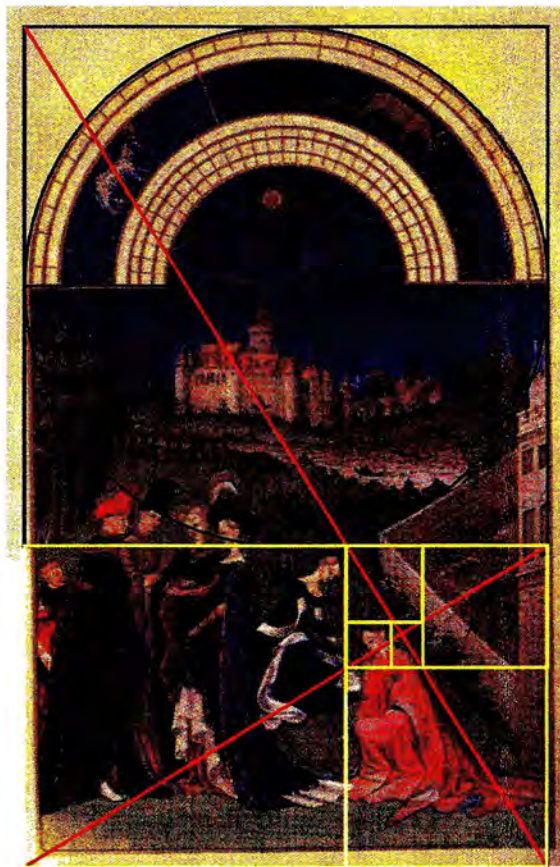


Constructing The Universe Activity Books
Volume 3

Fibonacci Numbers and The Golden Mean

Explore Harmony in Mathematics, Nature and Art



Michael S. Schneider

Author of

*A Beginner's Guide To Constructing The Universe:
The Mathematical Archetypes Of Nature, Art And Science*
(HarperPerennial)

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<http://www.constructingtheuniverse.com>

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Cover illustration:

"April" from *Les Très Riches Heures* of the Duc du Berry.

Illumination on parchment, 1405, Bibliothèque Nationale, Paris.

Geometric subdivision of the Golden Rectangle indicates the area of visual interest. See page 136.

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Introduction

After I had written *A Beginner's Guide To Constructing The Universe* I saw that Chapter Five was the largest of the ten. This was because the Pentagon is so closely related to the Fibonacci numbers, Golden Mean, Golden Spiral and nature that they required a great deal of space. In the dozen years since then I've observed and learned even more, in addition to the increased interest in these topics in pop culture from advertising, television and novels to movies and even religion. But the topics deserve further expression, and so this Activity Book was created.

This book's approach to the Fibonacci numbers and Golden Mean is active involvement, not just reading. It's for the self-learner and for teachers of children and adults to explore these topics through hands-on activities. It's hoped that it will guide people to become interested in "number theory", the wonderful connections 'behind' and among numbers. Artists, craftspeople, architects and designers will see how the Fibonacci numbers and Golden Mean have been applied by creative people across the centuries and cultures with most masterful excellence.

The main idea behind the Fibonacci numbers and Golden Mean is self-referential harmony. It builds upon itself and weaves through numbers, the designs and processes of nature, and proportions of great art. It's about interconnected unity, wholeness and stability through dynamic balance amidst diversity and change. The idea of harmony weaves through every page of this Activity Book. It's hoped that it provides a concrete basis for actually seeing and putting energy towards harmony and balance in our own lives, not just imagining it, especially during these times of discord and change.

This book has four parts:

- * **The Fibonacci Numbers**, demonstrating the underlying principles of this harmony;
- * **Fibonacci Numbers In Nature**, showing where, how and why these principles are expressed in living growth;
- * **Geometry of the Golden Mean**, teaching geometric skills for exploring patterns in nature and art;
- * **The Golden Mean In Art**, seeing how the Golden Mean was applied to art, and with hands-on examples to explore.

For a very long time, mathematics was considered to be a sacred language, not invented by anyone but discovered, eternally consistent and always true. Thus, it was used to design sacred works in every culture because it speaks this language of truth. Most of the geometric schemes underlying the art in this book have not been seen this way since the day they were created in antiquity.

The tools needed to explore this book include a good geometric compass and the skill to turn a circle whose end meets its beginning. A straightedge is also necessary, and while a ruler is good (and occasionally used) no markings on it are actually required. And colored pencils are highly recommended. If they can fit in the compass that's great, since it will make lines drawn on the art and other images easier to see. And colored pencils will be useful to shade the geometric patterns seen and created.

Finally, I hope that you are informed and inspired by this book, and enjoy it too!

Michael Schneider
San Anselmo, California

August 13, 2005 (8/13/05, quite coincidentally Fibonacci numbers!) © 2005 Michael S. Schneider
Constructing The Universe Activity Book Vol. 3

The Fibonacci Numbers

The Fibonacci numbers were named in the 20th century after the nickname of the 13th century merchant and mathematician Leonardo of Pisa, known as the "son of Bonaccio" or "filius Bonaccio", contracted to "Fibonacci". In his 1202 book *Liber Abbaci*, "The Book of Calculation" -- which introduced the ten digits 0123456789 to Europe -- he included a puzzle he learned in his travels.

The Rabbit Puzzle (after *Liber Abbaci* 1202)

It's January 1st and you have a newborn pair of rabbits, male and female.

If all rabbits...

...become mature in one month,

...mate immediately,

...then *gestate* for 1 month,

...and then give birth to *another pair like themselves*,

...and they, and every pair, continually do the same...

...and none ever die...

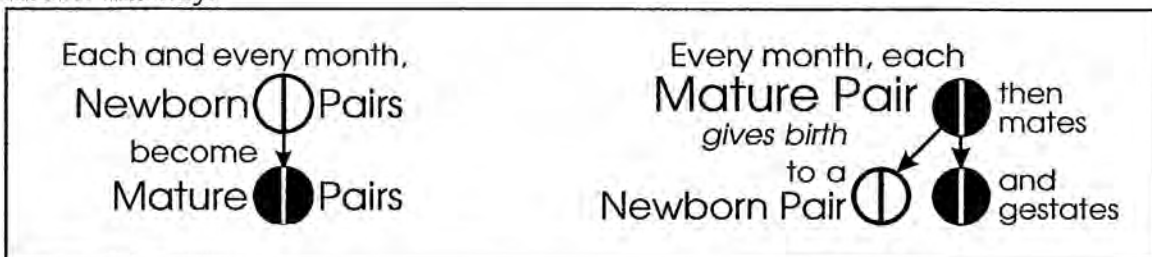


Leonardo of Pisa
"Fibonacci"
1175? - 1250?

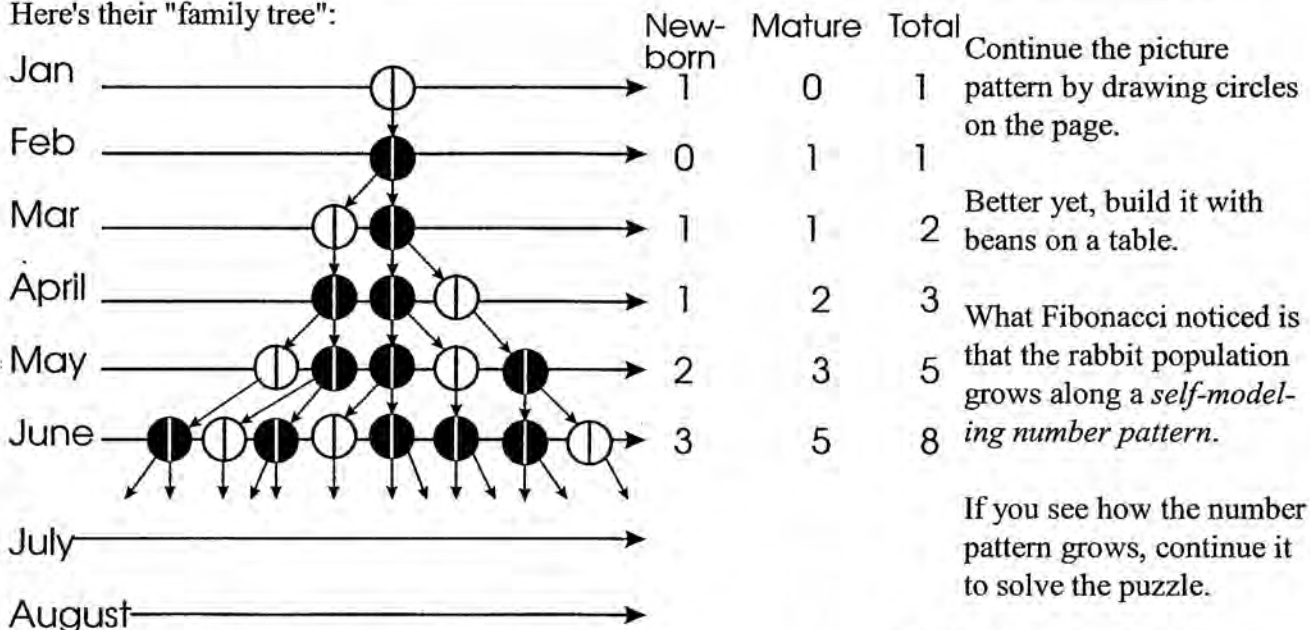
How many pairs of rabbits will there be one year later?

You can solve this without numbers if you have two types of beans or coins.

Think of it this way:



Here's their "family tree":



Simply put, the Fibonacci numbers are a sequence of numbers, beginning with 0 and 1, "nothing" and "something". Each two numbers add to make the next number, and so on, forever.

Fibonacci Numbers

$$\begin{aligned} \textcircled{0} + \textcircled{1} &= \textcircled{1} \\ 1 + 1 &= \textcircled{2} \\ 1 + 2 &= \textcircled{3} \\ 2 + 3 &= \textcircled{5} \\ 3 + 5 &= \textcircled{8} \\ 5 + 8 &= \textcircled{13} \end{aligned}$$

- 0
- 1
- 1
- 2
- 3
- 5
- 8
- 13
- 21
- 34
- 55
- 89
- 144
- 233
- 377
- 610
- 987
- 1597
- 2584
- 4181
- 6765
- 10946
- 17711
-

and so on...

Without looking at the list on the right, continue the Fibonacci number pattern above.

Sometimes it's easier to appreciate its growth with beans rather than with abstract numerals.

Start with zero and one bean, then count the number of beans in the last two piles, and put that many in the next pile, and so on Look at how it grows, appearing as a kind of waterfall or stairway. What do you see?

"He therefore who wishes to rejoice without doubt in regard to the truths underlying phenomena must know how to devote himself to experiment."
 — Roger Bacon (1220-1292, English philosopher and scientist)



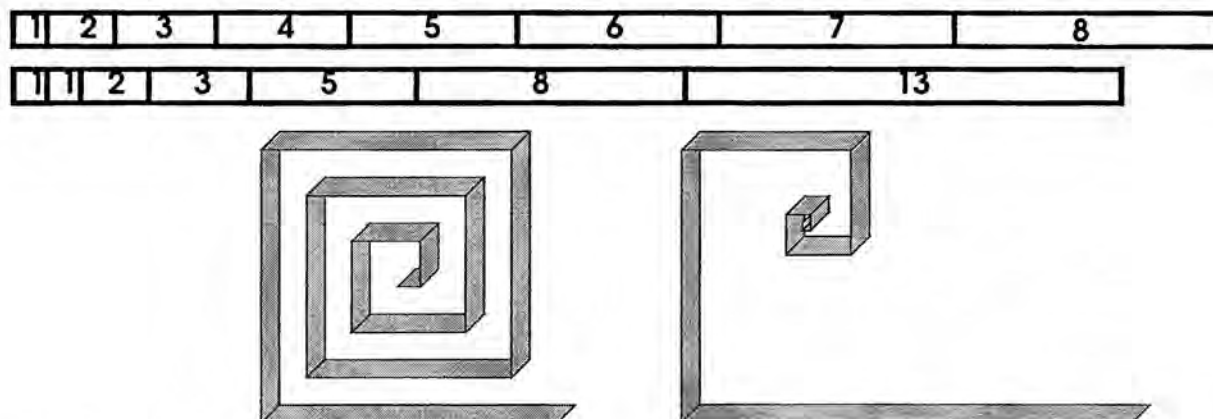
Expand this Fibonacci pattern of dots, or build it with beans

You can make number patterns visible another way.

Make long strips of paper. On the first, measure off 1 inch (or centimeter, or finger width), then 2 inches, 3 inches, 4 inches and so on, by the **ordinary counting numbers**. Fold the line between each measurement at a right angle to create a spiral. Notice how it grows, with the same distance between each coil.

On another strip, measure **Fibonacci number lengths**, as far as your strip will allow, and fold it. Notice how its spiral grows, getting larger and larger with each coil further apart.

Most spirals in nature, from embryos and seashells to hurricanes and galaxies, grow this way.



Try building these spirals with beans, or continue the dots on this page:



The Fibonacci numbers show us how growth can occur by accumulation, by parts built upon earlier parts.

Since each number in the Fibonacci series is related to all other numbers before and after it, we see that interrelated parts can fit together and create a harmoniously balanced whole.

This is a valuable method used throughout nature and by artists to compose whole, harmonious works.

First, we'll see more about the way these numbers weave wondrously together.

Weaving Fibonacci Number Patterns

Too often math is taught using abstract symbols like numerals and equations before we've had a chance to get our hands on the physical objects which those numerals and equations represent. There's something about doing activities with your hands that teaches something not available by just mentally knowing that it's true.

It wasn't always this way. Long ago, in a variety of locations and times, mathematics was introduced in a concrete manner, using pebbles. Numbers were treated as groups of pebbles which could be combined to add, subtract, multiply and divide, revealing visible patterns among them. The Greeks used limestone pebbles, called *calx* (since they were calcium) which eventually gave us our words "calculate" and "calculus". Manipulating pebbles to see patterns led to an understanding of "number theory", the principles behind the numerical relationships. What we call "arithmetic" today is not what it was called in ancient Greece where a distinction was made between *logistic* (ordinary computation) and *arithmetic* (number theory).

Sometimes the groups of pebbles were arranged as geometric forms, such as triangles, squares or rectangles. These are known as "figurate" or "polygonal" numbers. They were known to the ancient Chinese, Greeks and in Moorish Spain and North Africa, and appeared in Fifteenth century European arithmetic books. But they were of special interest to the Greeks in the teachings of Pythagoras (circa 520-500 BCE).

Two consecutive triangular numbers
always add to make a square number.

$$\begin{array}{c} \bullet \\ \bullet \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \bullet \\ \bullet \bullet \bullet \end{array} = \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{array}$$

$$3 + 6 = 9$$

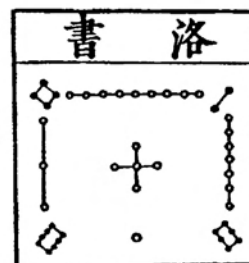
An equation is really the abstract form of a visible pattern, an abstract expression of a mathematical truth. We will explore the Fibonacci numbers through pebble patterns since there are many interesting equations which involve the Fibonacci numbers.

Each example of a Fibonacci idea will be presented four ways:

- * **In ordinary language of words**
- * **In the language of mathematical equation symbols**
- * **As examples with numbers**
- * **As patterns of beans**

This way, the ideas are not abstract, but can help us gain deep insight into the way numbers interrelate, "number theory". Look at each equation and build it with beans. After grasping the bean patterns, try to understand how it relates to the mathematical symbols. If you actually do these activities with your hands you'll surely gain insights into the marvelous interrelationships among Fibonacci numbers and better appreciate when they show up in the patterns of nature.

There are many interesting properties to the Fibonacci numbers, and all of these examples will show its remarkable interrelationships among parts and whole.



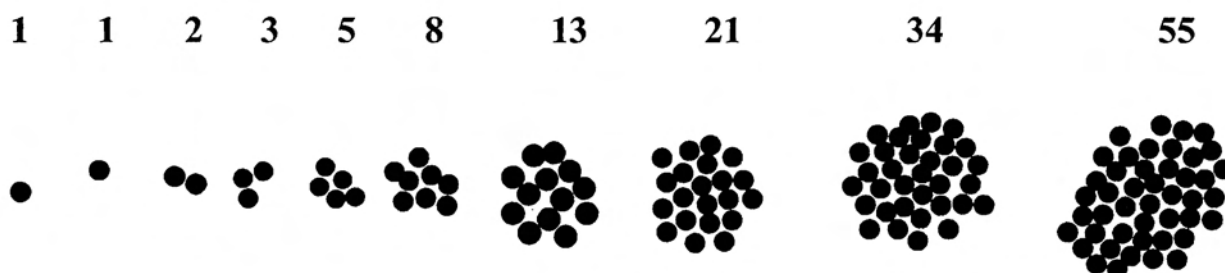
Chinese dots (the *Lo Shu*, or Lo River Document, above) depict a Magic Square (below) where each column, row and diagonal adds to 15.

4	9	2
3	5	7
8	1	6

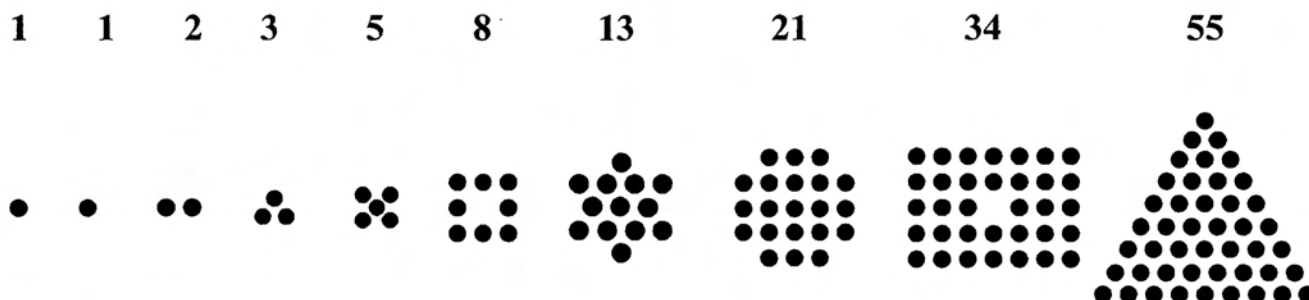
We'll look at the Fibonacci numbers both as abstract number symbols and as physical beans which we can manipulate in patterns. Each will help you to understand the other.

First, lets get our hands on this growing series of Fibonacci numbers using beans.

Just start with zero and one bean. The next pile is made by adding them: $0 + 1 = 1$ bean again. And the next is equal to these last two piles ($1 + 1 = 2$), and so on. Build the Fibonacci series by adding the same number in each two piles together to make each the next pile. Count the beans to be sure that they're Fibonacci numbers.



It's also fun to find symmetric geometric shapes which express each pile of Fibonacci numbers.



Can you find other symmetric patterns for these (and other) Fibonacci numbers?

Mathematicians abstractly symbolize the sequence of Fibonacci numbers using subscripts. Each Fibonacci number is represented by the letter "F" with a subscripted number in the ordinary sequence of consecutive integers 0, 1, 2, 3, 4, 5, 6, 7, 8, ...

$$F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, \dots$$

The first Fibonacci number, 0, is symbolized by

$$F_0 = 0$$

The next Fibonacci number, 1, is symbolized by

$$F_1 = 1$$

The following Fibonacci number, also 1, is symbolized by

$$F_2 = 1$$

The next Fibonacci number, 2, is symbolized by

$$F_3 = 2$$

... and so on.

This provides a convenient way to speak about Fibonacci numbers along their consecutive sequence. Continue writing the Fibonacci numbers (and their subscripts) in the blanks below.

0	1	1	2	3	5	8	13	21	34	55	89	144	233
F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}

377	610	987	_____	_____	_____	_____	_____	_____	_____	_____	_____	_____
F_{14}	F_{15}	F_{16}	F	F	F	F	F	F	F	F	F	F

It will be helpful if you learn the subscripts for the first ten Fibonacci numbers.

The Fibonacci rule can easily be stated in words:

Begin with 0 and 1. Each two numbers add together to make the next.

And this is how a mathematician uses abstract symbols to state this rule:

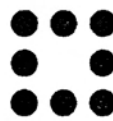
$$\underbrace{F_n + F_{n+1}} = F_{n+2}$$

Which means, in ordinary language:

Two consecutive Fibonacci numbers added together...

...Make the next Fibonacci number (for *any* value of "n")

In the particular case of $F_4 + F_5 = F_6$ or $3 + 5 = 8$, using beans we see :

	+		=	
3	+	5	=	8

Now let's use these letter-and-number symbols, and beans, to explore Fibonacci number ideas.

Dividing Fibonacci Numbers

Fibonacci numbers weave together in clear patterns. We can find them if we ask the question "Do Fibonacci numbers *divide into* one another?" The answer is yes, but only if you know the secret!

Here's the rule:

If any number called "n" divides neatly into another number "m", then F_n divides into F_m .

To "divide neatly" means to split into equal parts with none left over.

This means that since, say, 4 divides into 8, then F_4 divides into F_8 ; that is, 3 divides neatly into 21. Take the pile F_8 (21 beans) and split it into 3 equal piles with none left over.

Try this: Choose any two numbers where one of them divides neatly into the other, and see if the larger pile of beans it represents (F_m) can be divided into F_n equal parts.

Here's another question: Which Fibonacci numbers are *even numbers*?

Notice that the lowest even term, $F_3 (= 2)$ divides evenly into all the other even numbers: that is, F_3 divides into $F_6 (= 8)$, into $F_9 (= 34)$, into $F_{12} (= 144)$, and so on, all Fibonacci numbers having subscripts which are multiples of three ($F_3, F_6, F_9, F_{12}, F_{15}, F_{18}, F_{21}, \dots$) are even numbers.

Thus every *third* Fibonacci number is *even*, in this pattern:

odd odd **even** odd odd **even** odd odd **even** odd odd **even** ...

Fibonacci numbers: 1 1 2 3 5 8 13 21 34 55 89 144 ...

So every third pile of Fibonacci number beans can be split into two equal piles with no beans left over.

We can broaden our vision to see that **every n^{th} Fibonacci number is a multiple of F_n .**

$F_0 =$	0	No Fibonacci number is divisible by 0.		
$F_1 =$	1	divides every	1 st	(consecutive) Fibonacci number
$F_2 =$	1	also divides every	2 nd	Fibonacci number
$F_3 =$	2	divides every	3 rd	Fibonacci number (the even numbers)
$F_4 =$	3	divides every	4 th	Fibonacci number
$F_5 =$	5	divides every	5 th	Fibonacci number
$F_6 =$	8	divides every	6 th	Fibonacci number
$F_7 =$	13	divides every	7 th	Fibonacci number
$F_8 =$	21	divides every	8 th	Fibonacci number
$F_9 =$	34	divides every	9 th	Fibonacci number
$F_{10} =$	55	divides every	10 th	Fibonacci number
$F_{11} =$	89	divides every	11 th	Fibonacci number
$F_{12} =$	144	divides every	12 th	Fibonacci number ... and so on, forever.

We can translate some of these mathematical statements by creating and dividing Fibonacci piles of beans. For example, we can take every pile whose subscript can be divided by $F_5 (= 5)$, say, and thus divide piles F_{10} , F_{15} and F_{20} each into 5 equal piles (with none left over).

These patterns of divisibility reveal how the Fibonacci series weaves wonderfully through itself.

Adding Consecutive Fibonacci Numbers

Think about this rule:

Adding unity (one) to the sum of the consecutive Fibonacci numbers, starting from the beginning, will always equal the second Fibonacci number after where you stopped.

For example, the sum of the terms from F_0 to F_8 (from 0 to 21), plus one, equals F_{10} (55):

$$0 + 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 \overset{+1=}{\curvearrowright} 34 \quad 55 \quad 89 \quad 144 \dots$$

In mathematical symbols: $F_1 + F_2 + F_3 + \dots + F_n + 1 = F_{n+2}$

Look for the pattern and continue it below:

F_0	$= 0$	$+1$	$= 1$
F_1	$= 1$	$+1$	$= 2$
$F_1 + F_2$	$= 1 + 1$	$+1$	$= 3$
$F_1 + F_2 + F_3$	$= 1 + 1 + 2$	$+1$	$= 5$
$F_1 + F_2 + F_3 + F_4$	$= 1 + 1 + 2 + 3$	$+1$	$= 8$
$F_1 + F_2 + F_3 + F_4 + F_5$	$= 1 + 1 + 2 + 3 + 5$	$+1$	$= 13$
$F_1 + F_2 + F_3 + F_4 + F_5 + F_6$	$= 1 + 1 + 2 + 3 + 5 + 8$	$+1$	$= 21$
$F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7$	$= 1 + 1 + 2 + 3 + 5 + 8 + 13$	$+1$	$= 34$

You can also do this with beans.

Examine the expanding pattern below. You might want to write the number near each group. Fill in the blank spaces and continue the pattern.

•		$+ \bullet = \bullet \bullet$
• + •		$+ \bullet = \bullet \bullet \bullet$
• + • + •		$+ \bullet = \bullet \bullet \bullet \bullet$
• + • + • + • + •		$+ \bullet = \bullet \bullet \bullet \bullet \bullet$
• + • + • + • + • + • + •		$+ \bullet =$
• + • + • + • + • + • + • + • + •		$+ \bullet =$
• + • + • + • + • + • + • + • + • + • + •		$+ \bullet =$

Adding *Every Other* Fibonacci Number (The Odd Subscripts)

The sum of the *odd* subscripted Fibonacci terms ($F_1, F_3, F_5, F_7, \dots$), starting from the beginning, will always equal the Fibonacci number *following the largest term*.

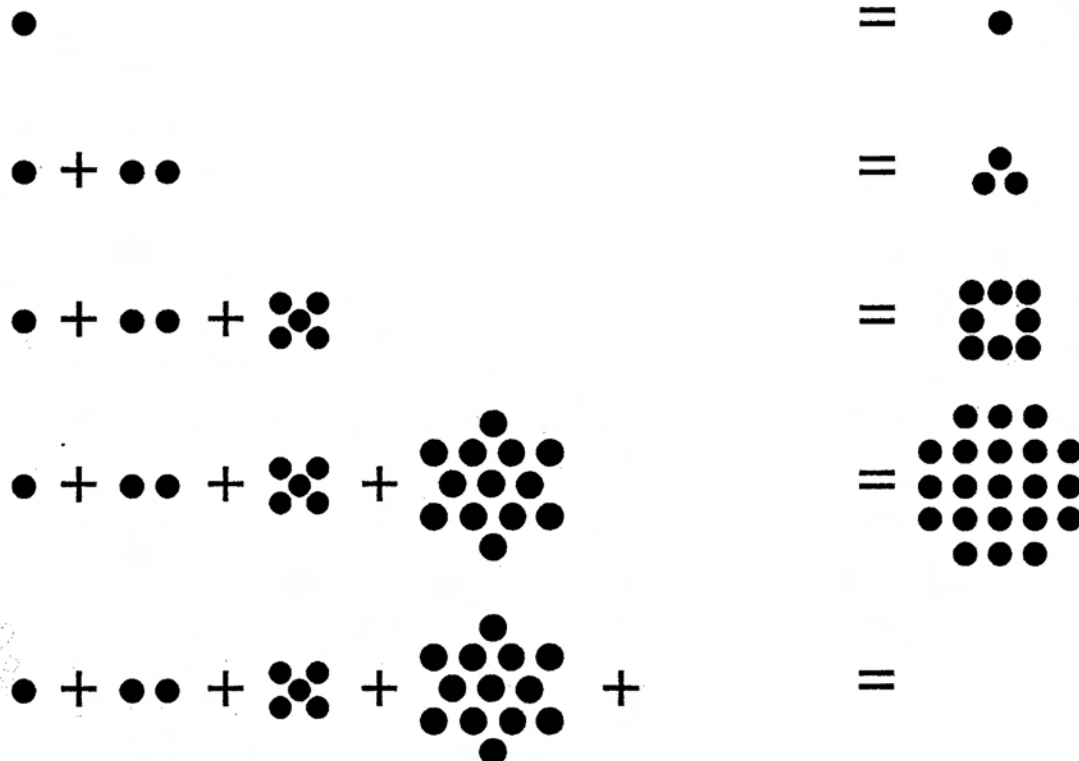
In doing so, the odd subscripted terms produce the series of *even* Fibonacci subscripts! This is reminiscent of Chinese philosophy where events, as they reach maturity, call their opposite into existence.

In mathematical symbols: $F_1 + F_3 + F_5 + F_7 + \dots + F_{2n-1} = F_{2n}$

If you see the pattern, continue this chart:

F_1	$= 1$	$= 1$	$= F_2$
$F_1 + F_3$	$= 1 + 2$	$= 3$	$= F_4$
$F_1 + F_3 + F_5$	$= 1 + 2 + 5$	$= 8$	$= F_6$
$F_1 + F_3 + F_5 + F_7$	$= 1 + 2 + 5 + 13$	$= 21$	$= F_8$

You can also see it with beans. Study this pattern. If it helps, write the numbers next to them. If you see it, fill in the blanks and continue drawing the pattern on paper, or with beans.



Adding *Every Other* Fibonacci Number (The Even Subscripts)

Starting with F_2 , the sum of any number of the *even* subscripted Fibonacci terms, plus 1, will always equal the Fibonacci number directly following where you stopped.

In doing so, the series of even subscripts calls forth the series of *odd* Fibonacci subscripts!

In mathematical symbols: $F_2 + F_4 + F_6 + F_8 + \dots + F_{2n} + 1 = F_{2n+1}$

If you see the pattern, continue the chart:

F_2	= 1	= 1	+1	= 2	= F_3
$F_2 + F_4$	= 1 + 3	= 4	+1	= 5	= F_5
$F_2 + F_4 + F_6$	= 1 + 3 + 8	= 12	+1	= 13	= F_7
$F_2 + F_4 + F_6 + F_8$	= 1 + 3 + 8 + 21	= 33	+1	= 34	= F_9

You can also see this with beans. Study and continue the pattern here and then with actual beans:



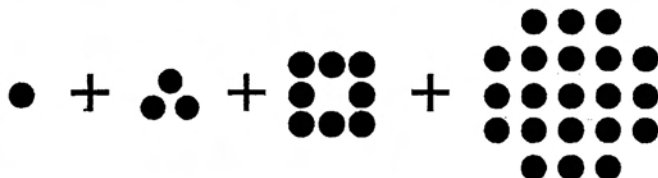
$$+ \bullet = \bullet \bullet$$



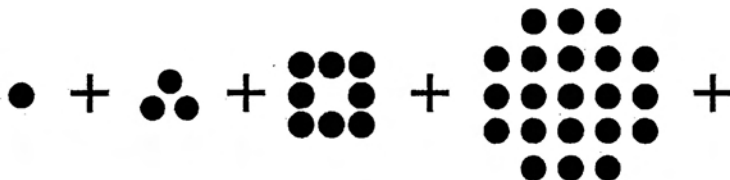
$$+ \bullet = \bullet \bullet \bullet \bullet$$



$$+ \bullet = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$$



$$+ \bullet =$$



$$+ \bullet =$$

Adding Fibonacci Number *Squares*

Mathematicians often speak of "squaring" a number. To "square" a number means *exactly that*: to arrange the beans in the shape of a *square*. Each square has the same number on each side. So to "square" a number means to multiply it by itself, since the area in a square is the product of its sides.

It's a unique fact that among the infinitely many Fibonacci numbers there's *only one* which can be arranged as a square *all by itself*: the 12th term, $F_{12} = 144$. Strangely enough, there are 12 beans on each side of that square.

We can use the idea of squares having Fibonacci number sides this way:
Choose any two consecutive Fibonacci numbers.

The sum of the squares of two consecutive Fibonacci numbers is always equal to the Fibonacci number which has the sum of their subscripts, the Fibonacci number two terms ahead.

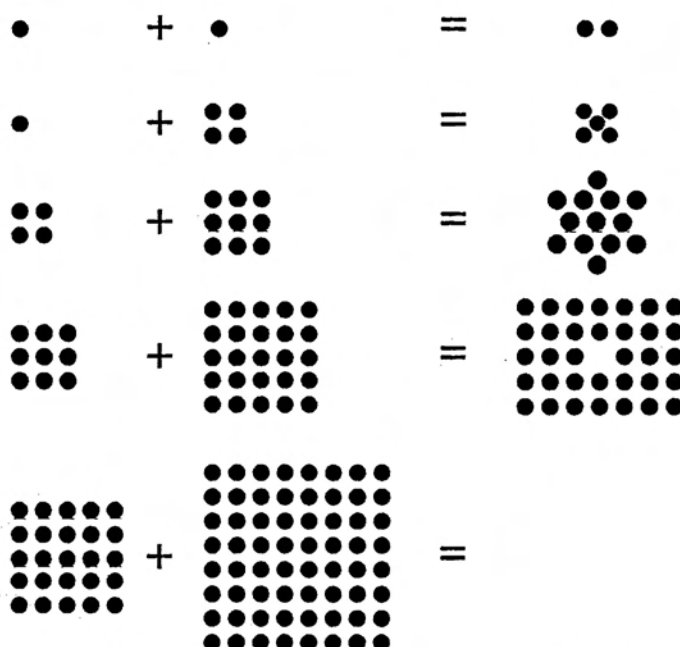
In mathematical symbols: $F_n^2 + (F_{n+1})^2 = F_{n+(n+1)} = F_{2n+1}$

Do you see the pattern? If you do, continue it below.

$(F_1^2 + F_2^2)$	$= (1^2 + 1^2)$	$= 1 + 1$	$= 2$	$= F_3$
$(F_2^2 + F_3^2)$	$= (1^2 + 2^2)$	$= 1 + 4$	$= 5$	$= F_5$
$(F_3^2 + F_4^2)$	$= (2^2 + 3^2)$	$= 4 + 9$	$= 13$	$= F_7$
$(F_4^2 + F_5^2)$	$= (3^2 + 5^2)$	$= 9 + 25$	$= 34$	$= F_9$

You can see it with piles of beans too. Make squares whose sides are Fibonacci numbers, and combine them to make a larger Fibonacci number.

If you see the pattern below, continue it, and build it with beans.



A Fibonacci Rhythm

Choose any 3 consecutive Fibonacci numbers: F_{n-1} , F_n and F_{n+1}

The product of the two outer Fibonacci numbers equals the square of the middle number, plus or minus *one* (alternating).

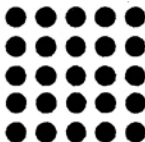
That is, if you take three consecutive Fibonacci numbers and multiply the smallest and largest terms together, and compare the result with the middle term multiplied by itself, they are always *one apart*.

In mathematical symbols: $F_{n-1} (F_{n+1}) = F_n^2 \pm 1$

Notice the pattern and extend the chart:

0, 1, 1	0 x 1	= 0	= (1 x 1)	- 1
1, 1, 2	1 x 2	= 2	= (1 x 1)	+1
1, 2, 3	1 x 3	= 3	= (2 x 2)	-1
2, 3, 5	2 x 5	= 10	= (3 x 3)	+1
3, 5, 8	3 x 8	= 24	= (5 x 5)	-1
5, 8, 13	5 x 13	= 65	= (8 x 8)	+1

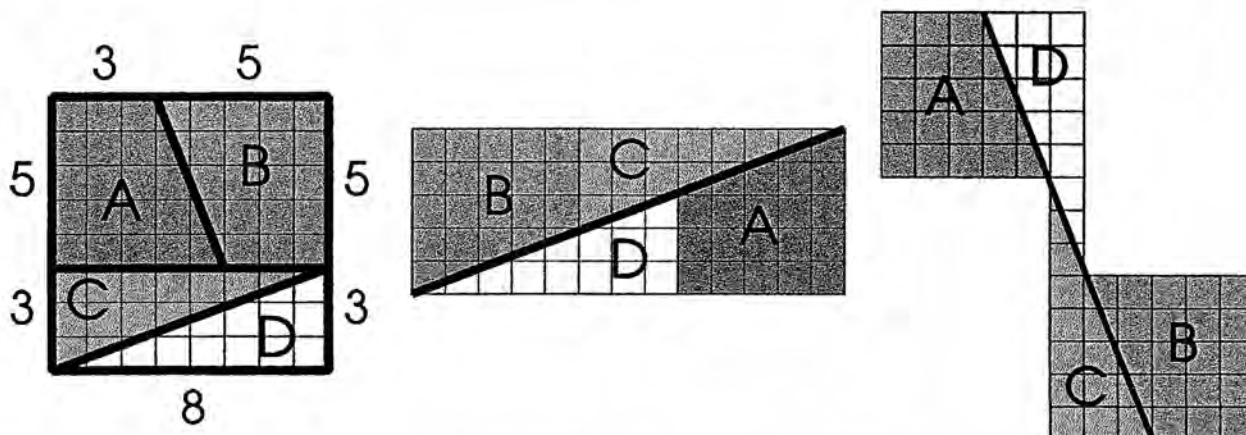
There is a kind of "breathing" going on in this alternating rhythm. You can also see it with beans. Continue the pattern below.

	=		-	
	=		+	
	=		-	
	=		+	
	=		-	
	=		+	

The Math Behind The *Checkerboard Puzzle*

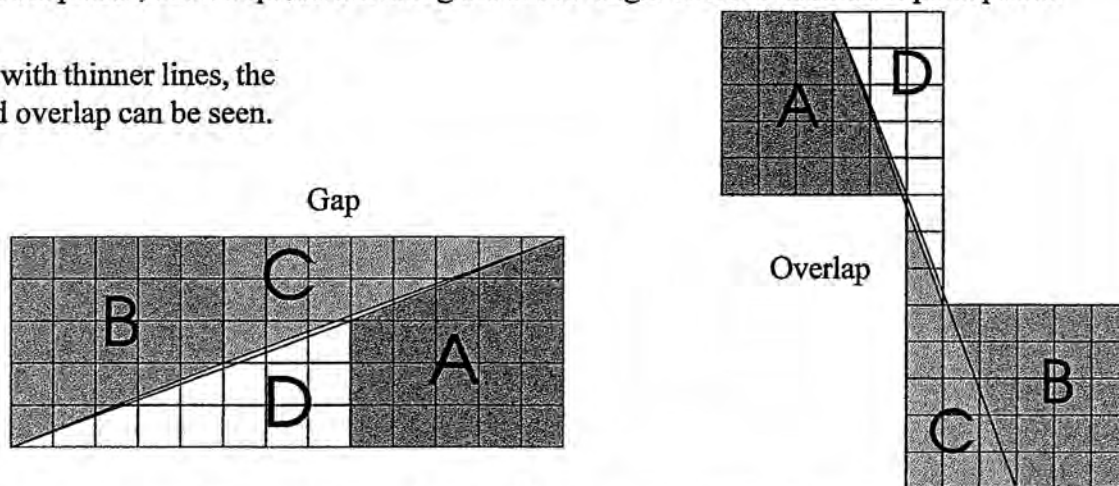
This mathematical pattern gives rise to a well-known puzzle which first appeared in the 1700's. In the late 19th century it was presented by the well-known puzzleist Sam Loyd (see next page).

Here's the puzzle: A normal $8 \times 8 = 64$ square checkerboard is cut with straight lines as shown below (left). Then the pieces are rearranged as a 5×13 rectangle. But $5 \times 13 = 65$ squares, one more than before! How is this possible?



Furthermore, the four pieces can be rearranged (above, right) to another configuration totaling 63 squares, one less than before! What is going on here? How can the exact same pieces rearrange to total such different areas? Well, they really don't. There's actually a very small gap (-1) or overlap (+1) between the pieces, but it's spread out along the entire length and so becomes imperceptible.

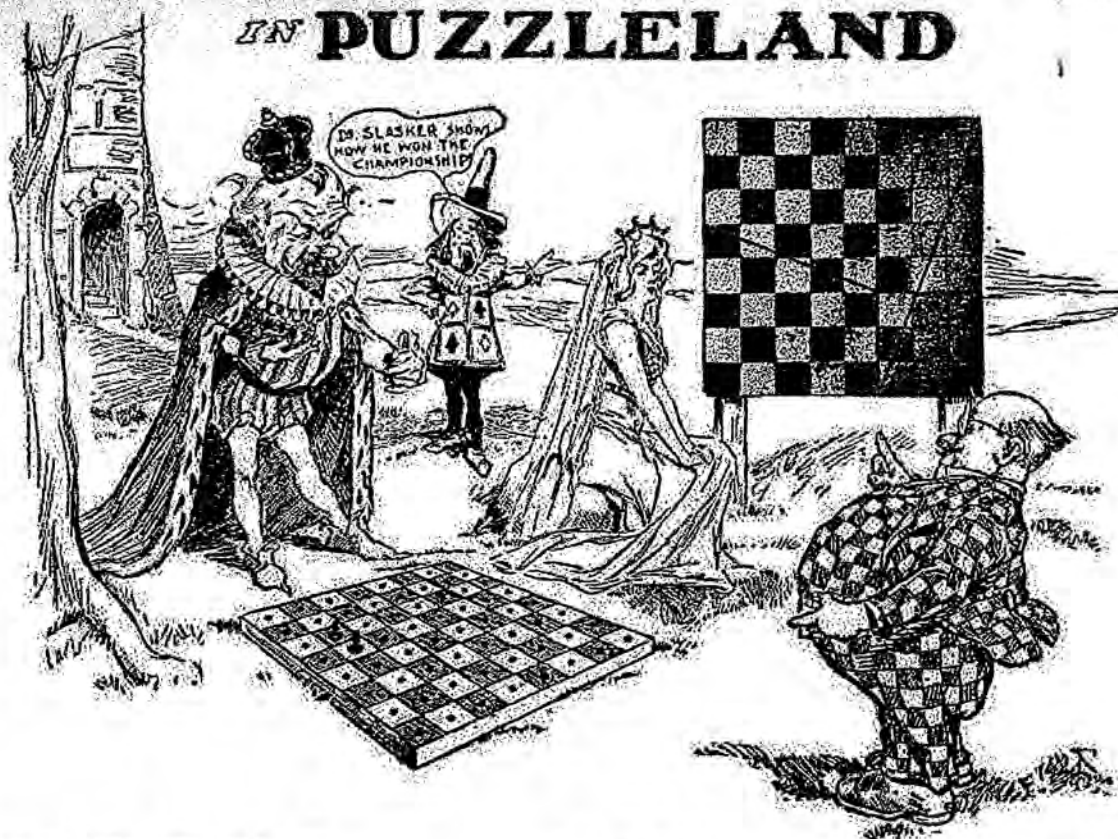
Drawn with thinner lines, the gap and overlap can be seen.



This happens because the pieces have sides which are consecutive Fibonacci numbers (3, 5, 8, 13). We saw with beans that the product of 3×8 is one less than 5×5 , and so we gain (+1) the area of one square spread out along the gap ($5 \times 13 = 65 = 64 + 1$) which makes it difficult to see. But arranged as a zig-zag, we "lose" one square in the slight overlap, or $(5 \times 6) + 3 + (5 \times 6) = 63 = 64 - 1$. This puzzle can be repeated using higher Fibonacci numbers, where the gap and overlap become even less visible as they are spread out over longer lengths.

It's a simple example of abstract mathematics made visible and tangible!

IN PUZZLELAND



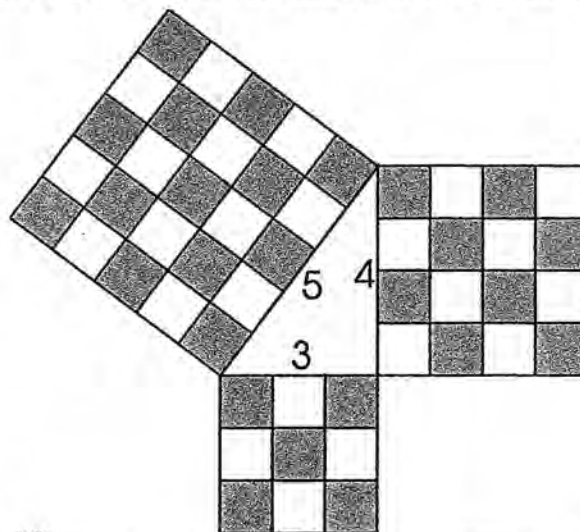
Tommy Riddles tells us that we need know nothing about checkers or chess to solve these puzzles. King Puzzlepate is trying to place the greatest number of men on a chess board without having three men in line in any possible direction. He has started by placing the first two men correctly; now it is up to you to assist him by adding as many men as possible without getting any three in line.

We are told that the first checkerboard ever constructed, which was made by a man by the name of Siesa, and is still preserved in the British Museum, is made of four pieces, as the one shown in the second puzzle. Now the four pieces of this board can be rearranged together so as to make three different puzzles: A square board of 64 squares, an oblong one of 65, or an odd-shaped one of but 63. It is said that Dr. Slasher won the championship by this marvelous coup of arranging the four pieces so as to reduce the board to 63 squares. See if you are able to do it. There has been so much discussion regarding this paradoxical problem that occasion is taken to say that Mr. Loyd presented it before the first American Chess Congress in 1858.

Here's the page showing the puzzle from

Sam Loyd's Cyclopedia Of 5000 Puzzles, Tricks and Conundrums With Answers (1914)

Speaking of checkerboards, in this famous proof of the Pythagorean Theorem, did you ever notice that the numbers of shaded squares are Fibonacci numbers: 5, 8 and 13?



Fibonacci Rectangles And Squares

Choose any three consecutive Fibonacci numbers: F_{n-1} , F_n , and F_{n+1}

The product of the two larger terms, minus the product of the two smaller terms equals the square of the middle term.

In mathematical symbols: $F_n (F_{n+1}) - F_{n-1} (F_n) = F_n^2$

Check for yourself that it's true. If you see the pattern, continue it below:

1, 2, 3	(2x3)	- (1x2)	= 6 - 2	= (2x2)	= 4	= F_3^2
2, 3, 5	(3x5)	- (2x3)	= 15 - 6	= (3x3)	= 9	= F_4^2
3, 5, 8	(5x8)	- (3x5)	= 40 - 15	= (5x5)	= 25	= F_5^2
5, 8, 13	(8x13)	- (5x8)	= 104 - 40	= (8x8)	= 64	= F_6^2
8, 13, 21	(13x21)	- (8x13)	= 273 - 65	= (13x13)	= 169	= F_7^2

You can also see it with beans. Two numbers multiplied together make a rectangle of beans.

Make a rectangle of beans whose sides are the numbers of a Fibonacci pair $F_3 \times F_4 = 2 \times 3$.

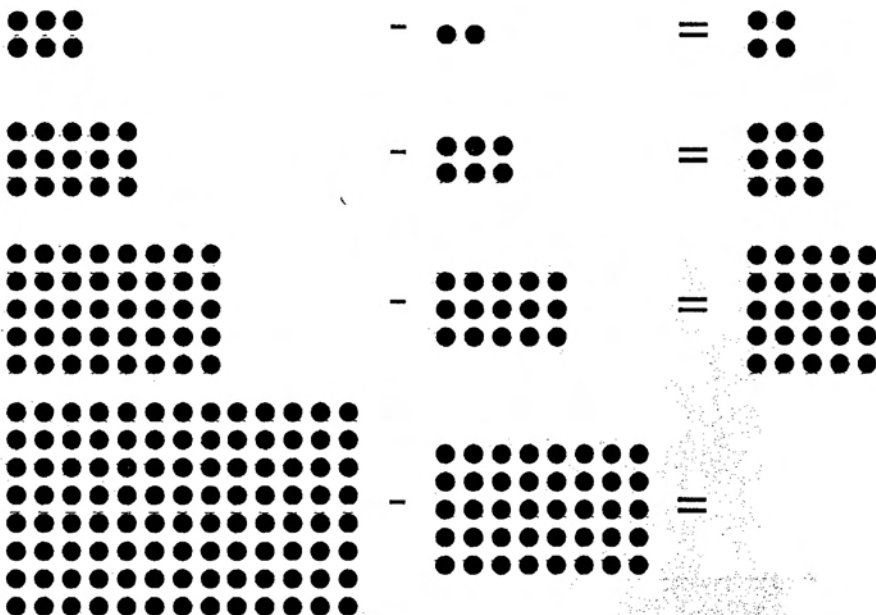
Make another rectangle whose sides are the numbers of the previous Fibonacci pair $F_2 \times F_3 = 1 \times 2$.

From the larger rectangle, *take away* the same number of beans which are in the smaller rectangle.

Using the remaining beans, make a square (whose side is the middle Fibonacci number of beans).

Whenever you subtract two such Fibonacci rectangles, you will always be able to make a square from what remains.

If you see the pattern, continue it below. Rectangles and squares represent the products of *different* or *same* numbers. It may help to write the numbers near the sides of each rectangle and square.



Subtracting From Fibonacci Rectangles

Choose any four consecutive Fibonacci numbers: $F_{n-2}, F_{n-1}, F_n, F_{n+1}$

Twice the third number minus the largest number equals the smallest (first) Fibonacci of the group.

In mathematical symbols: $2F_n - F_{n+1} = F_{n-2}$

If you see the pattern, continue it below:

0, 1, 1, 2	(2x1) - 2	= 2	- 2	= 0	= F_0
1, 1, 2, 3	(2x2) - 3	= 4	- 3	= 1	= F_1
1, 2, 3, 5	(2x3) - 5	= 6	- 5	= 1	= F_2
2, 3, 5, 8	(2x5) - 8	= 10	- 8	= 2	= F_3
3, 5, 8, 13	(2x8) - 13	= 16	- 13	= 3	= F_4
5, 8, 13, 21	(2x13) - 21	= 26	- 21	= 5	= F_5

Try this: Make four piles of beans to represent four consecutive Fibonacci numbers.

Double the number of beans in the third pile.

Then take away from it the same number of beans as in the largest pile.

What remains will always equal the number of beans in the smallest pile.

If you see the pattern, continue it below.

<u>Twice a Fibonacci number</u>	<u>Minus the next F #</u>	<u>Equals the smallest F # of the four.</u>
-------------------------------------	-------------------------------	---

● ●	-	● ●	=	
● ● ● ●	-	● ● ● ●	=	●
● ● ● ● ● ●	-	● ● ● ● ● ●	=	●
● ● ● ● ● ● ● ● ● ●	-	● ● ● ● ● ● ● ● ● ●	=	● ●
● ● ● ● ● ● ● ● ● ● ● ● ● ● ● ●	-	● ●	=	● ● ●
● ●	-		=	

Creating Fibonacci Rectangles

Choose any four consecutive Fibonacci numbers: $F_n, F_{n+1}, F_{n+2}, F_{n+3}$

The sum of the first term and last term is always double the third term.

In mathematical symbols: $F_n + F_{n+3} = 2 F_{n+2}$

Do you see the pattern? If so, continue it.

	<u>Sum of first and last terms</u>			<u>equals double the third term.</u>	
0, 1, 1, 2	$F_0 + F_3$	$= 0 + 2$	$= 2$	$= 2 \times 1$	$= F_2 + F_2$
1, 1, 2, 3	$F_1 + F_4$	$= 1 + 3$	$= 4$	$= 2 \times 2$	$= F_3 + F_3$
1, 2, 3, 5	$F_2 + F_5$	$= 1 + 5$	$= 6$	$= 2 \times 3$	$= F_4 + F_4$
2, 3, 5, 8	$F_3 + F_6$	$= 2 + 8$	$= 10$	$= 2 \times 5$	$= F_5 + F_5$
3, 5, 8, 13	$F_4 + F_7$	$= 3 + 13$	$= 16$	$= 2 \times 8$	$= F_6 + F_6$

You can see it with beans too.

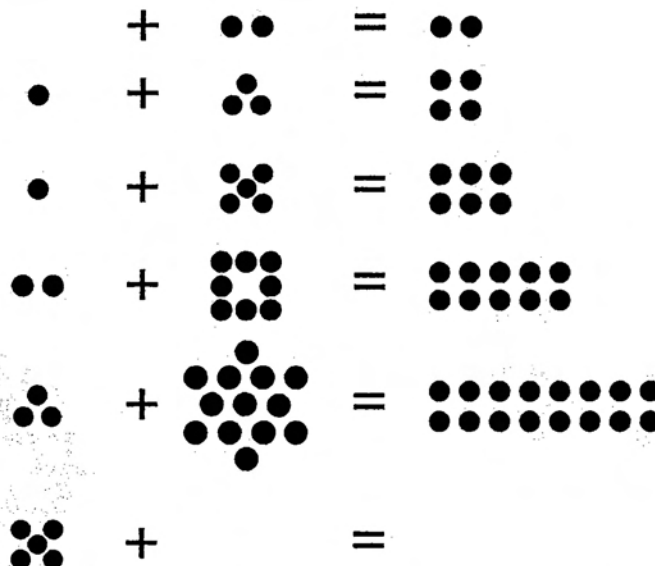
Start with beans in four piles of consecutive Fibonacci numbers.

Combine the smallest and largest piles.

Then, split that into two equal piles.

Each pile will always have the same number of beans as in the third pile.

If you see the pattern, continue it below.



Creating Fibonacci Squares

Choose any four consecutive Fibonacci numbers: $F_n, F_{n+1}, F_{n+2}, F_{n+3}$

The product of the first two terms, plus the product of the last two terms, plus double the second term *squared*, always equals the square of the largest term.

In mathematical symbols: $F_n (F_{n+1}) + F_{n+2} (F_{n+3}) + 2(F_{n+1})^2 = (F_{n+3})^2$

If you see the pattern, continue it below:

1, 1, 2, 3	$(1 \times 1) + (2 \times 3) + (2 \times 1^2)$	$= 1 + 6 + 2$	$= 9$	$= 3^2$	$= F_4^2$
1, 2, 3, 5	$(1 \times 2) + (3 \times 5) + (2 \times 2^2)$	$= 2 + 15 + 8$	$= 25$	$= 5^2$	$= F_5^2$
2, 3, 5, 8	$(2 \times 3) + (5 \times 8) + (2 \times 3^2)$	$= 6 + 40 + 18$	$= 64$	$= 8^2$	$= F_6^2$
3, 5, 8, 13	$(3 \times 5) + (8 \times 13) + (2 \times 5^2)$	$= 15 + 65 + 50$	$= 121$	$= 11^2$	$= F_7^2$

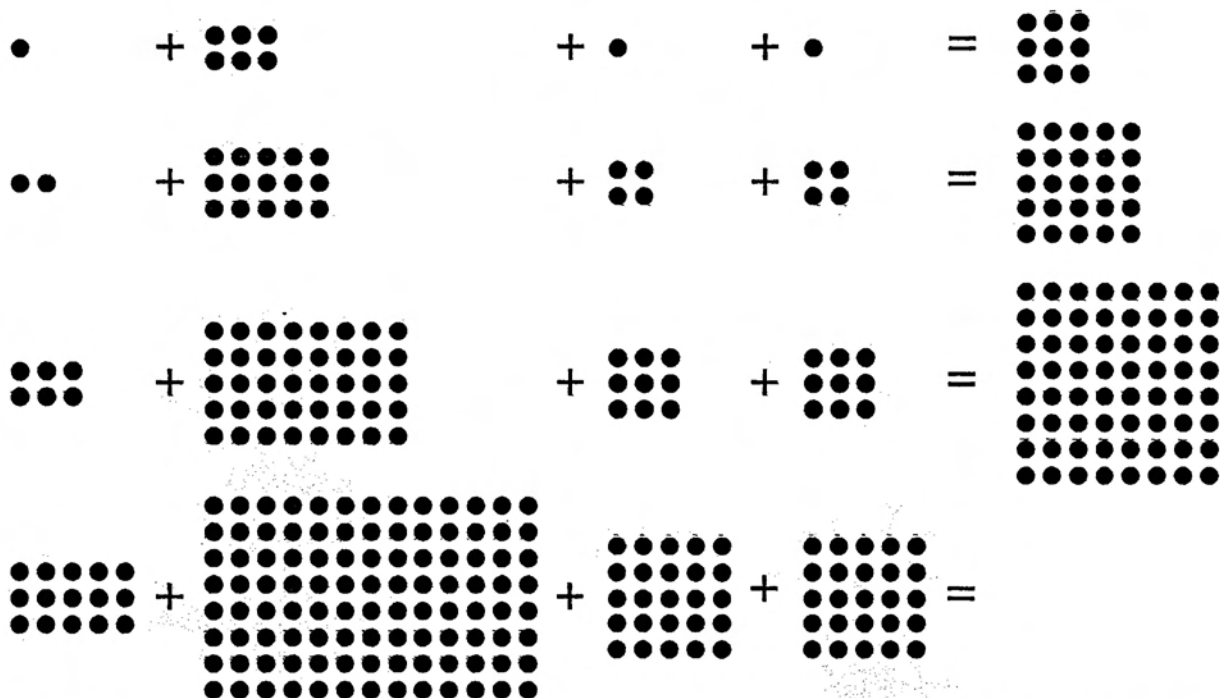
Try it with beans:

Make two rectangles whose sides are equal to the numbers of the first pair and the second pair of Fibonacci numbers.

Then make two squares whose sides are equal to the second term.

Add them all together into one big pile and then rearrange them to make a square whose side is the largest number of the group.

If you see the pattern, continue it below:



Square Fibonacci Symmetry

Choose any four consecutive Fibonacci numbers: $F_n, F_{n+1}, F_{n+2}, F_{n+3}$

Four times the product of the middle two terms, plus the square of the smallest term, always equals the square of the largest term.

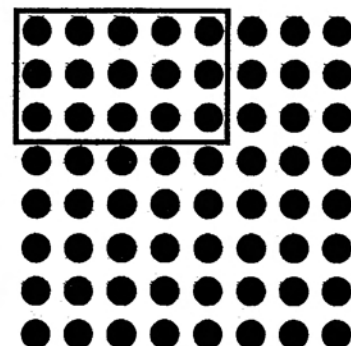
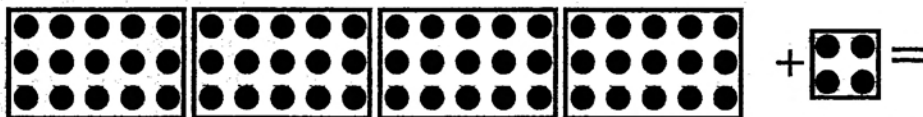
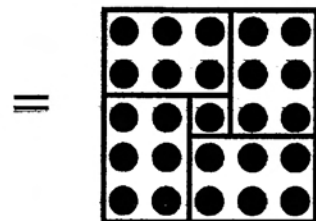
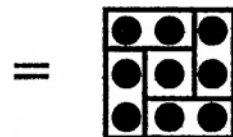
In mathematical symbols: $4(F_{n+1})(F_{n+2}) + (F_n)^2 = (F_{n+3})^2$

Mathematical formulas (and their solutions) may often be seen as groupings of numbers (beans) arranged in a symmetric geometric pattern. Here, the product of the two middle terms always makes a Fibonacci rectangle. Four of them can rotate around a central square made of the smallest term, creating a square whose side is the largest of the four Fibonacci numbers.

If you see the pattern in numbers, continue it below:

0, 1, 1, 2	$4 \times (1 \times 1) + 0^2$	$= 4 + 0$	$= 4$	$= 2^2$	$= F_3^2$
1, 1, 2, 3	$4 \times (1 \times 2) + 1^2$	$= 8 + 1$	$= 9$	$= 3^2$	$= F_4^2$
1, 2, 3, 5	$4 \times (2 \times 3) + 1^2$	$= 24 + 1$	$= 25$	$= 5^2$	$= F_5^2$
2, 3, 5, 8	$4 \times (3 \times 5) + 2^2$	$= 60 + 4$	$= 64$	$= 8^2$	$= F_6^2$

Here's the pattern expressed with beans. Each solution may be rearranged as squares and rectangles. If you see the pattern, draw boxes within the 8x8 square and continue the sequence using beans and string.



Triple Fibonacci Numbers

Choose any five consecutive Fibonacci numbers: $F_n, F_{n+1}, F_{n+2}, F_{n+3}, F_{n+4}$

The sum of the first term and last term is always triple the middle (third) term.

In mathematical symbols: $F_n + F_{n+4} = 3F_{n+2}$

Do you see the pattern? If so, continue it.

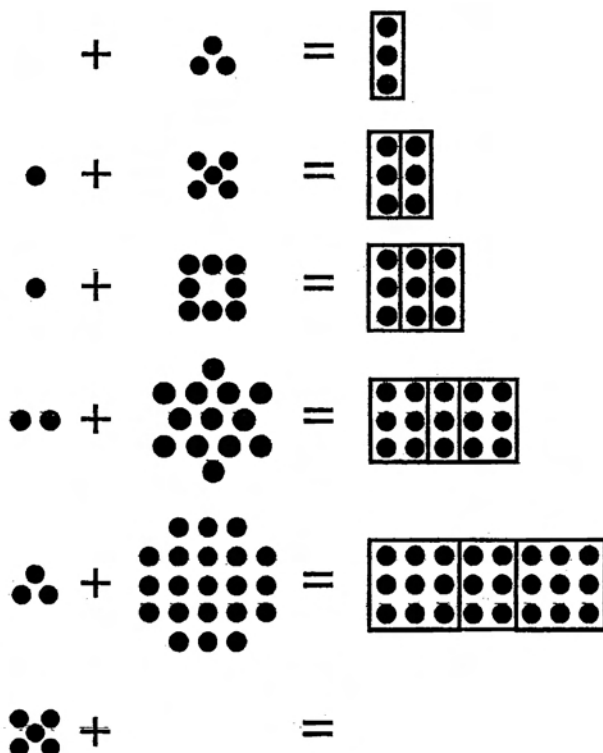
$F_0 + F_4$	$= 0 + 3$	$= 3$	$= 3 \times 1$	$= F_2 + F_2 + F_2$
$F_1 + F_5$	$= 1 + 5$	$= 6$	$= 3 \times 2$	$= F_3 + F_3 + F_3$
$F_2 + F_6$	$= 1 + 8$	$= 9$	$= 3 \times 3$	$= F_4 + F_4 + F_4$
$F_3 + F_7$	$= 2 + 13$	$= 15$	$= 3 \times 5$	$= F_5 + F_5 + F_5$
$F_4 + F_8$	$= 3 + 21$	$= 24$	$= 3 \times 8$	$= F_6 + F_6 + F_6$

You can see it with beans too. Start with beans in *five piles* of consecutive Fibonacci numbers.

Combine the smallest and largest piles. Then, split that into three equal piles.

Each pile should have the same number of beans in it as in the third pile, and so can be arranged as a rectangle having one side of 3.

Furthermore, each rectangle can be split into combinations of rectangles and squares having Fibonacci numbers on their sides (always 3 and then the series of Fibonacci numbers). Boxes have been drawn around them to show this pattern. Continue the pattern below:



Ten Fibonacci Piles Will Split Into Eleven Equal Parts

The sum of *any ten consecutive* Fibonacci numbers can be divided neatly by 11, and also by the seventh term.

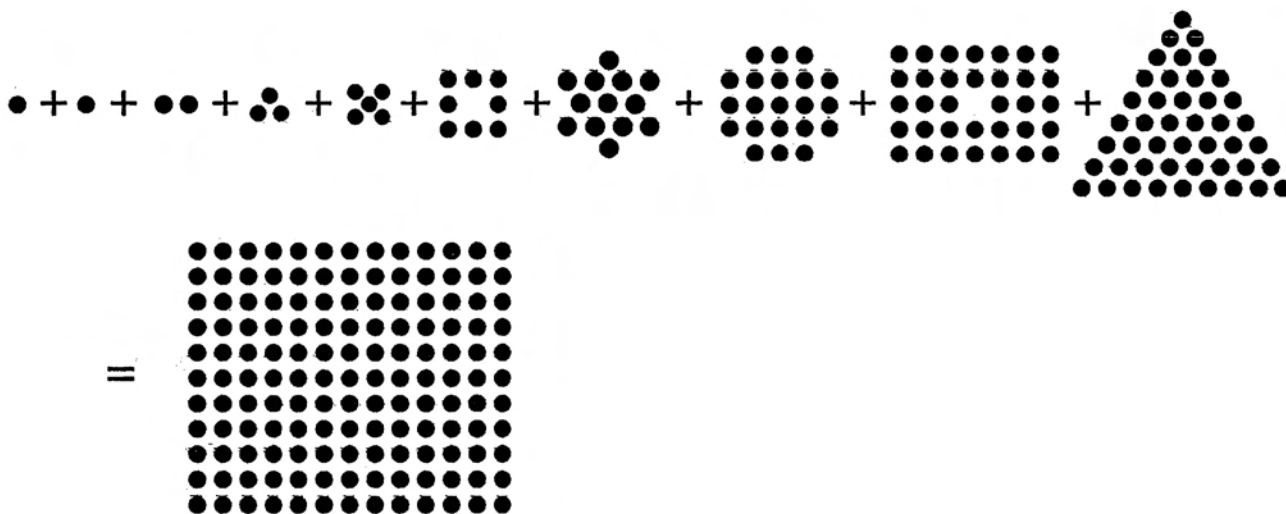
Arrange groups of beans as ten consecutive Fibonacci numbers, and gather them into one big pile. Count them. The number will always equal 11 times the seventh of the ten terms.

This means that you can always divide ten consecutive piles of Fibonacci numbers into eleven equal groups with none left over.

You can also divide them neatly into the number of piles of the seventh term.

For example, consider these ten consecutive Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

Added together $1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 = 143$ beans. (Notice that $143 = 11 \times 13$). This means it can be neatly divided into 11 equal piles, and also 13 equal piles (the seventh term).



Activity: Choose *another* ten Fibonacci numbers in a row and try this for yourself.

Combine them and divide it into eleven equal piles containing the seventh-term number of beans.

Adding Fibonacci Squares Makes Fibonacci Rectangles

Consider the Fibonacci sequence of any length, starting from its beginning.

The sum of squares of consecutive Fibonacci numbers, *starting from the beginning*, always equals the product of the last Fibonacci number times the Fibonacci number after it.

In other words, if each term of the Fibonacci becomes *the side of a square* of beans, and all the consecutive squares are put together into one big pile, that pile can be *rearranged as a rectangle* whose sides are the highest Fibonacci number chosen (F_n) and the Fibonacci number following it (F_{n+1}).

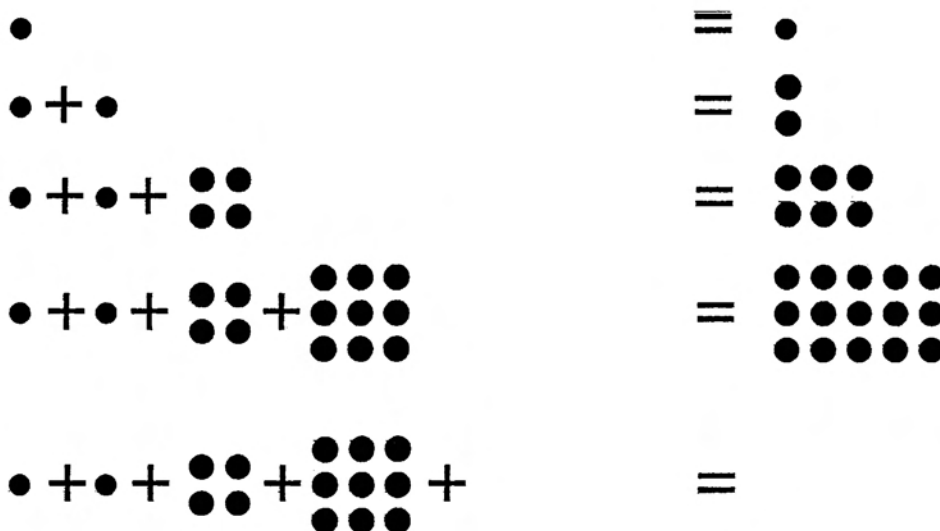
In mathematical symbols: $F_1^2 + F_2^2 + F_3^2 + F_4^2 + \dots + F_n^2 = F_n (F_{n+1})$

If you see the pattern, continue it below:

Sum of Fibonacci-sided SQUARES		= a Fibonacci-sided RECTANGLE
1^2	= 1	= 1 x 1
$1^2 + 1^2$	= 2	= 1 x 2
$1^2 + 1^2 + 2^2$	= 6	= 2 x 3
$1^2 + 1^2 + 2^2 + 3^2$	= 10	= 3 x 5
$1^2 + 1^2 + 2^2 + 3^2 + 5^2$	= 40	= 5 x 8
$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2$	= 104	= 8 x 13

You can demonstrate this with beans. Make consecutive squares of beans having consecutive Fibonacci numbers *on each side*.

Then add them together to see this growing pattern of squares and Fibonacci-sided rectangles. If you recognize the pattern, continue it below.



Fibonacci Numbers In Three Dimensions

Choose any three consecutive Fibonacci numbers: F_n , F_{n+1} , F_{n+2}

The sum of the *cubes* of the two greater numbers, minus the *cube* of the smallest number, always equals another Fibonacci number.

The subscript of this Fibonacci number equals the sum of the subscripts of the three original terms. This produces a pattern of Fibonacci terms whose subscripts are multiples of 3 (the sequence of even Fibonacci numbers).

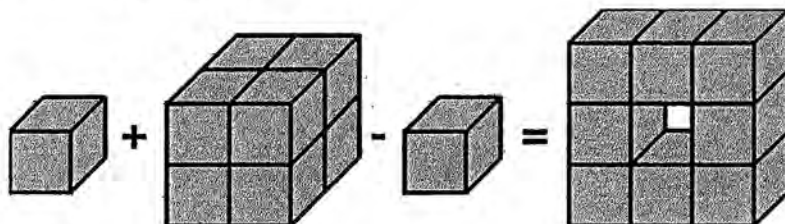
Written in mathematical symbols: $(F_{n+1})^3 + (F_{n+2})^3 - F_n^3 = F_{n+(n+1)+(n+2)} = F_{3n+3}$

What's the pattern here? If you see it, continue it below.

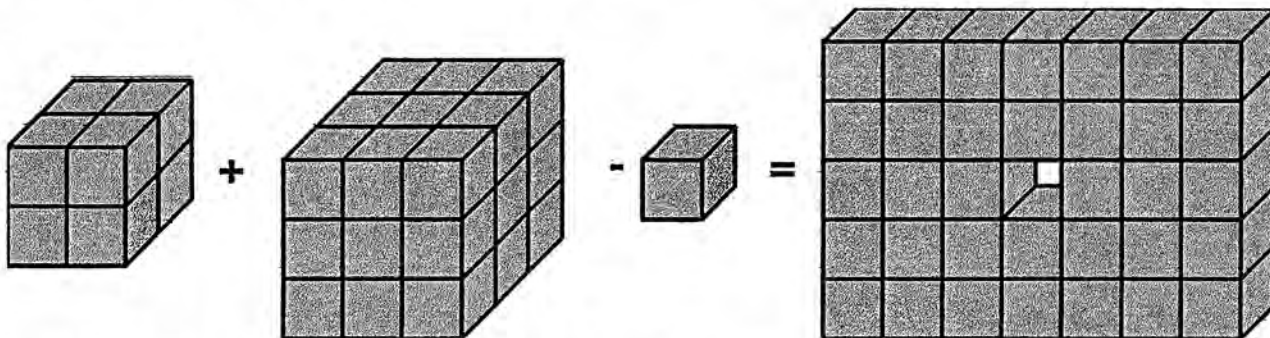
F_0, F_1, F_2	0, 1, 1	$1^3 + 1^3 - 0^3$	$= 1 + 1 - 0$	$= 2$	$= F_3$
F_1, F_2, F_3	1, 1, 2	$1^3 + 2^3 - 1^3$	$= 1 + 8 - 1$	$= 8$	$= F_6$
F_2, F_3, F_4	1, 2, 3	$2^3 + 3^3 - 1^3$	$= 8 + 27 - 1$	$= 34$	$= F_9$
F_3, F_4, F_5	2, 3, 5	$3^3 + 5^3 - 2^3$	$= 27 + 125 - 8$	$= 144$	$= F_{12}$
F_4, F_5, F_6	3, 5, 8	$5^3 + 8^3 - 3^3$	$= 125 + 512 - 27$	$= 610$	$= F_{15}$

Because this deals with the power of 3, or three-dimensions, we must be describing cubes. Cubes cannot easily be made with beans or pebbles, so this would have to be shown with Fibonacci-edged cubes. Perhaps you have some blocks and can build them:

The equation $1^3 + 2^3 - 1^3 = 8$ looks like this:



The equation $2^3 + 3^3 - 1^3 = 34$ looks like this:



Now, with this knowledge of some of the interweaving patterns of Fibonacci numbers, we can better appreciate and understand the patterns we'll find in living nature.

Fibonacci Numbers In Nature

“Come forth into the light of things;
Let Nature be your teacher.”

— William Wordsworth (1770-1850)

Why would we find Fibonacci numbers in nature? And where?

Nature's chief feature is that it expresses the ongoing creating process of ceaseless cycles of birth, growth and dissolution. Wherever there is **growth by accumulation**, Fibonacci numbers will be present.

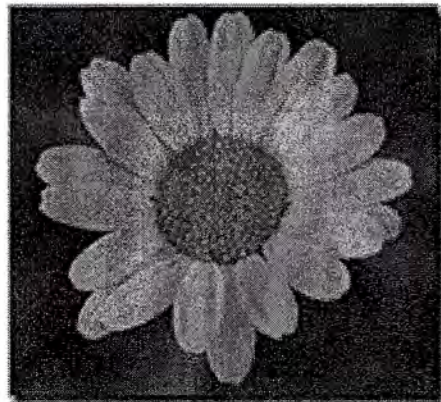
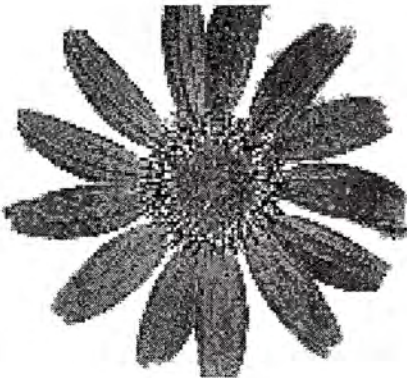
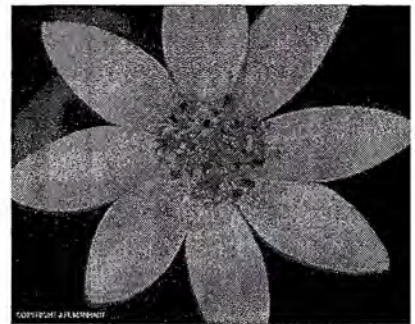
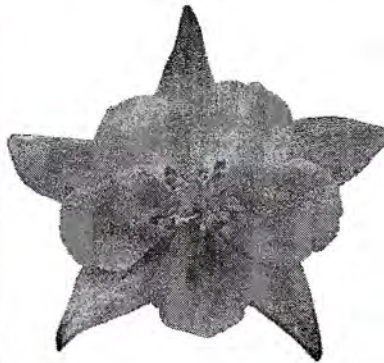
Nature is full of opportunities for observing Fibonacci numbers which translate into patterns of living nature. Perhaps the most obvious place to see them is in the numbers of petals of flowers.

It's important to realize that Fibonacci numbers are *an ideal* which nature aims for but cannot achieve perfectly. While many flowers in a field will happily display Fibonacci numbers of petals, some flowers will always have a few more or less. But the field's average will be Fibonacci numbers.

Count the numbers of petals of these flowers. Which have exact Fibonacci numbers, and which are very close?

“The mathematical rules of the universe are visible to men in the form of beauty.”

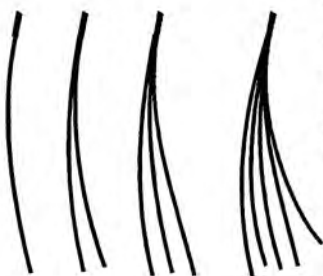
— John Michell



You will no doubt find flowers like poppies with four petals, irises with six, and passion flowers with ten petals. But look closely! Are the petals really all the same? Or are you seeing *pairs* of twos, threes and fives?

Another easy-to-find example of Fibonacci numbers are seen in the grouping of pine needle clusters, which appear as 1, 2, 3 and 5 needles together.

Look for them around pine trees and make a collection of the Fibonacci number pine needle groups you find. Can you associate different groupings with different types of pine?



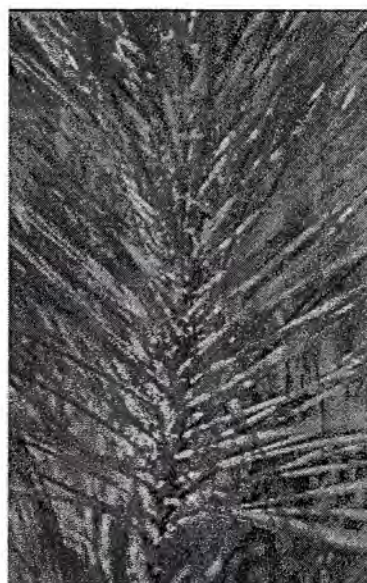
Also, look at the pine *branches* where the needles came from. Observe very carefully how the needles emerge *along spiral staircases* up and around the branch. The staircases go clockwise and counterclockwise. Learning to see "spiral staircases" crossing in two directions around a central column is a most valuable tool for discovering how nature grows. And it's what we must practice before we find how Fibonacci numbers play their

role.

Look around your natural environment. It doesn't have to be near woods or even in a park. Just notice where any plants grow, even small ones. Pay attention to whether they display a crossing "spiral staircase" pattern. Attune yourself to seeing spirals. With practice, you'll soon notice many examples. They're nature's "open secret".

Keep a notebook of your findings. Make sketches and detailed drawings of what you see. If you have a camera, take pictures with a close-up (macro) lens.

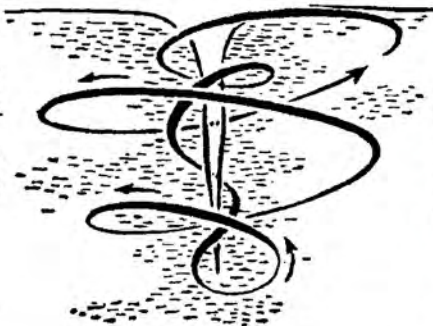
Look at these pictures for hints about some places that these parallel rows of winding staircases (called "parastichies") appear in the various parts of a growing plant. How many more examples can you find around you?



The Whirlpool Structure

You may have noticed how similar nature's living spirals are in appearance to whirlpools. They all have a central core of some kind, whether a solid trunk within whirling branches and leaves, or an empty hole within water. There is spiraling movement down around the core and then up again (or up then down) in the shape of a spiraling torus.

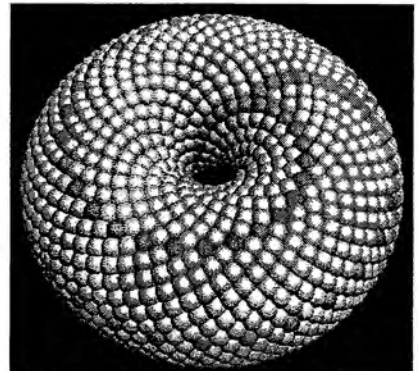
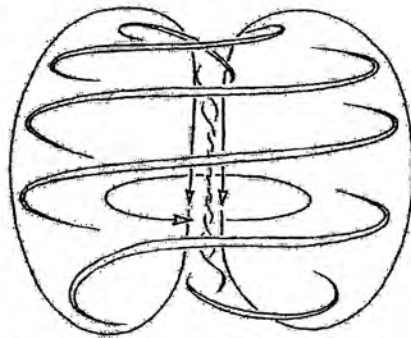
It is very useful to notice this structure in nature's forms, whether as plants or trees, flowers, fruits, vegetables, pinecones, tornados, hurricanes or any other spiraling events. It will change the way you see the world, into a dynamic whirling event of spirals within spirals within spirals without end.



Side-View

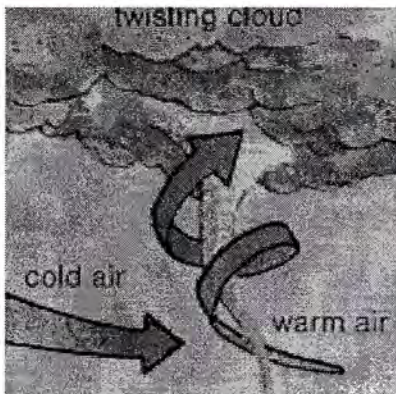


Top View

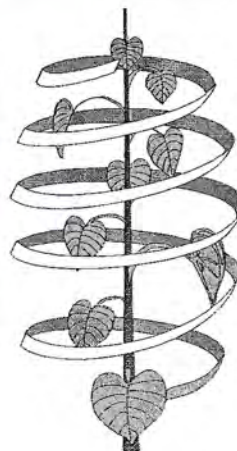
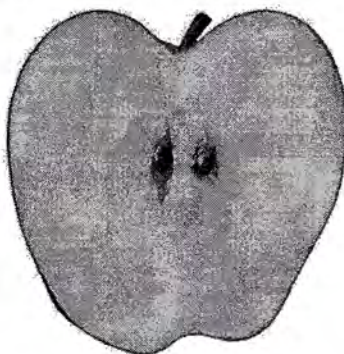
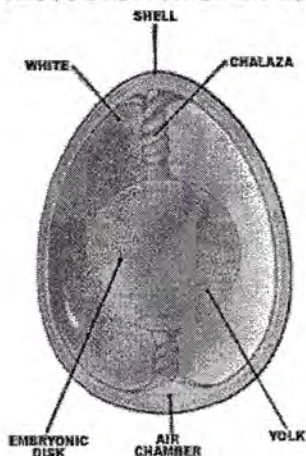


Spiraling Torus

Practice "seeing" spirals in nature, a pinecone, a plant, a tree or a forest of trees as slow whirlpools.



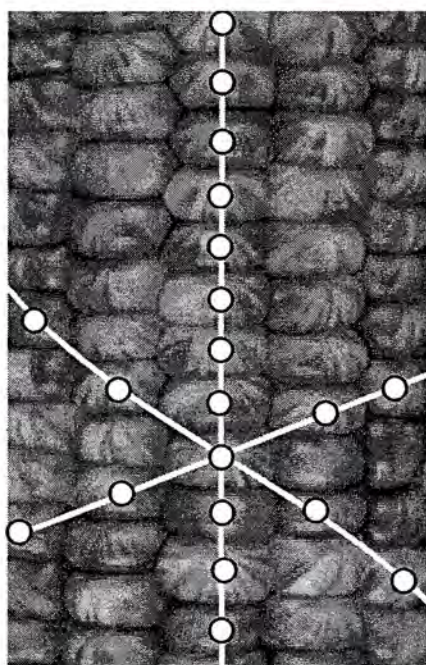
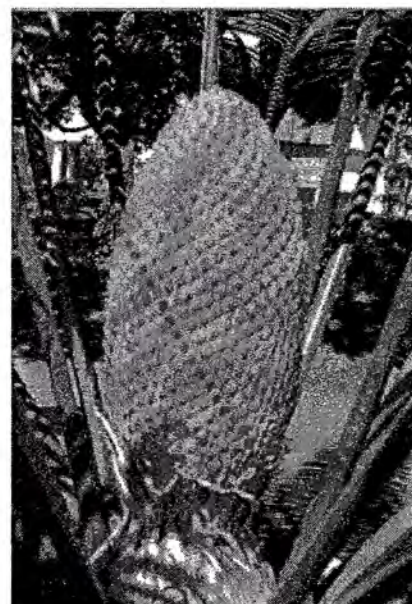
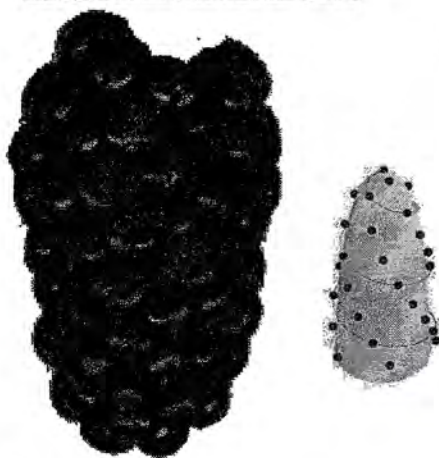
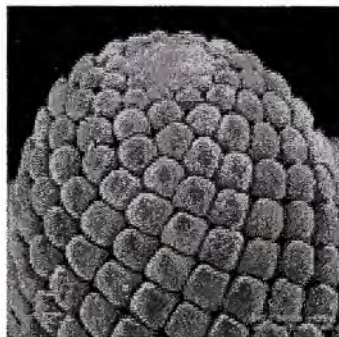
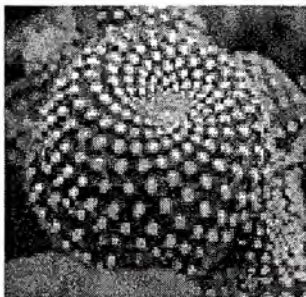
CROSS-SECTION OF AN EGG



Draw The Spirals

If you really pay attention, you'll find these spiral crossings nearly everywhere!
Look carefully at the plants around your home, and even the fruits and vegetables you eat.

Draw lines along the spiral paths on these pictures.
Draw them in both directions so they crisscross.
Sometimes the paths are very clear, sometimes they're not.



At first glance, in some plants and their parts, like ears of corn (left), the kernels seem to be arranged only in straight parallel rows up and down. But if you look carefully you'll see that each kernel is also the crossing point of two oppositely spinning spiral staircases, making three rows at once. Examine ears of corn of different varieties to see this, and whether the numbers of parallel rows in each direction is at (or near) three consecutive Fibonacci numbers.

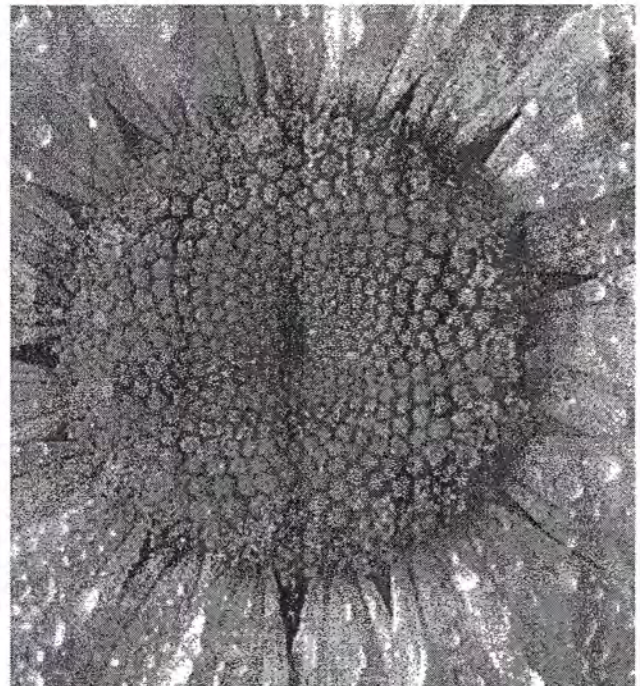
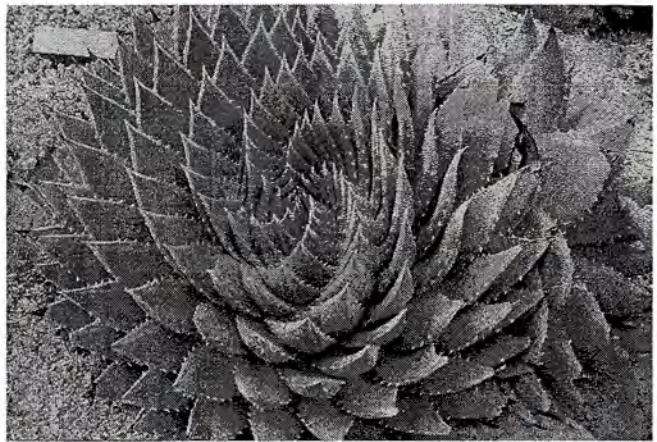
Learn to notice this triple-crossing in the plant world and you'll soon understand much about the ways by which nature is woven.

Draw a real or imaginary plant, fruit or vegetable having crossing "spiral staircases".



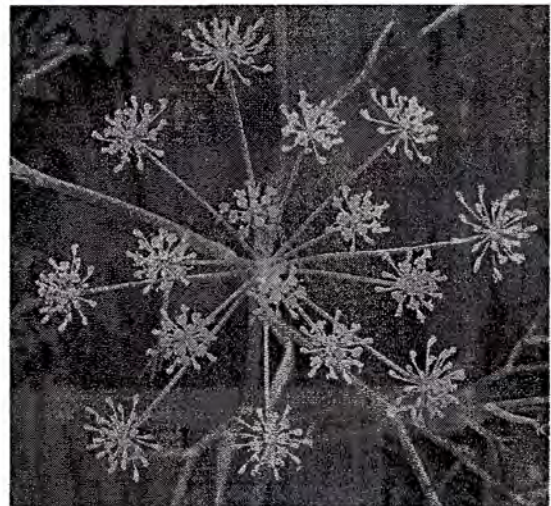
Draw spiral lines to connect the parallel rows in these plants and their parts. Then count the numbers of parallel rows in each direction. If you did it carefully, you'll find that *they are usually consecutive Fibonacci numbers!*

Spiral packaging with Fibonacci numbers is how nature packs the most parts of different and changing sizes into the least space during growth. This close-packing by crossing spirals is the most efficient way to have growing seeds, leaves, florets and fruits of *different sizes* fit together at once.



“The harmony of the world is made manifest in Form and Number, and the heart and soul and all the poetry of Natural Philosophy are embodied in the concept of mathematical beauty.”
— Sir D’Arcy Wentworth Thompson (1860-1948, Scottish zoologist and classical scholar)

“Numbers are the sources of form and energy in the world.”
— Theon of Smyrna (2nd Century AD Platonist mathematician)



Notice how each part is
a small model of the whole flower.

Fibonacci Floret Software

For free software which will animate spiraling Fibonacci numbers of florets, seeds and leaves in an ideal, growing flower, please go to my webpage www.constructingtheuniverse.com/spiral.htm . Scroll to the bottom and click on the link about the software (growth310.zip).

It runs in DOS which can work on Windows and Mac computers.

Use colored pencils to shade this screen-capture

GROWTH DYNAMICS OF
FLOWERS Ver 2.13

Seeds: 193
Radials: 34
Spirals: 21
Antispirals: 13

Seed 0 rad: 13
Seed 0 dist: 108

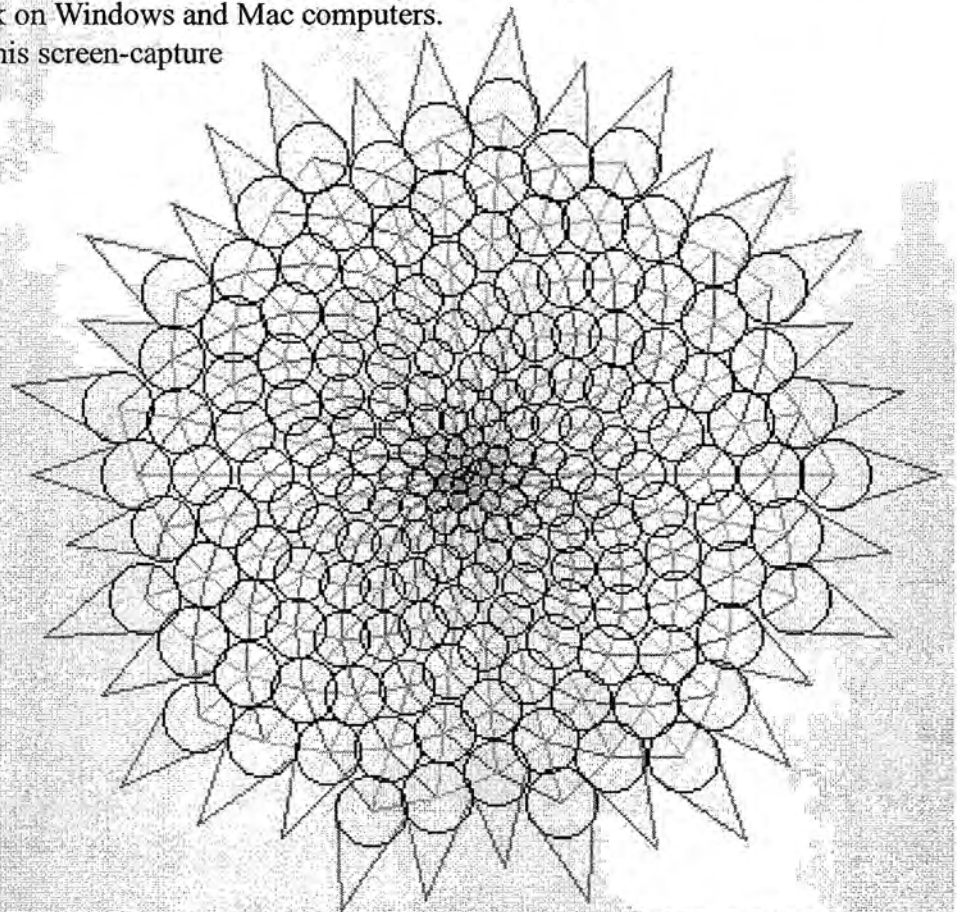
F1 Toggle help
F2 Toggle counts
F3 Toggle seeds
F4 Toggle leaves

F5 Toggle radials
F6 Toggle spirals 1
F7 Toggle spirals 2
F8 Toggle vortex

F9 Reverse growth
F10 Toggle pause
F11 Decrease series
F12 Advance series

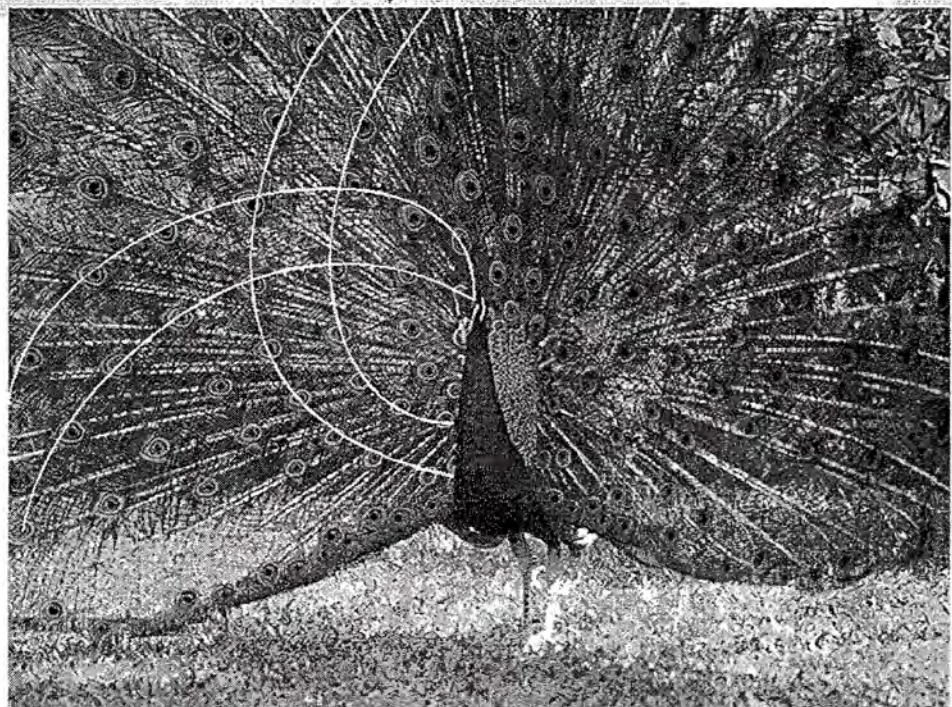
Any key to single step
ESC Exit

Seed rad = $\sqrt{n}$
Distance = n



Even the eyes of a
peacock's feathers form
this beautiful and
efficient pattern.

Connect them and see!



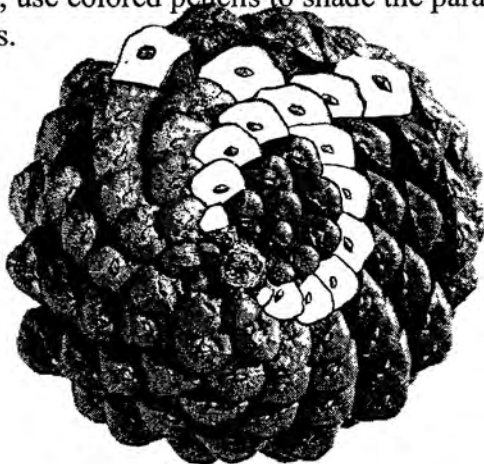
Painting Pine Cones

You can make more permanent examples of these spiraling Fibonacci numbers of the plant world by painting their spirals.

Cone bearing plants are very ancient, and different types may be found most everywhere on Earth. Collect examples which are as complete as possible, then clean and dry them. Use thin brushes with colored acrylic or other paints to make their spiraling patterns obvious. The direction of the rows you paint, and whether it is an even or odd number of parallel rows, may determine your color patterns.

Use them to start your own museum of local Fibonacci patterns.

Here are some examples. Until you can find a pine cone to paint, use colored pencils to shade the parallel rows of the cones.



“The distribution of energy follows definite paths which may be studied by means of geometric construction.”

— Samuel Coleman



Painting the bottom of a pinecone reveals that it is really a whirling *five-petaled flower*.

A Spiraling Leaf

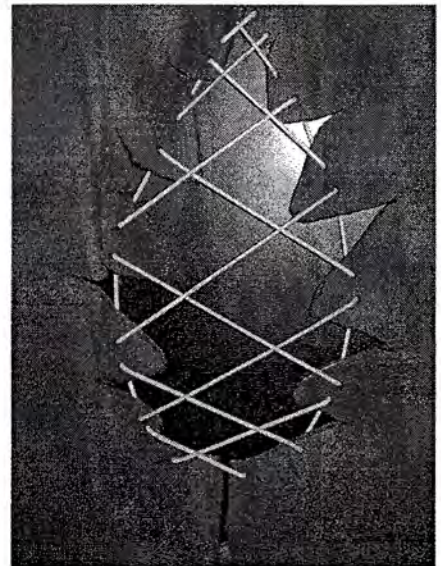
If you can find a large, stiff, dry leaf, like this oak leaf, you can weave a wire through it to show an invisible spiral whirling through it. Actually, you can weave in both directions, but the leaf may not be strong enough to hold the wires. A little white glue at the holes helps to hold it together.



Leaf front view

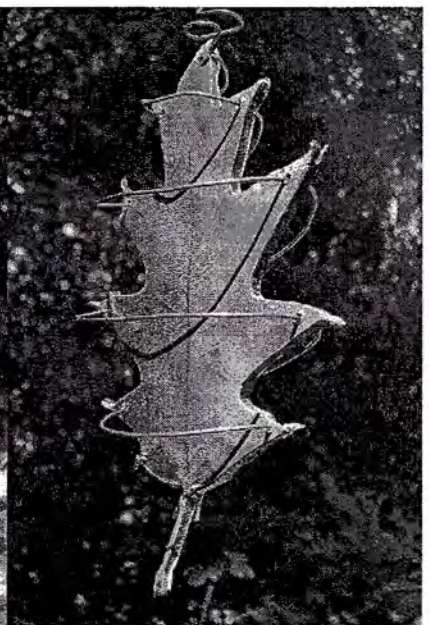
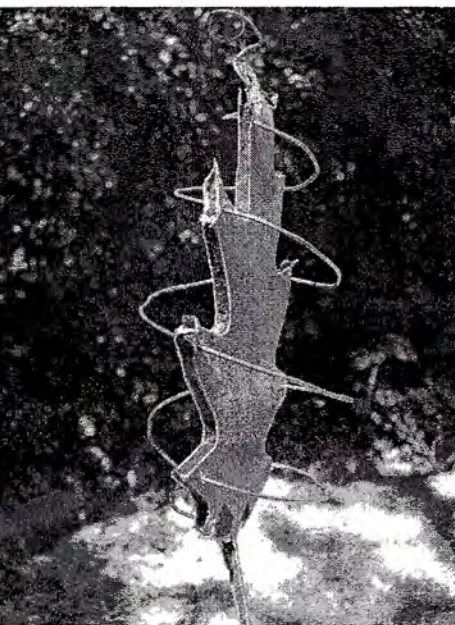
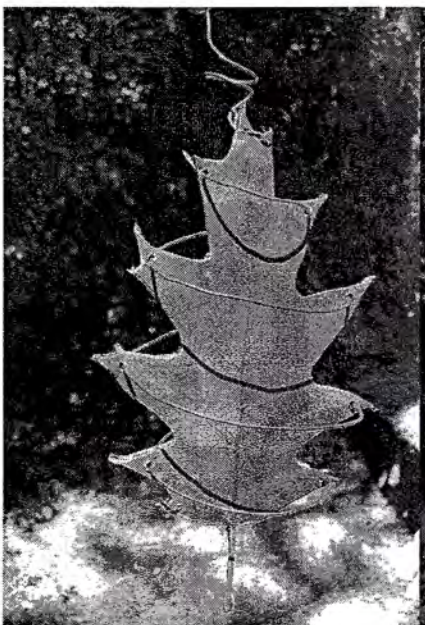


Leaf side view



Woven "negative space"
of the "leaf" below

The large "leaf" below is a copy of both sides of an actual leaf made on a scanner and printed on stiff paper. Both sides were glued onto a piece of foam core (or cardboard will do well too) cut to the shape of the "leaf". Holes were poked at the appropriate points, and stiffer wire was wound through to show the leaf's "spiral". The "negative space" cut into the foam core (above right) provided another opportunity to weave elastic thread and show the leaf's Fibonacci spirals crossing in both directions.



Painting Cauliflowers

The supermarket's produce section of fruits and vegetables is a great museum of Fibonacci patterns. Stroll through and you'll see what I mean. Many items found there can be dried and painted to reveal their spiraling Fibonacci row patterns.

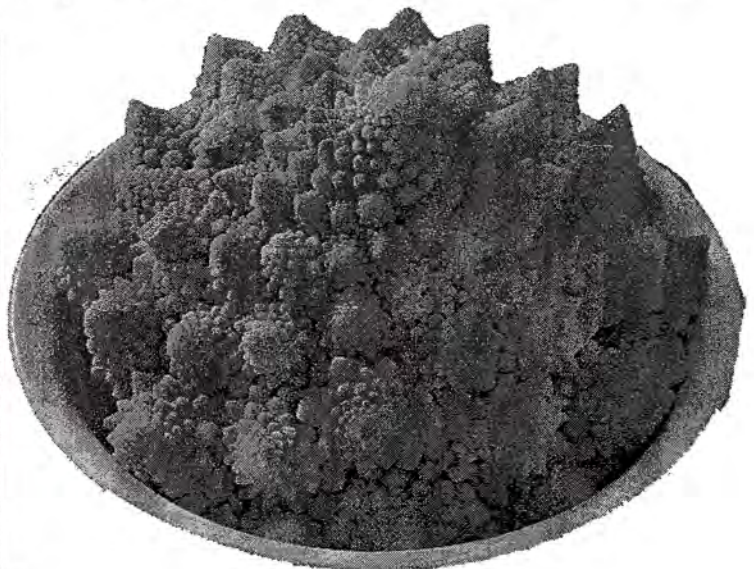
Various types of cauliflower are excellent for this. Look carefully at the head of one. As a whole it displays whirling spirals in each direction which can be painted with acrylic or vegetable color. Use different size brushes, and toothpicks for the fine work.

Paint the spirals in either direction and you're sure to find Fibonacci numbers, most likely five spiral rows in one direction and eight parallel rows spiraling in the other, or perhaps eight and thirteen rows.

Notice that the parts which make up the whole cauliflower, the bumps, are each a miniature but rougher version of the complete head! And each bump is made of even smaller spiraling bumps, each a replica of the others and of the whole. You can paint the main spiral of the whole cauliflower head, or paint the whorls of the smaller bumps within the whole, showing them to be all part of one grand design.

Due to its design of spirals within spirals within spirals, the cauliflower is the closest design on Earth I know of to the structure of a spiral galaxy. Our own Milky Way galaxy is filled with starry suns whose planets whirl around them, and moons whirl around planets. The remarkable fact about this spiraling Fibonacci arrangement is that all different sizes fit simultaneously to compose a whole, like a cauliflower, pinecone or tree with many branches.

A painted cauliflower won't last too long, so take pictures of it when it's painted, or few will believe it!



Is it a vegetable or a green galaxy?
Use a pencil to reveal some spirals!

Fibonacci Potatoes

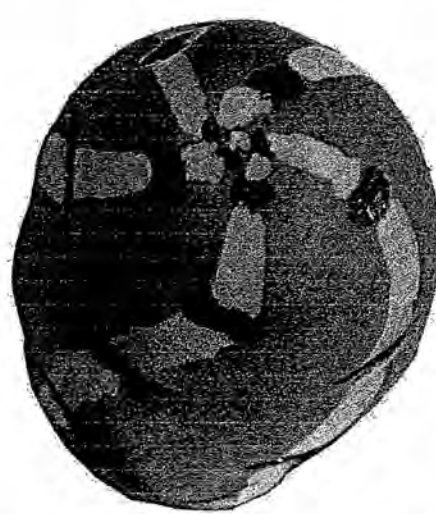
One of my favorite examples of Fibonacci numbers in the plant world, because it is so unexpected, is the potato! Usually considered, if at all, as a formless lump, the potato is overlooked and underrated. But it is noble and transforms to become a jewel, like a Faberge egg, when you make visible *the spiraling pattern of its eyes*.

Here's an activity to do: Choose a potato where you can clearly identify the eyes. Larger potatoes are usually best. The eye is the part which contains the whole where an entirely new plant can grow from, a fractal (the parts are models of the whole).

Clean and dry it well. Look closely to find the eyes, then paint them with dots. . You'll locate them if you follow their spiraling pattern around the potato, between its top and bottom poles. The eyes whirl quickly near the poles, but with close examination you'll be able to identify where they're aiming for.

Take photos, because sooner or later the potato will decay. Moisture travels through the eyes, so if they're painted with acrylic, which seals them, the decay process will be very much slower.

When you paint the eyes, you'll see them each spiraling simultaneously in two opposite directions, two parallel spirals in one direction, and three in the other. In addition, the same eyes may be connected in five spirals directly between the poles (below, right). Connect them with lines and you should find that each eye is part of two spirals in one direction, three spirals in the opposite direction, and five vertical lines between the poles connecting all the eyes.



Once the eyes are identified and painted with dots, they may be connected in any combination of three spiraling directions. The left photo shows one direction, the center shows around and also directly between poles, and on the right how five rows meet at one pole. Missing in the center photo are the near-horizontal spirals. Draw them.

A potato flower
has five petals.
Surprised?

“Where plants have five-fold patterns, a
consideration of their souls is in place.”
— Johannes Kepler

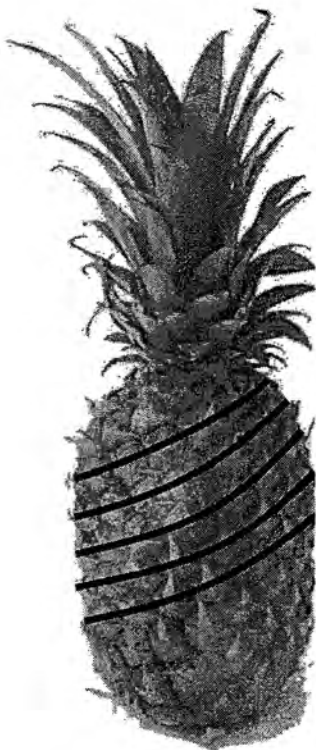
Pineapple Patterns

The pineapple provides another great example of the three Fibonacci numbers which occur at once connecting the poles: the two oppositely spinning parallel spiral rows, and vertical lines curving up the middle.

Growing pineapples are sensitive to their environment and so the parallel rows of fruit pods (see next page may grow into one another, confusing the Fibonacci count. But observe as you shop for one. Count the parallel spiral rows in each direction. If they're clear to count, most likely they'll show consecutive Fibonacci numbers (often 5 and 8, or 8 and 13, or 13 and 21).

You might want to paint the spirals on the outside (with vegetable coloring if you want to eat it). Or use toothpicks and colored yarn to connect them and see the weave by which nature's architecture is built.

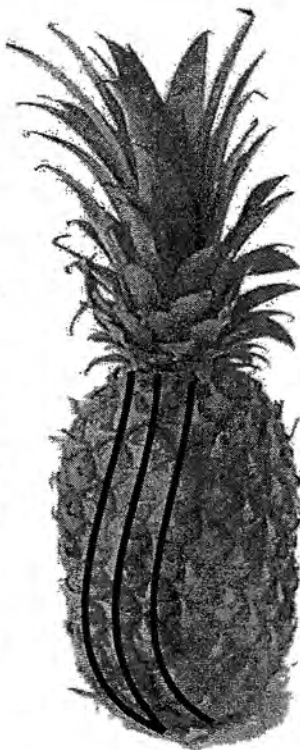
You can even grow a new pineapple by planting the top part in good soil. If you have patience, you'll see a stem emerge straight upward, with a tiny pineapple sprouting at its top!



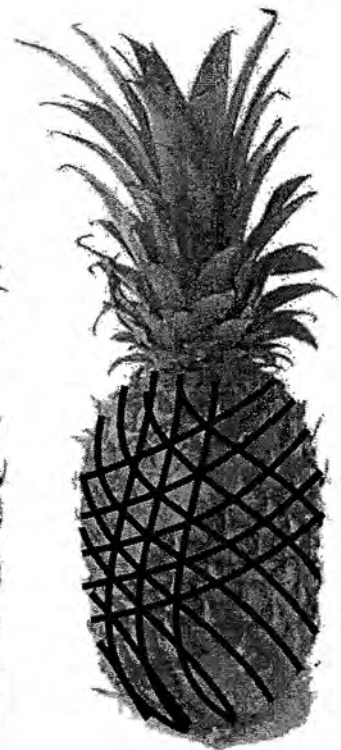
5 parallel rows of spirals in this direction



8 parallel rows of spirals in this direction



13 vertical curves connect all fruit pods around the whole pineapple



All three rows crossing at once.

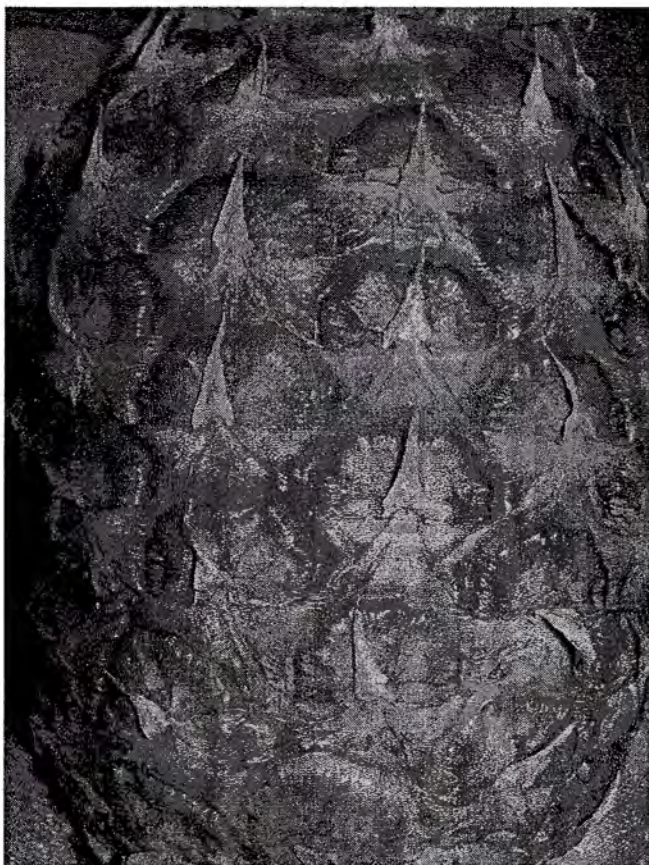
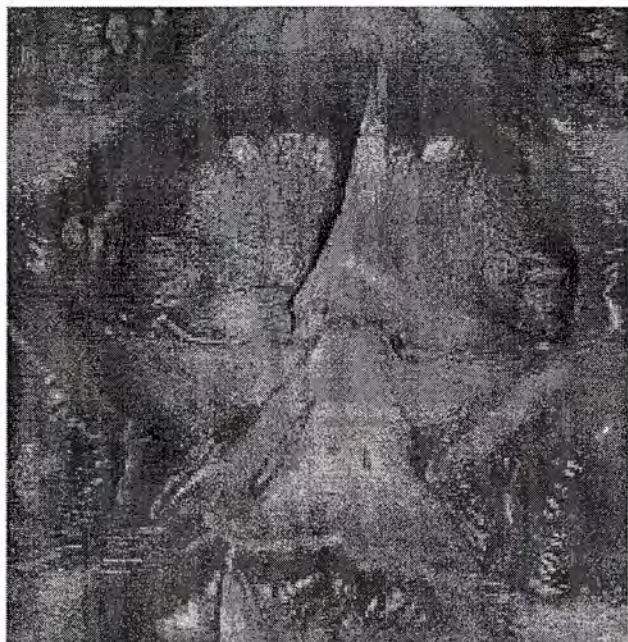
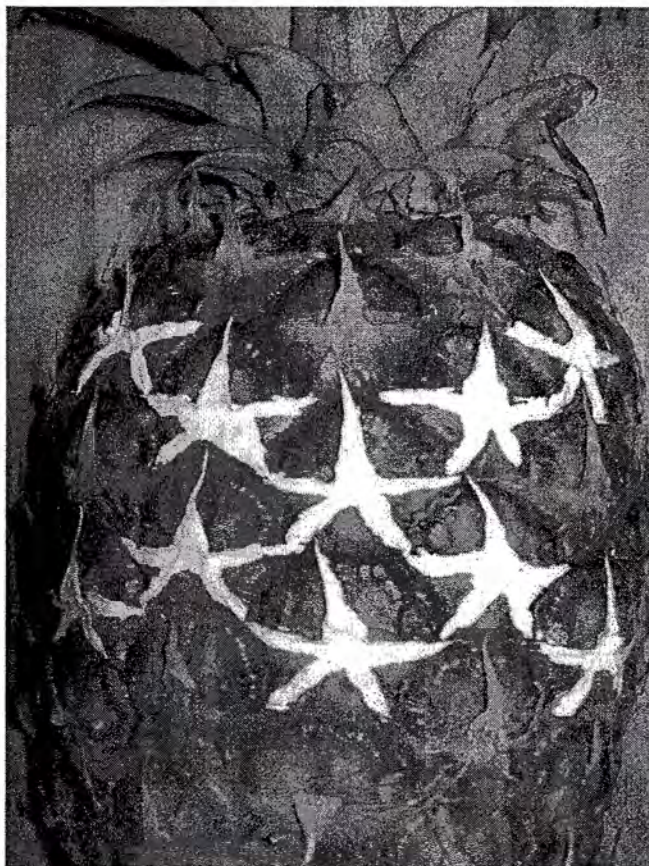
(Lines here indicate direction, not the same spiral.)

Pineapple Stars

Look closely at a pineapple and you can see the hexagonal fruit pods linking up and around it. Observe closer and see that within each Hexagon is a five-pointed star.

Use colored pencils to shade the stars on these pictures of pineapples. Or, better yet, use a fine brush to paint the stars on an actual pineapple.

They remind me of people joined in a spiraling web around the whole pineapple.



Weave a Pineapple Basket

Color and then cut out this overall design. (You may first want to enlarge it on a copier).

Cut the 13 individual strips along the dashed lines, all joined to the center strip.

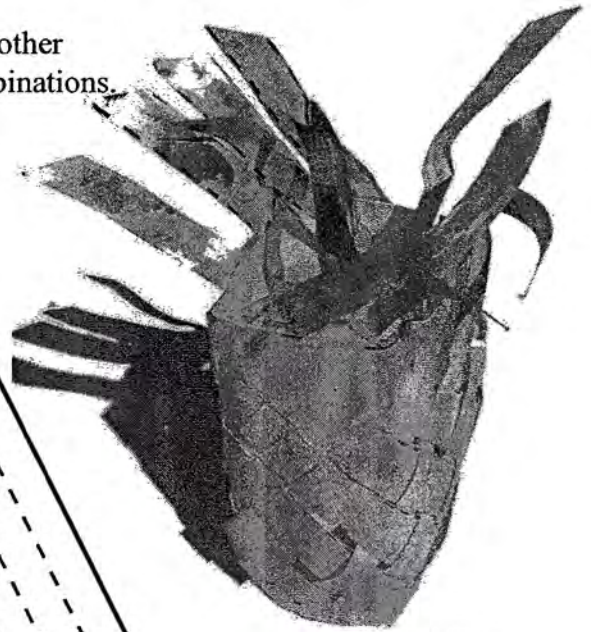
Cut *slots* along the dashes between the 8 strips, but all remain joined at both ends.

Fold up in half along the dotted line.

Roll into a cylinder and tape the bottom as a circle.

Weave the 13 strips over-under through the 8 slots.

Leaves should stick up at top. This may also be done with other consecutive Fibonacci number combinations.



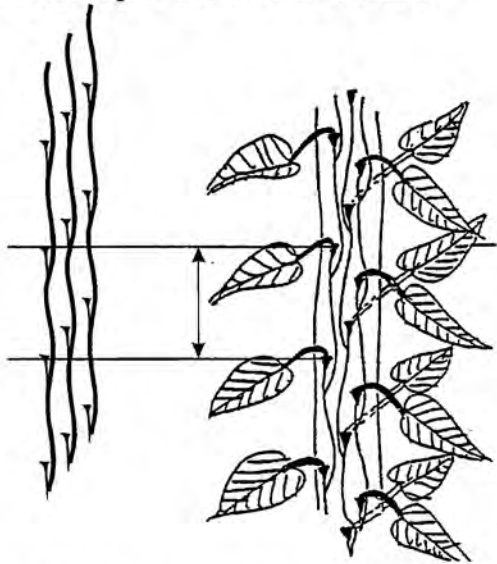
8 and 13 strips



13 and 21 strips

Fibonacci Branching

When you're around plants, especially vertical ones like sunflowers or beans, observe where the branches emerge from their central stem. Just above and below each connection point you might see the vein which feeds it running vertically along the stem. Notice how a vein goes through one branch and travels upward and downward, bypassing some branches until it reaches another branch to feed and continues past it in the same manner. If you find a dry, thin branch with other branches emerging, you might paint the different veins rising around the stem connecting different sets of branches. The number of veins will be a Fibonacci number, providing the most efficient arrangement for minimum overlap of branches and leaves!



This is the vein pattern for a plant with three veins, feeding every third branch or leaf.



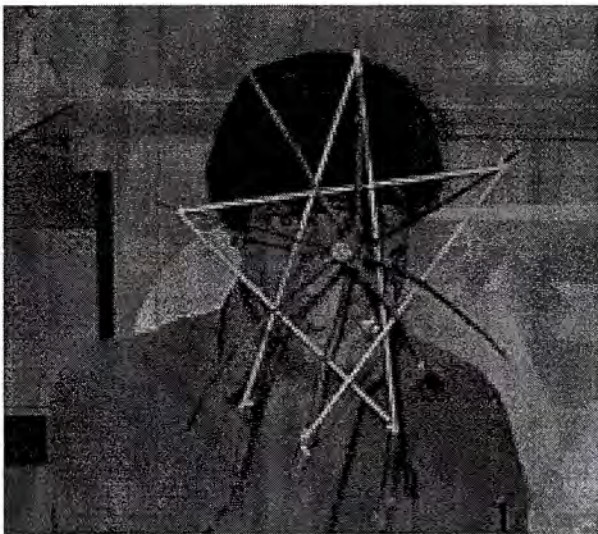
This dry branch has vein paths indicated by paint. They bypass some branches and feed others.

The study of branching patterns is called "phyllotaxis" ("leaf arrangement"). Research the topic to learn more!



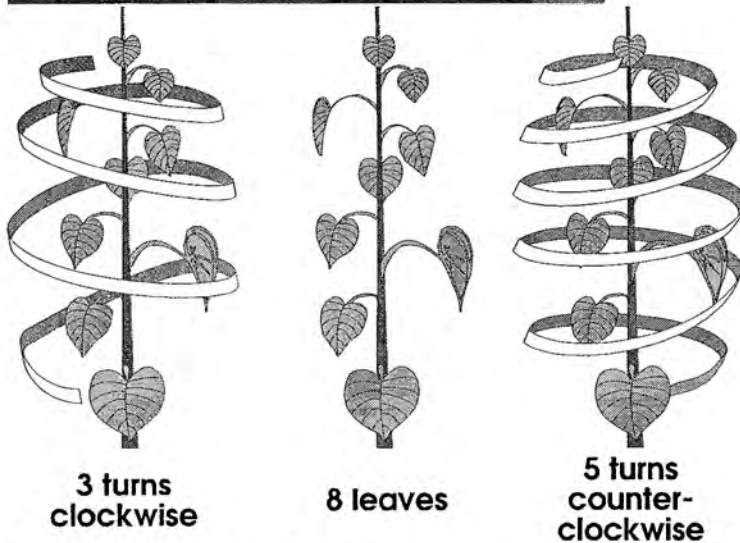
You can also vertically connect branches which share the same vein with colored yarn or ribbon. Shown here are ribbons on a plant having five veins.





Yet another activity is to connect the tips of consecutive branches in the order they emerge from the central stem. This will produce an ongoing star up the plant having a Fibonacci number of points, depending on the vein pattern.

A five-pointed star is shown here.



Finally, notice that as the branches emerge from the central stem, they form spiral staircases in two opposing directions. The number of spiral turns between corresponding vertical leaves sharing a vein will be Fibonacci numbers. These may be connected with a long chain of pipe cleaners or wire which will hold the spiral shape.

The number of spirals in each direction, along with the vertical lines, until the next vein-sharing branch or leaf is reached, will be three consecutive Fibonacci numbers.

Which Fibonacci formulas involving three consecutive Fibonacci numbers can teach us about what is going on here?



Using pipe cleaners to spirally connect successive branches in one or both directions.



A Spiraling Plant Model

An unusual model to make showing the growing spirals of plants can be built upon a screened kitchen strainer. It can be any size, but the larger ones hold more leaves.

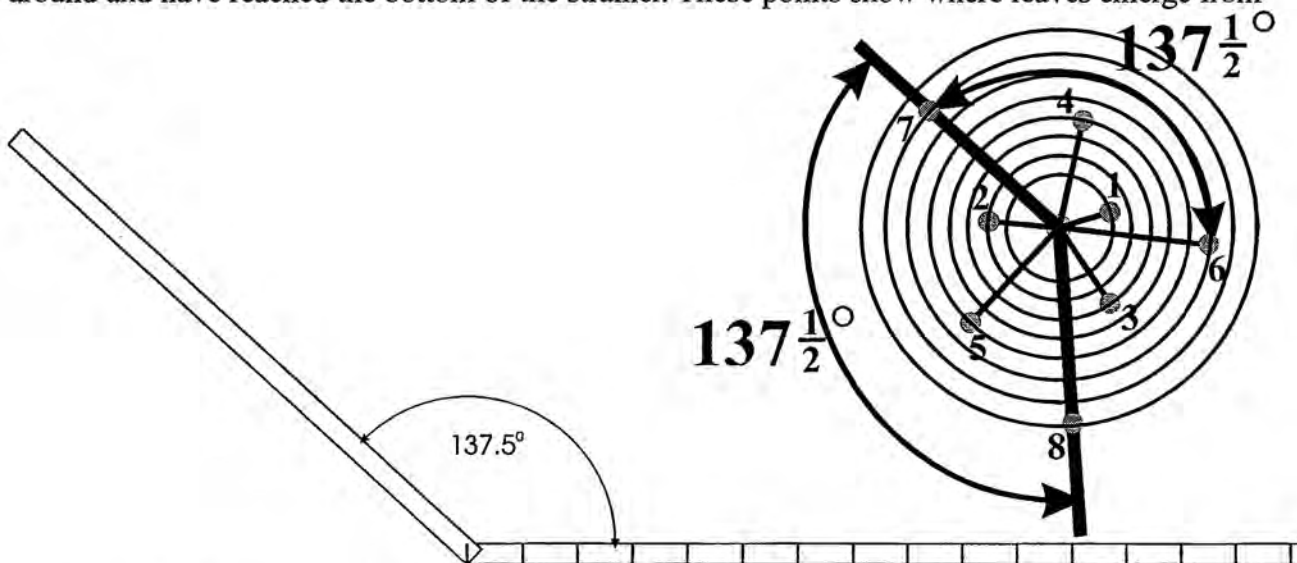
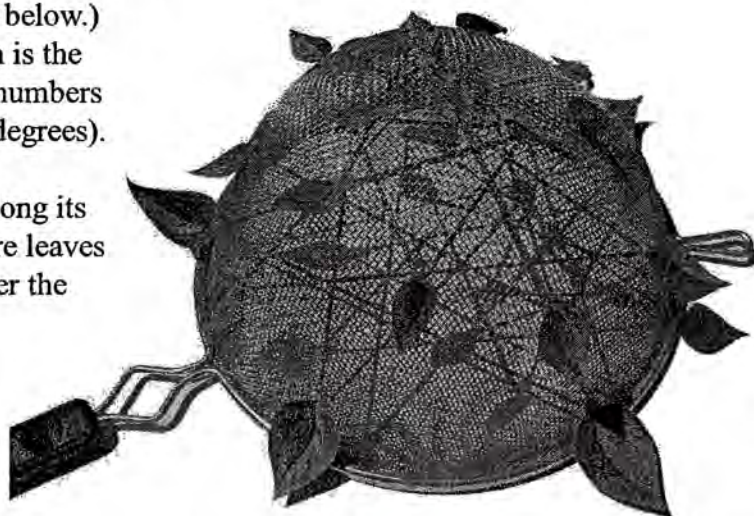
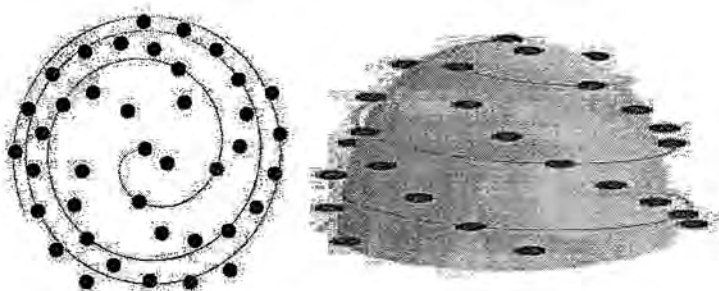
The first step is to tape two strips of paper together at an angle of 135.5 degrees. (See below.) Use a protractor to find this angle (which is the ratio of two consecutive high Fibonacci numbers times 360 degrees, subtracted from 360 degrees).

One strip has equally spaced markings along its length. The closer the marks are, the more leaves will appear around the strainer and clearer the spirals will be.

A thumbtack goes through the strips' meeting point into the *top center* of the strainer so that it can rotate. The length of the strips should reach the strainer's circular circumference.

With a marker pen, make a point on the strainer next to the first (top) mark on the strip. Then, rotate the strips so that the blank strip now rests on the first point marked. This has moved the marked strip ahead by 137.5 degrees.

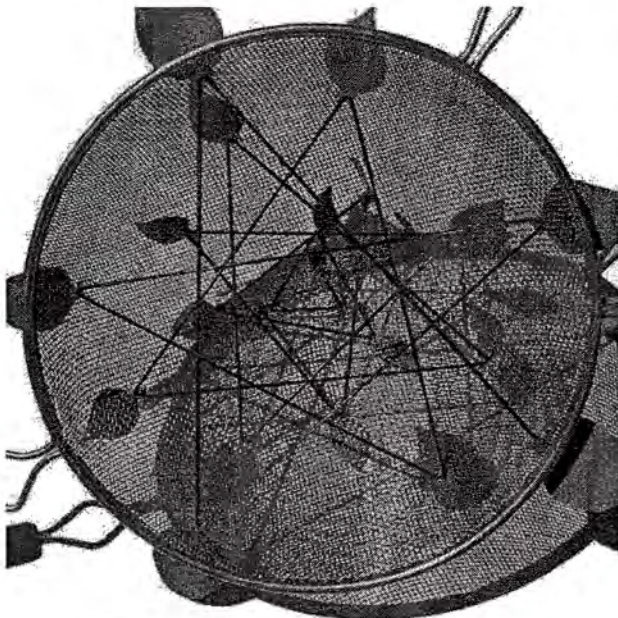
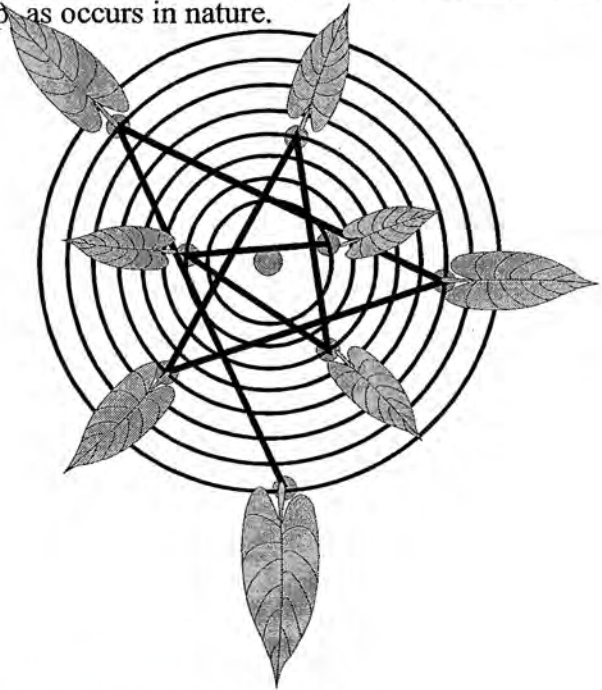
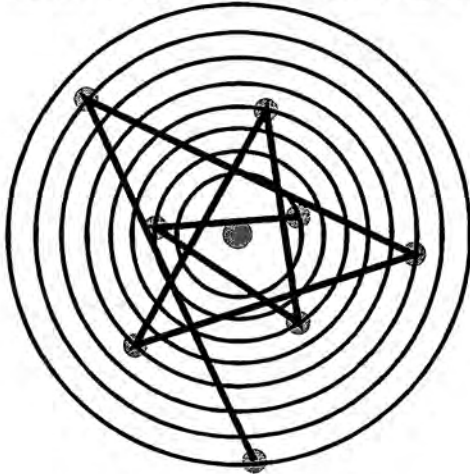
Then make a second point next to the second line on the strip. Again, rotate the strips by 137.5 degrees and make a third point on the strainer next to the third line on the strip. Repeat this procedure, making a point on each next line down, rotating the strips, until you have gone around and around and have reached the bottom of the strainer. These points show where leaves emerge from



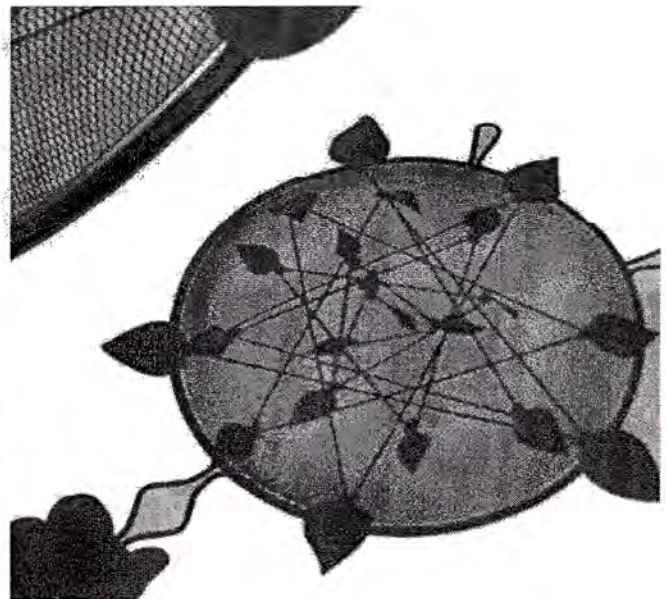
the strainer.

Before attaching the leaves, turn the strainer over and connect each point with a long thread through each marked hole. This will make an ongoing five-pointed star from the top center to the periphery. Then, turn the strainer back over and glue larger and larger "leaves" (which you have hand-drawn or printed on a computer, larger and larger in size) on each point around the strainer, with the largest leaves at the lower periphery.

This will produce a pattern of leaves spiraling in two directions at once. The closer the marks were on the strip and the more leaves you create, the more this becomes visible. This arrangement gives each leaf the most exposure with minimum overlap as occurs in nature.



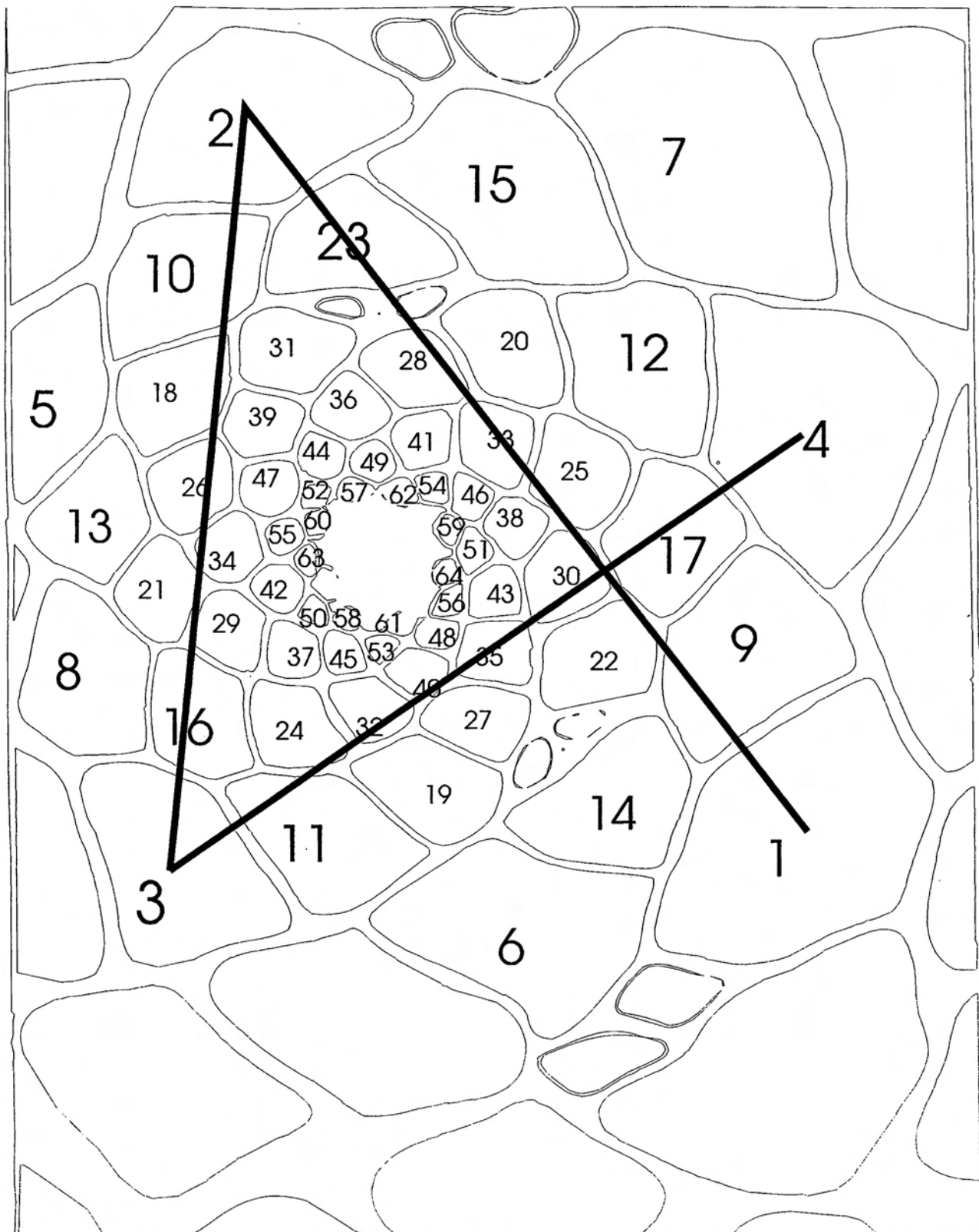
The inside of the strainer reveals an ongoing "nearly" five-pointed star.



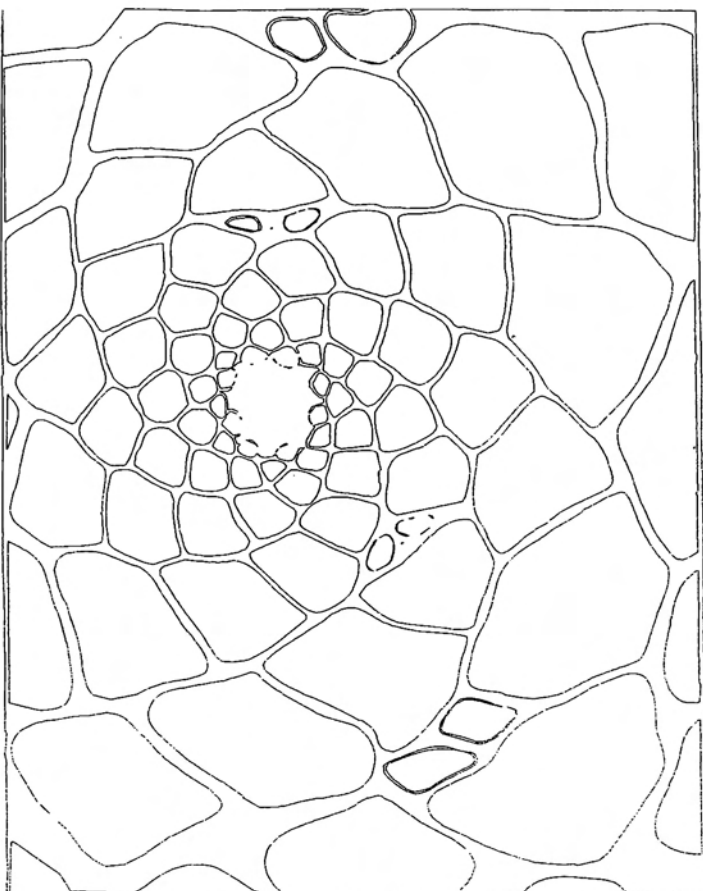
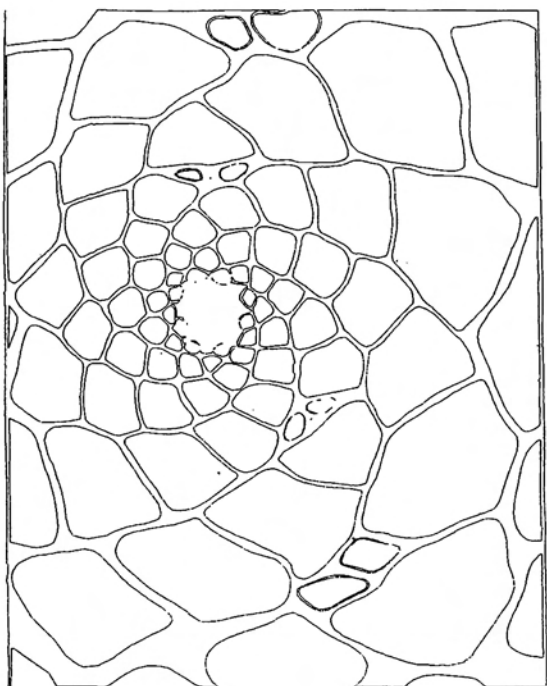
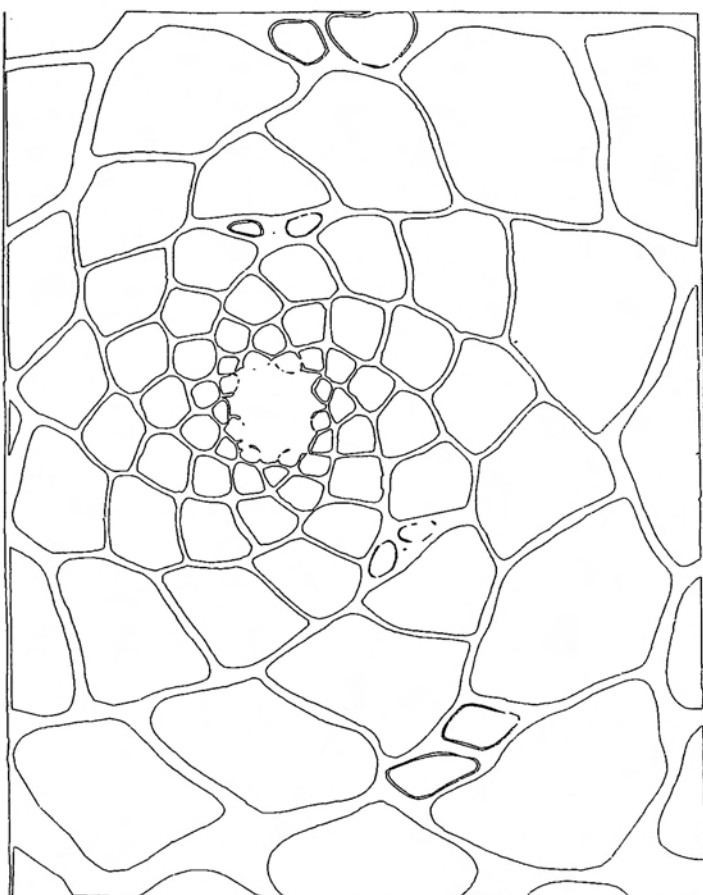
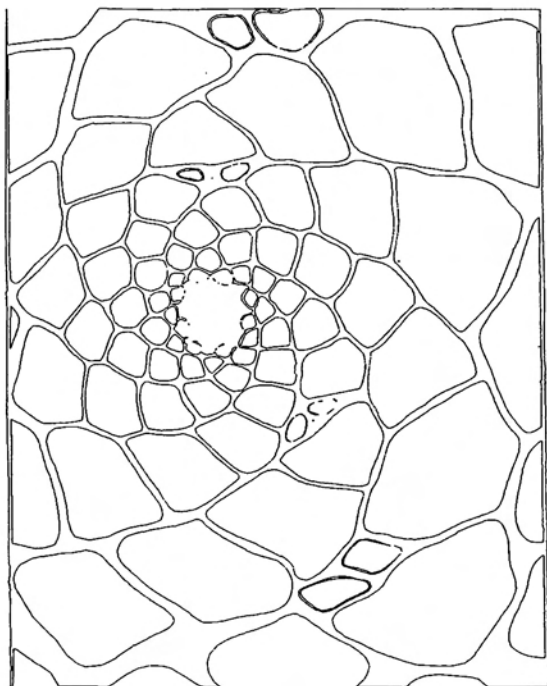
The strainer casts interesting shadows.

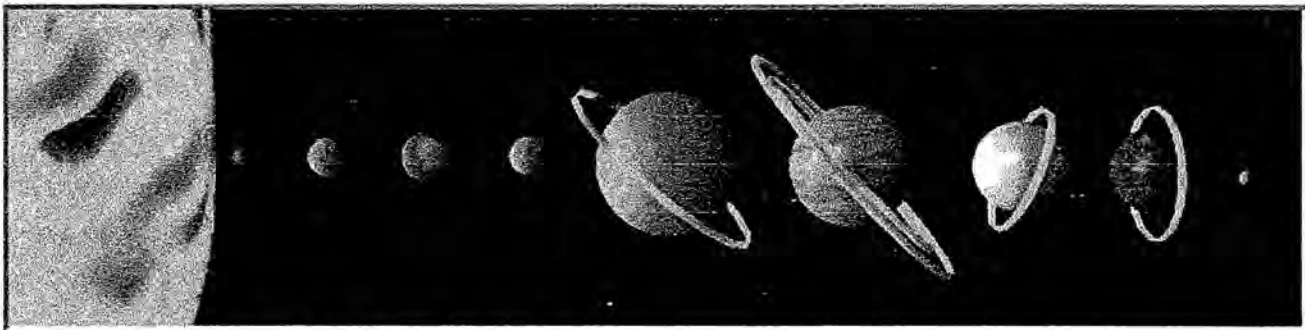
Pentagonal Growth Pattern

Here's the numbered growth sequence of a pinecone. Use a straightedge to connect each number in sequence (from 1 to 64) and you'll see the ongoing "nearly" five-pointed star by which it grows. Use colored pencils to shade the bract (island area) spirals.



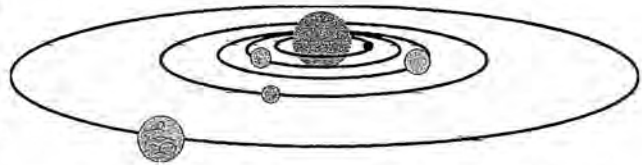
Use colored pencils to shade these 8 x 13
spiraling rows in each direction, or any
way you choose!





Fibonacci Numbers In The Solar System

Fibonacci number patterns appear not just on Earth but wherever accumulative relationships occur in the ongoing creating process known as the universe.

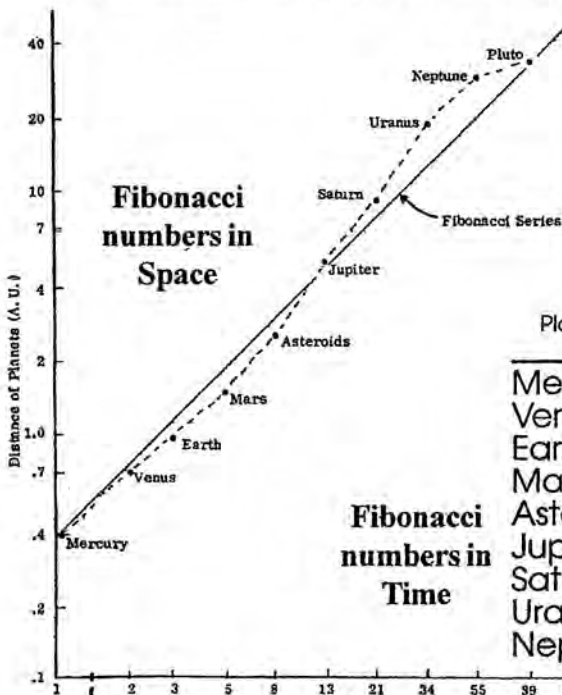


Enlarge your imagination from your local surroundings to the whole Earth among its family of planets, our solar system.

When depicted in one scene, our solar system must appear distorted. The sun is so large that when the planet *sizes* are shown relatively correctly, the distances between them must be foreshortened. Or when the *distances* are shown correctly, the planet sizes must be shrunk too small to see. Diameter and distance cannot both be shown easily or accurately in proper proportions on one page.

Yet Fibonacci numbers appear both in space and in time. First, consider the spacing of the planets. The graph below (left) plots the planet distances logarithmically (so they fit on one graph) in terms of distance from the sun (in Astronomical Units, where Earth = 1) on the Y-axis. Fibonacci Numbers are equally spaced along the X-axis. The solid line represents the ideal, which natural forms approximate, and planets strive for, but never perfectly reach.

Fibonacci numbers also appear in the way planets synchronizations in *time*. If we consider the time (in



"Earth days") planets take to revolve around the sun, the ratios for consecutive planets (below) approximate the sequence of consecutive Fibonacci numbers too, showing that the solar system is a harmoniously synchronized whole. Use your calculator to confirm the Fibonacci fractions below.

Planet	Orbital Period: Number of "Earth Days" in one revolution around the sun		Fibonacci Number Relationships
	Actual	Rounded Ideal	
Mercury	88	87	13
Venus	225	227	34
Earth	365	366	8
Mars	687	596	13
Asteroids	1,200-2,000	1,550	3
Jupiter	4,333	4,133	8
Saturn	10,760	10,333	1
Uranus	30,685	31,000	3
Neptune	60,190	62,000	2

Audible Fibonacci Rhythms

Traditionally, geometry is considered to be the expression of number in space, while music and song express numbers in time. This can occur as *meter*, the number of syllables or beats in each line of Proto-Limericks and Limericks which makes them "feel right". Consider these:

Proto-Limerick:

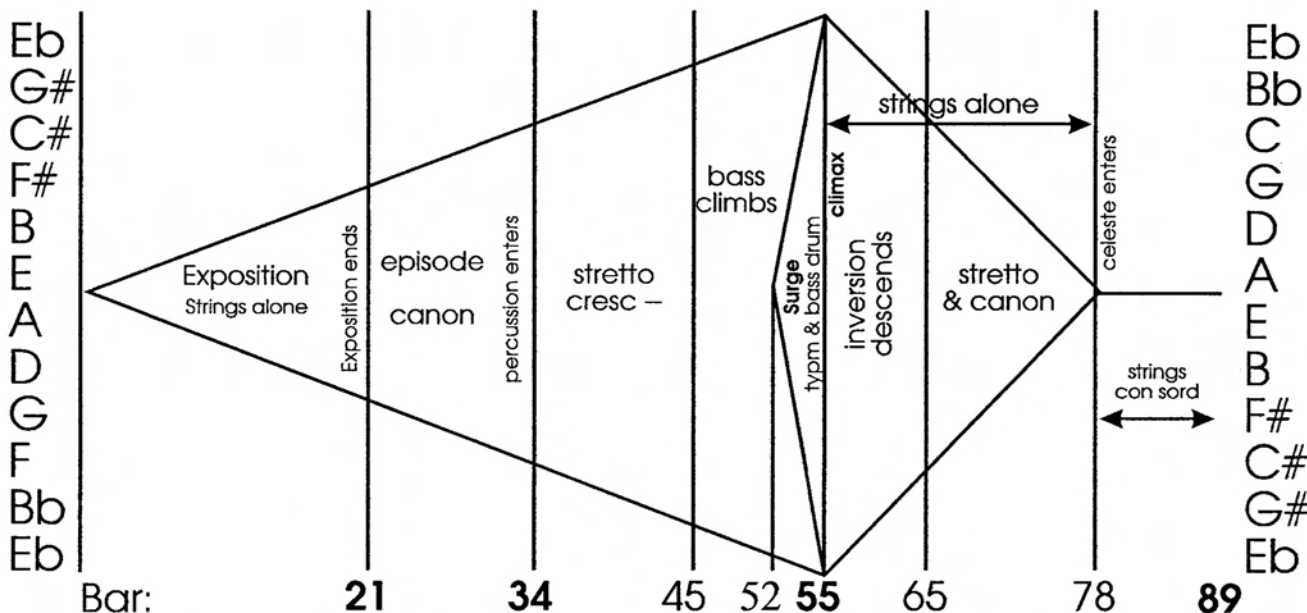
5 syllables	di dum di di dum	He skewered the slug
5 syllables	di dum di di dum	And gave it a tug.
3 syllables	di dum di	This cabbage,"
3 syllables	di dum di	said Babbage,
5 syllables	di dum di di dum	"Is hard to debug."

Limerick:

8 syllables	di di dum di di dum di dum	There was an Old Man with a beard,
8 syllables	di di dum di di dum di dum	Who said, "It is just as I feared! -
5 syllables	di dum dum di dum	Two Owls and a Hen,
5 syllables	di dum dum di dum	four Larks and a Wren,
8 syllables	di di dum di di dum di dum	Have all built their nests in my beard!"
		-- Edward Lear

Use Fibonacci number meter patterns to *make up your own* Proto-Limericks and Limericks!

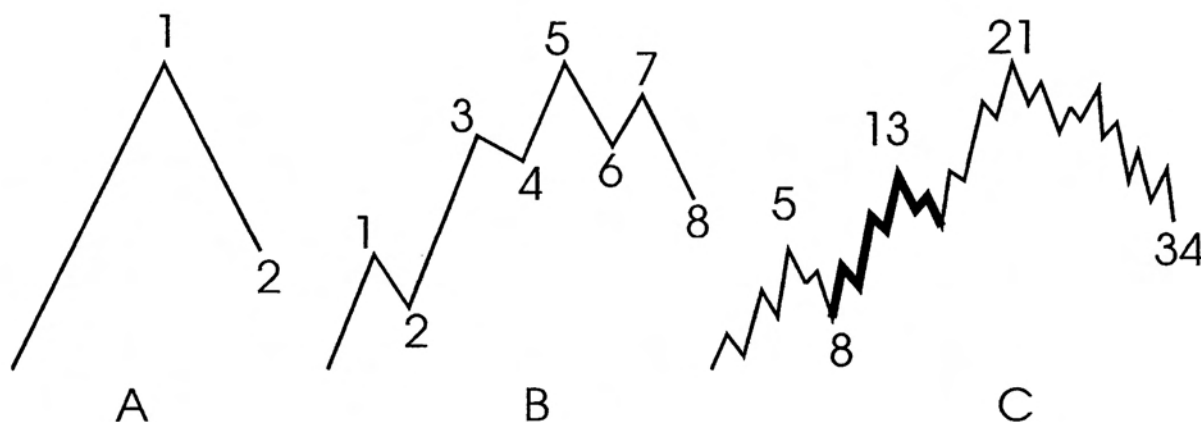
Fibonacci numbers have also been applied to music. Here, the composer Bela Bartók (who kept pinecones, nautilus shells, and other natural-world examples of Fibonacci numbers on his writing table) used Fibonacci numbers to time his piece called "*Music for String Instruments, Percussion and Celeste I. Fugue*". Notice how different instruments enter and leave at Fibonacci number bars, as well as the entire piece building to bar 55, then diminishing to the end at bar 89. Do research to find more musical details about this work, and about other composers and works in which Fibonacci numbers have been used. Listen to them and evaluate their "musical proportions" for yourself.



Fibonacci Numbers In The Stock Market

Some people say that as humans, being part of nature, we unconsciously manifest nature's rhythms in our decisions. Many stock market analysts see changes in stock prices as mass reactions to events, reflecting our opinions, valuations and expectations. Thus, Fibonacci numbers might be found in the patterns of high and low prices of stocks. Some analysts, students of W. D. Gann and R. N. Elliott, use them as a tool for helping to identify key turning points in the market and garner a profit.

In this theory, all major market moves may be described as a series of waves up and down (Figure A below). In Elliott Wave Theory, the basic wave (B) consists of an initial movement up, a second movement down, then the third wave (usually the largest) up again, followed by another retracement down, and finally the fifth movement which exhausts the momentum upward. As long as stock volume increases, another advance is due. These five are followed by three: a movement downward, a retracement up, and a final adjustment downward, making a total of $5 + 3 = 8$, the basic Fibonacci pattern.



In addition, each of the major waves (the numbered up and down movements) can themselves be separated into subwaves of the same pattern in miniature (C), and so on, always exhibiting Fibonacci relationships:

- Point 1 in Figure B is point 5 in Figure C
- Point 2 in Figure B is point 8 in Figure C
- Point 3 in Figure B is point 13 in Figure C
- Point 5 in Figure B is point 21 in Figure C
- Point 8 in Figure B is point 34 in Figure C

This is the idea of the fractal seen in the structure of trees and all through nature, where the parts model each other and the whole.

Look for yourself at the stock market pages of your newspaper. Find graphs showing how the market rises and falls and judge for yourself whether this is valid. If you see the Fibonacci pattern, then make a prediction. If you'd like to research the topic further, there's a lot of detailed information online and in many books.

Fibonacci Patterns In Art

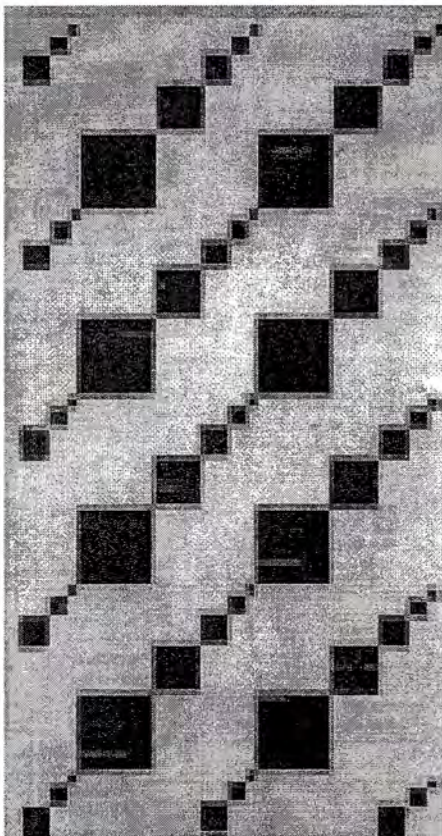
Modern artists sometimes make Fibonacci patterns visible. Here are a very few examples of weaving (quilts, rug and baskets), jewelry and furniture.

Can you find how Fibonacci numbers are used in each?

What can *you* think of designing with Fibonacci numbers?



Quilt
Kirsten Boucher

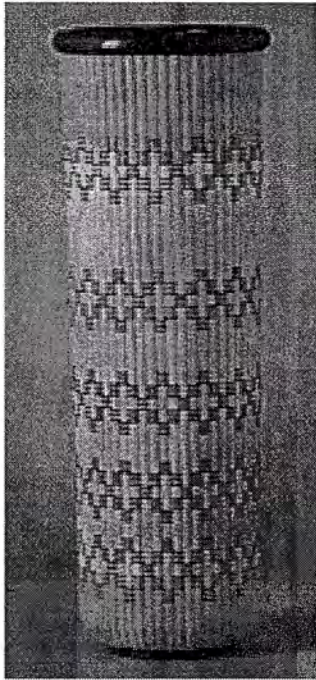


Rug
Nancy Kennedy

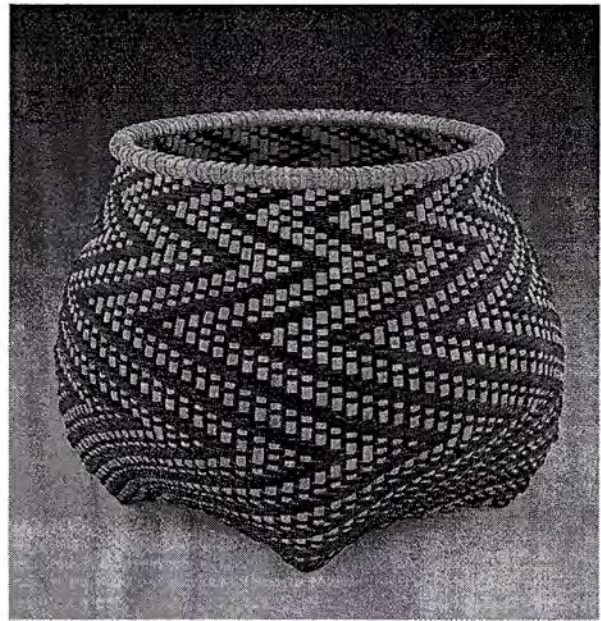


Quilt
Autumnal Equinox
Caryl Bryer Fallert

Basket
Joan Brink



Basket
Billie Ruth Sudduth

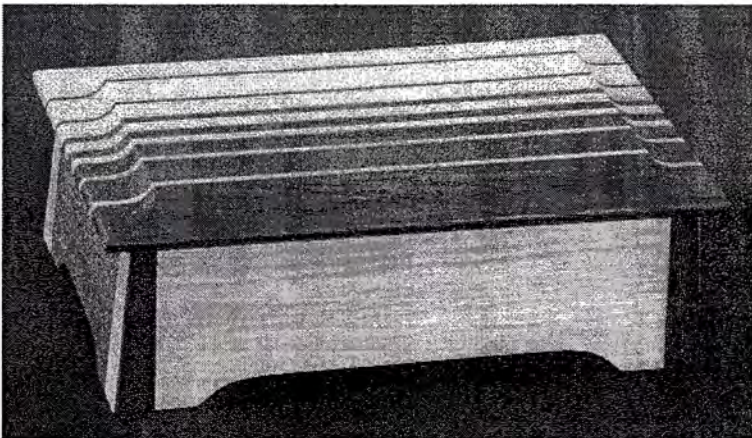


Neck Ring
Gretchen McPherson

Chest
Stephen Mildenhall

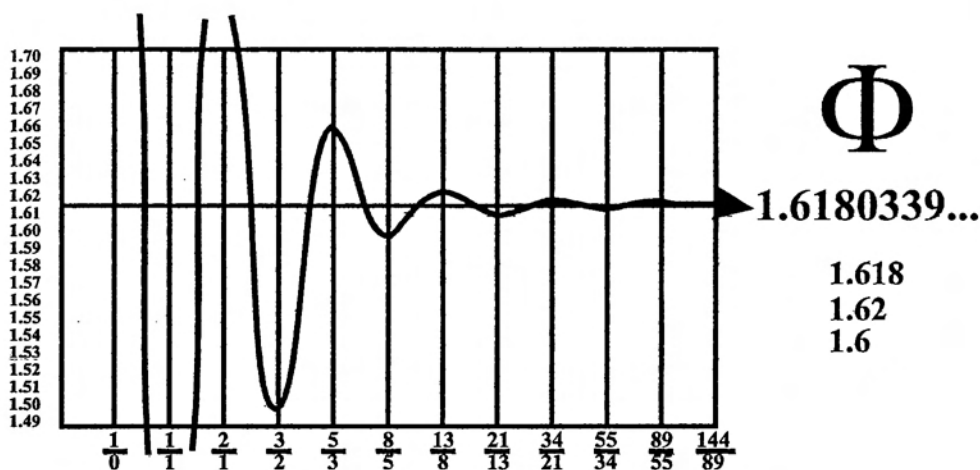


Box
Roger Gifkins



Phi -- The Golden Mean

The Fibonacci numbers accumulate forever. But the *ratio made of every two consecutive Fibonacci numbers* oscillates over and under an ideal value it always approaches but never reaches. You can see this by graphing the ratios of each two consecutive Fibonacci numbers:



The ratios of low Fibonacci numbers begin wildly high, but quickly zoom towards their ideal relationship. This is called by mathematicians an "asymptote", something approached but never actually reached. And while we call it a "number" we're actually looking at a *relationship*. This value, 1.618..., has been symbolized by the Greek letter Φ , "Phi", named after the Greek sculptor and designer of the Parthenon, Phidias, who used it in his works. It has also been called by other names, including the Golden Mean, Golden Ratio and Golden Section. The word gold indicates that it's highly regarded.

While this relationship of Fibonacci numbers is associated with Phi, don't be fooled by appearances. It's not the Fibonacci numbers themselves which makes this possible -- it's their *accumulative process* which funnels their ratios towards Phi.

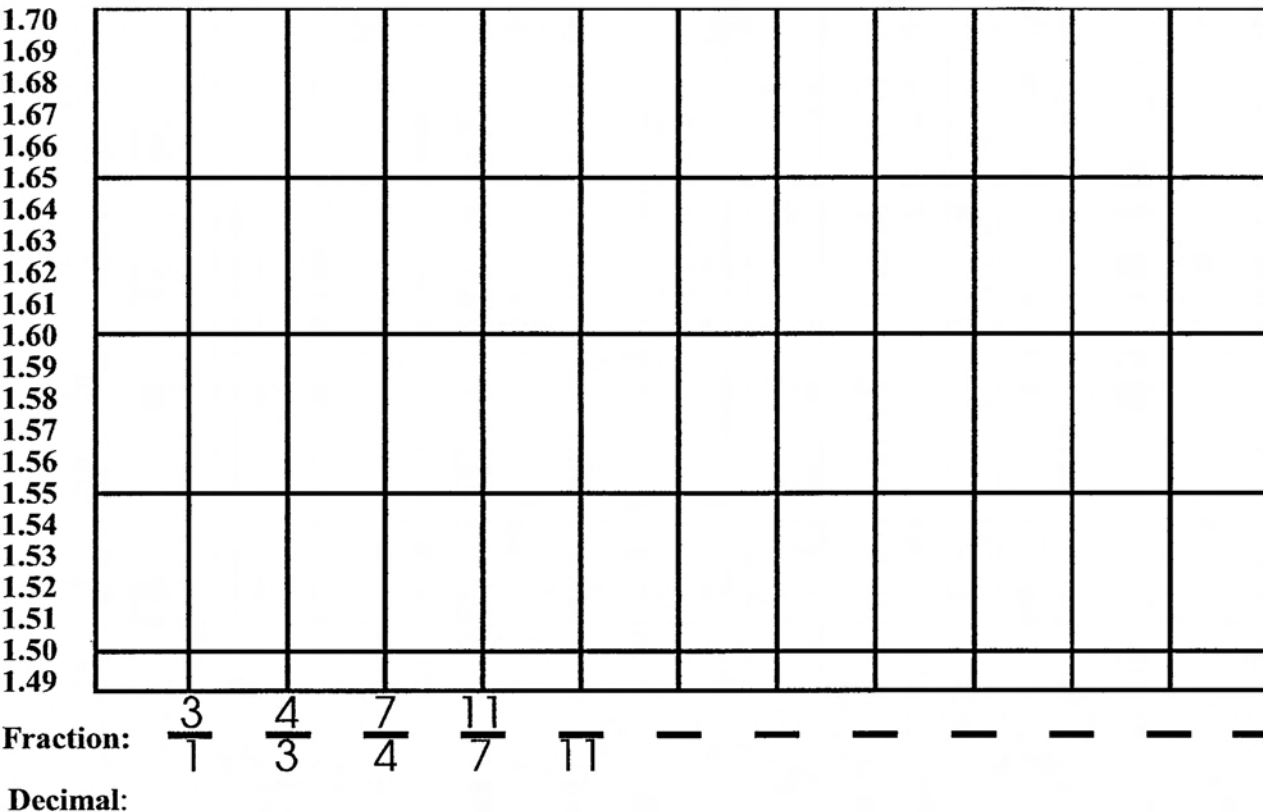
To prove it, pick *any* two numbers (and I do mean *any*), not just the zero and one which begin the Fibonacci sequence. If you apply the "recursive" (self-referential) rule that each two numbers add to make the next, you'll come up with a new series of numbers. When you graph the ratios made by these consecutive numbers they'll always approach 1.618... but, like the Fibonacci numbers, will never exactly reach it.

The Fibonacci sequence was the first recursive series known in the West. Another recursive series is called the Lucas series. It begins with 1 and 3, then follows the recursive rule about adding each two terms to make the next. Continue the series in the space below. Then, make ratios and graph them on the next page to see how they approach the same value of Phi that the pairs of Fibonacci numbers do.

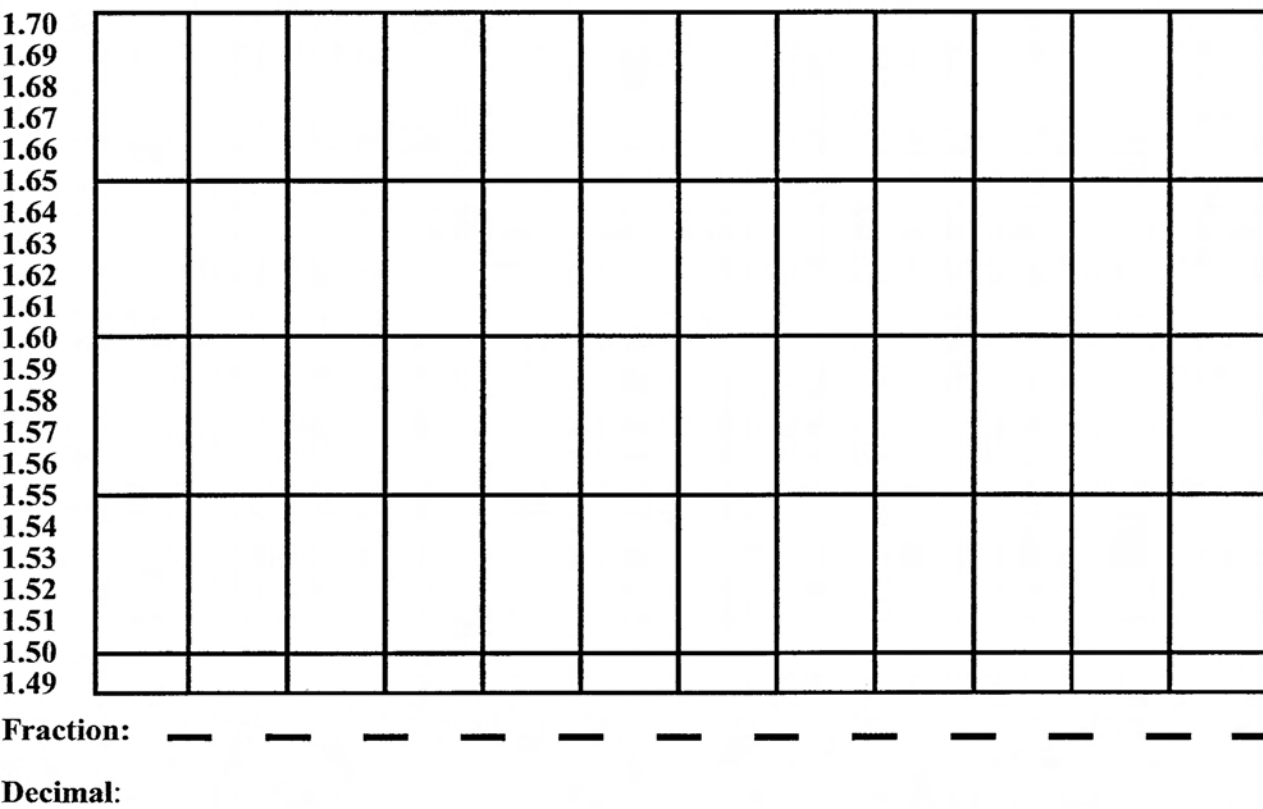
Continue the Lucas series here: 1, 3, 4, 7, 11,
Graph their ratios on the next page (top).

Make up your own number series. Start with *any* two numbers (your birthdate?) and add each two to make the rest. Then, graph their ratios on the next page (bottom) to see how quickly or slowly their fractions aim towards the Phi ideal!

Graph the ratios of terms of the Lucas series here. Use a calculator to determine the decimal value each ratio (write them below the fractions). Does it oscillate above and below the value of Phi?



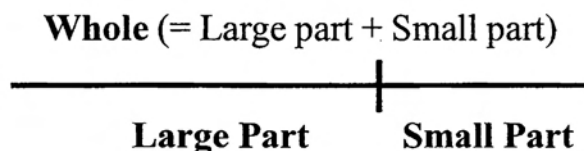
Graph the ratio of terms of a recursive number series that you make up:



Why would Phi be the "infinitely-ahead" goal of the ratios of *every* number series which grows by accumulating its past two terms onto itself? Why is that value so important?

As we saw earlier, the Fibonacci numbers weave interrelating patterns among each other and with the whole. So another way to find Phi is to ask a question about this special interrelationship, phrased as a mathematical equation and then solved. It will be a question about balancing the parts and the whole.

The question we would ask, in words, is this: Is there one mathematical balancing point on a line which divides it into two unequal parts in such a way that *the whole line relates to the large part in the same way that the large part relates to the smaller?*



Transformed into a "word equation":

$$\frac{\text{Whole}}{\text{Large Part}} = \frac{\text{Large Part}}{\text{Small Part}}$$

Now we'll convert the words into symbols, calling

the **Small part** = 1
the **Large part** = x (= the unknown balance)
and the **Whole** = $x + 1$ (= sum of both parts).

Finally, we can restate the "word equation" with abstract mathematical symbols, as

$$\frac{x + 1}{x} = \frac{x}{1} \quad \text{For what value(s) of } x \text{ is this true?}$$

If we multiply both sides by x , we get the equation $x + 1 = x^2$

That's interesting! It says that adding one to the balancing point is the same as multiplying it by itself!

One solution to this equation reveals that this unknown balancing point called x has a value equal to

$$\frac{1 + \sqrt{5}}{2} \text{ which equals Phi } 1.6180339\ldots! \text{ Try it on your calculator.}$$

What this mathematical investigation tells us is that **where Phi appears, some sort of balance is taking place among parts and whole**. Consider plants with their branches, leaves, flowers, vegetables and fruits whose designs involve two Fibonacci numbers spiraling in opposite directions. Each pair of Fibonacci numbers represents an approximate value of Phi. Plants and their parts since they obviously get what they need with low Fibonacci numbers to achieve balance and don't need a ratio any closer to the ideal value.

This value Phi, like its ancestors the Fibonacci numbers, has some interesting properties which derive from this internal balance of parts with each other and the whole. As we saw, **Phi is the only value where adding unity (one) is the same as multiplying it by itself** (squaring it):

$$\Phi + 1 = \Phi \times \Phi$$

$$(1.6180339... + 1) = (1.6180339... \times 1.6180339...) = 2.6180339...$$

Prove it on you calculator!

What about Phi's inverse? Of course, any number times its own inverse equals unity: $\Phi \times 1/\Phi = 1$
But Phi has another property too: it is **the only number that equals 1 plus its own inverse!**

$$\frac{1}{\Phi} = 0.618...$$

$$\Phi = 1 + \frac{1}{\Phi}$$

$$1.6180339... = 1 + 1 / 1.6180339... = 1 + 0.6180339...$$

Prove it on you calculator!

So Phi and its inverse have this unique and remarkable relationship with unity.

Adding 1 to Phi squares it. Subtracting 1 inverts it.

Phi is the only "number" which will do this.

It's worth pondering the equations which describe such balance.

Not only are the powers of Phi additive, but they're also multiplicative; that is, they're each 1.618... times larger than the previous power (and 0.618... times smaller than the next larger power of Phi). This simultaneous additive and multiplicative growth only happens with Phi. Resembling the growth of Fibonacci numbers, **each two powers of Phi add together to make the next power**. If you see the patterns, then continue them below. (Notice how the Fibonacci numbers weave through its structure!)

$\Phi^{-1} + \Phi^0$	$= \Phi^1$	$= 0 + 1\Phi$	$= 1 + 1/\Phi$
$\Phi^0 + \Phi^1$	$= \Phi^2$	$= 1 + 1\Phi$	$= 2 + 1/\Phi$
$\Phi^1 + \Phi^2$	$= \Phi^3$	$= 1 + 2\Phi$	$= 3 + 2/\Phi$
$\Phi^2 + \Phi^3$	$= \Phi^4$	$= 2 + 3\Phi$	$= 5 + 3/\Phi$
$\Phi^3 + \Phi^4$	$= \Phi^5$	$= 3 + 5\Phi$	$= 8 + 5/\Phi$

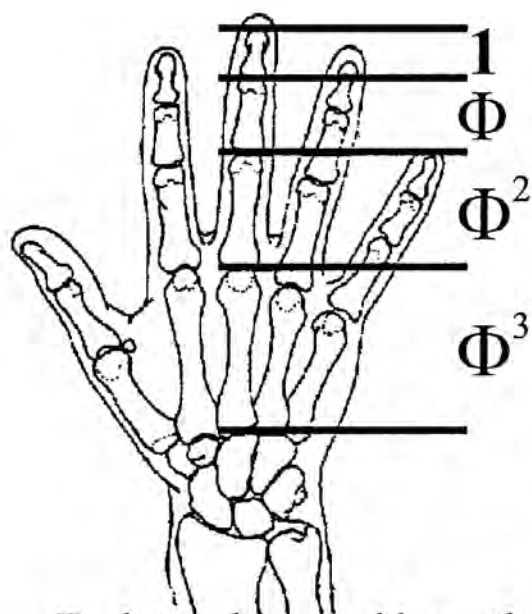
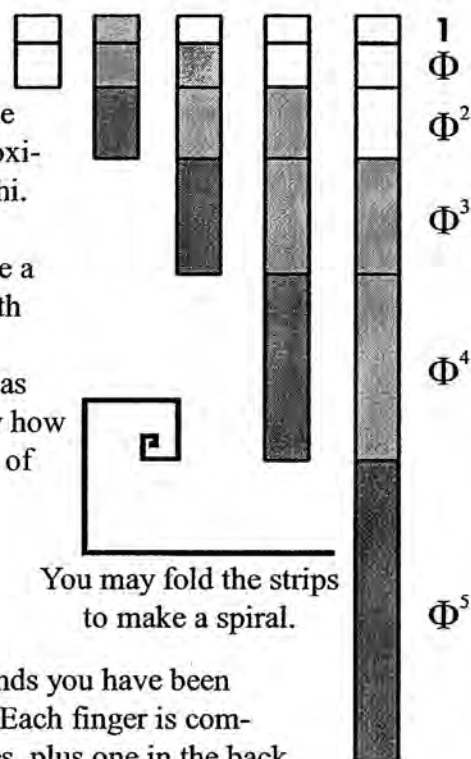
Use your calculator to prove these relationships.

These mathematical equations represent *eternally true ideas* about growth by accumulation, balance and wholeness. Such ideas have far-reaching effects when they manifest in nature and human affairs.

Phi Body Balance

Familiar examples of the balance of parts with each other and the whole are found in the structure of our own body, which is approximately built along the proportions of the expanding powers of Phi.

To see this type of growth, take a long strip of paper and measure a short distance from one end, and fold it there. Multiply this length times 1.618 and fold the strip at that point too. Then, each new section is found by folding each last two sections. Continue this as long as your paper strip will allow. These growing sections show how the different parts of the body, and the whole, expand by powers of Phi.



Each two bones add together to equal the next bone

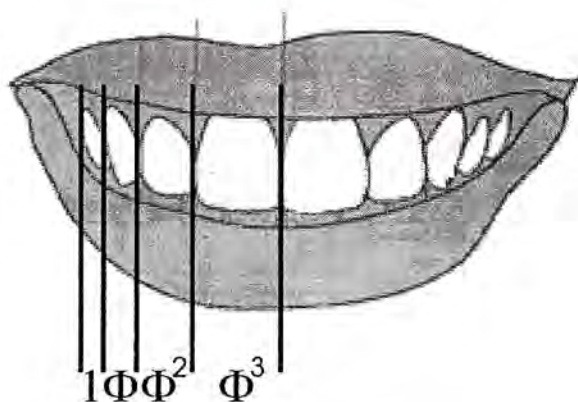
First, look at the hands you have been folding paper with. Each finger is comprised of three bones, plus one in the back of your hand. (Can you feel it?) Each bone is longer by the unfolding powers of Phi, and equals the sum of the previous two.

Prove it: match up the two smaller bones of one finger with the third bone of the *same finger on your other hand*. They should be equal. Then match the second and third bones on a finger of one hand to the fourth bone of your other hand. They should match too. This interconnected growth relationship among our bones allows us to perform delicate and gracefully balanced movements.

Dentists point out that the widths of our teeth, as seen from the front, ideally approximate the series of unfolding powers of Phi, where each two smaller widths add together to make the next larger one.

This arrangement of the parts (teeth) with the whole (mouth) is said to fit and look best together.

Smile!

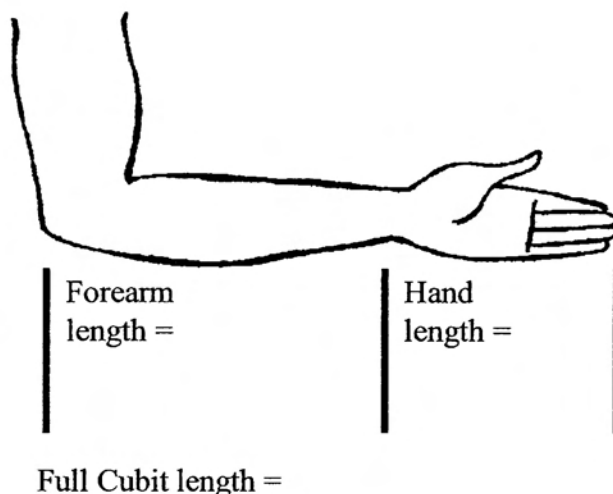


Our *cubit* is the ancient measure from our elbow to our middle finger tip.

Try this: Use a ruler to measure the length of your full cubit and write it under the cubit in the picture.

Then measure the length of your forearm from your elbow to your wrist. Write this distance under the picture's forearm.

The length of your cubit minus the length of your forearm will tell you the length of your hand.



Write this under the hand in the picture to the right.

Now, see how close their fractions come to the value of 1.618

$$\frac{\text{Cubit length}}{\text{Forearm length}} = \frac{\quad}{\text{Fraction}} = \quad \text{Decimal}$$

$$\frac{\text{Forearm}}{\text{Hand}} = \frac{\quad}{\text{Fraction}} = \quad \text{Decimal}$$

Remember: The Phi ideal is found *in the average of many people* and rarely is found to be exact in any one person.

Not just our arm and hand but our whole body is linked in an approximate Phi balance.

Have a friend help to measure your full height.

Use your calculator and multiply your height by the inverse of Phi = 0.618.

Height x 0.618 =

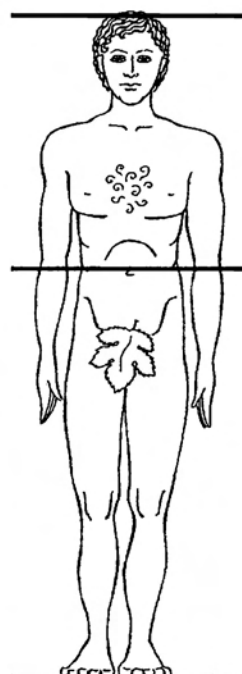
Does this come close to the distance from your feet to your navel?

Fill in these fractions and see how close you come to the ideal Phi ratio (which few people exactly equal):

$$\frac{\text{Height}}{\text{Feet to Navel}} = \frac{\quad}{\quad} =$$

Fraction

Decimal



Man by Libby Reid

$$\frac{\text{Feet to Navel}}{\text{Head to Navel}} = \frac{\quad}{\quad} =$$

Fraction

Decimal

Now multiply that distance from the floor to your navel by the inverse of Phi = 0.618.

Floor to navel x 0.618 =

Does this match the reach of your arms, the distance from the floor up to your fingertips with your arms at your side?

Measure and make fractions (and decimals) of these actual lengths to see if they are close to Phi.

$$\frac{\text{Feet to Navel}}{\text{Navel to fingertips}} = \frac{\quad}{\quad} =$$

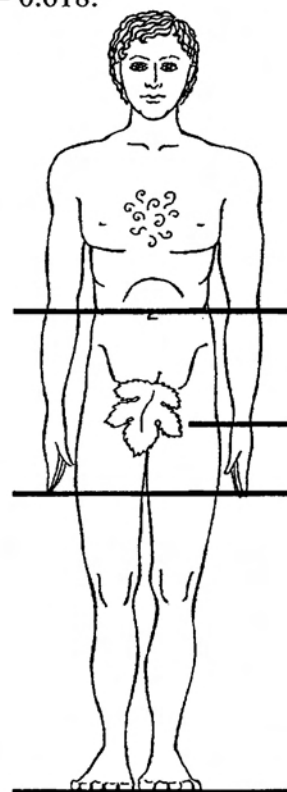
Fraction

Decimal

$$\frac{\text{Feet to Fingertips}}{\text{Navel to fingertips}} = \frac{\quad}{\quad} =$$

Fraction

Decimal



Next, multiply that distance from your feet to your fingertips by the inverse of Phi = 0.618 to see if that indicates the distance from the floor to the bottom of your knees.

Put your actual measurements into these fractions and see if they approach Phi:

$$\frac{\text{Feet to Fingertips}}{\text{Feet to Knees}} = \frac{\quad}{\quad} = \quad$$

Fraction Decimal

$$\frac{\text{Feet to Knees}}{\text{Knees to Fingertips}} = \frac{\quad}{\quad} = \quad$$

Fraction Decimal

Finally, we can see our entire body as one whole, connected in seven sections between eight levels by Phi balance.

Please see Volume 2, Chapter 5 for the construction of the levels of our body using a Pentagon.

Consider the distance from the tip of our nose up to the top of our eye socket as one unit. The distance from our eye socket up to the crown of our head should be about Phi times larger. And the distance from the tip of our nose down to our chin (or neck bottom) should also be Phi times the unit. From there downward, each two sections should equal the next, to reach the soles of our feet.

Measure these levels of your body and fill out the fractions and decimals to see how close your body comes to the ideal. Note: Rarely does anyone ever equal the ideal! And we don't usually reach our full proportions until the age of 21.

$$\frac{\text{Crown to eye}}{\text{Eye to nose}} = \frac{\quad}{\quad} = \quad$$

Fraction Decimal

$$\frac{\text{Nose to chin}}{\text{Eye to nose}} = \frac{\quad}{\quad} = \quad$$

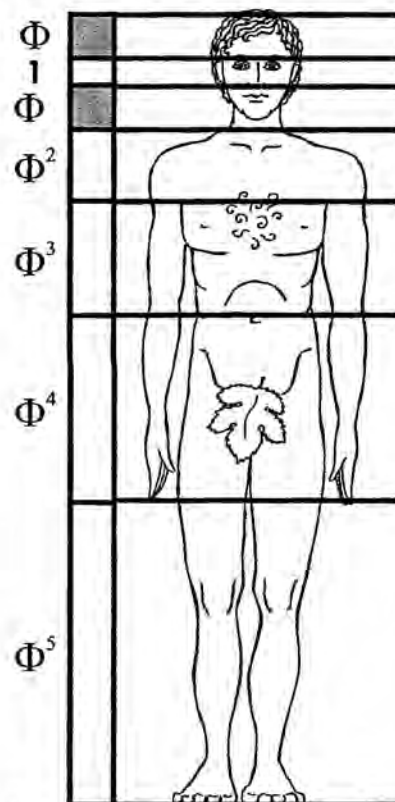
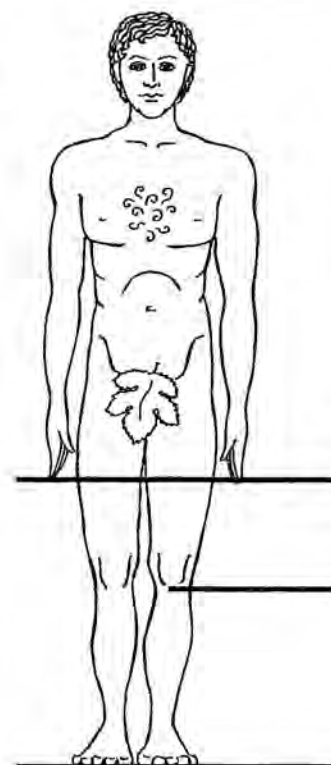
Fraction Decimal

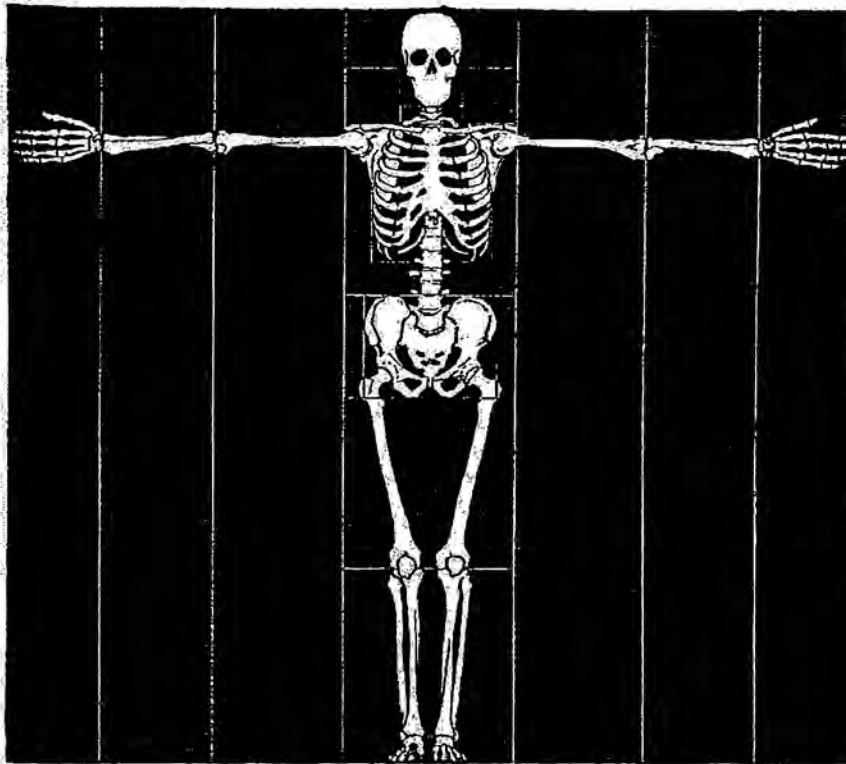
$$\frac{\text{Neck to Armpit}}{\text{Nose to Chin}} = \frac{\quad}{\quad} = \quad$$

Fraction Decimal

$$\frac{\text{Armpit to Navel}}{\text{Neck to Armpit}} = \frac{\quad}{\quad} = \quad$$

Fraction Decimal



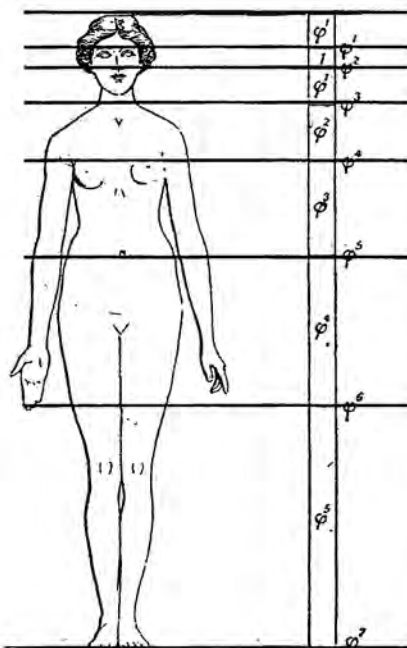


Projections frontale et latérale du squelette de Harvard.

by Jay Hambidge 1919

Extrait de « The Diagonal ».

Use a straightedge to draw lines from the levels of this body across the famous Stradivarius violin, and Jerry Garcia's "Rosebud" guitar to see how they appear to be designed along unfolding powers of Phi.



*Stradivarius
Violin*

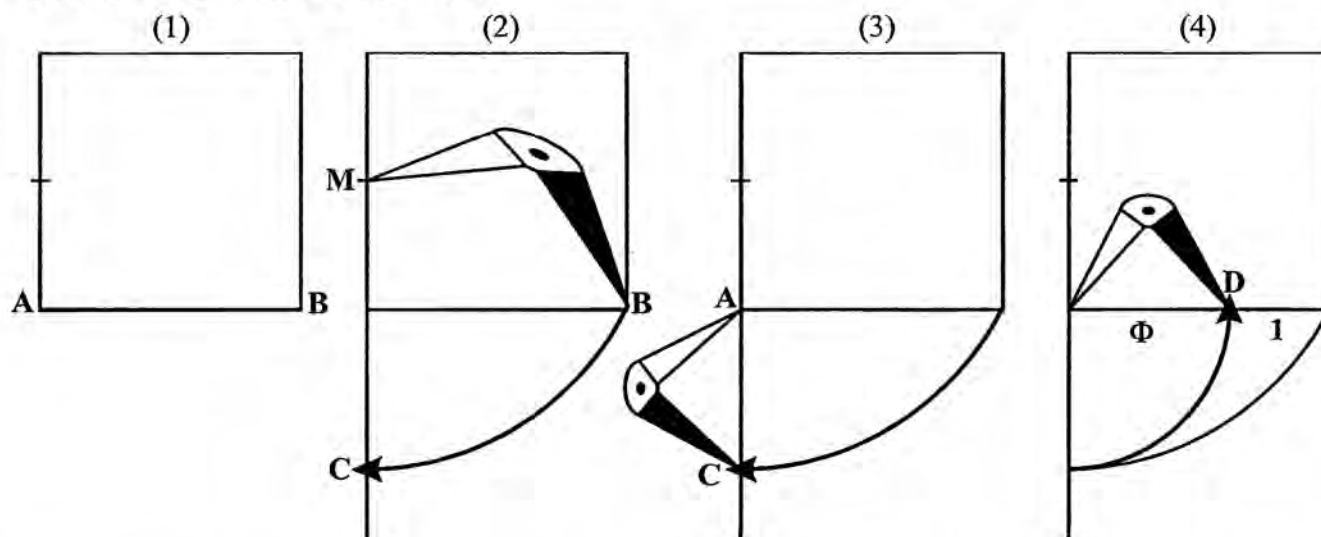
*Jerry Garcia's
"Rosebud" Guitar*

Is it possible to find the Golden Mean, the Phi balancing point on a line without doing any measuring? Of course it is, if you know how to use a compass and straightedge. In fact, there are many ways to do it geometrically, which will be valuable when we examine the appearance of Phi in worldwide art.

The first person known to have written about the Golden Mean, or "extreme and mean ratio" as it was called, was Euclid around 300 BCE. (Phi was called "extreme and mean ratio" referring to its unifying relationship between the extremes of whole and small part, mediated by its mean, the large part.) The construction is included in his famous work of thirteen books, *Elements*, at least twice: in Book 2 Proposition 11, and Book 6 Proposition 30.

Here in essence is Euclid's method "To cut a given finite straight line in extreme and mean ratio":

- (1) Start with a straight line segment (AB) and construct a square up from it. (See Vol.1 Ch. 4).
- (2) Find the midpoint (M) of the side above point A. Extend this side of the square downward. Place the compass point on midpoint M. Open the compass pencil to point B. Swing an arc so that it crosses the extended line at point C.

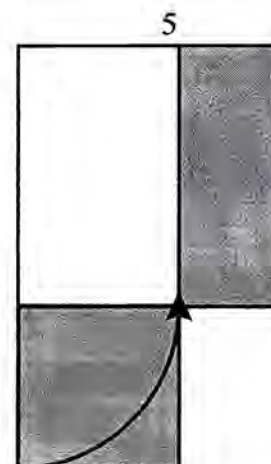


- (3) Place compass point at A and the pencil at C.

- (4) Swing the compass upward until it crosses the original line AB at D.

This is the Phi balancing point which divides the whole line AB into large AD and small DB parts, all in Golden Mean relationships.

- (5) Euclid also points out the remarkable fact that when the square is divided by this line, it creates a small square and rectangle (shaded) which have *equal areas*!



Divide A Line At The Golden Mean

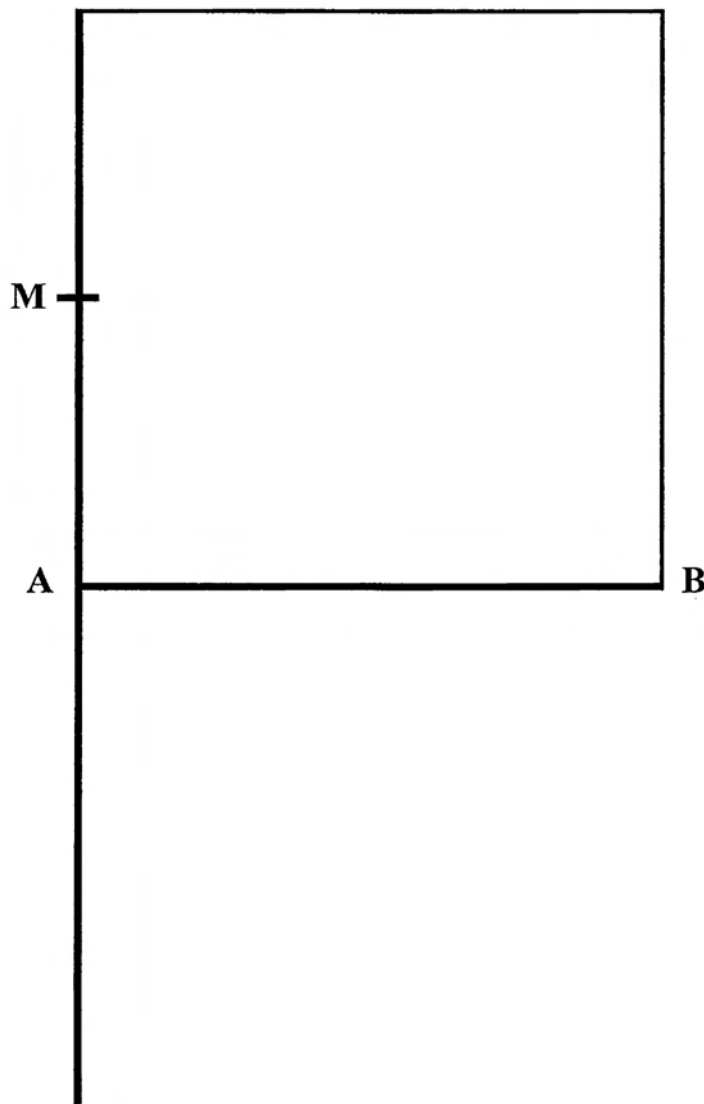
"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel."

-- Johannes Kepler

A square is constructed upon line AB.

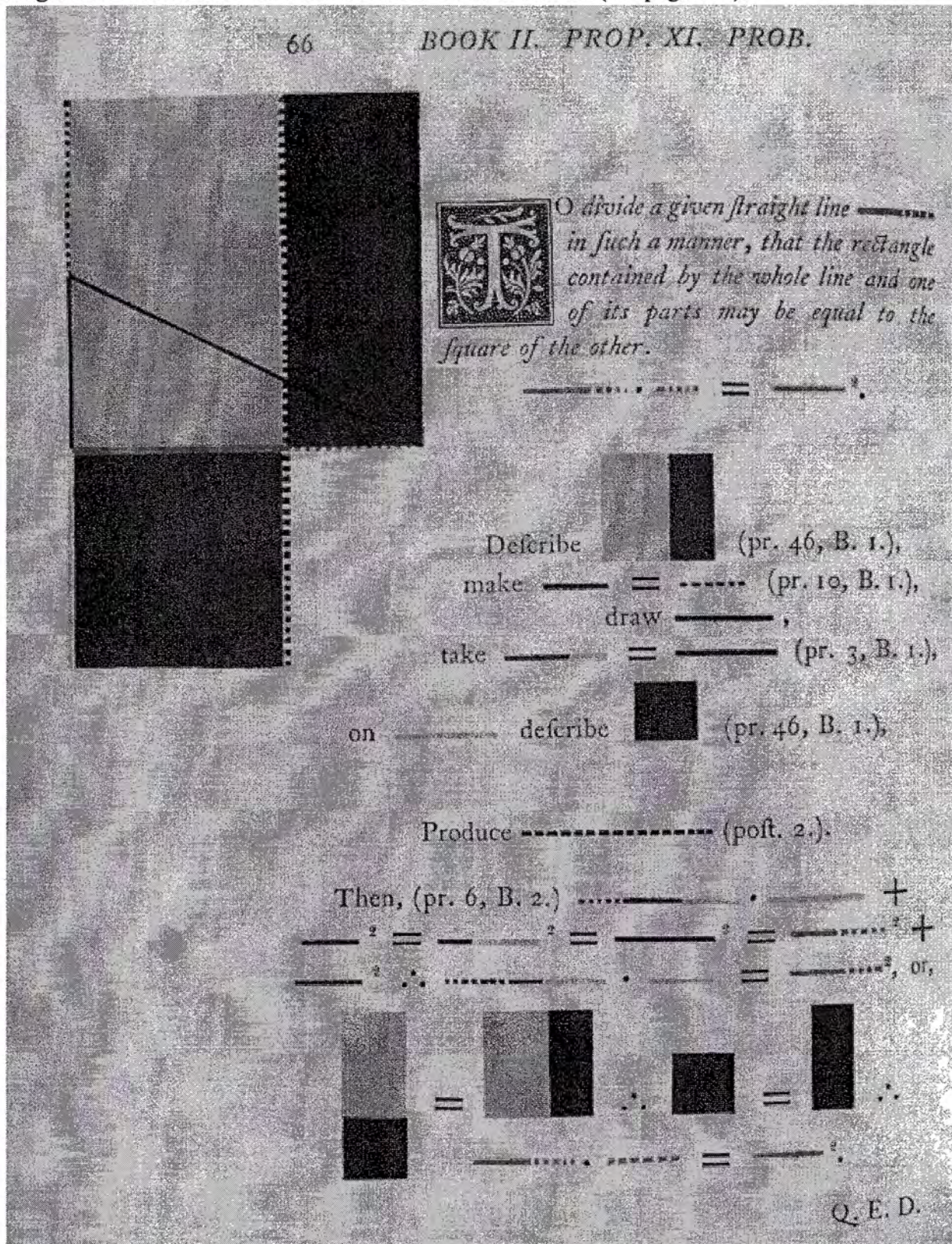
The midpoint M of its left side is marked, and that side is extended downward.

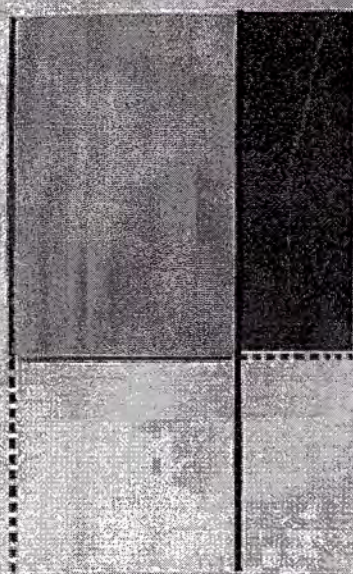
Use a compass and straightedge with the instructions on the previous page to divide AB at the Golden Mean, its balancing Phi point.



Byrne's Euclid

This and the following page are from Oliver Byrne's 1847 edition of Euclid's *Elements* related to the construction of the Golden Mean. The book has been called one of the oddest and most beautiful books of that century because Byrne attempted to present Euclid in colorful, step-by-step pictures using the fewest words. The entire work can be found online (see page 164).





Q cut a given finite straight line (—.....) in extreme and mean ratio.

On —..... describe the square

(B. I. pr. 46); and produce —....., so that

$$\text{—.....} \times \text{—.....} = \text{—.....}^2$$

(B. 6. pr. 29);

take —..... = —.....,

and draw —..... || —.....,

meeting —..... || —..... (B. I. pr. 31).

Then



$$= \text{—.....} \times \text{—.....}.$$

and is $\therefore =$ ; and if from both these equals

be taken the common part ,



, which is the square of —.....,

will be $=$ , which is $= \text{—.....} \times \text{—.....}$;

that is $\text{—.....}^2 = \text{—.....} \times \text{—.....}$;

$\therefore \text{—.....} : \text{—.....} :: \text{—.....} : \text{—.....}$,

and —..... is divided in extreme and mean ratio.

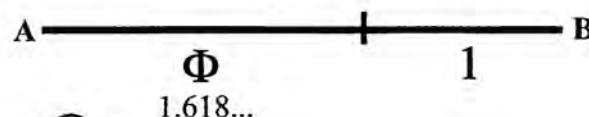
(B. 6. def. 3).

Q. E. D.

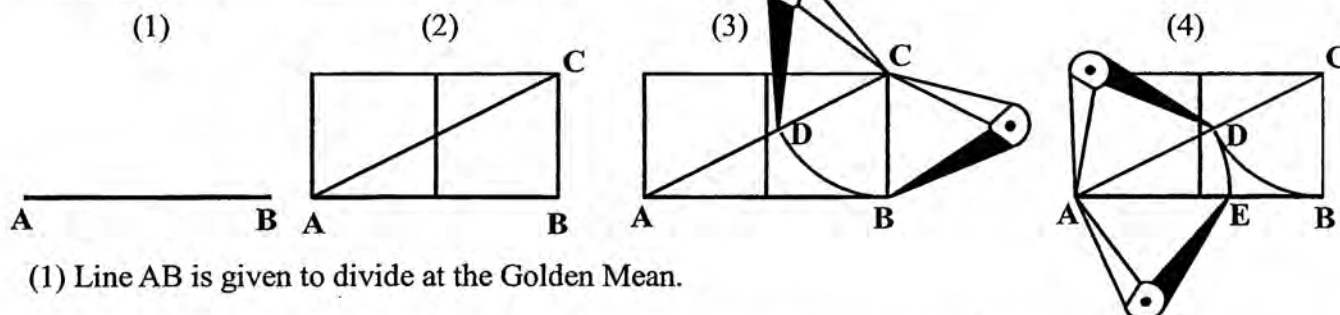
Finding The Golden Mean With Double Squares

There are a variety of ways to divide a line at its Golden Mean, the point balancing the parts with each other and the whole. What they have in common is that each method makes visible the abstract

mathematical value of Phi = $\frac{\sqrt{5} + 1}{2} = 1.618...$



Here's a useful construction which does it:



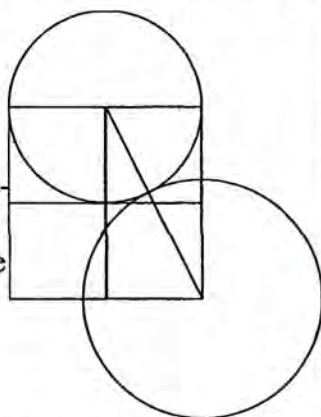
(1) Line AB is given to divide at the Golden Mean.

(2) Two small squares (which is equal to half a larger square) are constructed upon this line. A diagonal AC is drawn across the squares. Notice that this makes a right triangle ABC whose sides are lengths 1, 2 and $\sqrt{5}$, all the elements needed to build the mathematical value of Phi! The challenge is to transfer them to the line AB.

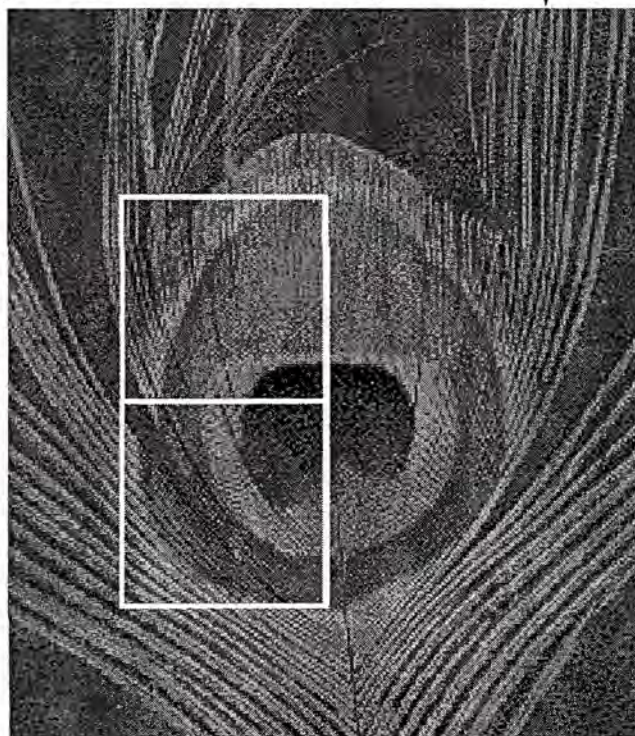
(3) Place your compass point at C, and the pencil to point B. Swing an arc upward until it crosses diagonal AC making point D. Line segment CD is equal to the side of the square (= 1).

(4) Now place your compass point on point A and open the pencil to point D. Swing an arc downward until it crosses the original line AB at point E, the Golden Mean balancing point = $(1 + \sqrt{5})/2$.

This is the same construction as above but rotated and showing a full square with completed circles instead of just arcs.



Here's an image of the "eye" of a peacock feather. A line is drawn vertically through its middle and two squares been drawn. Use your compass following the construction above to see where the Golden Mean appears in its design. Also do the construction in the opposite direction to see even more about it!



Phi In The *Vesica Piscis*

(1) Start with a line AB you'd like to divide into the Phi ratio at the Golden Mean.

(2) Open your compass points to A and B and turn a circle. Then, reverse the compass and turn another circle. You've created a *Vesica Piscis*, two circles which touch each others' center, symbol of creation through tension between polar complements. (See Volume 1, Chapter 2).

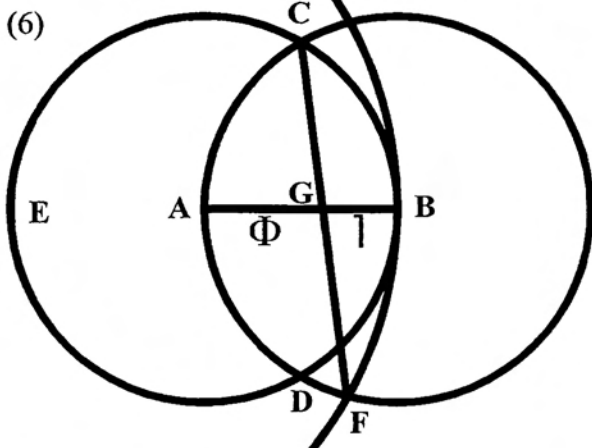
(3) We now wish to create point E on the circle to the left of the line. Some geometers will extend line AB with a straightedge alone, but it is more exact to place the compass points at the two crossing points (C and D) of the *Vesica Piscis*. Then swing an arc to the left where it crosses the circle at point E. Verify it by repeating with the compass point at D. The arcs should cross at E.

(4) Place the compass point at E and open the pencil to point B. Turn a circle so that it crosses the *Vesica Piscis* circle below point B at point F. We actually don't need the entire circle, only the part of the arc from B to F.

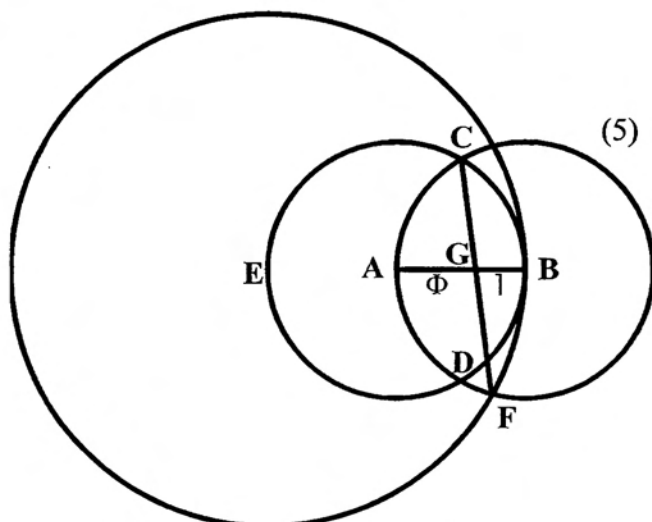
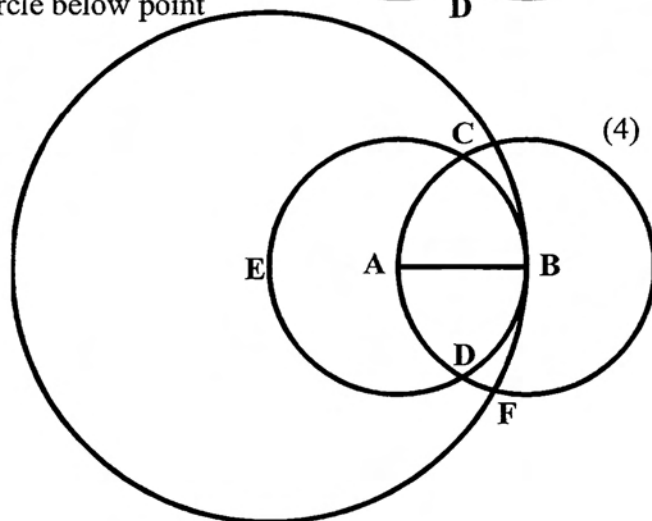
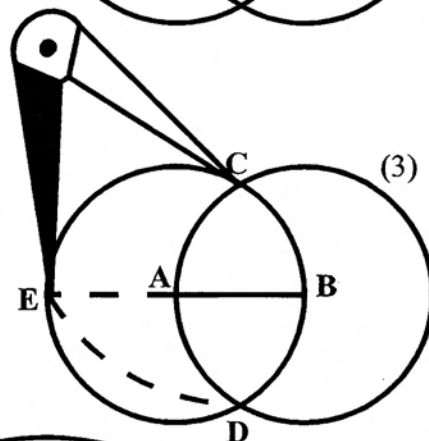
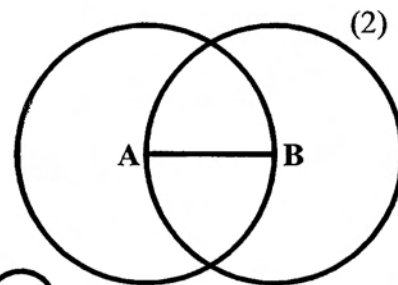
(5) With your straightedge, draw a line from point C to point F. This line crosses the original line AB at point G, the **Golden Mean point** which creates the Phi balance among the parts and the whole.

(6) Enlarged view.

Do this construction on the blank page to the left. Draw a line and construct a *Vesica Piscis* around it to find its Golden Mean division point.



A — B (1)



Phi In A Triangle

In Volume 2 Chapter 6 of these *Activity Books* we saw the mystery of how the radius of any circle will "walk" six times around its own circumference.

(1) Three points are connected to inscribe an equilateral triangle in the circle. Three diameters are drawn across the six points.

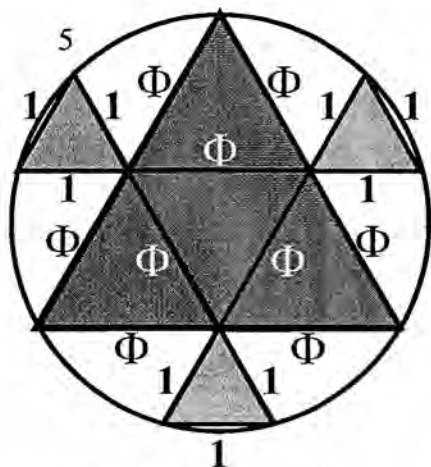
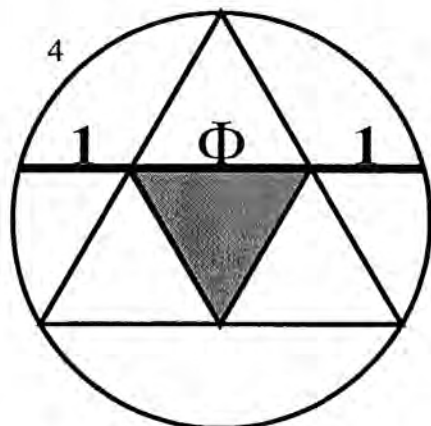
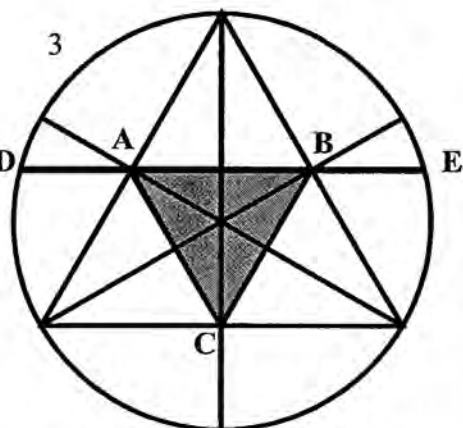
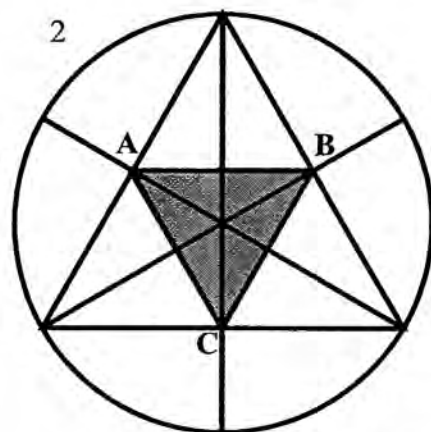
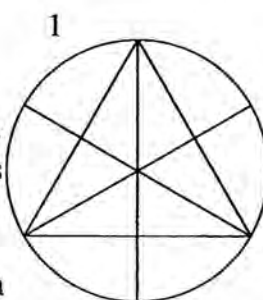
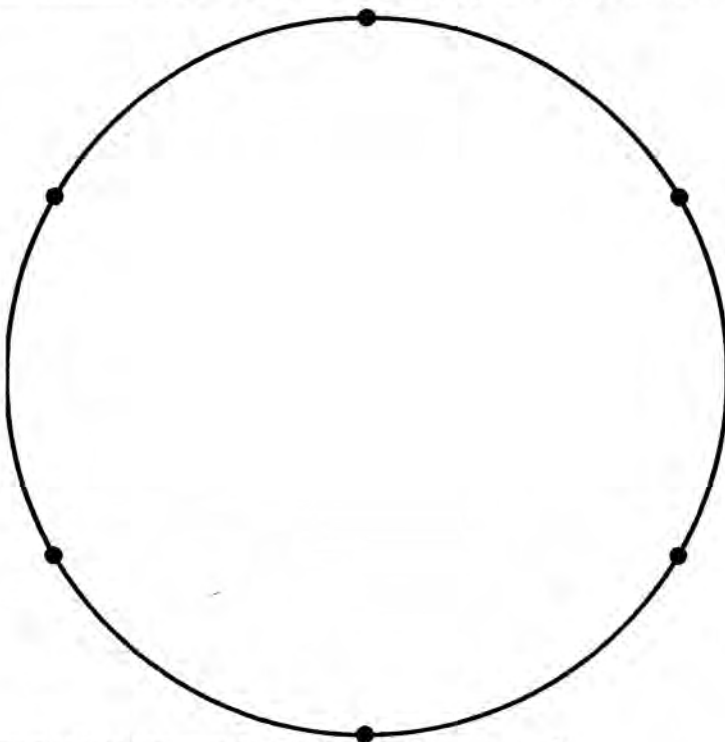
(2) The diameters cross the triangle at three points A, B and C. Connect them with straight lines to form a downward pointing equilateral triangle.

(3) Extend the line AB in both directions so that it reaches points D and E on the circle.

(4) Without realizing it, we've created the Golden Mean twice! The line segment AB is Phi times AD, and Phi times BE.

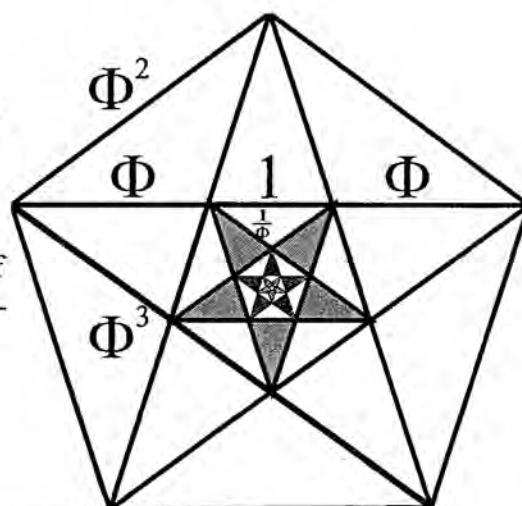
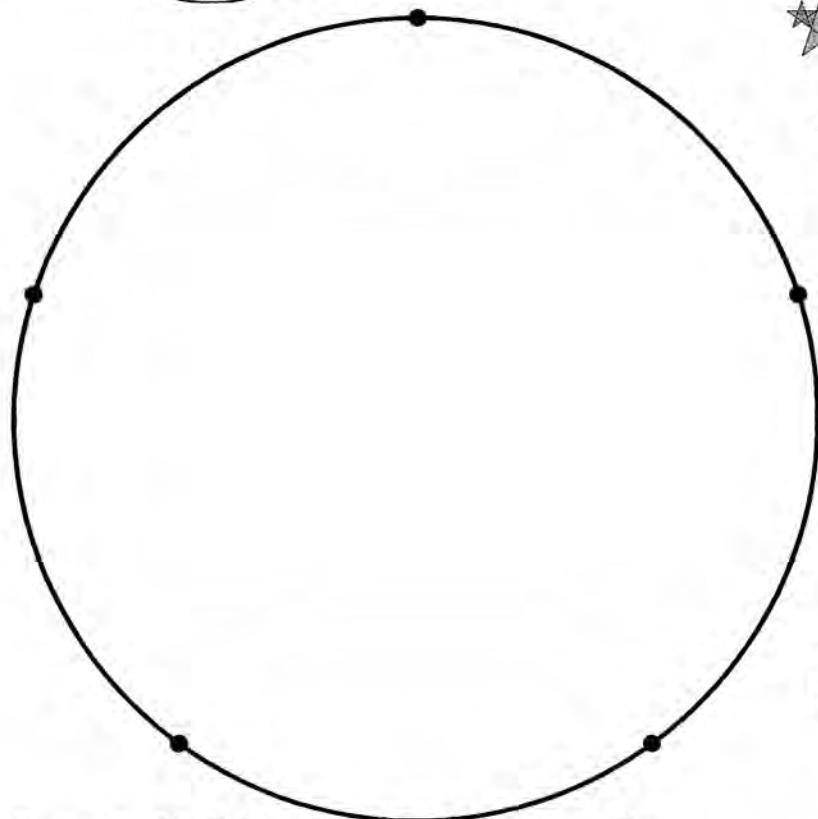
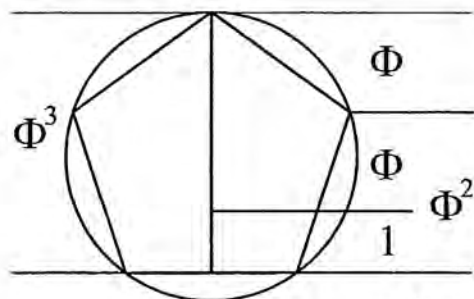
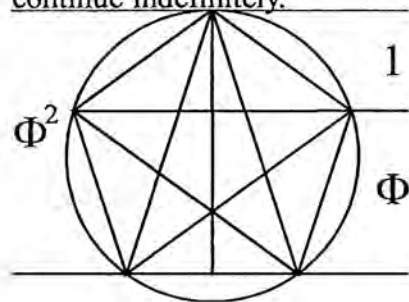
(5) Extending all three sides of triangle ABC will reveal more points on the circle which may be connected to make three small (light shaded) triangles. Any of the sides of the four larger dark triangles are each in Phi ratio to any of the sides of the small light triangles.

Do this construction in the circle with six points given here:



Phi In The Pentagon

In Volume 1 Chapter 5 we saw how to construct a Pentagon and subdivide its Pentagram star. Now we can add the remarkable fact that **everywhere the Pentagram star crosses itself it creates the Phi ratio!** Each segment is 1.618... times larger than others nearby it, and the inverse of Phi 0.618 times smaller than others. By subdividing a Pentagon into smaller and smaller stars, the Phi process will continue indefinitely.

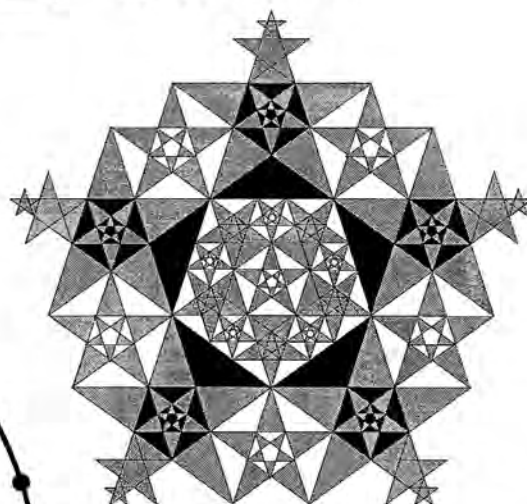


$$\frac{1}{\Phi} = \boxed{\Phi - 1} = 0.6180339...$$

$$1 = 1 = 1.0000000...$$

$$\frac{\sqrt{5} + 1}{2} = \Phi = 1.6180339...$$

$$\Phi^2 = \boxed{\Phi + 1} = 2.6180339...$$

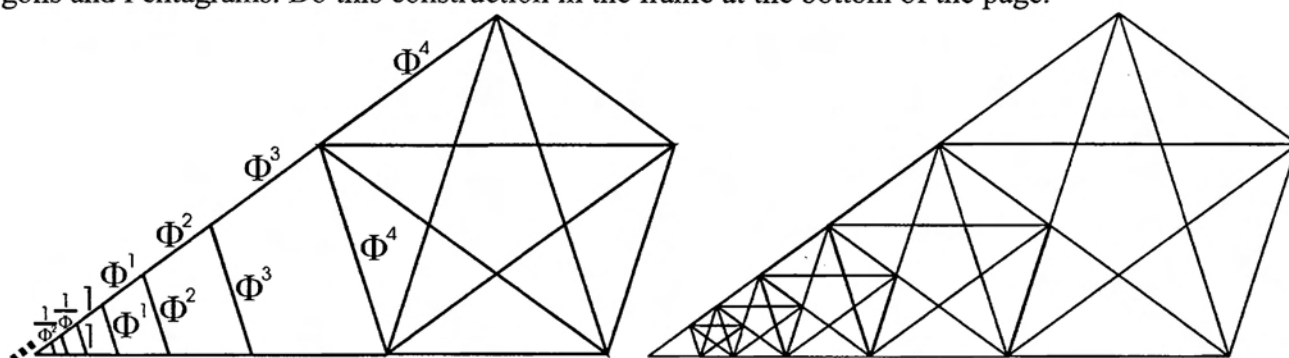


All these Pentagram stars of different sizes are connected to each other by powers of Phi, making one harmoniously diverse whole.

Use these five points around the circle to draw a Pentagon and Pentagram star. Subdivide it into the center (see top) as far as the sharpness of your pencil will allow.

Unfurling Powers Of Phi

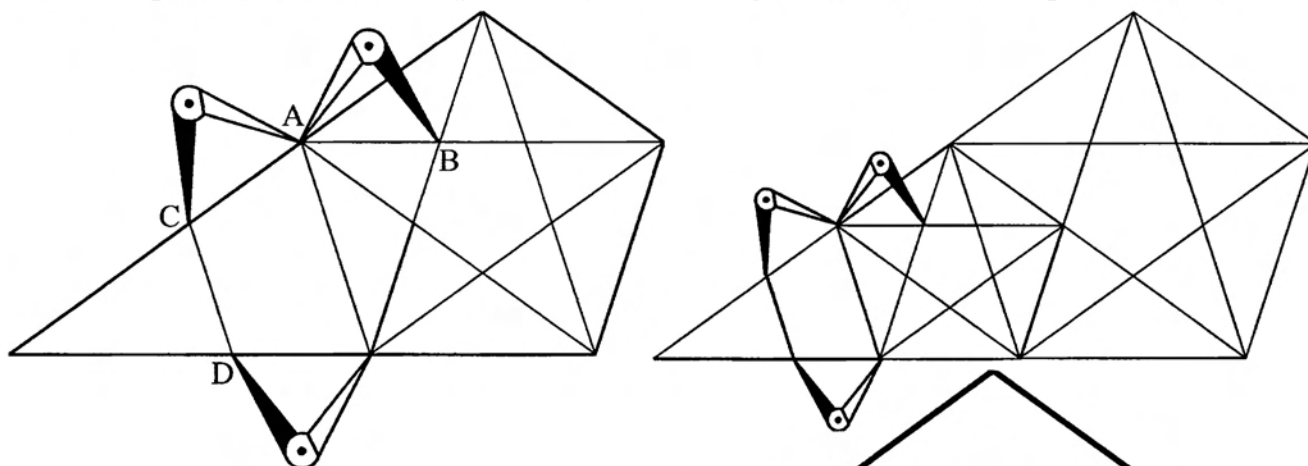
The powers of Phi folded within the Pentagram star can be unfurled as an ongoing chain of Pentagons and Pentagrams. Do this construction in the frame at the bottom of the page.



First, construct a Pentagon with its Pentagram star (done below). Use your compass to measure off the remaining two sides. Extend two sides so they meet at a point. (Already done below.)

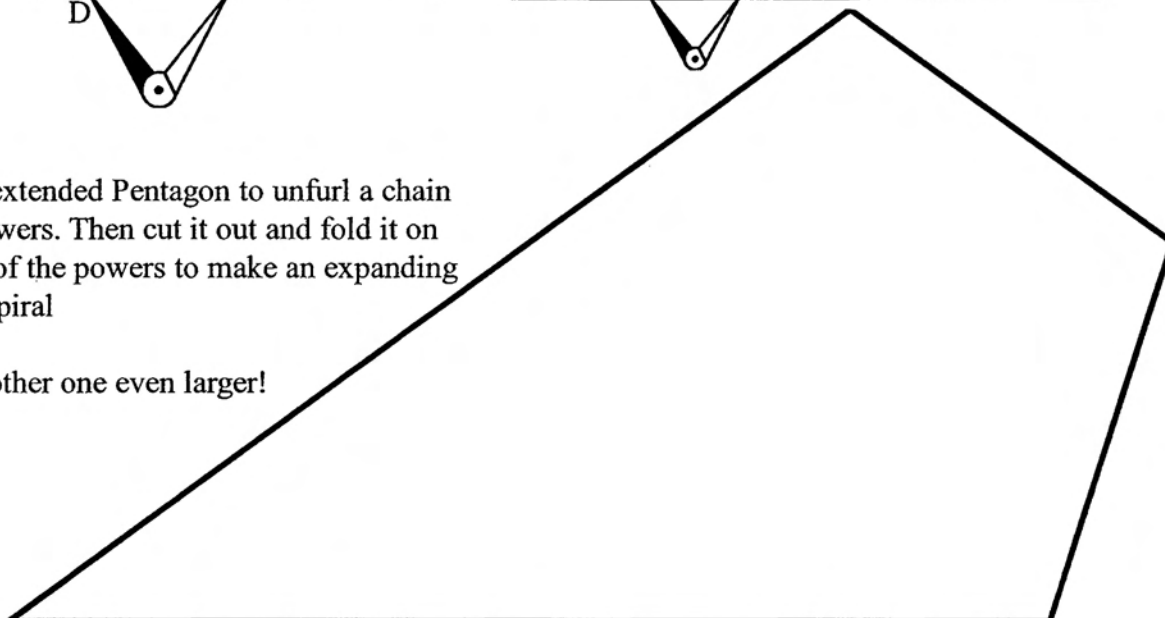
(1) Use your compass to measure between points A and B, then swing it to mark point C, and then point D. Draw line CD.

(2) Draw a Pentagram star in this smaller Pentagon. (You only have to draw two of its lines.) Then repeat the process described in step (1) as far as you can go, each step a smaller power of Phi.



Use this extended Pentagon to unfurl a chain of Phi powers. Then cut it out and fold it on the lines of the powers to make an expanding Golden Spiral

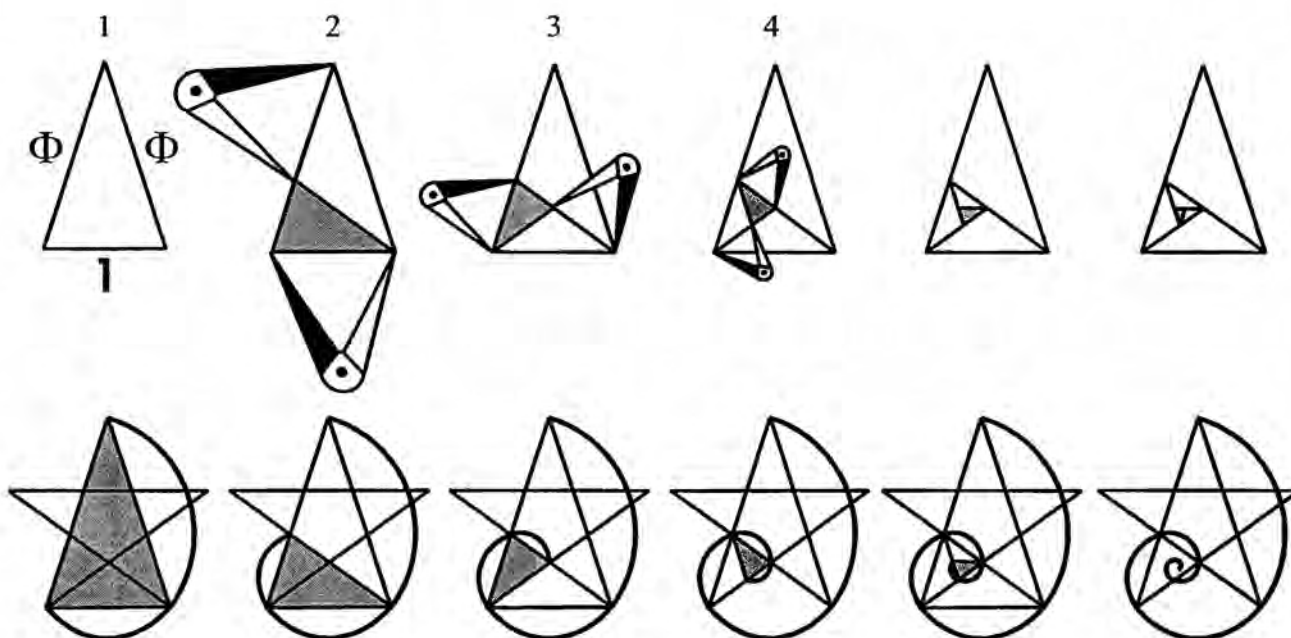
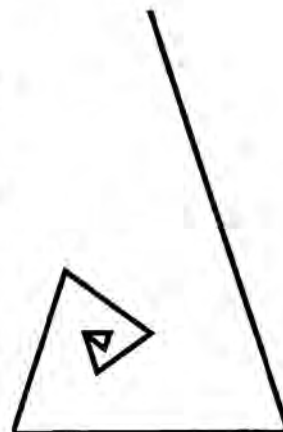
Make another one even larger!



Whirling Golden Triangles

The Golden Mean is prone to form a spiral, as we saw with a strip of paper folded along powers of Phi. So it should be no surprise to find that the ongoing powers of Phi found in the Pentagon and Pentagram star can be found to whirl around in this same type of spiral. Here's how to understand and construct it:

(1) Any side of a Pentagram star is Phi times longer than the side of the Pentagon around it. Here's the "Golden Triangle" within the Pentagram. It's possible to subdivide this Golden Triangle into smaller Golden Triangles whose sides (and areas) are ongoing powers of Phi.

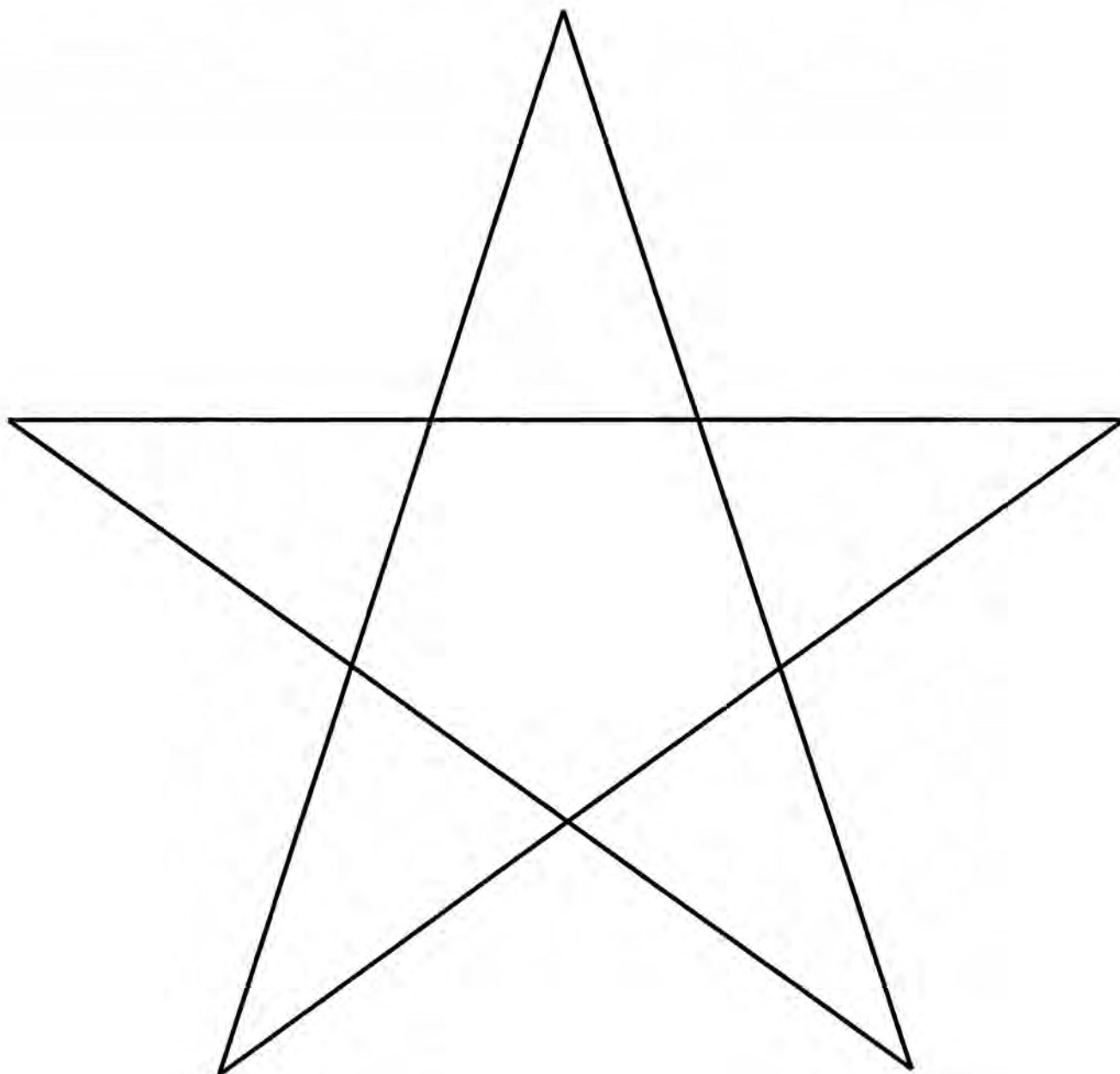
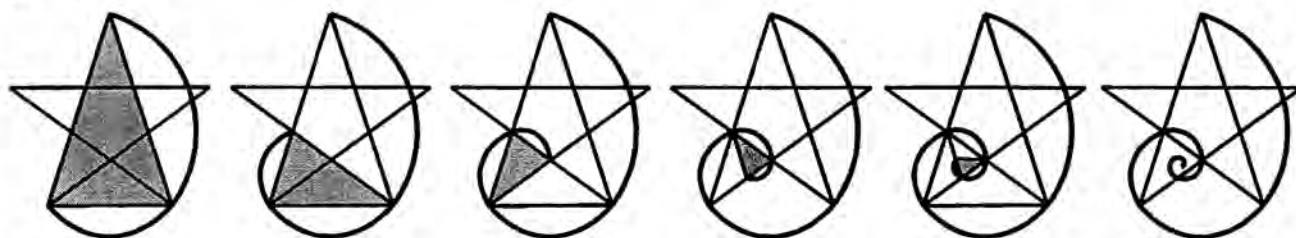


(2) Open your compass to the short side of the Golden Triangle. Then place your compass point at the top of the Triangle and mark that length upon one side. Connect this point to the Triangle's opposite corner and create a smaller Golden Triangle lying on its side at the base of the original Golden Triangle. Notice that a Pentagram star already creates this small Golden Triangle.

(3) For each new Golden Triangle, repeat this process of marking its bottom length onto its long side. If you do this you'll create many smaller Golden Triangles. Then you can draw a smooth curve around two sides of each smaller Golden Triangle resulting in the Golden Spiral.

Do this construction of whirling Golden Triangles on the next page.

Use colored pencils to construct whirling Golden Triangles and a Golden Spiral within this Pentagram.



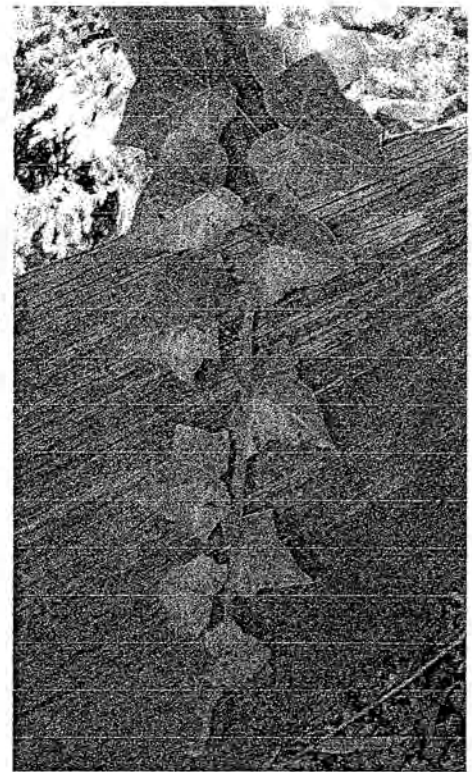
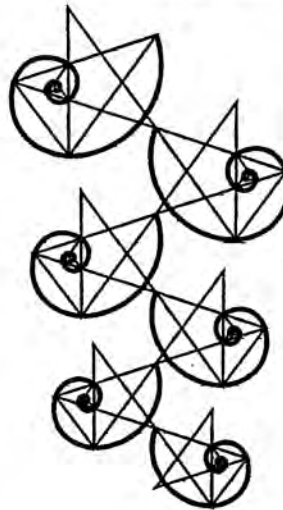
Flowing Vines

The arrangement of leaves and fruit on vines often remind us of flowing water and waterfalls.

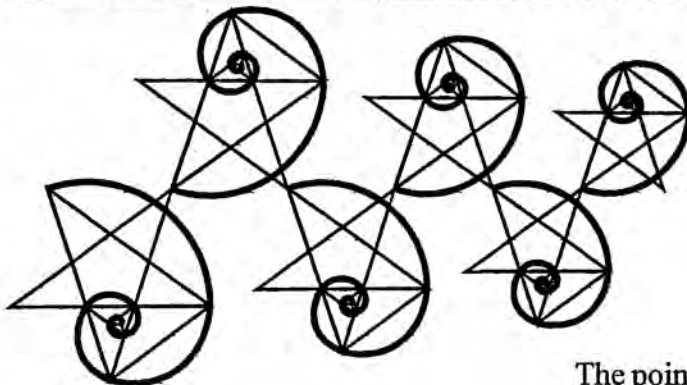
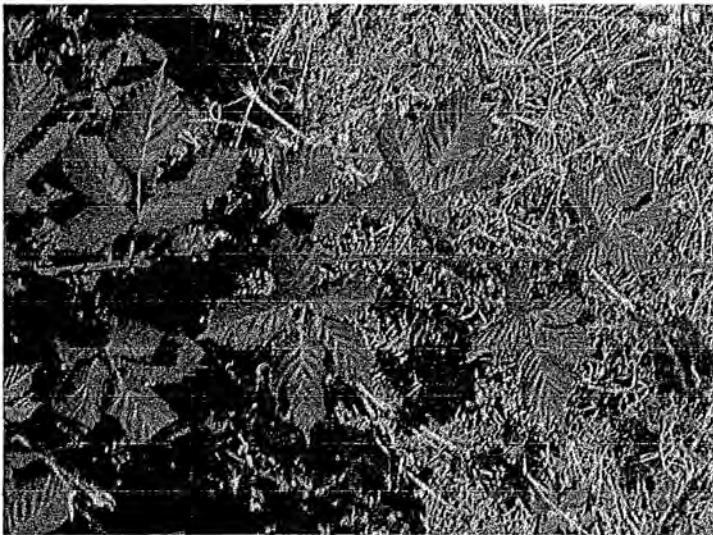
This should not surprise us since many types of leaves are based upon the Pentagon shape, which shares the spiral shape with flowing water and air.

Like leaves on a vine, linked Pentagram stars and their Golden Spirals resemble water and air flowing with a back-and-forth rhythm.

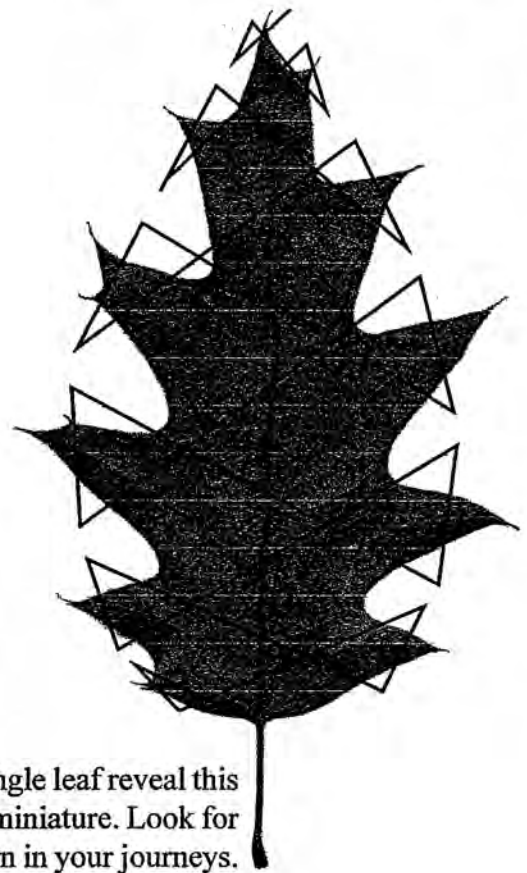
Ivy vine



Blackberry vine



The points of a single leaf reveal this whole pattern in miniature. Look for this pattern in your journeys.



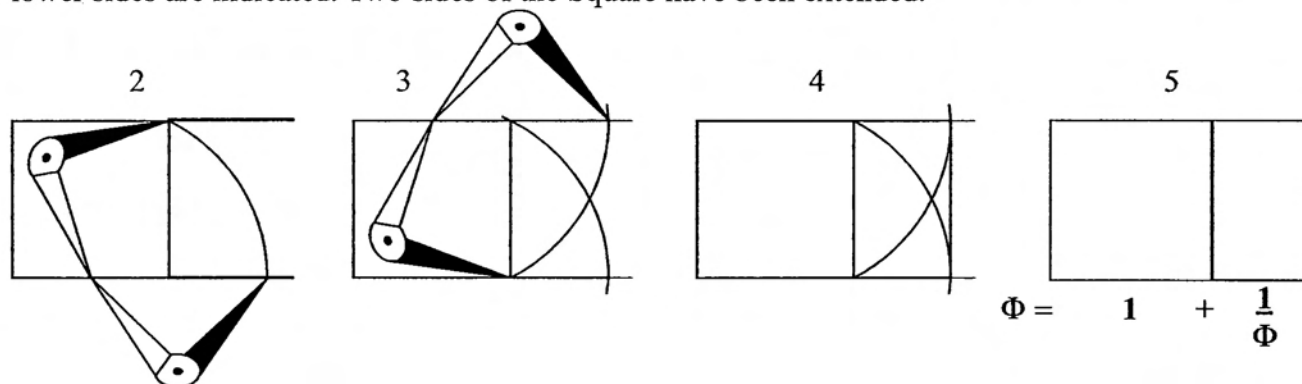
Oak leaf
with linked Pentagrams drawn

"Nature will reveal itself if we will only look."

-- Thomas A. Edison (1847-1941)

The Golden Rectangle

A Golden Rectangle is a rectangle whose long and short sides are in Phi ratio. One way to construct a Golden Rectangle from a square has been started for you below (1). Midpoints on the upper and lower sides are indicated. Two sides of the Square have been extended.

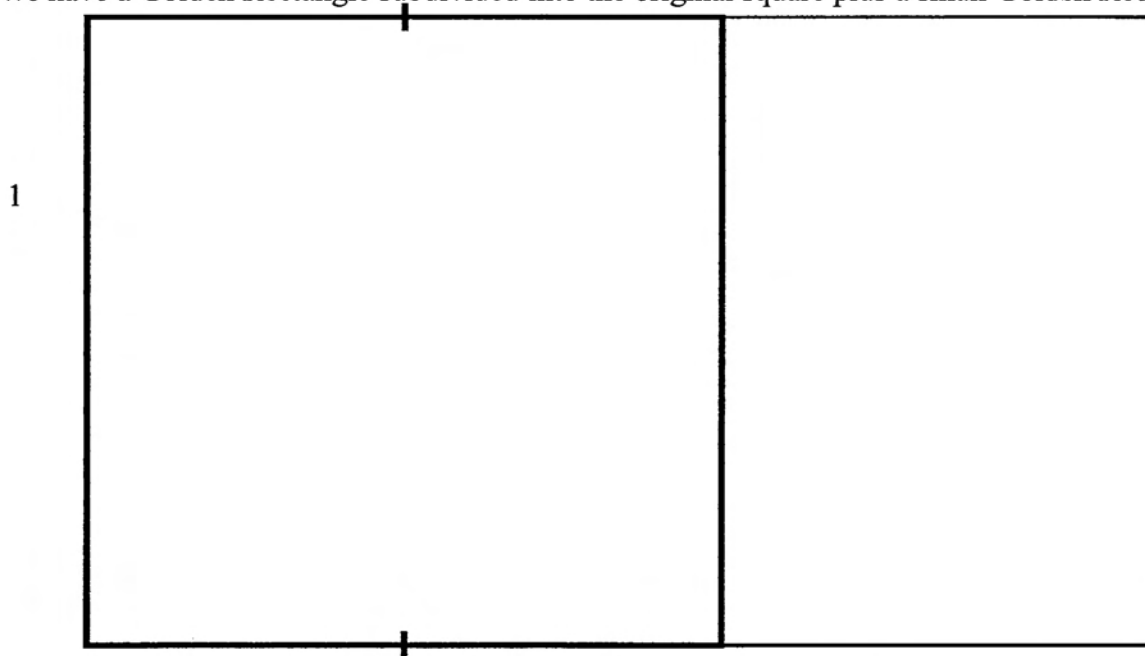


2) Place the compass point at the midpoint of the square's bottom side and open the pencil to the square's upper right corner. Swing the compass down to cross the lower extended line.

3) Repeat this with the compass point at top midpoint and the pencil open to the bottom corner.

4) Use your straightedge to connect the points where the arcs cross the extended lines.

(5) We have a Golden Rectangle subdivided into the original square plus a small Golden Rectangle.



Every Golden Rectangle is a three-in-one whole. It's a whole Golden Rectangle made of two unequal parts: a Square and a smaller, turned Golden Rectangle. The sides (and area) of each are 1.618... times larger or smaller than the next.

Constructing The Universe Activity Book Vol. 3

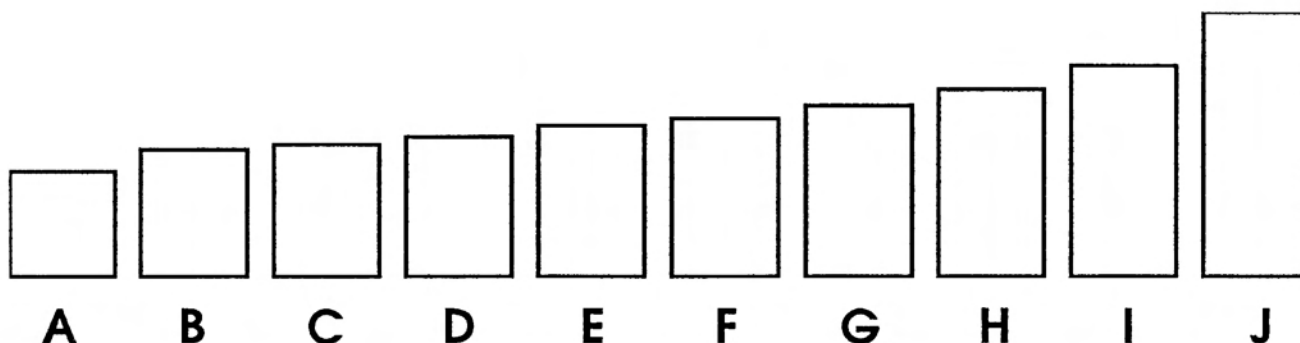
$$\boxed{\Phi} = \boxed{1} + \boxed{\frac{1}{\Phi}}$$

The areas of the whole and parts are in Phi relationship.

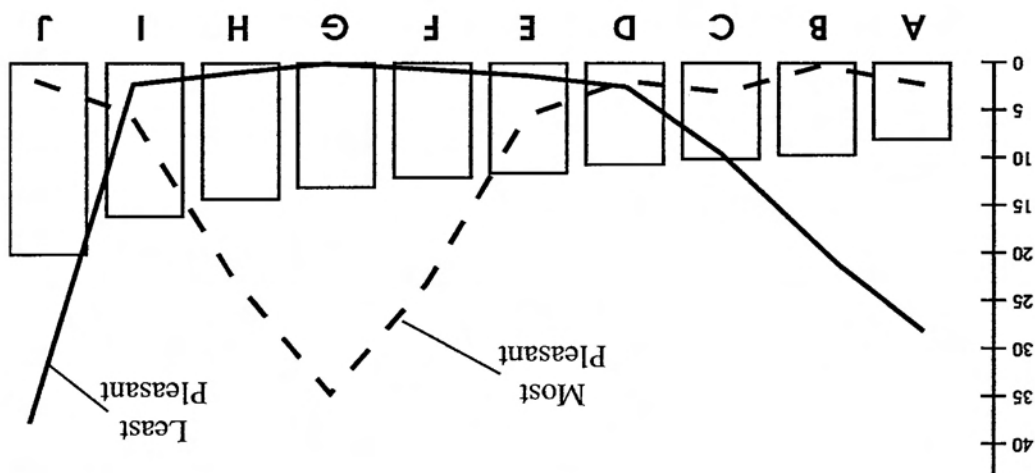
The "Most Pleasing Rectangle" Survey

What can you do with a Golden Rectangle? You might try to conduct this survey first asked by Gustav Fechner in the 1860's. He showed many people these ten rectangles having the same width and asked "Which is the most pleasing?" and "Which is the least pleasing?".

First, look at each rectangle yourself. Try to "feel" each one as having a different "pressure" on its sides. Which one seems to you to have the most pleasant balance of pressures? Which one appears least pleasant? When you have decided, turn the page around.



This is upsidedown so you won't be influenced by the information.
 The rectangle people often choose as the most pleasing, and least often chosen as the least pleasing, is the Golden Rectangle (G).
 Show the ten rectangles to as many people as you can. Keep a record of their answers. The more people you ask, the more accurate the survey will be. Compare your results with these in the graph.



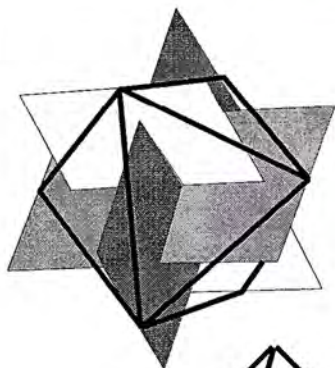
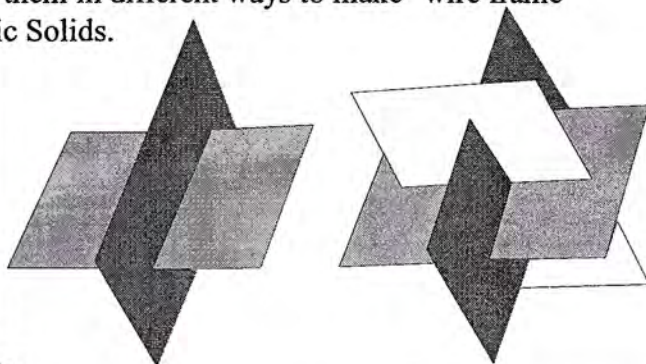
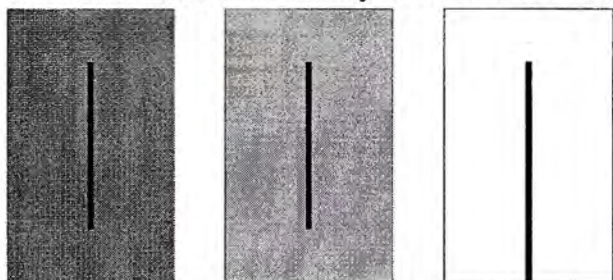
A Three-Dimensional Golden Frame

With three Golden Rectangles you can build a three-dimensional frame and geometric models on it.

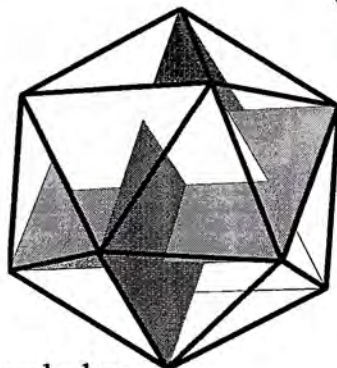
Make three Golden Rectangles of stiff cardboard, foam core, or even wood.

Along the long middle of each, measure and cut a slot equal to the width of the Golden Rectangle. Cut one slot to reach the end. Make each slot *as thick as the cardboard*. Then, simply slip one Rectangle into another, and the longest slot last, to make a three-dimensional frame with a Golden Rectangle in each plane.

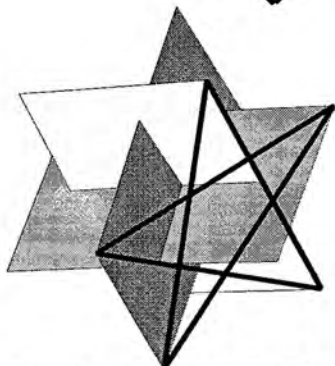
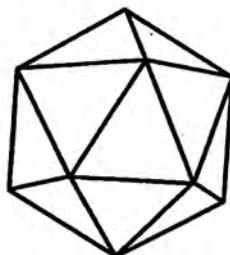
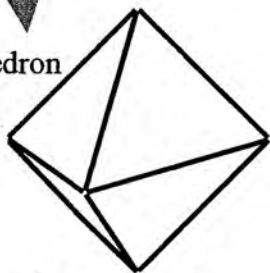
Poke holes near the corners (on the diagonals) near the middle of each short side. Then thread string, yarn or elastic thread through the holes to connect them in different ways to make "wire frame" models of different natural crystals and the Platonic Solids.



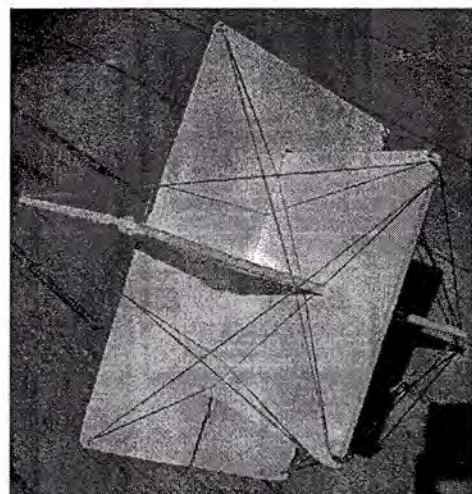
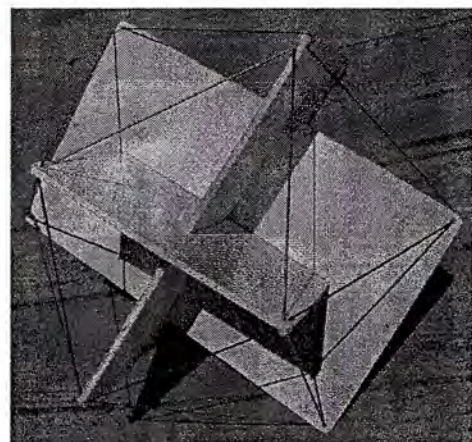
Octahedron



Icosahedron



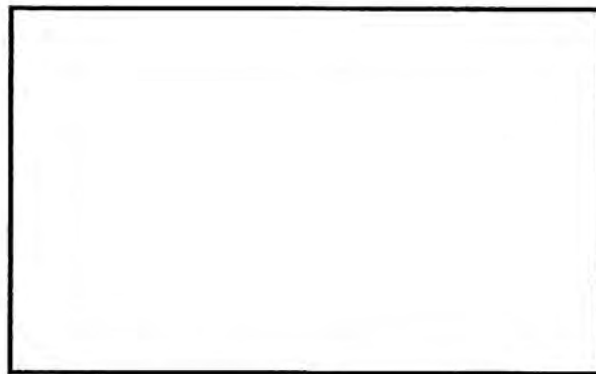
This frame divides space equally in all directions by Phi ratios. We can see this in the way that five corners may be connected to make a regular Pentagram star (and Pentagon). You can repeat this pattern all around to create an interesting structure.



Visual Points Of Interest

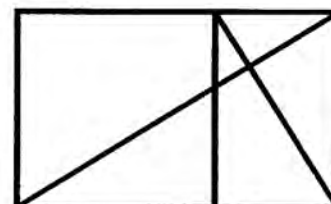
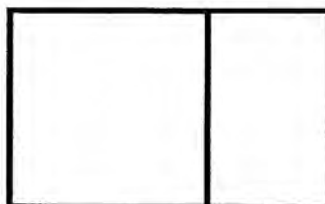
Whether we realize it or not, when we look at any enclosed space our gaze is encouraged to move or settle down by the "pressure" of the proportions of the sides. Our gaze may move around or perhaps be drawn to the central area, or towards the sides or a corner. Each shape has a different "feel".

Look at the space within this Golden Rectangle. Do your eyes tend to move anywhere, toward any particular points, sides or corners?



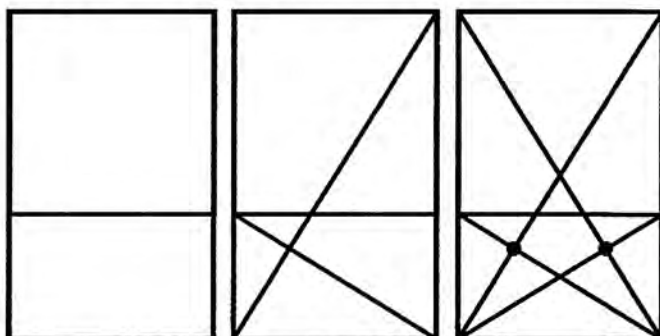
There is a long tradition of artists, architects and designers who have made intelligent use of the proportions of rectangles to compose their art. They knew how to find the "visual points of interest" or "hotspots" where our eye expects a point or line and is encouraged by the geometry to settle there.

Constructing a square on the short side in a Golden Rectangle reveals a square and smaller, upright Golden Rectangle. By drawing the "long diagonal" of the full Golden Rectangle, and also the "short diagonal" of the smaller Golden Rectangle, we see that they cross at a *right angle* near one corner. This point is called one "eye" of the Golden Rectangle, a "visual center of gravity" and focal point of balance and stillness to which our eye is unconsciously drawn. Do this construction to find the eye of the large Golden Rectangle above.



Look at the scene in this painting from *The Book Of Hours* made in the Fourteenth century for the Duc du Berry by the Limbourg brothers.

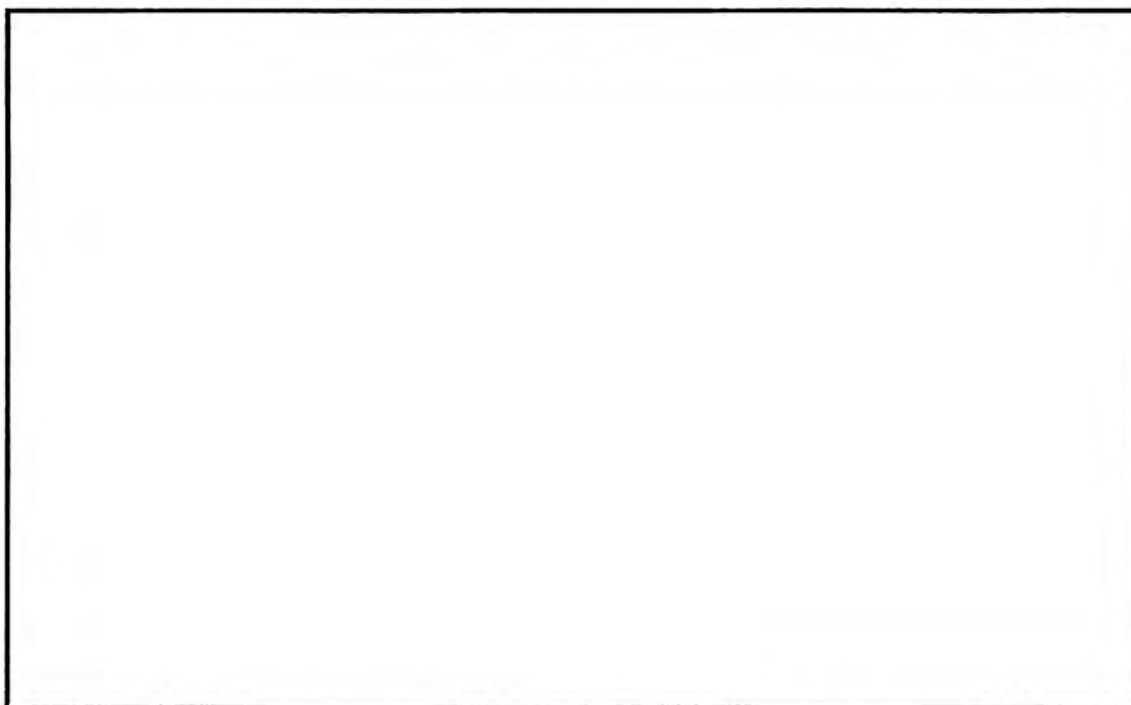
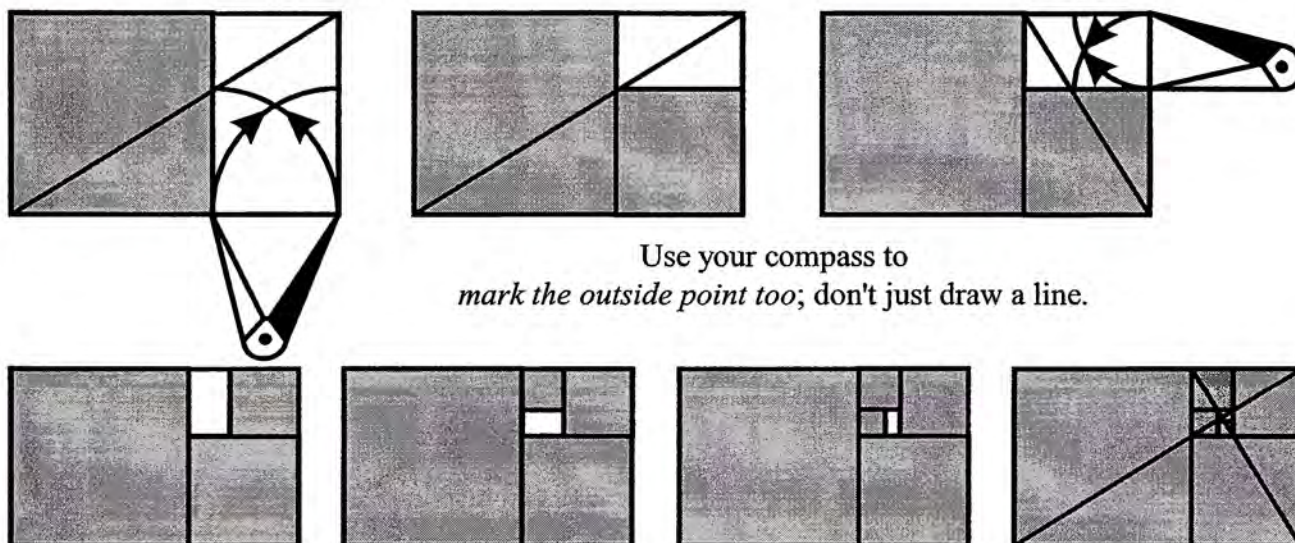
A Golden Rectangle has been drawn around the painting. Do the geometry shown below to find two of its "eyes". Does your gaze wander towards the man's fist, or to the person back in the woods? Is it only because people have been placed there?



Whirling Squares

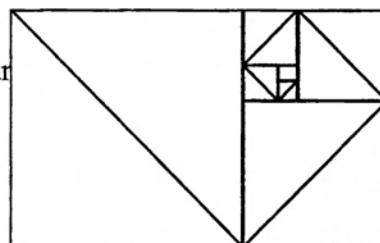
A unique fact about the Golden Rectangle is that when a square is marked off inside one end, what's left is always another, smaller, turned replica of the whole (a fractal). Also interesting is the fact that the diagonal of that full Golden Rectangle crosses the side of the large square at the corner point of another, smaller square. This continues forever, requiring only the long and short diagonals of the two largest Golden Rectangles to guide a vortex of whirling squares around and into its eye.

Use this approach to mark off both sides of each new square in the large Golden Rectangle below. Look carefully at these illustrations to see that each new square whirls counterclockwise inside the Rectangle, approaching its eye but never mathematically reaching it (an asymptote). The crossing of diagonals at the "visual center of gravity" predicts the eye towards which all the squares chase a diminishing Golden Rectangle forever.



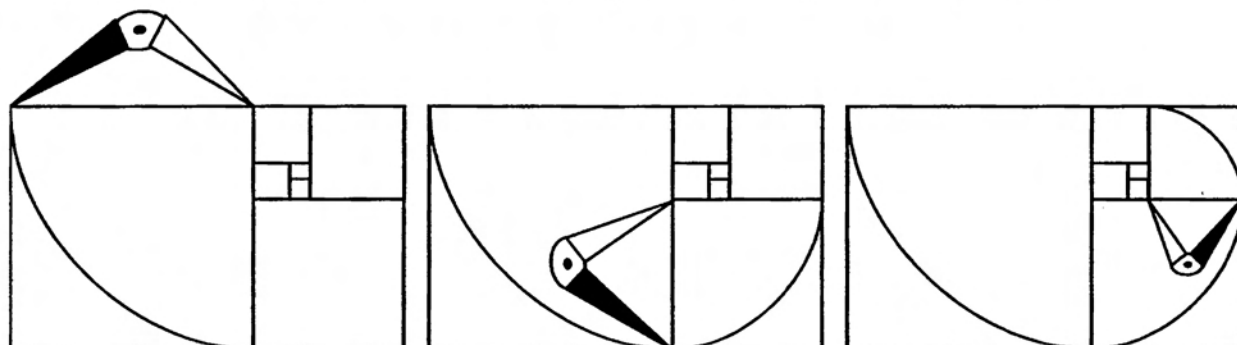
Another Turn At The Golden Spiral

Just as we saw a Golden Spiral appear among the self-similar Golden Triangles in the Pentagram star, it also appears among these self-similar whirling squares within the Golden Rectangle.



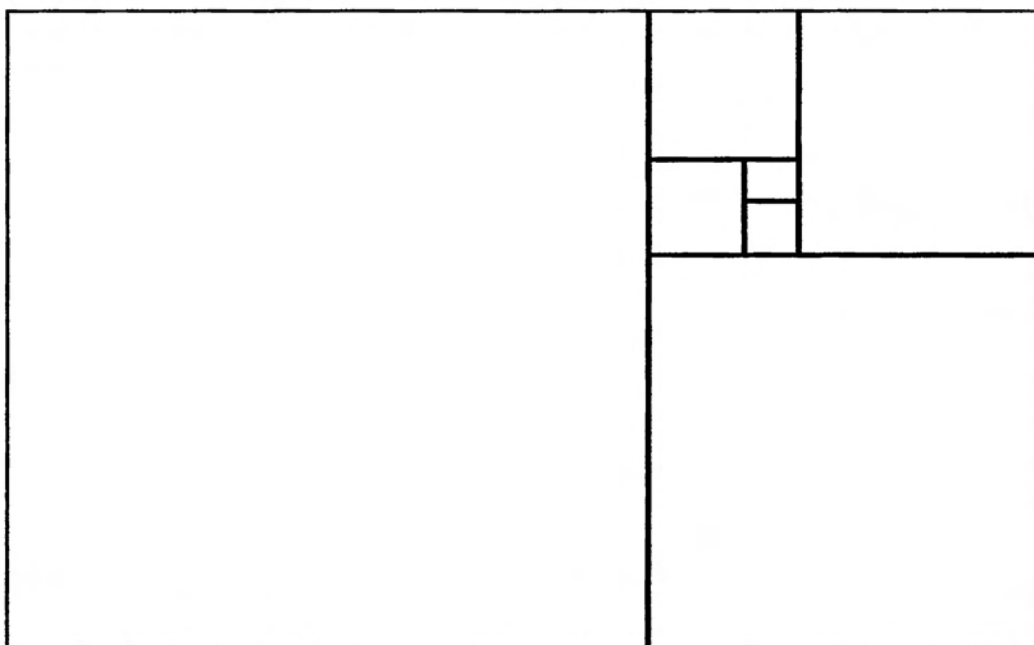
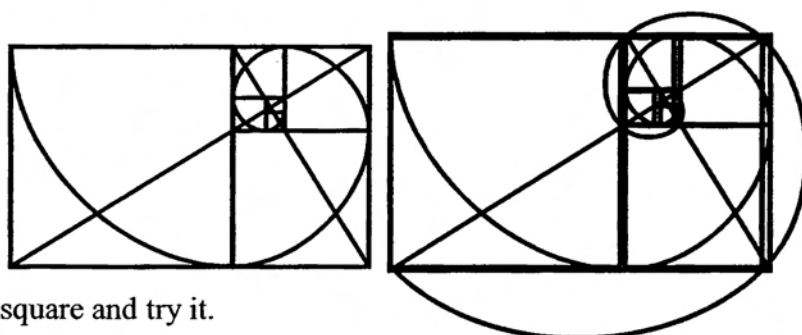
To see it, just draw a diagonal across each square. This makes a right-angled spiral like a folded strip of paper.

But to draw a *smooth* Golden Spiral, place your compass point on each square's corner and open the pencil to the length of its side. Then turn a quarter-circle in the square. Lift the compass and repeat this in each smaller square. Construct a Golden Spiral in the Golden Rectangle below.



The diagonals of the Golden Rectangles cross to predict the spiral's eye.

A Golden Spiral may be drawn inside or outside of the squares' corners. Or by connecting the center of each square. Put a dot at the center of each square and try it.



Balance Of The Golden Spiral

When we see spirals in nature, they often partake of the Golden Spiral's remarkable ability to maintain its balance at any size. Whether it grows or shrinks, this spiral maintains the same center of gravity, no matter its size or weight.

You can observe this by curling a piece of wire (or pipe cleaner) into a Golden Spiral along the spiral you constructed on the previous page. Use a stick or pencil to balance it at its eye. Then cut a section off the end and you'll see how the spiral *rotates back to the same angle it always makes* between eye and tip.



This mathematical characteristic allows seashells and an animal's horns to balance the same way no matter how large or heavy they become. It allows a plant to balance as it grows more branches, a flower to balance with increasing seeds, or an embryo to develop in a balanced way. From atoms to galaxies, the spiral maintains constancy amidst change, acknowledging unity underlying diversity.

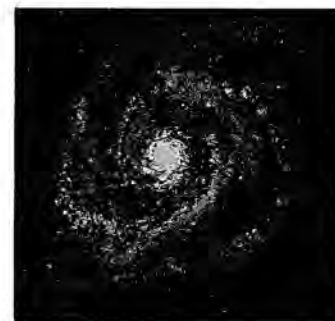
“The logarithmic spiral is as near as we can get to an accurate definition of a living process.”

— Sir Theodor Andrea Cook

How many of these spirals can you identify?

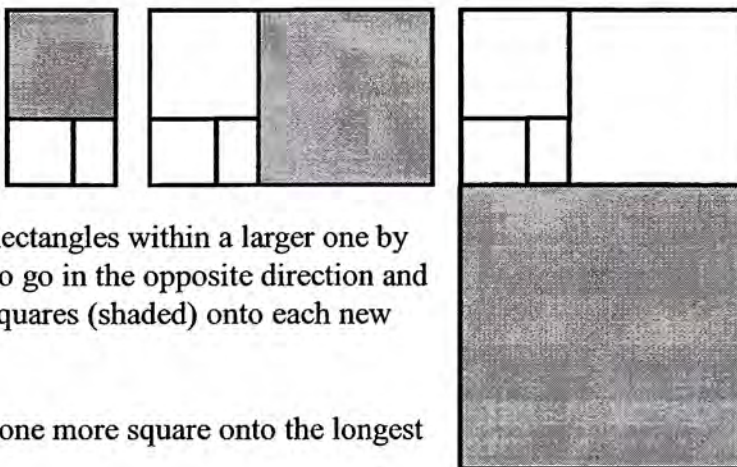
How might the spiral's remarkable balance might help each situation?

What other spirals do you know about?



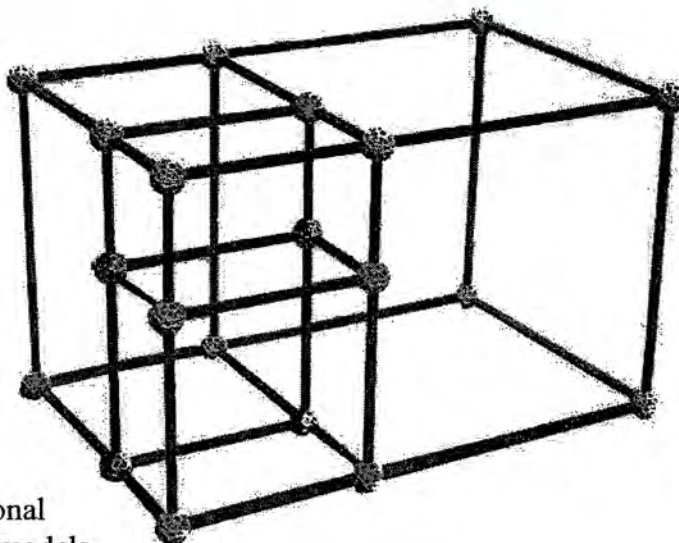
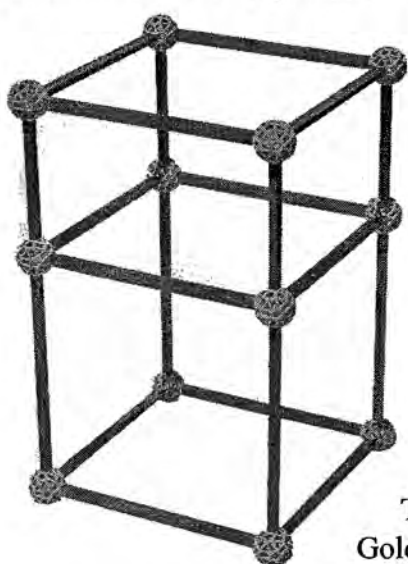
See Chapter 5 in *A Beginner's Guide To Constructing The Universe* for more examples.

Enlarging A Golden Rectangle

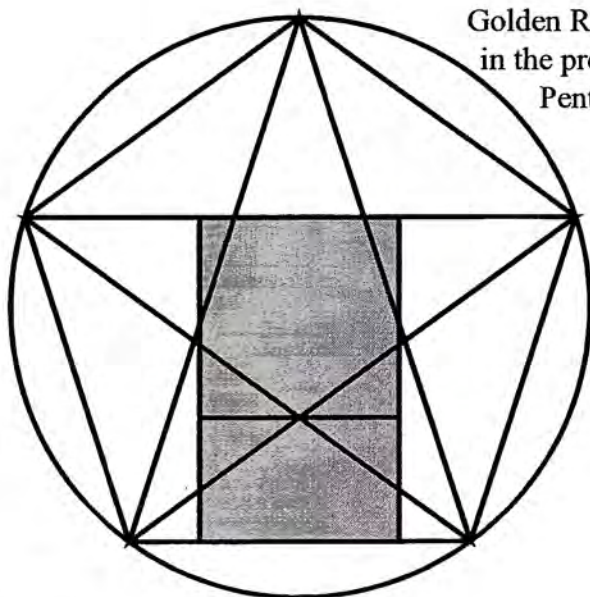


We've seen how to make smaller Golden Rectangles within a larger one by continually cutting out squares. We can also go in the opposite direction and *enlarge* any Golden Rectangle by *adding* squares (shaded) onto each new longer side.

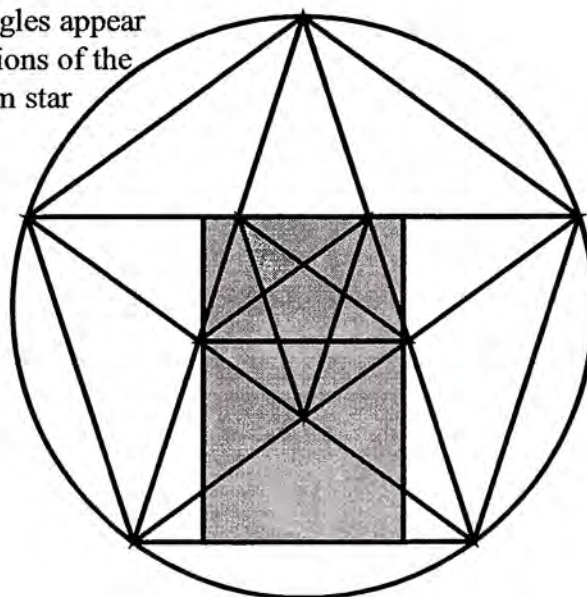
Use your straightedge and compass to add one more square onto the longest (left) side of the largest Golden Rectangle.



Three-dimensional
Golden Rectangle models
made with a *ZomeTool* Kit

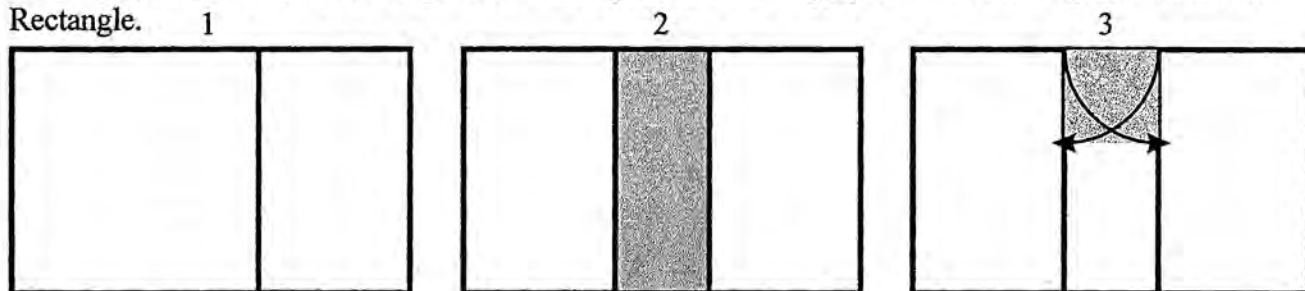


Golden Rectangles appear
in the proportions of the
Pentagram star

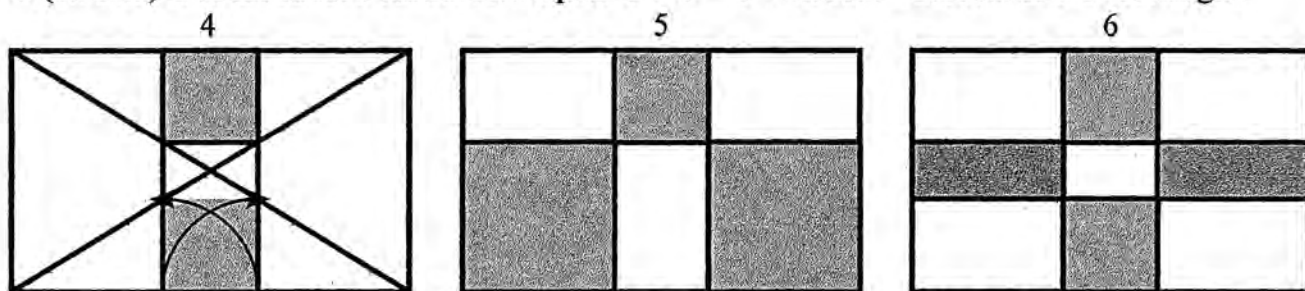


Rabatment

We've seen that when we draw a square inside the short side of a Golden Rectangle we're left with a smaller, turned Golden Rectangle. This technique of producing a small, turned replica of the whole on the inside is called "rabatment" after the French word rabat meaning "to fold" as a flap, although it's often taken simply as "rotate". It's an important geometric tool for exploring patterns within a Golden Rectangle, and has been used to plan works of great art. Let's apply rabatment to explore a Golden Rectangle.

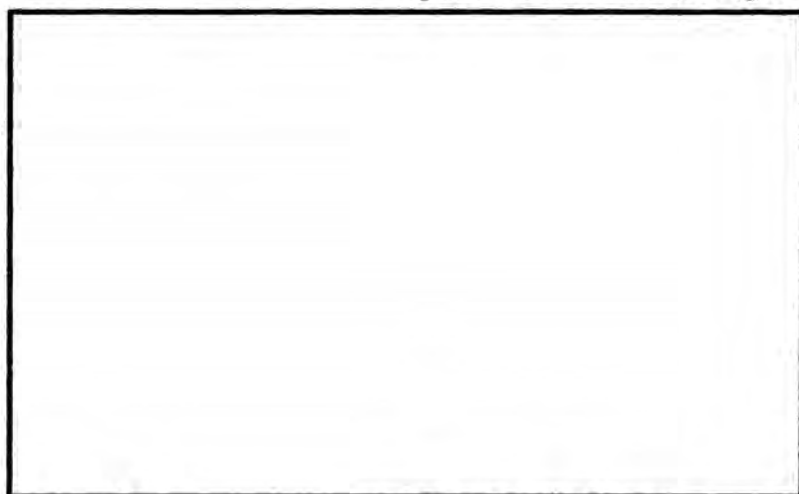
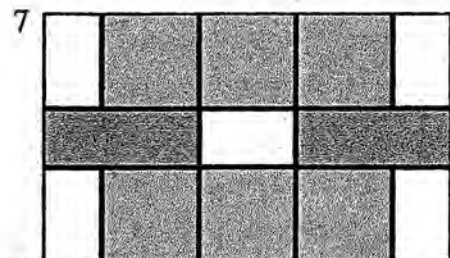


- (1) A square is drawn in one side of the Golden Rectangle, leaving a smaller version of itself.
 (2) Another square drawn *from the opposite end* overlaps with the first in the shaded area. On each side remains a Golden Rectangle. But what is the shaded area?
 (3) Marking a small square with our compass at the top end of the central area leaves a small Golden Rectangle below it. If the area of the square equals 1, and the area of the smaller Golden Rectangle is Φ (1.618...) then the shaded central area equals $1 + \Phi = 1 + 1.618... = 2.618...$ or a Φ^2 rectangle.



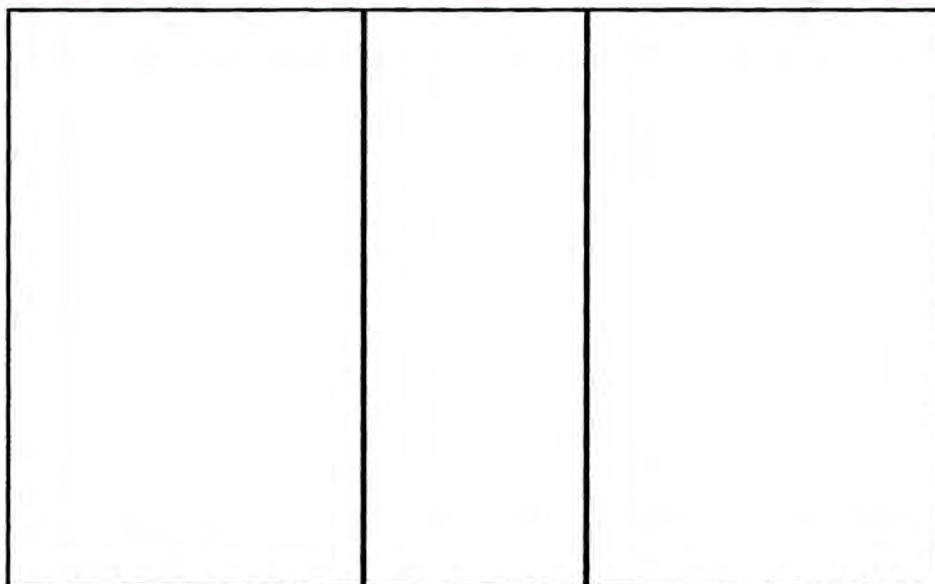
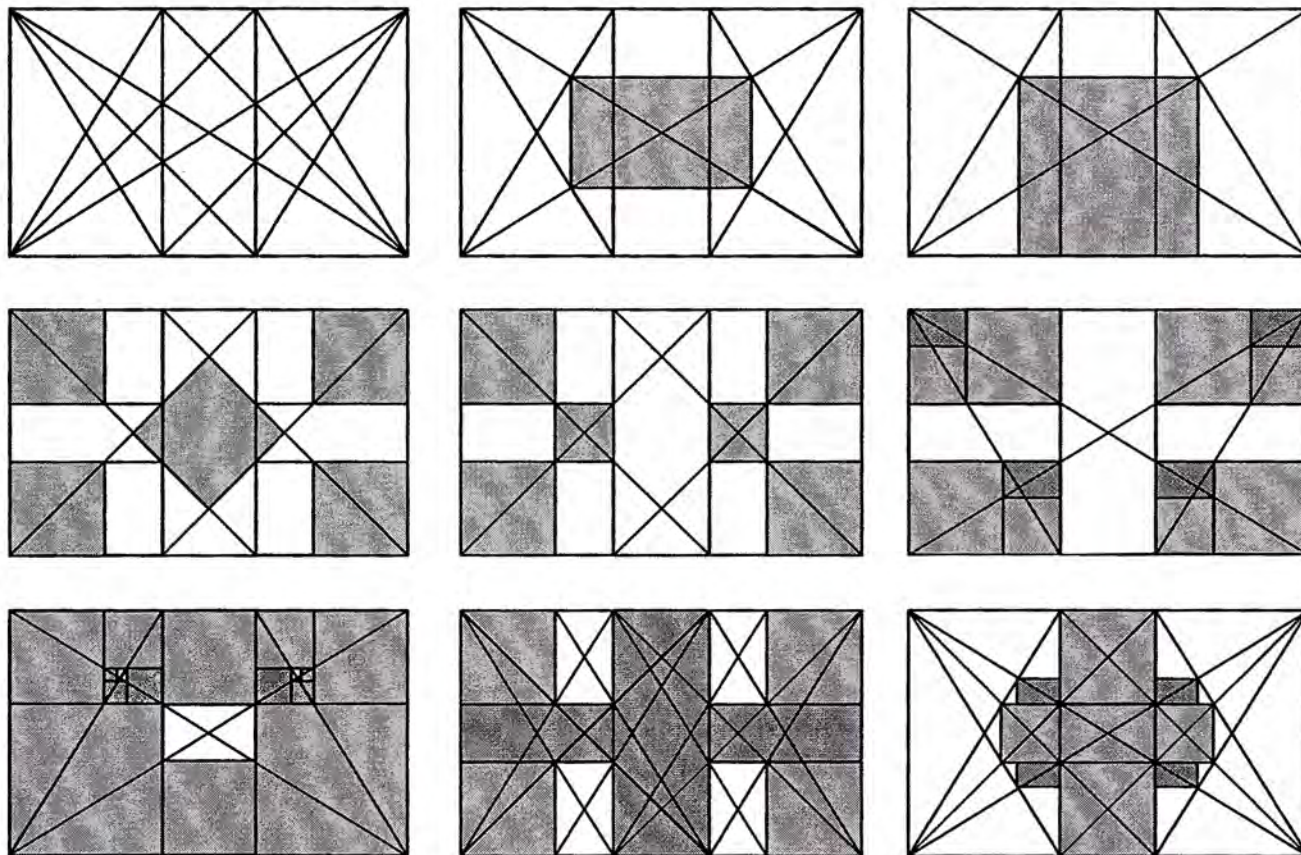
- (4) Another small square made at the bottom end of the Φ^2 rectangle leaves a small Golden Rectangle in the middle. Notice how the corners of this Golden Rectangle meet where the sides of the large rabatment squares cross the diagonals of the full Golden Rectangle.
 (5) Squares are drawn in the bottom of the two side Golden Rectangles. The whole is now comprised of three squares and three Golden Rectangles.
 (6) Squares are also made by rabatment at the top of each side Golden Rectangle, overlapping with the bottom squares creating two Φ^2 rectangles.
 (7) The four corner Golden Rectangles have each been divided further into squares and Golden Rectangles.

Using rabatment, copy one of these patterns in this Golden Rectangle.



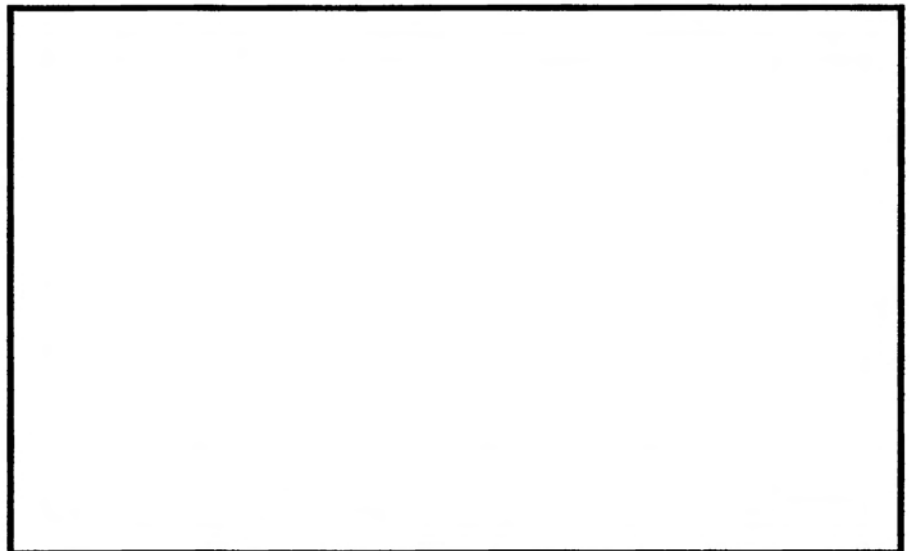
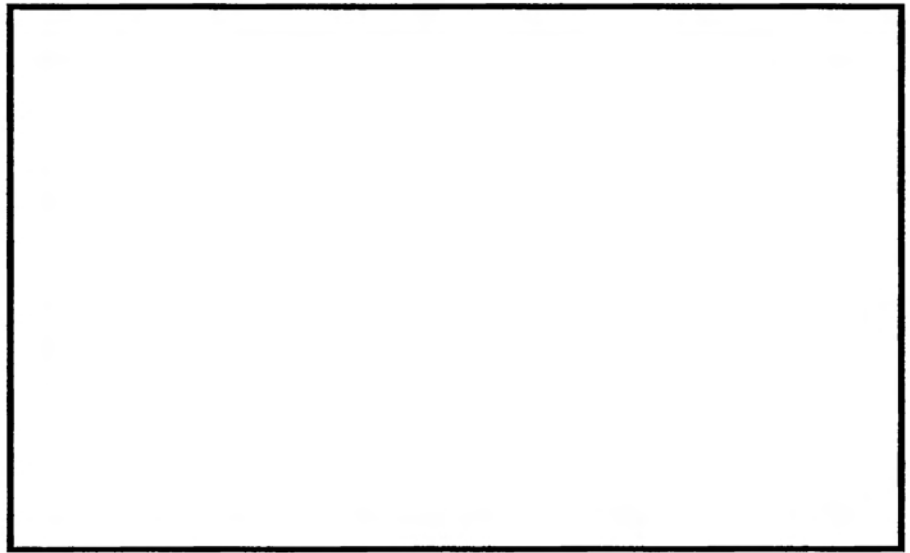
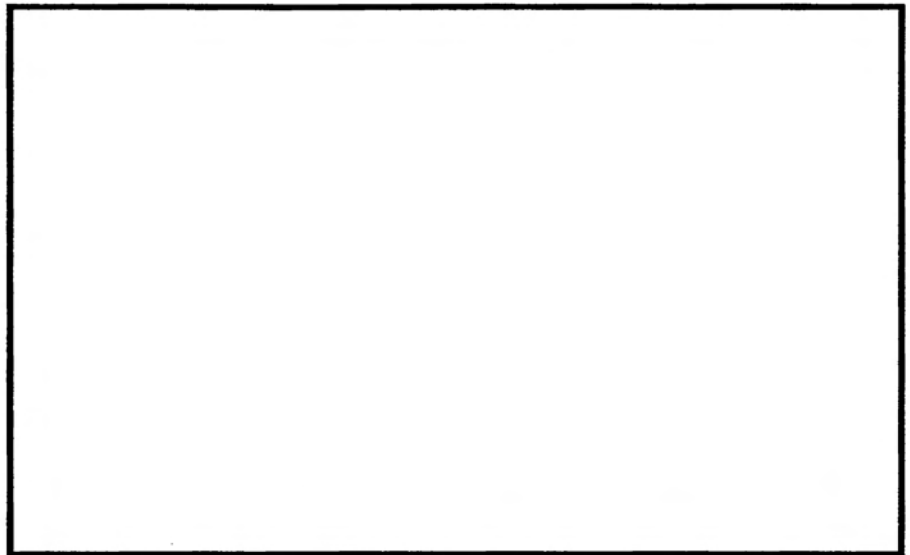
Diagonals drawn between various points of the rabatment cross to indicate natural subdivisions of the Golden Rectangle. They show us visual points, lines, angles and areas of "interest" which our eye unconsciously goes to and finds satisfying. Great artists of all times have known this and used rabatment in the Golden Rectangle (and other rectangles) in their compositions.

Look at each of these illustrations to see how diagonals within a Golden Rectangle may be used to highlight different points and areas within it. In the Golden Rectangle below, with squares by rabatment from each end, copy one of these patterns or explore your own using this simple technique. Use colored pencils to shade areas.



Here are three more
Golden Rectangles for
you to be creative with.
Use rabatment to create
lines and areas.

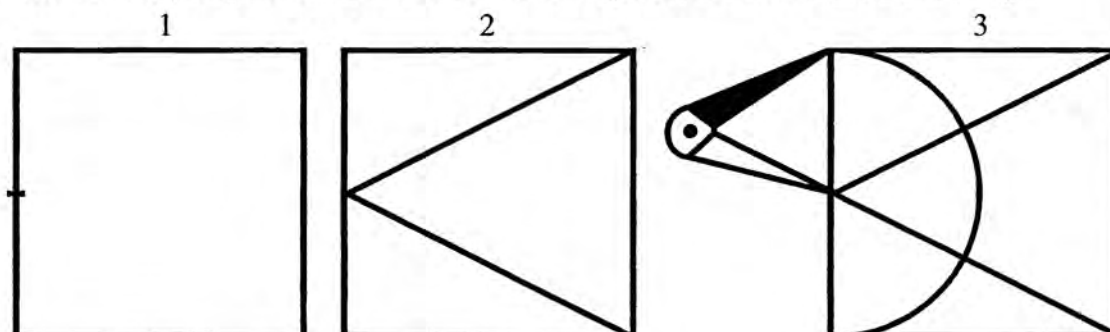
Draw and shade with
colored pencils.



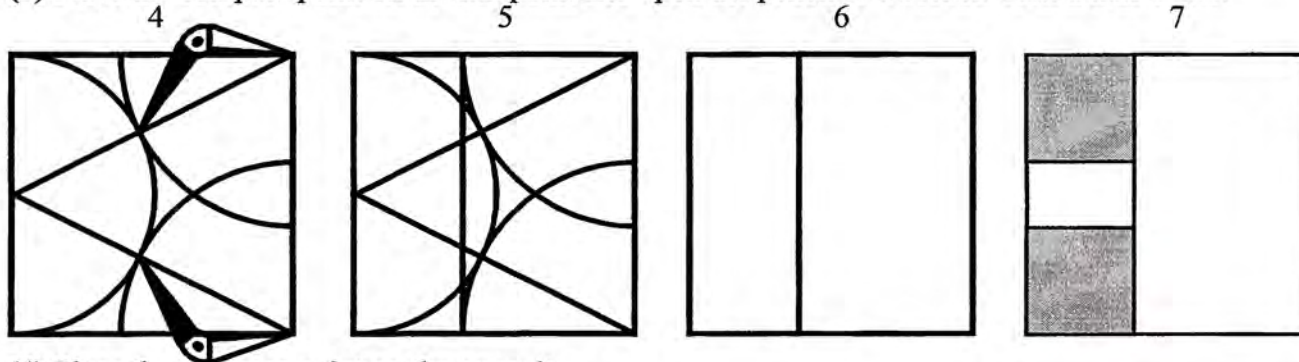
Subdivide A Square

We saw how to subdivide a Golden Rectangle into squares and smaller Golden Rectangles. But what about subdividing a *square* this way?

The method is based upon an earlier construction using two small squares to divide a line at the Golden Mean. Here, we just double them into a full square and create its mirror image.



- (1) Start with a square with the midpoint of one side identified.
- (2) Draw "half diagonals" from the midpoint to opposite corners.
- (3) Place the compass point on the midpoint and open the pencil to a corner. Turn a half-circle.

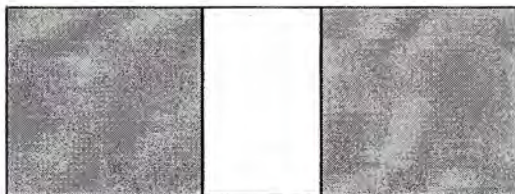


- (4) Place the compass point on the opposite corner of the square and open the pencil to where the half-circle crosses the diagonal. Turn an arc. Repeat this from the other corner.
- (5) Connect the two points where these arcs cross the square.
- (6) This line divides the square into a Φ^2 (again!) rectangle (left) and Golden Rectangle (right).
- (7) Drawing a small square inside each end of the Φ^2 rectangle leaves a small Golden Rectangle between them.

This square Medieval illumination depicts "The Founding Of Paris".

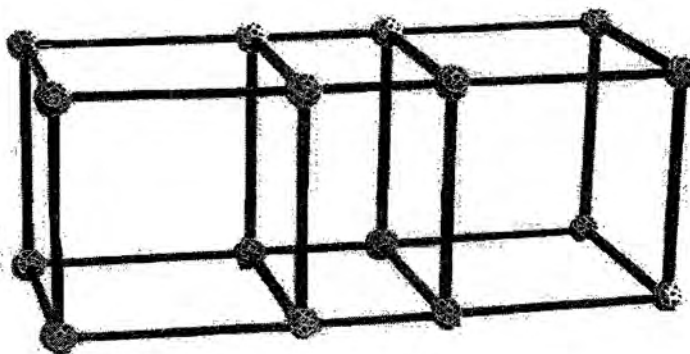
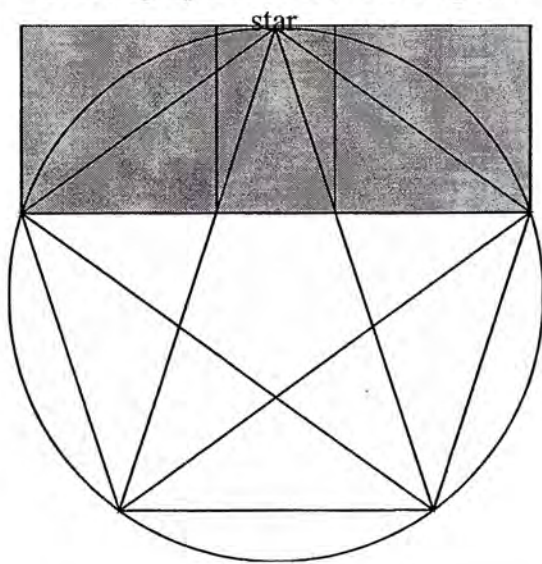
Subdivide it into a Golden Rectangle and a Φ^2 rectangle to see the hidden frame which guides its composition.



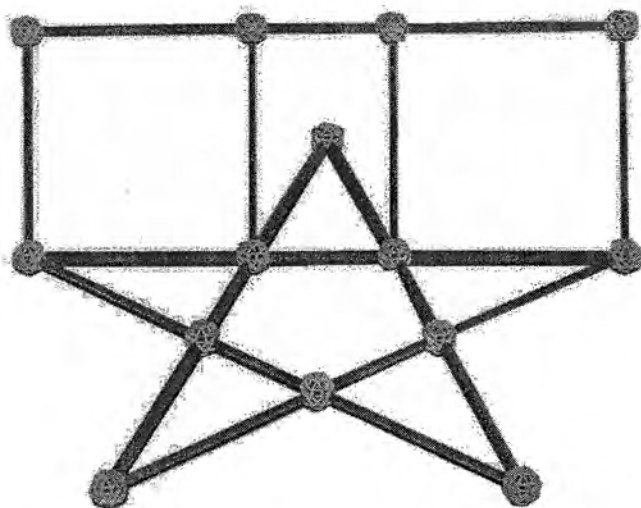


Φ^2 rectangle

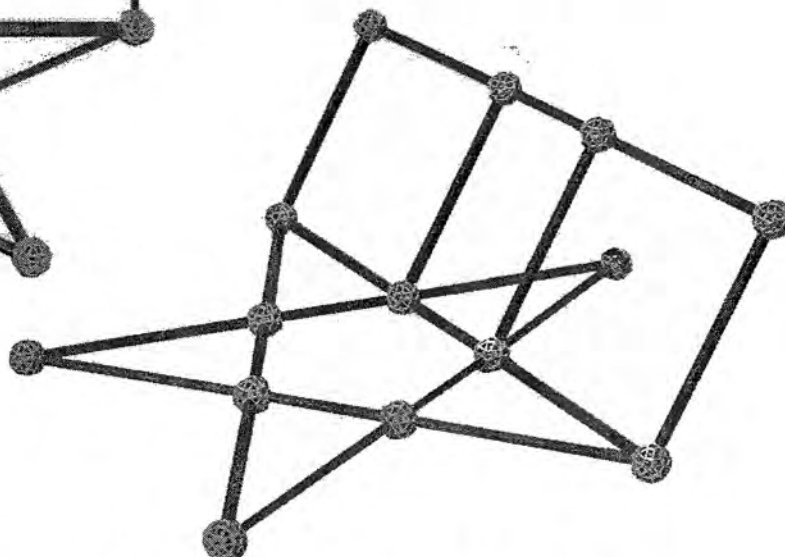
The Φ^2 rectangle is also found in the Golden proportions of the Pentagram



A three-dimensional model of the Φ^2 rectangle made with a *ZomeTool* Kit



Φ^2 rectangles are constructed from *ZomeTool* Pentagrams

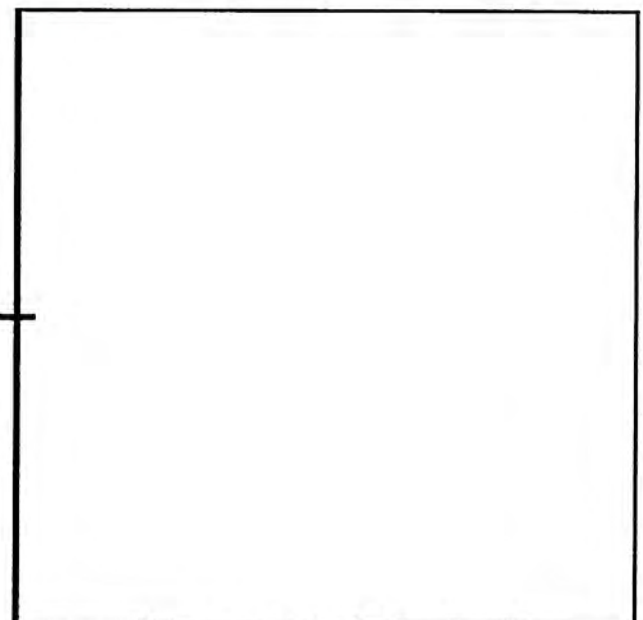
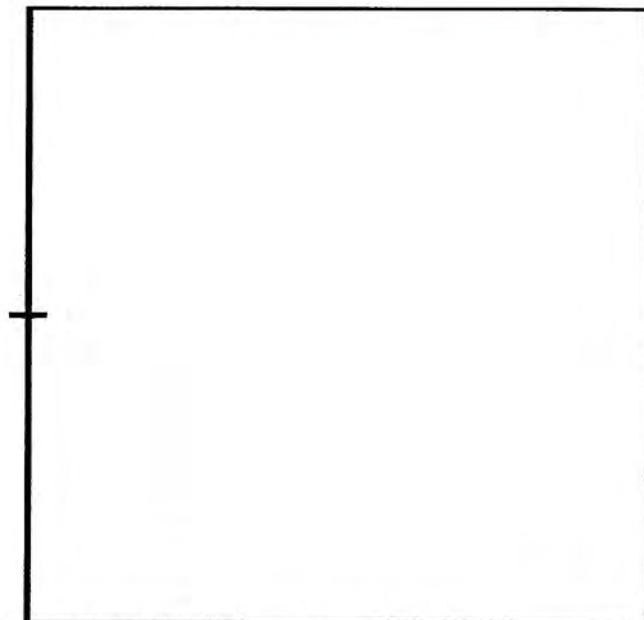
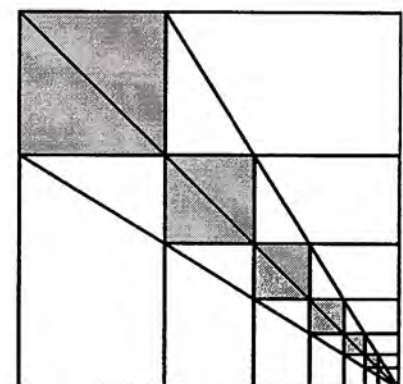
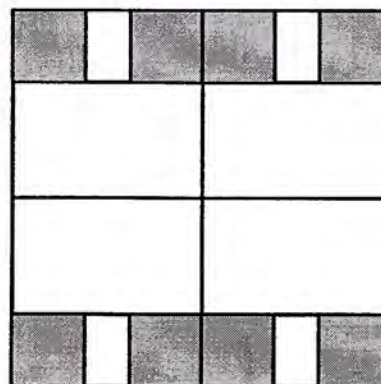
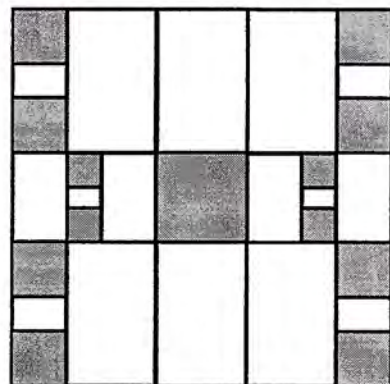
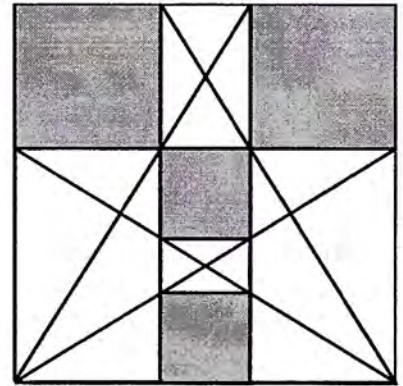
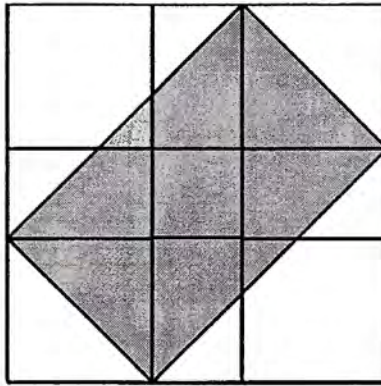
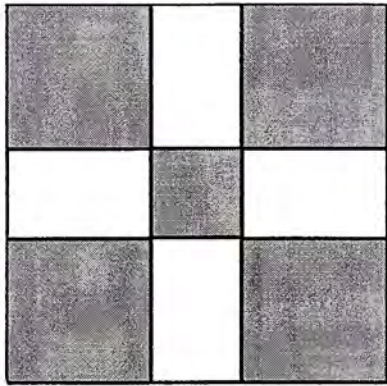


Below (and on the next page) are only a few of the very many ways that a square can be subdivided into smaller squares and Golden Rectangles. These also may be subdivided further the same ways.

Examine each so that you understand how it was divided.

Then, explore this symmetry by copying them or creating your own in the squares below.
Use colored pencils to draw lines and shade areas.

Notice the Golden Rectangle



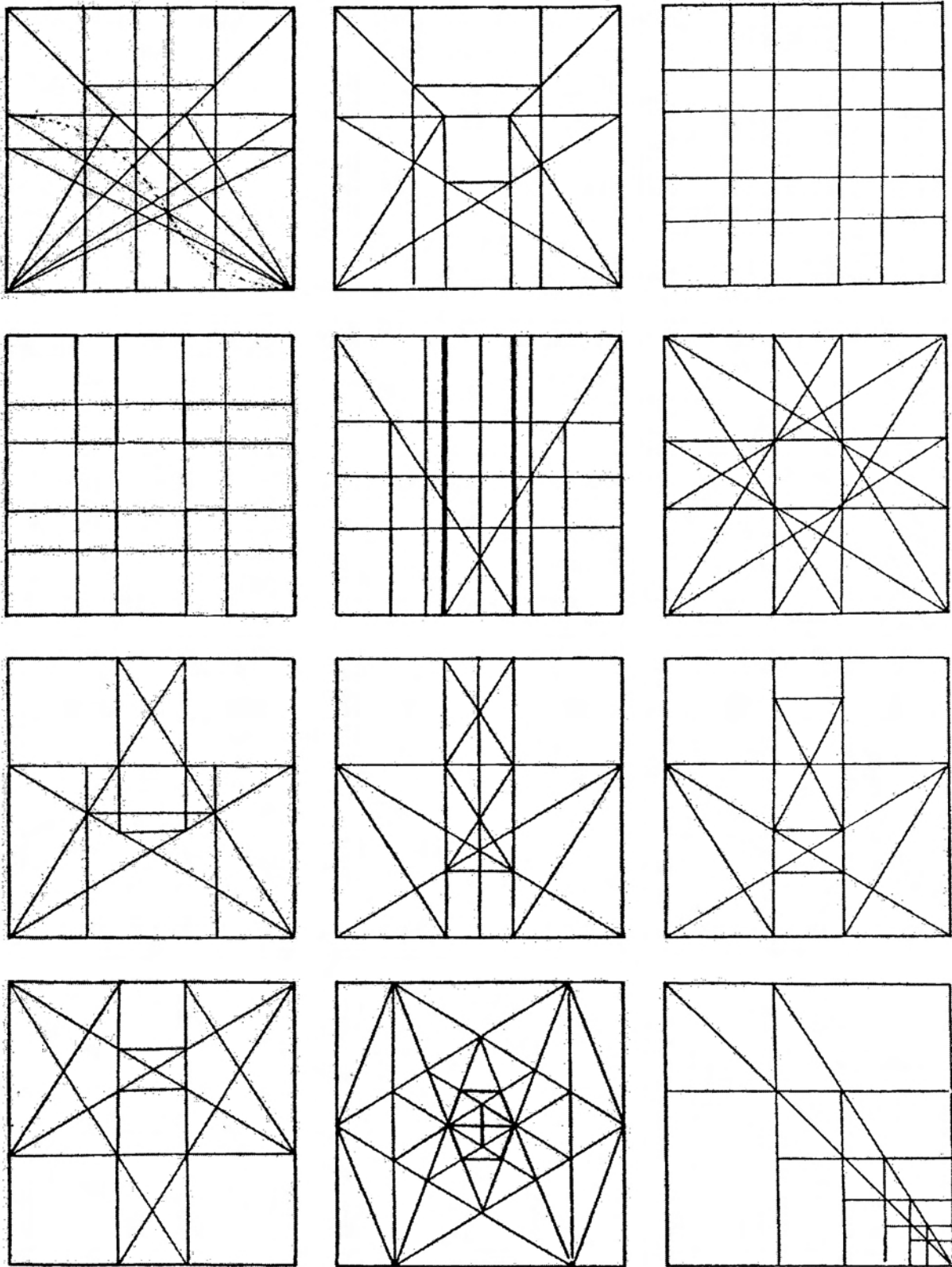
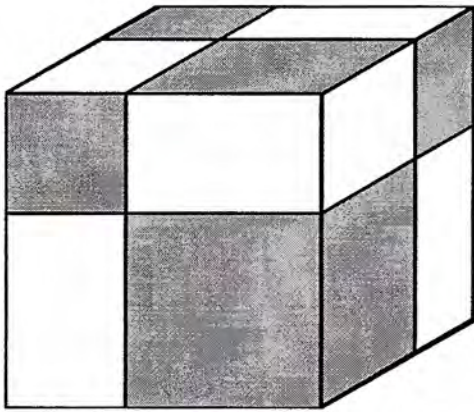


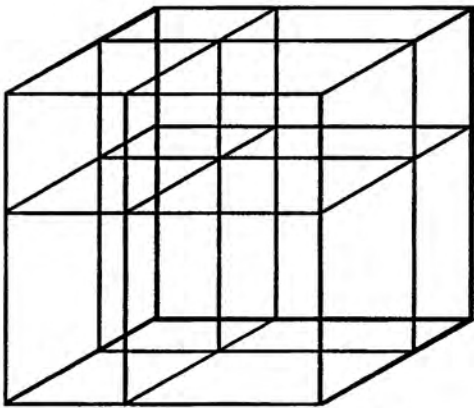
PLATE LVI

The Square, Harmonic Decompositions in the Φ Theme

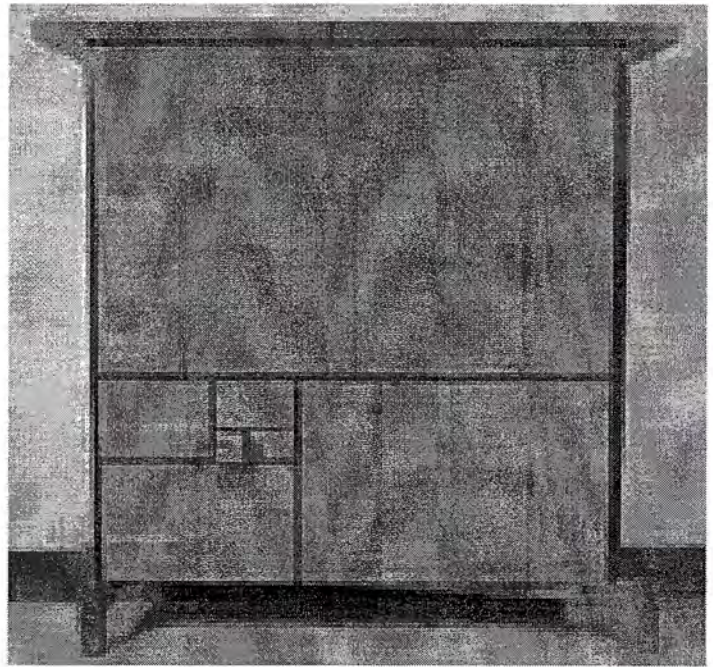
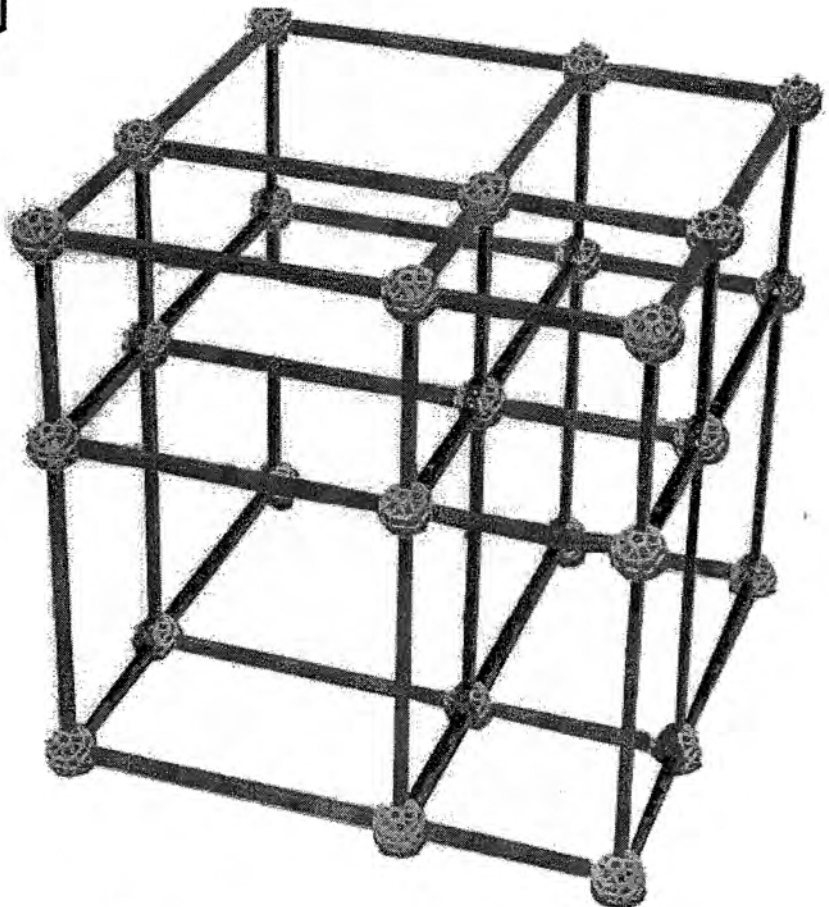
from "The Geometry Of Art And Life" by Matila Ghyka



In three dimensions a subdivided square becomes a subdivided cube.



A
Subdivided Cube
made with a
ZomeTool Kit



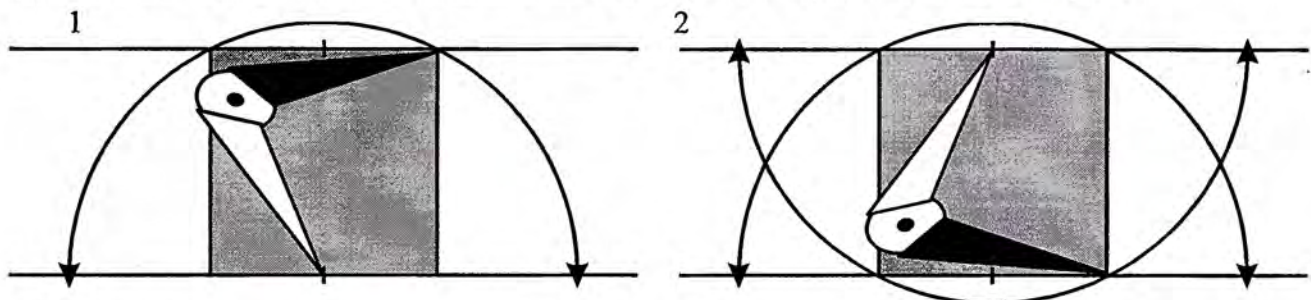
Fibonacci Framework Chest
Berthold Schwaiger

The Root-Five Rectangle

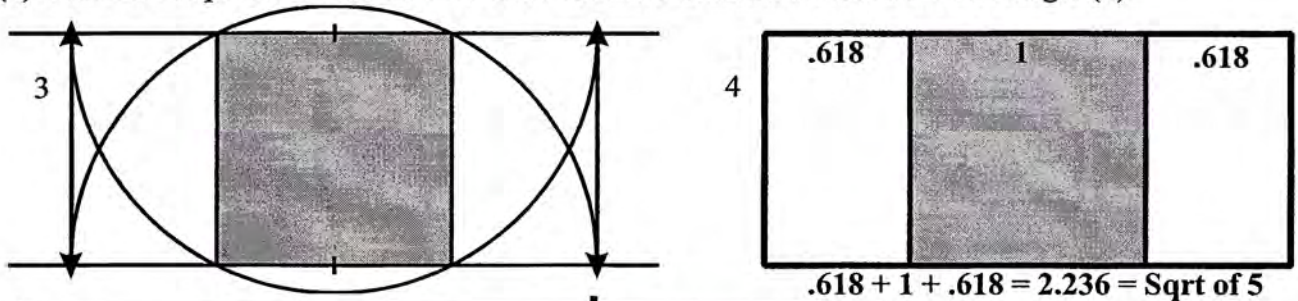
Another interesting rectangle associated with the Golden Rectangle is known as the "Square Root of Five Rectangle" or simply the "Root-Five Rectangle". (Recall that the mathematical value of Phi involves the square root of five.) It's composed of a square flanked by two small upright Golden Rectangles. There are many ways to construct it. An easy one starts with a square.

(1) Place the compass point at the midpoint at the bottom of the square (with extended sides) and open the pencil to an upper corner. Turn a half-circle arc to meet the extended sides.

(2) Place the compass point at the midpoint at the top of the square and repeat the process.

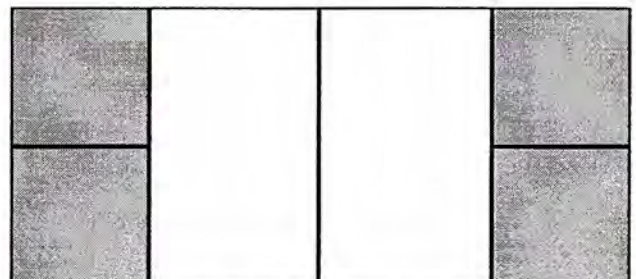


(3) Connect the points where the arcs cross the lines to create a Root-Five Rectangle (4).

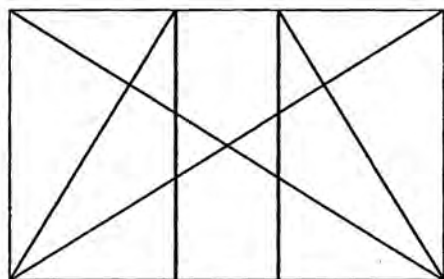


Construct a Root-Five Rectangle above.

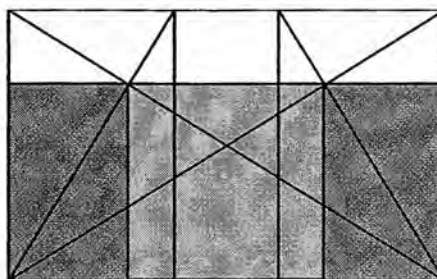
In this scheme, the central square of a Root-Five Rectangle is split in half and moved to the extremes of the Rectangle, while the two Golden Rectangles move to the center.



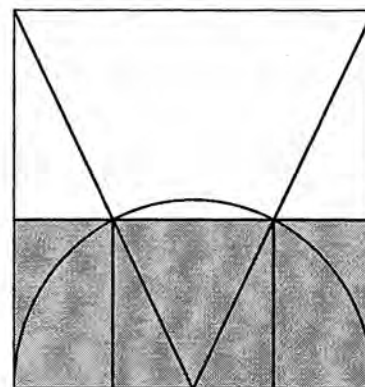
A Root-Five Rectangle can be found within a Golden Rectangle by the diagonals of its rabatment.



Golden Rectangle rabatment with squares drawn at each end. Diagonals are drawn across the whole Golden Rectangle and in each small Golden Rectangle.



The shaded Root-Five Rectangle, a square flanked by two small Golden Rectangles, is defined by the diagonals of the rabatment.

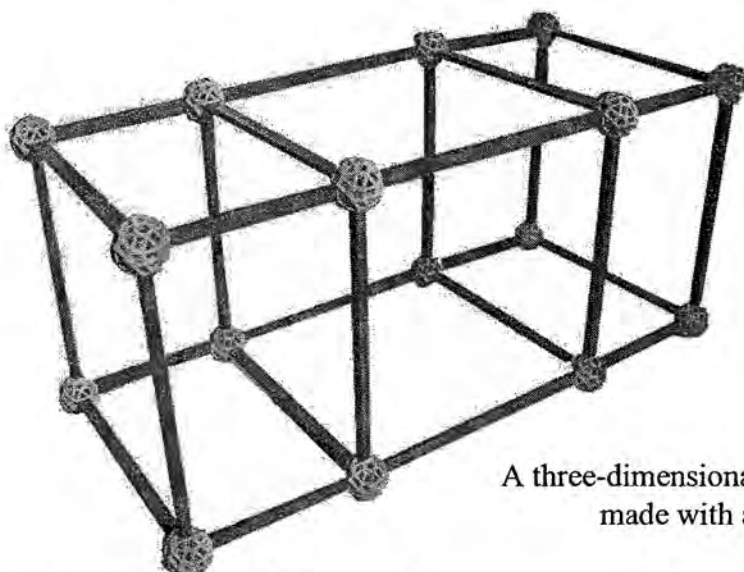
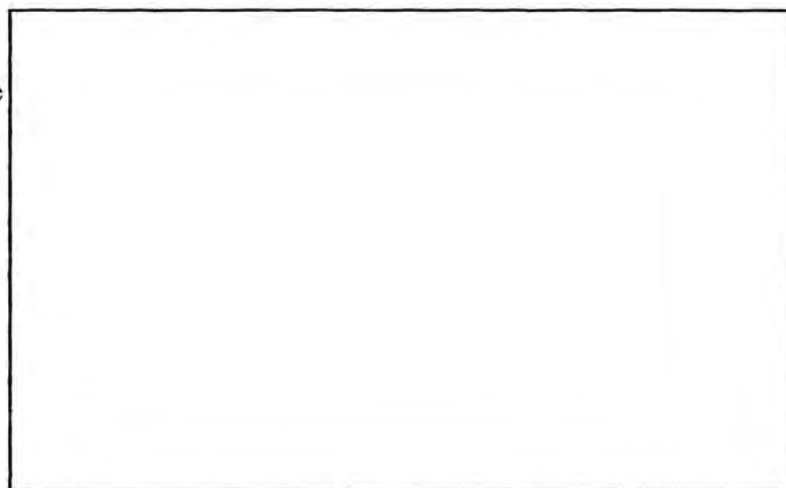


A Root-Five Rectangle can also be found within the Golden subdivision of a square.

Use rabatment to construct a Root-Five Rectangle in this Golden Rectangle.

Don't just approximate where a point should be on the bottom line.

"Measure" it with your compass and mark off the same distance on the opposite side before connecting them.



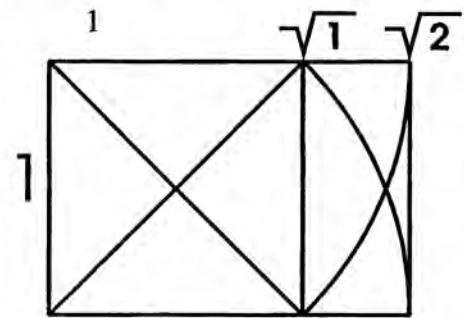
A three-dimensional **Root-Five Rectangle** made with a *ZomeTool* Kit.

The Root-Five Rectangle also emerges in the construction of the sequence of Square-Root Rectangles.

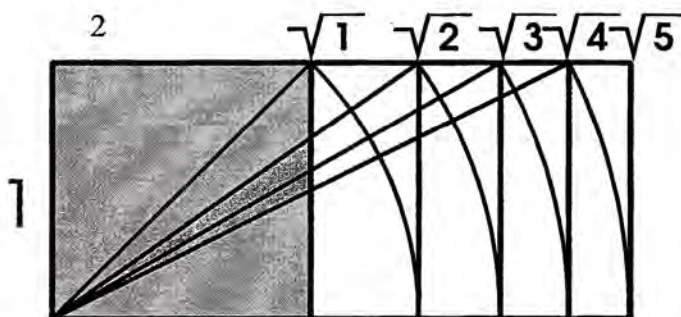
(1) Start with a square (a Square-Root-of-One Rectangle). Place the compass point at its bottom left corner and open the pencil to the upper right corner. Swing the compass down until it crosses the extended line. Then place the compass point at its the square's upper left corner and open the pencil to the lower right corner and swing an arc upward to the extended line. Connect the points where the arcs crossed the lines. Since the diagonal of a square equals the square-root of 2 ($= 1.414\dots$), the square extends to become a Root-2 Rectangle; that is, its sides are in the ratio of 1.414 to 1.

(2) Repeat this process with the compass point always on the square's lower or upper left corner, and pencil at the extreme right (opposite) corner of each newly-created Root-Rectangle. Do this a total of four times and the largest rectangle will be a Root-Five Rectangle whose sides are in the ratio 2.236... to 1.

(3) Notice that the area beyond the square may be divided into two equal, upright Golden Rectangles, and $1 + .618 + .618 = 2.236$

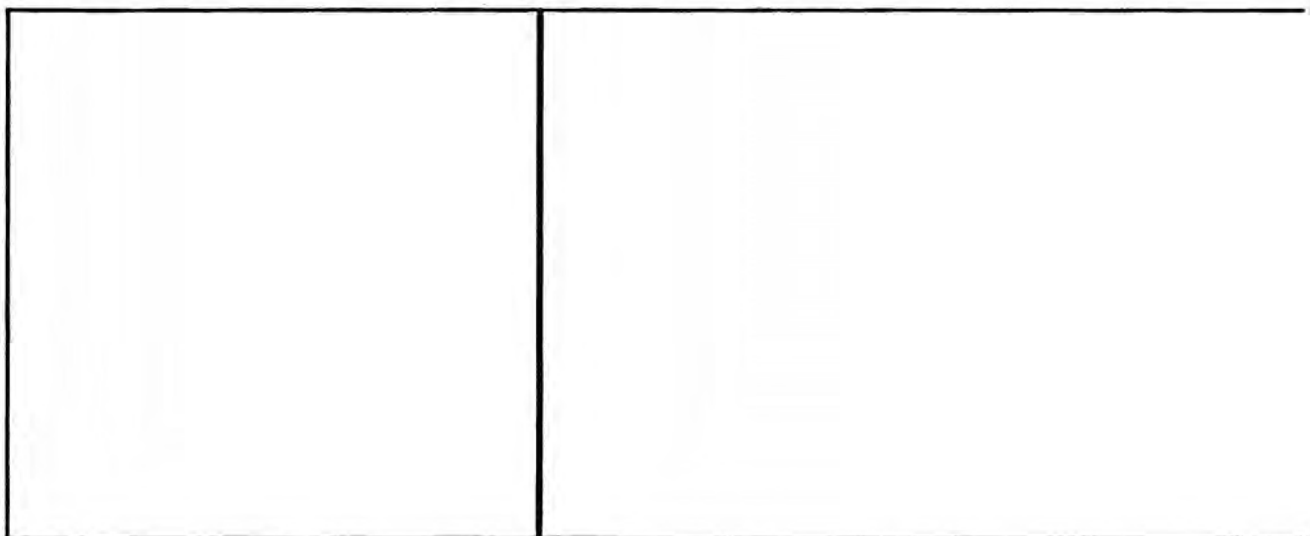


A Root-Two Rectangle

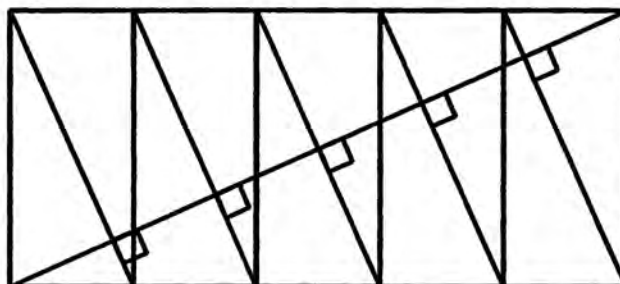
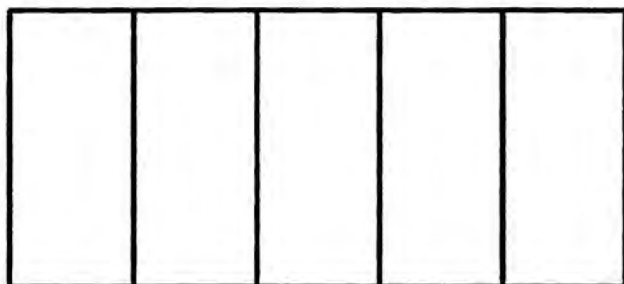


A Root-Five Rectangle

Do this Root-Five Rectangle construction in this extended square:

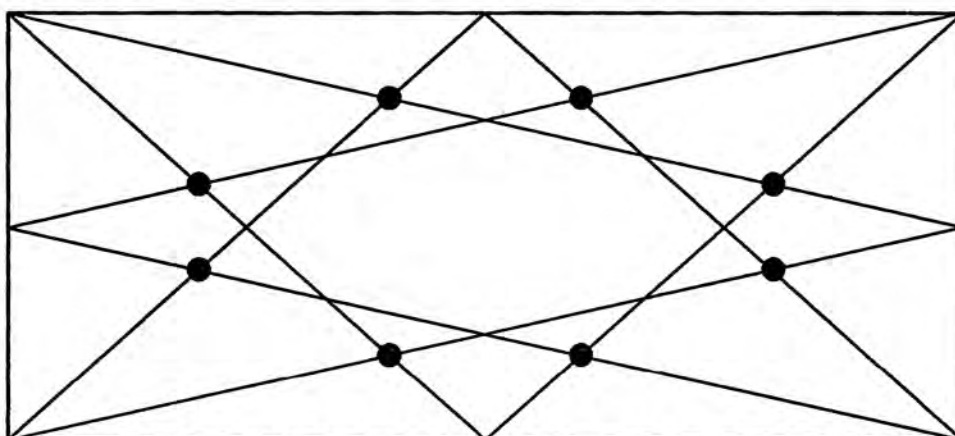


The Root-Five Rectangle has a remarkable property: when it is divided into five equal parts, each small upright rectangle is a Root-Five Rectangle. Also interesting is that the diagonal of the full Root-Five Rectangle crosses the diagonal of each small Root-Five Rectangle at a right angle.

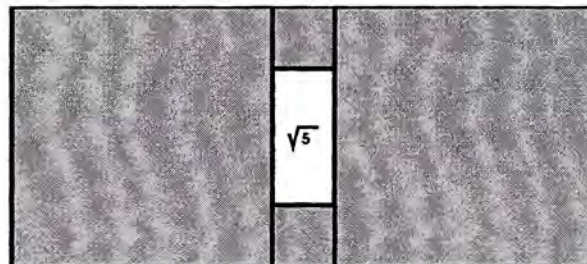
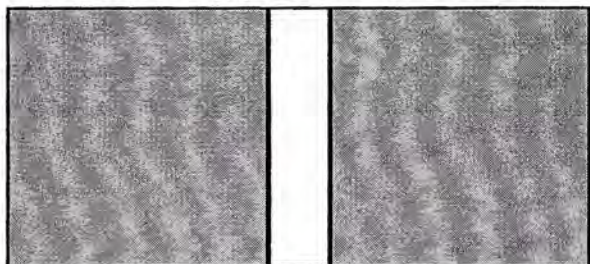


To divide a Root-Five Rectangle into five smaller, equal Root-Five Rectangles, first identify the midpoint of each side and connect them like this. Then, connect the crossings (indicated by dots) vertically and extend the lines. Try it here:

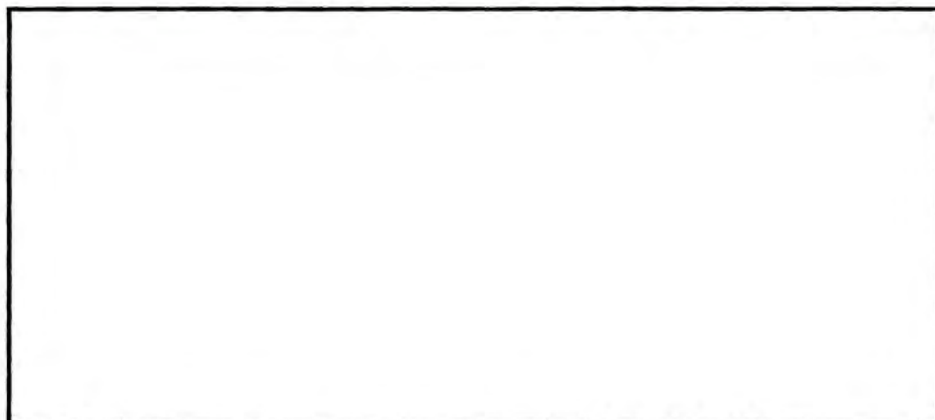
Then draw a diagonal and five small diagonals to see that they cross at right angles.



When a square is drawn inside each end of a Root-Five Rectangle, they leave a gap between them, too narrow to be another Root-Five Rectangle or a Phi-Squared Rectangle. But draw a square from each end of the gap and what is left *between them* is a small Root-Five Rectangle!

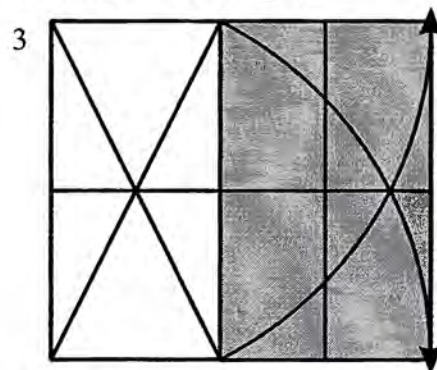
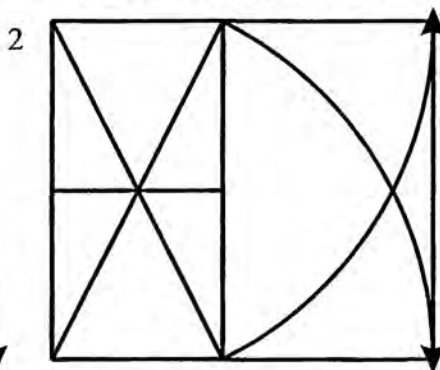
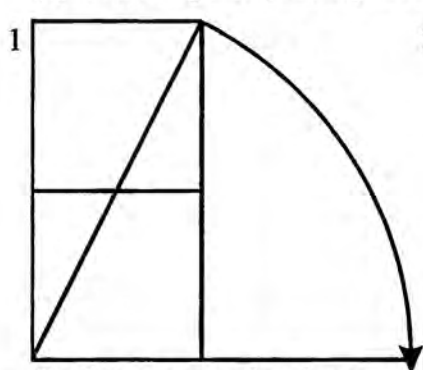


Do the construction in this Root-Five Rectangle:

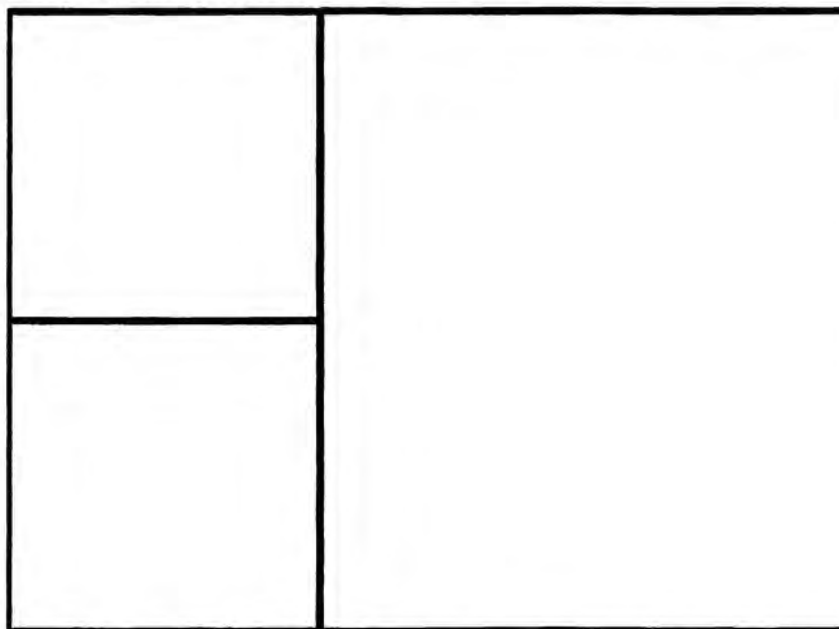


Artists and architects have sometimes found it useful to create two Root-Five Rectangles, one atop the other. Here's one method:

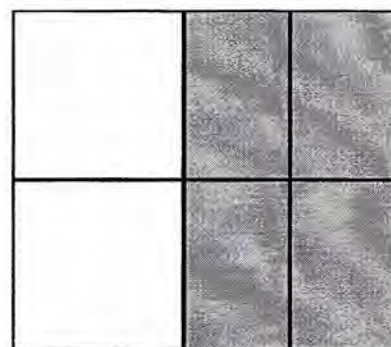
- (1) Start with two squares. Place your compass point at the bottom left corner and open the pencil to the upper right corner (of this 1, 2, Root-5 right triangle). Swing the compass down until it meets the line extended from the bottom square.
- (2) Invert the compass, placing the compass point at the top left corner and open the pencil to the lower right corner of the double squares. Swing an arc up to meet another extended line. Connect these crossing points.
- (3) The space to the right of the original squares may be divided into four small Golden Rectangles.
- (4) Taken together, the squares and Golden Rectangles comprise two Root-Five Rectangles.



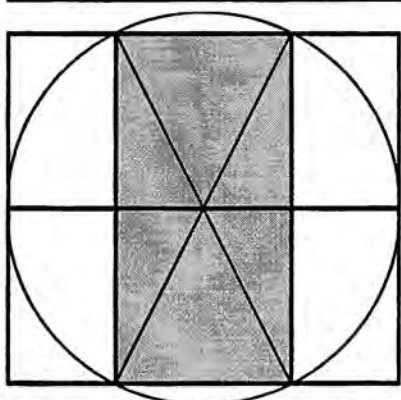
Do this construction here:



4

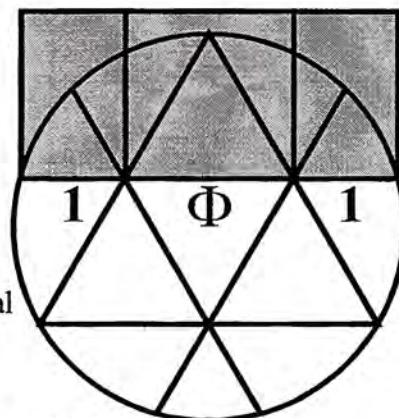


Hint: Construct a *Vesica Piscis* to find the midpoints of the top and bottom of the four Golden Rectangles.



Another method for constructing two Root-Five Rectangles begins by constructing two shaded squares in a circle.

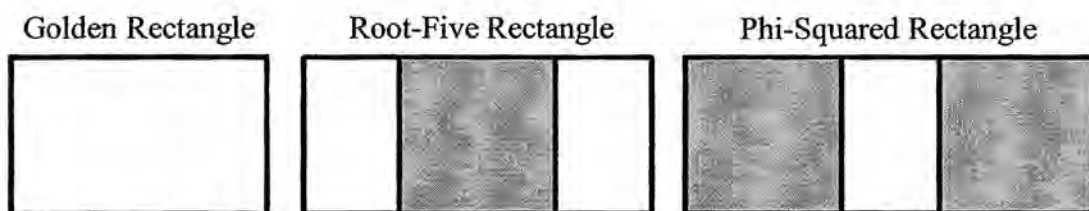
The appearance of the Golden Mean in an inscribed equilateral triangle naturally suggests a Root-Five Rectangle.



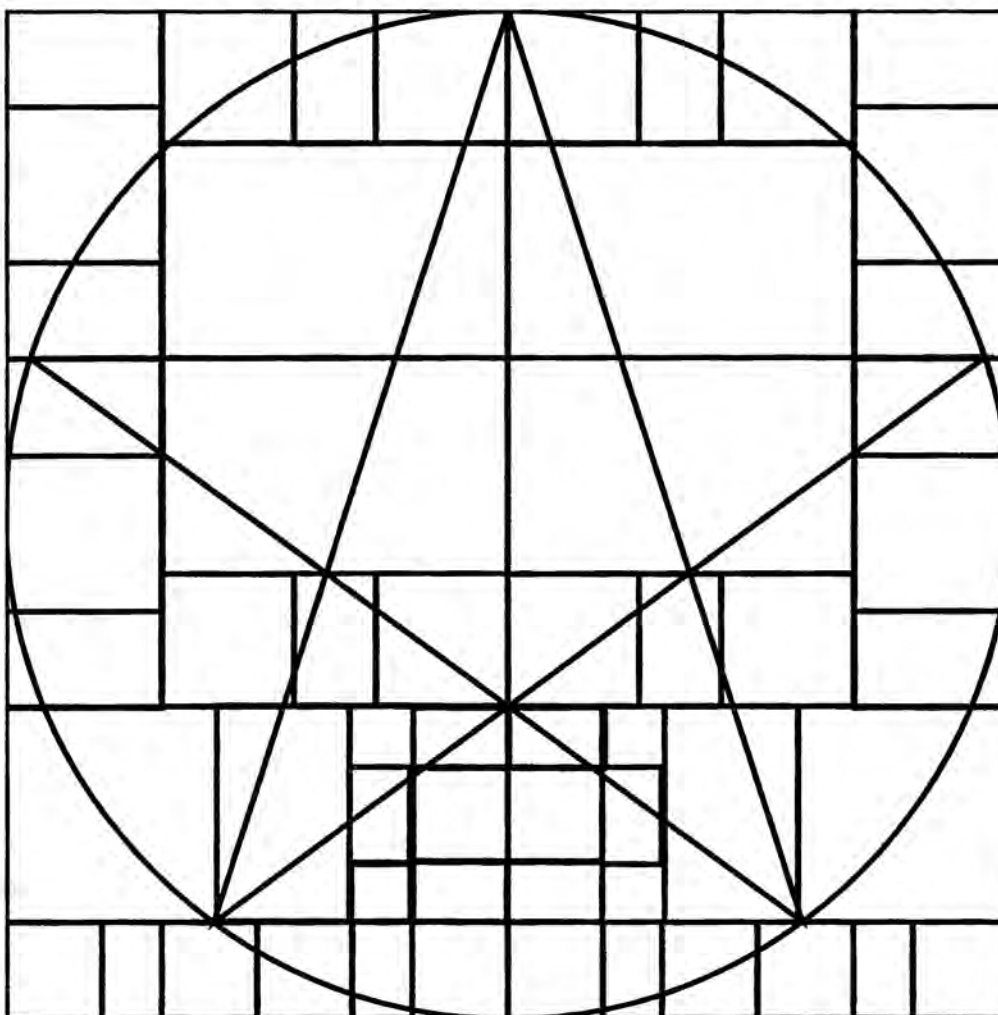
Phi Rectangles and the Pentagon

When a Pentagon is inscribed in a circle within a square, the opportunity arises for the appearance of the Golden Mean.

There are many ways to subdivide a Pentagon. Here's one which shows how it includes squares, Golden Rectangles, Root-Five Rectangles, and Phi-Squared Rectangles.



Use colored pencils to shade the different types of Phi-related Rectangles and/or their parts:

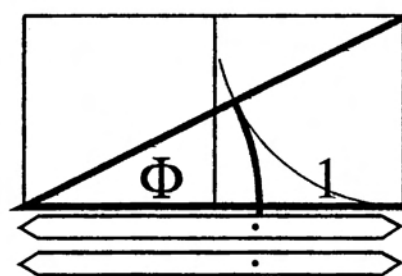


There are many rectangles related to Phi. This book's Appendix includes illustrations from Jay Hambidge's book "The Elements Of Dynamic Symmetry" showing these rectangles, their rabatments and halves of them for your appreciation. Artists, craftspeople, architects and designers can use them as grids for their compositions, applying rabatment and diagonals to develop them.

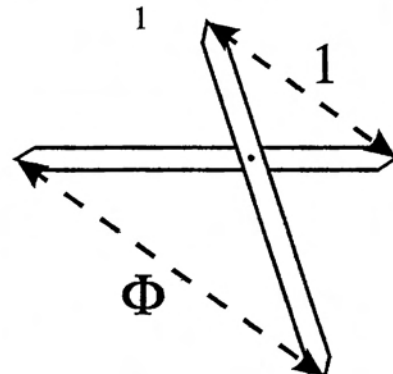
Build Golden Mean Calipers

The Golden Section Caliper appeared in Italy about 1575, presumably used by some Renaissance artists to automatically find Phi relationships in their canvas and sculpture.

(1) To build the simplest one, divide two flat sticks at the Golden Mean (or just multiply the length by 0.618 to find it). Use a screw and butterfly nut to join them. The space between their opposite points on each side will always relate to one another as Phi.

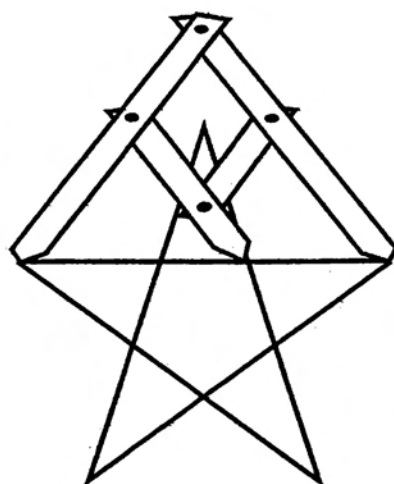
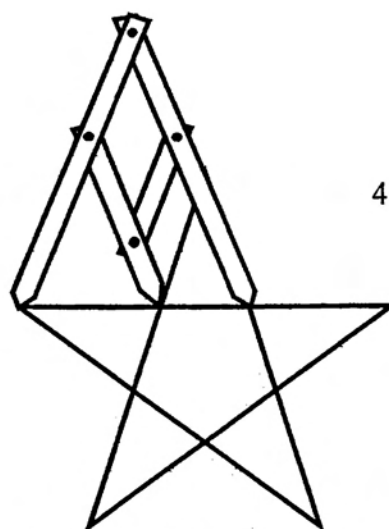
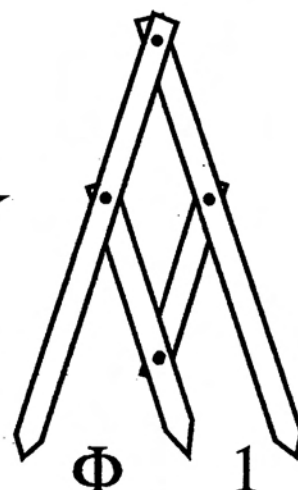
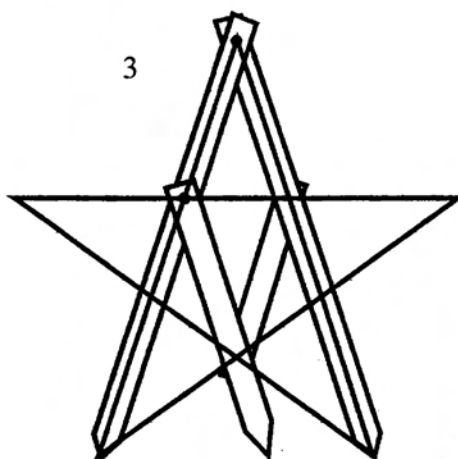
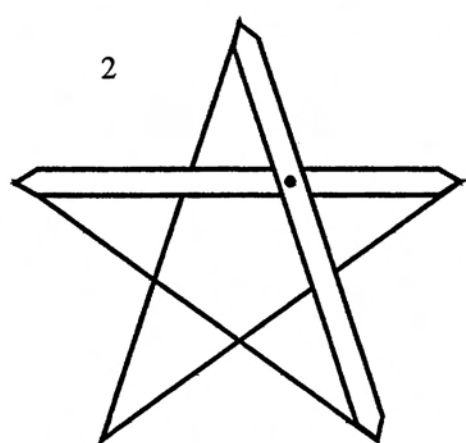


(2) Another way is to take advantage of the Phi ratios in the Pentagon.



(3) A second type of caliper is also based on the Pentagon, but its advantage is that the Phi ratio lengths lie in a straight line. To make it, first construct a large Pentagon. Then use its lines and crossings (as shown) to find the lengths of the Caliper's sections, and where to join them at Golden Section points with small bolts and wing nuts.

Alternatively, without a Pentagon you can simply measure the

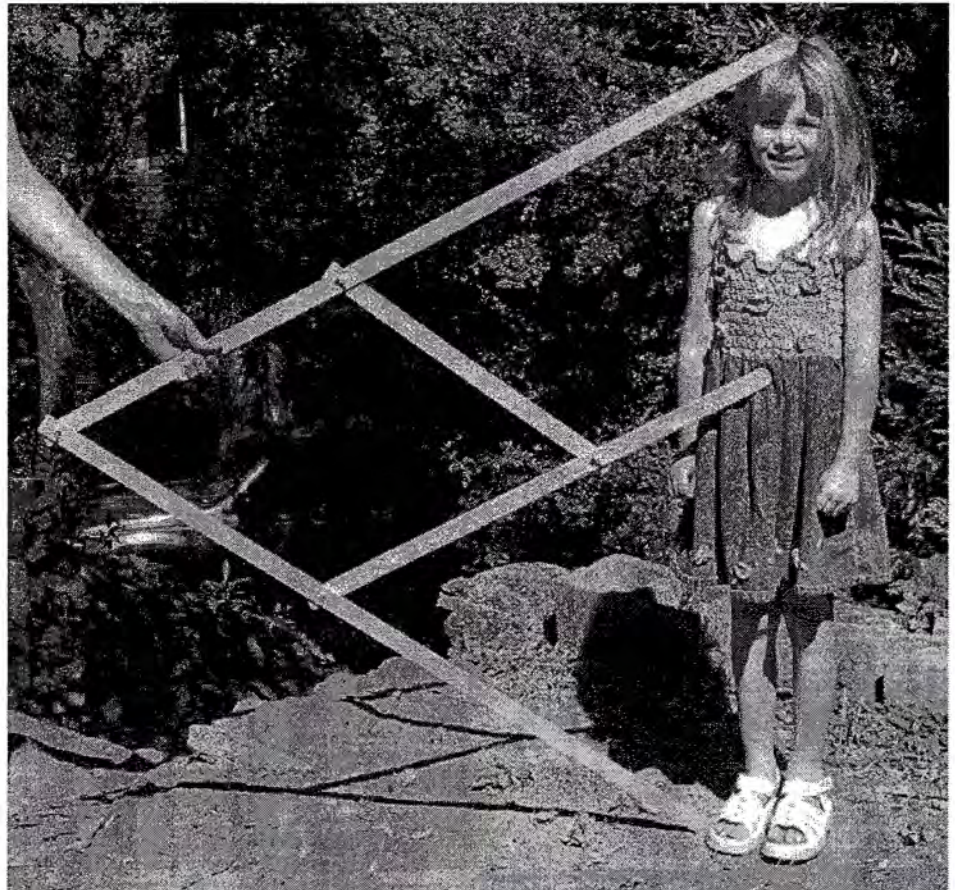
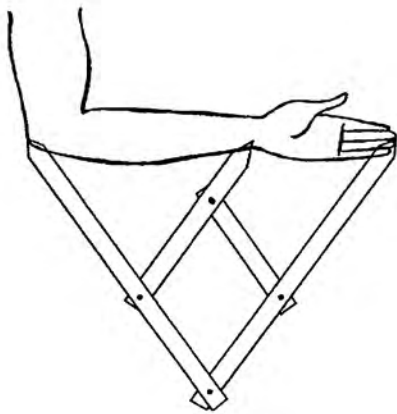
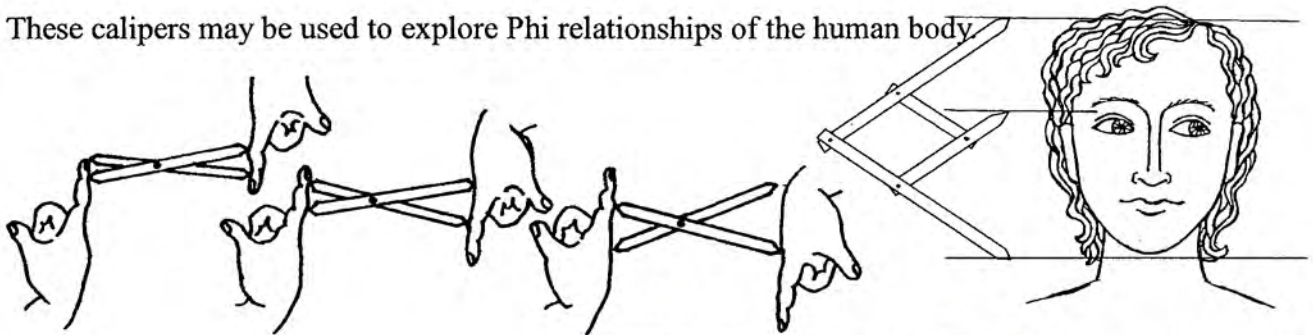


lengths between pivots with a ruler using the Phi proportion. The advantage of this caliper is that it measures the Phi ratio in a straight line (4).

You can use these calipers to explore the ongoing Phi proportions of a Pentagon and Pentagram, the dimensions of your own body, flowers, and the arts, crafts and architecture we'll explore in the next section. It's also useful for designing Phi into your own artistic creations.

Golden Mean Calipers And The Human Body

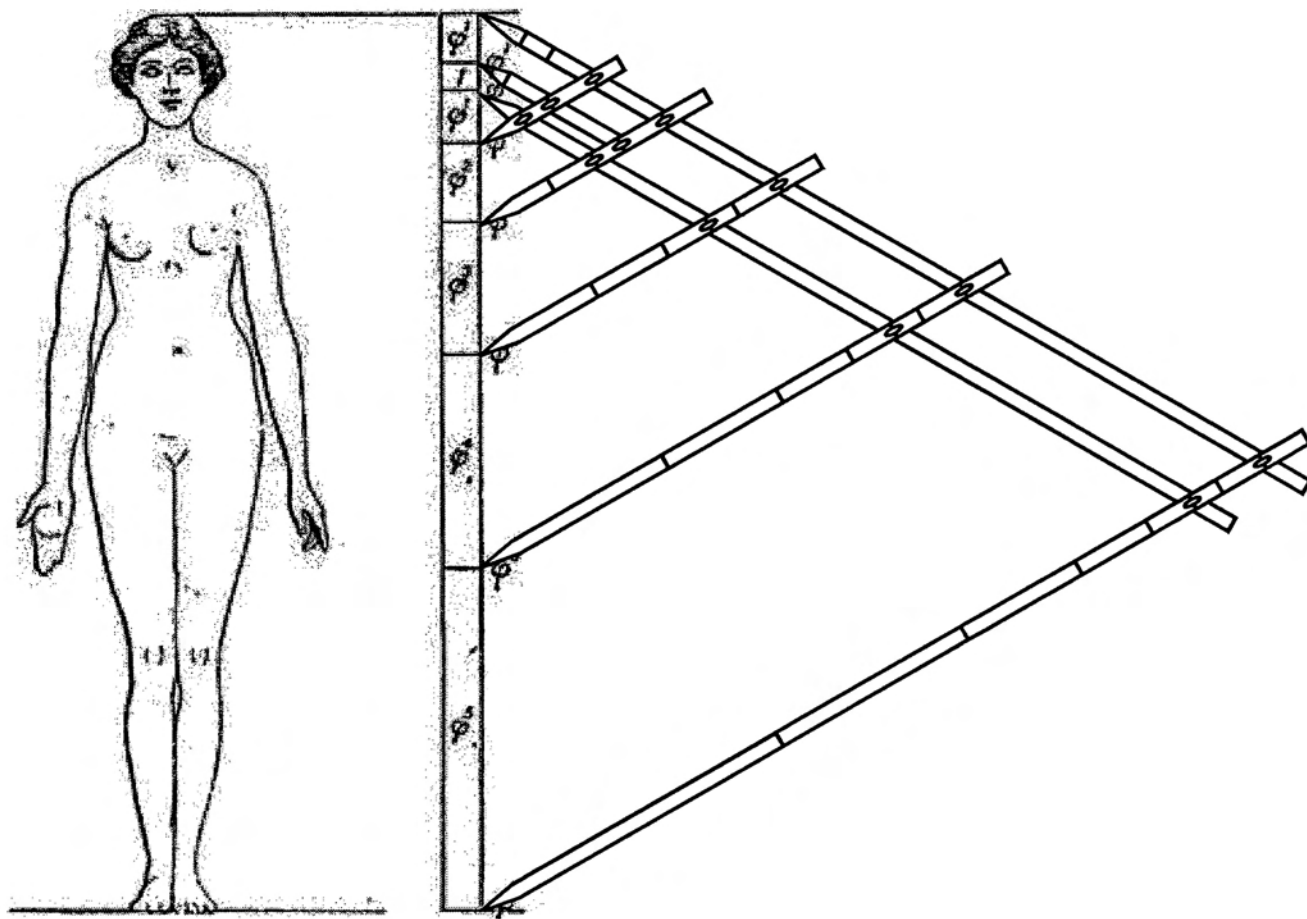
These calipers may be used to explore Phi relationships of the human body



These calipers can be made to show virtually any length of the sequence of the powers of Phi. In 1978 I made a Golden Mean Caliper which spanned the full height of a person (and well beyond!) indicating all eight levels (seven sections) of an ideal body related by unfolding powers of Phi.

Examine the picture and see if you can determine the distances between each pivot and line as sequences of powers of Phi. (Hint: define the shortest unit segment as Φ^1 and the longest segments as Φ^5 .)

These may also be used to design or explore the possible Phi dimensions of rooms, furniture, doorways and other large objects.



Geometry And Art

There's no doubt that geometric schemes have been applied to human creations, including art, for a very long time. Sometimes they're easy to see, as in ancient temples or the designs of cell phones. But geometry often goes unnoticed. Who but the manufacturer considers the geometry of a gum wrapper? Yet it's all around us in nature and the arts, crafts, architecture and the designs of ordinary objects, although, of course, not necessarily so.

Until the advent of the scientific age, the world was generally considered to be a sacred mystery. There was no "art" for enjoyment, entertainment, social commentary and commerce as we see today, but the intent was for the betterment of soul and spirit. The highest role of painting was to calm the emotions so that music could then purify them, and poetry could uplift them. The natural language to express this sacred vision was mathematics since it only speaks eternal truth. These truths of mathematics are not invented by anyone, but are discovered by examining relationships and patterns. Only the symbols to express them, the numerals and signs, are invented. The cross-cultural truths they express are not. And so the creators of sacred art used the eternal language of numbers and shapes to express the sacred truths they found within the harmonious cosmos and in themselves.

The Golden Mean holds a special place in this timeless language because it teaches about harmony, the just balance of parts with each other and the whole. The modern enchantment of the world presents a fragmented vision of seemingly separated parts. This contrasts with a confident vision of a great unity underlying apparent diversity, which was an important theme in sacred traditions worldwide, and was based on the study of mathematics and direct observation of nature and self. For this and other reasons, Plato admonished artists to

“...fix their eyes on perfect truth as a perpetual standard of reference, to be contemplated with the minutest care, before they proceed to deal with earthly canons about things beautiful.”

Many modern artists, art teachers and critics are loathe to apply any mathematical scheme to their creations out of fear that theoretical correctness will comprise spontaneity, constrict them and degrade their work. But the really great artists of all time know better, and I have to agree with their critics:

“Knowledge of a basic law gives a feeling of sureness which enables the artist to put into realization dreams which otherwise would have been dissipated in uncertainty.” — Jay Hambidge

“Regulating lines ... are ... a springboard and not a straightjacket. ... [T]hey satisfy the artist's sense ... and confer on the work a quality of rhythm.”
— Le Corbusier (1887-1965, Swiss architect, city planner, writer)

“Inspiration, even passion, is indeed necessary for creative art, but the knowledge of the Science of Space, of the Theory of Proportions, far from narrowing the creative power of the artist, opens form him an infinite variety of choices within the realm of symphonic composition.”
— Matila Ghyka (Romanian royalty, educator, writer)

“Sacred architecture is not, as our time chooses to see it, a ‘free’ art, developed from ‘feelings’ and ‘sentiment’, but it is an art strictly tied by and developed from the laws of geometry.” — Fredrik Macody Lund

It was emphatically summed in the statement:

“Mighty are numbers; joined with art, resistless!”
-- Euripides (c 484-406 BCE, Greek playwright)

Most extreme was Jean Mignot, the Gothic cathedral Master Builder from Paris who addressed the architects of Milan 1398 when he exclaimed

“Art without knowledge is nothing!”

But more calmly, Albert Einstein recognized the value of the Golden Mean when he wrote that it

“... is a scale of proportions which makes the bad difficult (to produce) and the good easy.”

Of course, harmony in art can appear without applying it intentionally. The American artist and teacher Jay Hambidge observed that while

“This symmetry cannot be used unconsciously although many of its shapes are approximated by designers of great native ability whose sense of form is highly developed.”

What can an intentional awareness of this knowledge do for an artist who uses it as a tool?

- * Proportion space to convey a particular feeling. Different ratios among sides convey different "pressure" of proportions on the space within.
- * Provide guidance for dividing space effectively. The Golden Mean in particular can maintain a feeling of harmonious relationship among the elements.
- * Identify eye pathways, showing how to take advantage of lines which the viewer's eye unconsciously moves along, including diagonals, arcs, loops and spirals, according to geometric pressure and visual elements.
- * Identify the locations of "points of visual interest" or "hotspots", taking advantage of certain nodes of calm where the viewer's eye is geometrically guided to expect a point or line on which to settle, and thus where to place detail most effectively.
- * Help to make decisions about design by showing points, lines and areas which form a rhythm.

Artists don't have to use this knowledge directly, but can work against it to *disturb* the feeling of interrelated harmony and wholeness, if that's what they really want to do. But the reader is encouraged to bear all this in mind when viewing the examples of the Golden Mean in art on the next pages. Astute artists and others can learn a great deal from the masters by activities involving themselves in the underlying geometry of these works.

It would indeed be best to study and measure the actual works of art, crafts and architecture. To be in

the presence of such creations itself teaches a great deal, but since it's not possible for everyone to travel the world for this purpose, we can only work with images on a page. There are certainly problems and risks with this, not the least of which is distortion caused by an uncentered photograph, which is often the case. This shifts geometric analysis to one side or another. Also, the mere thickness of lines drawn on an image can hide important distinctions. Even if we look to the published dimensions of works of art, we find that measurements of the same object from different sources often conflict and aren't always accurate either. Even if the source is the actual museum where the piece resides, measurements are usually given in both inches and centimeters, and simple calculations show them to often produce different ratios. Sometimes an artist will actually speak or write about their use of particular dimensions, as the American painter George Bellows did, so we know how to examine his use of geometry. Finally, there's a wide range in terms of accuracy by the artist. The most ancient pieces are often most geometrically accurate, as many Egyptian examples exhibit. Some pictures from European illuminated manuscripts have great geometric accuracy while others are drawn in a more "relaxed" manner. I have done my best to select examples which make sense as representations of the use of the Golden Mean in art. I've looked at each in a variety of ways. Most important are consistency and wholeness. Not just a part of it reveals Golden proportions, but each part must be related to other parts and the whole by the same scheme with no forcing of results. And any symbolism must be appropriate.

As I study this subject, I have to say that there appears to be a secret design language at work in great art, probably only taught within temples, guilds and among groups of artists, from mouth to ear and hand to eye. Some teachers are known, like Luca Pacioli who taught Leonardo da Vinci, but most will never be identified. We can only see what they and their students left for us. From studying great art one can discern a few of the "rules" at work when composing with this timeless language of geometry. In a painting, for example, one of the most interesting ideas has to do with the placement of objects. It seems clear that placing an element directly along a line will make it feel that it has no movement and is "dead". But when an object is moved slightly from the geometric ideal, either bent or shifted over from direct continuous contact with the line or point, it appears to have movement in that direction, and feels alive. The ancient Greeks were supreme masters of this, seen in the paintings on their pottery. They could accomplish this because they, like their Egyptian teachers, were great observers of nature, and students of the psychology of perception. Yet you will see what is called great "rigidity" of the placement of elements of Egyptian works. But this is because they were representing the realm of the gods, which is in the eternal stillness beyond time and movement. That's why Egyptian gods very rarely face the viewer: because they don't acknowledge us from their world of eternal "being", we who identify with time, movement, decay, and are continually "becoming". So an artist can use this knowledge different ways, since it is a language with which one can write anything. But whether secular or sacred it is best suited to conveying dynamic balance, harmony beauty and wholeness.

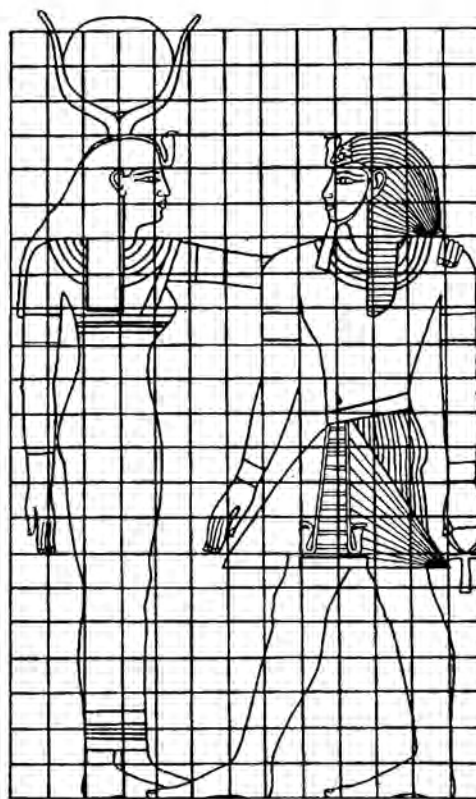
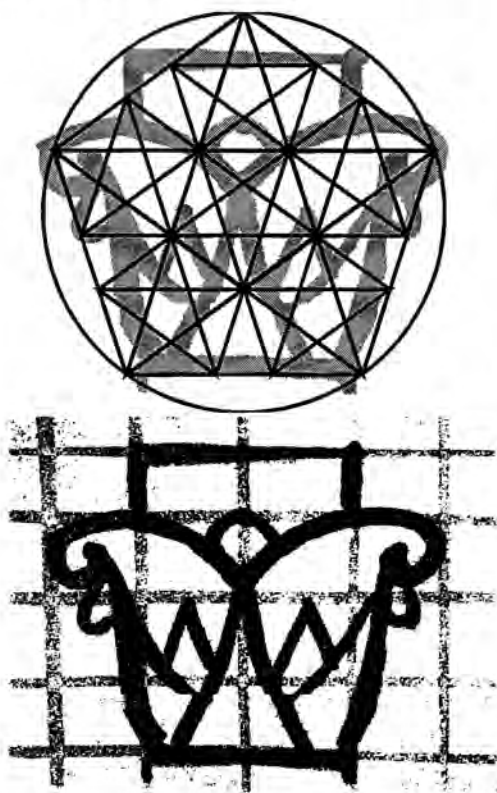
The examples of art presented here are intended for the reader to gain hands-on experience with geometry in the hands of the masters. Reader are encouraged to explore as many examples as interest them. Cultures are presented more or less in their chronological order, although they do overlap in time. Since this is a mathematics book and not a history book, the examples within each culture are grouped by an unfolding geometry from simple to complex, from the Golden Mean and single Golden Rectangle to combinations of Golden Rectangles, Root-Five rectangles and others in various combinations.

For each example, a geometric construction is shown near it which the reader is encouraged to repro-

duce upon the image. Drawn on the picture is a basic Golden Rectangle, double-squares or other geometry for you to start with. Doing a geometric construction on an image is something like solving a puzzle. What comes first? Where do I put the compass? I hope the directions for each example are clear. For more information, please consult an earlier page in the book showing the geometric construction intended. You are encouraged to use colored pencils to shade the patterns you notice (as you develop your geometric perception) in the geometric schemes provided for each picture. Use them as patterns in which to sketch your own designs.

The first examples are from Egypt. The earliest days of this great culture are mysterious, as simple desert dwellers with just a few hieroglyphs transformed to a complete system of art, philosophy, mathematics and measure, the sciences including medicine, chemistry, and metallurgy, philosophy, mythology and statecraft in about seventy-five years. Their civilization was born whole with scant period for development. Where and how they came by this knowledge so quickly is part of the mystery. Nevertheless they did, and their earliest works show great confidence in working with difficult materials, refined techniques and complex geometry.

Modern Egyptologists acknowledge their use of a coordinate grid for reproducing the same design accurately on any scale from a small piece of jewelry to the side of a cliff. There are references to lost "pattern books" cataloging these grid pictures (below) for the depiction of different objects and scenes, as exist today in India today from Vedic times. Previously, the proportions of the scene were composed using the sacred language of geometry a grid laid over it. The grid was the *final* stage of the design. This keeps the original geometry a secret while still allowing craftsmen not initiated into that level of the process to reproduce it easily.



Here are works of art which have Golden Ratio themes for you to enjoy, explore and learn from, and perhaps be inspired to design and create something yourself.

Gebel el-Arak Knife

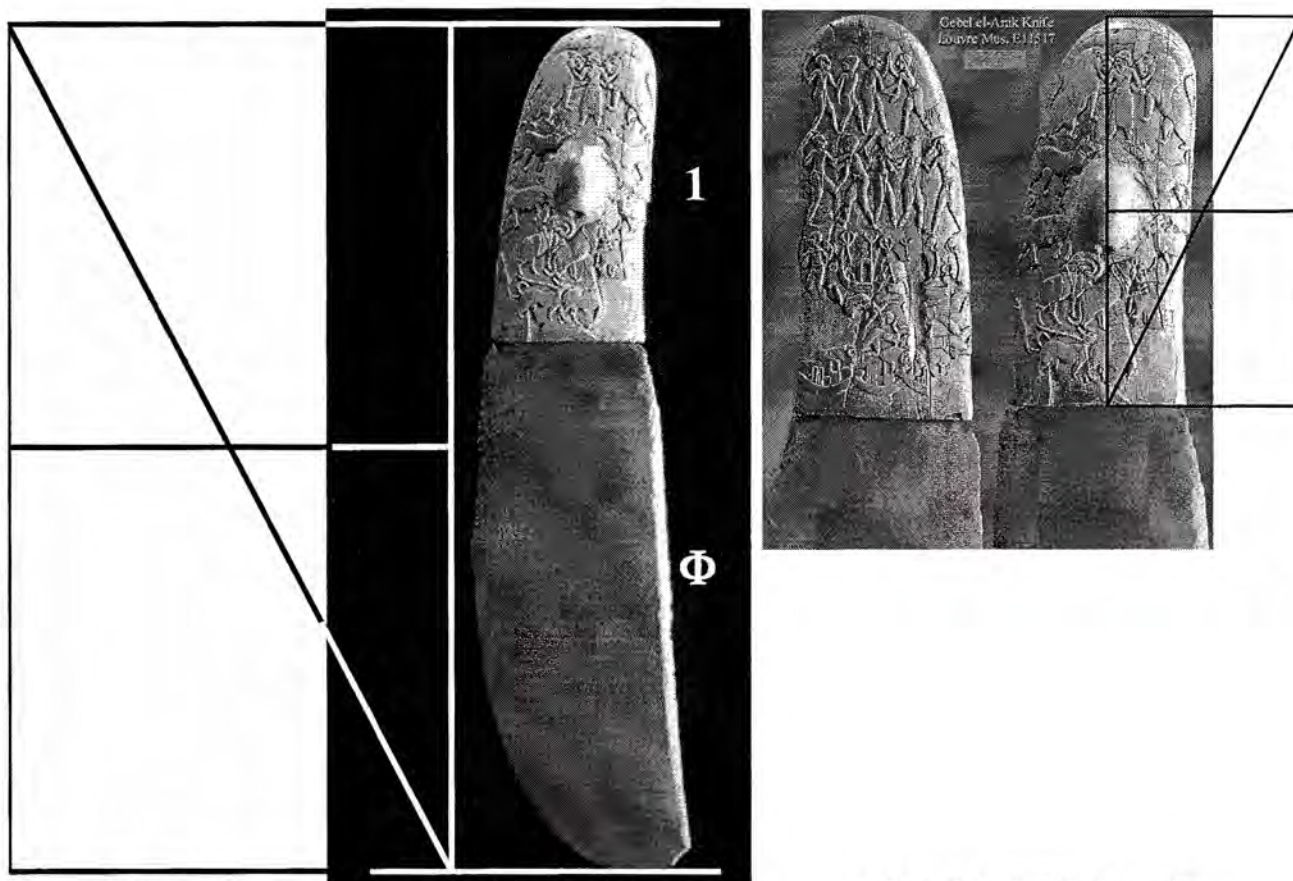
Prehistoric Egypt, Naqada II
c 3150 BCE
Louvre Museum

From the late Predynastic period, even before the first Egyptian Dynasty, comes this knife. It's one of the oldest prehistoric artifacts of ancient Egypt and its handle is one of the oldest known examples of bas-relief sculpture.

The blade is made of extremely high-quality, light ochre slate with a sharp serrated edge. The handle is made of a hippopotamus tooth and has a central knob with a hole for attaching a cord. It seems to have a symbolic, ritual purpose. One side of the handle depicts the "master of the beasts" motif, a figure wearing a Mesopotamian costume taming two lions. The other side depicts a battle scene showing groups of men fighting each other above several boats in the lower register.

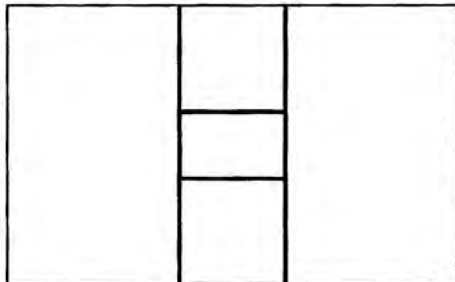
Different published dimensions of the knife range from 25 to 25.5 cm, with the blade from 9 to 10 cm, indicating a ratio of blade-to-handle between 1.55 to 1.83. But do the geometric construction to divide the knife's length at its Golden Mean (using the technique of two squares) and you'll see that the Phi division occurs precisely where the blade and handle meet. Furthermore, do this construction in both directions on the handle and see how the bump is defined by overlapping Golden Means.

This knife is an example of an object held in the hand which has the proportions of a hand.



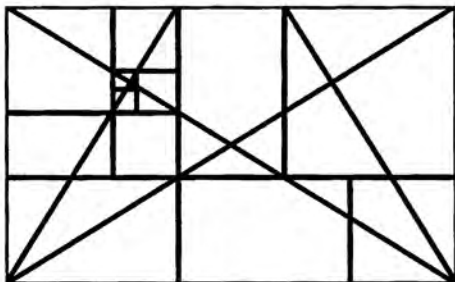
Dual Aker Lions

Under the long glyph signifying "the sky", the sun of today rises on the horizon at the center, between the two lions representing "yesterday" and "tomorrow".



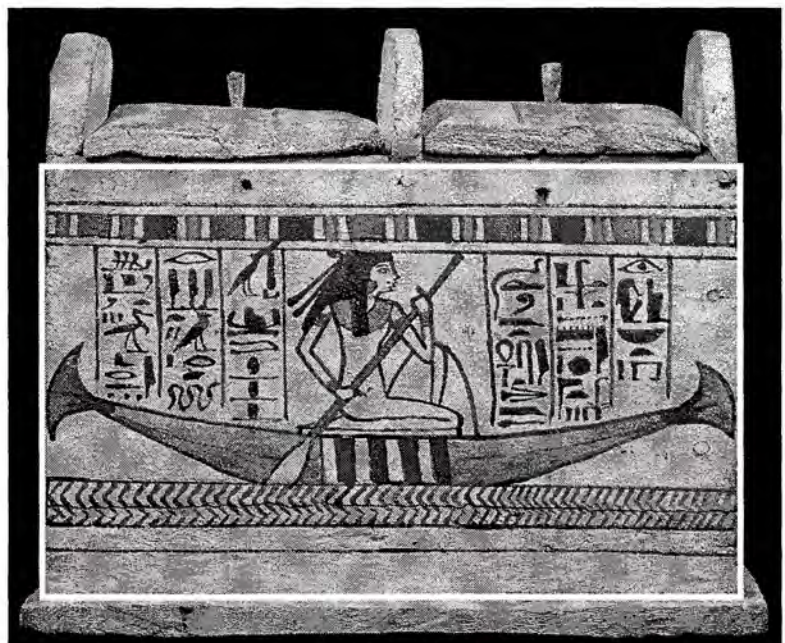
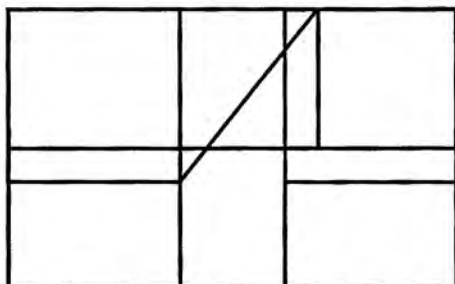
Tutankhamun's Ring With Solar Barque
18th Dynasty (1333-1323 BCE)
Cairo Antiquity Museum

This gold and faience ring depicts two baboons adoring the rising sun in its boat across the sky. The geometry shows visual points of interest at the baboons' raised hands.



Tutankhamun's *Ushabt* Box

Entombed with Tutankhamun was this box containing *Ushabt*, small mummiform figures placed in tombs to do agricultural work in the afterlife on behalf of the deceased.



Stela of Bai (Hearing Ear Stela)

New Kingdom 19th-20th Dynasty Deir el-Medina
Sculptured and painted limestone, Egyptian Museum, Cairo

Egyptian temples weren't like modern houses of worship in the sense that the public assembles to worship there. Egyptian temples were solely the domain of priests who maintained rituals to preserve the order of the universe in their land.

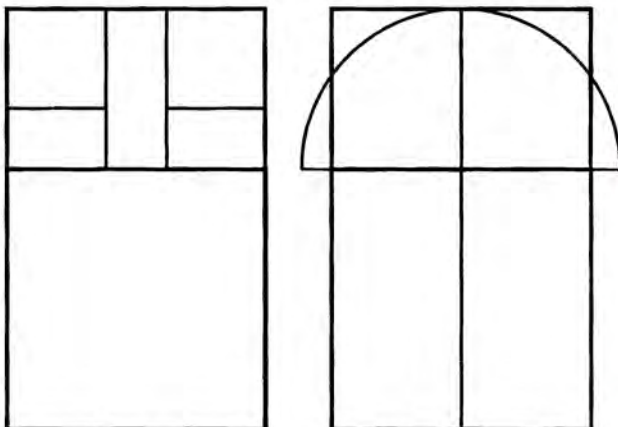
But those outside the priesthood, the common people who were not purified, could still have access to the inner sanctuary and the temple deity through "hearing ear" shrines, like this stela, that were placed in the outer walls of many temples. Since the sanctuary was often the last room in the back of the temple, a chapel of the "hearing ear" was located outside directly behind the sanctuary. This stela symbolically represented the god's ears to which the people could directly address their prayers.

This Stela of Bai is dedicated to the "good ram" Amon. Its upper part shows the god depicted in the form of a ram wearing dual feathers with symmetrical inscriptions identifying him as Amon-Re. The lower register is vertically divided into two equal areas. On one side, Bai is kneeling, while the other side has three pairs of ears incised and painted. To understand exactly whose ears they are, you have to know Egyptian color symbolism: black ears belong to Amon, yellow ears to Aset (Isis) and green ears to Asar (Osiris).

The whole stela is enclosed in a white Golden Rectangle. Add a square from the bottom to see how the whole stela is divided.

Do rabatment from both sides of the upper Golden Rectangle, and within each smaller Golden Rectangle as shown.

How was the top curved? Draw a vertical line connecting the top and bottom midpoints. Place your compass point where the vertical line crosses the top of the large square. Open the pencil to the top of the stela and turn an arc to reach both sides of the rectangle.



Mortuary Temple of Neb-Hepet-Ra Mentuhotep (Mentuhotep II)

Dynasty XI (2134-1991 BC), Luxor (ancient Thebes)

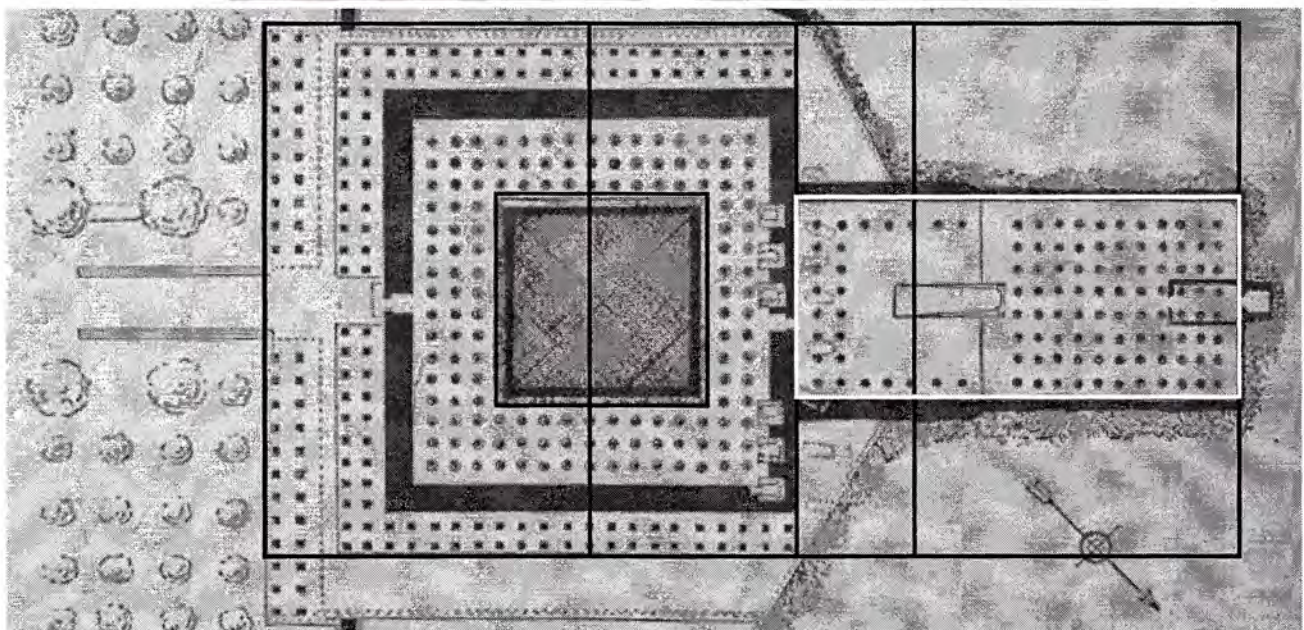
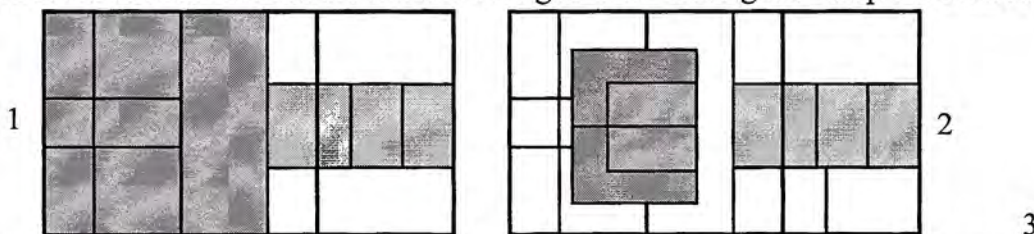
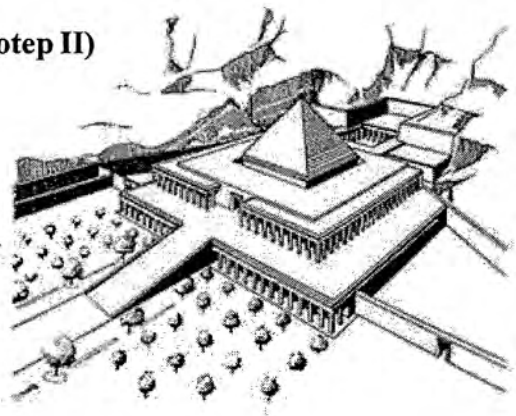
Mentuhotep II was the ruler who united Egypt at the end of the First Intermediate Period. The name of his mortuary temple is translated "Mentuhotep's (cult) sites shine blissfully". It was the first temple in western Thebes dedicated to Hathor, goddess of dance, music and love.

The complex consisted of a valley temple, a causeway, a stepped, terraced mortuary temple that is partially cut into the rock cliff face, and a subterranean burial chamber.

Inside the large ambulatory was a central core that Egyptologists believe was a symbolic version of the primeval mound of their creation myth, made of hard clay shaped as a cube. At the back of the courtyard (northwest end) stood the massive, terraced mortuary temple.

Its overall geometric plan appears to be designed within three side-by-side Golden Rectangles. By rabatment you can develop the geometry which guides the placement of walls and indicates important sites. Start by drawing a square (shaded, 1) inside the left end of the three Golden Rectangles. This encloses the main entrance and central ambulatory around the primeval mound. It can be subdivided into a Φ^2 rectangle and other squares and Golden Rectangles.

The adjacent temple (shaded right in 2 and in white in 3) is itself a Root-Five rectangle. The most sacred statue would have been in the small room to the right. The "hearing ear" chapel is furthest right.



Lintel of Sesostris III

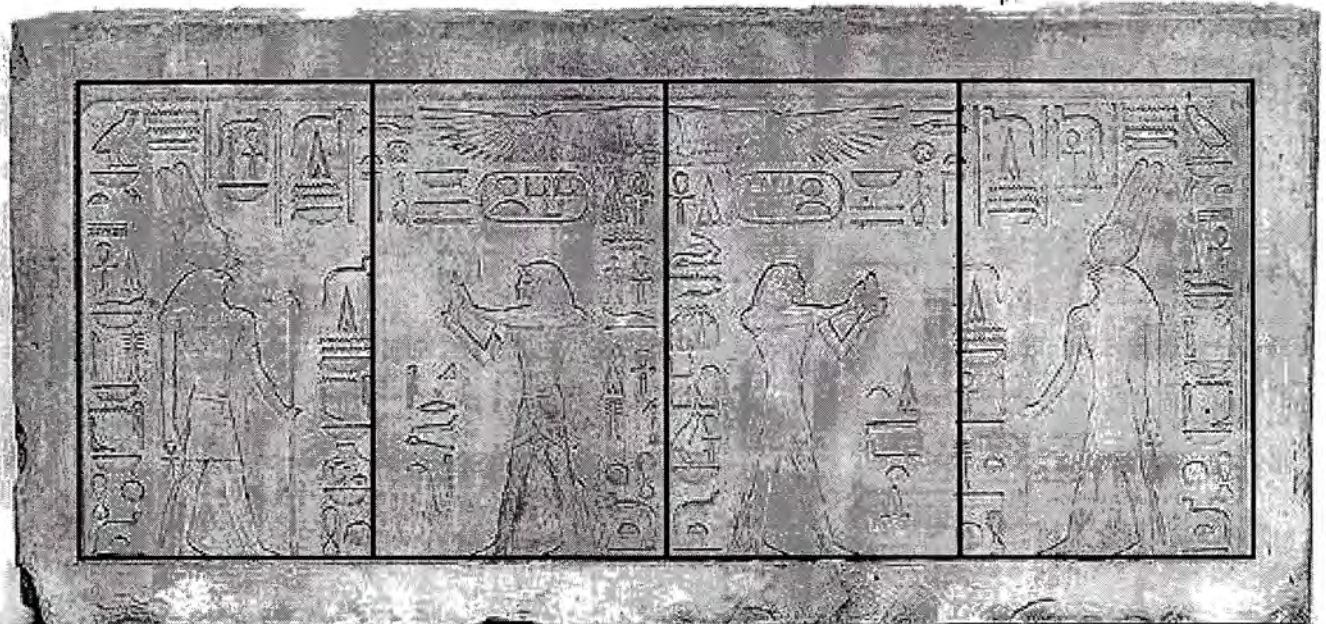
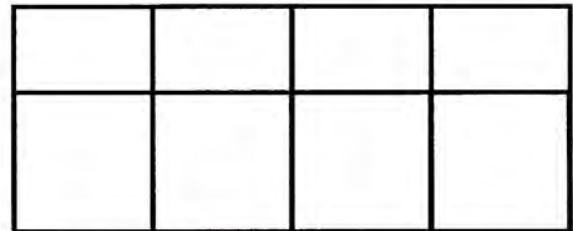
Middle Kingdom, 1878-1843 BCE (12th Dynasty), Medamoud Temple
Limestone bas-relief, Louvre Museum

This lintel originally surmounted the door to an offerings storeroom. It was found in the temple dedicated to the god Montu at Medamud, in Upper (southern) Egypt. It survived in good condition because it had been reused in the foundation of the current temple face down in the sand.

The stone contains scenes of a the pharaoh Sesostris III making offerings to a god, framed between the hieroglyphic signs representing earth and sky. In the center are two symmetrical representations of the pharaoh holding out offerings to the god "Montu, lord of Thebes." The pharaoh is protected by the winged sun disk identified as the god "Horus of Edfu".

The solar god is portrayed with a falcon's head, crowned with two tall feathers adorned with a double uraeus. He wears a pleated kilt with an animal's tail, and holds the *was* scepter (symbol of power) and the ankh sign (symbol of life). The pharaoh on the inner left, portrayed as a young man, offers a small conical loaf of bread. On the inner right, the pharaoh, with signs of age on his face, holds out a cake. The columns of text in front of the god represent his speech: "I give you all health and all joy as Ra", "I give you all life and power, all stability as Ra."

The geometry of the scene is framed on the stone in four adjacent upright Golden Rectangles, symbols of Phi and thus of the proportions of life. Draw a square at the bottom of each Golden Rectangle to see how they precisely define the height of the four subjects. Further rabatment and subdivided squares and Golden rectangles will reveal more details.



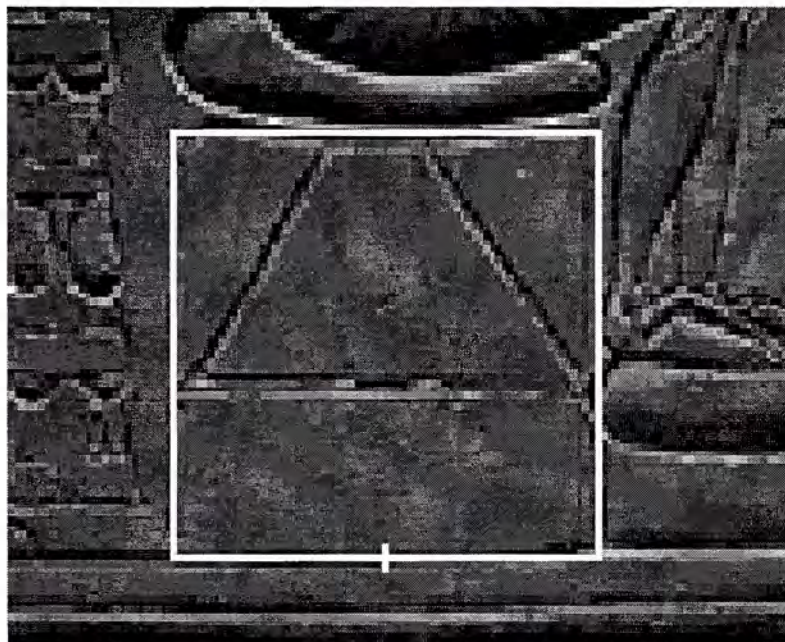
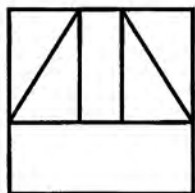
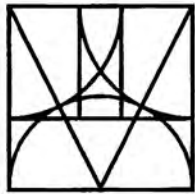
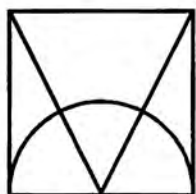
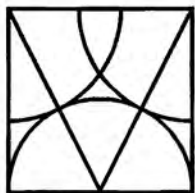
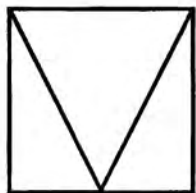
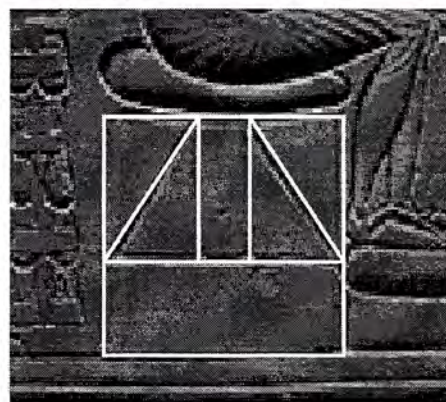
Household Stele of Akhenaten and Family

New Kingdom, 18th Dynasty c1350 BCE
Limestone, Egyptian Museum, Berlin

This limestone stele depicts the Pharaoh Akhenaten (Amenophis IV, 1379-1362 BCE) who may have been the uncle of Tutankhamen. He appears on the left, across from his wife Nefertiti, along with three of their daughters.

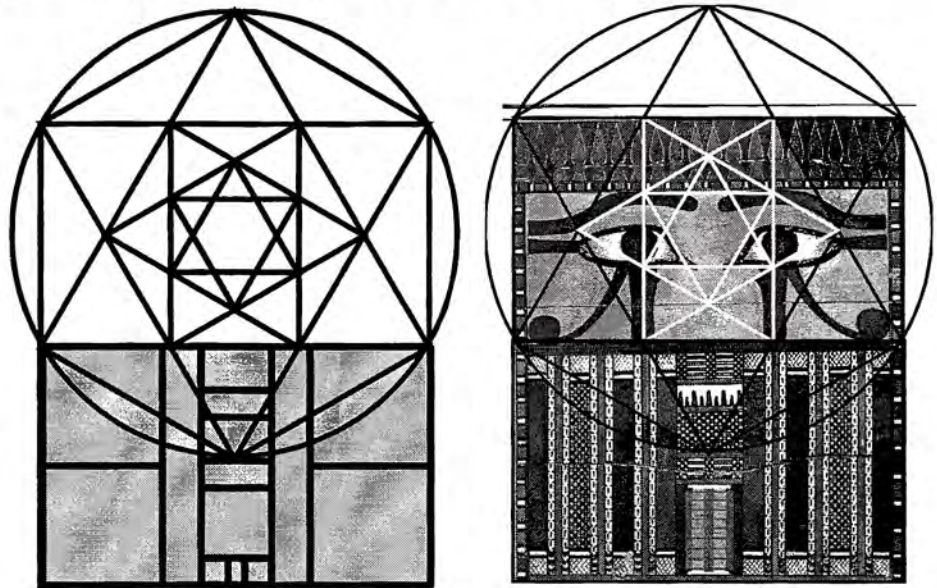
Akhenaten broke with twenty five centuries of tradition when he forbade the worship of any god but Aton, the literal sun disc. In this scene, the family receives blessings from the Aton disc. And since all worship of Aton had to go through Akhenaten and Nefertiti, this stele was located for that purpose in a private home.

The whole piece may have the dimensions of two Golden Rectangles. But Akhenaten's seat is of interest to a geometer. It depicts the geometric construction of the square subdivided into a Golden Rectangle above a Φ^2 rectangle. The Golden Rectangle is further subdivided by rabatment and diagonals. Do the geometry (below) on the picture starting with the square provided.



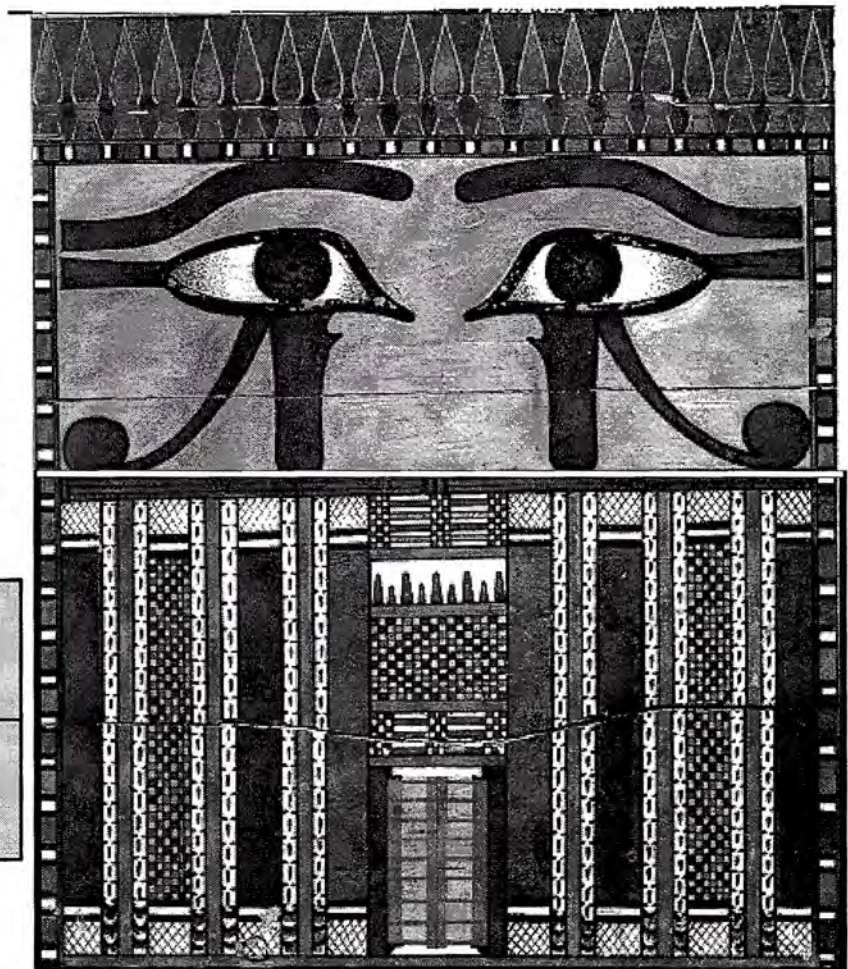
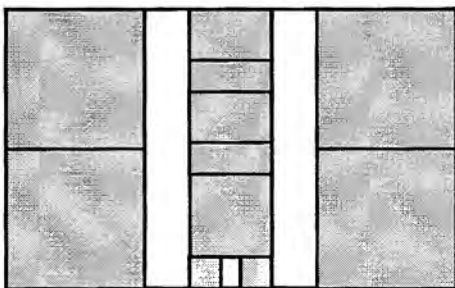
Eyes Painted On A Sarcophagus

This picture appeared in *A Beginner's Guide To Constructing The Universe* (and also in *Activity Book Volume 1, Chapter 6*) in the context of the Hexagon, Hexagram and Root-Three rectangle which provide the armature upon which the eyes, brows, lashes and tears align. But no mention was made of the rectangle below which depicts a magical door into (and out of) a sacred space.



It just so happens that this lower rectangle is a Golden Rectangle, appropriately a symbol of life.

The Golden Rectangle may be subdivided by rabatment into squares and smaller Golden Rectangles which guide the placement of the architectural features.



Stela of Ahmose

New Kingdom, early Dynasty 18, 1570-1504 BCE

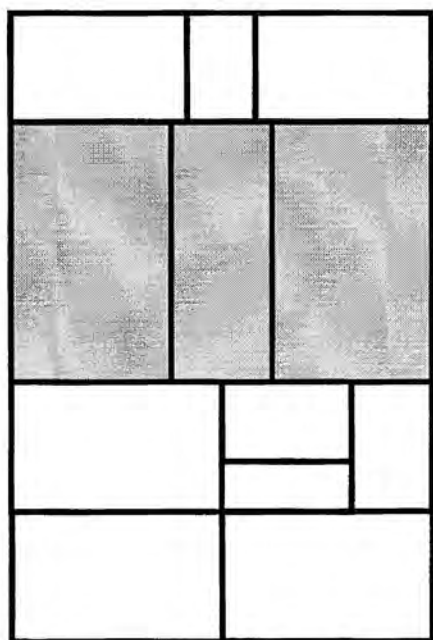
Painted limestone

Metropolitan Museum Of Art, New York

This stele is designed in four levels. At top are the eyes of the sun and moon aside the glyph shen signifying eternity. Below, we see Ahmose, chief of the metal workers, seated in a carved chair before a table filled with offerings. Beside him sits his wife, Werel, resting her hand on his shoulder. His son Meny (called "true of voice") recites a prayer, his arm raised in a gesture of invocation. The small boy Tjer crouches beside his father's chair and sniffs a lotus flower. Below that scene, we see Ahmose sitting in a chair (with a subdivided base) being honored by members of his family with written prayers for the well-being of his spirit, and offerings of food and flowers. One of the kneeling women turns to offer her companion a piece of fruit. This stele was probably placed in Ahmose's tomb chapel. The hieroglyphic text at the bottom is an offering formula that visitors recited for the benefit of his spirit.

The geometry of the stele is in four levels too. The lower two levels, the hieroglyphs and kneeling family members, are contained in four small Golden Rectangles. Above, the main scene is one large (shaded) Golden Rectangle. Draw squares from each end. At top are three Golden Rectangles of two sizes (although this stele was not made with perfect symmetry).

Explore it through rabatment.



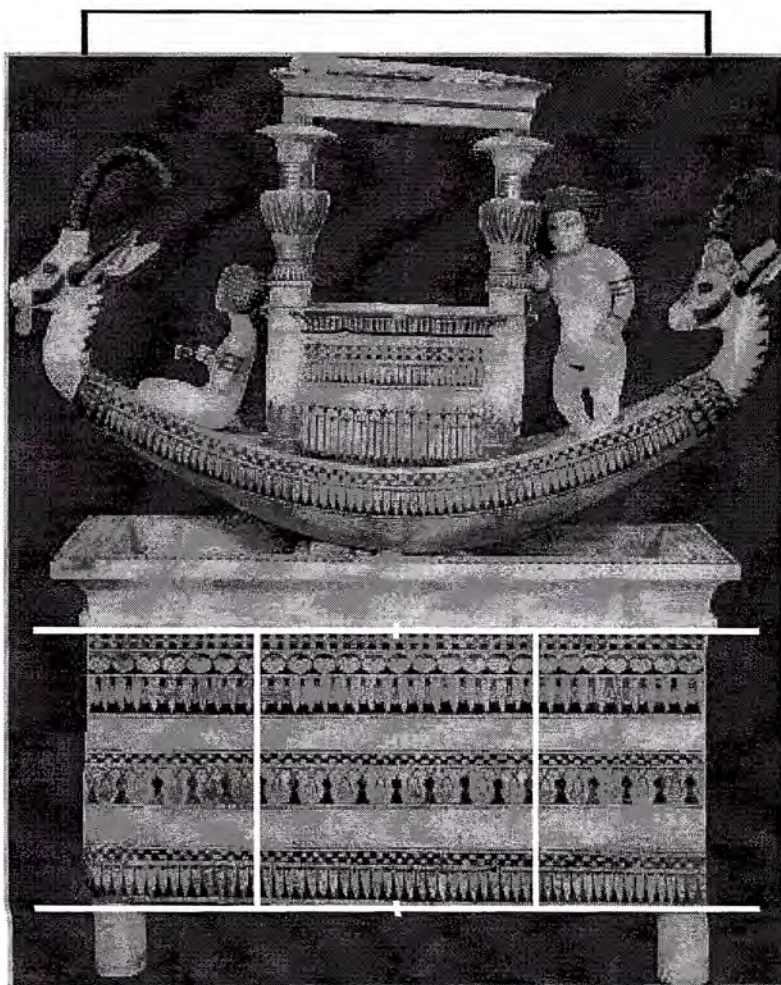
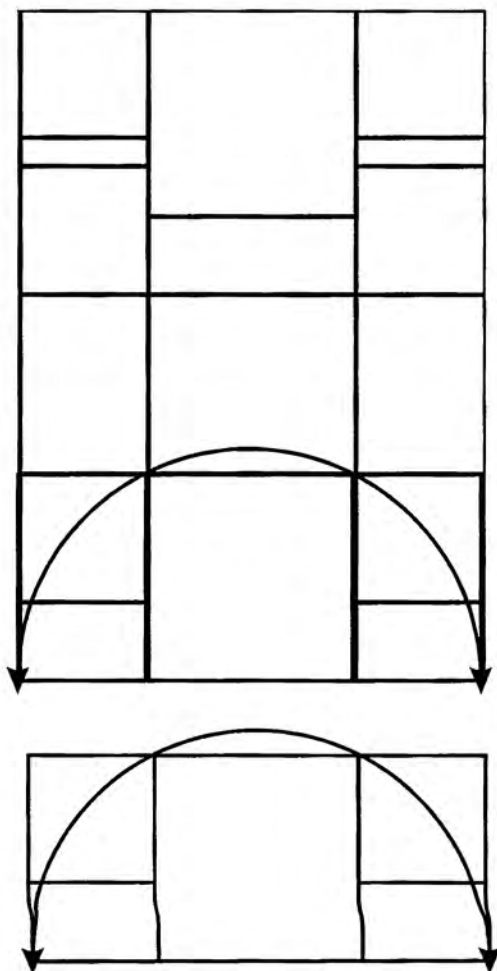
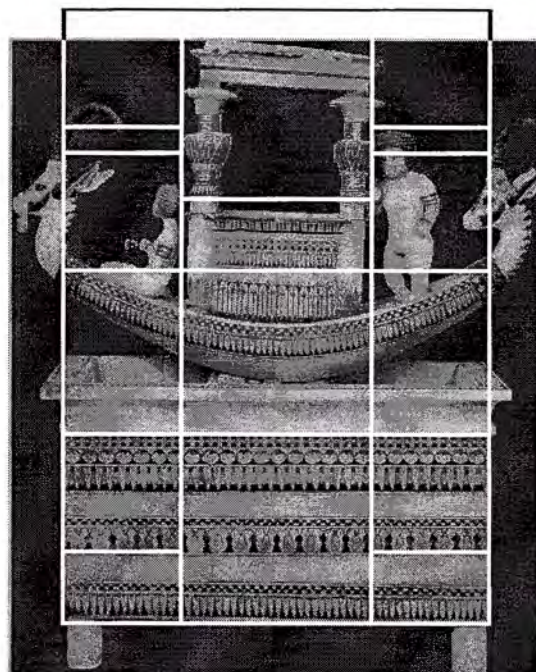
Tutankhamen's Alabaster Perfume Boat

Egyptian National Museum, Cairo

This alabaster box and boat probably was used to hold perfume. It appears to be a replica of his funerary barge. It's fourteen inches tall and inlaid with colored glass-paste and semiprecious stones highlighted with gold leaf.

The barge sits atop a box whose inlay design is a Root-Five rectangle. To prove it, a square with extended sides has been drawn on the picture below. The midpoints at the top and bottom of the square are indicated. Do the construction of the Root-Five rectangle. Then do a rabatment inside the side Golden Rectangles to see how the design is proportioned.

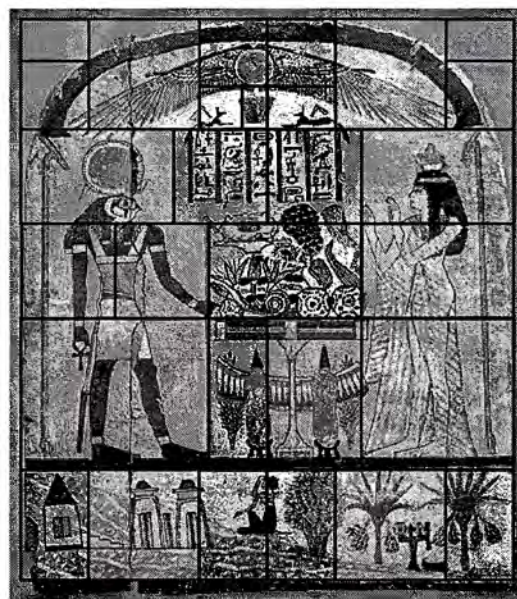
The geometry of the box is integrated into a larger whole design, which may be found by first extending the geometry of the Root-Five rectangle upward.



Stela of Djed-Amon-Iu-Ankh

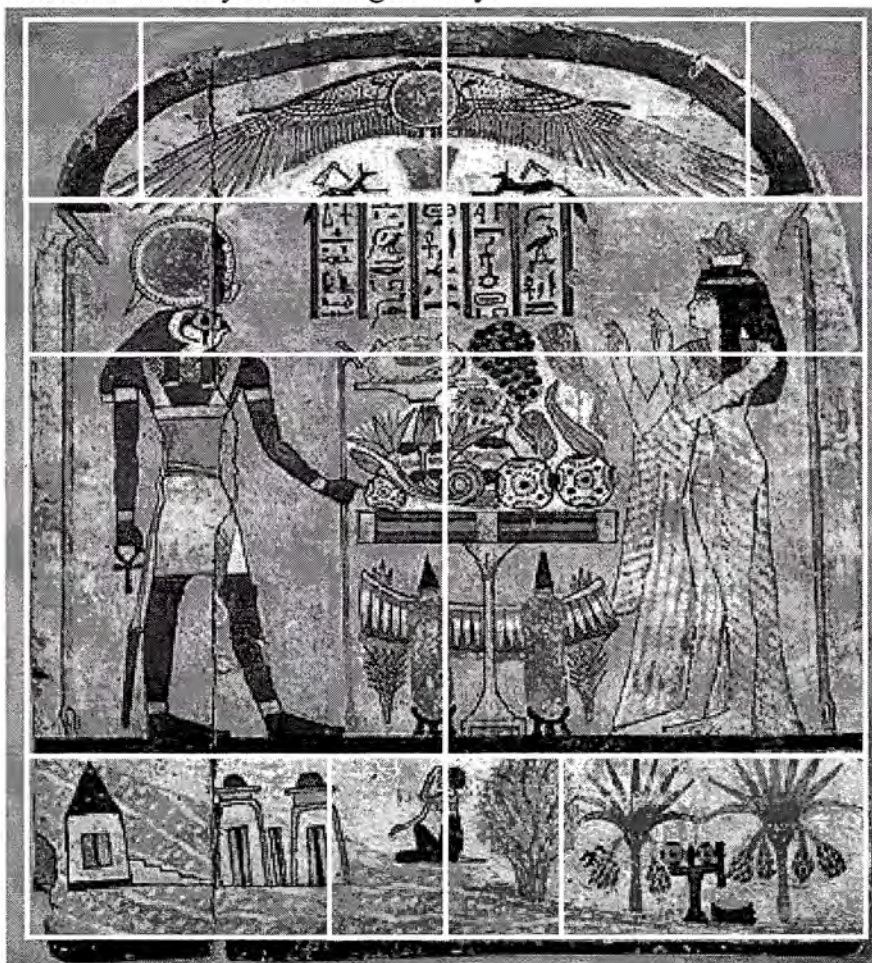
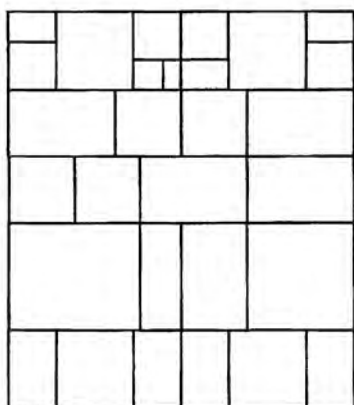
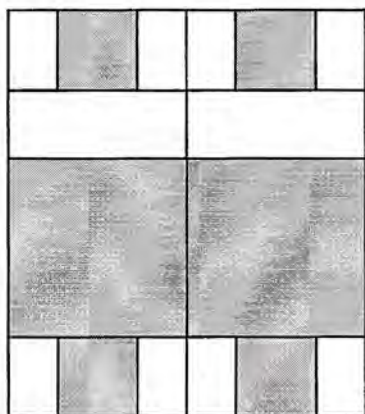
Third Intermediate Period, 22nd Dynasty (1075-715 BCE)
Thebes, Deir el-Bahari
Stuccoed and painted wood, Egyptian National Museum

This votive stela shows Djed-Amon-Iu-Ankh adoring and making an offering to Ra, the sun god. The lower register shows a necropolis where tombs have been cut into the pink slope with white specks. There's also a chapel surmounted by a pyramid and preceded by a staircase. A crouching woman mourns the dead, tearing her hair. Behind her a garden provides shade, water, air and offerings for the deceased. In the upper register is the winged sun god above Anubis and Wepwawet, two canine guides into and out of the underworld.



The geometry is symmetric: top and bottom registers are each pairs of Root-Five rectangles. The middle is proportioned by two squares topped by Φ^2 rectangles. It's interesting to find symmetry everywhere, except the table with the offering which is situated towards the right of center.

Start with this and apply rabatment to discover for yourself its geometry.



Karnak Temple Floorplan

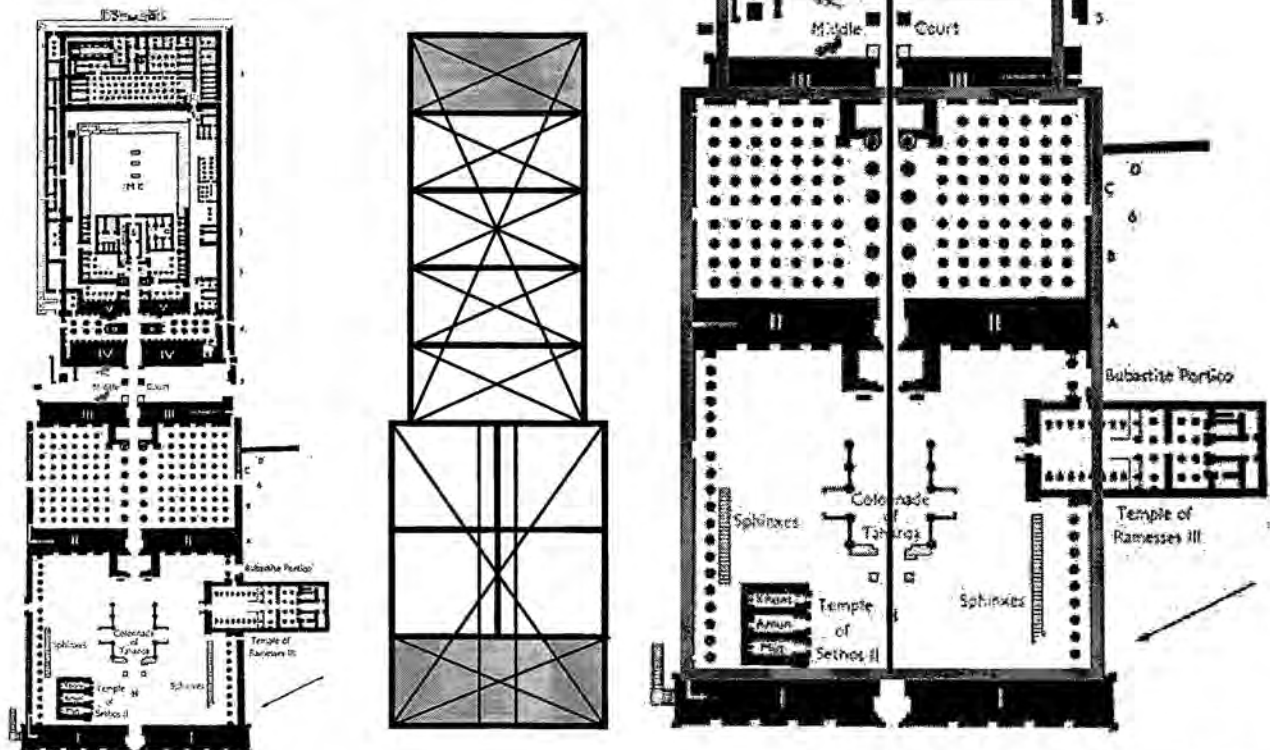
This is the floorplan of the Egyptian temple at Karnak (ancient Thebes).

Its overall geometry can be understood as two rectangles, one above the other. At top is a large Root-Five rectangle. Below is a square plus a small Root-Five rectangle. First, draw their two full diagonals. At what significant points do they cross?

Any Root-Five rectangle can be subdivided into five identical Root-Five rectangles. A small Root-Five rectangle has been drawn inside at the top. Walk its width down both sides of the long side of the large Root-Five rectangle to find the remaining four Root-Five subdivisions. Draw diagonals in each small Root-Five rectangle to find the special significance of each center.

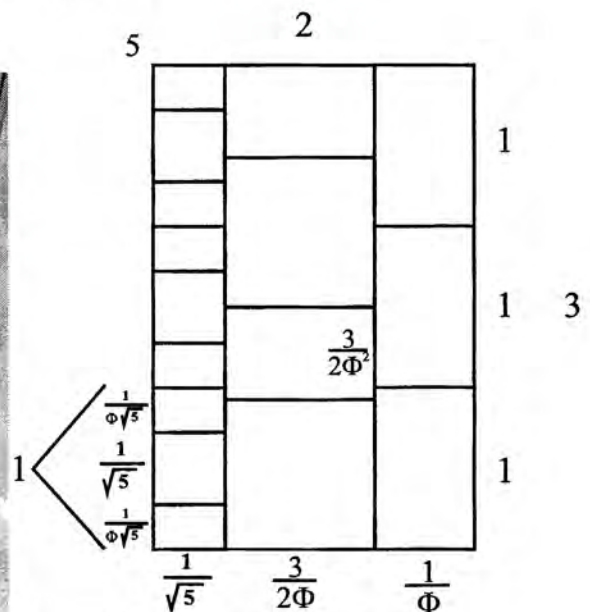
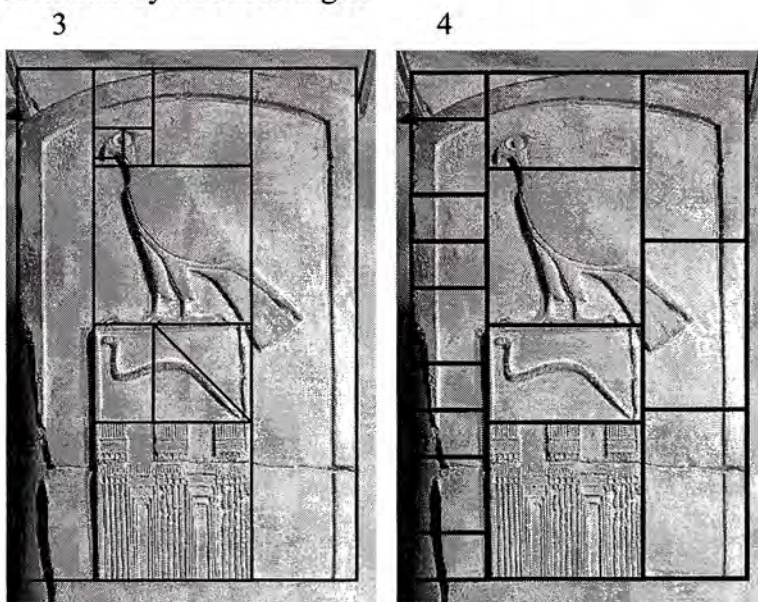
In the bottom rectangle, mark out a square from the top, leaving a small Root-Five rectangle at its bottom. Draw their diagonals. Also, mark off a square from each end of the Root-Five rectangle. Extend their lines to the top of the rectangle. What do they show?

Notice that the small Temple of Rameses III (bottom right) is also a small Root-Five rectangle. Like Phi, this is a symbol of life.



Dynasty 1, c 2900 BCE
Abydos
Louvre Museum

Regarding its geometry, we'll only consider the worked part from the lowest incised line to the stele's top. (1) We find it to be a two-by-three rectangle (the ratio of the "musical fifth", symbolically the "chord of triumph") which is easily subdivided into squares and the familiar Egyptian grid. But what is the geometry behind the design? (2) First, look at the obvious Golden Rectangle on which the falcon stands. No one seems to have noticed that this is a Golden Rectangle. It's divided into a square below a Golden Rectangle containing the snake. (3) Exactly two of these Golden Rectangles reach the stele's top. Applying rabatment frames the falcon's head. But how do the sides around these two Golden Rectangles expand to fill the two-by-three frame? (4) The space to their right contains exactly three vertical Golden Rectangles. The space to their left contains exactly three vertical Root-Five rectangles. (5) The overall result is a convenient two-by-three rectangle.

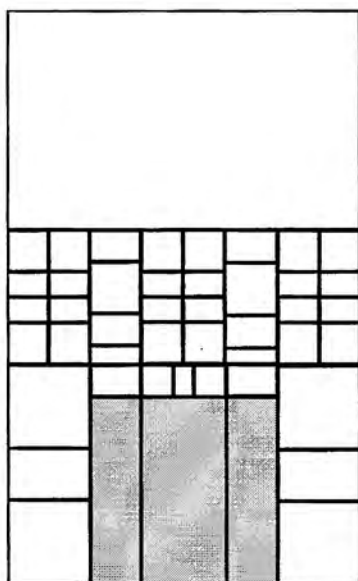
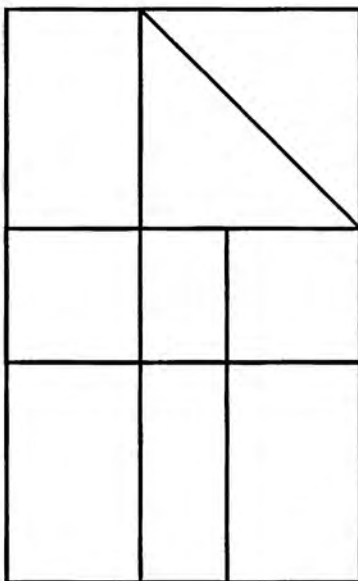


Notice how the Golden Rectangle with the snake and palace seems to lean a little to the left due to the falcon's weight on it? It's a surprising and subtle artistic touch. Let's explore it geometrically.

First, begin by applying rabatment. Draw a square from one end of the small Golden Rectangle. Notice how its diagonal guides the snake.

A square drawn from the top of the full Golden Rectangle lets us enter the geometry of the palace facade. Continue drawing squares from the ends of Golden Rectangles and replicate its geometric scheme with as much detail as possible. You will find squares, Golden Rectangles, Root-Five rectangles and Φ^2 rectangles.

So start by simple rabatment. Shade the patterns with colors.



The Great Pyramid and Squared Circle

4th Dynasty, 2600-2480 BCE, but perhaps much older.

Much has been written about this ancient wonder.

(1) Its dimensions seem to be based on a Φ to $\sqrt{\Phi}$ to 1 right triangle seen sliced down the pyramid's center. A circle whose radius is equal to the pyramid's height has a circumference (virtually) equal to the perimeter of its square base. (Do the math!) In other words, this pyramid "squares the circle".



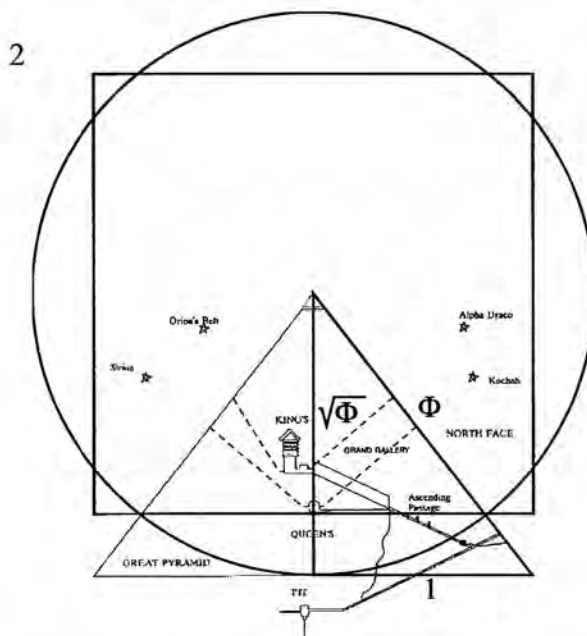
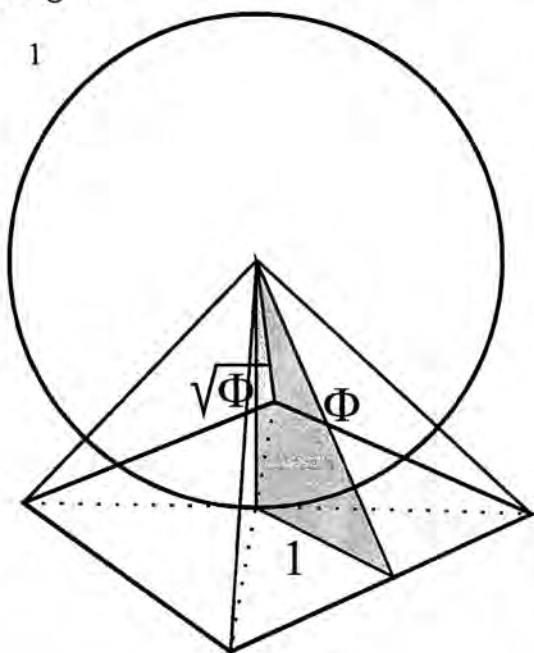
(2) If the square base is overlaid upon an image of the pyramid in its circle, notice how the sides of the pyramid cross the point where the square crosses the circle.

(3) Here's how to find this triangle within the pyramid. Start with two Golden Rectangles (provided on the next page). Open your compass to its long side and swing an arc inward to cross the central side. That point is Φ times the base of this Golden Rectangle.

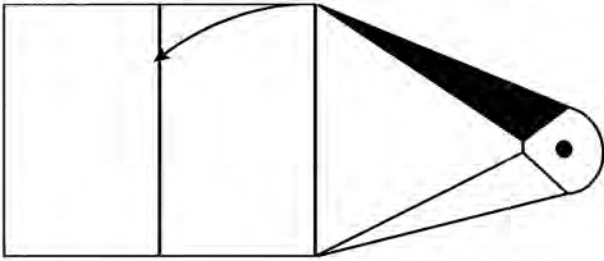
(4) Repeat the process on the other side.

(5) Draw lines from this point to each lower corner of the Golden Rectangles. These are the slanting sides of a central slide of the pyramid, equal to Φ ($= 1.618$) times the base.

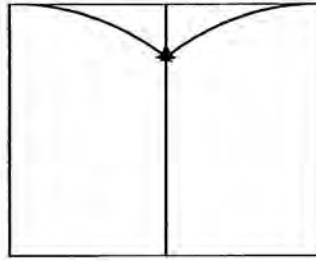
(6) Do this construction on the following page upon the large image of the squared-circle pyramid and show that the arcs cross at its top point. Two Golden Rectangles have been drawn (darker lines) for you to begin.



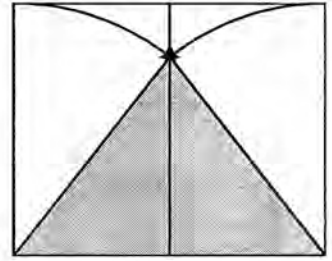
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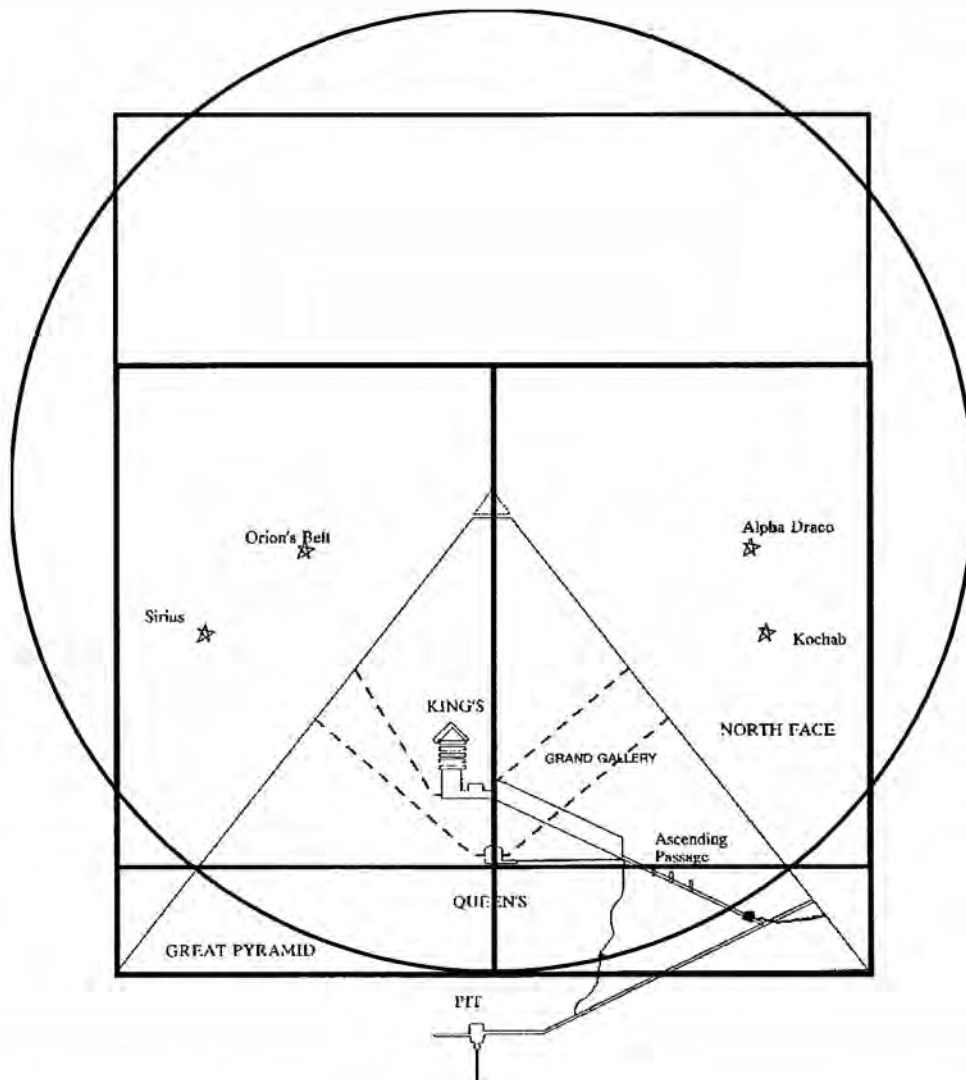
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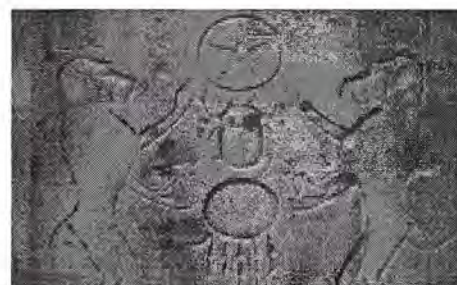
“Ebony Fan” of Tutankhamen

18th Dynasty

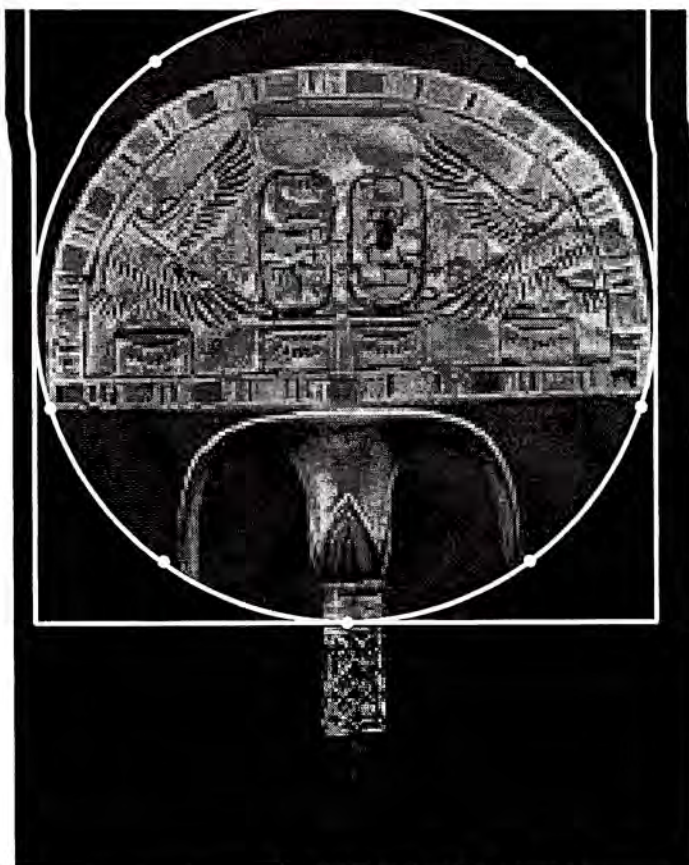
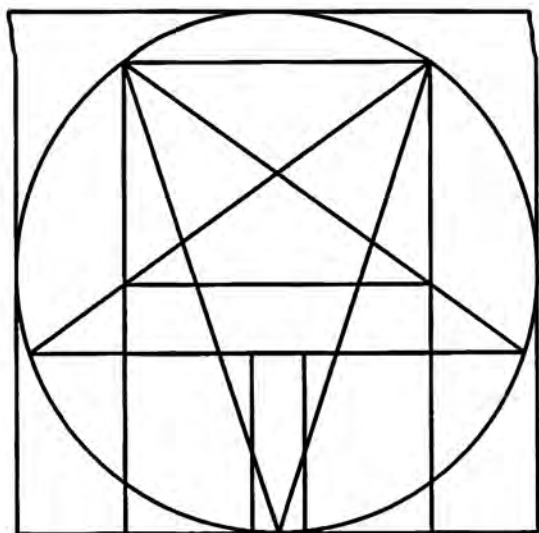
This Ebony Fan has the shape of a palm frond. It's probably one of a pair representing the two equinoxes when shadows are equal at the same hour of day. It contains Tutankhamen's twin cartouches (sacred names) with the vulture goddess Nekhbet on either side wearing the crowns of Upper and Lower Egypt.

A fan is related to shadows, which are due to the sun which normally appears circular. But in Egyptian art the sun is often depicted as flattened, as it actually appears when near the horizon at dawn and dusk, which anyone may observe. This indicates that it is entering or emerging from the "underworld" which the Egyptians called the *Duat*, symbolized by a five pointed star in a circle. Thus, pentagonal symmetry is the appropriate geometry for this fan. Notice how a line joining the top two points of the star flattens the fan to its symbolic and realistic proportions.

The fan is given below in a circle with five equally spaced points to draw a Pentagram. Find its details by rabatment, drawing squares and connecting points as seen in the geometric scheme.



*Baboons worshipping the sun and
Duat glyph*



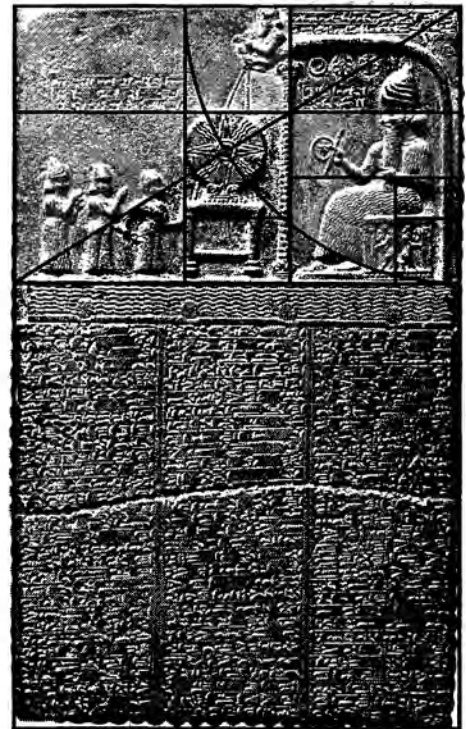
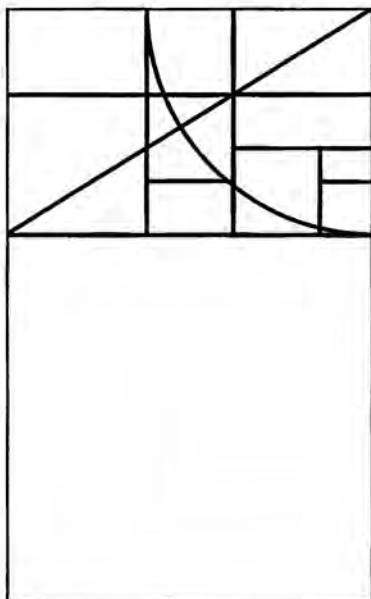
Tablet of Shamash

Babylonian, from Sippar, southern Iraq
9th century BCE
Length: 29.21 cm, Width: 17.78 cm
The British Museum, London

This stone tablet shows Shamash, the Mesopotamian sun-god, seated under a sacred awning and holding the rod and ring, symbols of divine authority. The symbols of the Sun, Moon and Venus are above him. In front of him is a large sun symbol supported by two divine attendants. On the far left is the Babylonian king Nabu-apla-iddina between two interceding deities who will introduce him to Shamash.

The cuneiform text below the scene describes how the Temple of Shamash at Sippar had fallen into decay and the image of the god had been destroyed. During the reign of Nabu-apla-iddina, however, a terracotta model of the statue was found on the far side of the Euphrates and the king ordered a new statue be constructed of gold and lapis lazuli. The text describes the restoration of the Sun-god's image and temple, and then confirms and extends the privileges of the temple.

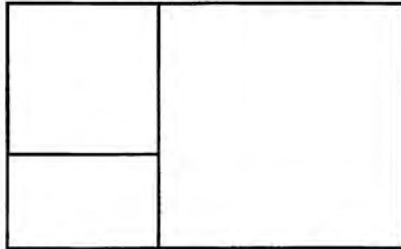
The whole tablet is a Golden Rectangle. Apply rabatment to see more of its details.



Meditation And Serpent

Indian painting

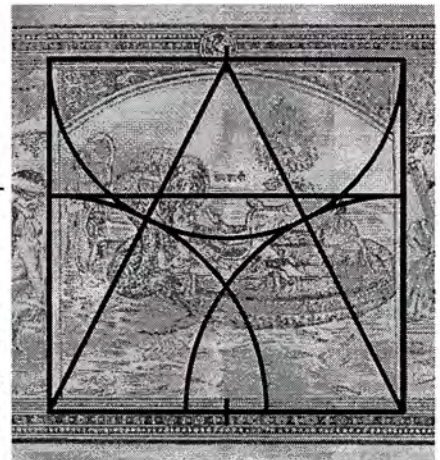
We can easily predict the Golden Rectangle geometry by the lines of paint.



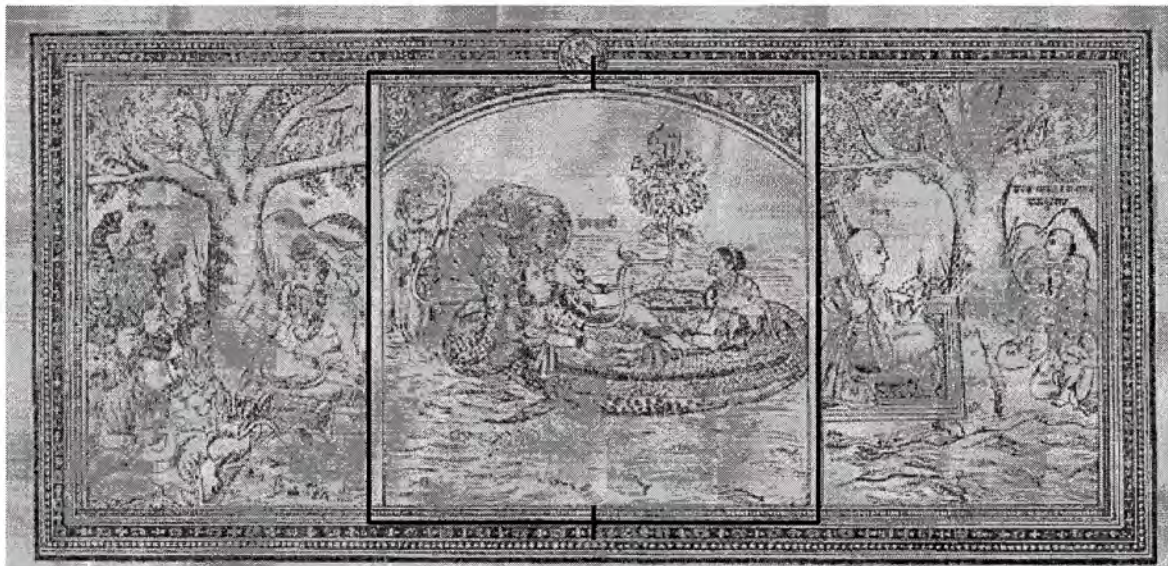
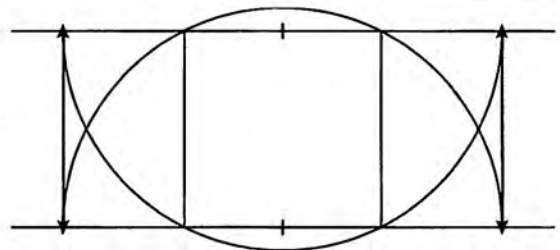
Narada's Great Ancient Tale (Atha Naradiyamahapuranam)
(also called "Vishnu Resting")

India, Mumbayyam: Sri Venkatesvara Stim-Yantragare, 1923.

In Hinduism, Vishnu is the deity responsible for maintaining the universe once it is created by Brahma. The entire universe is periodically destroyed by fire and dissolved into an infinite ocean before it begins again. We see here Vishnu resting upon this ocean until the universe is to be re-emanaed. He lies on the serpent Shesha, whose later function will be to hold up the world. Vishnu is attended by his wife, Lakshmi, and the monkey-faced sage Narada who tells the tale of creation. Brahma, the triple-faced deity responsible for beginning the world in each cycle, rests on a lotus coming out of Vishnu's navel.



The overall geometry is that of a Root-Five rectangle. A central square is provided, with the midpoint at its top, for you to subdivide it into a Golden Rectangle and Φ^2 rectangle. Use the lines and areas this creates to explore the rest of the scene.

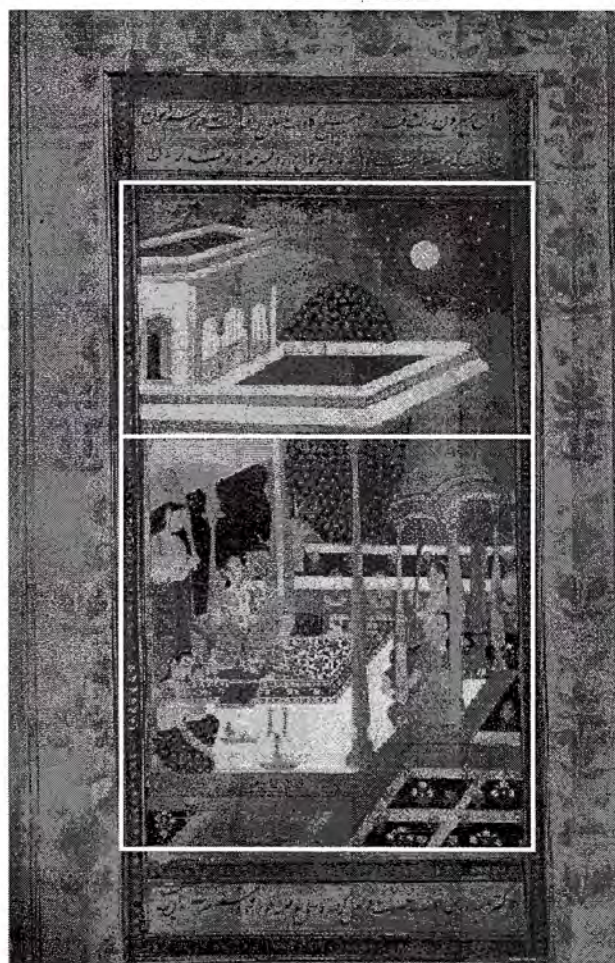
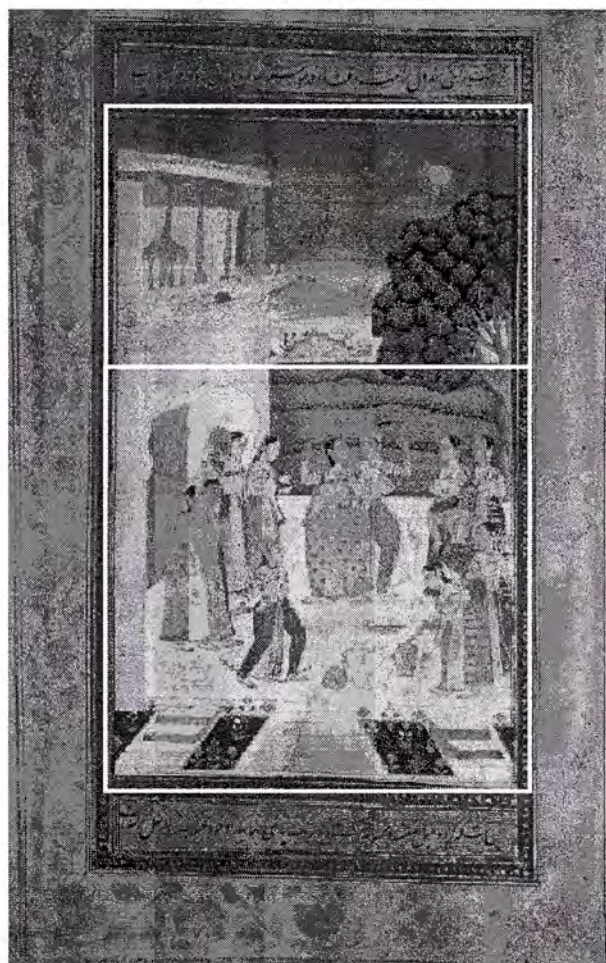
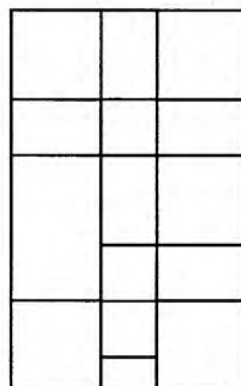
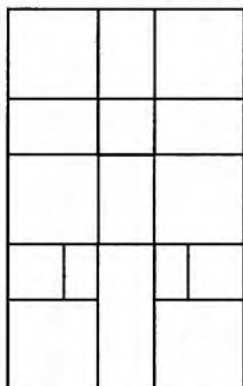


Ragmala Paintings

Opaque watercolor on paper, India

Ragmala (literally a “garland of melody”) refers to a cycle of Indian poems and accompanying paintings that portray mortals as they're affected by the emotion a particular raga (melody) expresses, which would be played in the presence of these paintings. The *Ragamala* melds three major art forms -- music, painting, and poetry, usually dealing with themes of love. The paintings are admired for their color balance, visual proportions and delicate line work, all of which convey the desired feeling so powerfully.

The harmonious geometry of each scene is framed in a single white Golden Rectangle. Apply rabatment to see more of its secret harmonious composition.



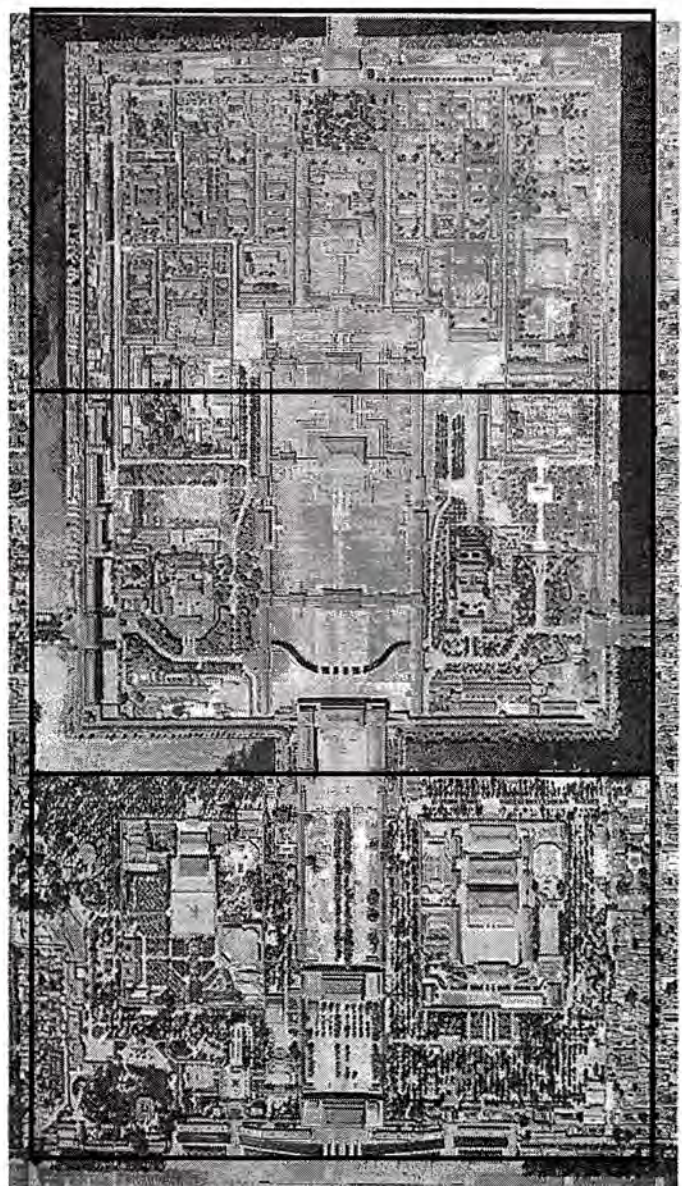
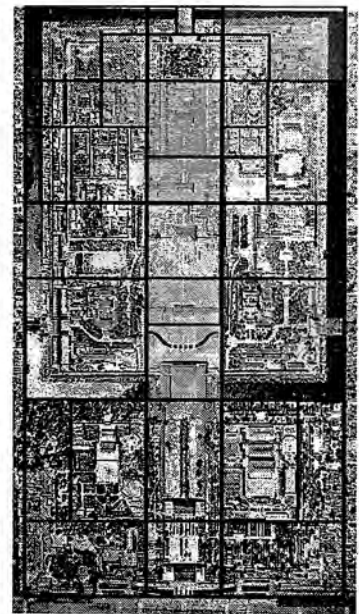
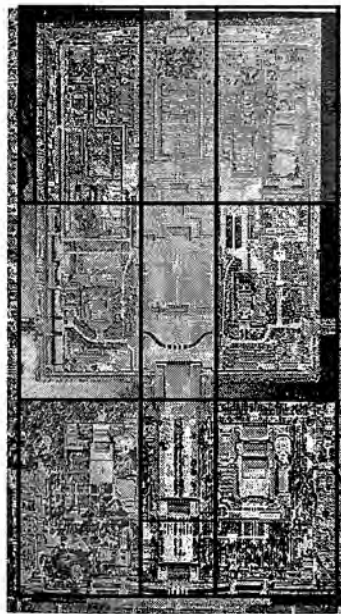
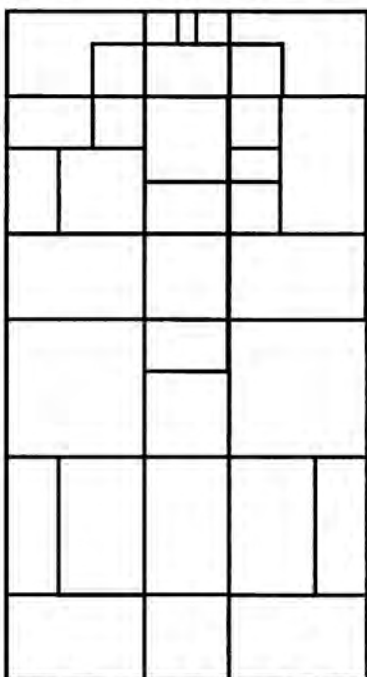
The Forbidden City Beijing, China

The Forbidden City (now known as the Palace Museum) was the former imperial palace complex, home to twenty-four Chinese emperors over 491 years from 1421 to 1911.

A wall encompasses its 9,999 rooms (a room being the space between four pillars). (And there's no doubt about the primacy of the number nine in China!) The palace consists of 12 million bricks. Another 20 million bricks comprise the walls of the pavilions and the surfaces of courtyards.

The wall around the sacred complex is surrounded by a moat. This is framed by two Golden Rectangles at whose center is the Hall Of Harmony Of The Center (*Chung Ho Tien*), the center not only of the Forbidden City but of the capitol Beijing and center point of the whole Chinese empire, the "Middle Kingdom" at the center of the entire cosmos. A third Golden Rectangle below them takes us to the ritual gate.

Rabatment subdivision of Golden Rectangles and squares, with diagonals, will reveal the placement of important buildings, walls, roads and sites.



Bonsai Trees

Bonsai (pronounced BONE-sigh) trees are trees which are dwarfed either naturally or artificially. They're grown in such way that they're small but look very old.

The source of inspiration for the art of Bonsai are the naturally occurring dwarf trees which grow in mountains on high rocky outcroppings and on the driest and poorest slopes, as well as on perpendicular cliffs of tiny islands and in wet bogs.

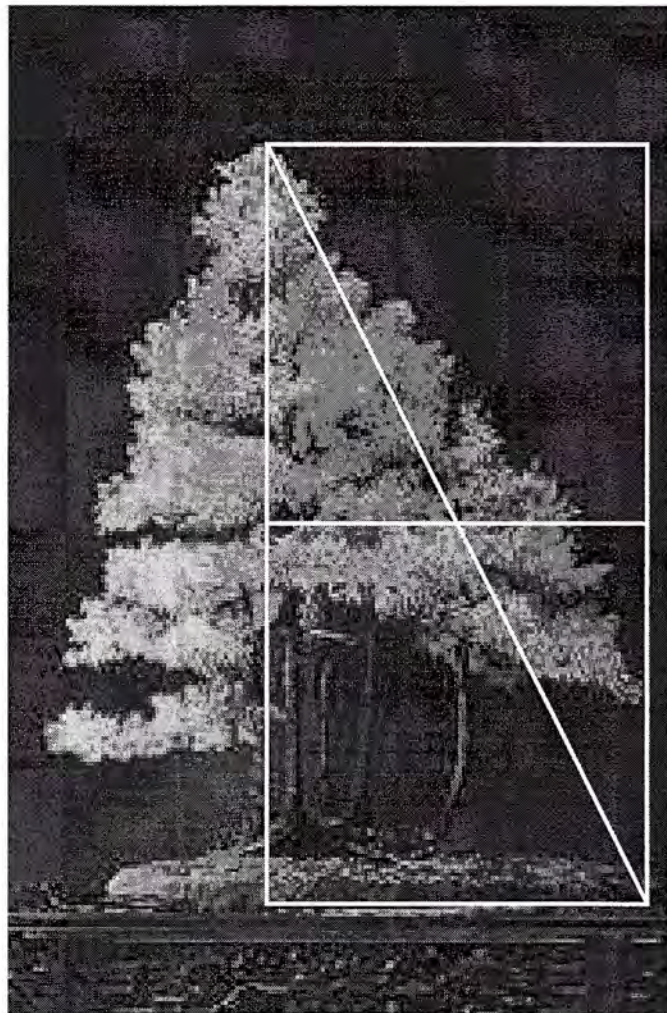
The history of Bonsai goes back countless ages, with the first evidence of small potted trees left in Egyptian tombs over 2000 years ago. The art first appeared in China more than 1700 years ago in the T'sun and Tang Dynasties, a golden age of learning. Records show Indian herbalists keeping prized dwarfed plants 1500 years ago, about the time it was adopted by the Japanese.

The harmonious proportions prized in Bonsai sometimes derive from the harmonious arrangement of the numbers one through nine of the *Lo Shu* magic square (see page 10). They are also derived from the Fibonacci numbers and Phi, as well as geometric constructions, including this example which applies the Golden Mean.

Use the double-squares to find the Phi division of the tree's height to see how the ratio occurs here.

Begin by opening your compass to the bottom side of the square and swing an arc up towards the right to cross the diagonal. Then place the compass point at the top of the triangle and open it to where the arc crossed the diagonal. Swing an arc down to the left.

What does it show?



The Parthenon

c 430 BCE Athens, Greece

The Parthenon on the Acropolis was home to Athena, the Greek goddess of war, wisdom and crafts. The structure was designed by the Athenian sculptor Phidias (born c 490 BCE) while Ictinus and Kallikrates were architect and contractor.

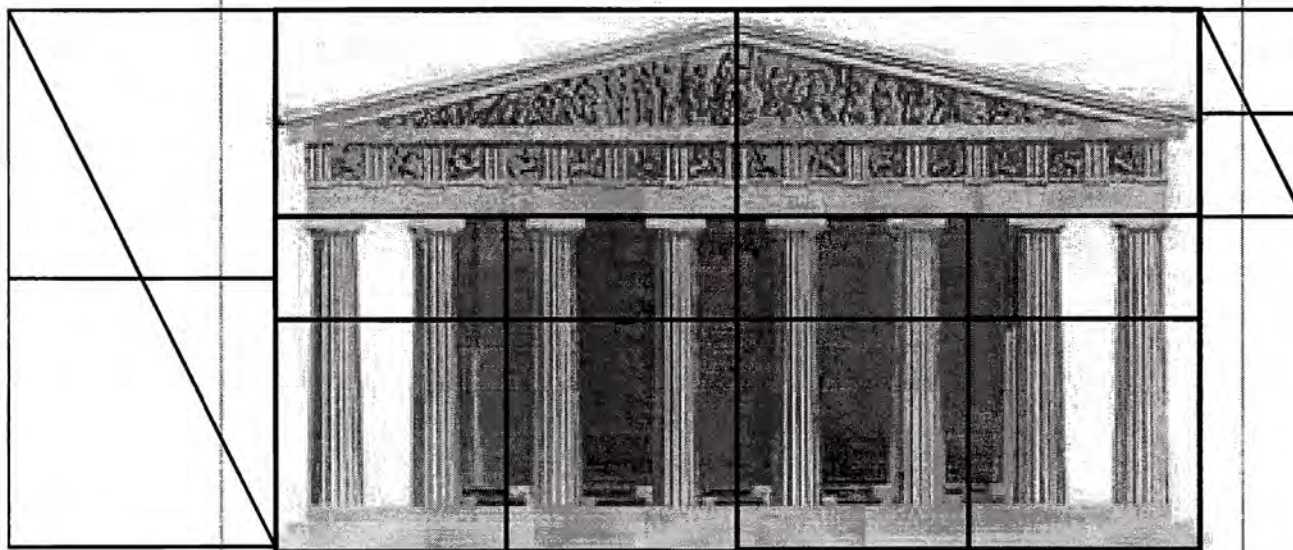
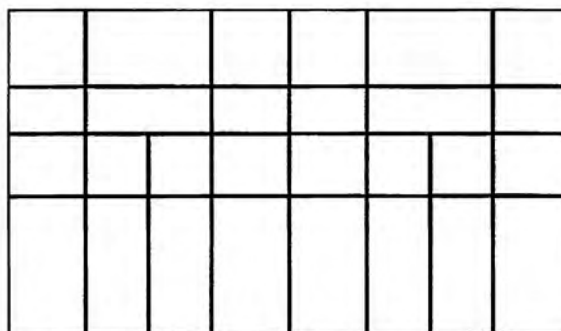
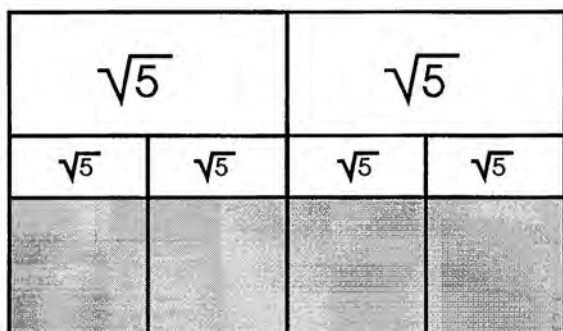
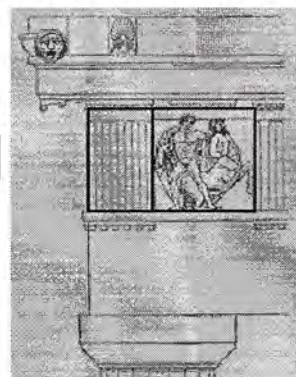
According to tradition, the overall scheme is a Golden Rectangle. Jay Hambidge saw it as subdivided into squares and Root-Five rectangles. In whole and in detail, every part is interrelated and unified by Phi's self-similar scheme. Perhaps this is why, even in ruins today, it is considered one of the most beautiful buildings in the world.

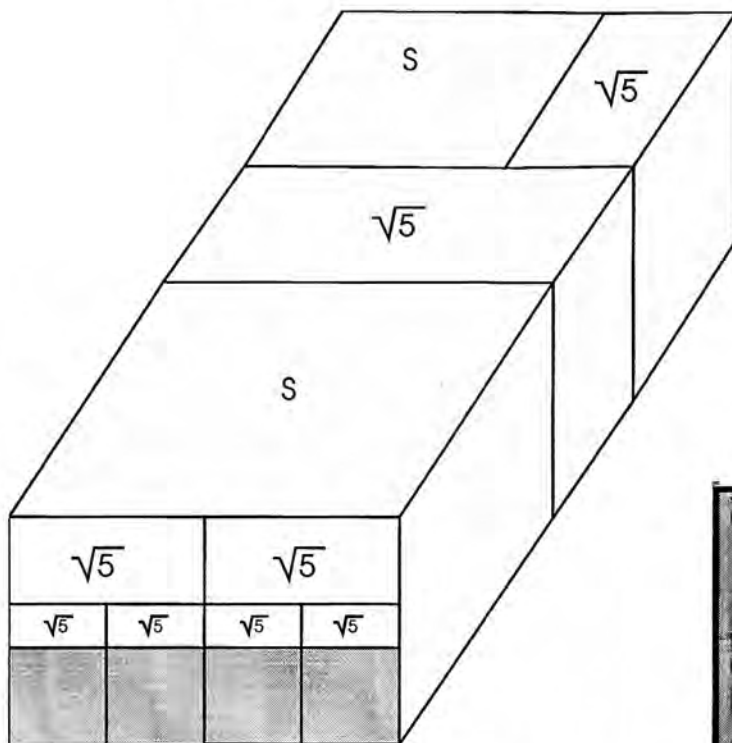
Notice the metopes above the columns which take the form of alternating squares and upright Golden Rectangles.



Apply rabatment to the geometry of the Parthenon's facade to see its scheme.

Also, on its sides are double-squares to find the Golden Mean of the Parthenon's full height, and where it's found above the columns.

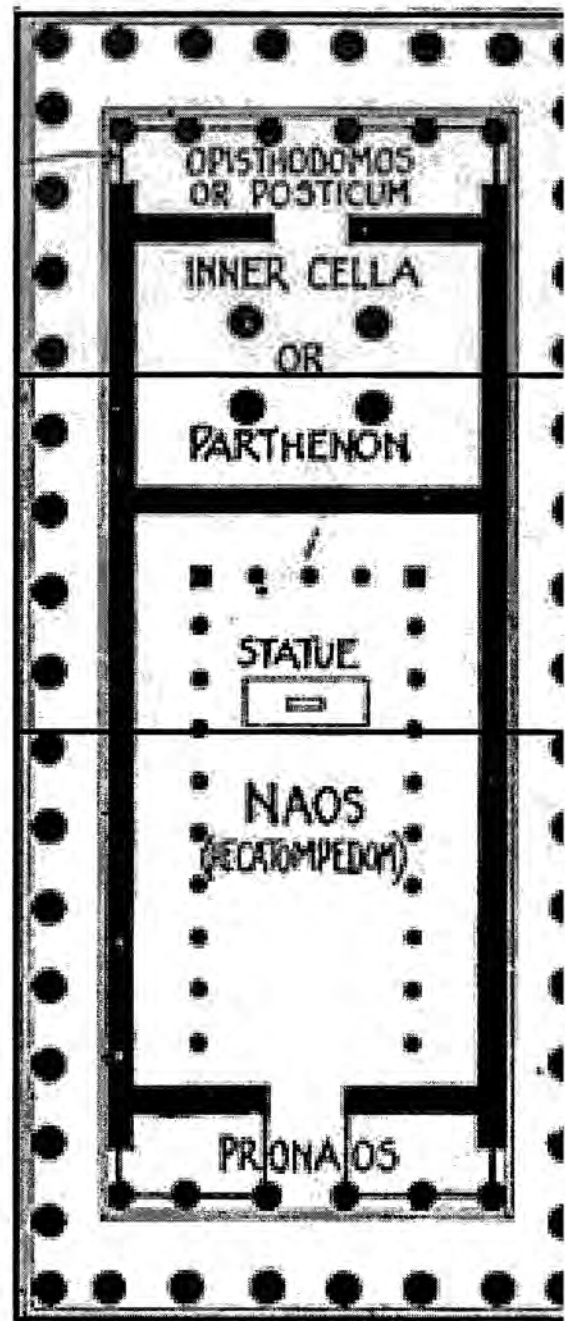
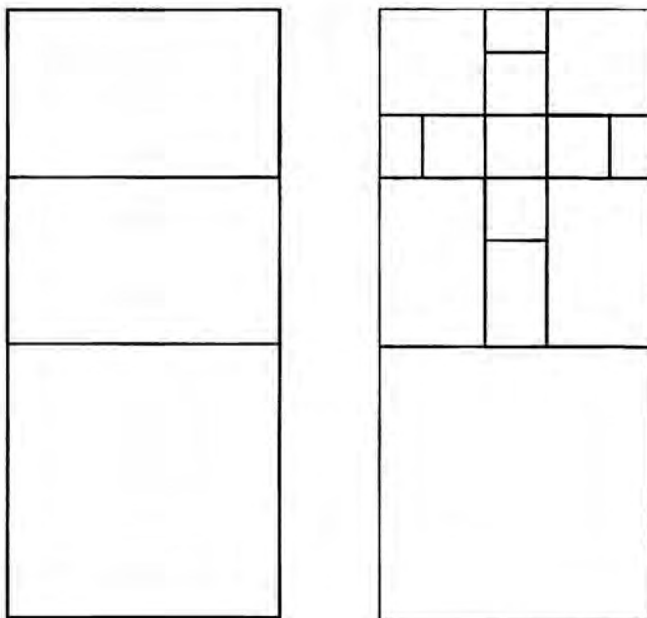




Three-dimensional proportions of the Parthenon according to Jay Hambidge.

Parthenon Floorplan

The floorplan of the Parthenon is a Root-Five rectangle, here divided into a square and two small Golden Rectangles. Use simple rabatment to bring out the proportions of its important details.



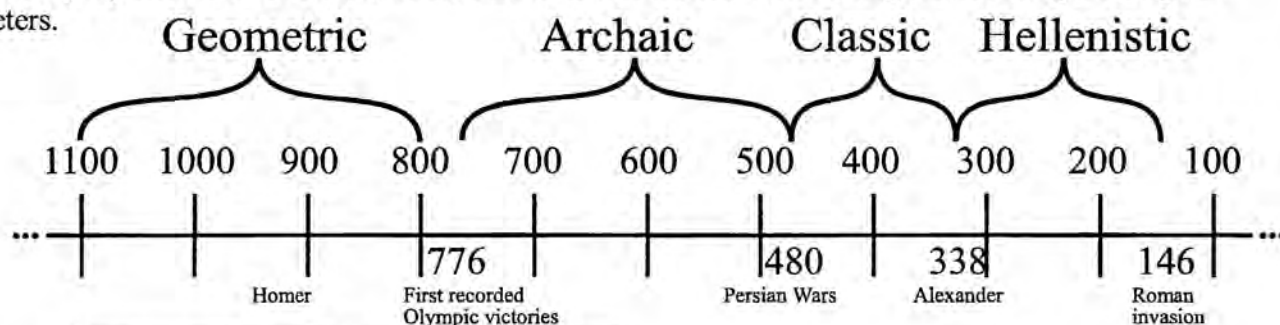
Greek Pottery

Greek pottery comes in a wide variety of shapes and sizes. In fact, the shape reveals its purpose, whether it was used to hold oil, perfume, wine, grain, or to mix liquids, transport them, or something else. It's a fascinating study.

Greek pottery is acclaimed worldwide for its beauty through all the different periods of its history. One era is even named "Geometric" for the patterns which prevail.



Greek writers said that they learned geometry and much else from the Egyptians, so it shouldn't be surprising to find highly developed geometry applied to practical concerns, including to pottery designs. One benefit to applying geometry to pottery design is that it provides an opportunity for the form to harmonize with the decoration if they share the same framework. This was a main thrust of the research of the writer and Yale professor of architecture Jay Hambidge, around the 1920's, and his student L. D. Caskey who was Curator of Classical Antiquities at the Museum of Fine Arts in Boston. Their research and writing about the "Dynamic Symmetry" geometry of Greek vases and related topics are inspiring to artists, architects, craftspeople and designers today who appreciate the benefit of this approach regarding unity in diversity. It cannot be coincidental that the times of the most wonderful pottery styles of ancient Greece coincided with the times of the most sage and accomplished geometers.



Pottery Styles

Geometers

The following examples in great part come from Hambidge and Caskey, and the reader is encouraged to use geometric skills to explore their timeless designs. We must recall that design philosophy of that time meant that everything from objects and buildings to towns and the nation were designed to be in tune with eternal verities as closely as possible. Mathematics served as the prime guide because through it eternal truths are made comprehensible. This was true around the world, and helps explain why we see so much intentional design and reverence for a geometry of the sacred.

Draw a square at each end of these Golden Rectangles and in the rectangles which result. Use colored pencils to draw and shade.

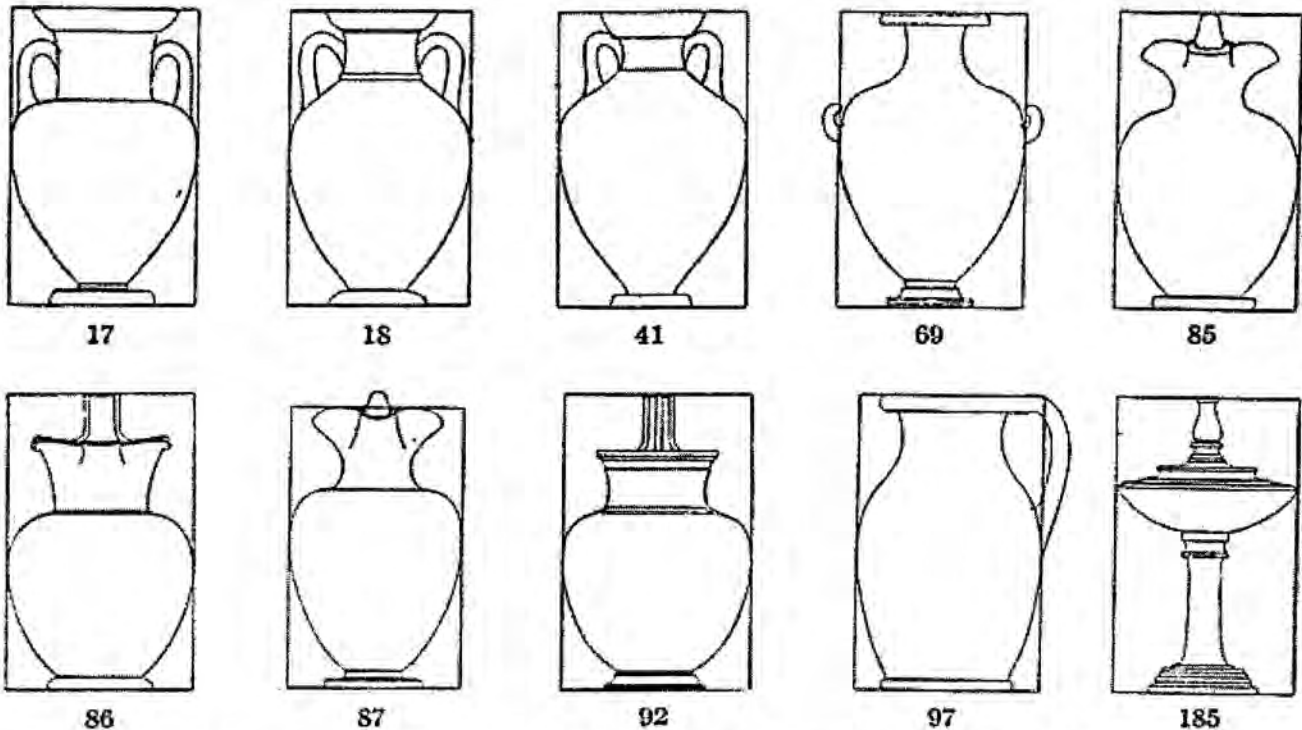


DIAGRAM VIII. Examples of the rectangle $1.618 = .618$

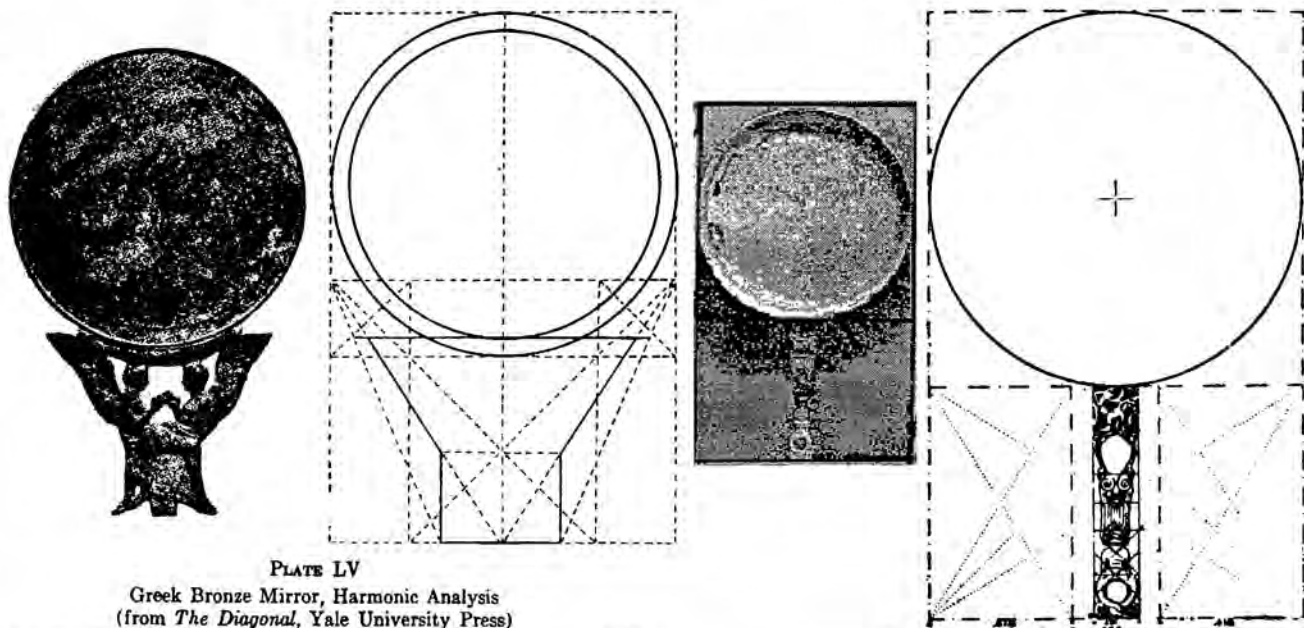


PLATE LV

Greek Bronze Mirror, Harmonic Analysis
(from *The Diagonal*, Yale University Press)

Here's pottery of different types all designed around two Golden Rectangles. Use rabatment to explore them.

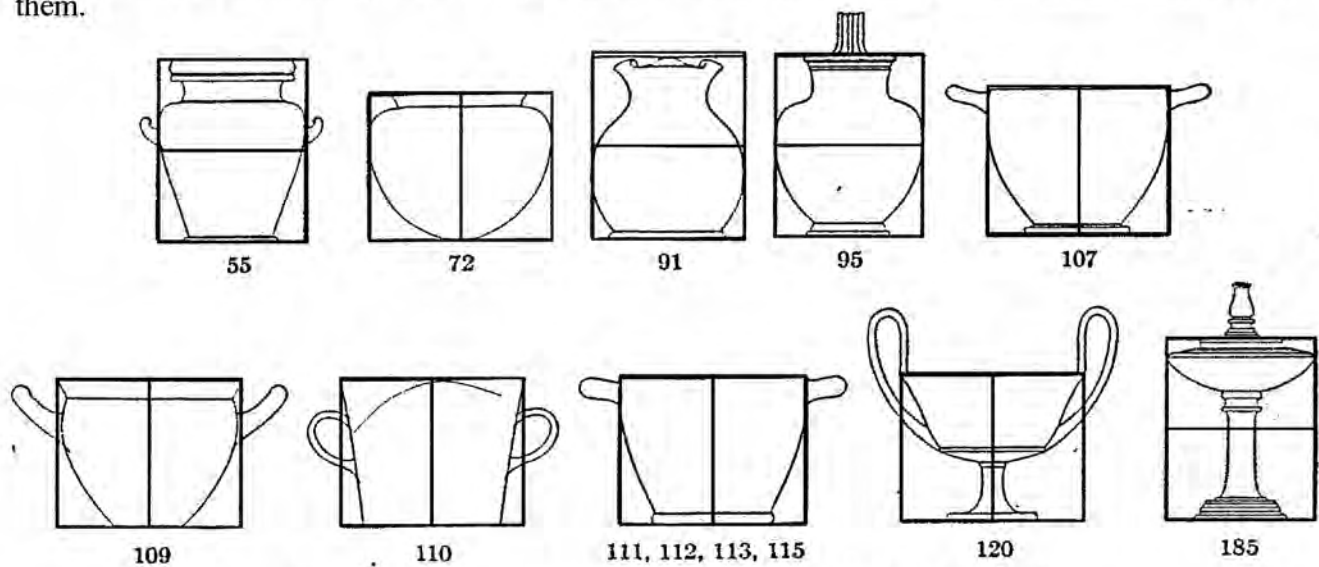
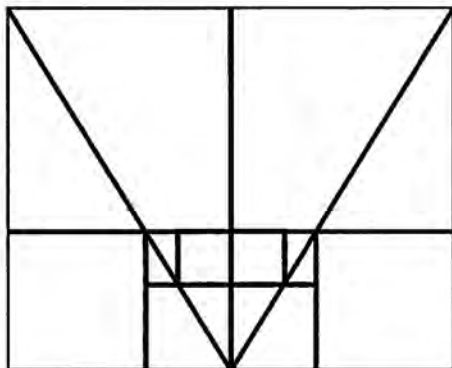


DIAGRAM XVI. Examples of the rectangle $1.236 = .809$

A Skyphos is large bowl for mixing water with dehydrated wine which had been lightened for transport. This one from Eleusis is a good example of how geometry was used to harmonize the decoration with the form. Start with the double Golden Rectangles and subdivide them by rabatment as shown.

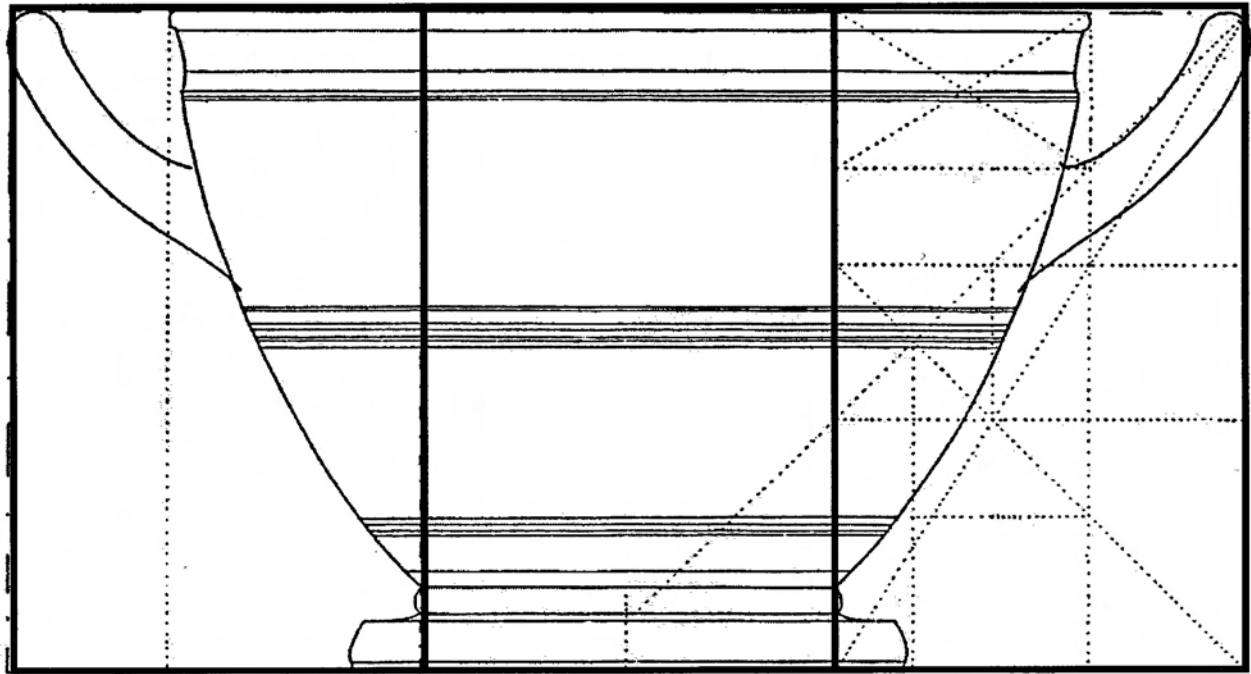
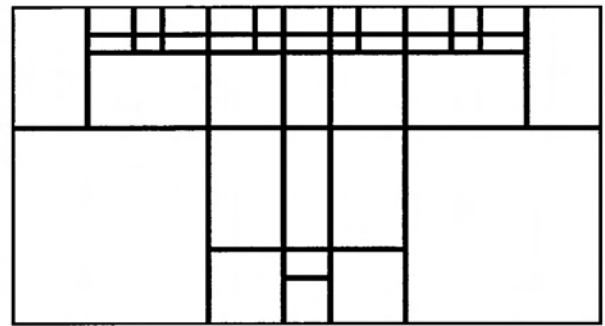
Notice how the chariot's wheel sits on a geometric groundline, and even its central axle is identified at a significant point. The diagonal of the left Golden Rectangle guides the wing of the magical chariot on its mission to teach the mysteries of agriculture around the world. This scene depicts the earth goddess Demeter before Triptolemos with her daughter Persephone behind him towards the end of their famous myth.



This Skyphos is based on three Golden Rectangles.

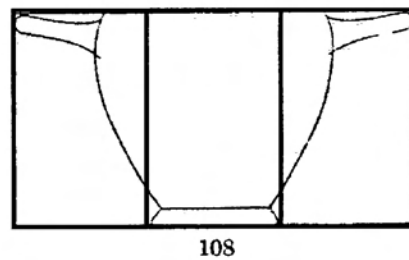
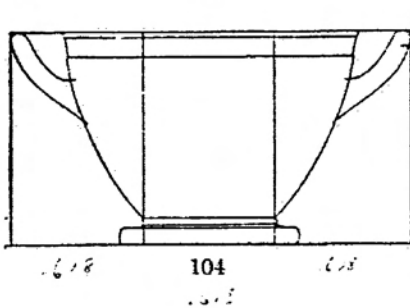
Do this rabatment on it to see inner realtionships.

Sketch illustrations on it guided by the lines and drawn diagonals.

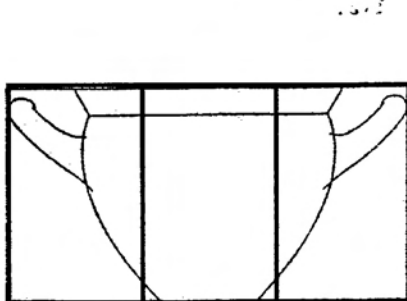


104

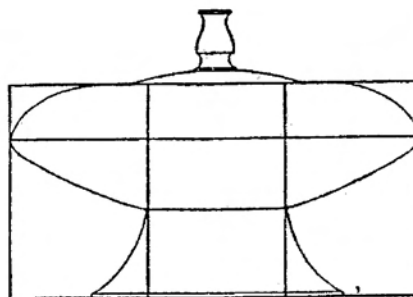
Explore any of these through rabatment.



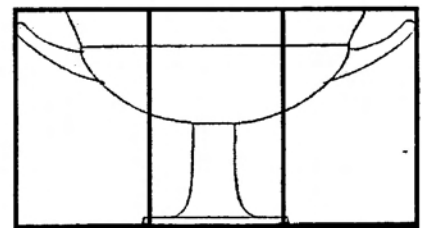
108



109



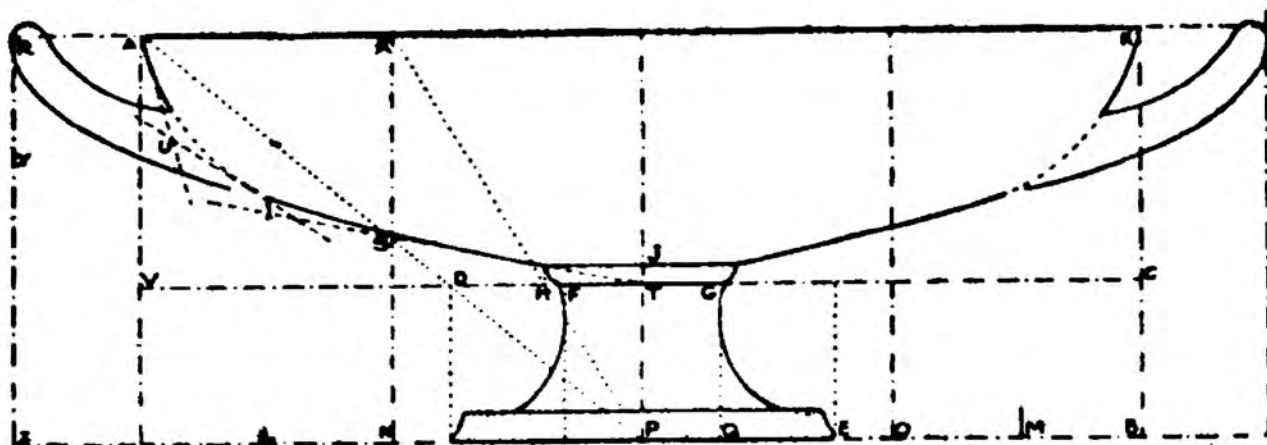
183



131

DIAGRAM XVII. Examples of the rectangle $1.854 = .5394$

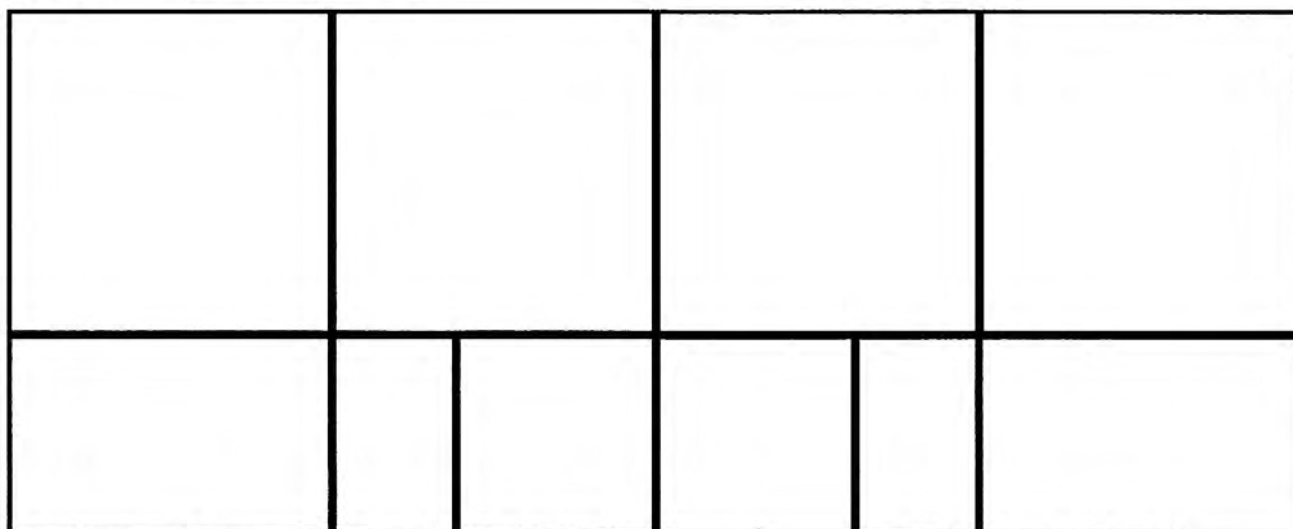
The proportions of this Kylix (wine drinking bowl) analyzed by Jay Hambidge are based on four Golden Rectangles:



Do rabatment to explore its scheme for yourself.



Design your own pottery in this grid.

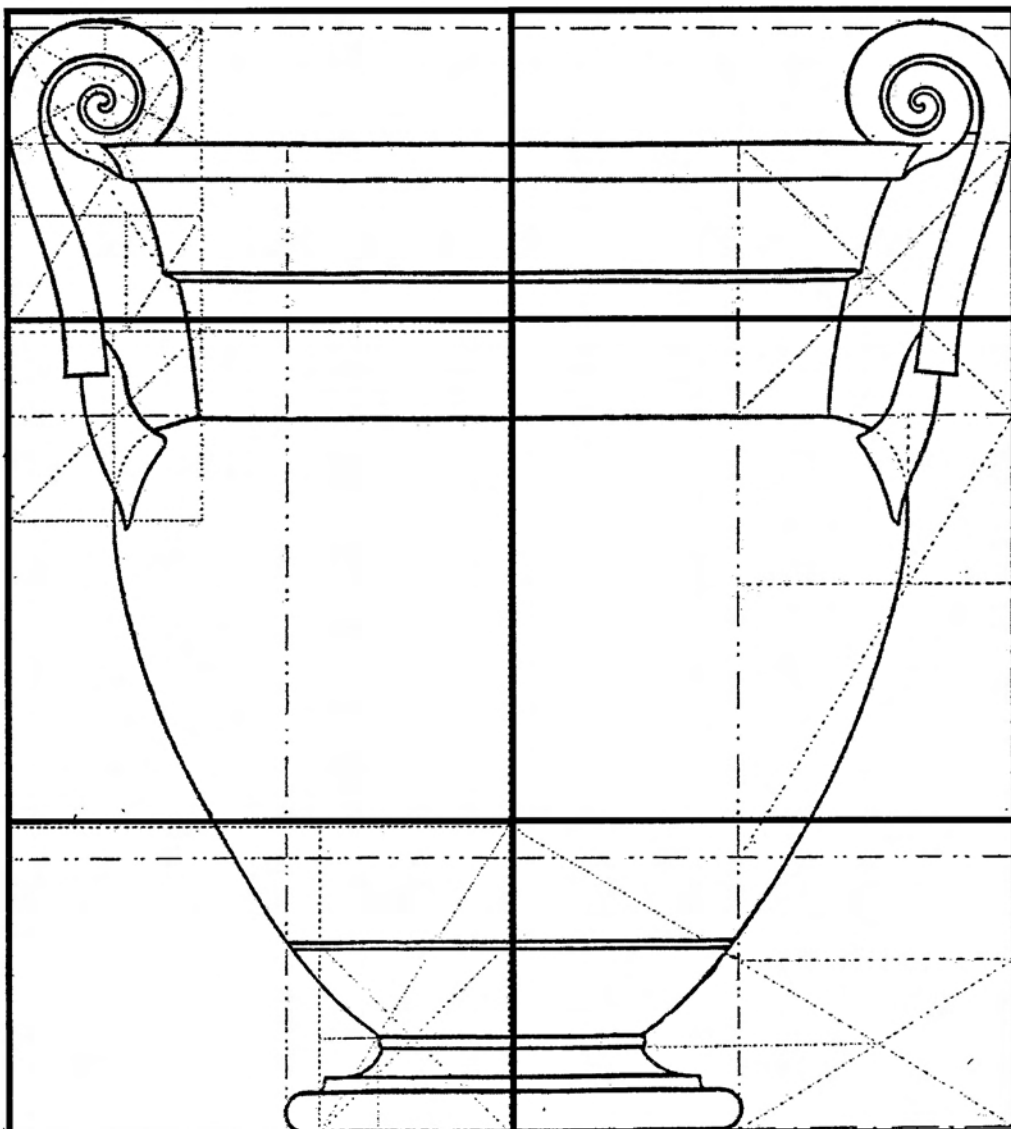
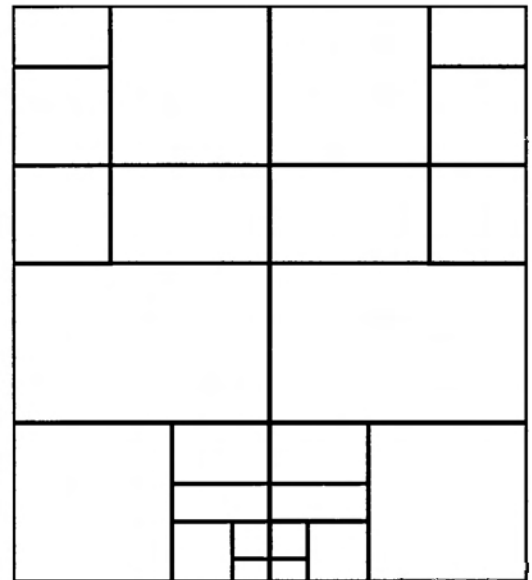


This vase is based upon two vertical Root-Five rectangles.

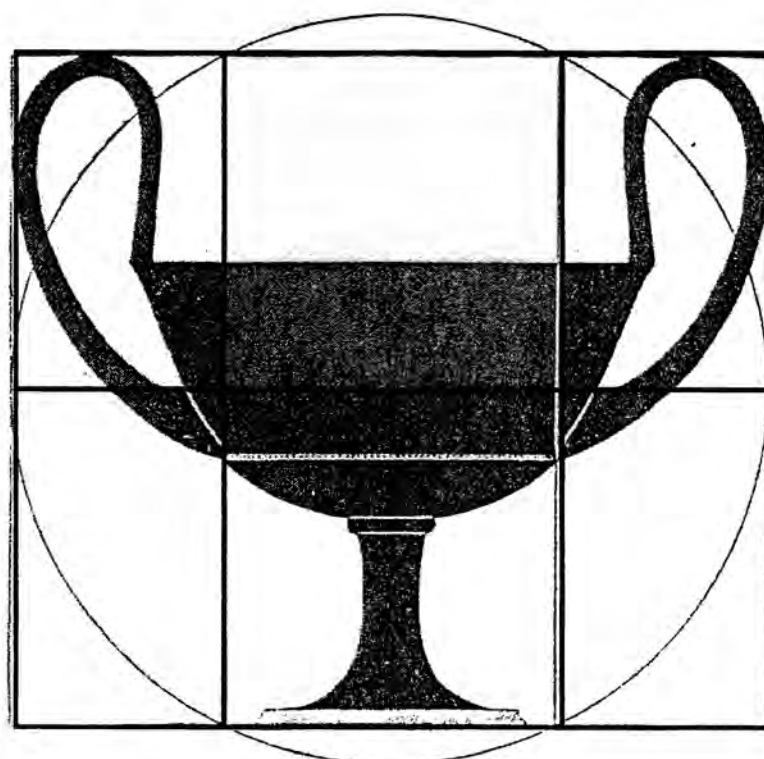
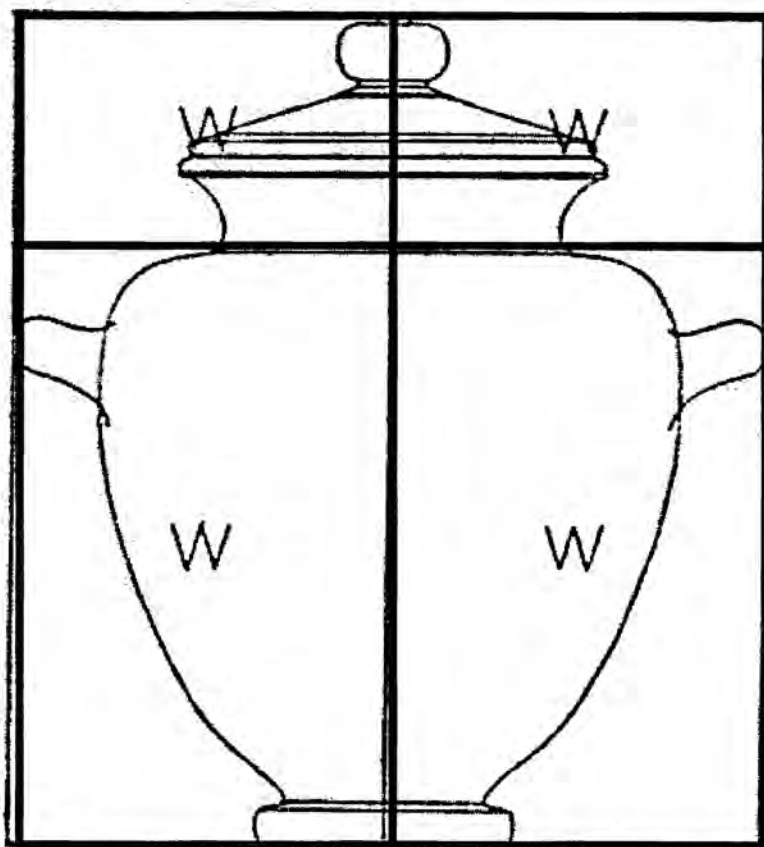
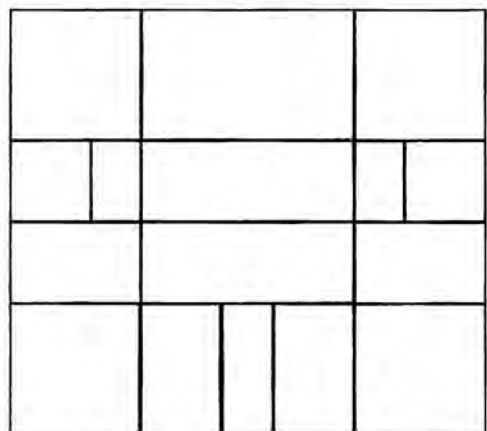
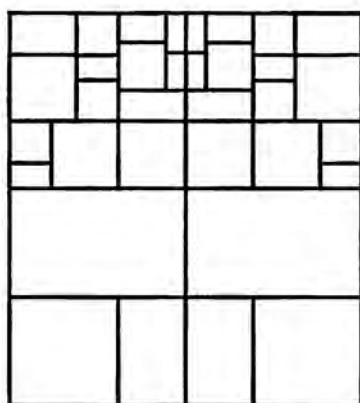
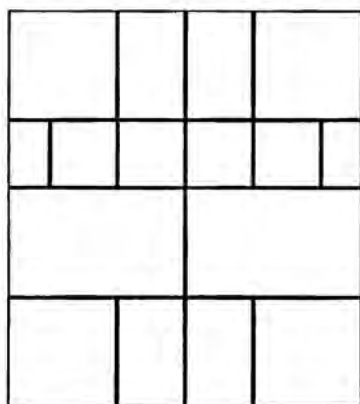
You can see the square between each set of small Golden Rectangles.

Apply rabatment and draw diagonals to explore its scheme.

Use colored pencils to design and sketch something original on it aligned with the proportions of its grid.



Apply rabatment to these double Root-Five rectangles to appreciate the proportions of the pottery.
 "W" is Hambidge's symbol for a Whirling Rectangle; that is, a Golden Rectangle.
 Design something original in the blank frames.



European Painting

European painters and architects inherited the tradition of using geometry to compose their works. Do these rabatments on the pictures.

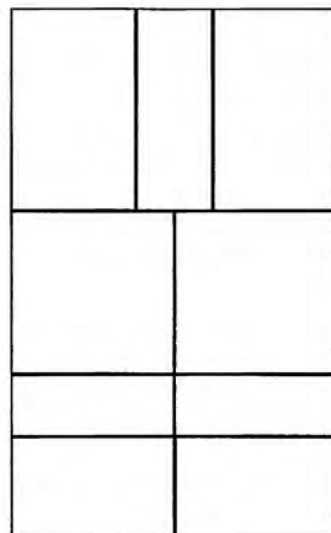
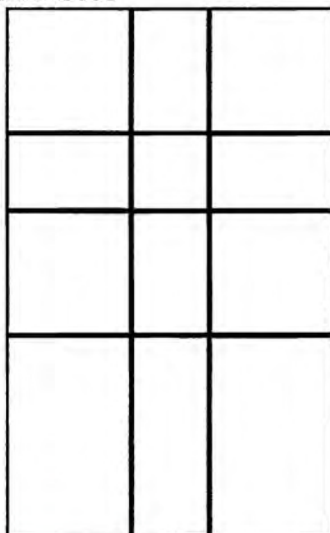
Jason and the Quest for the Golden Fleece

A Universal History, Acre, c1285

The British Library

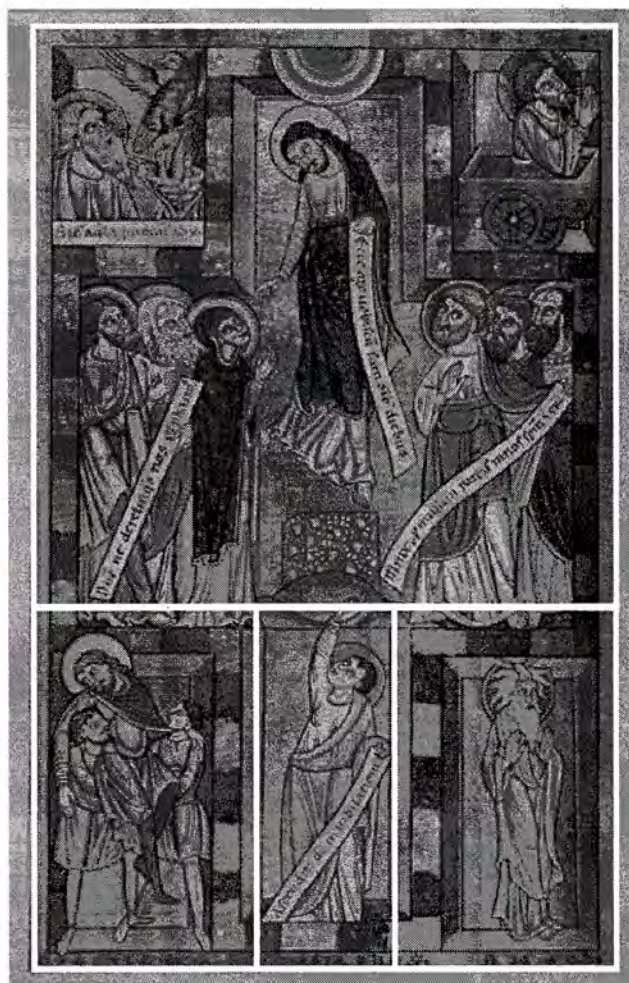
Pelias persuades Jason to build the Argo and depart on his quest to bring back the Golden Fleece

This scene at top is reminiscent of the Babylonian *Stele Of Shamash* (see page 121).



The Ascension

Unknown painter, German, Hildesheim, c1160
Tempera colors, gold leaf, silver leaf, and ink on parchment
11 1/8 x 7 7/16 in
Getty Museum



Les Très Riches Heures of the Duc du Berry

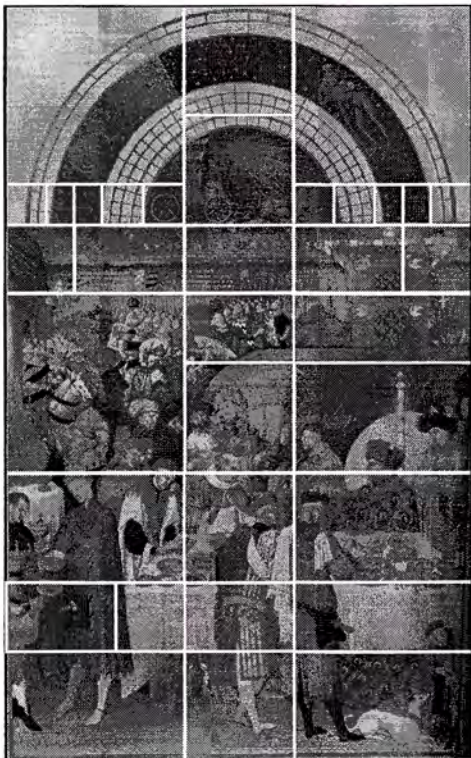
Illumination on parchment, 1405
Bibliothèque Nationale, Paris

Jean, the Duke of Berry, was a connoisseur of the arts, a bibliophile collector and a commissioner of extraordinary art treasures and invaluable painted manuscripts. In 1405 he commissioned the most famous book painters at this time Pol, Herman and Jehanequin Limbourg, to produce a *Book of Hours*, considered one of the most beautiful books published during the Middle Ages. Among its 172 luminous miniatures and glowing gold scroll borders, the book contains typical scenes from around and within the Duke's castle for each of the twelve months (see January below). Unfortunately, before they finished the book, the three Limbourg brothers died of the plague.

This scene below the January zodiac depicts the now-extinct custom of exchanging gifts on New Year's day.

The painting is framed in a Golden Rectangle.

Start by drawing a square in each end of the Golden Rectangle. Then, continue subdividing to replicate the geometric armature which frames the elements of this remarkable work.



The Baptism Of Jesus

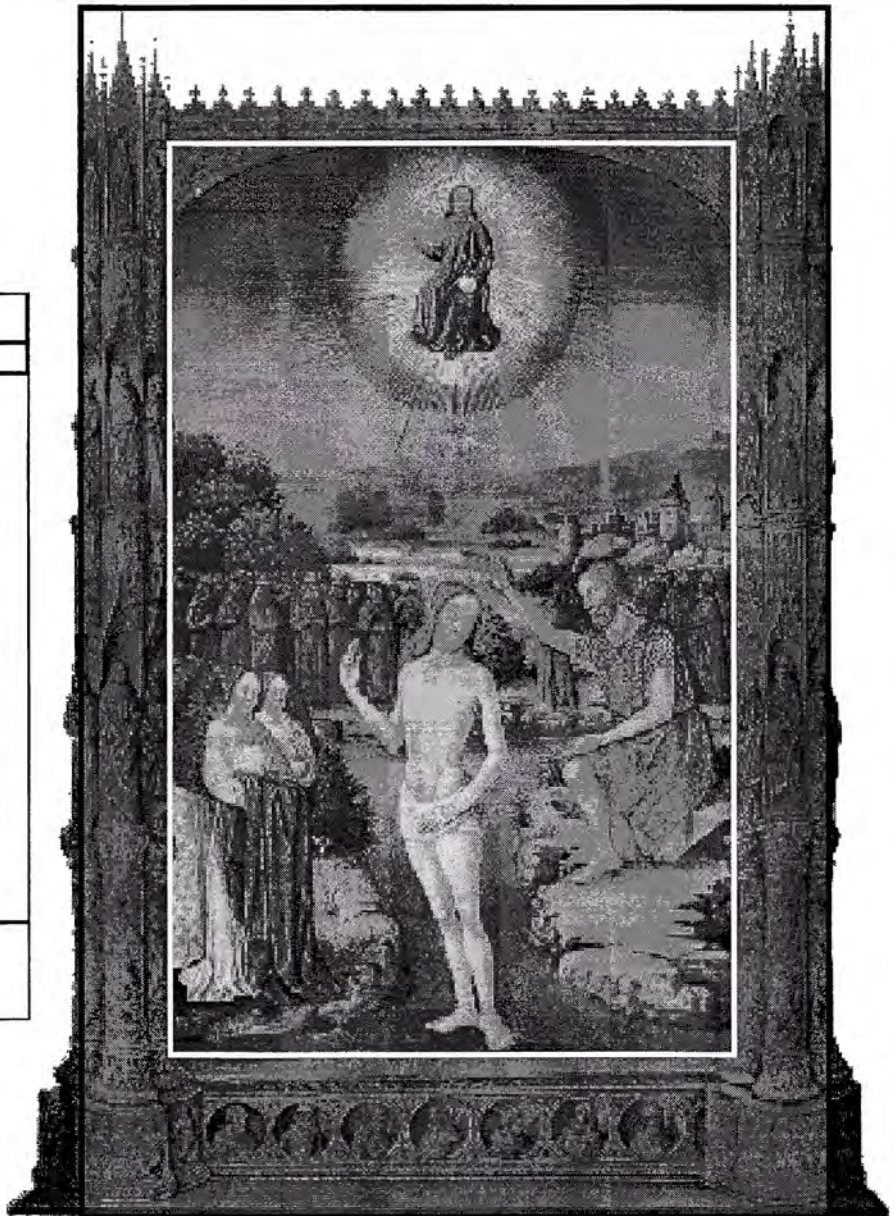
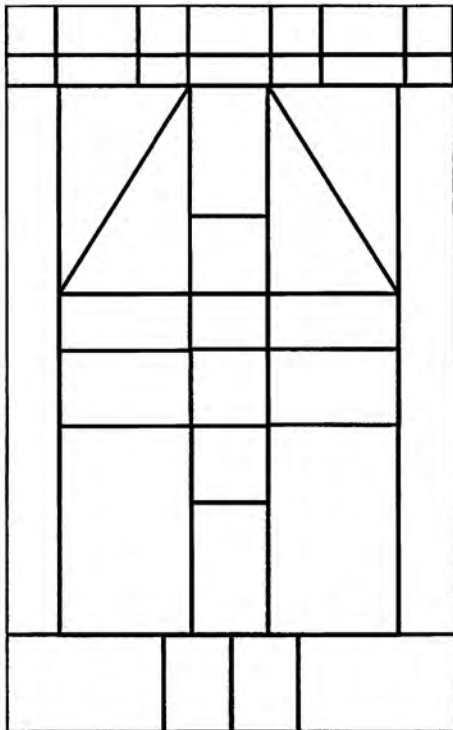
In *Les Très Riches Heures du Duc de Berry*.

Miniature on parchment painted 1485-1489 by Jean Colombe (c1440-1493) - France

Chantilly, France in the Musée Condé

After the Limbourg brothers died, sections of the book commissioned by the Duc de Berry were left unfinished. After seventy years passed many were completed by Jean Colombe. This illustration depicts Jesus standing in the waters of the River Jordan being baptized by John the Baptist with water from a shell. It illustrates Matthew 3, 16: "And Jesus being baptized, forthwith came out of the water, and lo the heavens were opened to him..."

The geometry of the outer "frame" is a large Golden Rectangle, and the picture itself is also a Golden Rectangle (white). Apply rabatment to see how the picture is organized and how the inner and outer frames relate.



The Virgin of the Rocks

(The Virgin with the Infant Saint John adoring the Infant Christ accompanied by an Angel)

Leonardo da Vinci

Painted 1483-86

Oil on panel

Published dimensions:

77 5/8 x 47 5/8 inches (ratio 1.6299...)

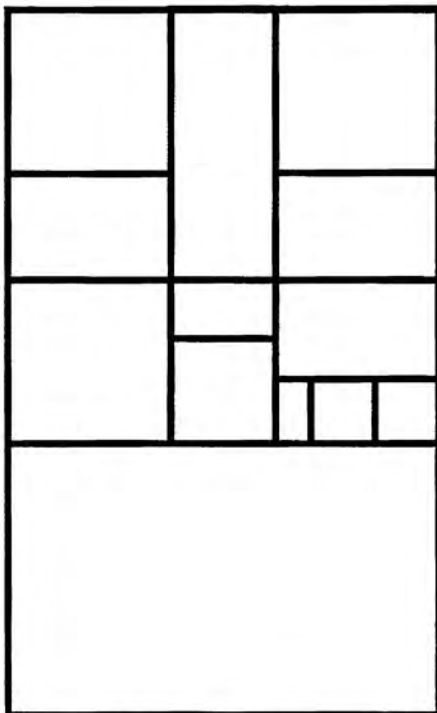
199 x 122 cm (ratio 1.6311...)

Louvre, Paris

(A similar but slightly different version, painted later, resides in the British Museum in London.)



Leonardo da Vinci's mathematical knowledge is well known, so it shouldn't be a surprise to find geometric schemes underlying his works. This first version of "Virgin Of The Rocks" is very close to being an exact Golden Rectangle. Try rabatment to explore it and decide for yourself whether this geometry may have been used.



The Haywain (Hay Cart)

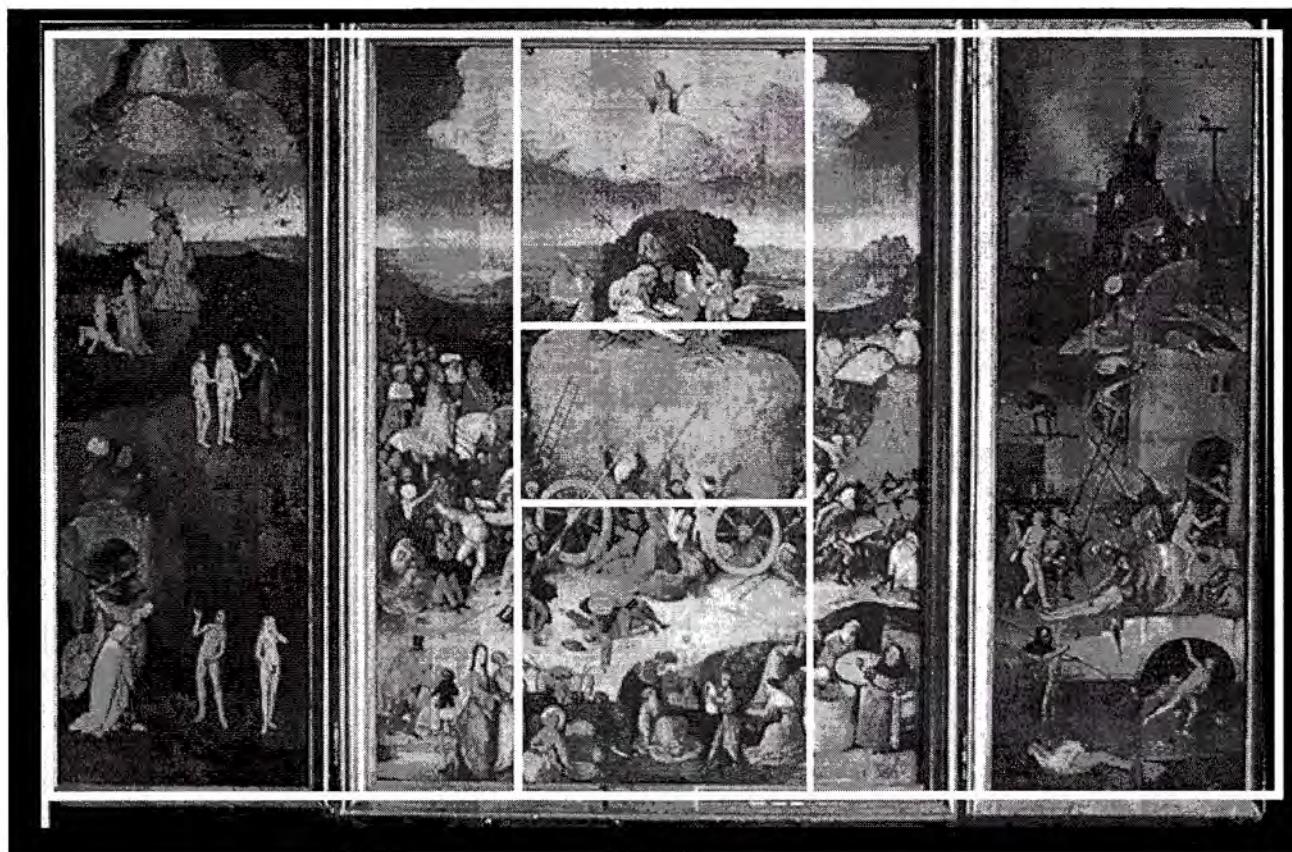
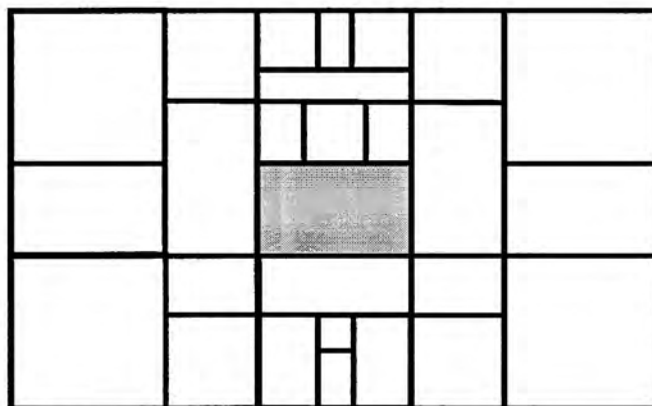
by Hieronymous (Jerome) van Aeken Bosch (c1450-1516)

Oil on panel, 1500-1502, Museo del Prado, Madrid

The subject of sins and its punishments was central to all of Bosch's art. *The Haywain* is a triptych which displays a progression from purity to sin across its panels. In the left panel we see the story of the Garden of Eden culminating with expulsion. In the central panel is a large wagonload of hay for which a greedy world grasps. All the while the wagon is being pulled by demons towards the right panel, a typical Boschian depiction of hell.

The geometry of the whole piece is that of a Golden Rectangle. The two side panels of the triptych are Phi-squared rectangles. Applying rabatment, squares were drawn from the left and right within the Golden Rectangle, and from the top and bottom of the central "overlap. What remains is a Golden Rectangle framing the hay itself, perhaps to enhance its allure.

Further subdivide Golden Rectangles and squares to see more of the armature of its composition.



The Birth Of Venus

Sandro Botticelli (1446-1510)

Painted c1485-86 for the villa of Lorenzo de' Medici at Castello

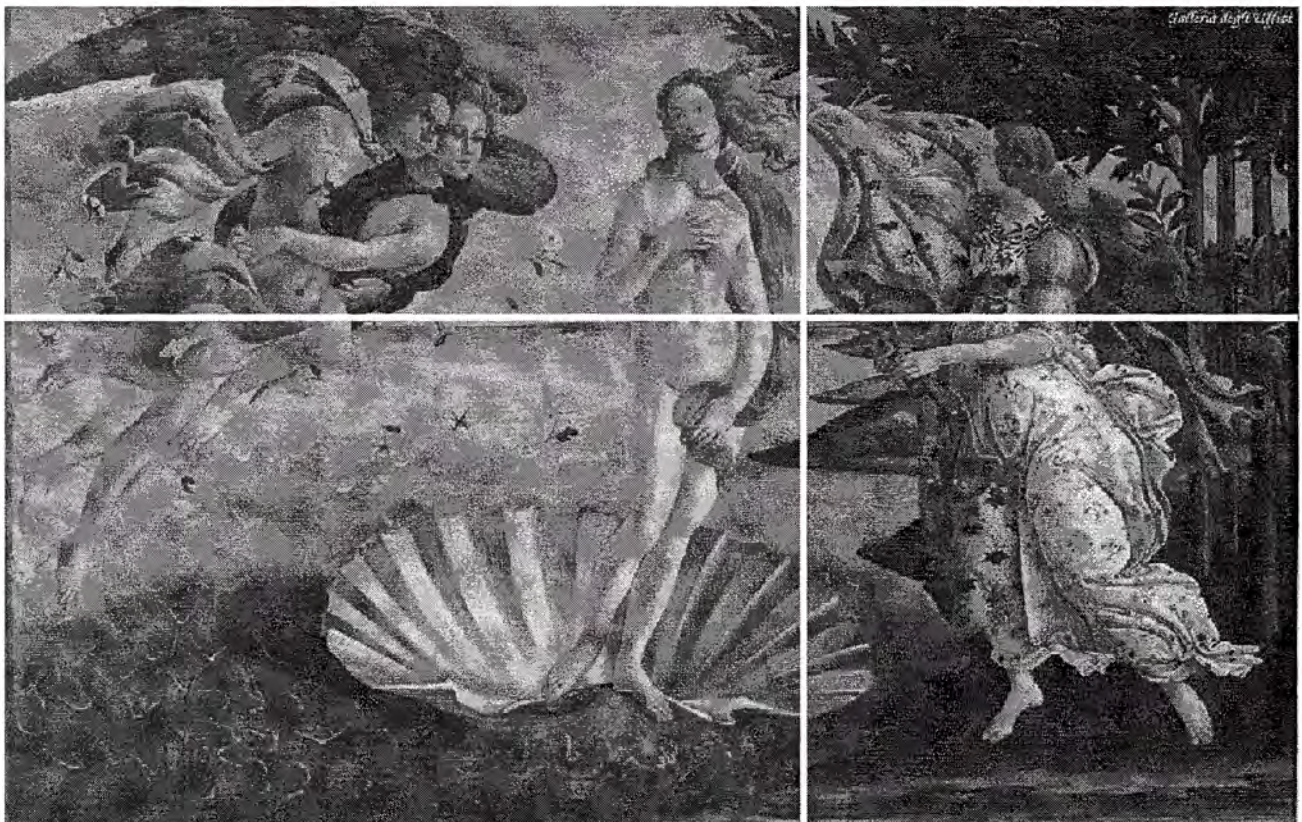
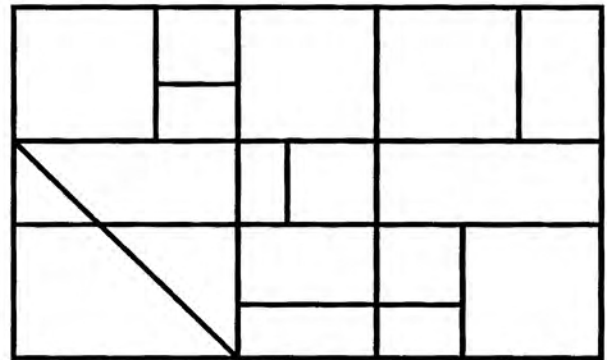
Tempera on canvas, 278.5 cm x 172.5

Galleria degli Uffizi, Florence, Italy

Botticelli evokes a classic myth showing Venus emerging from the sea on a gilded scallop shell. This isn't exactly her birth, but shows her being driven to the shore of Cyprus by flying wind-gods amidst a shower of roses -- each petal with a golden heart. She's about to step onto the land, where one of the three *Horae*, or "The Hours", Greek goddesses of the seasons and attendants to Venus, is about to receive her with a purple cloak. Venus is here a graceful and delicate being, epitome of beauty and harmony.

Botticelli must have intended to enhance her beauty with a Golden Rectangle (whose canvas is in the ratio = 1.61449...). Notice how the horizon line, which passes through her navel, is at the Golden Mean of the painting's height.

Explore it further through rabatment.

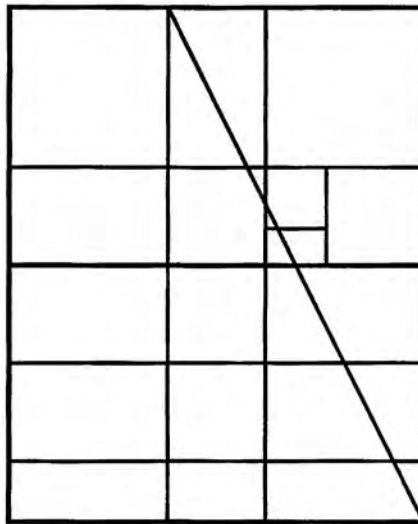


Saint Gregory the Great

Jean of Berry's *Petites Heures*
France, Paris 14th Century

(55 x 45 mm)

Bibliothèque Nationale France



This illuminated painting depicts Saint Gregory writing, inspired by the Holy Spirit.

Its dimensions are two Golden Rectangles. Midpoints are drawn at their sides.

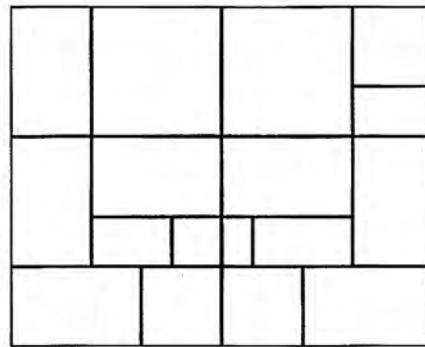
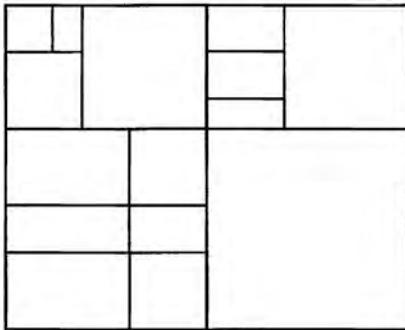
Use rabatment to create squares and Golden Rectangles and to see why the pen he's holding is pointed at the angle it is, and other features.



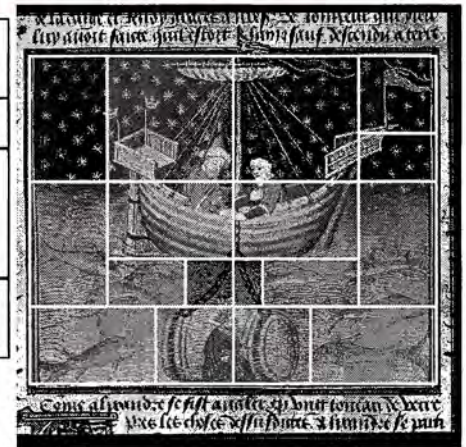
The Seventh Trumpet

From the Bamberg *Apocalypse*, c1020
Bamberg, Germany, Staatsbibliothek

Two Golden Rectangles frame the scene.
Notice how the line separating small Golden
Rectangle from the square centers the *Vesica
Piscis* and squares the corner of Christ's book.
Apply rabatment to see more.



**Alexander being lowered from
a ship in a glass barrel to view
the wonders of the sea**



From *The Alexander Romance* manuscript, Rouen, France, 1445
The British Library

Soon after the premature death
of Alexander the Great (356-
323 BCE) legends began to
develop that his military quests
were actually driven by an
ambition to investigate the
wonders of the world, seeking
true wisdom and eternal life. A
Greek text relating these tales
was translated into the French
prose manuscript from which
this illustration comes.

The geometry is that of two
Golden Rectangles. Do the
geometric rabatment and you'll
see that the boat is framed in
two smaller Golden Rectangles,
and Alexander's glass barrel sits
within a double-square.

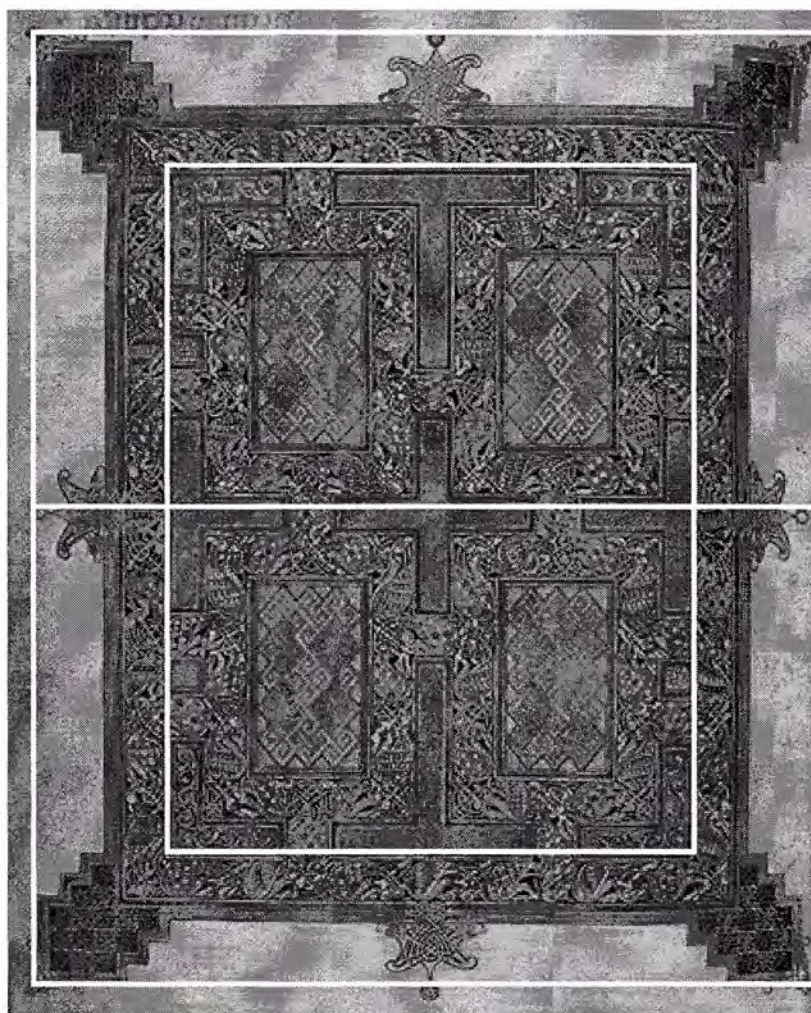
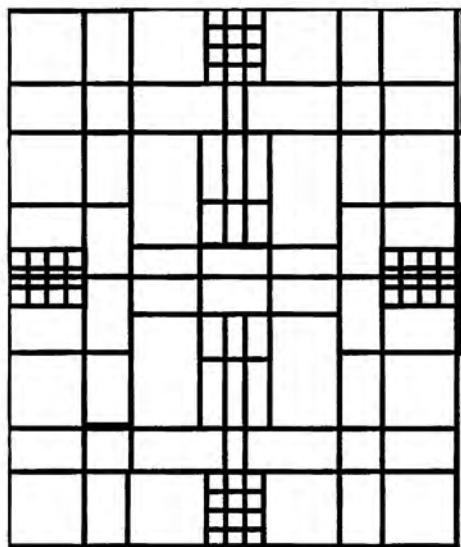
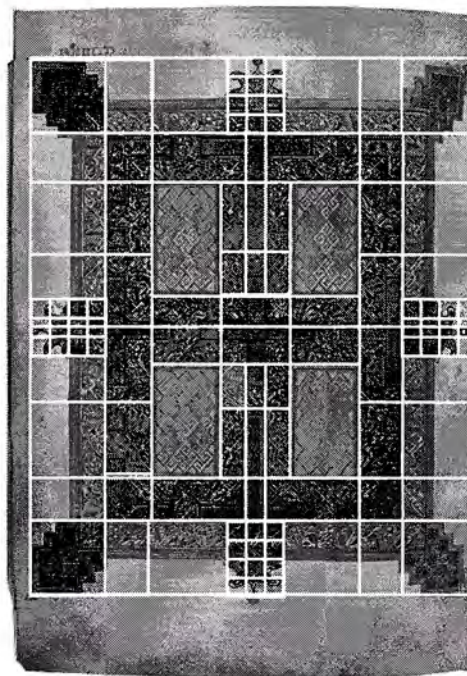


Lindisfarne Gospels

Carpet page preceding St. John's Gospel
between 710 and 721
The British Library

The magnificent book called the "Lindisfarne Gospels" (also "St. Cuthbert's Gospels" and the "Durham Book") was made and used at Lindisfarne Priory on Holy Island (northeast England), a major religious community, by the monk Eadfrid, Bishop of Lindisfarne between 698 and 721, "in honor of St. Cuthbert". It consists of 258 leaves of thick vellum, and contains the Four Gospels in the Latin of St. Jerome's Version, written in double columns.

The geometry of this "carpet" design seems to be framed within two Golden Rectangles. Within them, framing the central decoration, are another two Golden Rectangles. Notice the four upright Golden Rectangles on the page with squares and Golden Rectangle spaces between them. Apply rabatment to discover more of its framework.



The Annunciation

Leonardo da Vinci (1452 - 1519)
painted c1472-1475
Oil and tempera on wood
Uffizi Gallery, Florence, Italy.

98 x 217 cm (ratio = 2.21428)



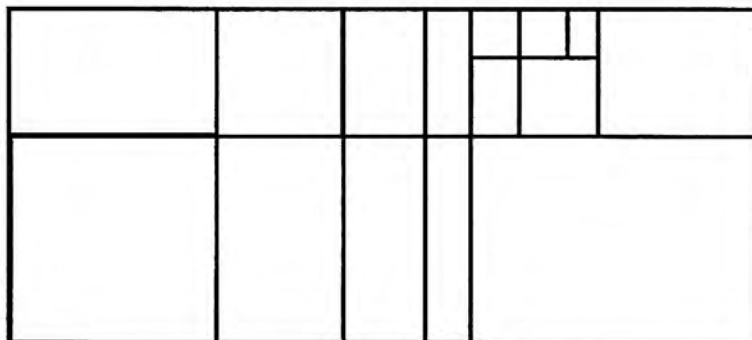
This masterpiece is considered to be a youthful work by Leonardo, executed while he was still an apprentice in the workshop of his master, Andrea del Verrocchio.

It depicts the archangel Gabriel kneeling in a dignified manner and raising his right hand in greeting to Mary, indicating her divine pregnancy. The Virgin has stopped reading and reacts to the Annunciation with an expression of deep respect and by gesturing with her left hand.

The painting appears to be a Root-Five rectangle and is divided (below) into a square and two small Golden Rectangles.

Notice the wall next to Gabriel is at the Golden Mean up the side of the painting, and how the edge of the building near Mary lies along the square from the painting's right edge.

Apply rabatment to find more details.



Philosophy Consoling Boethius and Fortune Turning the Wheel

Coëtivy Master

French, Paris, c1460-1470

Tempera colors, gold leaf, gold paint
on parchment, Getty Museum

"At the left, Philosophy, personified as a fashionably dressed young woman, visits Boethius, who had lost his exalted position as Roman envoy. She asks him why he is despondent, saying, "you are wrong if you think that Fortune has changed towards you. Change is her normal behavior." On the right, Fortune, a beautiful woman, spins her wheel, which represents the changes she brings about in men's lives. A king sits on top of the wheel, but a simple turn of the wheel can bring him down in station while it raises the fortunes of another. Philosophy demonstrates that Fortune rules the world and that the wise person ignores her ever-shifting ways, preferring eternal truths." -- Getty Museum Catalog description



Man: Body and Soul

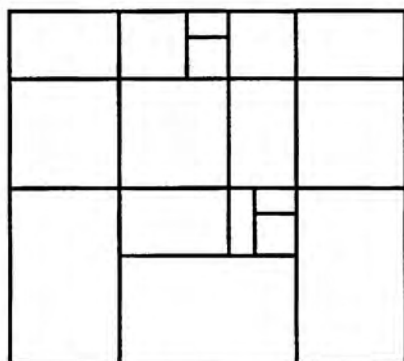
Bartholomaeus Anglicus

(Bartholomew the Englishman)

On the Properties of Things

France, Le Mans 15th Century

Bibliothèque Nationale France

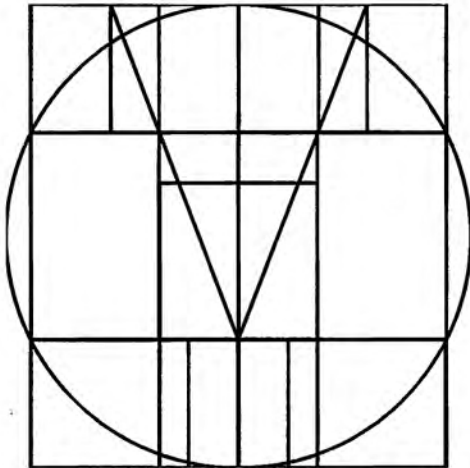


Trinity

Bartholomaeus Anglicus
(Bartholomew the Englishman)
On the Properties of Things

France, Le Mans 15th Century
Bibliotheque National France

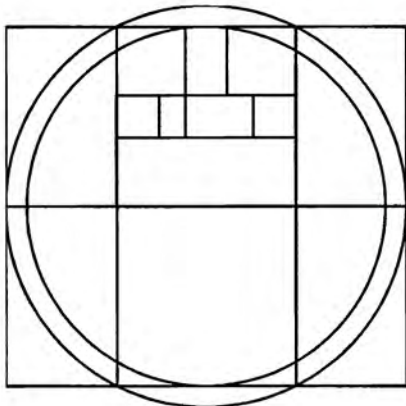
Notice the side-by-side Root-Five
rectangles of these two paintings.



Fall of the Rebel Angels

Bartholomaeus Anglicus
(Bartholomew the Englishman),
On the Properties of Things

France, Le Mans 15th Century
Bibliotheque National France



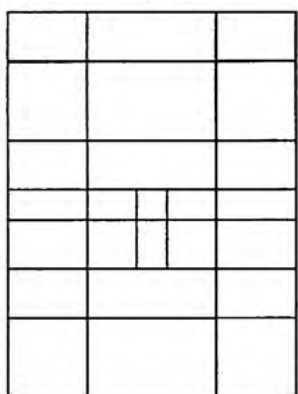
Abbot Vivian and the Monks of Tours Present the Bible to Charles the Bald

Vivian Bible Frontispiece

Gold and illumination on parchment
Tours, France c846 (Carolingian)
Paris, Bibliothèque Nationale

This scene is the Dedication page of the Vivian Bible, a lavishly illuminated 9th century manuscript Bible. It was made at the monastery of Tours. The scene shows the Abbot Vivian, surrounded by monks, presenting the Bible to Charles the Bold.

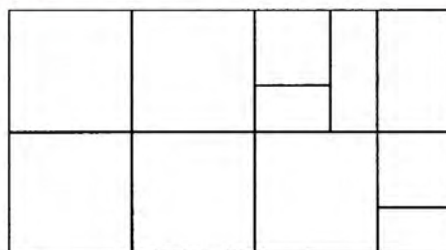
Three Root-Five rectangles frame these scenes.



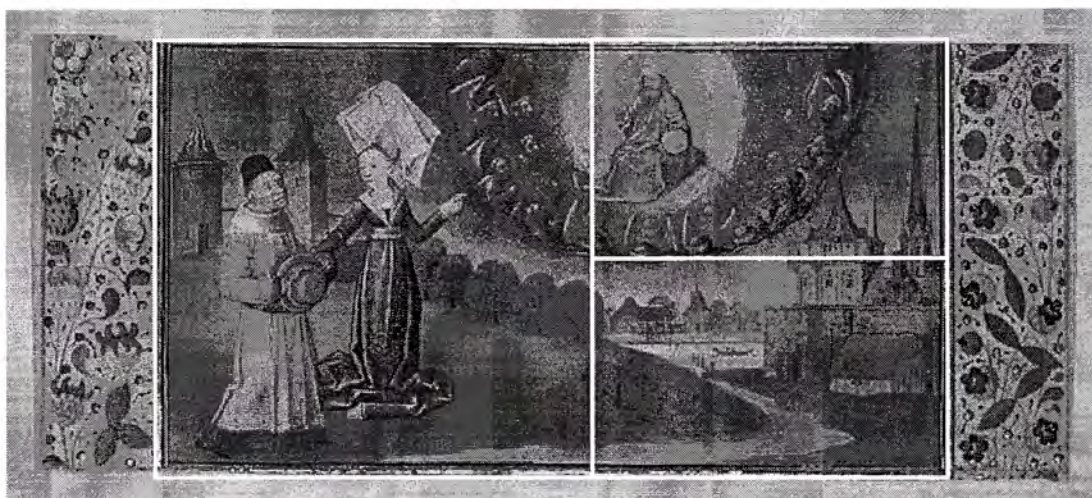
Philosophy Instructing Boethius On The Role of God

Coëtivy Master

French, Paris, c1460-1470
Tempera colors, gold leaf, and gold paint on parchment
Getty Museum



The painting within the borders seems to be a square plus two small Golden Rectangles, which Hambridge describes as a 1.809 rectangle (see Appendix).



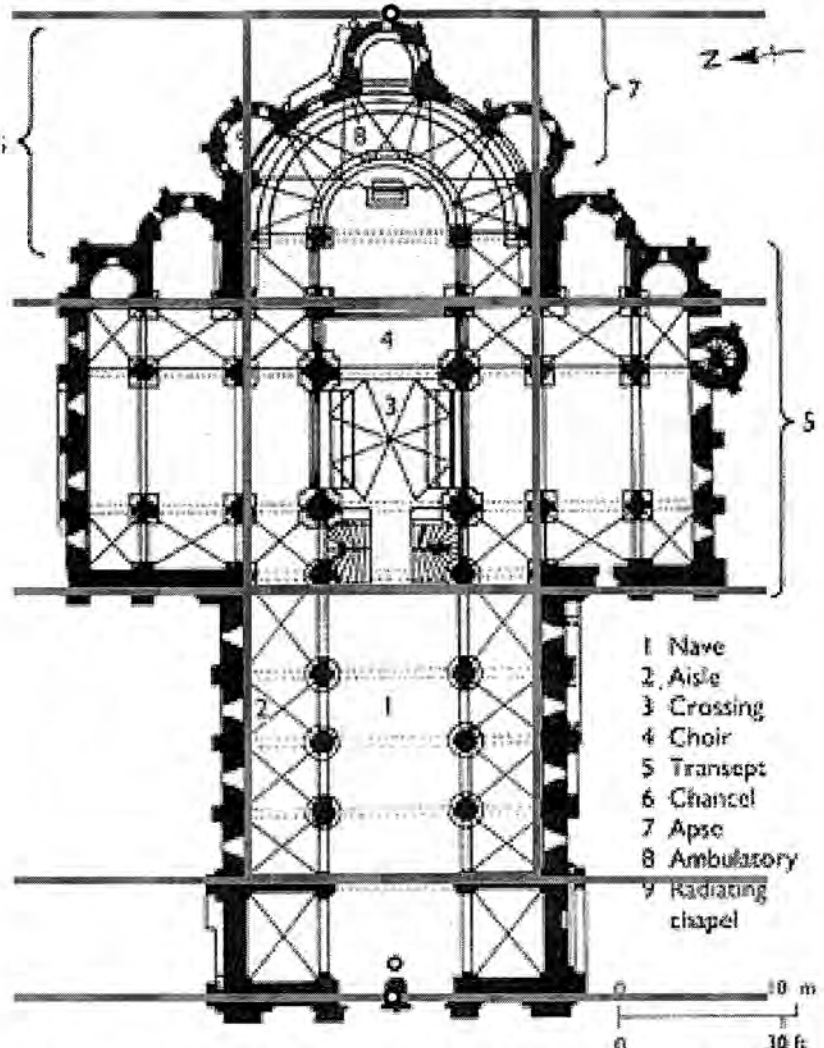
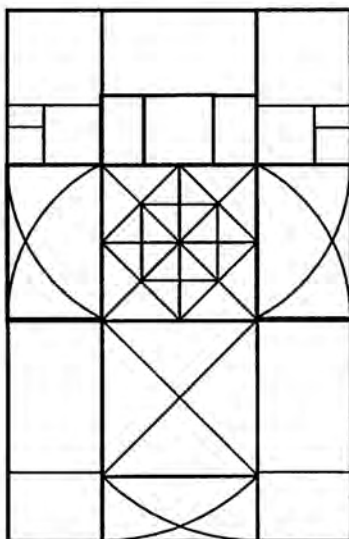
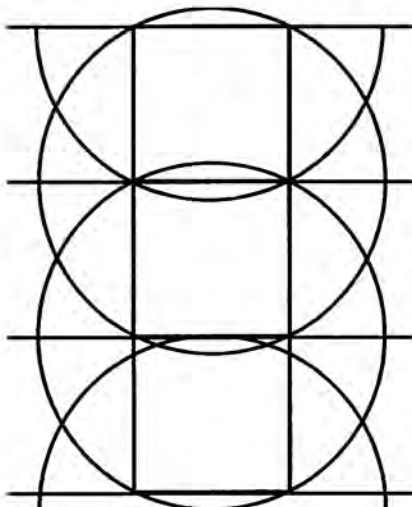
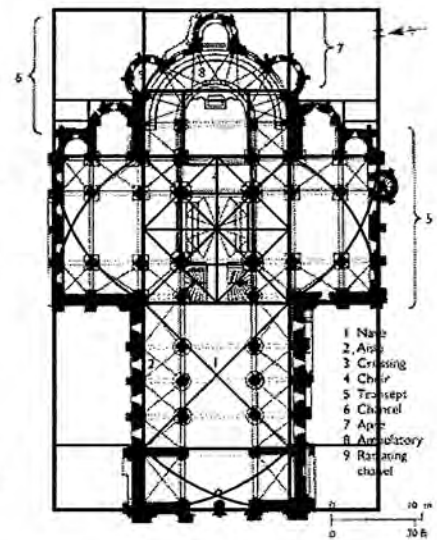
Saint Foy Abbey-Church

Conques, Aveyrou, France, built 1050 to 1120

Saint Foy abbey-church is Romanesque style architecture, the work of Conques' monks who had gained a solid reputation as builders.

The Saint Foy abbey-church in Conques was a popular stop for pilgrims on their way to Santiago de Compostela in Spain. Its construction was begun on the foundations of a smaller earlier basilica, directed by the abbot Odolric (1031-1065) completed around the year 1120. It was built in Romanesque style, using a warm-colored local limestone infilled with a local gray schist. The large dome that covered the crossing later collapsed and was replaced in the 15th century.

Its geometry is based on three Root-Five rectangles, and a Root-Two rectangle. Connect the midpoints of the upper and lower of the three squares to turn arcs and define the Root-Five rectangles. Arcs made by the full diagonal of the bottom square reveals the Root-Two proportions of the entrance. Then apply rabatment to see further details of its design.



Ka'aba Door

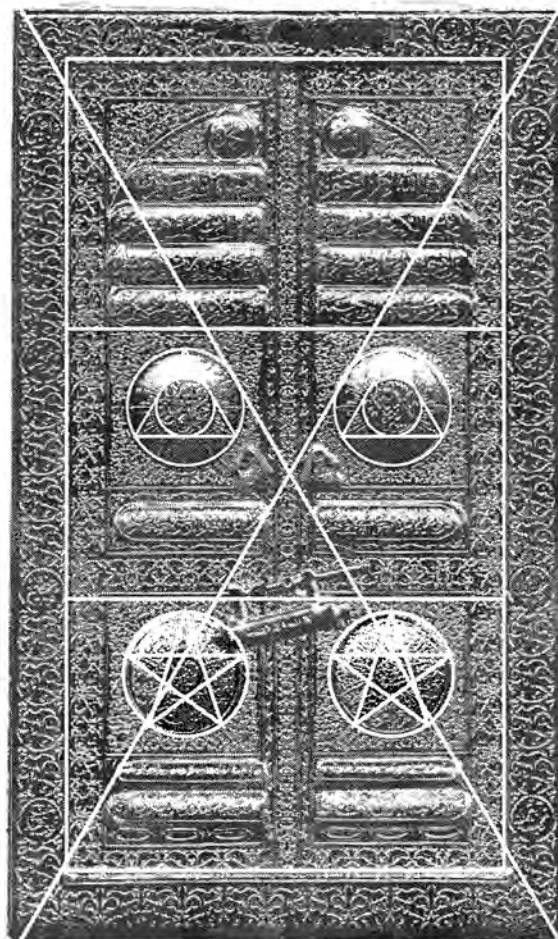
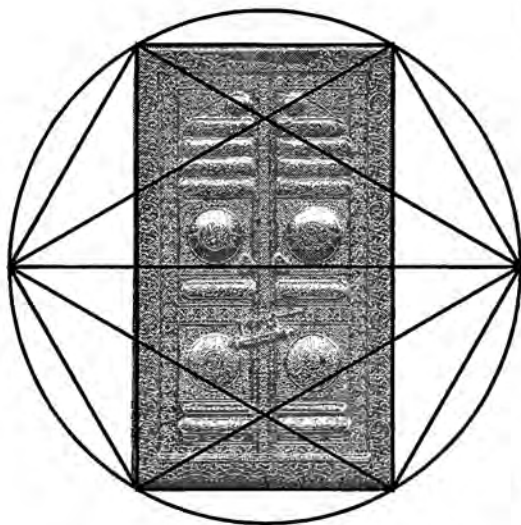
The Ka'aba is the name of the holiest site of Islam, a somewhat cubic structure located within the central courtyard of the Great Mosque in Mecca.

The word Ka'aba in Arabic literally means a "high place" of respect and prestige, but it is generally taken to mean a "cube" because the structure resembles one. The Ka'aba has been rebuilt many times over the centuries. The bottom of its single Sitarah (door) is over seven feet above the ground in the northeastern wall. It is usually covered by an embroidered fabric.

The door's overall form appears to be that of a Root-3 rectangle, found in the Hexagon. The Root-3 rectangle is often found in art and architecture to symbolize that it is sacred. The idea is reinforced within the border where we find the proportions of three Golden Rectangles, each containing symmetric, engraved designs. Two circles in the lowest Golden Rectangle contain smaller circles whose relative diameters indicate that they are related by the Pentagram star, symbol of humanity. The circles in the middle Golden Rectangle contain larger circles related to the whole circle by the equilateral triangle, symbol of divinity. Above that, in the top Golden Rectangle, are four pairs of bars and two small circles whose meaning requires a scholar to interpret. You might further explore the geometry of the door by rabatment and other constructions.

To construct the Hexagon around the door, place your compass point at its center and open the pencil to the end of one of the diagonals.

Turn a circle and, from the end of the diagonal, "walk" the radius around it to find six equally-spaced points (see Volume 2 Chapter 6). The width of the door will be seen to be equal to the radius of the circle.

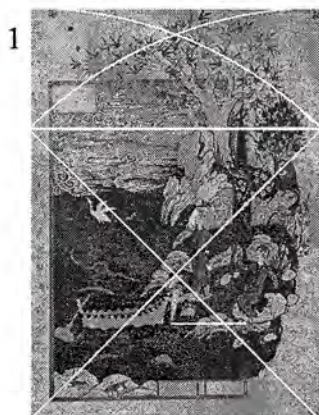


Salaman and Absal on the Heavenly Isle

This painting, called "Salaman and Absal on the Heavenly Isle" was made for the Persian Sultan Ibrahim Mirza in 1560. It illustrates one of the seven poems of the Haft Aurang ("Seven Thrones Of Grace") by the Persian Sufi poet Jami (1414-1492) called "Salaman and Absal" (translated by Edward FitzGerald in the 19th century). This story is actually older, the final part of a three-volume "visionary cycle" allegory by the great Persian Sufi philosopher, physician, astronomer, mathematician, alchemist and writer ibn Sina, called Avicenna in the west (980-1037).

The first part tells the story of a child stranded on a deserted island who grew up and recognized the Divine by contemplating nature. In the second part he accepts to undertake a sacred journey and awakens from his slumber of daily life. The third part describes the final stage of the mystic journey as an adept whose soul goes through trials and experiences until he is transformed into the angelic state. This painting depicts one stage of this journey, showing his arrival with his ill-fated lover on the Heavenly Isle intending to explore the pleasures of their senses.

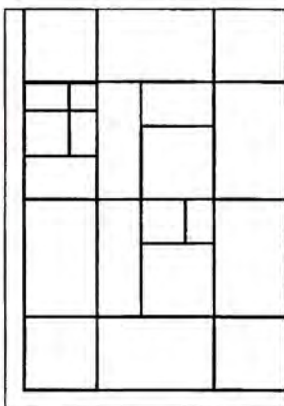
The geometry of the overall page is that of a Root-2 rectangle (Fig.1). You can prove it by marking a square from the bottom and use your compass to see that the diagonal of the square equals the length of the sheet. Within is a rectangle containing most of the scene, and it is a Golden Rectangle! (Fig.2) It extends to the right side of the page by constructing two Root-5 rectangles (Fig.3). Use this as a start to complete more of the construction shown in the geometry below (Fig.4). Notice that the small gold rectangle in the upper left of the scene is itself a Golden Rectangle!



3



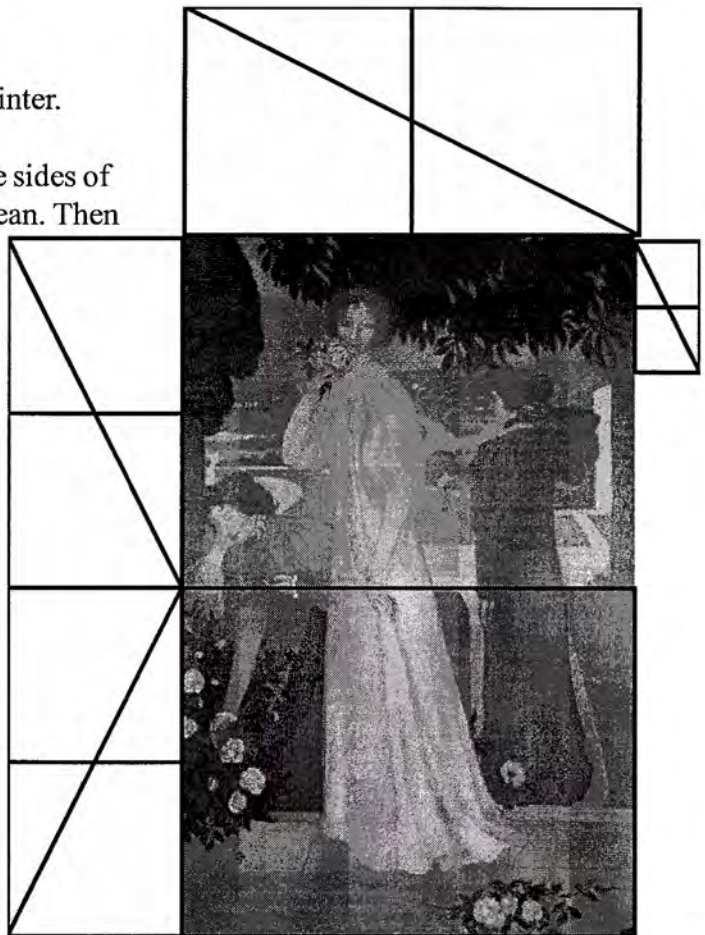
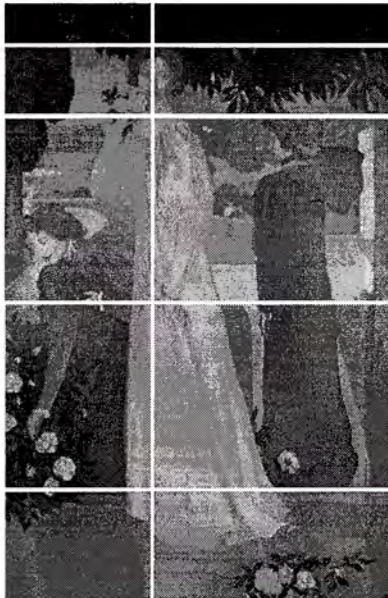
4



Woman

Maurice Denis (1870-1943), French *Nabi* Painter.

Double-squares have been provided along the sides of these paintings for you to find the Golden Mean. Then match points and draw lines across as shown to see consecutive powers of Phi unfold.



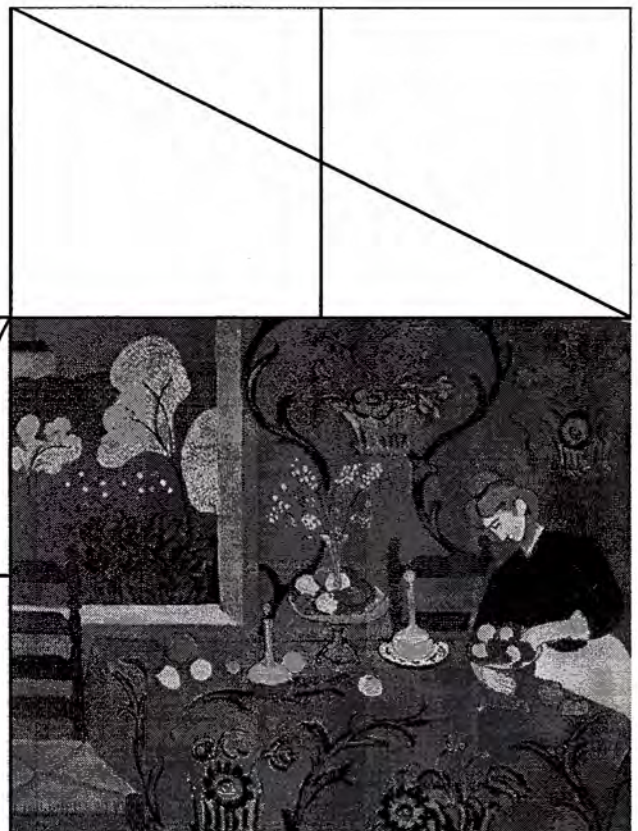
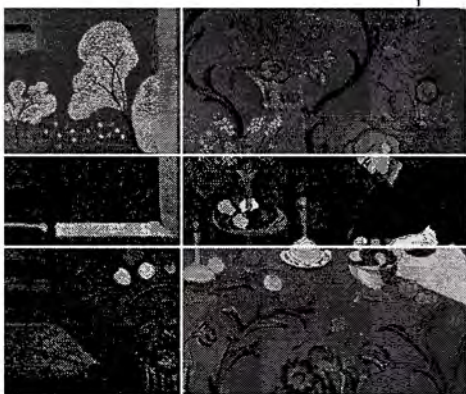
Harmony in Red/La desserte

Henri Matisse (1869-1954)

Spring 1908

Oil on canvas

Hermitage Museum, St. Petersburg



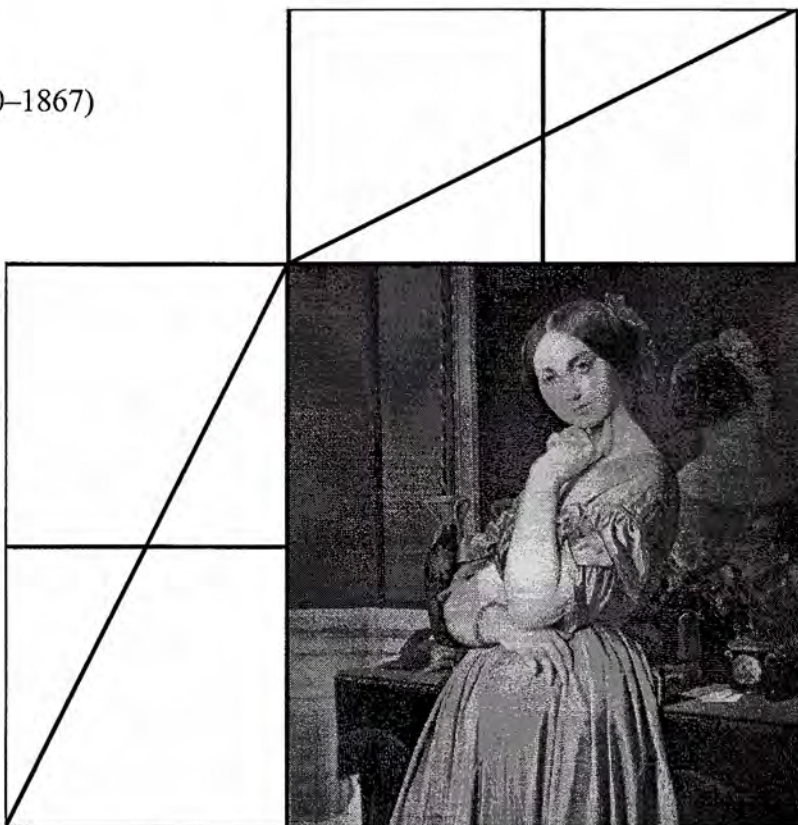
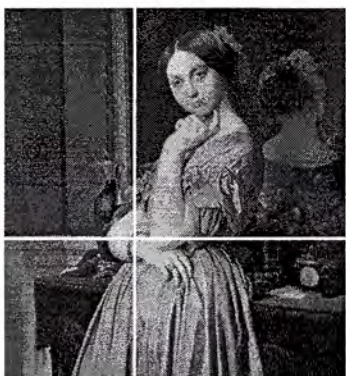
The Comtesse d'Haussonville

Jean-Auguste-Dominique Ingres (1780–1867)

Painted 1842-1845

Oil on canvas

In a traditional pose, we see the countess Louise, Princesse de Broglie (1818–1882) who is known for publishing a number of books including biographies of Robert Emmet and Lord Byron.



Anne In White

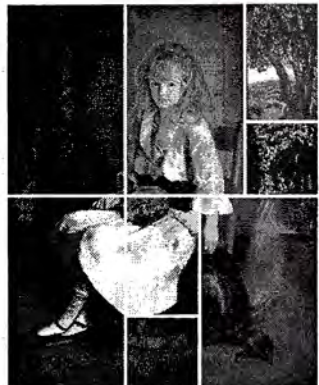
George Wesley Bellows (1882-1925, American Painter, Ashcan School)

Painted 1920, Oil on canvas

Dimensions 52 7/8 x 42 7/8 inches

George Bellows was a student of Jay Hambidge and the geometry of Dynamic Symmetry. Bellows had been aware of it since 1910, but he more seriously applied it when he spent time at the famous artists' colony in Woodstock, New York during 1920-1925.

This painting is based on two Golden Rectangles. Use rabatment to explore Bellows' use of Golden geometry in it.



Guernica

Pablo Picasso, 1937

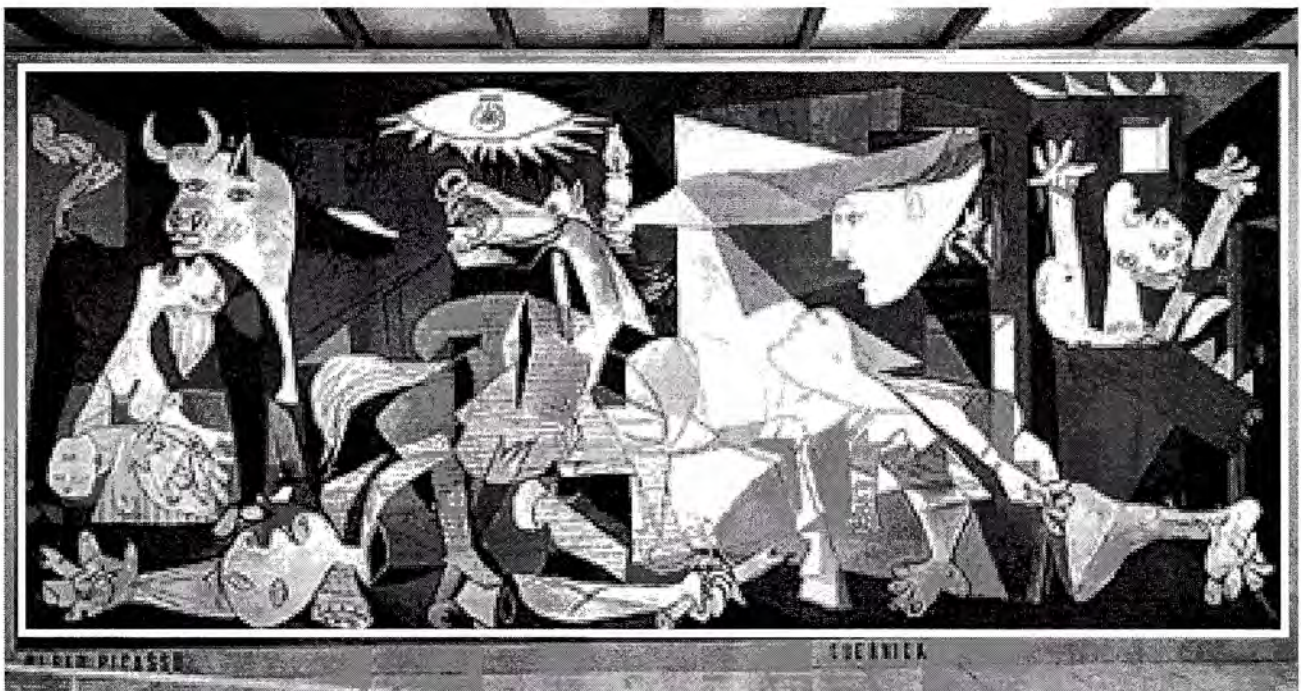
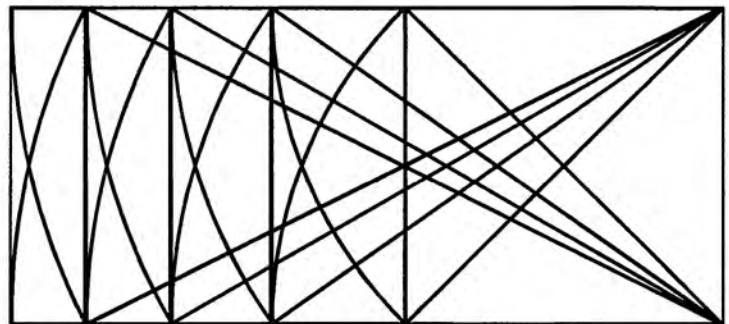
The Museo Nacional Centro de Arte Reina Sofía, Spain.

In April, 1937, a little Basque village in northern Spain was pounded with bombs for over three hours by the Germans on behalf of Spain's dictator Franco, resulting in sixteen hundred civilians killed or wounded. This shocked many people, including Picasso, who painted this anti-war icon in three months to be the centerpiece for the Spanish Pavilion of the 1937 World's Fair.

Its dimensions (349 x 776 cm) indicate a ratio = 2.22349... which reveals it to be a Root Five = 2.236... rectangle.

Draw a square in the right side and use the technique of unfolding Square-Root Rectangles (see page 94) to create vertical lines along the way. (You need not draw the diagonals or full arcs shown here, only the points with which to draw the vertical lines.)

Draw squares and diagonals to help identify some of the arrangement of its haunting elements.

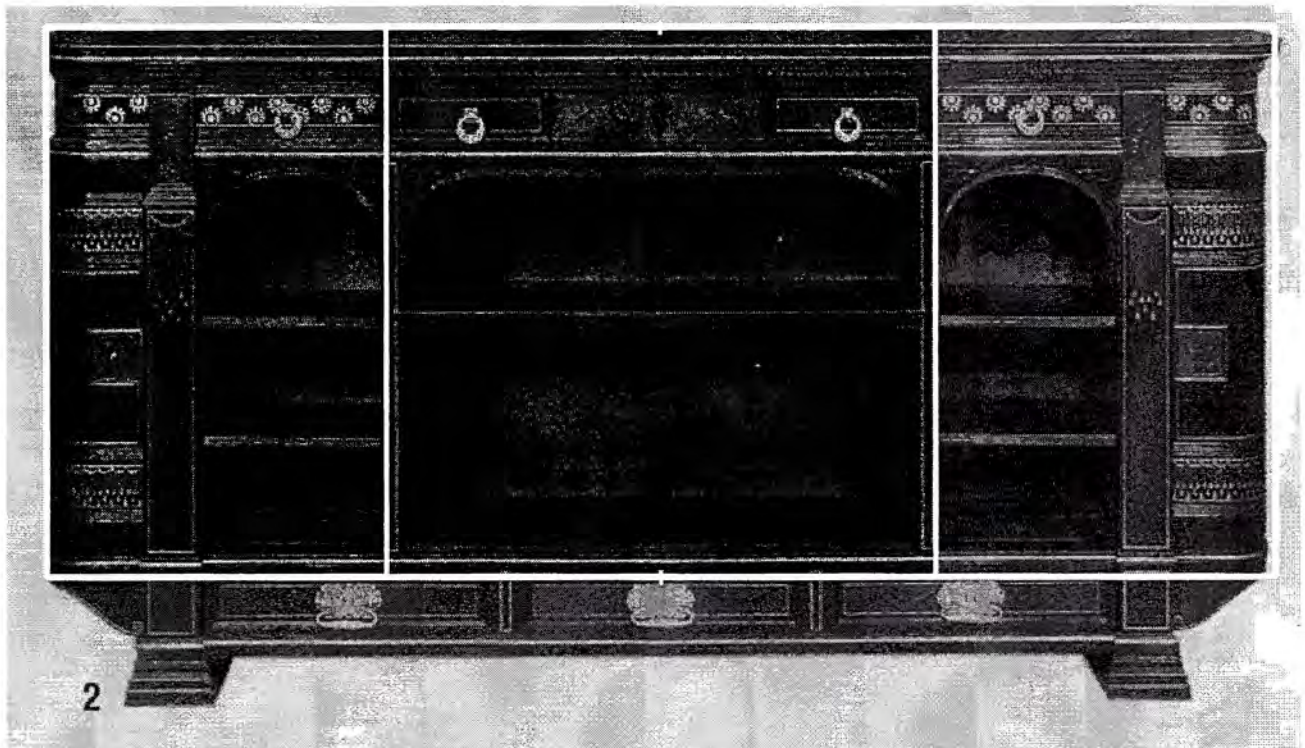
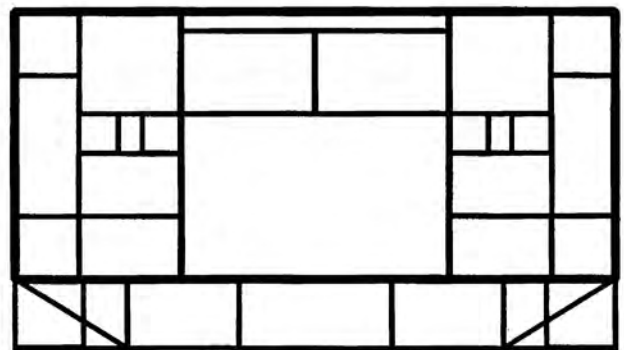
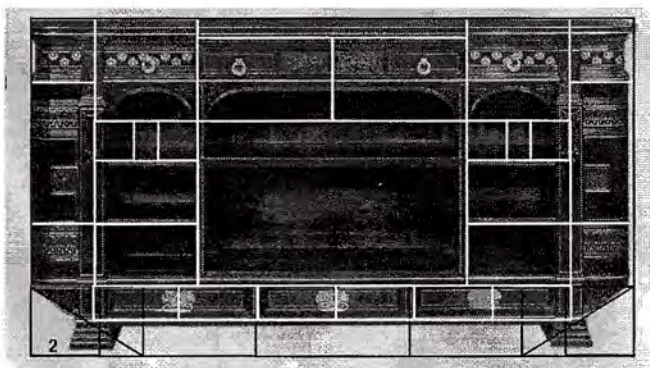


Herter Brothers Furniture

During the late nineteenth century (from 1864 to 1906), Herter Brothers, the New York firm of the German-born brothers Gustave and Christian Herter, was arguably the leading cabinetmaking and interior decorating firm in the United States. They excelled in complex marquetry, inlaid tile, gilding, and ebonization. Their outstanding work was in great demand by the most affluent clients of this opulent era.

They seem to have been educated in classical geometry, for all their work exhibits the most sophisticated and aesthetically pleasing proportions.

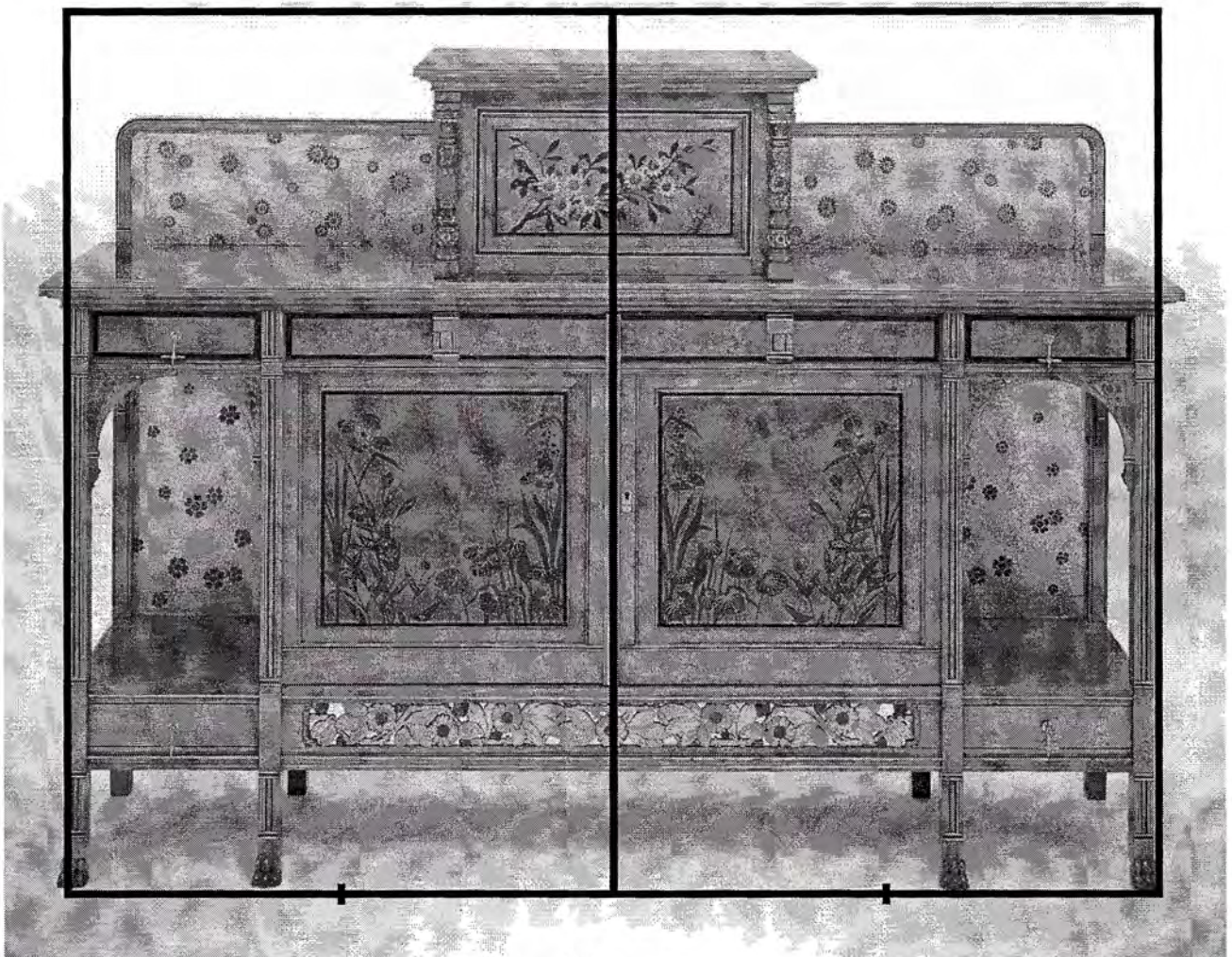
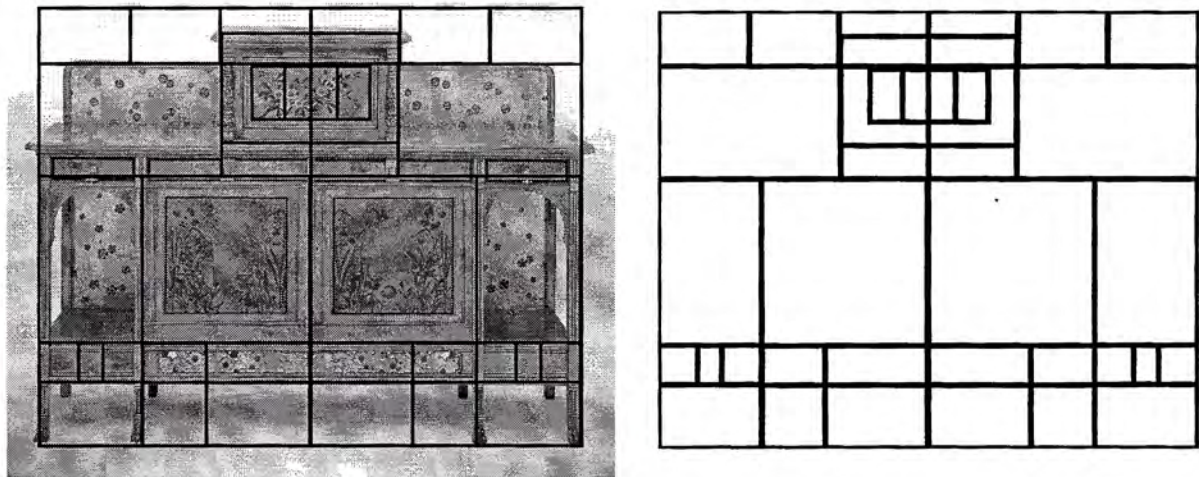
The body of this table, for example, is built along a Root-5 rectangle. The rectangle, central square and two side Golden Rectangles are given below. Use rabatment to discover the rest of the proportioning system applied here.



This Herter Brothers gilded cabinet (c1880, now in the Museum of Fine Arts, Boston) is made of maple wood, bird's-eye maple, oak or chestnut, stamped and gilt paper, with gilding, inlay, carved decoration, and brass pulls.

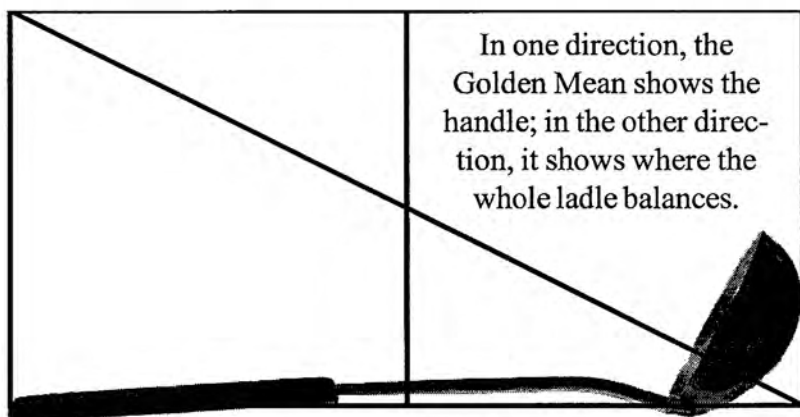
Its geometry can be understood within the frame of two upright Golden Rectangles.

Use rabatment and Golden subdivisions of the square to see the details of its design.



Everyday Household Objects

To this day, objects are still designed either intentionally or intuitively using Golden Mean harmony. Very often, they include objects and tools used by the hand and the whole body, becoming integrated with them. Find the Golden Mean and use rabatment to explore these familiar examples.

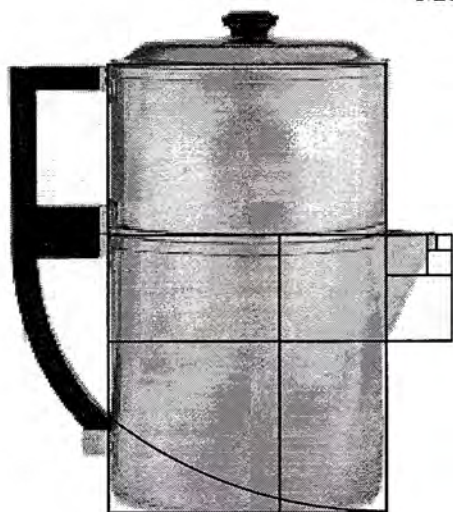


In one direction, the Golden Mean shows the handle; in the other direction, it shows where the whole ladle balances.

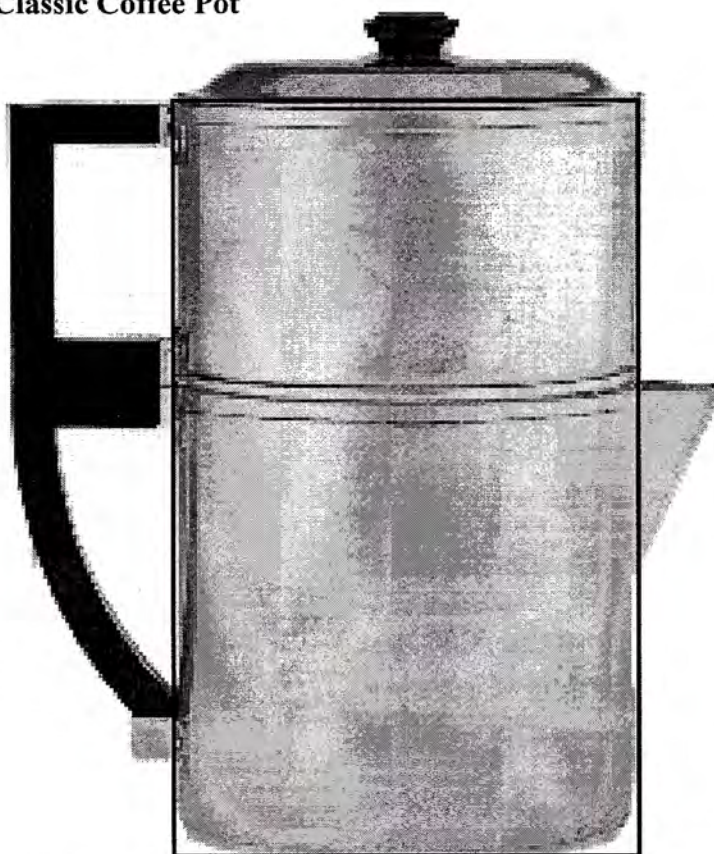
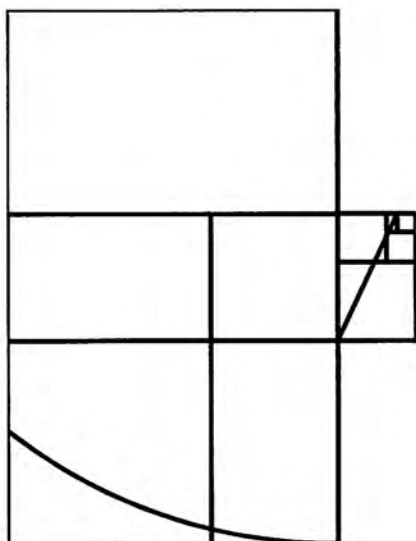
Modern Soup Ladle



Eames Chair
(the first to stack!)



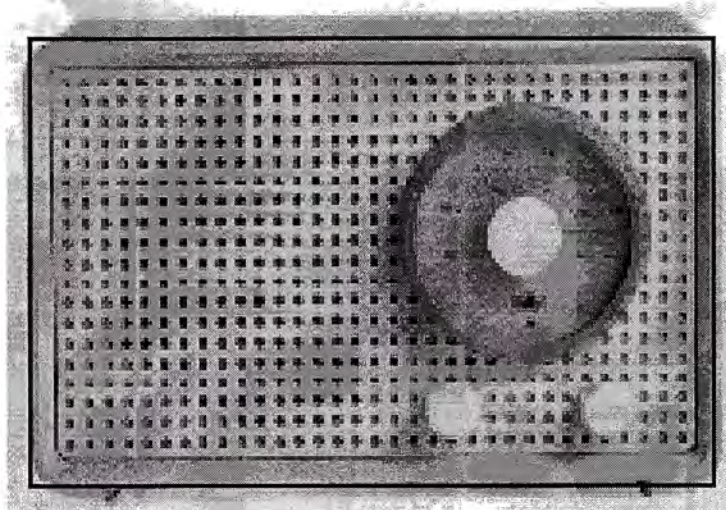
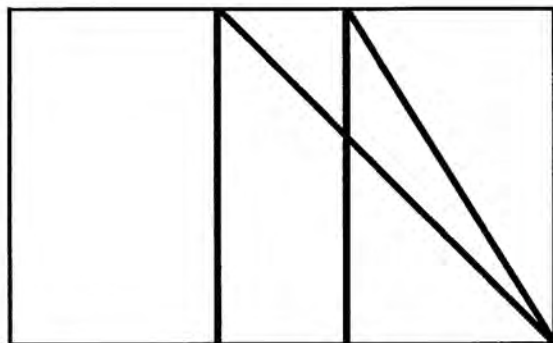
Classic Coffee Pot



Desktop Radio

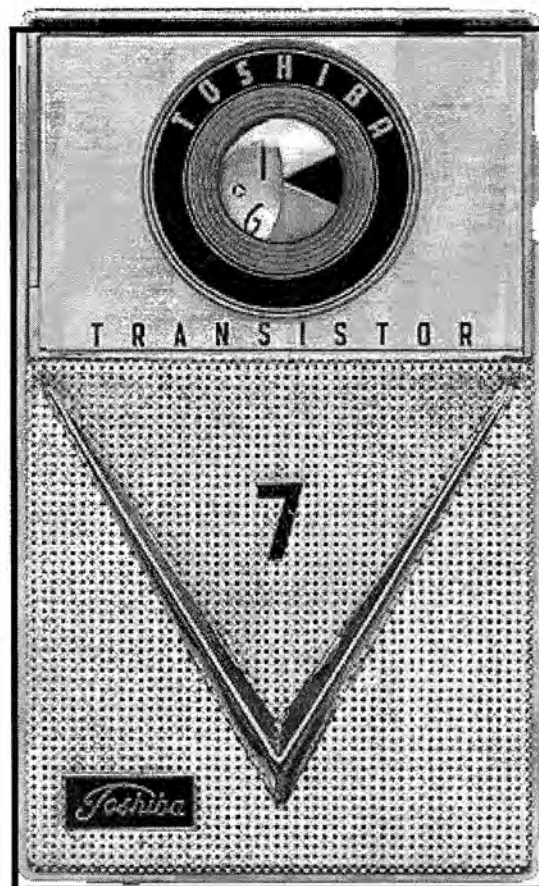
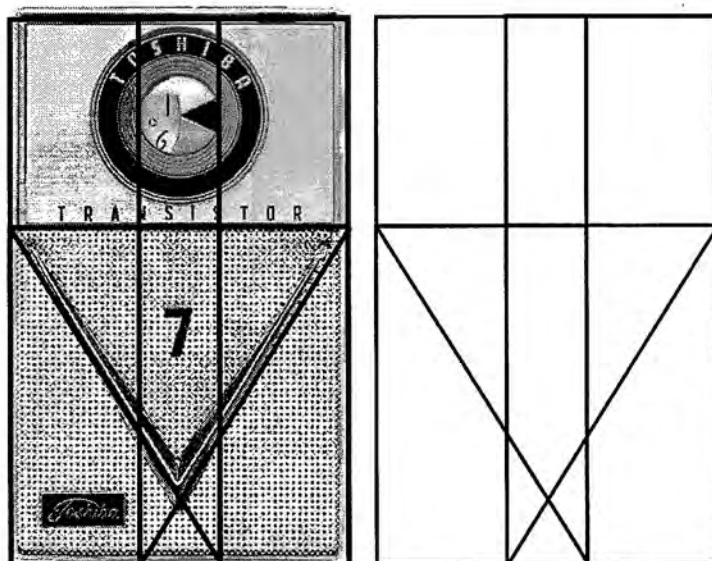
This simple, classic radio from the 1950's has a Golden Rectangle front.

Apply rabatment and diagonals to understand some of its design.



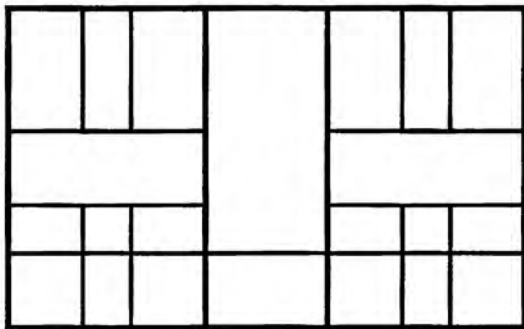
Early Transistor Radio

This early hand-held transistor radio from the late 1950's touts seven transistors. Its front is also a Golden Rectangle. Draw a square from the bottom over the speaker below. Use rabatment in the upper Golden Rectangle to mark out squares, then extend the lines downward and draw diagonals to help understand the placement of the "V" design.



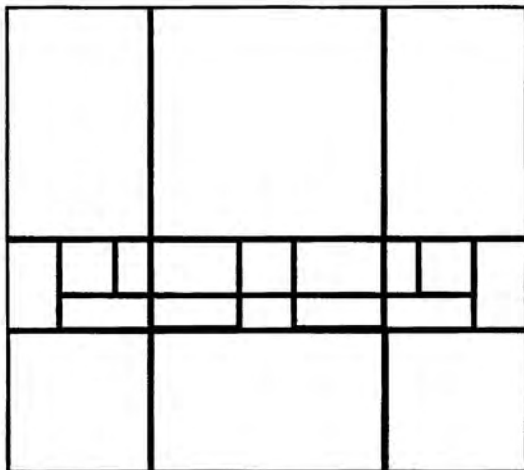
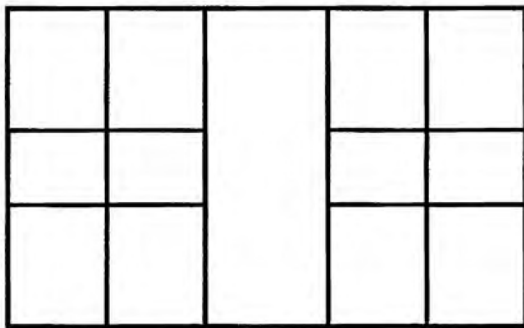
Credit Cards

Standard sized credit cards are 86mm by 54mm, creating a ratio of 1.592, about a millimeter from a Golden Rectangle of 1.618.

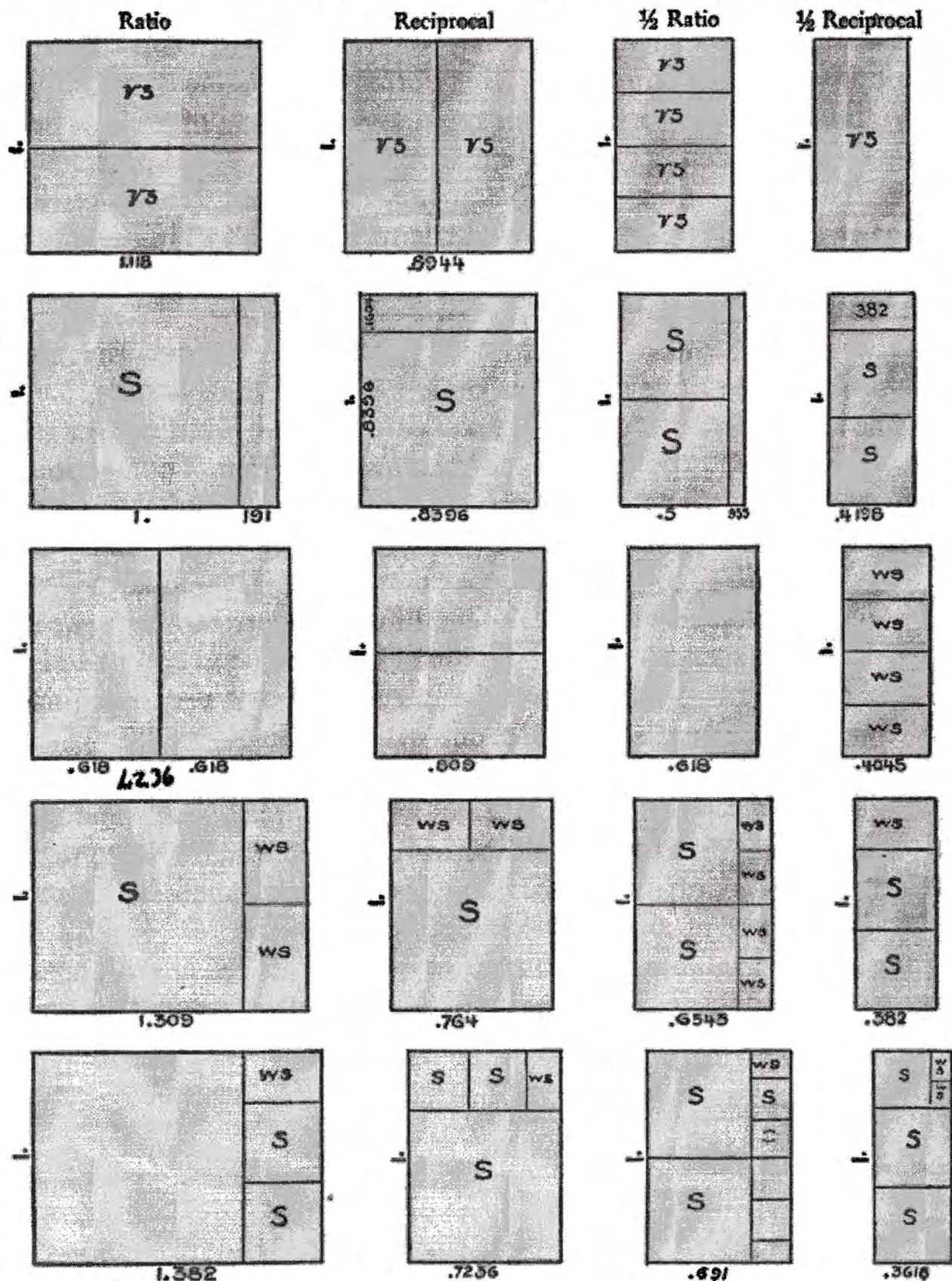


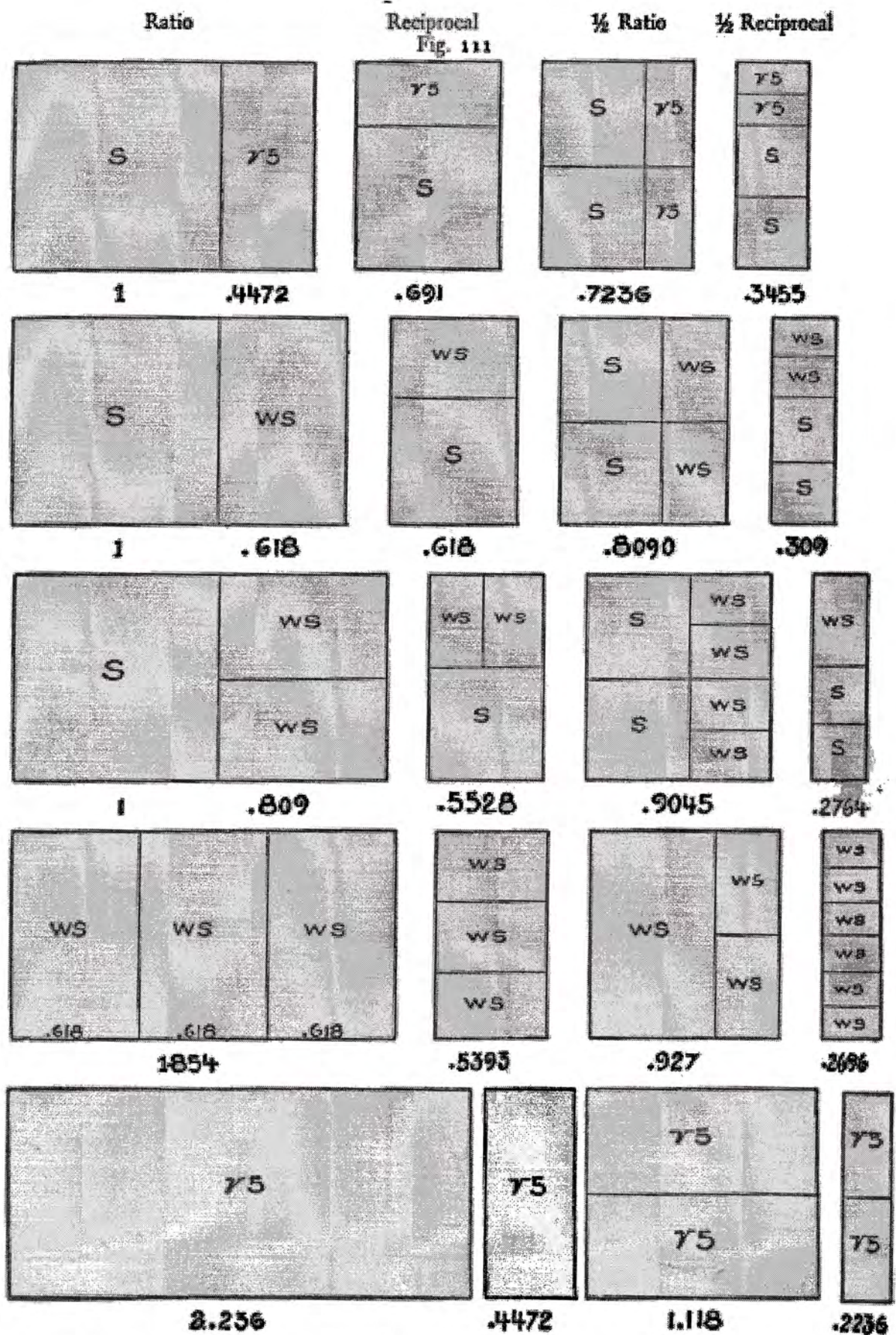
Candy Wrappers

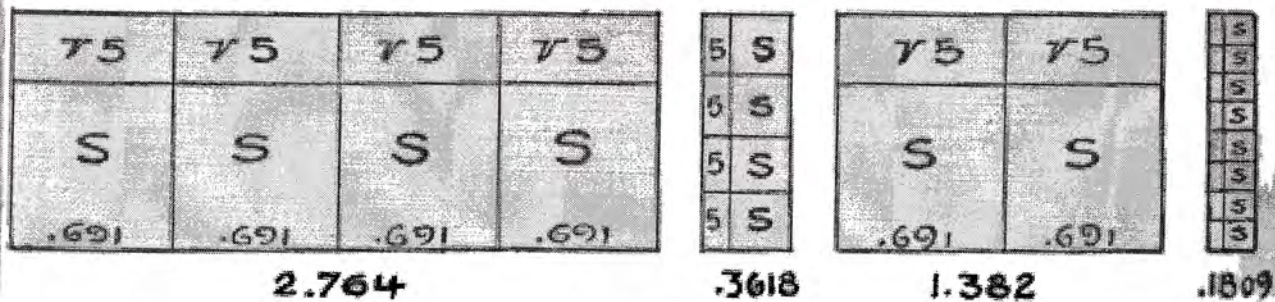
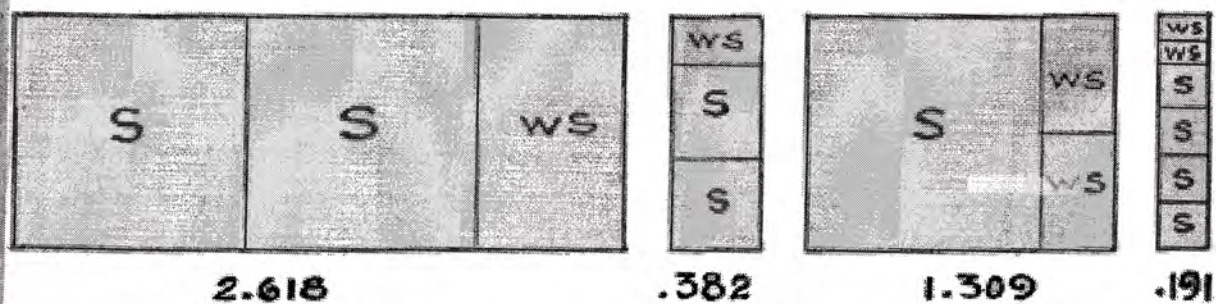
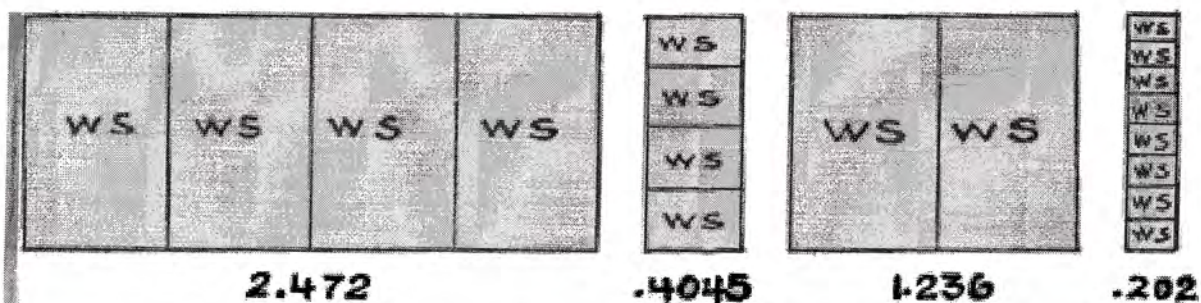
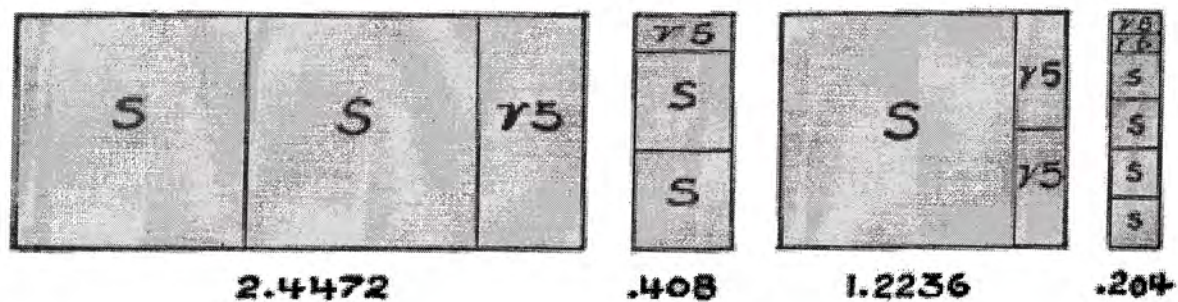
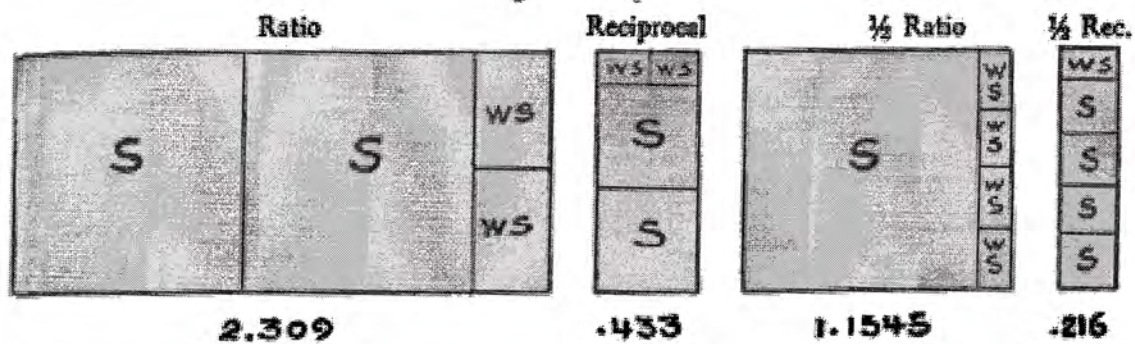
Use rabatment to discover how these wrappers integrate words into the design.

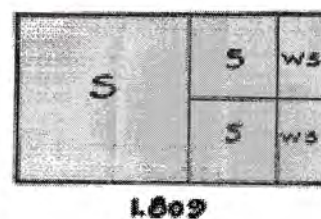
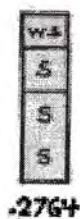
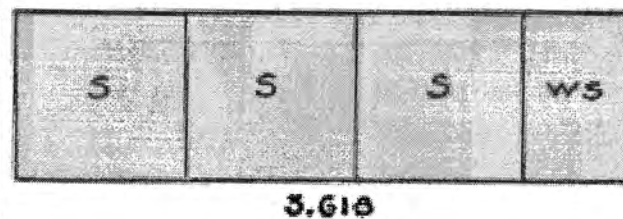
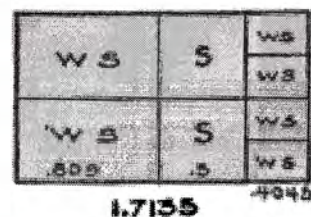
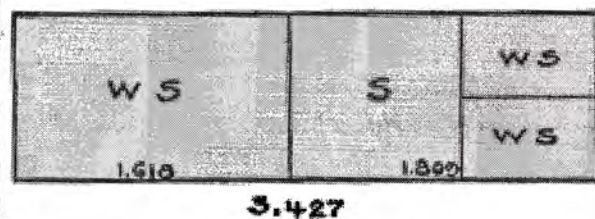
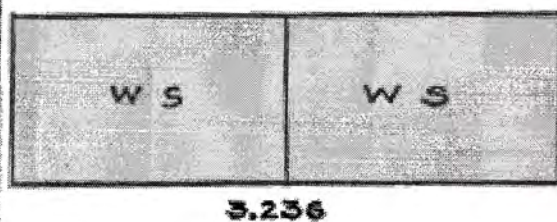
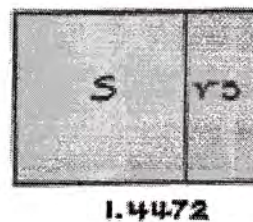
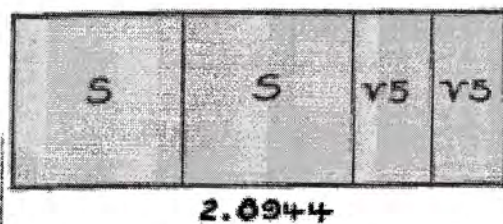
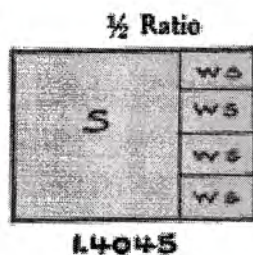
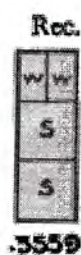
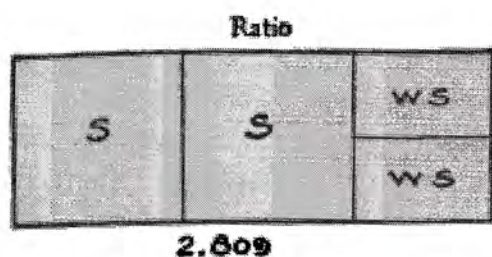


Here's a catalog of rectangles related to the Golden Rectangle (symbolized here by a WS for "Whirling Squares") from "The Elements Of Dynamic Symmetry" by Jay Hambidge. Many of these have been used by architects, craftspeople, architects and designers of various cultures and times.









Websites and Books

Fibonacci Numbers, the Golden Section and the Golden String

<http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html>

An award winning educational website. Very informative.

Fibonacci Numbers In Nature Slide Show

<http://britton.disted.camosun.bc.ca/fibslide/jbfbfslide.htm>

The Fibonacci Association

<http://www.mscs.dal.ca/Fibonacci/>

Byrne's Edition of Euclid

<http://www.sunsite.ubc.ca/DigitalMathArchive/Euclid/byrne.html>

ZomeTool Kits

<http://www.zometool.com/>

Michael S. Schneider's website:

<http://www.constructingtheuniverse.com>

Donald in Mathmagic Land (Disney 1959)

Buena Vista Home Video 1991

ASIN: 6301017129

Animation in VHS and possibly other formats

Very entertaining and educational! Donald is a tourist in the world of mathematics, includes the Golden Mean.

Here are some **books for continued research.**

A Beginner's Guide To Constructing The Universe -- Michael S. Schneider (HarperPerennial 1995)

A Collection of Manuscripts Related to the Fibonacci Sequence -- Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson [Editors] (The Fibonacci Association 1980)

A Mathematical History of the Golden Number -- Roger Herz-Fischler (Dover Publications 1998)

A Mathematical Mystery Tour -- Mark Wahl (Zephyr Press 1988)

A Primer for the Fibonacci Numbers -- Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson [Editors] (The Fibonacci Association)

Curves Of Life, The: Being an Account of Spiral Formations and Their Application to Growth in Nature, to Science and to Art; With Special Reference to the Manuscripts of Leonardo Da Vinci -- T. A. Cook (Dover Publications 1979)

Diagonal, The (Journal published at Yale 1919-21 by Jay Hambidge)

Divine Proportion, The -- H. E. Huntley (Dover Publications 1970)

Dynamic Symmetry: A Primer--Christine Herter (W. W. Norton & Co. 1966)

Dynamic Symmetry: The Greek Vase -- Jay Hambidge (Kessinger Publishing 2003)

Elements of Dynamic Symmetry, The -- Jay Hambidge (1919, Dover Publications 1967)

Esthetique Des Proportions Dans La Nature Et Dans Les Arts -- Ghyka, Matila (Gallimard, 1933)

Fascinating Fibonacci -- Trudi Hammel Garland (Dale Seymour Publications 1987)

Fibonacci and Lucas Numbers -- Verner E. Hoggatt, Jr. (Houghton Mifflin 1969)

Fibonacci Fun -- Trudi Hammel Garland (Dale Seymour Publications 1997)

Fibonacci Numbers -- N. N. Vorob'ev (Pergamon Press, Ltd. 1961)

Geometry of Art and Life, The -- Matila Ghyka (Dover Publications 1977)

Geometry of the Greek Vase -- L. D. Caskey (Museum of Fine Arts, Boston 1922)

The Golden Ratio: The Story of PHI, the World's Most Astonishing Number -- Mario Livio (Broadway Publications 2003) -- [mentions my discovery of the Golden Mean in the Babylonian Stele of Shamash]

Golden Section, The -- Garth E. Runion (Dale Seymour Publications 1990)

Nature's Harmonic Unity: A Treatise on Its Relation to Proportional Form -- Colman, Samuel (Benjamin Blom, Inc, 1912, 1971)

On Growth and Form (unabridged) -- d'Arcy W. Thompson (Dover Publications 1992)

The Painter's Secret Geometry: A Study of Composition in Art -- Charles Bouleau (Hacker Art Books 1980)

The Power of Limits: Proportional Harmonies in Nature, Art, and Architecture -- Gyorgy Doczi (Shambhala 1981)

Sacred Geometry -- Miranda Lundy (Wooden Books/Walker & Company 2001)

Sacred Geometry: Philosophy and Practice -- Robert Lawlor (Thames & Hudson 1989)