PART V

ON THE NON-ARISTOTELIAN LANGUAGE CALLED MATHEMATICS

Once a statement is cast into mathematical form it may be manipulated in accordance with these rules and every configuration of the symbols will represent facts in harmony with and dependent on those contained in the original statement. Now this comes very close to what we conceive the action of the brain structures to be in performing intellectual acts with the symbols of ordinary language. In a sense, therefore, the mathematician has been able to perfect a device through which a part of the labor of logical thought is carried on outside the central nervous system with only that supervision which is requisite to manipulate the symbols in accordance with the rules. (583) HORATIO B. WILLIAMS

The toughminded suggest that the theory of the infinite elaborated by the great mathematicians of the Nineteenth and Twentieth Centuries without which mathematical analysis as it is actually used today is impossible, has been committing suicide in an unnecessarily prolonged and complicated manner for the past half century. (22) E. T. BELL

The solution goes on famously; but just as we have got rid of the other unknowns, behold ! *V* disappears as well, and we are left with the indisputable but irritating conclusion—

$$0 = 0$$

This is a favourite device that mathematical equations resort to, when we propound stupid questions. (149) A. S. EDDINGTON

Who shall criticize the builders ? Certainly not those who have stood idly by without lifting a stone. (23) E. T. BELL

. . . let me remind any non-mathematicians . . . that when a mathematician lays down the elaborate tools by which he achieves precision in his own domain, he is unprepared and awkward in handling the ordinary tools of language. This is why mathematicians always disappoint the expectation that they will be precise and reasonable and clear-cut in their statements about everyday affairs, and why they are, in fact, more fallible than ordinary mortals. (529) OSWALD VEBLEN

CHAPTER XVIII

MATHEMATICS AS A LANGUAGE OF A STRUCTURE SIMILAR TO THE STRUCTURE OF THE WORLD

To-day there are not a few physicists who, like Kirchhoff and Mach regard the task of physical theory as being merely a mathematical description (*as economical as possible*) of the empirical connections between observable quantities, *i. e.* a description which reproduces the connection as far as possible, without the intervention of unobservable elements. (466) E. SCHRÖDINGER

But in the prevalent discussion of classes, there are illegitimate transitions to the notions of a 'nexus' and of a 'proposition.' The appeal to a class to perform the services of a proper entity is exactly analogous to an appeal to an imaginary terrier to kill a real rat. (578) A. N. WHITEHEAD

Roughly it amounts to this: mathematical analysis as it works today must make use of irrational numbers (such as the square root of two); the sense if any in which such numbers exist is hazy. Their reputed mathematical existence implies the disputed theories of the infinite. The paradoxes remain. Without a satisfactory theory of irrational numbers, among other things, Achilles does not catch up with the tortoise, and the earth cannot turn on its axis. But, as Galileo remarked, it does. It would seem to follow that something is wrong with our attempts to compass the infinite. (22)E. T. BELL

The map is not the thing mapped. When the map is identified with the thing mapped we have one of the vast melting pots of numerology. (604) E. T. BELL

The theory of numbers is the last great uncivilized continent of mathematics. It is split up into innumerable countries, fertile enough in themselves, but all more or less indifferent to one another's welfare and without a vestige of a central, intelligent government. If any young Alexander is weeping for a new world to conquer, it lies before him. (23) E. T. BELL

The present work—namely, the building of a *non-aristotelian system,* and an introduction to a theory of sanity and general semantics—depends, fundamentally, for its success on the recognition of mathematics as a language similar in structure to the world in which we live.

The maze of often unconnected knowledge we have gathered in the fields with which this part is dealing is so tremendous that it would require several volumes to cover the field even partially. Under such conditions, it is impossible to deal with the subject in any other way than by very careful selection, and so I shall, therefore, say only as much as is necessary for my present semantic purpose.

It is a common experience of our race that with a happy generalization many unconnected parts of our knowledge become connected; many 'mysteries' of science become simply a linguistic issue, and then the mysteries vanish. New generalizations introduce new *attitudes* (evaluation) which, as usual, seriously simplify the problems for a new generation. In the present work, we are treating problems from the point of view of such a generalization, of wide application; namely, *structure,* which is forced upon us by the denial of the 'is' of identity. ; so that structure becomes the only link between the objective and verbal levels. The next consequence is that structure alone is the only possible content of knowledge.

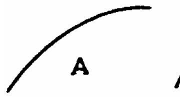
Investigating structure, we have found that structure can be defined in terms of relations; and the latter, for special purposes, in terms of multi-dimensional order. Obviously, to investigate structure, we must look for relations, and so for multi-dimensional order. The full application of the above principles becomes our guide for future enquiry.

In the recent past, we have become accustomed to such arguments as, for instance, that the theory of Einstein has to be accepted on ‘epistemological’ grounds. Naturally, the scientist or the layman who has heard the last term, but never bothered to ascertain that it means ‘according to the structure of human knowledge’, would recognize no necessity to accept something which violates all his habitual *s.r,* for reasons about which he does not know or care. But if we say that the Einstein theory has to be accepted, for the ‘time’ being, at least, as an irreversible *structural linguistic progress,* this statement carries for many quite a different verbal and semantic implication, and one worth considering.

Mathematics has, of late, become so extremely elaborate and complex that it takes practically a lifetime to specialize in even one of its many fields. Here and there notions of extreme creative generality appear, which help us to see relations and dependence between formerly non-connected fields. For instance, the arithmetization of mathematics, or the theory of groups, or the theory of aggregates, has each become such a supreme generalization. At present, there is a general tendency among all of us, scientists included, to confuse orders of abstractions. This results in a psycho-logical semantic blockage and in the impossibility of seeing broader issues clearly.

Some of the structural issues are still but little understood, and, in writing this chapter, I lay myself open to a reproach from the layman that I have given too much attention to mathematics, and from the professional mathematician that I have given too little. My reply is that what is said here is necessary for rounding up the semantic foundations of the system, and that I explain only enough to carry the main points of structure and as semantic suggestions for further semantic researches.

I have found that among some physicists and some mathematicians the thesis that mathematics is the only language which, in 1933, is similar in structure to the world, is not always acceptable. As to the second thesis, the similarity of its structure to our nervous system, some even seem to feel that this statement borders on the sacrilegious ! These objectors, apparently, believe that I ascribe more to mathematics than is just. Some physicists point out to me the non-satisfactory development of mathematics, and they seem to confuse the inadequacy of a given mathematical theory with the general *m.o* structure of mathematics. Thus, if some physical experimental investigation is conducted—for instance, on high pressure—and the older theories predict a behaviour exemplified by the curve (A), while the experimental new data show that the actual curve is (B), such a result would show unmistakably that the first theory is not structurally correct. But, in itself, this result does not affect the correctness of a statement about the general structure of mathematics which can account for *both curves.*



FIG

Until very lately, we had a very genuine problem in physics with the quantum phenomena which seem to proceed by discrete steps, while our mathematics is fundamentally based on assumptions of continuity. Here we had seemingly a serious structural discrepancy, which, however, has been satisfactorily overcome by the wave theory of the newer quantum mechanics, explained in Part X, where the discontinuities are accounted for, in spite of the use of differential equations and, therefore, of continuous mathematics.

But, if we start with fundamental assumptions of continuity, we always can account for discontinuities by introducing wave theories or some similar devices. Therefore, it is impossible, in our case, to argue from the wave theory (for instance) to the structure of mathematics, or vice versa, without a fundamental and *independent* general structural analysis, which alone can elucidate the problem at hand.

Mathematicians may object on the ground that the new revision of the foundation of mathematics, originated by Brouwer and Weyl, challenges the ‘existence’ of irrational numbers. , and, therefore, destroys the very foundations of continuity and the legitimacy of existing mathematics.

In answer to such a criticism, we should notice, first, that the current ‘continuity’ is of two kinds. One is of a higher grade, and is usually called by this name; the other continuity is of a lower grade and is usually called ‘compactness’. The new revision challenges the higher continuity, but does not affect compactness, which, as a result, will, perhaps, have to suffice in the future for all mathematics, since compactness is sufficient to meet all psycho-logical requirements, once the problems of ‘infinity’ are properly understood.

A structural independent analysis of mathematics, treated as a language and a form of human behaviour, establishes the similarity of this language to the undeniable structural characteristics of this world and of the human nervous system. These few and simple structural foundations are arrived at by inspection of known data and may be considered as well established.

The existing definitions of mathematics are not entirely satisfactory. They are either too broad, or too narrow, or do not emphasize enough the main characteristics of mathematics. A semantic definition of mathematics should be broad enough to cover all existing branches of mathematics; should be narrow enough to exclude linguistic disciplines which are not considered mathematical by the best judgement of specialists, and should also be *flexible* enough to remain valid, no matter what the future developments of mathematics may be.

I have said that mathematics is the only language, at present, which in structure, is similar to the structure of the world *and* the nervous system. For purposes of exposition, we shall have to divide our analysis accordingly, remembering, in the meantime, that this division is, in a way, artificial and optional, as the issues overlap. In some instances, it is really difficult to decide under which division a given aspect should be analysed. The problems are very large, and for full discussion would require volumes; so we have to limit ourselves to a suggestive sketch of the most important aspects necessary for the present investigation.

From the point of view of general semantics, mathematics, having symbols and propositions, must be considered as a language. From the psychophysiological point of view, it must be treated as an activity of the human nervous system and as a form of the behaviour of the organisms called humans.

All languages are composed of two kinds of words: (1) Of *names* for the somethings on the un-speakable level, be they external objects. , or *internal feelings*, which admittedly are *not* words, and (2) of *relational terms*, which express the actual, or desired, or any other relations between the un-speakable entities of the objective level.

When a ‘quality’ is treated physiologically as a reaction of an organism to a stimulus, it also becomes a relation. It should be noticed that often some words can be, and actually are, used in both senses; but, in a given context, we can always, by further analysis, separate the words used into these two categories. Numbers are not exceptions; we can use the labels ‘one’, ‘two’. , as numbers (of which the character will be explained presently) but also as names for anything we want, as, for instance, Second or Third Avenue, or John Smith I or John Smith II. When we use numbers as names, or labels for anything, we call them numerals; and this is *not* a mathematical use of ‘one’, ‘two’. , as these names do not follow mathematical rules. Thus, Second Avenue and Third Avenue cannot be added together, and do not give us Fifth Avenue in any sense whatever.

Names alone do not produce propositions and so, by themselves, say nothing. Before we can have a proposition and, therefore, meanings, the names must be related by some relation-word, which, however, may be explicit or implied by the context, the situation, by established habits of speech, . The division of words into the above two classes may seem arbitrary, or to introduce an unnecessary complication through its simplicity; yet, if we take modern knowledge into account, we cannot follow the grammatical divisions of a primitive-made language, and such a division as I have suggested above seems structurally correct in 1933.

Traditionally, mathematics was divided into two branches: one was called arithmetic, dealing with numbers; the other was called geometry, and dealt with such entities as ‘line’, ‘surface’, ‘volume’, . Once Descartes, lying in bed ill, watched the branches of a tree swaying under the influence of a breeze. It occurred to him that the varying distances of the branches from the horizontal and vertical window frames could be expressed by numbers representing measurements of the distances. An epoch-making step was taken: geometrical relations were expressed by numerical relations; it meant the beginning of analytical geometry and the unification and arithmetization of mathematics.

Further investigation by the pioneers Frege, Peano, Whitehead, Russell, Keyser, and others has revealed that ‘number’ can be expressed in ‘logical terms’—a quite important discovery, provided we have a valid ‘logic’ and structurally correct *non-el* terms.

Traditionally, too, since Aristotle, and, in the opinion of the majority, even today, mathematics is considered as uniquely connected with quantity and measurement. Such a view is only partial, because there are many most important and fundamental branches of mathematics which have nothing to do with quantity or measurement—as, for instance, the theory of groups, analysis situs, projective geometry, the theory of numbers, the algebra of ‘logic’, .

Sometimes mathematics is spoken about as the science of relations, but obviously such a definition is too broad. If the only content of knowledge is structural, then relations, obvious, or to be discovered, are the foundation of all knowledge and of all language, as stated in the division of words given above. Such a definition as suggested would make mathematics co-extensive with all language, and this, obviously, is not the case.

Before offering a semantic definition of mathematics, I introduce a synoptical table taken from Professor Shaw’s *The Philosophy of Mathematics*, which he calls only suggestive and ‘doubtless incomplete in many ways’. I use this table because it gives a modern list of the most important mathematical terms and disciplines necessary for the purpose of this work, indicating, also, in a way, their evolution and structural interrelations.

CENTRAL PRINCIPLES OF MATHEMATICS*

[figure]

*This table differs slightly from the one printed in *The Philosophy of Mathematics*. The corrections have been made by Professor Shaw and kindly communicated to me by letter.

A semantic definition of mathematics may run somehow as follows: Mathematics consists of limited linguistic schemes of multiordinal relations capable of exact treatment at a given date.

After I have given a semantic definition of number, it will be obvious that the above definition covers all existing disciplines considered mathematical. However, these developments are not fixed affairs. Does that definition provide for their future growth ? By inserting as a fundamental part of the definition ‘exact treatment at a given date’, it obviously does. Whenever we discover any relations in any fields which will allow exact ‘logical’ treatment, such a discipline will be included in the body of linguistic schemes called mathematics, and, at present, there are no indications that these developments can ever come to an end. When ‘logic’ becomes an ∞-valued ‘structural calculus’, then mathematics and ‘logic’ will merge completely and become a general science of *m.o* relations and multi-dimensional order, and all sciences may become exact.

It is necessary to show that this definition is not too broad, and that it eliminates notions which are admittedly non-mathematical, without invalidating the statement that the content of all knowledge is structural, and so ultimately relational. The word ‘exact’ eliminates non-mathematical relations. If we enquire into the meaning of the word ‘exact’, we find from experience that this meaning is not constant, but that it varies with the date, and so only a statement ‘exact at a given date’ can have a definite meaning.

We can analyse a simple statement, ‘grass is green’ (the ‘is’ here is the ‘is’ of predication, not of identity) , which, perhaps, represents an extreme example of a non-mathematical statement; but a similar reasoning can be applied to other examples. Sometimes we have a feeling which we express by saying, ‘grass is green’. Usually, such a feeling is called a ‘perception’. But is such a *process* to be dismissed so simply, by just calling it a name, ‘perception’ ? It is easy to ‘call names under provocation’, as Santayana says somewhere; but does that exhaust the question ?

If we analyse such a statement further, we find that it involves comparison, evaluation in certain respects with other characters of experience. , and the statement thus assumes relational characteristics. These, in the meantime, are non-exact and, therefore, non-mathematical. If we carry this analysis still further, involving data taken from chemistry, physics, physiology, neurology. , we involve relations which become more and more exact, and, finally, in such terms as ‘wave-length’, ‘frequency’. , we reach structural terms which allow of exact 1933 treatment. It is true that a language of ‘quality’ conceals relations, sometimes very effectively; but once ‘quality’ is taken as the reaction of a given organism to a stimulus, the term used for that ‘quality’ becomes a name for a very complex relation. This procedure can be always employed, thus establishing once more the fundamental character of relations.

These last statements are of serious structural and semantic importance, being closely connected with the $\bar{A}$, fundamental, and undeniable negative premises. These results can be taught to children very simply; yet this automatically involves an entirely new and modern method of evaluation and attitude toward language, which will affect beneficially the, as yet entirely disregarded, *s.r.*

We must consider, briefly, the terms ‘kind’ and ‘degree’, as we shall need them later. Words, symbols. , serve as forms of representation and belong to a different universe—the ‘universe of discourse’—since they are not the un-speakable levels we are speaking about. They belong to a world of higher abstractions and not to the world of lower abstractions given to us by the lower nerve centres.

Common experience and scientific investigations (more refined experience) show us that the world around us is made up of absolute individuals, each different and unique, although interconnected. Under such conditions it is obviously optional what language we use. The more we use the language of diverse ‘kind’, the sharper our definitions must be. Psycho-logically, the emphasis is on difference. Such procedure may be a tax on our ingenuity, but by it we are closer to the structural facts of life, where, in the limit, we should have to establish a ‘kind’ for every individual.

In using the term ‘degree’, we may be more vague. We proceed by similarities, but such a treatment implies a fundamental interconnection between different individuals of a special kind. It implies a definite kind of metaphysics or structural assumptions—as, for instance, a theory of evolution. As our ‘knowledge’ is the result of nervous abstracting, it seems, in accordance with the structure of our nervous system, to give preference to the term ‘degree’ first, and only when we have attained a certain order of verbal sharpness to pass to a language of ‘kind’, if need arises.

The study of primitive languages shows that, historically, we had a tendency for the ‘kind’ language, resulting in over-abundance of names and few relation-words, which makes higher analysis impossible. Science, on the other hand, has a preference for the ‘degree’ language, which, ultimately, leads to mathematical languages, enormous simplicity and economy of words, and so to better efficiency, more intelligence, and to the unification of science. Thus, chemistry became a branch of physics, physics, a branch of geometry; geometry merges with analysis, and analysis merges with general semantics; and life itself becomes a physico-chemical colloidal occurrence. The language of ‘degree’ has very important *relational, quantitative,* and *order* implications, while that of ‘kind’ has, in the main, qualitative implications, often, if not always, concealing relations, instead of expressing them.

The current definition of ‘number’, as formulated by Frege and Russell, reads: ‘The number of a class is the class of all those classes that are similar to it’.[1] This definition is not entirely satisfactory: first, because the multiordinality of the term ‘class’ is not stated; second, it is *A*, as it involves the ambiguous (as to the order of abstractions) term ‘class’. What do we mean by the term ‘class’ ? Do we mean an extensive array of absolute individuals, un-speakable by its very character, such as some *seen aggregate,* or do we mean the *spoken definition* or *description* of such un-speakable objective entities ? The term implies, then, a fundamental confusion of orders of abstractions, to start with—the very issue which we must avoid most carefully, as positively demanded by the non-identity principle. Besides, if we explore the world with a ‘class of classes’. , and obtain results also of ‘class of classes’, such procedure throws no light on mathematics, their applications and their importance as a tool of research. Perhaps, it even increases the mysteries surrounding mathematics and conceals the relations between mathematics and human knowledge in general.

We should expect of a satisfactory definition of ‘number’ that it would make the semantic character of numbers clear. Somehow, through long experience, we have learned that numbers and measurement have some mysterious, sometimes an uncanny, importance. This is exemplified by mathematical predictions, which are verified later empirically. Let me recall only the discovery of the planet Neptune through mathematical investigations, based on its action upon Uranus, long before the astronomer actually verified this prediction with his telescope. Many, a great many, such examples could be given, scientific literature being full of them. Why should mathematics and measurement be so extremely important ? Why should mathematical operations of a given Smith, which often seem innocent (and sometimes silly enough) give such an unusual security and such undeniably practical results ?

Is it true that the majority of us are born mathematical imbeciles ? Why is there this general fear of, and dislike for, mathematics ? Is mathematics really so difficult and repelling, or is it the way mathematics is treated and taught by mathematicians that is at fault ? If some light can be thrown on these perplexing semantic problems, perhaps we shall face a scientific revolution which might deeply affect our educational system and may even mark the beginning of a new period in standards of evaluation, in which mathematics will take the place which it ought to have. Certainly, there must be something the matter with our epistemologies and 'psychologies' if they cannot cope with these problems.

A simple explanation is given by a new $\bar{A}$ analysis and a *semantic* definition of numbers. What follows is written, in the main, for non-mathematicians, as the word 'semantic' indicates, but it is hoped that professional mathematicians (or some, at least) may be interested in the *meanings* of the term 'number', and that they will not entirely disregard it. As semantic, the definition seems satisfactory; but, perhaps, it is not entirely satisfactory for technical purposes, and the definition would have to be slightly re-worded to satisfy the technical needs of the mathematicians. In the meantime, the gains are so important that we should not begrudge any amount of labour in order to produce finally a mathematical and, this time, $\bar{A}$ semantic definition of numbers.

As has already been mentioned, the importance of notation is paramount. Thus the Roman notation for number—I, II, III, IV, V, VI. ,— was not satisfactory and could not have led to modern developments in mathematics, because it did not possess enough positional and structural characteristics. Modern mathematics began when it was made possible by the invention or discovery of positional notation. We use the symbol '1' in 1, 10, 100, 1000. , in which, because of its place, it had different values. In the expression '1', the symbol means 'one unit'; in 10, the symbol '1' means ten units; in 100, the symbol '1' means one hundred units, .

To have a positional notation, we need a symbol '0', called zero, to indicate an empty column and, at least, one symbol '1'. The number of special symbols for 'number' depends on what base we accept. Thus, in a binary system, with the base 2, our 1 is represented by 1; our 2, by a unity in the second place and a zero in the first place, thus by 10; our 3, by 11; our 4, by a unity in the third place and two zeros in the first and second places; namely, 100; our 5, by 101; our 6, by 110; 7, by 111; 8, by 1000; 9, by 1001; . . . 15, by 1111; 16, by 10000, . In a binary system, we needed only the two symbols, 1 and 0. For a system with the base 3, we would need three symbols, 1, 2, 0: our 1 would be represented by 1; our 2, by 2; 3, by 10; 4, by 11; 5, by 12; 6, by 20, . In our decimal system, obviously, we need 10 symbols, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0.

For more details on notation, the interested reader is referred to the fascinating and elementary book of Professor Danzig, *Number the Language of Science*. Here we only emphasize what is necessary for our purpose. Every system has its advantages and drawbacks. Thus, in the binary system, still used by some savage tribes, of which we retain traces when we speak of couples, or pairs, or braces, we get an enormous simplicity in operations by using only two symbols, 1 and 0. It should be remembered that, in every system, the tables of addition and multiplication must be memorized. In the binary system, these tables are reduced to $1+1=10$ and $1\times1=1$, while in our decimal system, each table has 100 entries. But what we gain in simplicity by a low base-number is offset very seriously by the cumbersomeness of the notation. As Danzig tells us, our number 4096 is represented in a binary system by 1,000,000,000,000. That we adopted the decimal system is probably a physiological accident, because we have ten fingers. The savage, with his binary system, did not reach even the finger stage; he is still in the fist stage.

For practical purposes, it is simpler to have a base which has many divisors, as, for instance, 12. We still use this duodecimal system when we divide a foot into twelve inches, or a shilling into twelve pennies, or count by dozens or gross. It seems that mathematicians would probably select a prime number for a base, but the gain would be so slight and the difficulty of offsetting a physiological habit so tremendous, that this will probably never happen.[2]

From what was said already, it is, perhaps, clear that mathematics requires a positional notation in which we must have a symbol for ‘1’ and zero, at least. For these and other reasons, the two numbers 1 and 0 are somehow unusually important. Even in our decimal system we generate numbers by adding 1 to its predecessor. Thus $1+1=2$, $2+1=3.$, and we must enquire into the semantic character of these numbers.

The notions of matching, comparing, measuring, quantity, equality. , are all interwoven and, by necessity, involve a circularity in definitions and implications if the analysis is carried far enough. The interested reader may be referred to the chapter on equality in Whitehead’s *The Principle of Relativity* to learn more on this subject.

In the evolution of mathematics, we find that the notions of ‘greater’, ‘equal’, and ‘less’ precede the notion of numbers. Comparison is the simplest form of evaluation; the first being a search for relations; the second, a discovery of exact relations. This process of search for relations and structure is inherent and natural in man, and has led not only to the discovery of numbers, but also has shaped their two aspects; namely, the cardinal and the ordinal aspects. For instance, to ascertain whether the number of persons in a hall is equal to, greater than, or less than the number of seats, it is enough to ascertain if all seats are occupied and there are no empty seats and no persons standing; then we would say that the number of persons is equal to the number of seats, and a *symmetrical relation* of equality would be established. If all seats were occupied and there were some persons standing in the hall, or if we found that no one was standing, yet not all seats were occupied, we would establish the *asymmetrical relation* of greater or less.

In the above processes, we were using an important principle; namely, that of *one-to-one* correspondence. In our search for relations, we assigned to each seat one person, and reached our conclusions without any counting. This process, based on the *one-to-one* correspondence, establishes what is called the cardinal number. It gives us specific relational data about this world; yet it is not enough for counting and for mathematics. To produce the latter, we must, first of all, establish a definite system of symbolism, based on a definite relation for generating numbers; for instance, 1+1=2, 2+1=3. , which establishes a definite *order*. Without this ordinal notion, neither counting nor mathematics would be possible; and, as we have already seen, order can be used for defining relations, as the notions of relation and order are interdependent. Order, also, involves asymmetrical relations.

If we consider the two most important numbers, 0 and 1, we find that in the accepted symbolism, if $a=b$, $a-b=0$; and if $a=b$, $a/b=b/b=1$; so that both fundamental numbers express, or can be interpreted as expressing, a *symmetrical relation* of equality.

If we consider any other number—and this applies to all kinds of numbers, not only to natural numbers—we find that any number is not altered by dividing it by one, thus, 2/1=2, 3/1=3. , in general, $N/1=N$; establishing the *asymmetrical* relation, *unique* and *specific* in a given case that N is N times more than one.

If we consider, further, that 2/1=2, 3/1=3, *and so on,* are all *different, specific,* and *unique,* we come to an obvious and $\bar{A}$ semantic definition of number in terms of relations, in which 0 and 1 represent *unique* and *specific symmetrical* relations and all other numbers also *unique* and *specific asymmetrical* relations. Thus, if we have a result '5', we *can always say* that the number 5 is five times as many as one. Similarly, if we introduce apples. Five apples are five times as many as one apple. Thus, a number in any form, 'pure' or 'applied', can always be represented as a relation, *unique* and *specific* in a given case; and this is the foundation of the exactness of dealing with numbers. For instance, to say that *a is* greater than *b* also establishes an asymmetrical relation, but it is *not unique* and *specific;* but when we say that *a* is five times greater than b, this relation is *asymmetrical, exact, unique,* and *specific.*

The above simple remarks are not entirely orthodox. That 5/1=5 is very orthodox, indeed; but that numbers, in general, represent indefinitely many *exact, specific,* and *unique,* and, in the main, *asymmetrical* relations is a structural notion which necessitates the revision of the foundations of mathematics and their rebuilding on the basis of new semantics and a future structural calculus. When we say 'indefinitely many', this means, from the reflex point of view, 'indefinitely flexible', or 'fully conditional' in the semantic field, and, therefore, a prototype of human semantic reactions (see Part VI). The scope of the present work precludes the analysis of the notion of the lately disputed 'irrational'; but we must state that this revision requires new psycho-logical and structural considerations of fundamental 'logical' postulates and of the problems of 'infinity'. If, by an arbitrary process, we *postulate* the existence of a 'number' which alters all the while, then, according to the definition given here, such expressions should be considered as functions, perhaps, but not as a number, because they do not give us *unique* and *specific relations.*

These few remarks, although suggestive to the mathematician, do not, in any way, exhaust the question, which can only be properly presented in technical literature in a postulational form.

It seems that mathematicians, no matter how important the work which they have produced, have never gone so far as to appreciate fully that they are willy-nilly producing an ideal human relational language of structure similar to that of the world *and* to that of the human nervous system. This they cannot help, in spite of some vehement denials, and their work should also be treated from the semantic point of view.

Similarly with measurement. From a functional or actional and semantic point of view, measurement represents nothing else but a search for *empirical structure* by means of extensional, ordered, symmetrical, and asymmetrical relations. Thus, when we say that a given length measures five feet, we have reached this conclusion by *selecting* a unit called 'foot', an *arbitrary and un-speakable* affair, then laying it end to end five times in a definite *extensional order* and so have established the asymmetrical, and, in each case, *unique* and *specific* relation, that the given entity represents, in this case, five times as many as the arbitrarily selected unit.

Objection may be raised that the formal working out of a definition of numbers in terms of relations, instead of classes, would be very laborious, and would also involve a revision of the foundations of mathematics. This can hardly be denied; but, in the discussions of the foundations, the confusion of orders of abstractions is still very marked, thereby resulting in the manufacture of *artificial semantic difficulties.* Moreover, the benefits of such a definition, in eliminating the mysteries about mathematics, are so important that they by far outweigh the difficulties.

As the only possible content of knowledge is structural, as given in terms of relations and multiordinal and multi-dimensional order; numbers, which establish an endless array of exact, specific, and, in each case, unique, relations are obviously the most important tools for exploring the *structure* of the world, since structure can always be analysed in terms of *relations.* In this way, all mysteries about the importance of mathematics and measurement vanish. The above understanding will give the student of mathematics an entirely different and a very natural feeling for his subject. As his only possible aim is the study of structure of the world, or of whatever else, he must naturally use a *relational tool* to explore this complex of relations called ‘structure’. A most spectacular illustration of this is given in the internal theory of surfaces, the tensor calculus. , described in Part VIII.

In all measurements, we select a unit of a necessary kind, for a given case, and then we find a *unique* and *specific* relation as expressed by a number, between the given something and the selected unit. By relating different happenings and processes to the same unit-process, we find, again, *unique* and *specific* interrelations, in a given case, between these events, and so gather structural (and most important, because uniquely possible) wisdom, called ‘knowledge’, ‘science’, .

If we treat numbers as relations, then fractions and all operations become relations of relations, and so relations of higher order, into the analysis of which we cannot enter here, as these are, of necessity, technical.

It should be firmly grasped, however, that some fundamental human relations to this world have not been changed. The primitive may have believed that words *were* things (identification) and so have established what is called the ‘magic of words’ (and, in fact, the majority of us still have our *s.r* regulated by some such unconscious identifications); but, in spite of this, the primitive or ‘civilized’ man’s words *are not*, and never could be, the things spoken about, no matter what semantic disturbances we might have accompanying their use, or what delusions or illusions we may cherish in respect to them.

At present, of all branches of mathematics the theory of numbers is probably the most difficult, obscure. , and seemingly with the fewest applications. With a new $\bar{A}$ definition of numbers in terms of relations, this theory may become a relational study of very high order, which, perhaps, will some day become the foundation for epistemology and the key for the solution of all the problems of science and life. In the fields of cosmology many, if not the majority, of the problems, by necessity, cannot be considered as directly experimental, and so the solution must be epistemological.

At present, in our speculations, we are carried away by words, disregarding the simple fact that speaking about the 'radius of the universe', for instance, has *no meaning*, as it cannot possibly be observed. Perhaps, some day, we shall discover that such conversations are the result of our old stumbling block, identification, which leads to our being carried away by the sounds of words applicable to terrestrial conditions but meaningless in the very small, as discovered lately in the newer quantum mechanics, and, in the very large, as applied to the cosmos. An important illustration of the retardation of scientific progress, blocked by the confusion of orders of abstractions, is shown in the fact that the newer quantum mechanics were slow in coming, and though astronomers probably know about it, yet they still fail to grasp that expressions such as the 'radius of the universe', the 'running down of the universe'. , are meaningless outside of psychopathology.

In this connection, we should notice an extremely interesting and important semantic characteristic; namely, that the term 'relation' is not only multiordinal but also *non-el*, as it applies to 'senses' and 'mind'. Relations are usually found empirically; so in a language of relations we have a language of similar structure to the world and a unique means for predictability and rationality.

Let me again emphasize that, from time immemorial, things have *not* been words; the only content of knowledge has been structural; mathematics has dealt, in the main, with numbers; no matter whether we have understood the character of numbers or not, numbers have expressed relations and so have given us structural data willy-nilly, . This explains why mathematics and numbers have, since time immemorial, been a favorite field, not only for speculations, but also why, in history, we find so many religious semantic disturbances connected with numbers. Mankind has somehow felt instinctively that in numbers we have a potentially endless array of *unique* and *specific exact relations*, which ultimately give us structure, the last being the only possible content of knowledge, because words are not things.

As relations, generally, are empirically present, and as man and his ‘knowledge’ is as ‘natural’ as rocks, flowers, and donkeys, we should not be surprised to find that the unique language of exact. , relations called mathematics is, by necessity, the natural language of man and *similar in structure* to the world *and* our nervous system.

As has already been stated, it is incorrect to argue from the structure of mathematical theories to the structure of the world, and so try to establish the similarity of structure; but that such enquiry must be *independent* and start with quite ordinary structural experiences, and only at a later stage proceed to more advanced knowledge as given by science. Because this analysis must be independent, it can also be made very simple and elementary. All exact sciences give us a wealth of experimental data to establish the first thesis on similarity of structure; and it is unnecessary to repeat it here. I will restrict myself only to a minimum of quite obvious facts, reserving the second thesis—about the similarity of structure with our nervous system—for the next chapter.

If we analyse the silent objective level by objective means available in 1933, say a microscope, we shall find that whatever we can see, handle. , represents an *absolute individual*, and *different from anything else in this world.* We discover, thus, an important *structural* fact of the external world; namely, that in it, everything we can see, touch. , that is to say, all lower order abstractions represent absolute individuals, different from everything else.

On the verbal level, under such empirical conditions, we should then have a language of *similar structure;* namely, one giving us an indefinite number of *proper names, each different.* We find such a language *uniquely* in numbers, each number 1, 2, 3. , being a unique, sharply distinguishable, *proper name* for a relation, and, if we wish, for anything else also.

Without some higher abstractions we cannot be human at all. No science could exist with absolute individuals and no relations; so we pass to higher abstractions and build a language of say x_i, (i=1, 2, 3, . . . n), where the x shows, let us say, that we deal with a variable x with many values, and the number we assign to i indicates the individuality under consideration. From the structural point of view, such a vocabulary is similar to the world around us; it accounts for the individuality of the external objects, it also is similar to the structure of our nervous system, because it allows generalizations or higher order abstractions, emphasizes the abstracting nervous characteristics, . The subscript emphasizes the differences; the letter x implies the similarities.

In daily language a similar device is extremely useful and has very far-reaching psycho-logical semantic effects. Thus, if we say ‘pencil_1’, ‘pencil_2’, . . . ‘pencil_n’, we have indicated structurally two main characteristics: (1) the absolute individuality of the object, by adding the indefinitely individualizing subscript 1, 2, . . . *n*; and (2) we have also complied with the nervous higher order abstracting characteristics, which establish similarity in diversity of different ‘pencils’. From the point of view of *relations,* these are usually found empirically; besides, they may be invariant, no matter how changing the world may be.

In general terms, the structure of the external world is such that we deal always on the objective levels with absolute individuals, with absolute differences. The structure of the human nervous system is such that it abstracts, or generalizes, or integrates. , in higher orders, and so finds similarities, discovering often invariant (sometimes relatively invariant) relations. To have ‘similar structure’, a language should comply with both structural exigencies, and this characteristic is found in the mathematical notation of x_i, which can be enlarged to the daily language as ‘Smith_i’, ‘Fido_i’, . , where i=1, 2, 3, . . . *n*.

Further objective enquiry shows that the world and ourselves are made up of *processes,* thus, ‘Smith_{1900}’ is quite a different person from ‘Smith_{1933}’. To be convinced, it is enough to look over old photographs of ourselves, the above remark being structurally entirely general. A language of ‘similar structure’ should cover these facts. We find such a language in the vocabulary of ‘function’, ‘propositional function’, as already explained, involving also four-dimensional considerations.

As words are not objects—and this expresses a structural fact—we see that the ‘is’ of identity is unconditionally false, and should be entirely abolished as such. Let us be simple about it. This last semantic requirement is genuinely difficult to carry out, because the general *el* structure of our language is such as to facilitate identification. It is admitted that in some fields some persons identify only a little; but even they usually identify a great deal when they pass to other fields. Even science is not free from identification, and this fact introduces great and artificial semantic difficulties, which simply vanish when we stop identification or the confusion of orders of abstractions. Thus, for instance, the semantic difficulties in the foundations of mathematics, the problems of ‘infinity’, the ‘irrational’. , the difficulties of Einstein’s theory, the difficulties of the newer quantum theory, the arguments about the ‘radius of the universe’, ‘infinite velocities’, the difficulties in the present theory. , . , are due, in the main, to semantic blockages or commitment to the structure of the old language—we may call it ‘habit’—*which says structurally very little,* and which I disclose as a semantic disturbance of *evaluation* by showing the *physiological mechanism in terms of order.*

If we abolish the 'is' of identity, then we are left only with a functional, actional. , language elaborated in the mathematical language of function. Under such conditions, a *descriptive* language of ordered happenings on the objective level takes the form of 'if so and so happens, then so and so happens', or, briefly, 'if so, then so'; which is the prototype of 'logical' and mathematical processes and languages. We see that such a language is again similar in structure to the external world descriptively; yet it is similar to the 'logical' nervous processes, and so allows us, because of this similarity of structure, predictability and so rationality.

In the traditional systems, we did not recognize the complete semantic interdependence of differences and similarities, the empirical world exhibiting differences, the nervous system manufacturing primarily similarities, and our 'knowledge', if worth anything at all, being the *joint product* of both. Was it not Sylvester who said that 'in mathematics we look for similarities in differences and differences in similarities' ? This statement applies to our whole abstracting process.

The empirical world is such in structure (by inspection) that in it we can add, subtract, multiply, and divide. In mathematics, we find a language of similar structure. Obviously, in the physical world these actions or operations alter the relations, which are expressed as altered unique and specific relations, by the language of mathematics. Further, as the world is full of different shapes, forms, curves. , we do not only find in mathematics special languages dealing with these subjects, but we find in analytical geometry unifying linguistic means for translation of one language into another. Thus any 'quality' can be formulated in terms of relations which may take the 'quantitative' character which, at present, in all cases, can be also translated into geometrical terms and methods, giving structures to be *visualized.*

It is interesting, yet not entirely unexpected, that the activities of the higher nervous centres, the conditional reflexes of higher order, the semantic reactions, time-binding included, should follow the exponential rules, as shown in my *Manhood of Humanity.*

In our experience, we find that some issues are additive—as, for instance, if one guest is added to a dinner party, we will have to add plates and a chair. Such facts are covered by additive methods and the language called 'linear' (see Part VIII). In many instances—and these are, perhaps, the most important and are strictly connected with sub-microscopic processes—the issues are not additive, one atom of oxygen ‘plus’ two atoms of hydrogen, under proper conditions, will produce water, of which the characteristics are not the sum of the characteristics of oxygen and hydrogen ‘added’ together, but entirely *new* characteristics *emerge.* These may some day be taken care of by non-linear equations, when our knowledge has advanced considerably. These problems are unusually important and vital, because with our present low development and the lack of structural researches, we still keep an additive *A* language, which is, perhaps, able to deal with additive, simple, immediate, and comparatively unimportant issues, but is entirely unfit structurally to deal with principles which underlie the most fundamental problems of life. Similarly, in physics, only since Einstein have we begun to see that the primitive, simplest, and easiest to solve linear equations are not structurally adequate.

One of the most marked structural characteristics of the empirical world is ‘change’, ‘motion’, ‘waves’, and similar dynamic manifestations. Obviously, a language of similar structure must have means to deal with such relations. In this respect, mathematics is unique, because, in the differential and integral calculus, the four-dimensional geometries and similar disciplines, with all their developments, we find such a perfect language to be explained more in detail in the chapters which follow.

It will be profitable for our purpose to discuss, in the next chapter, some of the mathematical structural characteristics in connection with their similarity to the human nervous system; but here I will add only that, for our purposes, at this particular point, we must specially emphasize *arithmetical* language, which means numbers and arithmetical operations, the theory of function, the differential and integral calculus (language) and different geometries in their two aspects, ‘pure’ and applied. Indeed, Riemann tells us bluntly that the *science* of physics originated only with the introduction of differential equations, a statement which is quite justified, but to which I would add, that physics is becoming scientific since we began to eliminate from physics semantic disturbances; namely, identification and elementalism. This movement was originated, in fact, although not stated in an explicit form, by the Einstein theory and the new quantum theories, the psychological trend of which is formulated in a *general semantic theory* in the present work.

It is reasonable to consider that metric geometry, and, in particular, the [E]-system, was derived from touch, and, perhaps, the ‘kinesthetic sense’, and projective geometry from sight.

Although the issues presented here appear extremely simple, and sometimes even commonplace, yet the actual working out of the verbal schemes is quite elaborate and ingenious, and impossible to analyse here more fully; so that only one example can be given.

The solution of mathematical equations is perhaps to be considered as the central problem of mathematics. The word ‘equation’ is derived from Latin *aequare*, *to* equalize, and is a statement of the symmetrical relation of equality expressed as $a=b$ or a-b=0 or a/b=1. An equation expresses the relation between quantities, some of which are known, some unknown and to be found. By the solution of an equation, we mean the finding of values for the unknowns which will satisfy the equation.

Linear equations of the type $ax=b$ necessitated the introduction of fractions. Linear equations with several variables led to the theory of determinants and matrices. , which underwent, later, a tremendous independent development; yet they originated in the attempt to simplify the solution of these equations.

Quadratic equations of the type $x^2+ax+b=0$ can be reduced to the form $z^2=A$, the solution of which depends on the extraction of a square root. Here, serious difficulties arose, and seemingly necessitated the introduction of ‘irrational’ numbers and ordinary complex numbers, involving the square root of minus one ($\sqrt{-1} = i$), a notion which revolutionized mathematics.

Cubic equations of the form $x^3+ax^2+bx+c=0$ necessitated the extraction of cube roots, in addition to the problems connected with the solution of quadratic equations, and involved more difficulties, which have been analysed in an extensive literature.

Biquadratic equations of the type $x^4+ax^3+bx^2+cx+d=0$ involve the problems of quadratic and cubic equations. When we consider equations of a degree higher than the fourth, we find that we cannot solve them by former methods; and mathematicians have had to invent theories of substitutions, groups, different special functions and similar devices. The solution of differential equations introduced further difficulties, allied with the theory of function.

The linear transformations of algebraic polynomials with two or more variables in connection with the theory of determinants, symmetrical functions, differential operations. , necessitated the development of an extensive theory of algebraic forms which, at present, is far from being complete.

In the above analysis, I have refrained from giving details, most of which would be of no value to the layman, and unnecessary for the mathematician; but it must be emphasized that the theory of function and the theory of groups, with their very extensive developments, involving the theory of invariance, and, in a way, the theory of numbers, rapidly became a unifying foundation upon which practically the whole of mathematics is being rebuilt. Many branches of mathematics have become, of late, nothing more than a theory of invariance of special groups.

As to practical applications, there is no possibility to list them, and the number increases steadily. But, without the theory of analytic function, for instance, we could not study the flow of electricity, or heat, or deal with two-dimensional gravitational, electrostatic, or magnetic attractions. The complex number involving the square root of minus one was necessary for the development of wireless and telegraphy; the kinetic theory of gases and the building of automobile engines require geometries of *n* dimensions; rectangular and triangular membranes are connected with questions discussed in the theory of numbers; the theory of groups has direct application in crystallography; the theory of invariants underlies the theory of Einstein, the theory of matrices and operators has revolutionized the quantum theory; and there are other applications in an endless array.[3]

In Part VIII, different aspects of mathematics are analysed, but the interested reader can be referred also to the above-mentioned book of Professor Shaw for an excellent elementary, yet structural, view of the progress of mathematics.