	**EXTENDED COXETER GRAPH**	**SU(2) SUBGROUP**	**CATASTROPHE GERM**
A_n	1–1– . . . –1 1* (joined to both end nodes)	$^{c}n + 1$	$A^{n+1} + B^2 + C^2$
D_n	1–2– . . . –2–1 1 (below the first 2), 1* (below the last 2)	$^{d}n - 2$	$A(B^2 - A^{n-2}) + C^2$
E_6	1–2–3–2–1 2 (below the 3) 1* (below the 2)	TD	$A^4 + B^3 + C^2$
E_7	1*–2–3–4–3–2–1 2 (below the 4)	OD	$A^3 + AB^3 + C^2$
E_8	2–4–6–5–4–3–2–1* 3 (below the 6)	ID	$A^5 + B^3 + C^2$

Fig. A7 E_7 & (extended) $^\wedge E_7$ Coexeter Graphs

(Where the underlined numbers are indexing numbers)

```
1   2   3   5   6   7        0    1   2   3   5   6   7
0 – 0 – 0 – 0 – 0 – 0        1* – 2 – 3 – 4 – 3 – 2 – 1
        |                                 |
        0                                 2
        4                                 4
```

about the supposed correspondence with finite subgroups of *SU(2)?* The following relations are due to the mathematician John McKay (1980), so we call these finite groups *M cKay groups.* For a given extended graph:

1. The sum of the squares of the balance numbers is the number of elements in the associated McKay group.

For E_7:

$1^2 + 2^2 + 3^2 + 4^2 + 3^2 + 2^2 + 1^2 + 2^2 = 48$

where 48 is the number of elements in the McKay group *OD.*

2. The balance numbers are the dimensions of fundamental vector spaces on which the McKay group acts. Such vector spaces are called *inequivalent irreducible representation* spaces (or *iirep* spaces). Thus the number of balance numbers *b* (i.e., the number of nodes in the extended graph) is equal to the number of iireps of the McKay group.

For E_7, the iirep dimensions of *OD* are: 1, 2, 3, 4, 3, 2, 1, 2.

3. The number of *classes* within the McKay group is also equal to *b.* (Note: Two group elements are in the same *class* if their action on a vector space differs only by a change of basis, or coordinate system, for the vector space.)

For E_7, the number of classes in the McKay group *OD* is eight.

4. If we make the McKay-group elements into basis vectors of a vector space, this vector space becomes an algebra called a *group algebra.* We can also make the classes into basis vectors of an algebra which is called the *center* of the group algebra. (Note: The center is a commutative subalgebra which commutes with the entire group algebra.) The dimensionality of the center is *b,* since *b* is the number of classes.

For E_7, the McKay-group algebra is **C**[*OD*], each of whose elements is a complex sum over the group elements *OD*— i.e., any element of **C**[*OD*] can be written as $c_1 x_1 + c_2 x_2 + \ldots + c_{48} x_{48}$, where c_1 through c_{48} are complex numbers, and x_1 through x_{48} are the elements of *OD.* There are eight classes in *OD.* Thus, if we partition the 48 elements into these eight classes and assign to each element of a class the same complex number, we will have an element of the 8-d center of **C**[*OD*].

5. Just as rectangular basis vectors define a grid structure whose vertices are called a "lattice" on a vector space, so the basis roots define a nonrectangular lattice (called a *root lattice* L_r) in the reflection space. The intersection points of the *alcoves* generated by the extended graph also define the vertices of a lattice, called a *weight lattice* L_w (see Fig. A4).

Fig. A8 $^\wedge E_7$ (Extended) Cartan Matrix

C = -A + 2E

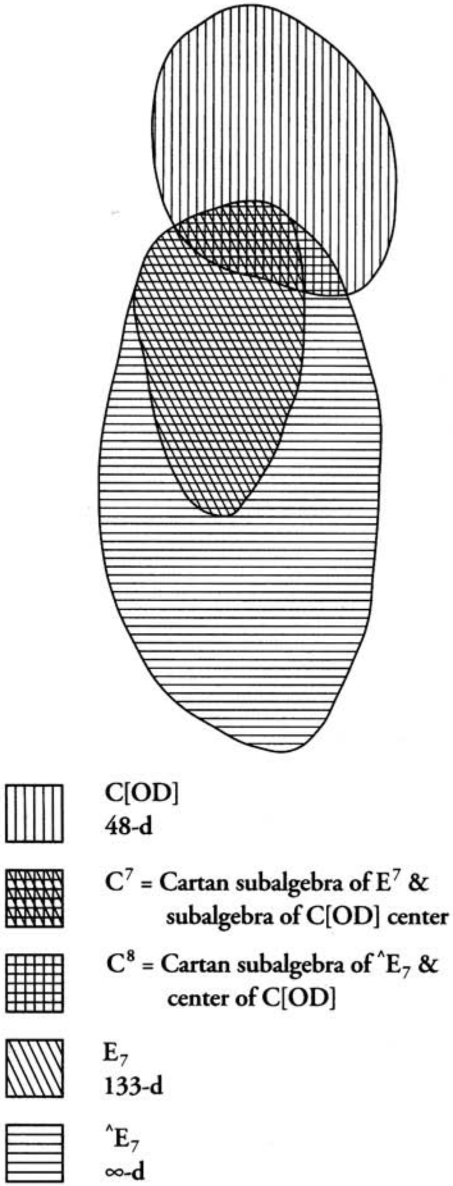
	0	1	2	3	4	5	6	7
0	2	-1	0	0	0	0	0	0
1	-1	2	-1	0	0	0	0	0
2	0	-1	2	-1	0	0	0	0
3	0	0	-1	2	-1	-1	0	0
4	0	0	0	-1	2	0	0	0
5	0	0	0	-1	0	2	-1	0
6	0	0	0	0	0	-1	2	-1
7	0	0	0	0	0	0	-1	2

Note: If we remove the 0'th row and the 0'th column, we have the ordinary Cartan matrix for E_7.

In unified field theory, the roots correspond to eigenvalues of *force particles,* whereas the weights correspond the eigenvalues of *matter particles.* If we think of the reflection space (where these roots and weights reside) as a space acted on by an operator, the most important such operator is one changing a weight basis to a root basis. This is called the *Cartan matrix.* The Cartan matrix has as columns the coordinates of the basic roots and thus can be derived directly from the Coxeter graph. In like manner, the *extended Cartan matrix* can be derived from the extended Coxeter graph (see Fig. A7).

The Cartan matrix can be defined as

FIG. A9 THE INTERSECTION OF E_7 WITH C[OD]



$C = -A + 2E$, where A is the *adjacency matrix* of the graph, and E is the identity matrix consisting of ones written along the primary diagonal of the matrix.

Note: In the adjacency matrix A (of either the ordinary or extended graph), the component corresponding to the i'th row and the j'th column has a 1 if the i'th node is adjacent to the j'th node, and 0 otherwise (see Fig. A8).

6. McKay (1980) has proved the amazing fact that eigenvectors (i.e., the vectors changing only in length) under the action of an extended Cartan matrix are the columns of *the character table* of the McKay group associated to the extended graph.

Note: The character table of a finite group is a square array of numbers, which are the *characters* (i.e., the sums of the diagonal numbers) of the iirep matrices for each class of elements of the group. The rows of the character table correspond to iireps; the columns correspond to the classes.

McKay's theorem implies that the columns of the character table of the McKay group provide a rectangular basis for the n-dimensional reflection space embedded in the **C**[OD] center of dimension $n + 1$ (see Figs. A9 and A10).

Let us look at these facts from the point of view of quantum mechanics in the context of unified field theory. We pick a Lie algebra which we hope encodes the basic symmetries of the world. The basis operators of its Cartan subalgebra h are a complete set of commuting operators. The eigenvectors correspond to eigenstates of the quantum system — the world. There are two types of eigenvalue in the reflection space h^*: force eigenvalues corresponding to the root lattice and matter eigenvalues corresponding to the weight lattice.

The Cartan matrix acts on reflection space h^* and transforms its weights into roots: i.e., matter eigenvalues into force eigenvalues. Thus it is a kind of super-operator. Think of the extended Cartan matrix as a super measurement operator whose eigenvectors are the classes of the McKay group associated to the Lie algebra. In general, Eigenvectors correspond to particle states. In this case (since the

FIG. A10 THE OD CHARACTER TABLE

(8 classes C_i & 8 iireps R_j)

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
R_0	1	1	1	1	1	1	1	1
R_1	2	0	1	0	$\sqrt{2}$	-2	-1	$-\sqrt{2}$
R_2	3	-1	0	-1	1	3	0	1
R_3	4	0	-1	0	0	-4	1	0
R_4	2	2	-1	0	0	2	-1	0
R_5	3	-1	0	1	-1	3	0	-1
R_6	2	0	1	0	$-\sqrt{2}$	-2	-1	$\sqrt{2}$
R_7	1	1	1	-1	-1	1	1	-1

Note: The ordering of iireps accords with the indexing of the $\hat{}E_7$ (extended) graph (see Fig. A7). The first column, which contains the E_7 balance numbers, corresponds to the identity element.

operator is a super-operator), we expect the eigenvectors to correspond to classes of observable particle states.

Is there an assignment of particle classes to the eight classes of the *OD* group? Actually, the most straightforward assignment is to the five classes of the *O* group. But since $O = OD/\pm 1$, the elements of *O* are pairs of *OD* elements, so the classes of *O* are derived from the classes of *OD*. The classes of *O* are equivalent to the classes of permutation *cycle pattern*. For example, there are six ways to permute four objects by *swapping* two and leaving the other two untouched. If we name the objects 1, 2, 3, and 4, the six such permutations can be written as: (1 2)(3)(4), (1 3)(2)(4), (1 4)(2)(3), (3 4)(1)(2), (2 4)(1)(3), and (2 3)(1)(4). These are called cycle patterns because the numbers in brackets are considered as a cycle — e.g., (1 2)(3)(4) is read as 1 goes to 2 and 2 goes to 1; 3 goes to 3; 4 goes to 4.

The five permutation classes partition the 24 elements of *O* into three classes of *even* permutations and two classes of *odd* permutations.

Note: An *even* permutation can be considered as a combination of an even number of *swaps;* an *odd* permutation is equivalent to an odd number of swaps.

The number of elements in the three even classes are: one, three, and eight.

The number of elements in the two odd classes are: six and six.

If we call an even permutation *e* and an odd permutation *o,* we can write the general rules for combining permutation as:

$$e \times e = e$$

$$o \times o = e$$

$$e \times o = o$$

$$o \times e = o$$

We notice that this matches the quantum field rules for force bosons *b* and matter fermions *f*: *f* interacts with *f* by exchanging *b* ($f \times b = f$), while *b* interacts with *b* by exchanging *b* ($b \times b = b$).

This suggests that the class numbers one, three, and eight correspond, somehow, to the numbers of force particles in each of three classes. Moreover, it would seem that the class numbers six and six correspond to the two classes of matter particle: quarks and leptons.

If we also partition the 24 elements of *O* into six *cosets* of the subgroup K_4, then three of the cosets correspond to even permutations, and the three remaining cosets correspond to three families of fermions, with two quarks and two leptons in each family.

We display the *O multiplication table* and define the 24 permutations with assignments to particle labels as Figure A11.

Since the fermion family structure is generally regarded as the deepest mystery of particle physics (cf. Georgi, 1982), we are justified in thinking that the finite group solution to this problem is a fundamental clue to the structure of unified field theory (cf. Sirag, 1982).

To test this idea further, we must construct an interpretation of the multiplication table which makes sense of the particle labels. First we note that every particle interaction fundamentally entails three particles. Such an interaction is represented by a vertex in a *Feynman diagram,* i.e., three lines meeting at a point. There are only two patterns to this vertex: two fermion lines and a boson line, three boson lines. This matches the odd-even structure of the *O* multiplication table, as mentioned above (see Fig. A12).

The three lines of any interaction vertex can be labeled: forward, lateral and forward; and the multiplication table can be interpreted so that $ab = c$ means forward *a* and lateral *b* and forward *c* meet at a vertex. In short, forward x lateral = forward. When this rule is postulated, the deep structure of weak and strong interactions is displayed (see Fig. A13). To distinguish these *O* group bosons from the standard force bosons, we use the particle labels *identon, kleinon* and *familon,* rather than *photon, weakon* and *gluon,* respectively. The standard force bosons are recovered by embedding the gauge groups $U(1) \times SU(2) \times SU(3)$ in the group algebra $\mathbf{C}[O]$, which is a subalgebra of $\mathbf{C}[OD]$.

It may seem that we have strayed far from the idea of a reflection space in defining particle classes corresponding to finite group classes. However, although the McKay group *OD* is not a reflection group, its factor group *O* is a reflection group. In fact, *O* is the reflection group of the A_3 reflection space. And A_3 can be used to unify the eigenvalue structure of the

FIG. A11

Multiplication table for O (i.e. S_4):

						Permutation:	Particle	Coset:
ABCD	**EFGH**	**IJKL**	**MNPQ**	**RSTV**	**WXYZ**	**Even:**	**Bosons:**	
ABCD	EFGH	IJKL	MNPQ	RSTV	WXYZ	**A** (1) (2) (3) (4)	identon	1st
BADC	FEHG	JIKL	PQMN	TVRS	XWZY	**B** (12) (34)	kleinon	(=K_4)
CDAB	GHEF	KLIJ	MNQP	VTSR	YZWX	**C** (13) (24)	kleinon	
DCBA	HGFE	LKJI	QPNM	SRVT	ZYXW	**D** (14) (23)	kleinon	
EGHF	IKLJ	ACDB	WZYX	MPNQ	RVST	**E** (124) (3)	familon	2nd
FHGE	JLKI	BDCA	XYZW	PMQN	TSVR	**F** (234) (1)	familon	
GEFH	KIJL	CABD	YXWZ	NQMP	VRTS	**G** (143) (2)	familon	
HFEG	LJIK	DBAC	ZWXY	QNPM	STRV	**H** (132) (1)	familon	
ILJK	ADBC	EHFG	RTSV	WYZX	MQPN	**I** (142) (3)	familon	3rd
JKIL	BCAD	FGEH	TRVS	XZVW	PNMQ	**J** (134) (2)	familon	
KJLI	CBDA	GFHE	VSTR	YWXZ	NPQM	**K** (123) (4)	familon	
LIKJ	DACB	HEGF	SVRT	ZXWY	ZMNP	**L** (243) (1)	familon	
						Odd:	**Fermions:**	
MQNP	RSVT	WZYX	ACDB	EFHG	ILKJ	**M** (24) (1) (3)	quark	4th
NPMQ	VTRS	YXWZ	CABD	GHFE	KJIL	**N** (13) (2) (4)	quark	
PNQM	TVSR	XYZW	BDCA	FEGH	JKLI	**P** (1234)	lepton	
QMPN	SRVT	ZWXY	DBAC	HGEF	LIJK	**Q** (1432)	lepton	
RVTS	WYXZ	MNPQ	IJKL	ADCB	EGFH	**R** (14) (2) (3)	quark	5th
STVR	ZXYW	QPNM	KLJI	DABC	HFGE	**S** (23) (1) (4)	quark	
TSRV	XZWY	PQMN	JILK	BCDA	FHEG	**T** (1342)	lepton	
VRST	YWZX	MNQP	KLIJ	CBAD	GEHF	**V** (1243)	lepton	
WXZY	MPQN	RTSV	EHFG	IKJL	ABDC	**W** (12) (3) (4)	quark	6th
XWYZ	PMNQ	TRVS	FGEH	JLIK	BACD	**X** (34) (1) (2)	quark	
YZXW	NQPM	VSTR	GFHE	KILJ	CDBA	**Y** (1423)	lepton	
ZYWX	QNMP	SVRT	HEGF	LJKI	DCAB	**Z** (1324)	lepton	

electromagnetic and strong color forces, as emphasized by Georgi (1981) (see Fig. A14).

The exact interpretation of the relationship between *O* as a factor group of *OD* and *O* as A_3 reflection space has yet to be worked out. Since the eigenvalues associated with the measurement of particles reside in reflection space, it is reasonable to expect that it will be intimately related to the fundamental fact of quantum mechanics: only eigenvalues are observable.

Furthermore, because there seems to be no fundamental distinction between observation and what is observed, I propose that the reflection space (i.e., eigenvalue space) is universal consciousness.

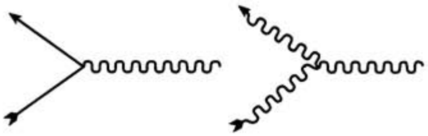
Consciousness as Reflection Space

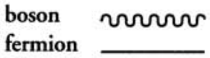
As we have seen, all of the magical structure of unified field theory is in the reflection space. Because of the McKay theorem, we can view this reflection space via the Lie algebra, or the McKay group, or both. In fact, we will argue that the McKay-group algebra corresponds to the physical aspect of the world, while the Lie algebra corresponds to the mental aspect. The reflection space exists in the intersection of the McKay-group algebra and the Lie algebra (actually the infinite-dimensional Lie algebra defined by the extended Coxeter graph) (see Fig. A9 on p. 339). Since this intersection space mediates between the two algebras, and since it is so active, I propose to identify it with *universal consciousness.*

It might be supposed that the idea of

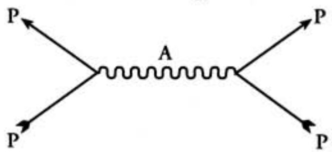
FIG. A12 FEYMAN DIAGRAMS

2 Types of Vertices





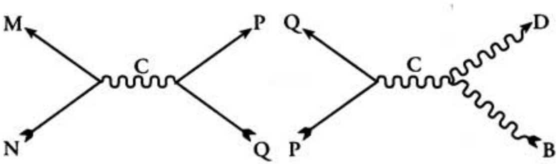
Electromagnetism



PA = P

P = electron
A = identon
(~photon)

Weak Interaction



NC = M
QC = P

PC = Q
CB = D

Q = electron neutrino
P = electron
M= up quark
N = down quark
C = kleinon (~weakon)
B = kleinon (~weakon)
D = kleinon (~weakon)

Feyman Diagrams display two types of vertices. The second type of vertex shows up in the weak interaction because weakons can interact with each other.

Actually A is somewhat different from a photon, so we call it an identon, because it corresponds to the identity element of the group. Also B, C, and D are somewhat different from the weakon so we call them kleinons, after Felix Klein for whom the Klein group K_4, consisting of A, B, C, & D, is named.

We recover the standard gauge theory of these force particles by the embedding of U (1) x SU (2) x SU(3) in the algebra C[OD].

consciousness as reflection space is simply a bad pun. After all, mathematicians have taken an ordinary word *reflection* and given it a technical meaning which is different from, but similar to, the usual meaning. The ordinary meaning has also generated the figurative meaning, by which reflection is a kind of thinking. It may be, however, that there is something deeper in this figure of speech. Perhaps thinking of all kinds is a kind of reflection. And perhaps this kind of reflection is akin to the mathematician's use of the word. Of course, there is more to consciousness than thinking. But there is also much more to reflection space than reflection. It is often assumed that a theory of consciousness is impossible because consciousness is so complicated. Perhaps the few aspects of reflection space which I have so far described will suggest that mathematical complexities may be rich enough to describe so rich an experience as consciousness.

Figures of speech have a habit of cutting two ways. We say that certain molecules have a "memory" if they return to an original shape after being deformed. This is a figure of speech (in fact, a gross anthropomorphism), of course. However, it is not entirely foolish to imagine that our own memories may, in part, be describable by using molecules with a "memory."

In the philosophical debates over the mind-body problem, this scheme of consciousness as reflection space could be considered either dualistic or monistic, since we are proposing that the world is a single space consisting of two different (but intersecting) algebras. This intersection is identified as universal consciousness. Thus the scheme is monistic geometrically, but dualistic algebraically.

Moreover, there is no problem here with the notion of the mental realm acting on the physical real and vice versa, because both realms in this theory are spaces. In fact, the action of each of these realms upon the other is mediated by the overlapping space — the reflection space.

The question now becomes: is there a reflection space with the appropriate qualities to fulfill this very demanding role? And does it have further properties which confirm it in this role?

Because of the theoretical successes of the

FIG. A13 THE STRONG INTERACTION MODELED BY THE FAMILON PRODUCTS OF THE O GROUP

MG =V
VJ = M
MF = S
SL = M
NL = Z
ZF = N
NJ = X
XG = N
NH = S
SK = N
NL = Z
ZF = N
FM = S
SL = M
NK = W
WH = N
NG = R
RJ = N
MK = Y
YH = M
MJ = Z
ZG = M
NG = R
RJ = N

LF = A
FL = A
JG = A
GJ = A
KH = A
HK = A

These products are derived from the multiplication table of the O group. They model the labeling of this diagram via the rule:
forward x lateral = forward.

The strong force binding the proton and neutron is mediated by the exchange of a pi meson, which changes the identity of these particles. According to the quark model, the pi meson (consisting of an up quark (u) and an anti-down quark ($\bar{d}$)) transfers an up quark (u) from the proton to the neutron, and a down quark (d) from the neutron to the proton. The quarks are bound to each other by gluons, each of which consists of a color and an anticolor.

Here the "gluons" are replaced by a complex of familons, which changes the family label of the quarks.

The standard theory of quarks and gluons is recovered by the embedding of the color gauge group SU (3) in C[OD], but here we're examining the underlying structure of OD itself via = OD/±1.

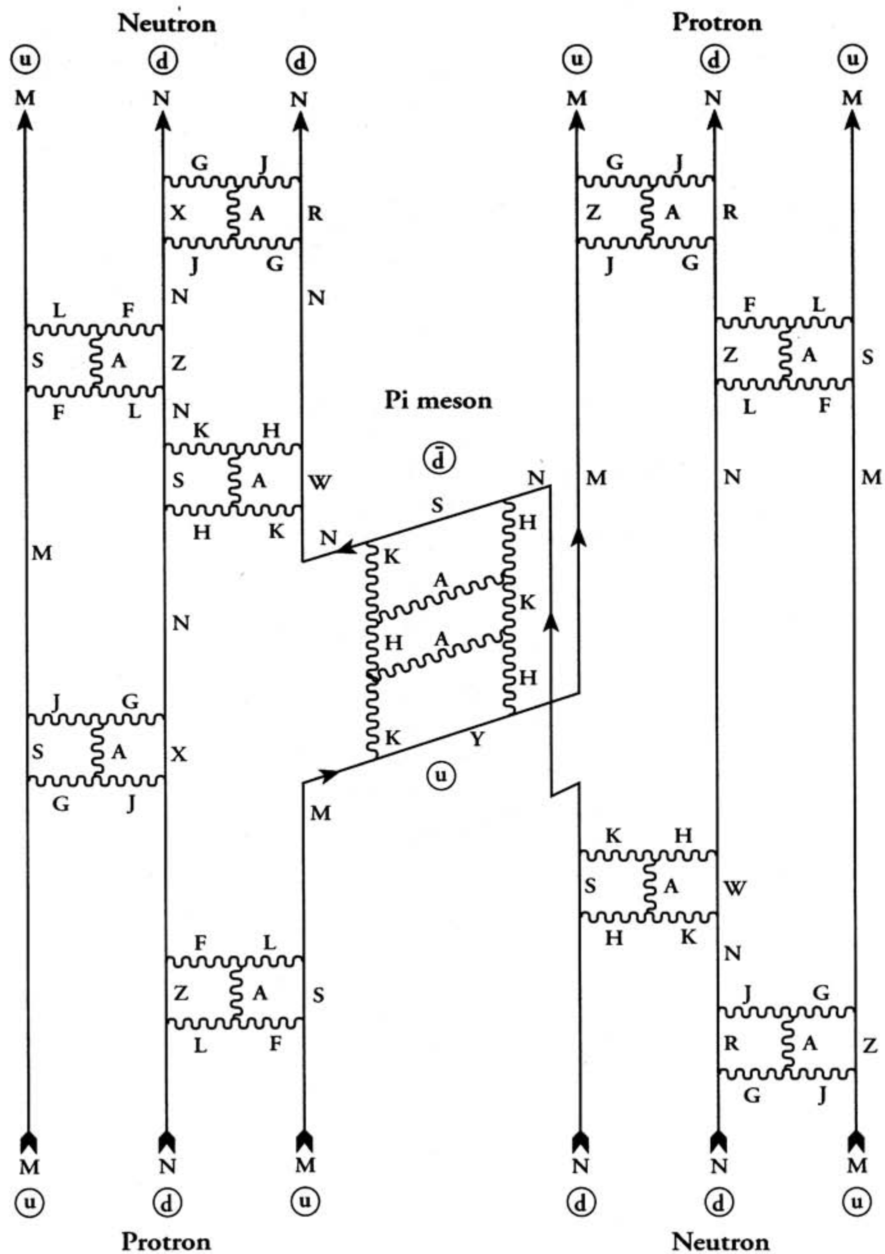
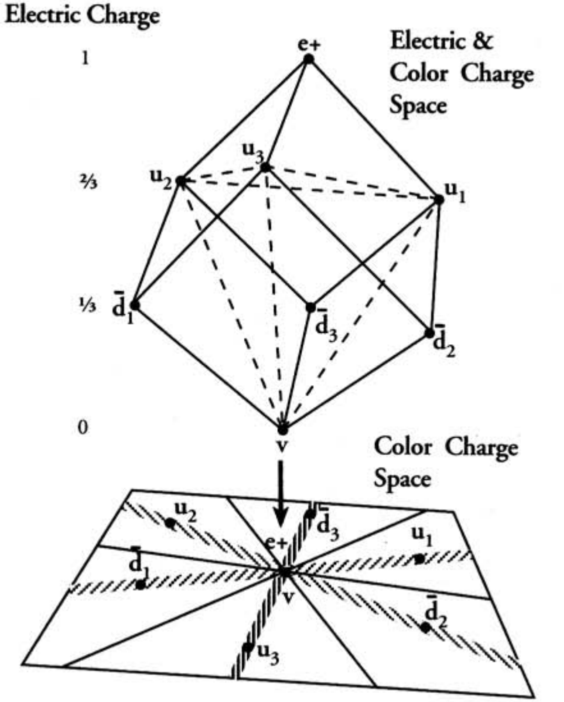


FIG. A14



THE A_3 REFLECTION SPACE projects down the A_2 reflection space, so that a tetrahedron (inscribed in a cube) is projected down to a triangle. The vertices of the tetrahedron carry the labels of three colored up quarks u_1, u_2, and u_3, and a neutrino *v*. A reciprocal tetrahedron carries the anti-down quark labels $\bar{d}_1$, $\bar{d}_2$, and $\bar{d}_3$, and the anti-electron e^+.

Thus, the eight vertices of the cube in A_3 reflection space encode the charge and color eigenvalues of a fermion family.

Note that the dotted lines in the A_2 reflection space correspond to mirrors, while the solid lines correspond to roots (or mirror vectors). Cf. Fig. A4.

E_8 x E_8 superstring theory, we might first try out the E_8 reflection space. However, an examination of the McKay groups *ID, OD,* and *TD* of the exceptional Lie algebras makes E_7 with its McKay group *OD* more plausible.

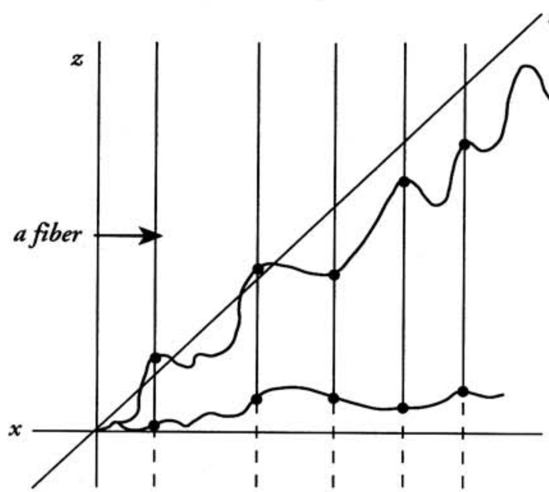
Remember that the balance numbers of the E_7 graph are 1, 2, 3, 4, 2, 3, 2, 1, which are the dimensions of the iirep vector spaces on which *OD* acts. According to the theory of complex group algebras, each *n*-dimensional iirep space corresponds to a unitary Lie group *U(n)* embedded in the group algebra. This, along with other facts of group theory, implies that the maximal compact subspace of the E_7 McKay-group algebra, which we label **C**[*OD*], is the 48-d unitary Lie group (Note: $1^2 + 2^2 + 3^2 + 4^2 + 2^2 + 3^2 + 2^2 + 1^2 = 48$):

$$P = U(1) \times U(2) \times U(3) \times U(4) \times U(2) \times U(3) \times U(2) \times U(1)$$

which we call *P* because it plays the role of the *principal fiber bundle* of our scheme.

A *fiber bundle* is a total space *B* which projects down to a subspace *X* (called the base space) in such a way that all the points of *B* which project to a point of *X* constitute a fiber *F;* moreover, there is a Lie group which acts on each fiber *F* as a symmetry group. In the case of a principal fiber bundle, the fibers are all copies of the symmetry group, so that the action of the symmetry group on a fiber is the action of the Lie group on itself. For any fiber bundle, we can write $b = f + x$, where *b, f,* and *x* are dimensions of *B, F,* and *X,* respectively. In fact, each point of *X* can be considered to be a

FIG. A15



x axis = time
y axis = space
z axis = internal space

A FIBER BUNDLE with 2–d spacetime as base space, and a 1–d fiber. More complicated fiber bundles are not picturable.

If the 1–d fibers were replaced by circles of unit radius — e.g. by connecting the ends of the 1–d fibers — we would have an example of a principal fiber bundle, because a unit circle is equivalent to the U(1) group, and thus each fiber would be a copy of U(1).

The bundle pictured above is a vector bundle, because each fiber is a vector space (1–d, in each case).

The path in the bundle is projected onto the path in the base space via the bundle projection, which projects each fiber onto a point of the base space.

copy of the Lie group F (see Fig. A15).

Every unified field theory is specified, in part, by constructing a principal fiber bundle whose base space is spacetime, and whose basic fiber is the symmetry group of the world (cf. Bleecker, 1981; Duff, 1986).

For example, the principal fiber bundle of the $E_8 \times E_8$ superstring theory would be a 506-d bundle, with a 10-d base space and the 496-d basic fiber, $G(E_8 \times E_8)$ — i.e., the Lie group generated by the Lie algebra $E_8 \times E_8$. This fiber bundle is a rather large structure, and much ingenuity has been expended in masking most of it in order to make contact with ordinary (low-energy) particle physics which corresponds to the product Lie group $U(1) \times SU(2) \times SU(3)$.

In the scheme of this paper, E_7 is the high-energy symmetry group, but because of the McKay correspondence between E_7 and *OD*, the unitary Lie group P in $\mathbf{C}[OD]$ contains the low-energy symmetry group. P is a 48-d principal fiber bundle projecting down to a 10-d base space S, each point of which is a copy of the 38-d basic fiber G. We write these spaces out as the following Lie groups:

$S = U(2) \times T^6$;

$G = U(1) \times SU(2) \times SU(3) \times SU(4) \times SU(2)$

where $U(2)$ is a 4-d spacetime called *conformally compactified Minkowski space;* T^6 is a 6-d torus = $U(1) \times U(1) \times U(1) \times U(1) \times U(1) \times U(1)$.

We consider S as the 10-d spacetime of superstring theory and G as the symmetry group of the following six forces, which we identify in sequence as: electromagnetism, weak, strong (color), hyperweak, gravity, and perhaps the feeble force.

We can write: $P = S \times G$. Since $U(2) = U(1) \times SU(2)$, we can rearrange S as $S = SU(2) \times T^7$. We regard $SU(2)$, i.e., *spherical 3-space* S^3, as the space of cosmology — the space in which we as macroscopic bodies appear to live. Every point of a macroscopic body is a point of S^3. Thus, if we view S as a fiber bundle, every point of a macroscopic body is actually a 7-d space which is a copy of T^7. Note that the 7-torus T^7 incorporates the factor $U(1)$ from the $U(2)$ spacetime, and thus includes time.

Now T^7 corresponds (via McKay's theorem) to the 7-d reflection space of E_7 as follows:

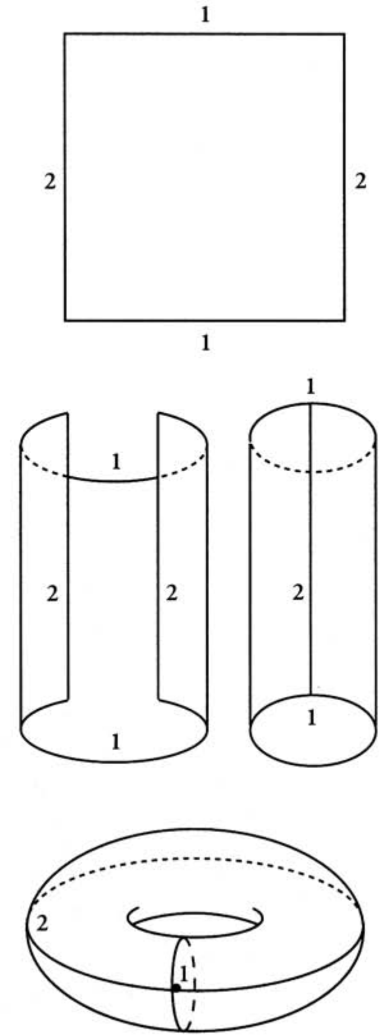
$$T^7 = \mathbf{R}^7/L_r$$

where $\mathbf{R}^7$ is the *real* part of the E_7 complex reflection space $\mathbf{C}^7$, and L_r is the E_7 root lattice. This means that all the points of the lattice are identified as a single point, the identity element of T^7, and every other point of T^7 is a copy of L_r.

To see how this works, you can generate a 2-torus (the surface of a donut) from a rectangular grid on a 2-d plane. First take one square of this grid, e.g., a sheet of paper, and glue together two opposite edges. You now have a tube. If you connect the ends of the tube to each other, you will have a 2-torus. Notice that the four corners of the sheet of paper are at the same point of the 2-torus (see Fig. A16).

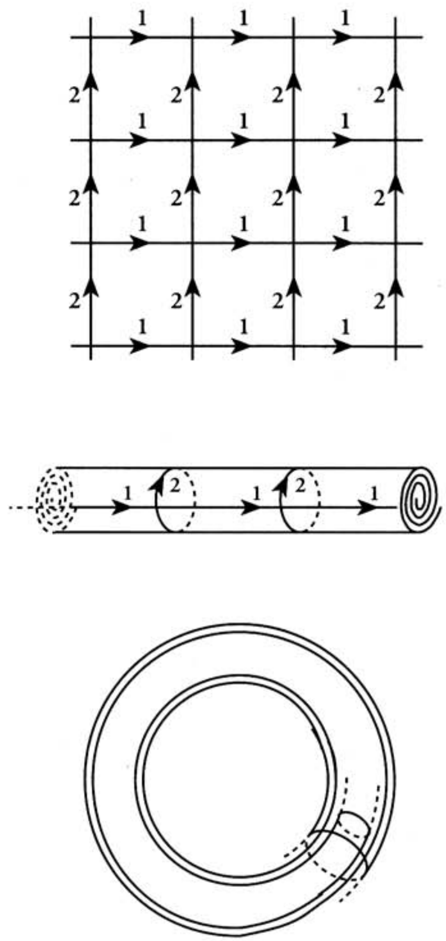
If the sheet of paper with a rectangular grid

FIG. A16



A TORUS can be derived from a square by connecting up the sides labeled 2, and then the sides labeled 1. Note that all four corners of the square become a single point of the torus.

FIG. A17



A TORUS can be derived from a lattice on a plane by rolling the plane an infinity of times in one direction so that an infinite tube (of circumference equal to the lattice spacing 2) is produced. Then wrap the tube an infinity of times around a circle of circumference equal to the lattice spacing 1. (*Drawing after Stillwell*)

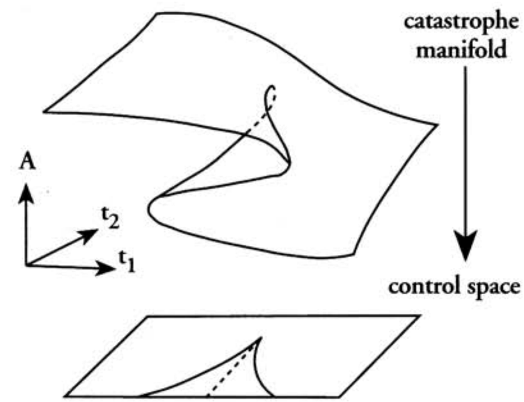
were infinite in extent, you could still get a 2-torus by rolling the paper an infinity of times along one grid direction and then doing the same along an orthogonal grid direction. In doing this, all the vertices of the grid (the lattice points) become identified as one point in T^2. In fact, since the positioning of the grid of paper is arbitrary, we can consider every point of T^2 as a copy of all the grid lattice points (see Fig. A17).

This procedure for generating tori works even if the grid is not rectangular and is hyperdimensional, as is the case of the E_7 root lattice L_r.

Since, by our fiber bundle construction, each point of the base space S^3 is a copy of T^7, we can consider each point of a macroscopic body to be a copy of $\mathbf{R}^7/L_r$. Since $\mathbf{R}^7$ is the (real) E_7 reflection space, it is the home of the fundamental eigenvalues of our unification scheme. If we identify consciousness with $\mathbf{C}^7$ space, it would appear that every point of any macroscopic body has access to consciousness. This is why we must consider $\mathbf{C}^7$ as space of universal consciousness. From this point of view, the long evolution of larger and larger brains is the evolution of richer and richer access to the universal consciousness.

Our experience suggests that consciousness is also causal. For example, I can choose to raise my arm. Even though most of what I do occurs unconsciously, these unconscious happenings seem to be ultimately under my control as a conscious entity. Raising my arm, after all, requires a host of activities only some of which

FIG. A18



THE A_3 CATASTROPHE is pictured via the real catastrophe manifold, which is the net of critical points of the A_3 catastrophe polynomial:

$$K = A^4 + t_1A^2 + t_2A$$

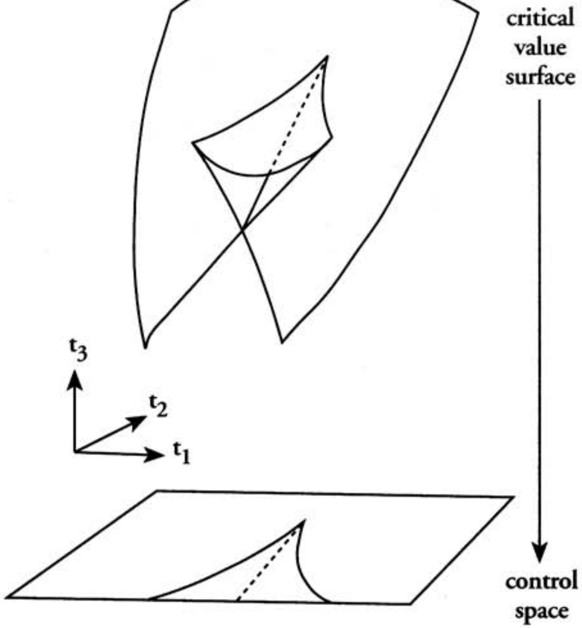
The catastrophe manifold is projected down to the control space, here 2–d, and the folds of the manifold become cusp lines in the control space.

Such a control space diagram is called a separatrix. The same separatrix occurs if we add any number of squared terms to K.

are understood by physiologists, cell biologists, molecular biologists, chemists and physicists. Nevertheless, all this unconscious activity is coordinated by my desire to lift my arm.

I believe that the causal aspect of consciousness derives also from our access to the E_7 reflection space $\mathbf{C}^7$. In this case it must be viewed as the *critical value space* of a catastrophe (see Figs. A18 and A19).

FIG. A19



THE A_3 CATASTROPHE is pictured via the real critical value surface embedded in the 3–d base space R^3 (with parameters t_1, t_2, t_3) of the A_3 catastrophe bundle B^5. B^5 is the zero set of the A_3 polynomial:

$$K = A^4 + B^2 + C^2 + t_1A^2 + t_2A + t_3$$

which is the value of K on the catastrophe manifold of critical points of K (Cf. Fig. A18).

K is a complex polynomial. The corresponding polynomial for E_7 is:

$$K = A^3 + AB^3 + C^2 + t_1C^2 + t_2B^3 + t_2B^4 + t_3B^4 + t_4AB + t_5AB^2 + t_6A^2 + t_7$$

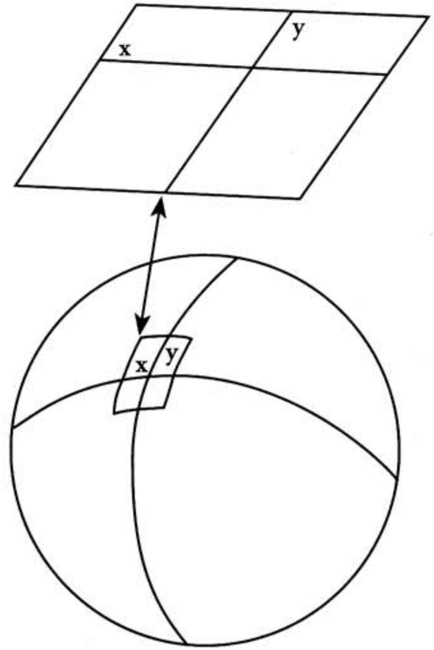
A *catastrophe* is a large change in a dynamic system caused by a small change in the parameter space on which the system depends. The mathematician Rene Thom (1975) invented catastrophe theory in order to provide a framework for a theory of biological development. Such a theory has not yet emerged. However, catastrophe theory has been successful in describing dynamic systems in the physical sciences. Certainly the mathematics of catastrophe theory is sound and very fruitful. In fact, the Russian mathematician Vladimir Arnold (1981) has been able to define Thom's seven elementary catastrophes via the Lie algebra reflection spaces. Moreover, Arnold showed that there is an infinite hierarchy of catastrophes such that there is a 1:1 correspondence between the simple catastrophes and the *A-D-E* complex reflection spaces.

Note: A *complex catastrophe* is modeled by a catastrophe bundle, a fiber bundle which is a complex manifold — i.e., each small piece of the manifold looks like a complex vector space of dimension equal to that of the manifold. For example, S^2 can be regarded as a 1-d complex manifold, because each small piece of S^2 looks like a 2-d real plane which can be viewed as the plane of complex numbers, called a 1-d complex plane. Since a complex number is an ordered pair of real numbers, an n-dimensional complex space can be considered a $2n$-dimensional real space (see Fig. A20).

The *A-D-E* classification is very deep since it also classifies simple singularities of maps, degenerate critical points of functions, caustics,

FIG. A20

THE MANIFOLD S^2: THE 2–SPHERE



A small piece of S^2 looks like a flat plane. S^2 can be regarded as a 2–d real manifold or as a 1–d complex manifold. In this case, we relabel the y-axis on the plane as iy, where i = ✓-1

wave fronts, quivers, crystallographic reflection groups, all the finite subgroups of *SU(2)* and therefore the regular polyhedra. In fact, Arnold (1986) hypothesizes that the *A-D-E* scheme classifies all "simple" objects in mathematics.

In other words, there is a complex hierarchical structure, specified by the hierarchy of the *A-D-E* Coxeter graphs. All the mathematical structures listed by Arnold (and probably many yet to be discovered) are different ways of viewing this structure. The advantage of this discovery is that what is almost impossible to see in one view is easy in another view. By using all these views, a deep understanding of this structure will emerge. Arnold calls this field of mathematics by the suggestive name *Platonics* (cf. Arnold, 1986).

The correspondence between reflection spaces and catastrophes is via the *catastrophe germs* listed in Figure A6 on page 337.

As we will see, for a given Coxeter graph, the actions of the McKay group and the Lie algebra interact in a truly marvelous way in the structure of a catastrophe.

In the case of E_7, we generate the following construction:

$$E_7 \Rightarrow M^9 \Rightarrow \mathbf{C}^7/W \Leftarrow \mathbf{C}T^7 \Leftarrow \mathbf{C}[OD]$$

This means that the 133-d complex Lie algebra E_7 projects down to the 9-d complex-catastrophe bundle M^9, which in turns projects down to a 7-d *complex vector space* $\mathbf{C}^7/W$, the final base space, which is the *critical value space* for M^9.

(Note: W is the E_7 *Weyl reflection group*, and $\mathbf{C}^7/W$ is the *orbit* space consisting of the set of W orbits in the E_7 reflection space $\mathbf{C}^7$.)

Since the reflection group W acts on complex as well as real spaces, $\mathbf{C}^7$ can be regarded as the complex reflection space of E_7. Remember that $T^7 = \mathbf{R}^7/L_r$, where T^7 is the real E_7 torus, $\mathbf{R}^7$ is the real reflection space of E_7, and L_r is the E_7 root lattice. Moreover, $\mathbf{C}^7$ is the intersection of the Lie algebra E_7 and $\mathbf{C}[OD]$.

Let us focus on the *catastrophe projection:* $M^9 \Rightarrow \mathbf{C}^7/W$. This is a *fiber-bundle projection* where each point of the base space $\mathbf{C}^7/W$ is an image of a complex 2-d *fiber*. The fiber F_0 at the origin (the zero point) of $\mathbf{C}^7/W$ is a space with a *singularity* (i.e., a point which is not smooth, such as a cusp) (cf. Arnold, 1981). This singularity space F_0 is the *zero set* of the E_7 *catastrophe germ:* $g = A^3 + AB^3 + C^2$. Thus F_0 is the set of solutions derived by setting this polynomial g equal to zero. This polynomial is called a catastrophe germ because each fiber of the bundle M^9 is a *deformation* of the fiber F_0. In fact, we can write out the full E_7 catastrophe bundle polynomial K as:

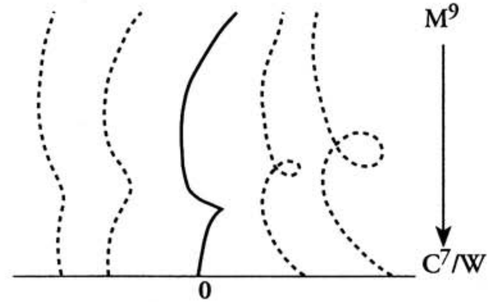
$$K = A^3 + AB^3 + C^2 + t_1B^2 + t_2B^3 + t_3B^4 + t_4AB + t_5AB^2 + t_6A^2 + t_7.$$

It is interesting that the *degrees* of those *monomials* in K which are *coefficients* of the t-parameters (i.e., 2, 3, 4, 2, 3, 2 and 0 for the constant 1) match the balance numbers on the extended Coxeter graph — if we exclude the node marked with a star (see Fig. A6) — and we have used this fact to make the indexing of the t-parameters match the indexing of the Coxeter graph (see Figs. A7 and A10 on pages 338 and 339).

There are 10 complex variables in this polynomial: $A, B, C, t_1, \ldots, t_7$. Thus the zero set is a 9-d complex space, i.e., the *complex manifold* M^9.

(Note: in general, the zero set of an n-variable polynomial is an $(n-1)$-dimensional space (cf. Kendig, 1977).

FIG. A21



A 2–d fiber bundle analogous to the 9–d complex fiber bundle M^9. The fiber at the origin 0 of the 1–d base space has a cusp singularity. The fiber at the origin of the 7–d complex base space C^7/W is the 2–d complex space C^2/OD. As we pick fibers further from the origin, the singularity becomes more benign (to the right), and even disappears (to the left). The change in the singularity structure of the fiber C^2/OD is determined by the critical surface Σ in C^7/W.

If we substitute a complex number for each of the parameters t_1 through t_7, the solution set of the equation, $K = 0$, is a 2-d complex space. Note: Once we specify the seven parameters, there remain only the three variables: A, B, C. This 2-d solution set is the fiber attached to the base $\mathbf{C}^7/W$ at the point specified by the seven complex coordinate numbers. This is why we call $\mathbf{C}^7/W$ a parameter space. Note that if we set all seven parameters to zero, i.e., choose the origin of $\mathbf{C}^7/W$, all the terms beyond the germ become zero. Thus the fiber at the origin of $\mathbf{C}^7/W$ is the zero set of the germ itself. There is a smooth *deformation* (or *unfolding*) of the fiber as one traces a path in $\mathbf{C}^7/W$. This smoothness does not preclude "rapid" changes in the fiber corresponding to "slow" changes in the parameter space. This is what is meant by the word *catastrophe* (see Fig. A21).

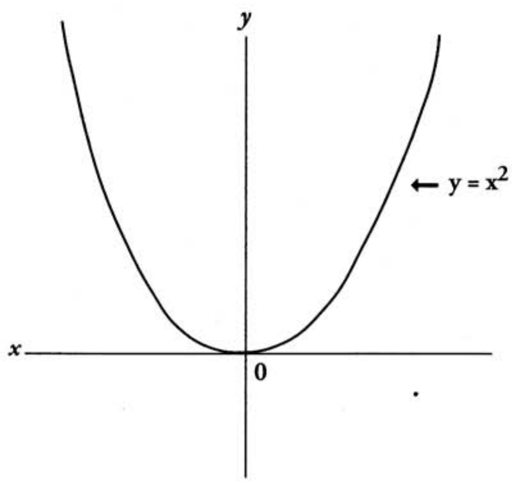
In the language of catastrophe theory, the six parameters $t_1, \ldots, t_6$ are the parameters of the *control space* of the E_7 catastrophe. This implies that the E_7 control space is a subspace of the E_7 base space $\mathbf{C}^7/W$. This base space is a *critical value space* in the following sense:

> Every mapping has a set of *critical points*, i.e., points on the graph of the mapping which locate qualitative changes. For example, $Y = X^2$ is a mapping from X to Y (each 1-d spaces); the graph of this map is a *parabola* with the lowest point at the origin (0,0) of X, Y space. This point is the only *critical point* of the mapping, since only at this point does the graph change direction from going down to going up. The value of the graph at the critical point is called the *critical value* (e.g., the critical value of the parabola is zero; see Fig. A22).

In the much more complicated case of a mapping from $\mathbf{C}^{10}$ to $\mathbf{C}^1$, as described by the E_7 catastrophe polynomial K, the set of all values of the t parameters $\{t_1, \ldots, t_7\}$ for which K has zero as a critical value forms a 6-d hypersurface Σ in $\mathbf{C}^7/W$. This set of critical values is called the *critical value hypersurface* (see Fig. A19).

The hypersurface Σ contains a great deal of information about the *singularities* of the fibers attached to $\mathbf{C}^7/W$. Any point of Σ in $\mathbf{C}^7/W$ is attached to a fiber which has a singularity. The type of singularity is determined by the nested Coxeter graphs. The most severe singularity occurs at the origin and is described by the E_7 graph. This singular point when *resolved* (i.e., pulled apart to form a non-singular structure) looks like a "bouquet" of seven 2-spheres just touching each other according to the *adjacency pattern* of the nodes of the E_7 graph (see Fig. A23).

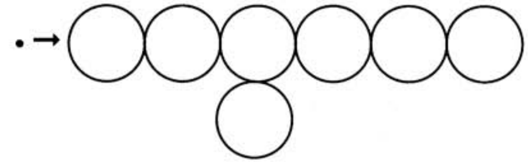
Fig. A22



A PARABOLA is the graph of the equation $y = x^2$. The point at the origin 0 is a critical point.

Fig. A23

The Resolution of the Singular Point in C^2/OD



The seven 2–spheres are in contact in accordance with E_7 Coxeter graph.

As one moves along a path away from the origin in the critical value surface, one picks out fibers whose singularity structures become simpler and simpler just as their Coxeter graphs become simpler and simpler. Since Σ is a 6-d space, it is "thin" within the 7-d space $\mathbf{C}^7/W$ — just as a 2-d surface is "thin" within the 3-d space. A path crossing this thin Σ will correspond to a rapid change in fiber. Thus a small change along this crossing path corresponds to a large change in the fiber (and therefore in the catastrophe manifold; see Fig. A21).

This kind of activity is the essence of mental control of the body: a small change in mental activity corresponds to a large change in bodily activity. But how is the fiber in this catastrophe bundle related to bodily activity?

Remember that E_7 corresponds to the mental world (both conscious and unconscious), whereas $\mathbf{C}[OD]$ corresponds to the physical world, the overlap $\mathbf{C}^7$ being universal consciousness.

Amazingly, one can also describe the singular fiber F_0 at the origin of $\mathbf{C}^7/W$ as $\mathbf{C}^2/OD$. In other words, each point of F_0 is actually a set of 48 points and is a copy of OD. This is because OD acts on $\mathbf{C}^2$ by its representation via 2-by-2 matrices. In this action certain polynomials on $\mathbf{C}^2$ remain invariant.

According to the mathematics of *invariant theory* (Springer, 1977) there are three fundamental invariant polynomials of the group OD acting on $\mathbf{C}^2$. This means that any polynomial which is invariant under OD is a polynomial of these fundamental polynomials — i.e., any OD-invariant polynomial can be written as a polynomial whose variables are the three fundamental invariant polynomials. These three polynomials are not independent, but form a relation called a *syzygy*. The OD syzygy can be written in the form $A^3 + AB^3 + C^2 = 0$. Thus A, B, and C (which define the fiber F_0) are not simply variables. They are also invariant polynomials of OD. In order to describe A, B, and C, we first describe the fundamental invariant polynomials of the *tetrahedral* group TD, which is a subgroup of OD.

There are three fundamental invariant polynomials of TD:

$f = x^5y - xy^5$;
($f = 0$: 6 vertices of octahedron)

$h = x^8 + 14x^4y^4 + y^8$;
($h = 0$: 8 vertices of cube)

$j = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$;
($j = 0$: 12 vertices of cuboctahedron)

where x and y are complex variables, so that the elements of TD act on (x,y)-vectors.

Note: A vector in an n-dimensional space is an ordered set of n numbers. Here the vector space is $\mathbf{C}^2$, so a vector is an ordered pair of complex numbers.

Using the rules of matrix multiplication, the action of c, one particular element of TD, on the general vector $\mathbf{V} = (x,y)$ is $c\mathbf{V} = \mathbf{V}'$:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Then by substituting $-y$ for x and x for y in the polynomial f, we have:

$f' = (-y)^5x - (-y)x^5 =$

$-y^5x + yx^5 =$

$x^5y - xy^5 = f$

Thus f is invariant under the action of element c. Similarly, f is invariant under all 12 elements of TD. In the same way, h and j are invariant under TD.

Since TD is a subgroup of OD, we expect a close relationship between the fundamental invariant polynomials f, h, and j of the group TD and the fundamental invariant polynomials A, B, and C of the group OD. In fact, A, B, C are constructed from f, h, j as follows:

$A = f^2$;

$B = h$;

$C = fj$.

The syzygy between the fundamental OD invariants A, B, and C, expressed by $A^3 + AB^3 + C^2 = 0$, has the consequence that $\mathbf{C}^2/OD$ is a space which is equivalent to the zero set of $A^3 + AB^3 + C^2$.

Furthermore, the syzygy between A, B, and C makes the *OD-invariant algebra* finite dimensional. In fact, this invariant algebra, the set of all OD-invariant polynomials $\mathbf{C}[A,B,C]$ — i.e., the set of polynomials in A, B, and C— is also the *coordinate algebra* of $\mathbf{C}^2/OD$. This coordinate algebra is 7-dimensional and has as basis elements the seven *coefficients* of the seven parameters $t_1, \dots, t_7$. These coefficients are listed in the catastrophe polynomial K and are: $B^2, B^3, B^4, AB, AB^2, A^2, 1$.

Note: The *coordinate algebra* of a space is the set of all distinct polynomials which live on that space. Thus $\mathbf{C}[A,B,C]$ is the set of distinct polynomials living on $\mathbf{C}^2/OD$.

In *string theory*, the spacetime in which the string vibrates is generated as the set of *scalar fields* living on the 2-d string manifold. Since a *scalar field* on a space is a continuous assignment of a number (a scalar) to each point of the space, a polynomial is a good example of a scalar field. This suggests that we should interpret $\mathbf{C}[A,B,C]$ as a set of scalar fields living on $\mathbf{C}^2/OD$. In other words: the deformations of $\mathbf{C}^2/OD$ are analogous to the vibrations of the string surface.

An important difference between our theory and string theory, however, is that a string surface is a 2-d real space, whereas $\mathbf{C}^2/OD$ is a 2-d complex space, which suggests that our theory entails a complexification of string theory.

Thus we start with 10 complex variables: $A, B, C, t_1, \ldots, t_7$. There is a polynomial K in these variables. The zero set of K is a complex space M^9, called the catastrophe manifold. There is a fiber bundle projection, $M^9 \Rightarrow \mathbf{C}^7/W$, such that the fiber F_0 at the origin of $\mathbf{C}^7/W$ is a singular space $\mathbf{C}^2/OD$ (see Fig. A21).

Remember: $\mathbf{C}^7/W$ is the set of points (an orbit) reached by acting on one point of $\mathbf{C}^7$ with all the elements of the E_7 reflection group W.

A, B, and *C* are the fundamental invariants of the *OD* group acting on $\mathbf{C}^2$; $t_1, \ldots, t_7$ are fundamental invariants of the E_7 reflection group W acting on $\mathbf{C}^7$.

The actions of *OD* and *W* are very different because *OD* is not a reflection group, whereas *W* is a reflection group. The action of a finite *nonreflection* group on a complex vector space is to create a space of the same dimension but different *topology.* The action of a *reflection* group on a complex vector space is to create another copy of that space — i.e., we regard $\mathbf{C}^7/W$ as a copy of $\mathbf{C}^7$ because they are isomorphic vector spaces.

The action of *OD* on $\mathbf{C}^2$ creates the complex string surface $\mathbf{C}^2/OD$ which is the zero set of the germ $A^3 + AB^3 + C^2$. The deformations of $\mathbf{C}^2/OD$ are parameterized by $t_1, \ldots, t_7$. Thus for each point t of $\mathbf{C}^7/W$ a different deformation of $\mathbf{C}^2/OD$ is selected.

Since the action of *W* on $\mathbf{C}^7$ creates another copy of $\mathbf{C}^7$, $\mathbf{C}^7/W$ is also a 7-d complex vector space. However, $\mathbf{C}^7/W$ is related to $\mathbf{C}^2/OD$ in a wonderful and useful way:

> $\mathbf{C}^7/W$ contains the E_7 critical value surface Σ, which is a 6-d hypersurface cutting through itself in a complicated way and thus dividing the reflection space into many regions. Every point of Σ picks out a deformation of $\mathbf{C}^2/OD$ which contains a singularity. Since $\mathbf{C}^7/W$ is the set of orbits of *W* in $\mathbf{C}^7$, *W* creates Σ in $\mathbf{C}^7/W$ in the sense that Σ is the set of *nonregular W* orbits in $\mathbf{C}^7$.

Note: *W* has 288 x 7! x 2 = 2,903,040 elements. If the number of distinct points in an orbit of *W* is equal to this element number, the orbit is called *regular;* otherwise it is called *nonregular.* As it turns out, each point of Σ in $\mathbf{C}^7/W$ is a nonregular element — i.e. consists of fewer than 2,903,040 distinct points of $\mathbf{C}^7$.

In case the number 288 x 7! x 2 looks too abstract, we can remember that 288 = 1 x 2 x 3 x 4 x 3 x 2 x 1 x 2, which are the node numbers appearing on the extended E_7 Dynkin diagram and are thus also the dimensions of the iireps of the *OD* group. (This is one more way in which the structure of *OD* and *W* inter-twine). The 7! (7-factorial) is 7 x 6 x 5 x 4 x 3 x 2 x 1, which is the number of elements in the symmetric-7 group S_7, which is the group of all permutations of seven objects (e.g., the basic mirror planes of $\mathbf{C}^7$), and the factor 2 corre-sponds to the bilateral symmetry of the extended E_7 Coxeter graph itself. Thus we have a geometric meaning for all of the factors in the number of *W* elements 288 x 7! x 2.

We can use geometric reasoning to clarify further the orbit mapping $\mathbf{C}^7 \Rightarrow \mathbf{C}^7/W$ as follows (cf. Bott, 1979):

> Consider that the 63 mirror planes in $\mathbf{C}^7$ cut this reflection space into 2,903,040 Coxeter chambers, since each chamber can be reached from the fundamental chamber by an action of one of the elements of the Weyl group, each of whose elements is a series of reflections. Thus any point inside any of these Coxeter chambers is copied by reflections into all the other Coxeter chambers, thereby generating a regular orbit of that point. Moreover, in the orbit mapping $\mathbf{C}^7$ fi $\mathbf{C}^7/W$, this regular orbit of the Coxeter chamber point is mapped onto a point outside the critical value surface Σ.

In contrast to this regular orbit mapping, if a point belonging to one of the 63 mirrors of $\mathbf{C}^7$ is chosen and acted on by reflections, that mirror point will be reflected only onto other mirrors. Thus the biggest mirror orbit will be a reflection onto all 63 mirrors. These orbits of mirror points are clearly nonregular orbits. In effect, the orbit mapping $\mathbf{C}^7 \Rightarrow \mathbf{C}^7/W$ trans-forms a mirror point into a point of the critical surface Σ, and a chamber point into a point outside this critical surface.

To summarize: Within E_7 there is a fiber bundle projection

$$M^9 \Rightarrow \mathbf{C}^7/W$$

where M^9 is a compact, complex manifold, which is the zero set of a polynomial K in the 10 variables $A, B, C, t_1, \ldots, t_7$. The fiber at the origin 0_7 of $\mathbf{C}^7/W$ is $\mathbf{C}^2/OD$, which is the zero set of $A^3 + AB^3 + C^2$. Each point of $\mathbf{C}^7/W$, parameterized by $t_1, \ldots, t_7$, picks out a deformation of $\mathbf{C}^2/OD$— i.e., a different fiber in M^9.

A, B, and *C* are fundamental invariants of *OD* acting on $\mathbf{C}^2$.

$t_1, \ldots, t_7$ are fundamental invariants of *W* acting on $\mathbf{C}^7$.

OD is a nonreflection group acting on $\mathbf{C}^2$; thus this action creates an orbit space $\mathbf{C}^2/OD$ which has a singularity: the zero set of $A^3 + AB^3 + C^2$, which is a 2-d compact, complex surface.

W is a reflection group on $\mathbf{C}^7$; thus this action creates an orbit space $\mathbf{C}^7/W$, which is isomorphic to $\mathbf{C}^7$. Under this orbit mapping, the 63 mirrors in $\mathbf{C}^7$ get mapped onto the critical value surface Σ in $\mathbf{C}^7/W$, and the space between the mirrors (the Coxeter chambers) gets mapped onto the space outside the critical surface Σ.

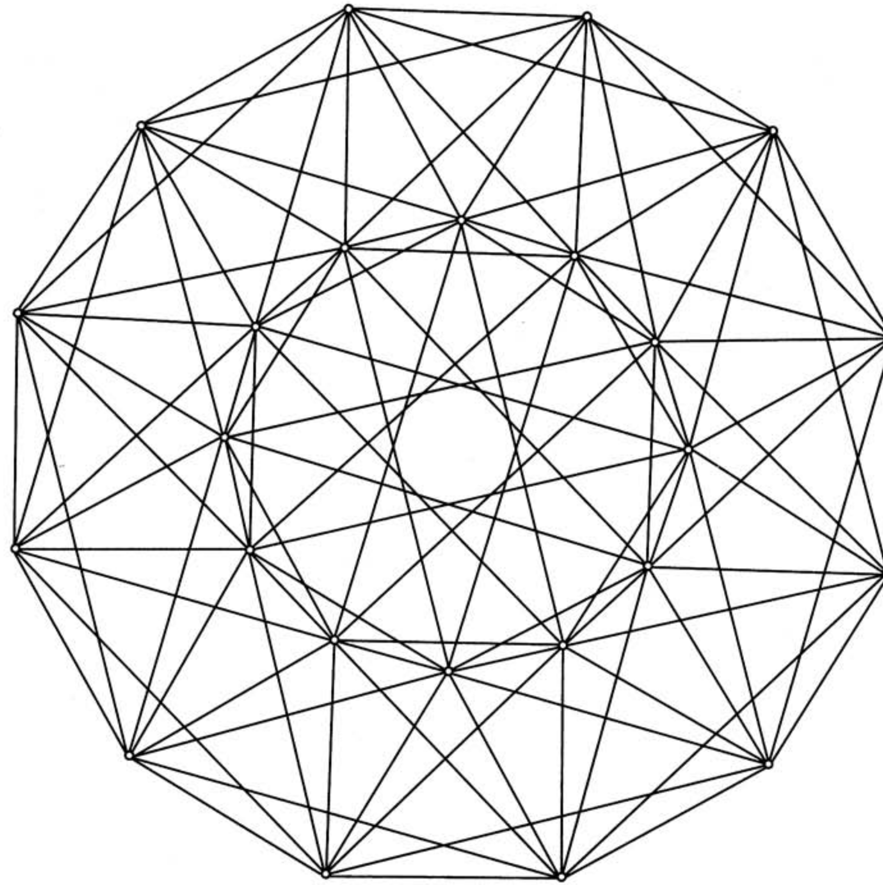
Σ is called a critical value surface because crossing it corresponds to a rapid change in the catastrophe bundle M^9. Moreover, the points of Σ pick out singularity-containing deformations of $\mathbf{C}^2/OD$.

All of this activity is going on inside the 133-d Lie algebra E_7, because M^9 is a subspace of E_7; *W* is the Weyl reflection-group of E_7; and $\mathbf{C}^7/W$ is the action of *W* on the largest commutative subalgebra $\mathbf{C}^7$ of E_7.

With this summary of the E_7 side of the theory, we must now look more closely at the $\mathbf{C}[OD]$ side.

FIG. A24

THE 24–CELL



This is a 2–d projection of a 4–d figure which has 24 vertices and 24 three–d faces (called cells), each of which is an octahedron. (Coxeter, 1973)

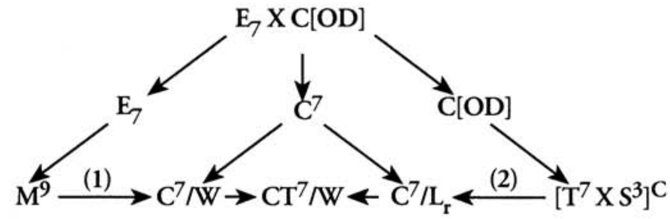
Since $\mathbf{C}^2/OD$ is the identity fiber in M^9, we are reminded that OD plays a fundamental role within the E_7 algebra; in fact, $\mathbf{C}^7$ is the intersection of E_7 with $\mathbf{C}[OD]$. As we have seen, $\mathbf{C}^7$ (universal consciousness) in the form of $\mathbf{C}^7/W$ controls the deformation structure of $\mathbf{C}^2/OD$ in the space E_7, which we have identified with universal mind.

We have yet to identify $\mathbf{C}^2/OD$ and its deformations with anything other than the fibers of the E_7 catastrophe bundle M^9. Having identified $\mathbf{C}[OD]$ with the physical world, we expect $\mathbf{C}^2/OD$ to have an intimate connection with the physical world. One could say that $\mathbf{C}^2/OD$ and its perturbations are mental images of the physical world. This is because the deformations of $\mathbf{C}^2/OD$ are controlled by the critical space $\mathbf{C}^7/W$, which is the set of W orbits in $\mathbf{C}^7$, the intersection of the mental world E_7 and the physical world $\mathbf{C}[OD]$. In effect, $\mathbf{C}^2/OD$ is a counterpart to ordinary physical space — i.e., cosmic space S^3. This is suggested by the mathematical properties of these spaces:

> $\mathbf{C}^2$ can be viewed as the set of quaternions **Q**, since a quaternion is an ordered pair of complex numbers. S^3 can be viewed as the set of unit quaternions. OD is a finite subgroup of S^3. And in fact, it is well known that S^3/OD = the intersection of $\mathbf{C}^2/OD$ with S^5 (the unit sphere in $\mathbf{C}^3$, the space parameterized by *A, B,* and *C,* the fundamental invariants of *OD*; cf. Milnor, 1968,1975). In other words, S^3/OD provides a good approximation to the topology of $\mathbf{C}^2/OD$ near the singular point.

FIG. A25

THE SPLICED BUNDLE (SB^{181}) MAPPINGS



Incidentally, OD is a very symmetric set of 48 points in S^3. OD is two copies of the *24-cell* (each reciprocal to the other). One 24-cell is the subgroup TD.

Note: The 24-cell is a unique regular polytope in $\mathbf{R}^4$ space, of which there is no analog in spaces of higher or lower dimension. The 24 vertices of the 24-cell (i.e., the points TD in S^3) are the points at which the 24 S^3-spheres touch the central S^3 sphere in the most efficient *sphere packing* in $\mathbf{R}^4$ (see Fig. A24).

Thus the 24-cell as a sphere-packing figure has analogs in other spaces, such as $\mathbf{R}^7$, where the 126 sphere-packing points are the E_7 roots. Because of the hierarchy of the reflection space structures, the 24-cell is a substructure of the 126 vertex E_7 root figure V_{126}.

This suggests that the 24-cell is a structure mediating between V_{126} and the V_{48}, the double 24-cell or OD.

Moreover, all these considerations suggest that the cosmic space S^3 as embedded in $\mathbf{C}[OD]$ must be a fiber similar to the fiber $\mathbf{C}^2/OD$.

In order to describe the relationship of $\mathbf{C}^2/OD$ to S^3, we can display Figure A25, where (1) is a fiber bundle projection with identity fiber $\mathbf{C}^2/OD$, and (2) is a fiber bundle projection with identity fiber $S^3 \times S^3$. (Remember that $\mathbf{C}^7/L_r$ is the complex 7-torus CT^7.) Therefore the *spliced bundle* $SB^{181} = E_7 \times \mathbf{C}[OD]$ contains the *spliced sub-bundle* $M^9 \times [T^7 \times S^3]^C$ with the projection $M^9 \times [T^7 \times S^3]^C \Rightarrow \boldsymbol{CT^7/W}$ with identity fiber $(\mathbf{C}^2/OD) \times S^3 \times S^3$. As different points along a path in $\mathbf{C}T^7/W$ are chosen, the identity fiber undergoes deformations. Since $\mathbf{C}T^7/W$ is the Lie group version of $\mathbf{C}^7/W$, these perturbations are described by the critical value surface Σ in $\mathbf{C}^7/W$.

Thus we have identified the physical counterpart of $\mathbf{C}^2/OD$ with $S^3 \times S^3$. Notice that both of these structures are complex, and that the real part of $\mathbf{C}^2/OD$ is a 2-d surface while the real part of $S^3 \times S^3$ is the spherical cosmic space S^3. We can regard $S^3 \times S^3$ as complexified cosmic space.

This implies that every deformation of $\mathbf{C}^2/OD$ is linked to a deformation of the cosmic space S^3. We can view each point of S^3 as a copy of $\mathbf{C}^2/OD$, and each point of a deformed S^3 as a copy of a perturbed $\mathbf{C}^2/OD$. The deformations are controlled by the points of $\mathbf{C}^7/W$. Thus each path through $\mathbf{C}^7/W$ corresponds to a different evolution of the fiber $\mathbf{C}^2/OD \times S^3 \times S^3$.

What about *time?* Remember that S^3 is the space part of *cosmic spacetime U(2),* because $U(2) = T^1 \times S^3$. The one-torus (a circle) T^1 is part of the T^7 in the *superstring spacetime* $S^3 \times T^7$. Thus time becomes complexified as a parameter t_1 in the seven-parameter basis of $\mathbf{C}^7/W$. This implies that *time is a part of universal consciousness.*

The fact that we must identify t_1 with time raises the question: What is the interpretation of the fibration $M^9 \Rightarrow \mathbf{C}^7/W$ such that t_1 as a deformation parameter of $\mathbf{C}^2/OD$ can be considered as time?

We can consider the critical surface Σ in $\mathbf{C}^7/W$ to be a family of *6-d wave fronts* evolving in time. Under this interpretation t_1 does indeed become a time parameter, so that $\mathbf{C}^7/W$ is a complex 7-d spacetime. The singularities of the wave fronts are described by the E_7 singularity structure (cf. Arnold, Gusein-Zade, Varchenko, 1985).

Note: The *same* time parameter t_1 plays three roles:

1. *Cosmic time* in the 4-d spacetime: $U(2) = T^1 \times S^3$

2. *Supertime* in 10-d superstring spacetime: $T^7 \times S^3$

3. *Complex time* in the 7-d consciousness spacetime $\mathbf{C}^7/W$.

The picture we now have is that (corresponding respectively to the three roles of the time parameter) every spacetime path in $\mathbf{C}^7/W$ synchronizes three evolutions:

1. An evolution of cosmic space S^3

2. A series of deformations of the $\mathbf{C}^2/OD$ fiber

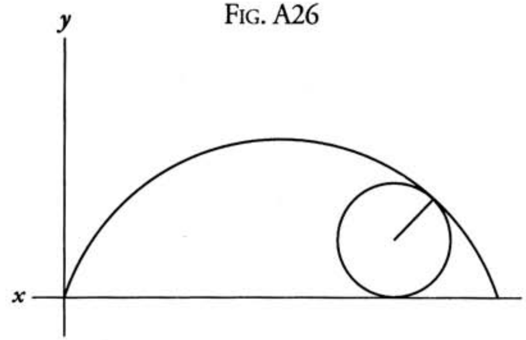
3. An evolution of wave fronts in $\mathbf{C}^7/W$.

Because of the fact that $\mathbf{C}T^7/W$ is the base space of the SB^{181} fiber bundle, $\mathbf{C}T^7/W$ contains an *image* of all the most essential structure of the entire SB^{181} space. Remembering that $\mathbf{C}T^7/W$ is the Lie group version of the Lie algebra space $\mathbf{C}^7/W$, we can say that the paths in $\mathbf{C}^7/W$— i.e., the set of evolutions of wave fronts in $\mathbf{C}^7/W$— are the images of the deformations of $\mathbf{C}^2/OD$, as well as the evolutions of S^3.

Every point of S^3 is a location in the space of the macroscopic world. However, the identity-subfiber s_0 in SB^{181} is the direct product of $(\mathbf{C}^2/OD) \times S^3 \times S^3$. Thus every point of S^3 can be regarded as a copy of $\mathbf{C}^2/OD$. This view is especially appropriate in the present state of the universe — i.e., S^3 is of "cosmic" size, approximately 10^{28} cm in radius. Thus, as we look at smaller and smaller regions of S^3, we will see the structure of the 4-(real)-dimensional $\mathbf{C}^2/OD$ emerge, rather than a mere 0-d point.

Remember that every point along a path in

FIG. A26



THE CYCLOID CURVE is the line traveled by a point on the rim of a rolling wheel.

Applied to cosmology:

x = time

y = the radius of a spherical universe S^3

$\mathbf{C}^7/W$ selects a different deformation (or unfolding) of $\mathbf{C}^2/OD$. The critical value surface Σ in $\mathbf{C}^7/W$ (i.e., the family of wave fronts) determines what happens along the path (once the path is chosen). The zero point 0_7 (the origin) of $\mathbf{C}^7/W$ selects $\mathbf{C}^2/OD$ itself. This a complicated space with one singular point (rather like a line crossing itself three times at one point). As one moves along a chosen path away from 0_7, the deformations of $\mathbf{C}^2/OD$ change this 2-d fiber so that it becomes less and less complicated. If the path selects a fiber outside Σ, the singularity in the fiber goes away.

It is natural to suppose that 0_7 corresponds to the "big bang" singularity of cosmology. The cosmological singularity, however, refers to a condition of infinite density of the cosmos.

This is a different meaning of "singularity" than that associated with $\mathbf{C}^2/OD$.

The study of spaces such as $\mathbf{C}^2/OD$ is a branch of mathematics called *singularity theory* (cf. Arnold, 1981), and it deals with "singular" points such as cusps and nodes, where the space becomes unsmooth. (For the sake of readers who have studied calculus, let me say that at such unsmooth points on a curve the derivative becomes undefined, and we say that the curve is undifferentiable at a singularity; analogous statements about partial derivatives hold for higher dimensional unsmooth spaces, which are undifferentiable at a singular point.)

It is striking, however, that the curve which represents the *scale factor* R of the universe as a function of time has a *cusp singularity* at $t = 0$, just when the *cosmological singularity* occurs. For a compact universe (e.g., S^3) this curve is the *cycloid curve* (i.e., the curve traced out by a point on the rim of a rolling wheel); there is also a cusp singularity corresponding to the cosmic singularity of the "big crunch" (see Fig. A26).

This cosmic cycloid curve would be a subspace of the parameter space $\mathbf{C}^7/W$, which includes time. It is possible to regard the scale factor R as a measure of a variable gravitational "constant" G (cf. Sirag, 1983). In other words, G is effectively one of the seven parameters in $\mathbf{C}^7/W$.

More correctly, we should regard G as a *toroidal* parameter of $\mathbf{C}T^7/W$. As we have said, $\mathbf{C}T^7/W$ is the Lie group version of the Lie algebra space $\mathbf{C}^7/W$. The relationship between a Lie group LG and Lie algebra LA can be expressed via the *exponential map:*

$$LG = \exp(LA)$$

This implies that in our case the parameter g in $\mathbf{C}^7/W$ which corresponds to G in T^7 has the following relationships:

$$e^g = G;$$

$$e^0 = 1;$$

$$e^{-0.69} = \tfrac{1}{2}.$$

where e is the base of the *natural logarithms:*

2.71828 . . . etc.

If we set $G = 1$ (i.e., $g = 0$ at the time $t = 0$, then as the cosmos S^3 expands to its maximum R, the gravitational parameter G goes to ½ (i.e., g goes to –0.69). Similarly, as S^3 contracts to its minimum size, $G = ½$ ($g = -0.69$) goes back to $G = 1$ ($g = 0$). In other words, as G oscillates between 1 and ½ (i.e., g oscillates between 0 and –0.69), S^3 oscillates between $R = 1$ and $R = 10^{41}$. Thus a small change in the parameter space $\mathbf{C}^7/W$ corresponds to a large change in the fiber S^3 (cf. Sirag, 1983; Marciano, 1984; Appelquist et al., 1985).

Since each point of S^3 is a copy of $\mathbf{C}^2/OD$, we should also consider the changes in the radius d of $\mathbf{C}^2/OD$. According to string theory arguments, d is approximately 10^{-30} cm, which in natural units is $137G^{1/2}$. Thus, by the construction considered here, as G goes from 1 to ½, d goes from 10^{-30} cm to $.707(10^{-30})$ cm. This small change in d (less than one order of magnitude) corresponds to a change of 41 orders of magnitude in R, radius of S^3. Moreover, the changes go in the opposite direction: d gets smaller as R gets bigger, and vice versa.

We are now in a position to describe the relationship of mind to body. There is a *universal body,* $\mathbf{C}/[OD]$. There is a *universal mind,* E_7. The intersection of the two — $\mathbf{C}^7$ — is *universal consciousness.*

There is a *universal geometric entity,* the spliced bundle SB^{181}, which is the direct product $E_7 \times \mathbf{C}[OD]$.

Notice that from the point of view of SB^{181}, the system is monistic; but from the point of view of the constituent algebras E_7 and $\mathbf{C}[OD]$, the system is dualistic.

SB^{181} projects down to the base space $\mathbf{C}^7/W$. The subfiber attached to the identity element of $\mathbf{C}T^7/W$ is $s_0 = (\mathbf{C}^2/OD) \times S^3 \times S^3$.

Every observer corresponds to a different path in $\mathbf{C}^7/W$— i.e., each observer has his own cosmos, his own copy of s_0 carried along the $\mathbf{C}^7/W$ path. Each point along the path is, in effect, a new observation since a new copy of s_0 is selected for each point of the path.

In one sense, each observer is separate, having his own path in reflection space $\mathbf{C}^7$. However, in $\mathbf{C}^7/W$ many paths coincide part of the way. The coincidence of two or more paths over any distance is called the *contact* between the paths. Remember, these are not paths in spacetime, but paths in critical space $\mathbf{C}^7/W$ (which includes complexified time). Moreover, since we have identified the reflection space $\mathbf{C}^7$ with universal consciousness, the contact structure of $\mathbf{C}^7/W$ must play an important role in coordinating the observations of separate observers. For example, even though two separate observers live in separate "universes" via paths in $\mathbf{C}^7$, they can communicate with each other meaningfully if they are in contact via paths in $\mathbf{C}^7/W$— i.e., while the paths are in contact, the observers see the same universe, because they experience the same eigenvalues.

In fact, since eigenvalues are observations, and the reflection space has been identified with eigenvalue space, we can make the following postulate:

> Within $\mathbf{C}^7$, point *a* is aware of point *b* if and only if point *a* is in contact with point *b* in $\mathbf{C}^7/W$— i.e., *a* and *b* in $\mathbf{C}^7$ are mapped onto the same point of $\mathbf{C}^7/W$.

This postulate justifies, in a fundamental way, our calling the space $\mathbf{C}^7$ consciousness (cf. Culbertson, 1982).

Notice that we are not explaining consciousness here. Rather we are defining the conditions under which awareness (an aspect of consciousness) exists between two points. We are assuming the existence of consciousness as fundamental. As some philosophers (e.g., Descartes) have emphasized, the existence of consciousness is the safest starting point for any theory of reality — i.e., if the philosopher knows nothing certain about the world, he at least knows that he is aware of his uncertainty, so he can begin his speculations on reality by assuming his own awareness.

Given the above postulate, we can see that the "purpose" of the projection from $\mathbf{C}^7$ onto $\mathbf{C}^7/W$ is to make points which are separate in $\mathbf{C}^7$ identical in $\mathbf{C}^7/W$— i.e., bring the points into contact and thus into mutual awareness.

Moreover, it becomes clear that a brain must be a set of points in cosmic space S^3 attached to a set of paths in $\mathbf{C}^7/W$ in such a way as to increase its contact structure. The entire evolution of biological entities is in general an evolution of increasing the richness of contact (and thus awareness).

The nature of the contact structure of all the paths in $\mathbf{C}^7/W$ is determined by the orbit structure of the E_7 reflection group W acting on $\mathbf{C}^7$. Remember that a single point of $\mathbf{C}^7/W$ is an orbit, which is a set of (as many as 2,903,040) points in $\mathbf{C}^7$. Thus the reflection group W weaves together sets of points in $\mathbf{C}^7$ to create points in $\mathbf{C}^7/W$, so that paths in contact in $\mathbf{C}^7/W$ can be lifted to separate paths in $\mathbf{C}^7$, while separate paths in $\mathbf{C}^7$ can be *lowered* to paths in contact in $\mathbf{C}^7/W$. (Note: Lifting and lowering are the standard mathematical terms for these mappings.)

Moreover, there is a close relationship between the contact structure of $\mathbf{C}^7/W$ and the wave front structure Σ, because Σ in $\mathbf{C}^7/W$ is the set of nonregular W orbits in $\mathbf{C}^7$.

Remember: The 63 mirror planes in $\mathbf{C}^7$ are mapped onto Σ in $\mathbf{C}^7/W$; the point where all the mirrors intersect is the origin of $\mathbf{C}^7/W$, to which is attached the identity fiber $\mathbf{C}^2/OD$ (cf. Bott, 1979).

Because of the relationship between the contact structure and the wave front structure in $\mathbf{C}^7/W$, we can see that there must be a close relationship between the contact structure and the causal properties of both the *mental fiber* $\mathbf{C}^2/OD$ and the *physical fiber* S^3. This means that there is an intimate relationship between *awareness* (contact) and *volition* (wave fronts).

At this point the reader may feel that the choice of $\mathbf{C}^7$ as the space of primary consciousness is arbitrary. Why not $\mathbf{C}^8$ or $\mathbf{C}^9$ or infinite dimensional $\mathbf{C}$-space?

The answer is that we have identified $\mathbf{C}^7$ as the space of consciousness because it is the intersection of E_7 with $\mathbf{C}[OD]$; in turn, these algebras are brought into the picture by the structure of a unified field theory which unites the forces as we know them in the physical world. The physical world is decisive in this discussion because we know that physical factors affect consciousness (e.g., close your eyes) and that consciousness affects the physical world (e.g., you can decide to close, or not to close, your eyes).

But from a purely mathematical point of view, our choice of $\mathbf{C}^7$ is arbitrary. We could have chosen any of infinity of finite subgroups of *SU(2)* as our starting point. Call that finite subgroup (a McKay group) g; then we could write:

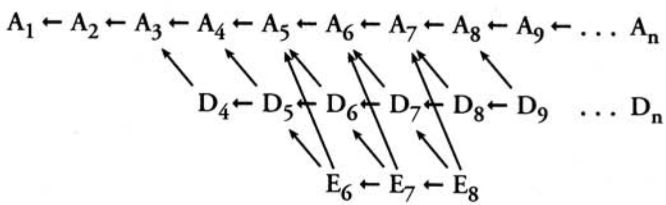
$$\mathbf{C}^n \text{ is the intersection of } X_n \text{ with } \mathbf{C}[g]$$

where X_n is some Lie algebra: A_n, D_n, or E_6, E_7, E_8. And $\mathbf{C}[g]$ is the group algebra of (McKay) group g. Thus we can form a spliced-bundle projection:

$$X_n \times \mathbf{C}[g] \Rightarrow \mathbf{C}^n$$

FIG. A27

THE A–D–E HIERARCHY



In fact, as we have pointed out early in this paper, there is a hierarchy of these *A-D-E* algebras, so that, for instance, we have the projections:

$$E_8 \Rightarrow E_7 \Rightarrow E_6 \Rightarrow D_5 \Rightarrow D_4 \Rightarrow A_3 \Rightarrow A_2 \Rightarrow A_1$$

The structure of this hierarchy derives from the structure of the Coxeter graph of these algebras. The subscripts are the ranks of the algebras and correspond to the number of nodes in the algebra's Coxeter graph. Not only is the algebra of lower rank a subalgebra, but, more important, these projections entail a hierarchy of singularity structure — i.e., the deformations of the higher rank singularity contain the deformations of the lower rank singularity.

The difference between any of the *A-D-E* spliced-bundle schemes and the scheme of this paper is that quite different unified field theories would be entailed — i.e., different from the physical forces we know about.

It is conceivable that the entire *A-D-E* set of schemes is active, and that we have here described only one scheme E_7. If this were so, there would be an infinite set of consciousness structures (*realms*), all intimately tied together by the hierarchical structure of the *A-D-E* classification based on the Coxeter graphs (see Fig. A27).

There are many possible applications of such a hierarchy. Since the actual existence of such a hierarchy of realms is speculative, we shall content ourselves with the observation that it is suggested by the mathematical structure of our unification scheme E_7.

This is the same position that Maxwell found himself in when (in 1864) he unified electricity and magnetism and discovered the electromagnetic theory of light. The mathematics of his unification suggested to him the speculative idea that visible light is only a small part of an infinite spectrum of light frequencies. In due course, radio waves, x-rays and other forms of light verified his speculation.

It may be that there is an infinite set of consciousness realms, hierarchically organized according to the *A-D-E abutment* scheme as depicted in Figure A27. It should be mentioned that the hierarchy of abutments corresponds as well to hierarchy of control in the sense that higher catastrophe structures embed and control lower catastrophe structures (cf. Gilmore, 1981).

This means that, if the hierarchy is actual (not merely mathematical), there is a whole hyperphysical world with its own set of forces above us, the E_8 realm, and also a world directly below us, the E_6 realm.

The connections between these separate realms is via the spaces of consciousness:

$$\mathbf{C}^8 \Rightarrow \mathbf{C}^7 \Rightarrow \mathbf{C}^6$$

because the Dynkin diagrams are nested according to this hierarchy, and the lattices and Weyl groups are embedded according to this hierarchy.

At this point it is useful to mention that there is indeed something special about the three algebras E_8, E_7, E_6. For one thing, the corresponding McKay groups are symmetry groups of the Platonic solids: icosahedron (and dodecahedron), octahedron (and cube), tetrahedron. Perhaps more important for a theory of consciousness is the fact that the root lattices corresponding to these three algebras are generated by error-correcting codes (cf. Conway and Sloane, 1988).

This is a large subject with much connection to other aspects of the theory presented here (including unified field theory). I will just point out that the E_7 lattice is generated by the Hamming-7 code and that the E_8 lattice is generated by the Hamming-8 code. Moreover, *h*, a fundamental invariant polynomial of the *OD* group, is the *weight polynomial* of the Hamming-8 code. Thus there is a connection between E_7 and E_8 via coding theory, in addition to the ones we have already considered. Coding theory is an application of *information* theory, so that it is natural to suppose a connection between coding theory and cognitive aspects consciousness. Needless to say, this is an active area of my own research (cf. Sirag, 1984, 1986).

There is much room for further speculation and comparison with philosophical and mystical ideas. I will mention only one such mystical idea (cf. Chang, 1971):

> Fa Tsang (643–712), the Chinese master of Hwa Yen Buddhism, prepared for the Empress Wu an octagonal room completely covered with mirrors, including the floor and ceiling. In the center he placed an image of the Buddha with a burning torch. He brought the Empress into this room and said (in part):
>
> Your Majesty, this is a demonstration of Totality in the Dharmadhatu. In each and every mirror within this room you will find the reflections of all the other mirrors with the Buddha's image in them. And in each and every reflection of any mirror you will find all the reflections of all the other mirrors, together with the specific Buddha image in each, without omission or misplacement. The principle of interpenetration and containment is clearly shown by this demonstration. Right here we see an example of one in all and all in one — the mystery of realm embracing realm ad infinitum is thus revealed. The principle of simultaneous arising of different realms is so obvious here that no explanation is necessary. These infinite reflections of different realms now simultaneously arise without the slightest effort; they just naturally do so in a perfectly harmonious way. . . .
>
> As for the principle of the nonobstruction of space, it can be demonstrated in this manner. . . . (saying which, he took a crystal ball from his sleeve and placed it in the palm of his hand). Your Majesty, now we see all the mirrors and their reflections within this small crystal ball. Here we have an example of the small containing the large, as well as of the large containing the small. This is a demonstration of the nonobstruction of "sizes," or space.
>
> As for the nonobstruction of times, the past entering the future and the future entering the past cannot be shown in this demonstration, because this is, after all, a static one, lacking the dynamic quality of the temporal elements. A demonstration of the nonobstruction of times, and of time and space, is indeed difficult to arrange by ordinary means. One must reach a different level to be capable of witnessing a "demonstration" such as that. But in any case, your Majesty, I hope this simple demonstration has served its purpose to your satisfaction.

Notice that the suggestion of a need for a "different level" to explain the temporal aspects of the mystical experience is a suggestion that reality is hyperdimensional. Moreover, we have found in the catastrophe and wave front evolution structure a dynamical description of this hyperdimensional realm.

Let us quote from the *Hwa Yen Sutra* itself (cf. Chang, 1971):

> The Indescribable-Indescribable
> Turning permeates what cannot be described. . . .
> It would take eternity to count
> All the Buddha's universes.
> In each dust-mote of these worlds
> Are countless worlds and Buddhas....
> An excellent mathematician could not enumerate them
> But a Bodhisattva can clearly explain them all. . . .

Perhaps a seventh-century mathematician could not cope with the mystic vision, but Fa Tsang appears to have had a method that can be enlarged upon significantly by today's mathematics. Moreover, the mathematics of singularity theory and the *A-D-E* hierarchy is rather recent, and therefore many aspects of this hierarchy remain to be discovered. I will close with a speculation of one of the most perceptive mathematicians who has contributed to the study of the *A-D-E* hierarchy, V. I. Arnold (1986):

> At first glance, functions, quivers, caustics, wave fronts and regular polyhedra have no connection with each other. But in fact, corresponding objects bear the same label not just by chance: for example, from the

icosahedron one can construct the function $x^2 + y^3 + z^5$, and from it the diagram E_8 and also the caustic and wave front of the same name.

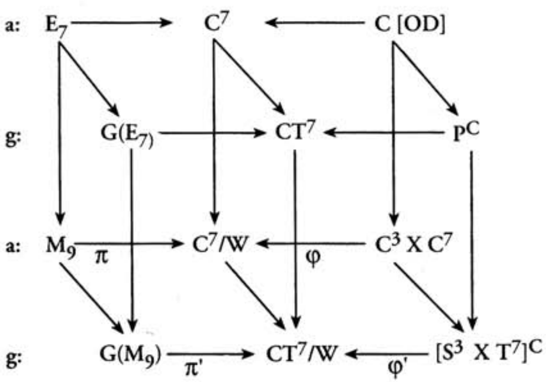
To easily checked properties of one of a set of associated objects correspond properties of the others which need not be evident at all. Thus the relations between all the *A, D, E* classifications can be used for the simultaneous study of all simple objects, in spite of the fact that the origin of many of these relations (for example, of the connections between functions and quivers) remains an unexplained manifestation of the mysterious unity of all things.

Coda

In order to clarify the plethora of algebraic and geometrical relationships entailed in our overall scheme, let us lay out Figure A28.

FIG. A28

E_7 MAPPING BOX DIAGRAM



a = algebra level
g = group level

Note that the diagram forms a box divided into two compartments with the following features:

1. The back panel maps algebras to each other
2. The front panel maps groups to each other
3. The upper panel is projected down to the lower panel
4. The left compartment contains the Lie (algebra and group) structures
5. The right compartment contains the finite group-algebra structures
6. The panel dividing the box into the two compartments contains the intersections between the Lie structures and the finite group-algebra structures.

In particular:

ALGEBRA STRUCTURES

E_7 is the 133-d (complex) Lie algebra.

C[*OD*] is the 48-d (complex) *OD* group algebra.

$\mathbf{C}^7$ is the intersection of E_7 with **C**[*OD*].

$\mathbf{C}^7$ is also the Cartan subalgebra (maximal commutative subalgebra) of E_7.

$\mathbf{C}^7$ is also a subalgebra of $\mathbf{C}^8$, which is the center of **C**[*OD*].

M^9 is the 9-d (complex) E_7 catastrophe bundle.

$\mathbf{C}^7/W$ is the critical space of the E_7 catastrophe bundle.

$\mathbf{C}^7 \times \mathbf{C}^3$ is the **C**[*OD*] subalgebra whose Lie group is $[T^7 \times S^3]^{\mathbf{C}}$.

GROUP STRUCTURES

$G(E_7)$ is the 133-d (complex) Lie group whose Lie algebra is E_7.

P^C is the 48-d (complex) Lie group which consists of the set of invertible elements of **C**[*OD*].

P is the 48-d (real) manifold consisting of all unitary elements of **C**[*OD*] — i.e., P is the unitary Lie group in **C**[*OD*].

P is also the maximal compact subspace of **C**[*OD*].

$\mathbf{C}T^7$ is the intersection of $G(E_7)$ and P^C.

$\mathbf{C}T^7$ is also the Cartan subgroup of $G(E_7)$.

$\mathbf{C}T^7$ is also a (complex) 7-torus, which is a maximal torus in $G(E_7)$ — i.e., the maximal commutative subgroup of $G(E_7)$.

$G(M^9)$ is the (complex) 9-d catastrophe bundle embedded in the Lie Group $G(E_7)$.

$\mathbf{C}T^7/W$ is the Lie-group view of the critical space $\mathbf{C}^7/W$ of the E_7 catastrophe bundle.

$[T^7 \times S^3]^{\mathbf{C}}$ is the (complex) 10-d superstring spacetime, which is a base space of the (complex) principal fiber bundle P^C.

MAPPINGS

$G = \exp(A)$ is the *exponential map* which transforms Lie algebra structures A into Lie group structures G.

$A = \log(G)$ is the *logarithmic map* which transforms Lie group structures G into Lie algebra structures A.

π = the fiber bundle projection from M^9 to $\mathbf{C}^7/W$, with $\mathbf{C}^2/OD$ as the fiber at the origin of $\mathbf{C}^7/W$.

π' = the fiber bundle projection from $G(M^9)$ to $\mathbf{C}T^7/W$, also with $\mathbf{C}^2/OD$ as the fiber at the origin of $\mathbf{C}T^7/W$.

Φ = the fiber bundle projection from $\mathbf{C}^7 \times \mathbf{C}^3$ to $\mathbf{C}^7$, with $\mathbf{C}^3$ as the fiber at the origin of $\mathbf{C}^7$.

Φ' = the fiber bundle projection from $[T^7 \times S^3]^{\mathbf{C}}$ to $\mathbf{C}T^7$, with $(S^3)^{\mathbf{C}}$ as fiber. (Note: $(S^3)^{\mathbf{C}} = S^3 \times S^3$.)

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Glossary

Algebra. the study of the structure of numbers and the systems of entities abstracted from and generalized upon this structure. The most basic structure in this study is the *group* From groups one can define *rings,* and from rings and groups one can define *modules,* and *vector spaces* and *associative algebras.* We can construct the following hierarchy of structures:

Semigroup. $\{S, \bullet\}$ closure and associativity defined on S with composition of elements under $\sum$.

Group. $\{G, \bullet\}$ identity element and inverses also defined on G.

Commutative group. $\{G, +\}$ commutativity ($a + b = b + a$) defined on G. Note: We conventionally regard composition of elements of a commutative as addition.

Ring. additive group $\{R, +\}$; multiplicative semigroup (i.e., closure and associativity) $\{R, \bullet\}$.

Module. Scalar ring $\{R, \bullet\}$, i.e. only the multiplicative aspect of the ring is used; vector commutative group $\{V, +\}$; scalar multiplication distributes over vector addition : $r \bullet (v_1 + v_2) = r \bullet v_1 + r \bullet v_2$.

Vector space. the multiplicative structure of the ring (with zero omitted) becomes a *commutative group* $\{R-0, \bullet\}$; vector commutative group $\{\mathbf{V}, +\}$.

Associative algebra. multiplicative semigroup (with composition denoted by •) in the vector space $\{\mathbf{V}, +, \bullet\}$.

Note: Mathematicians used to call *algebra* as defined here *hypercomplex numbers,* or *abstract algebra.* Now they simply call it algebra. *High school algebra* is essentially the study of the associative algebras **R** (the set of real numbers) and $\mathbf{C} = \mathbf{R} + \sqrt{-1}\mathbf{R}$ (the set of complex numbers) because every *polynomial* in one unknown has a root (or solution) in **C**. The multiplicative structure of **C** (if we leave out 0) is a commutative group. For these reasons, **C** is a very special case of an algebra.

Both **R** and **C** are examples of commutative algebras. The first noncommutative algebra was discovered by Hamilton in 1842 and was called by him the *quaternions,* which we designate **Q**.

An associative algebra is also called a *linear algebra,* a mathematical structure *A* which closely imitates the essential features of the real number line **R**. These features are: an additive structure on *A, a multiplicative structure on A,* and a distributive relationship between these two structures. The additive structure is provided by requiring *A* to be a *vector space,* which by definition is a commutative group. The multiplicative structure is not necessarily a group, but must obey the properties of closure and associativity. Thus not every element of *A* has a multiplicative inverse, nor does the multiplicative structure necessarily have an identity element. Moreover, the additive identity element, *zero,* when used multiplicatively has the effect of making any element of *A* into zero: $a0 = 0a = 0$, for all elements *a* of *A*.

Examples of associative algebras are: the real numbers **R**; the complex numbers **C**; the *total matrix algebra M (n),* the set of all $n \times n$ matrices. These algebras are called *associative algebras* because their multiplicative structure obeys the associative law: $a(bc) = (ab)c$.

Any associative algebra can be made into a *Lie algebra* by defining a new multiplicative structure based on the underlying associative multiplicative structure. Conventionally, this new structure is called the *Lie bracket:* $[a,b] = ab - ba$, where ab and ba are defined by the underlying associative multiplication structure. This is usually done via matrix representations of the underlying associative algebra. It is a basic theorem of algebra that any finite dimensional associative algebra is equivalent to a matrix algebra and thus can be faithfully represented by a matrix algebra. The dimension of the algebra is, of course, the dimension of the vector space which provides the additive structure for the algebra. For example, the dimension of the matrix algebra $M(n)$ is n^2, the number of components of an $n \times n$ matrix. This is obvious to anyone who remembers that two matrices are added by componentwise addition.

Basis. a set **B** of vectors in a vector space such that any vector in the space is equivalent to a sum over **B**. The number of vectors in **B** is equal to the *dimension* of the vector space. Thus, in an *n*-dimensional vector space **V**, if **B** is the set $\{b_1, b_2, \ldots, b_n\}$, then any vector **v** of **V** can be written as $\mathbf{v} = c_1b_1 + c_2b_2 + \ldots + c_nb_n$, where the set $\{c_1, c_2, \ldots, c_n\}$ consists of numbers which are called *coordinates.* Thus a basis provides a *coordinate system* for the vector space.

Note: When a vector is regarded as an ordered set of numbers, these numbers are the coordinates and are usually called the *components* of the vector.

Bell's Theorem. a statement proved by John S. Bell (1964) that if a quantum state vector corresponds to some objective reality, that reality must have nonlocal effects.

Consciousness. awareness — not necessarily self-awareness. Awareness entails *perceptual* and *cognitive* aspects of reality. Self-awareness entails also the *volitional* aspect. (For example, in a state of muscular paralysis due to anesthesia, it is possible to have awareness without self-awareness.)

Component. see Basis.

Coordinate system. see Basis.

Deformation. a transformation of a mathematical structure which shrinks, twists or otherwise changes the structure without tearing.

Dimension. the number of degrees of freedom in a space. The number of coordinates necessary to locate a point in a space. The coordinates are not necessarily rectangular, e.g., the surface of the earth is 2-dimensional; customarily it has two coordinates called longitude and latitude.

Dual space. the set of *linear functions* of a vector space. An n-dimensional space has an n-dimensional dual space. The dual space of a dual space is the original vector space.

Eigenvalue. the numerical solution x to the operator equation:

$$Av = xv$$

where A is the operator, and v is a vector called the *eigenvector* solution to this operator equation. Geometrically, this means that A is acting upon the vector v and does not change its direction but only its length; this length change is by the factor x. E.g., if x is 2, v is doubled in length by A; if x is ½, v is cut in half by A; if x is 1, v doesn't change at all.

Usually A is represented by an $n \times n$ matrix where n is a dimension of v, i.e., the number of components in v. In many cases, the solutions can be found by transforming the matrix M representing A into *diagonal form;* i.e., M is transformed into $\text{diag}(M) = gMg^{-1}$ (where g is an $n \times n$ matrix representing an element of a group, and g^{-1} is the inverse of g). In $\text{diag}(M)$ all components are zero except for the components of the primary diagonal. These components are the eigenvalues of M. It should be noted that M and gMg^{-1} represent the same transformation, but with differing coordinate systems, on the vector space in which the v live. In fact, the *eigenvectors* v are the basis vectors of the coordinate system in which $\text{diag}(M)$ is defined.

Eigenvector. a vector solution v to the operator equation described under *eigenvalue.* In general, there are several eigenvector solutions to an operator equation. The number n of solutions is the dimensionality of the space the vectors live in. The matrix representing the operator is an $n \times n$ matrix. If there are n eigenvectors belonging to an operator acting on an n-dimensional space, the n eigenvectors form a *basis* for this space.

Field. a smooth assignment of some type of mathematical object to each point of some space; e.g., a *scalar* field assigns a scalar (a number) to each point of a space; a *vector* field assigns a vector to each point of a space.

Force field. (also called a *gauge field*) a smooth assignment of an element of a Lie algebra to each point of spacetime. The standard correspondence of Lie algebras to forces is as follows (using Lie group labels): *U(1),* electromagnetism; *SU(2),* weak force; *SU(3),* strong (color) force.

Function. a mapping from an n-dimensional space to a 1-dimensional space. This means that n numbers go into the function and one number comes out; e.g., given the equation $z = x^2 + 3y$, we can say that z is a function of x and y (see Mapping).

A *linear function* is an additive entity in the sense that the sum of two linear functions is also a linear function on the space on which the function is defined.

Hyperspace. a space of more than three dimensions.

Lie algebra. a nonassociative *algebra* which obeys two extra rules:

1. Anticommutativity: $a \bullet b = -b \bullet a$

2. The Jacobi identity: $a \bullet (b \bullet c) = (a \bullet b) \bullet c + b \bullet (a \bullet c)$, which replaces the associative law $a(bc) = (ab)c$.

Since most Lie algebras can be faithfully represented by sets of matrices, the Lie product $a \bullet b$ can be faithfully represented via the Lie bracket: i.e., $a \bullet b = [a, b] = ab - ba$, where the product ab is simply the ordinary product of two matrices. In fact, any matrix algebra is an associative algebra which can be made into a Lie algebra by defining Lie brackets on the underlying matrix algebra.

The building blocks of Lie algebras are called *simple Lie algebras.*

Mapping. from an n-dimensional space N to an x-dimensional space X is a rule which assigns to each element of N a unique element of X. Note that N and X may be the same space. A *function* is a special case of a mapping, in which X the space being mapped to is 1-dimensional.

Matter field. a smooth assignment of a complex vector (of appropriate dimension) to each point of spacetime. The vector space is acted upon by the Lie group associated with the force field or fields involved.

Mind. the realm of all mental events (such as beliefs, memories, images, thoughts) including *subconscious* events.

Nonlocality. independence from spatial and temporal constraints. A nonlocal effect is instantaneous and undiminished over distance. Note: A nonlocal effect does not necessarily transfer energy or information from one point to another. In particular, transfer of energy or information is not implicit in the nonlocality referred to in *Bell's Theorem*. The understanding of a nonlocal effect which does not transfer energy or information is an open problem in physics.

Orbit. the set of points in a space selected by the action of all the elements of a group on a single point of the space. If the group is continuous (a Lie group), the orbit will be continuous; e.g., *SO(3)* acting on a single point of ordinary 3-d space creates an ordinary orbit — a circle. If the group is discrete, the orbit of a point will be a discrete set of points; e.g., the orbit of a reflection group acting on a point between the mirrors (i.e., in a reflection chamber) is a point in each reflection chamber; thus the number of points in such an orbit is equal to the number of elements in the reflection group.

Operator. a mathematical entity that transforms a space; e.g., on a *vector space,* an operator may rotate, stretch, translate or perform some combination of these transformations on the vectors of the space. An operator which acts on an n-dimensional space is represented by an $n \times n$ *matrix.*

Parameter. a variable which selects members of a family of structures. For example, a and b are parameters in the equation for a straight line: $y = a + bx$; a selects the point where the line intercepts the y axis; b selects the slope of the line. In this paper, $t_1, \ldots, t_7$ are parameters which select a fiber of the catastrophe manifold X^9; these fibers are called the *deformations* of the identity fiber C^2/OD (see Unfolding).

Quaternions. an ordered pair of complex numbers, or equivalently, an ordered quartet of real numbers. Although quaternions are not commutative, they do have multiplicative inverses, which means that the set of quaternions (with zero omitted) forms a group under multiplication. If we think of the quaternions as an ordered quartet of real numbers, we will regard the quaternions as forming a 4-d vector space. The set of all quaternions of unit length form a sphere S^3, which is also a Lie group with the usual label *SU(2).*

Realm. reality structure. In this paper we use the word realm to refer to the combination of a universal mind with a universal body. Thus there is a hierarchical spectrum of realms corresponding to the *A-D-E* hierarchy.

Space. a set with a *continuous infinity* of elements. These elements are called *points.* The simplest space is a line.

Unfolding. a family of functions F which contains a particular function f is called an unfolding of f. If F contains all the functions close to f, the folding is called a *universal unfolding* (see Deformation).

Unified field theory. a theory combining two or more force fields into a single *force field.* The theory must also give an adequate description of the *matter fields* which feel the force fields entailed in the unified theory, because matter fields interact with each other by exchanging force fields.

Universal body. the physical realm in all its aspects. Individual bodies are substructures of the universal body. In this paper, the universal body is identified with the group algebra, C[*OD*]. It is possible that there is a universal body corresponding to each of the McKay-group algebras, so that there would be a hierarchical spectrum of universal bodies matching the spectrum of universal minds. One could view the entire spectrum of universal bodies as a Supreme Body.

Universal consciousness. that consciousness of which individual consciousnesses are substructures. In this paper, universal consciousness is identified with C^7, the intersection of the Lie algebra E_7 with the group algebra C[*OD*].

Universal mind. the mental realm in all its aspects, including the *subconscious.* Individual minds are viewed as substructures within the universal structure. In this paper, the universal mind is described as the 133-dimensional E_7 group. It may be necessary to extend this description to the infinite-dimensional Kac-Moody group $^\wedge E_7$. Also, it is possible that there is a universal mind corresponding to each of the *A-D-E* Lie groups, so that there would be a hierarchical spectrum of universal minds comprising a Supreme Mind.

Vector. a set of ordered numbers called components. The number of these components is the dimension of the vector. A real vector has real numbers as components; a complex vector has complex numbers as components. Since a complex number is an ordered pair of real numbers, an n-dimensional complex vector can be regarded as a $2n$-dimensional real vector.

Geometrically, a vector is an entity having magnitude and direction. We can recover the algebraic definition, if we imagine the vector tied to the origin of a coordinate system. Then the components of the vector are the coordinates of the tip of the vector.

Vector space. a space whose elements, called vectors v, form a commutative group, which by convention is considered additive so that the 0-vector is the identity element. It must also be possible to multiply v by a scalar s (i.e., a number); a law of *distribution* holds for this scalar multiplication:

$$sv_1 + sv_2 = s(v_1 + v_2).$$

We say that a vector space is defined over the set of scalars. If this set is the *real numbers,* we call the space a real vector space. If this set is the *complex numbers,* we call the space a complex vector space. Since a complex number is an ordered pair of real numbers, it is always possible to consider an n-dimensional complex vector space as a $2n$-dimensional real vector space.

Wave function. a function ψ which provides a description of a quantum system. It is a function of the observables of the system in such a way that the square of the amplitude of the wavefunction is the probability for seeing a particular value of an observable. For example, if we consider only the observable of position x, then the square of the amplitude of $\psi(x)$ is the probability of seeing the system (say, a particle) at x. The word "wavefunction" is really a misnomer in the sense that most wavefunctions are not wavelike. Like waves, however, wavefunctions are additive: i.e., two wavefunctions (for the same system) can be added to produce another wave function. An alternative view of the wavefunction is a vector called the **state vector** of the system. Whenever a system is observed a particular aspect of the wavefunction, called an eigenfunction, is observed; alternatively a projection of the state vector, called an eigenvector, corresponds to an observed state.