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Preface

The range of intended readership for logic books is wide, and includes computer scientists, philosophers, mathematicians and the lay reader. The aims of these books can vary widely too: some are meant to be read for general interest; others are intended to develop logical thinking; some are meant to give the reader an understanding of logic sufficient to support their professional activities; others are intended to develop the deeper understanding required by the professional logician. There are many different systems of logic, including classical systems based upon natural language and a wide range of systems of symbolic logic. Finally, there are the pedagogical issues to be considered. How should the logical systems be presented and explained in a way that will best achieve the purposes of the book for the intended readership?

Where does this book fit into this scheme? The complete answer to this question can only be ascertained by reading the book in its entirety; nevertheless it is possible to give an overview. The book arose out of the need for a text which would be suitable for graduates from a wide range of disciplines studying on a conversion M.Sc. in computing. Little prior mathematical ability is assumed; furthermore there is a little more emphasis on the relationship to language than many other books of this level, whilst still retaining the importance of formalism. Thus the book is suitable not only for computer scientists beginning a study of logic, but also to those studying logic in philosophy or mathematics. Its purpose is to develop an understanding of the nature and application of symbolic logic, and also of its relationship to language. It sets out to develop the skills of reasoning and an ability to work with abstract formalism; as well as being important in their own right, these will help improve skills of program design and development in the computer scientist. The main logical system developed in the book is that of natural deduction, though necessarily truth semantics are also developed. In addition, there is a brief introduction to automated reasoning and logic programming.

There are a number of distinctive pedagogical features to the book. The material is presented in a carefully explained step-by-step approach with copious worked examples and exercises. The solutions to these exercises are considered to be an integral part of the exposition and so all the solutions have been included in an appendix. One difficulty of teaching any symbolic subject, whether mathematics or formal logic, is that many students feel uncomfortable unless they can attach a name and a meaning to all symbols they encounter; this problem is particularly acute for students who are just beginning their studies. In logic, this problem manifests itself most notably with the conditional connective $\Rightarrow$, and is aggravated by the almost universal habit of introducing this connective at the very beginning. In this book, the conditional is not introduced until after many of the important concepts of logic have already been introduced. At this stage it is now possible to present a deeper exploration of the concept of the conditional connective.

Another difficulty that can arise for many students is understanding the notion of 'term'. Part of the difficulty of this may lie in the confusion surrounding the different uses of the word 'variable'. This book attempts to alleviate this difficulty by introducing the concept of arbitrary constant, as distinct from that of proper constant; and by defining the concept of term using only constants and functions. Although this may seem a little unconventional, there is precedent for it in the books by Lemmon (1965) and Galton (1990). The advantage of this approach, as pointed out by Lemmon, is pedagogical. The disadvantage is that if any readers move on to more advanced work in logic, they will encounter a different definition (and one that is arguably better from a technical point of view). However, any such reader should be able to cope with a variety of formalisms and to see the relationships between them, without being thrown by alternative definitions and use of symbols.

The material is presented in eight chapters. The first chapter considers the need for rules of reasoning and the concept of symbol. Chapter 2 introduces symbolic connectives for conjunction, disjunction and negation, and considers how these can be used in the symbolic representation of compound propositions. A central concept of this book is that of propositional forms (or schemas); this concept is introduced in Chapter 3, together with truth tables and properties of propositional forms. Chapter 4 considers the notion of argument and what constitutes a valid argument; this in turn leads to a consideration of a system of natural deduction involving the three connectives $\neg$, $\wedge$ and $\vee$. The conditional connective is introduced in Chapter 5, where it is seen to be an essentially abstract concept for which any meaning can be given only in terms of logic itself. Chapters 6 and 7 develop first order logic, as distinct from the logic of propositional forms, and how this may be used to build theories. A brief introduction to logic programming is presented in Chapter 8; the intention here is not to give a basic understanding of logic programming, but simply to illustrate that there are alternative systems of logic, and that some of these may be more suitable for automation than natural deduction.

Throughout the book, copious worked examples and exercises are used to aid understanding of the concepts and to develop logical skills. Since the exercises are an integral part of the book, it was felt important to include solutions to all the exercises (Appendix A). Appendices B–E provide useful summaries of the content of the book. Finally a short bibliography is included in Appendix F; this should prove helpful to anyone wishing to study logic further.

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Language, Logic and Symbols 1

1.1 Information and technology

Information and technology are very important in the modern world. Of course, information has been important throughout history, but the way that it has been handled has changed. Initially, the 'technology' was purely oral, but gradually technology developed which enabled greater use of information. Major developments in this technology include: the invention of writing; the invention of printing; and the invention of the electronic computer.

Another major development was that of reasoning; reasoning enables us to start with given facts and to deduce other facts. We do not know how or when reasoning started. However, we do know that to a large extent reasoning depends upon language, and that for most of us it is an essential part of being human. Yet not everyone will agree with an argument which someone else has put forward. This disagreement may arise in two ways. Firstly, there may be disagreement with the given facts; if we start arguing from false information, then any other facts we deduce may not necessarily be true (though they could be). The resolution of such a disagreement requires us to ascertain whether the given facts are true. Secondly, even if the given facts are correct, there may be disagreement as to whether the methods of deduction are valid. Since ancient times it has therefore been found necessary to lay down rules for what is acceptable reasoning. If such rules sometimes seem obvious, remember that we have grown up with these rules as part of our culture and value system. Cultures other than our own may well have different systems of reasoning.

One set of rules for reasoning was laid down by the ancient Greeks over two thousand years ago. They called this system of reasoning 'logic'. This logic forms the basis of reasoning throughout much of Western civilization. It should be noted, however, that the term **logic** is often applied to any system of reasoning; in such circumstances, the term **classical logic** may then be used to refer to the logic of the ancient Greeks.

Until the nineteenth century, reasoning was purely verbal and used normal, everyday language; we shall refer to such language as **natural language**. Since then, much progress has been made on developing **symbolic logic** in which information is represented using letters and special symbols, rather like algebra. Symbolic logic enables a certain degree of automation of reasoning; indeed, its original motivation was a desire to be able to decide a logical problem by 'calculation', just as the answer to a numerical problem can be calculated using arithmetic. For this reason, symbolic logic is closely associated with computers. Indeed, the development of the electronic computer in the twentieth century depended upon developments on symbolic logic. Furthermore computers can be used not only for numerical computation but also for logical 'computation' – such logical computation is important in areas of computing such as artificial intelligence. Thus an understanding of logic is essential for a thorough understanding of modern computing.

This book is about symbolic logic, and how reasoning can be expressed as a set of rules for handling the strings of symbols. We shall consider **rules of deduction** that reflect a natural style of reasoning; the resulting system of logic is often called **natural deduction**. This system of logic will be justified by consideration of **truth values**. Before beginning to look at symbolic logic, however, it will be useful to briefly explore the use of natural language.

1.2 Characteristics of natural language

The language of everyday life we call natural language. Natural language serves many purposes: to express feelings; to give commands; to ask questions; to convey information. Often these purposes are intricately intertwined: '*I love you*' may superficially seem like a mere statement of fact, but when spoken it may convey feeling directly in the way in which it is spoken: '*Do you know that rabbits were introduced to Britain by the Romans?*' is, from a purely grammatical point of view, a question, but yet conveys information. Natural language is often ambiguous, a feature put to good use in literature.

The subtle complexity of everyday language is both a weakness and a strength. Without it, language would be greatly impoverished and much less expressive; yet it hinders the precise handling of information. Where correct information is important, ordinary language is often abandoned. Thus, lawyers use a form of language which most other people find difficult to understand, and which somehow seems lacking in humanity.

Language also influences the way we think. Perhaps a difficulty that many people have with mathematics is that it often conflicts with everyday thinking. A good example is the need to *reduce* the speed of vehicles to *increase* the rate of traffic flow when there is very heavy traffic. This result follows from a branch of mathematics known as queuing theory. Yet ordinary thinking might suggest the opposite to be true.

Logic is concerned with the accurate representation of facts and the correct reasoning about these. There are various ways in which we might represent information precisely. These include:

- the more careful use of everyday language;

- the use of specially adapted language such as legal language or technical jargon;
- the use of symbolic notation such as that used in arithmetic and (more generally) in mathematics.

In logic all three approaches have been used.

Note that the accurate representation of facts does not mean that they are true. Furthermore, even if we reason correctly we can only be sure of a true conclusion if the facts from which we start are true. For example, using precise symbolic notation we could write down

$$2 + 3 = 7$$

to represent the (false) fact that adding two to three gives seven. Using mathematical reasoning we may conclude that

$$(2 + 3)^2 = 7^2$$

Although we have used reasoning that is logically correct, the conclusion is false because we started with false information.

1.3 Connectives

Natural language usually consists of sentences. A sentence may be a command ('Come here and sit down!'), a question ('How many legs does Rex have?') or a **statement** ('Rex has four legs.'). In logic we are concerned with the use of statements to represent information, and reasoning about statements.

Now a statement may represent a simple fact such as '*Rex has four legs*'. Or it may represent several facts connected together in some relationship. For example, the statement

'*Rex has four legs and Fido has three*'

may be analysed into two facts:

- '*Rex has four legs*';
- '*Fido has three legs*'

and a relationship, represented by the word '*and*', which connects these two facts. The word '*and*' is an example of a **connective**.

Note that we have begun to a develop the language needed to talk about logic. In this case we have introduced the word 'connective'. There is, however, a potential source of confusion between this language of logic and that used in other fields. For example, in traditional grammar, the word '*conjunction*' is used where we have used '*connective*'; to make matters worse, the word '*conjunction*' is used in logic with a different meaning. In talking about logic, we shall frequently need to use an everyday word with a new meaning. Care therefore needs to be taken to understand such words in their technical sense when appropriate.

1.4 Semantics

Semantics is concerned with meanings. For example, we need to be able to give meaning to the connective '*and*'. Before we do this, however, we need to look a little more closely at the idea of '*meaning*' itself.

One approach to the meaning of a word is in terms of **denotation** and **connotation.** In grammatical terms, the denotation of a word or a phrase is its literal meaning, its primary meaning – literally, what the word denotes. Thus the denotation of '*sunshine*' is that radiation from the sun that we can see and feel. The precise definition can given in terms of physics, perhaps something along the lines of:

> Electromagnetic radiation in the visible and near infra-red regions of the spectrum which emanates from the sun

Yet to most people, many other ideas are associated with the word; feeling good, relaxation, holidays, summer.... These associated ideas of a word or a phrase are called the connotation of that word or phrase. Connotation plays an important part in the ordinary use of language, in giving emotive power to words beyond their literal meaning. In logic, however, we strip away such additional meanings; the result is that the subject of logic can seem somewhat cold and artificial.

1.5 Truth values

In logic, semantics is based upon **truth values.** Usually when we make a statement of fact, there is an implicit assumption that the fact is correct; that it is true. However this may *not* be the case. Perhaps we set out to deliberately lie as I did above when writing down

$$2 + 3 = 7$$

Perhaps we are mistaken; many people outside Australia might write

> '*Sydney is the capital of Australia*'

believing this statement to be true. Perhaps we are *proposing* a fact for which we do not know the truth value.

> '*Humanity will be destroyed during the next 500 years*'

In order to emphasize the possibility that a stated fact may be either true or false, we introduce the word **proposition.** Usually we shall talk about propositions rather than statements. A proposition which does not include any connective is called an **atomic proposition**, A proposition which is built up from atomic propositions using one or more connectives is called a **compound proposition.**

Suppose we want to decide the truth value of the compound proposition

‘*Rex has four legs and Fido has three legs*’.

In logic the connective ‘*and*’ is such that the compound proposition is true whenever the separate propositions are themselves true; furthermore, whenever the separate propositions are true, then so is the compound proposition. As far as logic is concerned this *is* the meaning of ‘*and*’.

We accept the fact that in logic, we lose some of the subtler nuances of natural language. For example, the sentences

- ‘*Rex has four legs and Fido has three*’
- ‘*Rex has four legs but Fido has three*’

convey subtly different shades of meaning, yet in logical terms they have the same semantics: Consider the use of ‘*but*’. In

‘*Rex has four legs but Fido has three*’

we can identify two components:

- ‘*Rex has four legs*’;
- ‘*Fido has three legs*’.

Furthermore, we can see that

- whenever the compound proposition is true, each component must be true;
- whenever each component is true, the compound proposition must be true.

Consideration of the truth values tells us that we have a conjunction of the two components, and hence that we can write down

‘*Rex has four legs and Fido has three legs*’.

As far as logic is concerned ‘*but*’ is just another way of saying ‘*and*’; subtle nuances of natural language are frequently lost when we work with logic.

1.6 Conjunction, Disjunction and Negation

Conjunction is just one connective that we use in logic. In this section we shall define two other connectives – disjunction and negation – in terms of truth values. To begin with, however, we shall recap on the definition of conjunction.

Conjunction

Definition 1.1

The conjunction of two propositions is true if and only if the individual propositions are both true. □

If a compound proposition with two components is true precisely when the components are both true, then we have a conjunction. We could write the complete proposition as the first component, followed by the word '*and*', followed by the second component. Alternatively we could use some other conjunction word such as '*but*' to make the English sound reasonable.

Disjunction

The situation frequently arises where any one of a number of factors can lead to a particular situation. For example, suppose that a doctor believes that a particular disease is caused either by too much salt in the diet or by too much alcohol, and that there is no other cause. Then if a patient has the disease, the doctor will assert

'*Your diet contains too much salt or you drink too much*'.

This proposition can be analysed as

- '*Your diet contains too much salt*'
- '*Your drink too much*'

connected by '*or*'. Now if the doctor's assertion is true, then clearly *at least one* of the two component propositions must be true, possibly both. From another point of view, we can see that the compound proposition will only be false if both the component propositions are false. (Presumably because there is another possible cause of the disease not known to the doctor.) This truth analysis is different to that for conjunction; the relationship between the component propositions in this case is called a **disjunction.**

Definition 1.2

The disjunction of two propositions is false if and only if the individual propositions are both false. Alternatively, the disjunction of two propositions is true if and only if at least one of the individual propositions is true. □

Great care must be taken with using '*or*' in a sentence. The proposition '*The disease is caused by too much salt or too much drink*' could be analysed in terms of the two component propositions:

- '*The disease is caused by too much salt*'
- '*The disease is caused by too much drink*'

which are connected by '*or*'. But do we have a disjunction? The answer depends upon what meaning we intend. Suppose, for example, that research had narrowed down the possible causes of the disease to '*too much salt*' and '*too much drink*', then we are asserting that at least one of the two component propositions is true; in these circumstances, we would indeed have a disjunction.

Now consider the compound proposition

'*Mr. Smith went to see his doctor on Thursday or Friday*'

This proposition can be analysed into two atomic propositions:

- '*Mr. Smith went to see his doctor on Thursday*'
- '*Mr. Smith went to see his doctor on Friday*'

What is meant by the original statement? The English is rather ambiguous. It *could* mean that the doctor was possibly seen on both days; in that case the compound proposition would be true if at least one of the atomic propositions were true, and hence the compound proposition would be a disjunction. But there is another possibility, namely that the doctor was seen on one day only. With this second interpretation, the original statement would be false if both the atomic propositions were true. Clearly in this case we would not have a disjunction. This latter use of the word '*or*' is said to be the **exclusive or**; disjunction is often referred to as the **inclusive or.**

Negation

In language we often want to express that a certain fact is wrong; that the opposite holds. For example we might want to say

'*Sydney is not the capital of Australia*'.

We say that this proposition is the **negation** of the proposition

'*Sydney is the capital of Australia*'.

Negation can be expressed by the words

'*It is not the case that* ...'

so in this case a more formal statement of the negated proposition might be

'*It is not the case that Sydney is the capital of Australia*'.

Extending the ideas of conjunction and disjunction, we refer to negation as a connective (even though there is nothing to connect). Negation represents a relation between the original and the negated propositions: if the original is false, the negated proposition will be true. Indeed, because the negated proposition is asserting that original statement is incorrect, it can *only* be true if the un-negated proposition is false.

Definition 1.3

The negation of a proposition is true if and only if that proposition is false.

□

1.7 Reasoning in natural language

Language not only enables us to express known facts, it also enables us to reveal further facts by reasoning. For example, suppose we know

- '*Shouting is bad for opera singers*'
- '*Amelia is an opera singer*'

then we could reason that '*Shouting is bad for Amelia*'. We call the two original facts **premisses**; the additional fact we have obtained is called the **conclusion**. This ability to reveal facts by reasoning is a powerful feature of language. Now, although most people would probably accept the above argument as valid, there may be dispute over other arguments. For example suppose we also know that

- '*Shouting is bad for opera singers*'
- '*Shouting is bad for Bert*'.

Using everyday language and thinking we might possibly infer that

'*Bert is an opera singer*'.

Is this argument valid? Many people would think not; they would argue that there might be some other reason why shouting is bad for Bert.

A major purpose for logic is to set down rules for determining whether or not a particular argument is **valid**. The first argument would be considered valid, while the second would be considered invalid. It was this kind of difficulty which led to the development of **rules of deduction**; use of such rules ensures that arguments are valid. Rules of deduction in symbolic logic will be introduced and explained in the following chapters, but for the moment we shall consider a few rules expressed in terms of natural language. Even when reasoning in natural language it is still important to have such rules to ensure that our arguments are valid.

Reasoning with conjunction

Suppose we have the following *separate* propositions:

- '*Rex has four legs*';
- '*Fido has three legs*'.

From these two separate propositions we can argue that

'*Rex has four legs and Fido has three legs*'.

We could justify this argument by invoking common sense. However, we can also justify the argument by using the semantics of conjunction. From definition 1.1 we see that the conjunction of two propositions is true whenever both the component propositions are true.

In general, the conjunction of two premisses can always be deduced from those premisses. This last sentence is in fact a rule of deduction stated in ordinary language. To make such rules of deduction easier to understand, however, logicians have for a long time made use of letters. For example, the rule of deduction we have just met could more clearly be written as:

Rule:
Given premisses '*P*' and '*Q*' then we can deduce '*P and Q*'. □

Note that after applying a rule of deduction, we often modify the wording of the resulting sentence in order to make it sound better. For example we could shorten

'*Rex has four legs and Fido has three legs*'

to something like

'*Rex has four legs and Fido has three*'.

Another possibility is to use the word '*but*' instead of '*and*':

'*Rex has four legs but Fido has three*'.

Thus we can begin to make simple deductions involving conjunction. There are however two more rules of deduction for conjunction, which we shall now look at. Suppose we have as premiss

'*Rex has four legs and Fido has three legs*'

then common sense tells us that we can argue that

'*Rex has four legs*'.

This conclusion is simply the first proposition in the original compound proposition. Likewise we can also deduce

'*Fido has three legs*',

which is simply the second proposition in the original compound proposition.

We know from the semantics of conjunction that when a compound proposition of the form '*P and Q*' is true, then both '*P*' and '*Q*' are also true. Hence we see that we must have the following two rules of deduction.

Rule:
Given premiss '*P and Q*' then we can deduce '*P*' □

Rule:
Given premiss '*P and Q*' then we can deduce '*Q*' □

Reasoning with disjunction

What are the rules of deduction for disjunction? There are two relatively straightforward rules for introducing disjunction '*or*'.

Rule:
Given the premiss '*P*' then we can deduce '*P or Q*' □

Rule:
Given the premiss '*P*' then we can deduce '*Q or P*' □

Again, these rules can be justified in terms of the semantics of disjunction. From definition 1.2, both '*P or Q*' and '*Q or P*' are true when at least one of '*P*' and '*Q*' is true; in particular, both '*P or Q*' and '*Q or P*' are true when '*P*' is true.

For example, if we know that '*Rex has four legs*' then we can deduce

'*Rex has four legs or Rex has a wet nose*'.

At first such a deduction might seem a little strange, and perhaps is not so obvious as a common sense deduction. Nevertheless, it does fit with our definition of disjunction in terms of truth values. If '*Rex has four legs*' is true then it is certainly the case that at least one of

- '*Rex has four legs*'
- '*Rex has got a wet nose*'

is true.

Rules of logical deduction are a powerful tool and enable us to reason correctly. By contrast, common sense reasoning may, on the one hand, lead us to make invalid arguments yet may, on the other hand, prevent us from seeing valid conclusions. The application of rules of deduction is not easy however. The rules can be made a little clearer by using letters to stand for propositions, but more is needed. Following the success of symbolism in mathematics, it was perhaps inevitable that people would try to find a way of reasoning by calculation. It is these attempts which led to the development of what we now call symbolic logic.

1.8 Symbols

In the preceding section we have met the connectives of conjunction and disjunction. Conjunction can be represented by the word '*and*', but we have seen that it may correspond to other words such as '*but*' in natural language. Disjunction can be represented by the word '*or*', but only in the sense of the inclusive or. The use of words for connectives can therefore lead to misunderstandings. In this book we shall use special symbols for connectives.

Mathematics is an area that has long used special symbols such as the numerals 0,1,2,3,... and arithmetic operators $+$, $-$, $\times$, $\div$. Although many people regard mathematics as difficult, mathematical notation was developed over a long period of time in order to make life easy! Most people would find '347' easier to read than '*three hundred and forty seven*'. Likewise, given the problem '*add three hundred and forty seven to two hundred and twenty one*', most people would convert this into symbolic form, $221 + 347$, before obtaining the answer, possibly with the help of a calculator or computer. With practice and familiarity, the same will become true of logic.

The difficulty at first with symbolic logic, however, is that it is necessary to learn fancy new symbols such as $\wedge$ and $\forall$. In order to circumvent the necessity of learning such new symbols, the first attempts at symbolic logic were based upon ordinary arithmetic, using familiar symbols. This led to a form of symbolic logic known as **boolean logic**. In boolean logic, familiar symbols are given new meanings. For example, the symbol 1 is used to represent truth, while the symbol + is used to represent disjunction. The beauty of this approach is that logic can be reduced to calculation using rules that are very similar to ordinary arithmetic and algebra, for example: $1+0 = 0$; and $a(b+c) = ab+ac$. Further details of this approach can be found in Chapter 7.

There are however disadvantages with boolean logic. The rules of boolean logic are not quite the same as ordinary arithmetic; for example, in boolean logic we have $1 + 1 = 1$. Furthermore, we sometimes use ordinary arithmetical equations as propositions; thus we might have occasion to write

$$(2 + 3 = 5) + (2 + 3 = 6)$$

to represent the compound proposition which says that either $2 + 3 = 5$ or $2 + 3 = 6$. This could be confusing. The biggest problem with boolean logic, however, is that it can only be used to tackle the simplest of logical problems. For these reasons, logicians soon decided to introduce new special symbols for logic. This is the approach adopted in this book.

In order to fully understand the notation used in this book, however, it will be useful to consider some particular aspects of mathematical notation.

Constants and variables

Letters can be used in various ways in mathematical formulae. For example, consider the formula which relates the circumference of a circle to its diameter:

$$C = \pi \times D$$

The symbol π is in fact a letter of the Greek alphabet, corresponding to p in the familiar Latin alphabet; in English it is normally pronounced to sound like '*pie*'. It represents a specific number, approximately equal to 3.1415962. And it always has this same value, irrespective of the circle under consideration. We say that π is a **constant**. Numbers are also constants and are represented by special symbols such as 1, 2.7, -0.33. Now the letters C and D represent the circumference and diameter of the circle respectively. Neither letter, however, represents a specific fixed value; the values taken depend upon the particular circle. The letter D take any positive value, and C then has the value given by the formula. These two letters are said to represent **variables.**

It is often important to know whether any letter represents a constant or a variable. For example, if we are writing a computer program then it is often necessary to declare an identifier as a constant or a variable – compilers handle the two situations very differently. One approach is to declare each letter as a constant or variable before it is first used in a mathematical formula. Although this would work, it can be rather tiresome; furthermore, it relies upon the reader to remember which is which. Another approach is to rely upon the context in which the letters are used. For example, in the formula for the circumference of a circle, it is fairly straightforward to decide which are the variables and which is the constant. But in talking about mathematical formulae in general, we may not know what the letters represent. Thus suppose we have the formula

$$y = a \times x$$

in which we have not ascribed meanings to the letters y, a and x. How do we know whether each letter represents a constant or a variable? One approach is to introduce conventions about what letters represent constants and what letters represent variables. A common convention used in mathematics is that letters near the beginning of the alphabet represent constants, while those near the end represent variables. Thus a represents a constant (even though we do not know *what* constant a represents), while x and y represent variables.

Note that in the formula for the circumference of a circle, the Greek letter π is used to represent a special constant. Traditionally, Greek letters are used for special purposes, as are Latin letters in fancy typefaces such as calligraphic ($\mathcal{A}$ and $\mathcal{B}$), or blackletter ($\mathfrak{A}$ and $\mathfrak{B}$). This a a useful tradition, and one that is adopted in this book.

In this book we shall use several conventions for the use of letters. Each convention will be introduced at an appropriate point, but all of them are summarized in appendix B.

Sets and lists

Often we want to refer to a collection of several items. One way of doing this is simply to list the items concerned. For example we might want to refer to the continents of the world:

Africa, Australia, Antarctica, Asia, Europe, N. America, S. America

A problem with this approach is that a list places the items in an order of some kind. Sometimes the order has some significance. Thus in the example above, the continents are listed in alphabetical order. But it is often the case that no significance is intended by the order in which the items are placed; what *is* important is the **set** of items. To indicate that we are concerned only with the set of items and not with the order in which they are listed, we enclose the list in curly braces {...} . Thus

{ *Africa, Australia, Antarctica, Asia, Europe, N. America, S. America* }

represents the *set* of continents. Note that changing the order of items or repeating items results in a different list, but the corresponding set stays the same. Thus

N. America, S. America, Africa, Australia, Antarctica, Asia, Europe

is not the same list as before, while

Africa, Australia, Antarctica, Asia, Asia, Europe, N. America, S. America

is different yet again. However,

{ *N. America, S. America, Africa, Australia, Antarctica, Asia, Europe* }

{ *Africa, Australia, Antarctica, Asia, Asia, Europe, N. America, S. America* }

both represent the same set as before, namely the set of continents.

Often we want to combine the elements of two sets. For example, we can combine $\{London, Paris, Berlin\}$ with $\{Madrid, Paris, Rome, Lisbon\}$ to give the set $\{London, Paris, Berlin, Madrid, Rome, Lisbon\}$. Note that in the combination set, it is necessary for the item *Paris* to appear once only. We can indicate the combination of the two original sets by means of the symbol $\cup$:

$$\{London, Paris, Berlin\} \cup \{Madrid, Paris, Rome, Lisbon\}$$

The combination of two sets in this manner is called **union.**

Compound Propositions 2

2.1 Symbolic connectives

In this section we shall introduce the basic notation for expressing compound propositions using letters and symbolic connectives. Various notational conventions are introduced and some examples given of their use. An exercise follows, which will give you chance to become familiar with the notation.

Notation: $\wedge$

Conjunction is represented by the symbol $\wedge$ (pronounced '*and*'). □

Notation: $\vee$

Disjunction is represented by the symbol $\vee$ (pronounced '*or*'). □

Notation: $\neg$

Negation is represented by the symbol $\neg$ (pronounced '*not*'). □

Thus, for example, we could write $2 + 3 = 5 \vee 3 > 4$ to represent the disjunction of $2 + 3 = 5$ and $3 > 4$. Note that parentheses may be used to make the expression a little clearer: $(2 + 3 = 5) \vee (3 > 4)$. Alternatively, we can use letters to represent the propositions.

Notation: $p, q, r, s, \ldots$

Often, it is convenient to represent propositions by letters; for this purpose we use lower case letters p, q, r, s, perhaps with subscripts as in p_3, q_1 and so on. □

Example 2.1

Suppose p_1 represents $2 + 3 = 5$ and p_2 represents $3 > 4$. Then $p_1 \vee p_2$ represents the disjunction of $2 + 3 = 5$ and $3 > 4$. □

Example 2.2

If p represents '*Rex has four legs*' and q represents '*Fido has three legs*', how could the compound proposition '*Rex has four legs but Fido only three*' be represented symbolically?

Solution

The compound proposition is the conjunction of '*Rex has four legs*' and '*Fido has three legs*'. We can therefore represent it as $p \wedge q$. □

Exercise 1: Symbolic connectives

1. Suppose p_3 represents $\frac{6}{8} = \frac{3}{4}$ and p_4 represents $\frac{12}{16} = \frac{6}{8}$, then what does each of the following represent?

   (a) $\neg p_3$
   (b) $p_4 \wedge p_3$

   Assuming the normal laws of arithmetic, which of these are true?

2. Suppose r represents the proposition '*Edinburgh is the capital of Scotland*' and s represents the proposition '*Cardiff is the capital of Wales*'. Represent symbolically the compound proposition

   '*Edinburgh is the capital of Scotland and Cardiff is the capital of Wales.*'

3. Identify the atomic proposition(s) in each of the following:

   (a) '*Two is either a prime number or an even number.*'
   (b) '*One and one equals two.*'

   Choose suitable letters to represent the atomic propositions and hence write down a symbolic representation of each statement.

2.2 Operators

So far we have been dealing with fairly straightforward examples of compound propositions. For these simple cases, it is reasonable to regard a symbolic connective as a shorthand for a word in natural language. For example:

'*Roses are red*'$\wedge$'*Violets are blue*'

can be read simply as

'*Roses are red and violets are blue.*'

Unfortunately, this approach causes difficulty later on when we come to make deductions from more complex propositions. An alternative way of viewing connectives is needed.

A useful analogy for this alternative approach is to consider the use of arithmetical symbols. The plus symbol $+$ was originally just a shorthand for the word '*and*' (or more precisely, the Latin equivalent '*et*'). This usage is still evident whenever we read an **arithmetic expression** like $1 + 1$ as '*one and one*'. Yet in order to carry out arithmetical calculations or algebraic manipulations, we regard the symbol + as more than just a linguistic shorthand. In an arithmetical expression such as 2 + 3, the + combines two simpler expressions (in this case the numbers 2 and 3) to form a more complex expression. Furthermore, if we have values for the component expressions, then there is a unique value for the compound expression. Thus in this case the compound expression 2 + 3 has a value of 5; normally we would write this as $2 + 3 = 5$. We say that the + is an **operator**, while the operation itself is called **addition**. The component expressions upon which addition operates are called **operands.** For addition, the two operands are placed either side of the + sign; thus we can think of addition as

$$\Box + \Box$$

where the two squares mark the places into which the operands may be placed. What we now have is a **schema** for the addition operator. Other words that are used instead of schema are **form** and **scheme.**

Exercise 2: Arithmetic operators

1. Substitute operands 3 and 1 + 4 into the addition schema $\Box + \Box$. What are the values of the two operands and the resulting expression?
2. The subtraction operator has schema $\Box - \Box$. What expression results from substituting operands of 2 + 3 and 7 − 5 into the first and second squares respectively? What is the value of this expression? What expression results from substituting the same operands into the other squares? Does this new expression have the same value?

2.3 Conjunction as an operator

The same principles can be applied to a connective such as conjunction. Conjunction is an operator which combines two propositions (the operands) to give a more complex proposition, with the two operands place either side of the $\wedge$ symbol:

$$\Box \wedge \Box$$

For example, substituting '*Rex has four legs*' and '*Fido has three legs*' into the conjunction schema gives

'*Rex has four legs*' $\wedge$ '*Fido has three legs*'

Definition 2.1

The two operands of conjunction are known as **conjuncts.** □

Example 2.3

The two conjuncts of $2 + 3 = 5 \wedge 5 > 6$ are $2 + 3 = 5$ and $5 > 6$. □

Now we can associate a truth value with any proposition, just as we can associate a numerical value with an arithmetic expression. In this book we shall use special symbols T and F to represent truth and falsity respectively. (We shall, however, need to define these symbols more carefully, which we shall do in section 3.4.) Furthermore, just as in arithmetic we use an equals sign to indicate that two expressions have the same numerical value, so in logic we have a special symbol to indicate that two logical expressions have the same truth value.

Notation: $=_T$

The symbol $=_T$ is used to indicate that two logical expressions have the same truth value. □

Note that we take the term **logical expression** to include truth values T and F as well as propositions.

Example 2.4

Suppose we know that '*Rex has four legs*' has truth value T but that '*Fido has three legs*' has value F. We could write these facts as

$$\begin{aligned} \text{'}\textit{Rex has four legs}\text{'} &=_T T \\ \text{'}\textit{Fido has three legs}\text{'} &=_T F \end{aligned}$$

Now from our understanding of conjunction, the compound proposition has truth value F. Hence we can write down

'*Rex has four legs*' $\wedge$ '*Fido has three legs*' $=_T F$ □

In general, the truth value of a conjunction depends entirely on the truth values of conjuncts. In the above example, the truth value is F because the conjunction of *any* true proposition with *any* true false proposition is false. We can represent this fact by writing $T \wedge F =_T F$.

Care must be taken in interpreting an expression such as $T \wedge F$. So far, we have use connectives such as $\wedge$ to construct a compound proposition from atomic propositions. Now T and F are truth values, not propositions. Therefore $T \wedge F$ is not a compound proposition, although it does have a truth value. For this reason, some logicians carefully avoid writing such expressions. In section 3.4, however, we shall define the symbols T and F more carefully, and will then be able to justify the use of connectives with truth values. For the time being it is best to imagine that a connective can be used either to form a compound proposition from simpler propositions, or to operate on truth values.

With this understanding, we can now list all the possible cases for the conjunction of truth values as follows:

$$\begin{aligned} T \wedge T &=_T T \\ T \wedge F &=_T F \\ F \wedge T &=_T F \\ F \wedge F &=_T F \end{aligned}$$

This table captures the fact that the conjunction of two propositions is true only when both the component propositions are true.

Example 2.5

Substitute '4 *is an even number*' and '3 *is a prime number*' as the left and right second conjuncts respectively into the conjunction schema $\square \wedge \square$. What are the truth values of each conjunct, and of the resulting compound proposition?

Solution

Substituting '4 *is an even number*' into the first place of the schema $\square \wedge \square$ gives '4 *is an even number*' $\wedge\ \square$. Then substituting '3 *is a prime number*' into the second place gives '4 *is an even number*' $\wedge$ '3 *is a prime number*'. Now '4 *is an even number*' $=_T T$ and '3 *is a prime number*' $=_T T$ so

$$\text{`4 \textit{is an even number}'} \wedge \text{`3 \textit{is a prime number}'} =_T T \wedge T =_T T$$

The compound proposition formed by the conjunction of '4 *is an even number*' and '3 *is a prime number*' is true. □

2.4 Schematic letters

We have so far presented the conjunction schema as $\square \wedge \square$, in which each square represents a place where a proposition may be substituted. This notation is not always convenient however. For example we often need to distinguish between the left and right conjuncts. To overcome this difficulty, we use special letters instead of squares to mark the places where propositions may be substituted.

Notation: $\mathcal{P}, \mathcal{Q}, R \ldots$

Upper case calligraphic letters such as $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$, possibly with subscripts, indicate places in a schema where propositions may be placed. We shall refer to these letters as **schematic letters**. □

For example, we could write the conjunction schema as $\mathcal{P} \wedge \mathcal{Q}$. Now placing $2 + 3 = 5$ into the place marked $\mathcal{P}$ and $7 > 0$ into the place marked $\mathcal{Q}$ would yield $2 + 3 = 5 \wedge 7 > 0$. We say that $2 + 3 = 5$ is an **instance** of $\mathcal{P}$ and that $\mathcal{P}$ is **instantiated** to $2 + 3 = 5$. Likewise $7 > 0$ is an instance of $\mathcal{Q}$ and $2 + 3 = 5 \wedge 7 > 0$ is an instance of $\mathcal{P} \wedge \mathcal{Q}$.

Definition 2.2

If $\mathcal{P}$ and $\mathcal{Q}$ are propositions then $P \wedge \mathcal{Q}$ is the conjunction of $\mathcal{P}$ and $\mathcal{Q}$. The conjunction schema $\mathcal{P} \wedge \mathcal{Q}$ has the following **truth table**:

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$
T	T	T
T	F	F
F	T	F
F	F	F

Thus the only instances of $\mathcal{P} \wedge \mathcal{Q}$ which are true are those for which the instances of both $\mathcal{P}$ and $\mathcal{Q}$ are true. □

Note that we use letters in two distinct ways. Broadly speaking these two ways refer to the concepts of variable and constant discussed in section 1.8.

- Letters such as $\mathcal{P}$ or $\mathcal{Q}$ are schematic and represent places into which propositions may be placed. Schematic letters may be regarded as **propositional variables** which range over all possible propositions.
- Letters such as p and q are used to represent propositions. These letters may be regarded as being **propositional constants**; in any given context, such a letter will refer to a fixed proposition (even though we may not know what that proposition is).

In this book, capital calligraphic letters such as $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ will represent schematic letters, while propositions may be represented using letters such as p, q, r and s. Note that according to this convention, it is possible to take a letter such as p as an instance of a schematic letter such as $\mathcal{P}$.

Example 2.6

Instantiate $\mathcal{P}$ to p_1 and $\mathcal{Q}$ to p_2 in the conjunction schema $\mathcal{P} \wedge \mathcal{Q}$.

Solution

To 'instantiate $\mathcal{P}$ to p_1' means to replace $\mathcal{P}$ by p_1. Replacing $\mathcal{P}$ by p_1 in $\mathcal{P} \wedge \mathcal{Q}$ yields $p_1 \wedge \mathcal{Q}$. Replacing $\mathcal{Q}$ by p_2 in $p_1 \wedge \mathcal{Q}$ then yields $p_1 \wedge p_2$ □

Example 2.7

Suppose we substitute $2 + 3 = 5$ for $\mathcal{P}$ and $7 < 1$ for $\mathcal{Q}$ in the conjunction schema. What is the resulting compound propositions, and what is its truth value? (Assume the normal laws of arithmetic apply.)

Solution

Replacing $\mathcal{P}$ by $2 + 3 = 5$ and $\mathcal{Q}$ by $7 < 1$ in the conjunction schema gives $(2+3 = 5) \wedge (7 < 1)$. Now $2+3 = 5$ has truth value T and $7 < 1$ has truth value F. From the truth table for conjunction, we see that when the truth value of the left conjunct is T and the truth value of the right conjunct is F, the truth value of the conjunction is F. Hence $(2 + 3 = 5) \wedge (7 < 1) =_T F$. □

Exercise 3: Conjunction

In each question, use the given conjuncts in $\mathcal{P} \wedge \mathcal{Q}$. Find the corresponding truth value of the conjunction for the given truth values.

1. $\mathcal{P}$ '*My cat is black*'; truth value F;
   $\mathcal{Q}$ '*Your cat is white*'; truth value F.
2. $\mathcal{P}$ '*Shakespeare wrote Hamlet*'; truth value T;
   $\mathcal{Q}$ '*Shakespeare wrote MacBeth*'; truth value T.
3. $\mathcal{P}$ q; truth value T;
   $\mathcal{Q}$ p; truth value F.
4. $\mathcal{P}$ $2 \times 7 = 27$; truth value F;
   $\mathcal{Q}$ $3^2 = 9$; truth value T.
5. $\mathcal{P}$ $3 > 2$; truth value T;
   $\mathcal{Q}$ $3 > 2$; truth value T.

2.5 Negation as an operator

Definition 2.3

If p is a proposition then $\neg p$ is its negation. The negation schema $\neg\mathcal{P}$ has the following truth table:

$\mathcal{P}$	$\neg\mathcal{P}$
T	F
F	T

Thus for every instance of $\mathcal{P}$ which is true, the corresponding instance of $\neg\mathcal{P}$ is false; and for every instance of $\mathcal{P}$ which is false, the corresponding instance of $\neg\mathcal{P}$ is true. □

Example 2.8

Replace $\mathcal{P}$ by '*Rex has four legs*' in $\neg\mathcal{P}$. If '*Rex has four legs*' $=_T T$, what is the truth value of the resulting negation?

Solution

Replacing '*Rex has four legs*' by $\mathcal{P}$ in the negation schema $\neg\mathcal{P}$ yields $\neg$'*Rex has four legs*' as the negation of '*Rex has four legs*'. Note that we would normally word this negation as '*Rex does not have four legs*'. Now corresponding to $\mathcal{P} =_T T$ in the truth table for negation, we see that $\neg\mathcal{P} =_T F$. Hence $\neg$'*Rex has four legs*' $=_T F$ □

Exercise 4: Negation

In each question, use the given proposition to replace $\mathcal{P}$ in the negation schema $\neg\mathcal{P}$. Find the truth value of the resulting negation.

1. '*The moon is made of blue cheese*'; truth value F.
2. '1234 *is an even number*'; truth value T.
3. q_3; truth value T.

2.6 Disjunction as an operator

Definition 2.4

If p and q are two propositions then $p \vee q$ is the disjunction of p and q. The propositions p and q are called the **disjuncts**. The disjunction schema $\mathcal{P} \vee \mathcal{Q}$ has the following truth table:

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \vee \mathcal{Q}$
T	T	T
T	F	T
F	T	T
F	F	F

Thus the only instances of $\mathcal{P} \vee \mathcal{Q}$ which are false are those for which the instances of both $\mathcal{P}$ and $\mathcal{Q}$ are false. □

Example 2.9

Replace $\mathcal{P}$ by '*Fido has three legs*' and $\mathcal{Q}$ by '*Rex has four legs*' in the the disjunction schema $\mathcal{P} \vee \mathcal{Q}$. What is the truth value of the resulting disjunction if '*Fido has three legs*' $=_T F$ and '*Rex has four legs*' $=_T T$?

Solution

The resulting disjunction is '*Fido has three legs*'$\vee$'*Rex has four legs*'. From the truth third row of the truth table, we see that this disjunction has truth value of T. □

Exercise 5: Disjunction

In each question, use the given propositions to replace $\mathcal{P}$ and $\mathcal{Q}$ in the disjunction schema $\mathcal{P} \vee \mathcal{Q}$. Find the truth value of the resulting disjunction.

1. $\mathcal{P}$ '*You broke the window*'; truth value F.
   $\mathcal{Q}$ '*I'm a Martian*'; truth value F.
2. $\mathcal{P}$ '*Shakespeare wrote Hamlet*'; truth value T.
   $\mathcal{Q}$ '*Francis Bacon wrote Hamlet*'; truth value F.
3. $\mathcal{P}$ r_1; truth value F.
   $\mathcal{Q}$ r_2; truth value T.
4. $\mathcal{P}$ '2 *is an even number*'; truth value T
   $\mathcal{Q}$ '3 *is an odd number*'; truth value T.

2.7 Use of schematic letters

In the preceding sections we have given schemas for conjunction, negation and disjunction as $\mathcal{P} \wedge \mathcal{Q}$, $\neg\mathcal{P}$ and $\mathcal{P} \vee \mathcal{Q}$ respectively. It must be remembered, though, that the schematic letters $\mathcal{P}$ and $\mathcal{Q}$ are used simply to mark places where propositions may be placed; other letters would serve equally as well. For example, we could have represented the conjunction schema $\square \wedge \square$ as $\mathcal{Q} \wedge \mathcal{R}$ or $\mathcal{P}_1 \wedge \mathcal{P}_2$. This use of schematic letters $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ ... contrasts with the use of propositional constants p, q, r ...: $\mathcal{P} \vee \mathcal{Q}$ and $\mathcal{Q} \vee \mathcal{P}$ both represent the same thing, namely the conjunction schema; but $p \vee q$ and $q \vee p$ represent different propositions (even though we do not know what these are).

Example 2.10

Instantiate $\mathcal{Q}_1$ to '*Grass is green*' and $\mathcal{Q}_2$ to '*Elephants are pink*' in $\mathcal{Q}_2 \vee \mathcal{Q}_1$.

Solution

Replacing $\mathcal{Q}_1$ by '*Grass is green*' in $\mathcal{Q}_2 \vee \mathcal{Q}_1$ gives $\mathcal{Q}_2 \vee$ '*Grass is green*'. If we now replace $\mathcal{Q}_2$ by '*Elephants are pink*' in $\mathcal{Q}_2 \vee$ '*Grass is green*' the result is '*Elephants are pink*' $\vee$ '*Grass is green*'. □

Example 2.11

Instantiate both $\mathcal{R}$ and $\mathcal{S}$ to q in $\mathcal{S} \wedge \mathcal{R}$.

Solution

Replacing $\mathcal{R}$ by q in $\mathcal{S} \wedge \mathcal{R}$ gives $q \wedge \mathcal{R}$. Replacing $\mathcal{S}$ by q in $q \wedge \mathcal{R}$. gives $q \wedge q$. □

Example 2.12

Suggest a connective schema of which the following proposition is an instance: '2 *is an even number but it is prime*'

Solution

In finding an appropriate connective schema, the first task is to identify the basic facts. They are '2 *is an even number*' and '2 *is a prime number*'. Notice how it is necessary to expand '*it is prime*' into '2 *is a prime number*'.

The next task is to identify the relationship between these two facts. In this case the relationship is that of ... *but*.... As discussed in section 1.5, the truth properties of '*but*' are the same as '*and*'. Thus we need a conjunction schema. One possibility would be to use schematic letters $\mathcal{P}_1$ and $\mathcal{P}_2$ and write the conjunction schema as $\mathcal{P}_1 \wedge \mathcal{P}_2$. □

Note that, in choosing schematic letters for a connective schema, the same letter cannot be used more than once. (The reason for this will be explained in section 3.1.)

Exercise 6: Connective schemas

1. In each of the following, use the given instantiations to obtain a proposition.

   (a) '*Rex has a wet nose*' for $\mathcal{Q}$ in $\neg\mathcal{Q}$.
   (b) $2+3=8$ for $\mathcal{R}_1$ in $\neg\mathcal{R}_1$.
   (c) p for $\mathcal{Q}$ in $\neg\mathcal{Q}$.
   (d) q for $\mathcal{P}_1$ in $\neg\mathcal{P}_1$.
   (e) r_2 for $\mathcal{S}$ in $\neg\mathcal{S}$.
   (f) '*Rex is black*' for $\mathcal{R}$ and '*Rover is white*' for $\mathcal{Q}$ in $\mathcal{R} \wedge \mathcal{Q}$.
   (g) $2+3=8$ for $\mathcal{R}_1$ and $3^2=9$ for $\mathcal{R}_2$ in $\mathcal{R}_1 \vee \mathcal{R}_2$.
   (h) p for $\mathcal{Q}_1$ and q for $\mathcal{Q}_2$ in $\mathcal{Q}_1 \wedge \mathcal{Q}_2$.
   (i) p_1 for $\mathcal{P}$ and p_2 for $\mathcal{Q}$ in $\mathcal{P} \wedge \mathcal{Q}$.
   (j) r_2 for $\mathcal{S}$ and q for $\mathcal{Q}_1$ in $\mathcal{S} \vee \mathcal{Q}_1$.

2. For each of the following compound propositions, suggest a connective schema of which the proposition is an instance.

   (a) '*This book is long but I read it quickly*'.
   (b) '*Either there is a hole in the exhaust or a bracket has worked loose*'.
   (c) '*Lunch or dinner will be served during the flight*'.
   (d) $3 \times 6 \neq 7$

2.8 More complex compound propositions

So far we have been considering compound propositions containing just a single connective. Each compound proposition has been an instance of a connective schema in which each schematic letter has been replaced by a simple atomic proposition. For example, $\neg\, 2+3=8$ is an instance of $\neg\mathcal{R}_1$ in which $\mathcal{R}_1$ has been replaced by the atomic proposition $2+3=8$. Yet there is no reason why we should restrict ourselves to using atomic propositions as instances of schematic letters. A schematic letter can be instantiated to *any* proposition, possibly a compound proposition.

Example 2.13

Instantiate $\mathcal{P}$ to $\neg\, 0<3$ and $\mathcal{Q}$ to $\neg\, 3<2$ in the connective schema $\mathcal{P} \vee \mathcal{Q}$.

Solution

$\neg\, 0<3 \;\vee\; \neg\, 3<2$ □

Example 2.14

Instantiate $\mathcal{P}_1$ to $\neg p$ and $\mathcal{P}_2$ to $q \wedge r$ in the connective schema $\mathcal{P}_1 \vee \mathcal{P}_2$.

Solution

$\neg p \vee q \wedge r$ □

Exercise 7: Instantiation to compound propositions

1. Instantiate $\mathcal{Q}$ to ¬‘*Rover is a brave dog*’ in $\neg\mathcal{Q}$.
2. Instantiate $\mathcal{P}_1$ to $2 \leq 3 \wedge 2^2 \leq 3^2$ and $\mathcal{P}_2$ to $2 \geq 3 \wedge 2^2 \geq 3^2$ in $\mathcal{P}_1 \vee \mathcal{P}_2$
3. Instantiate $\mathcal{R}$ to $p_1 \wedge \neg p_2$ and $\mathcal{S}$ to $\neg p_3 \wedge p_4$ in $\mathcal{R} \wedge \mathcal{S}$.

2.9 Parse trees

Unfortunately, there is a difficulty when we are presented with a compound proposition having more than one connective: the compound proposition may be an instance of more than one connective schema. For example, $p \vee q \wedge r$ may be obtained in two different ways:

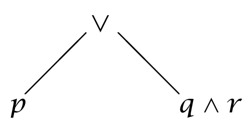
- from $\mathcal{P} \vee \mathcal{Q}$ with $\mathcal{P}$ instantiated to p and with $\mathcal{Q}$ instantiated to $q \wedge r$;
- from $\mathcal{P} \wedge \mathcal{Q}$ with $\mathcal{P}$ instantiated to $p \vee q$ and with $\mathcal{Q}$ instantiated to r.

In order to make clear which of these two possible interpretations is meant we can place parentheses (round brackets) around any compound proposition which is used.

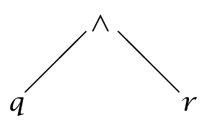
- Replacing $\mathcal{P}$ by p and $\mathcal{Q}$ by $q \wedge r$ in $\mathcal{P} \vee \mathcal{Q}$ gives $p \vee (q \wedge r)$.
- Replacing $\mathcal{P}$ by $p \vee q$ and $\mathcal{Q}$ by r in $\mathcal{P} \wedge \mathcal{Q}$ gives $(p \vee q) \wedge r$.

It may perhaps be easier to understand the two possible interpretations by thinking of connectives as operators. This also makes it possible to represent the different interpretations diagrammatically.

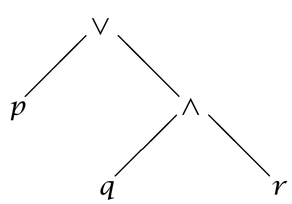
- The compound proposition $p \vee (q \wedge r)$ is obtained by $\vee$ operating on the disjuncts p and $q \wedge r$. This can be represented diagrammatically as

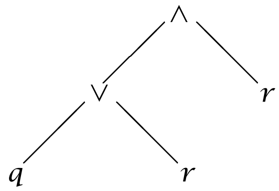
  Now $q \wedge r$ is the result of $\wedge$ operating on q and r.

  Replacing $q \wedge r$ in the first diagram by this second diagram gives

- The compound proposition $(p \vee q) \wedge r$ is obtained by $\vee$ operating on the disjuncts $p \vee q$ and r. Now the compound proposition $p \vee q$ is itself obtained from $\vee$ operating on p and q. This can be represented diagrammatically as



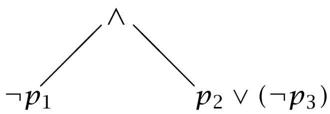
The diagrams obtained are called **parse trees.** In a parse tree, each operator is linked to its operands by straight lines. The process of analysing an expression to see how it is built up from its components is called **parsing.** To parse a compound proposition, we first determine the top level connective. We shall call this top level connective the **main** connective of the compound proposition. In $p \vee (q \wedge r)$, the main connective is $\vee$, while in $(p \vee q) \wedge r$ it is $\wedge$. This process is then repeated for any operand that is itself a compound proposition.

Example 2.15

Parse $(\neg p_1) \wedge (p_2 \vee (\neg p_3))$. What is the main connective?

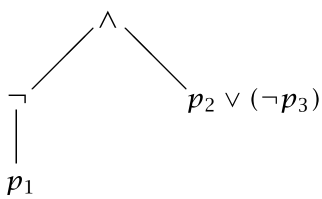
Solution

Start with the first left parentheses and find its matching right parenthesis. The proposition sandwiched between this pair of parentheses is $\neg p_1$. The connective immediately following is $\wedge$. Immediately after this connective, the proposition $p_2 \vee (\neg p_3)$ is sandwiched between a pair of parentheses. We can now start to build the parse tree, with $\wedge$ as the main connective.

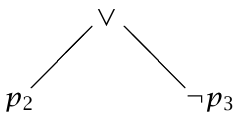


Each conjunct of $\wedge$ is a compound proposition and must be analysed as a parse tree. In $\neg p_1$, there is only one connective $\neg$, which must be the top level connective. The parse tree for $\neg p_1$ is drawn as

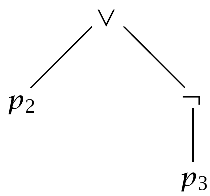
Replace $\neg p_1$ in the previous diagram by this second tree. We say that the tree for $\neg p_1$ is a **subtree** of the complete tree for $(\neg p_1) \wedge (p_2 \vee (\neg p_3))$.



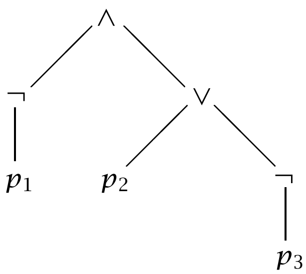
We now have to find the subtree corresponding to $p_2 \vee (\neg p_3)$. In this case, there is no initial left parenthesis, but instead we have an atomic proposition p_2. The connective immediately following p_2 is $\vee$, which must therefore be the top level connective for the subtree.



But this subtree must itself have a subtree corresponding to the compound proposition $\neg p_3$. Replacing $\neg p_3$ by its subtree yields



as the subtree for $p_2 \vee (\neg p_3)$. Thus the complete parse tree for $(\neg p_1) \wedge (p_2 \vee (\neg p_3))$ is



$\square$

Exercise 8: Constructing parse trees

For each of the following compound propositions, construct the parse tree. What is the main connective in each case?

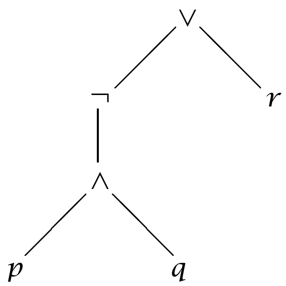
1. $(\neg p) \wedge q$
2. $\neg (p \wedge q)$
3. $(q_1 \wedge q_2) \wedge q_3$
4. $q_1 \wedge (q_2 \wedge q_3)$
5. $\neg ((p \vee q) \wedge r)$
6. $((\neg p) \wedge (\neg q)) \vee (\neg r)$

2.10 Compound propositions from parse trees

It is also possible to reverse the process in order to write down the expression corresponding to a given parse tree. We start with the propositions at the lowest points of the parse tree (the **leaves**) and work upwards, replacing subtrees by the corresponding compound propositions.

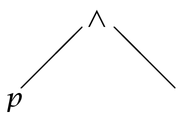
Example 2.16

Find the compound proposition for the following parse tree.

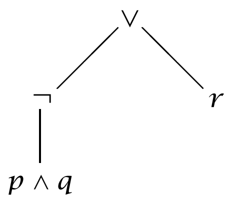


Solution

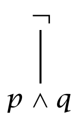
The lowest level propositions are p and q. The subtree



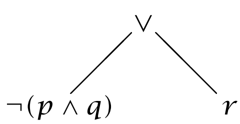
can be replaced by $p \wedge q$ in the original parse tree to give



The lowest level proposition is now $p \wedge q$. The subtree



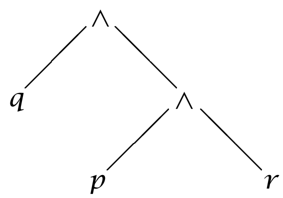
can now be replaced by $\neg(p \wedge q)$ to give

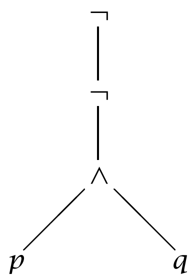


We now have just the main connective $\vee$ and the two disjuncts $\neg(p \wedge q)$ and r. The complete compound proposition is thus $(\neg(p \wedge q)) \vee r$. Notice how it is necessary when using a compound proposition as an operand to enclose it in parentheses. □

Exercise 9: Compound propositions from parse trees

For each of the following parse trees, find the corresponding compound proposition.

1. 

2. 

2.11 Connective priorities

We have seen how it is necessary to use parentheses in compound propositions with more than one connective in order to avoid ambiguity. The difficulty with this, however, is that a large number of parentheses can make the expression difficult to read. For example, $(\neg p_1) \vee ((\neg(\neg p_2)) \wedge (\neg p_3))$ can seem rather confusing. There is a way, however, in which we can often avoid having to write down parentheses. This relies on using a convention of connective **priority**. We must therefore first understand the concept of priority. This concept will, however, be familiar already to most readers.

Operator priority in arithmetic

The concept of priority can be demonstrated by reference to the calculation of arithmetic expressions.

Example 2.17

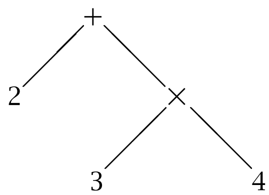
Calculate $2 + 3 \times 4$.

Solution

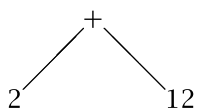
Clearly 2 is to be added to something, while 4 is to be multiplied by something. The difficulty arises with the 3 since it has two adjacent operators, $+$ and $\times$. Which operator applies to the 3? Now the convention in arithmetic is that multiplications are calculated before additions. Multiplication is said to have have a higher priority than addition. Another way of expressing priority is to say that multiplication **binds** more strongly than addition. In this case the first operation is to multiply 3 by 4. The result of this calculation is 12, which is then added to 2 to yield a final answer of 14. This sequence of operations may be made explicit as follows:

$$2 + 3 \times 4 = 2 + (3 \times 4) = 2 + 12 = 14$$

Note how parentheses are used to emphasize that the multiplication is done first; operations within brackets always have priority to those outside. Alternatively we can use a parse tree to show the way in which the calculation is performed:



In evaluating the parse tree, we work up from the lowest operations. The multiplication is the lowest operation and is done first. The corresponding subtree is replaced by its value of 12.



Finally, the addition of 2 and 12 gives the final answer of 14. □

Example 2.18
Calculate $(2 + 3) \times 4$.

Solution
We make use of the rule, referred to above, that **subexpressions** operations within brackets have a higher priority. In this case we have a pair of parentheses $(\ldots)$ enclosing $2 + 3$ so this addition is performed first. Our arithmetical expression is thus equal to 5×4. The multiplication is performed second to give the final answer: $(2+3)\times 4 = 5\times 4 = 20$. (Note that $(2+3)\times 4$ and $2 + (3 \times 4)$ have different values.) □

Example 2.19
Parse $3 \times 5 \times 2$, and hence evaluate it.

Solution
The two possibilities for parsing the expression are $(3\times 5)\times 2$ and $3\times(5\times 2)$. To decide which is the correct one, we use another rule of priority: when the choice is between two multiplications, the left hand one has the higher priority (that is, binds more strongly). Thus the 5 is bound to the first multiplication rather than the second. The correct parsing is therefore $(3\times 5)\times 2$, from which

we can calculate $(3 \times 5) \times 2 = 15 \times 2 = 30$. Note that $3 \times (5 \times 2)$ also gives the same value of 30; this is the result of a special property of multiplication known as association. □

Priority rules for logic

In logic we can also stipulate rules for determining priorities of connectives. In this book, the following convention is used:

- Negation $\neg$ has a higher priority than conjunction $\wedge$, which itself has a higher priority than disjunction $\vee$.

  Highest $\quad \neg \quad \wedge \quad \vee \quad$ Lowest

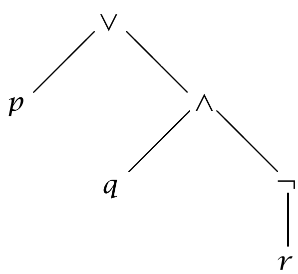
  Alternatively, we say that $\neg$ binds more strongly than $\wedge$, which in turn binds more strongly than $\vee$.[1]
- Connectives within parentheses bind more strongly than those outside – this is the equivalent of calculating brackets first in arithmetic.
- If a proposition has the same connective either side, then the left hand connective has the higher priority. Essentially this means that, unless parentheses indicate otherwise, the left hand occurrence of two conjunctions $\wedge$ has the higher priority; likewise, the left hand occurrence of two disjunctions $\vee$ has the higher priority.

Example 2.20

Parse $p \vee q \wedge \neg r$. What is the main connective?

Solution

Of the three connectives, negation binds the most strongly, so we can introduce parentheses around $\neg r$ to give $p \vee q \wedge (\neg r)$. Of the remaining two connectives, conjunction now binds most strongly, so we can introduce parentheses around $q \wedge (\neg r)$ to give $p \vee (q \wedge (\neg r))$. There is now just one connective left, namely disjunction; this is the main connective. The parsed expression is therefore $p \vee (q \wedge (\neg r))$ and the parse tree is



□

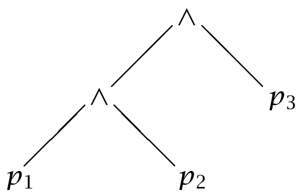
[1]Note that some authors adopt the convention that $\wedge$ and $\vee$ have equal priority.

Example 2.21

Parse $p_1 \wedge p_2 \wedge p_3$. What is the main connective?

Solution

There are two connectives, both conjunction. Using our rules of priority, the left hand connective binds more strongly. Hence the compound proposition can be parsed as $(p_1 \wedge p_2) \wedge p_3$, with the main connective being the second conjunction $\wedge$.



□

Example 2.22

Parse $2 + 3 = 5 \;\wedge \neg\, 2 \le 1 \;\vee\; 9 \times 6 = 42$. Represent the answer by introducing appropriate parentheses.

Solution

In this case there is a mixture of arithmetical and logical symbols. A useful first step is to clearly identify the atomic propositions; they are

- $2 + 3 = 5$
- $2 \le 1$
- $9 \times 6 = 42$

We can now parse the compound proposition. As before, negation has highest priority:

$$2 + 3 = 5 \;\wedge\; (\neg\, 2 \le 1) \;\vee\; 9 \times 6 = 42$$

Next comes conjunction:

$$(2 + 3 = 5 \;\wedge\; (\neg\, 2 \le 1)) \;\vee\; 9 \times 6 = 42$$

This is the parsed expression, with the single occurrence of disjunction as the main connective. □

Example 2.23

What is the main connective in $\neg(p \wedge (q \vee r)) \vee p \wedge r \vee s$?

Solution

In any compound proposition, the main connective is the one with lowest priority; that is, the connective which binds least strongly. Thus our problem is to determine which connective has the lowest priority. Now connectives nested inside parentheses have higher priority than those outside. Thus neither the $\wedge$ nor the $\vee$ connective in $(p \wedge (q \vee r))$ can be the main connective. Of the three remaining connectives outside the parentheses, the right hand $\vee$ has lowest priority. We can represent this by means of a simplified parse tree:

□

Exercise 10: Connective priorities

1. Parse each of the following compound propositions. Represent the answer by introducing appropriate parentheses.

   (a) $\neg\neg p$
   (b) $\neg p \wedge q$;
   (c) $q \wedge \neg r$
   (d) $\neg p \vee q$;
   (e) $p \wedge q \wedge \neg r$.
   (f) $p \vee q \vee \neg r$;
   (g) $q \wedge \neg p \vee q$
   (h) $p_1 \vee p_2 \vee \neg p_3$
   (i) $3 > 0 \;\vee \neg 1 + 1 = 2 \;\wedge\; 2 + 3 = 5$
   (j) $\neg$'*Fido has three legs*' $\vee$ '*Rex has four legs*'
   (k) $p \wedge q \vee \neg r \wedge p$.
   (l) $\neg p \wedge (\neg q \vee p \wedge r)$
   (m) $\neg\neg(\neg p_2 \wedge \neg p_1)$

2. What is the main connective in each of the following compound propositions?

   (a) $\neg(p_1 \wedge p_2) \vee p_3$
   (b) '*Rex has four legs*' $\vee$ '*Fido has three legs*' $\vee$ $\neg$'*Rover has a wet nose*'
   (c) $\neg\neg q$
   (d) $\neg\, 1 + 1 = 2 \;\vee\; (1 + 1)^2 = 2^2$
   (e) $\neg(p \vee q \wedge r \vee \neg s)$
   (f) $\neg(\neg(\neg p_1 \vee p_2 \wedge p_3) \vee \neg p_4) \wedge \neg(p_5 \wedge p_6 \vee \neg\neg p_7)$

2.12 Removing parentheses

In writing down compound propositions symbolically, we may need to decide whether to include parentheses or not. Although there is nothing logically incorrect in always putting in parentheses to indicate the parsing of an expression, removing unnecessary parentheses can make for easier reading. Now parentheses are unnecessary whenever we would parse the expression identically without them. They can therefore be removed when the connective to which they correspond binds more strongly than other adjacent connectives, even in the absence of the parentheses.

Example 2.24

Remove unnecessary parentheses in each of the following expressions.

1. $(p \vee q) \vee r$
2. $p \vee (q \vee r)$
3. $r \vee ((\neg p) \wedge q)$

Solution

1. The left hand disjunction in $p \vee q \vee r$ still binds more strongly than the right hand disjunction, even though we have removed the parentheses. Therefore dropping the parentheses from $(p \vee q) \vee r$ does not alter the parsing. We can safely write $p \vee q \vee r$ instead of $(p \vee q) \vee r$.
2. The parentheses in $p \vee (q \vee r)$ mean that the right hand disjunction binds the more strongly. As we have just seen, dropping these parentheses would result in the left hand disjunction binding more strongly. The parentheses cannot be removed wihout altering the parsing.
3. Since negation binds more strongly than conjunction we can drop the corresponding parentheses to give $r \vee (\neg p \wedge q)$. Now conjunctiopn binds more strongly than disjunction, hence we can again drop parentheses to give $r \vee \neg p \wedge q$.

□

Exercise 11: Removing parentheses

Remove unnecessary parentheses in each of the following expressions.

1. $(\neg p) \vee q$
2. $(p_1 \vee (\neg p_2)) \vee p_3$
3. $(p \wedge (\neg q)) \vee (\neg r)$
4. $((\neg q_1) \wedge (q_1 \vee q_2)) \vee (q_1 \wedge q_2)$
5. $r \wedge ((\neg p) \vee q)$

2.13 Truth values of compound propositions

In sections 2.4, 2.5 and 2.6, we have seen how the truth value of a compound proposition with just one connective can be found using truth tables. This can be readily adapted to finding the truth value of a compound proposition with more than one connective. First parse the proposition. Next, replace each atomic proposition by its truth value. Finally, evaluate the resulting expression starting with the highest priority connectives and finishing with the main connective.

Example 2.25

If $p =_T T$, $q =_T F$ and $r =_T T$, what is the truth value of $\neg p \wedge (q \vee p \wedge r)$?

Solution

The original expression $\neg p \wedge (q \vee p \wedge r)$ parses to $(\neg p) \wedge (q \vee (p \wedge r))$ with the first conjunction as the main connective. Substituting truth values for propositions we get

$$(\neg T) \wedge (F \vee (T \wedge T)) =_T F \wedge (F \vee (T \wedge T)) =_T F \wedge (F \vee T) =_T F \wedge T =_T F$$

The truth value of $\neg p \wedge (q \vee p \wedge r)$ is F. Note that since $\neg T$ and $F \vee (T \wedge T)$ occur within different pairs of parentheses, we can evaluate them in either order. Thus an alternative sequence for evaluation is

$$(\neg T) \wedge (F \vee (T \wedge T)) =_T (\neg T) \wedge (F \vee T) =_T (\neg T) \wedge T =_T F \wedge T =_T F \quad \square$$

Example 2.26

What is the truth value of $\neg\ 3 > 0\ \vee\ 1 + 1 = 2$, assuming the normal laws of arithmetic?

Solution

$\neg\ 3 > 0\ \vee\ 1 + 1 = 2$ parses to $(\neg\ 3 > 0)\ \vee\ 1 + 1 = 2$. Hence we have $(\neg\ 3 > 0)\ \vee\ 1 + 1 = 2 =_T (\neg T) \vee T =_T F \vee T =_T T$ $\square$

Exercise 12: Truth values of compound propositions

1. If $p =_T T$, what is the truth value of $\neg\neg p$?
2. If $p_1 =_T T$ and $p_2 =_T F$, what is the truth value of $\neg p_1 \vee p_2$?
3. If $q_1 =_T F$, $q_2 =_T F$ and $q_3 =_T F$, what is the truth value of $q_1 \vee q_2 \vee q_3$?
4. If $r_1 =_T F$, $r_2 =_T T$ and $r_3 =_T F$, what is the truth value of
$$\neg\neg(r_2 \vee \neg r_3) \wedge \neg(r_1 \vee r_3)?$$
5. If '*Rex has four legs*' $=_T T$ and '*Fido has three legs*' $=_T F$, what is the truth value of $\neg$'*Fido has three legs*' $\vee$ '*Rex has four legs*'?
6. What is the truth value of $\neg\ 7 < 0\ \wedge (7 = 3 + 4\ \vee \neg\ 7 = 2 + 5)$?

Propositional Forms 3

3.1 Compound propositions from propositional forms

Consider the proposition $\neg\, 3 < 2 \wedge (2 \times 3 = 7 \;\vee\; 0 < 5)$. In Chapter 2 we saw that it can be obtained as an instance of $\mathcal{P} \wedge \mathcal{Q}$ by substituting $\neg\, 3 < 2$ for $\mathcal{P}$ and $2 \times 3 = 7 \;\vee\; 0 < 5$ for $\mathcal{Q}$. However, it can also be obtained as an instance of $\neg\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$ with $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ replaced by the atomic propositions $3 < 2$, $2 \times 3 = 7$ and $0 < 5$ respectively. Yet another possibility is to start with $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$, and to replace $\mathcal{P}$ by the negation $\neg\, 3 < 2$. Expressions such as $\neg\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$ and $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$ play an important part in logic and are known as **propositional forms.**

Example 3.1

What proposition is obtained as an instance of $\neg\mathcal{P} \wedge \mathcal{Q}$ if $\mathcal{P}$ is instantiated to '2 *is an odd number*' and $\mathcal{Q}$ to '2 *is a prime number*'? How might this compound proposition be expressed in natural language?

Solution

Making the replacements in the propositional form gives the compound proposition $\neg$'2 *is an odd number*' $\wedge$ '2 *is a prime number*'. This might be expressed as '2 *is not an odd number but it is prime*'. □

Exercise 13: Compound propositions from propositional forms

1. What proposition is obtained as an instance of $\mathcal{P} \wedge (\mathcal{Q} \vee \neg\mathcal{R})$ if $\mathcal{P}$ is replaced by '*Rex has four legs*', $\mathcal{Q}$ is replaced by '*Fido has three legs*' and $\mathcal{R}$ is replaced by '*Rover has a wet nose*'?

2. Obtain a meaningful statement in English for each of the following.

   (a) $\neg\mathcal{P} \wedge \mathcal{Q}$ if $\mathcal{P}$ is replaced by '*It is cold*' and $\mathcal{Q}$ is replaced by '*It is snowing*'.

   (b) $\neg\mathcal{P} \vee \neg\mathcal{Q} \wedge \mathcal{R}$ if $\mathcal{P}$ is replaced by '*The water pump is working*', $\mathcal{Q}$ is replaced by '*There's anti-freeze in the radiator*' and $\mathcal{R}$ is replaced by '*Last night was very cold*'.

3.2 Propositional forms for compound propositions

Obtaining a compound proposition as an instance of a propositional form is fairly straightforward. If, however, we try to reverse the process and find the propositional form of which a given compound proposition is an instance, we encounter a problem: there is more than one possible answer.

Example 3.2

Derive a list of propositional forms for each of which the compound proposition $\neg\, 3 < 2 \wedge (2 \times 3 = 7 \ \vee\ 0 < 5)$ is an instance.

Solution

We have already seen that the compound proposition is an instance of the propositional form $\neg\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$ in which all the schematic letters are instantiated to *atomic* propositions. Such a form represents an extreme case in that we have the maximum number of connectives (in this case three). Now within this form are the connective schemas $\neg\mathcal{P}$ and $\mathcal{Q} \wedge \mathcal{R}$. We can obtain forms with fewer connectives by replacing these connective schemas with schematic letters. Thus we can obtain $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$, with two connectives, as another propositional form by replacing $\neg\mathcal{P}$ with $\mathcal{P}$. Note however that in the new form the schematic letter $\mathcal{P}$ needs to be instantiated to a different proposition. Similarly we can obtain the form $\neg\mathcal{P} \wedge \mathcal{Q}$, again with two connectives. Going one stage further, we can obtain the propositional form $\mathcal{P} \wedge \mathcal{Q}$ with just one connective – the main connective schema. But this is not all. We can continue the process of reducing the number of connectives to zero to obtain $\mathcal{P}$ as the simplest propositional form. Thus the compound proposition can be obtained as an instance of each of the of the following propositional forms.

1. $\neg\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$ with $\mathcal{P}$ replaced by $3 < 2$, $\mathcal{Q}$ by $2 \times 3 = 7$ and $\mathcal{R}$ by $0 < 5$.
2. $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$ with $\mathcal{P}$ replaced by $\neg\, 3 < 2$, $\mathcal{Q}$ by $2 \times 3 = 7$ and $\mathcal{R}$ by $0 < 5$.
3. $\neg\mathcal{P} \wedge \mathcal{Q}$ with $\mathcal{P}$ replaced by $3 < 2$ and $\mathcal{Q}$ by $2 \times 3 = 7 \ \vee\ 0 < 5$.
4. $\mathcal{P} \wedge \mathcal{Q}$ with $\mathcal{P}$ replaced by $\neg\, 3 < 2$ and $\mathcal{Q}$ by $2 \times 3 = 7 \ \vee\ 0 < 5$.
5. $\mathcal{P}$ with $\mathcal{P}$ replaced by $\neg\, 3 < 2 \wedge (2 \times 3 = 7 \ \vee\ 0 < 5)$.

Note that there are other, similar lists using different schematic letters. For example we may construct a list starting with $(\neg\mathcal{Q}_1) \wedge (\mathcal{Q}_2 \vee \mathcal{Q}_3)$ □

Example 3.3

List all the propositional forms for each of which the compound proposition $\neg\, 3 < 2 \wedge\ 2 \times 3 = 7 \vee\ 0 < 5$ is an instance.

Solution

Replacing atomic propositions by schematic letters gives $\neg\mathcal{P} \wedge \mathcal{Q} \vee \mathcal{R}$. Note that there are no parentheses in this form. The parsing is thus determined by the priorities of the connectives to be $((\neg\mathcal{P}) \wedge \mathcal{Q}) \vee \mathcal{R}$, with the main connective being disjunction. The propositional forms are therefore:

1. $((\neg\mathcal{P}) \wedge \mathcal{Q}) \vee \mathcal{R}$, that is $\neg\mathcal{P} \wedge \mathcal{Q} \vee \mathcal{R}$;
2. $(\mathcal{P} \wedge \mathcal{Q}) \vee \mathcal{R}$, that is $\mathcal{P} \wedge \mathcal{Q} \vee \mathcal{R}$;
3. $\mathcal{P} \vee \mathcal{R}$;
4. $\mathcal{P}$. □

In both the preceding examples, the first step was to find a propositional form having the greatest number of connectives; the given proposition can be obtained as an instance of this form by replacing each schematic letter with an atomic proposition.

Definition 3.1

A **maximal form** for any proposition is a propositional form which has the greatest number of connectives and of which the given proposition is an instance. The given proposition can be obtained from the maximal form by instantiating each schematic letter to an atomic proposition. □

Example 3.4

What is a maximal form for $\neg\, 3 < 2 \wedge (2 \times 3 = 7 \ \vee\ 0 < 5)$?

Solution

In Example 3.2 we obtained a list of propositional forms of which the given proposition is an instance. The form with most connectives, namely three, was seen to be $\neg\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$. The given proposition can be obtained from this maximal form by instantiating $\mathcal{P}$ to the atomic proposition $3 < 2$, $\mathcal{Q}$ to the atomic proposition $2 \times 3 = 7$ and $\mathcal{R}$ to the atomic proposition $0 < 5$. To obtain the given proposition from any other form in the list, however, involves instantiating one or more of the schematic letters to compound propositions. □

Note that there is more than one maximal form for any proposition. However, any one maximal form can be obtained from any other maximal form by a simple replacement of schematic letters. For example, replacing $\mathcal{P}$ by $\mathcal{Q}_1$, $\mathcal{Q}$ by $\mathcal{Q}_2$ and $\mathcal{R}$ by $\mathcal{Q}_3$ in $\neg\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$ yields $(\neg\mathcal{Q}_1) \wedge (\mathcal{Q}_2 \vee \mathcal{Q}_3)$ as an alternative maximal form for $\neg\, 3 < 2 \wedge (2 \times 3 = 7 \ \vee\ 0 < 5)$. Essentially, all maximal forms for any proposition are the same apart from a different choice of schematic letters. We shall therefore speak of *the* maximal form to emphasize this fact.

If we are given the compound proposition in natural language form, the first step in finding the maximal form is to identify the atomic propositions and to rewrite the compound proposition using symbolic connectives. In doing this, we must give some thought to the choice of atomic propositions.

Example 3.5

Find the maximal propositional form of '*Jupiter and Saturn and Uranus are giant planets*'.

Solution

From a purely linguistic point of view we could regard the given proposition as a single atomic proposition. Yet the intended meaning would seem to be that there are three separate facts:

- '*Jupiter is a giant planet*';
- '*Saturn is a giant planet*';
- '*Uranus is a giant planet*'.

We take these to be the three atomic propositions, from which we can build up the original proposition by conjunction.

'*Jupiter is a giant planet*' $\wedge$ '*Saturn is a giant planet*' $\wedge$ '*Uranus is a giant planet*'

The corresponding propositional form is $\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R}$. □

Example 3.6

Find the maximal propositional form for '*Either there are nine planets or Pluto is not a planet*'.

Solution

We can analyse the given proposition into the disjunction of two propositions: '*There are nine planets*' and '*Pluto is not a planet*'. We may thus write

'*There are nine planets*' $\vee$ '*Pluto is not a planet*'

However, the proposition '*Pluto is not a planet*' can itself be regarded as the negation of '*Pluto is a planet*'. Thus our original proposition can be written as

'*There are nine planets*' $\vee$ $\neg$'*Pluto is not a planet*'

The maximal propositional form is $\mathcal{P} \vee \neg\mathcal{Q}$. □

It is also possible for the same schematic letter to be repeated in a propositional form. When a letter is repeated, the same proposition must be substituted into all occurrences of that letter. For this reason it was necessary in section 2.7 to insist that schematic letters in any connective schema be used no more than once. Thus, for example, instances of $\mathcal{P} \wedge \mathcal{P}$ include $2 + 3 = 5 \;\wedge\; 2 + 3 = 5$ and $7 > 0 \;\wedge\; 7 > 0$ but *not* $2 + 3 = 5 \;\wedge\; 7 > 0$; hence $\mathcal{P} \wedge \mathcal{P}$ cannot represent the conjunction schema.

Example 3.7

Replace $\mathcal{P}$ by $1 + 1 = 2$ and $\mathcal{Q}$ by $2^2 = 4$ in $\neg\mathcal{P} \vee \mathcal{Q} \wedge \mathcal{P}$.

Solution

Both occurrences of $\mathcal{P}$ must be replaced by the proposition $1 + 1 = 2$. Thus we get $\neg\, 1 + 1 = 2 \;\vee\; 2^2 = 4 \;\wedge\; 1 + 1 = 2$. □

Note that all instances of $\neg\mathcal{P} \vee \mathcal{Q} \wedge \mathcal{P}$ will also be instances of $\neg\mathcal{P} \vee \mathcal{Q} \wedge \mathcal{R}$ in which $\mathcal{P}$ and $\mathcal{R}$ are both replaced by the same proposition. However, there are also instances of $\neg\mathcal{P} \vee \mathcal{Q} \wedge \mathcal{R}$ in which $\mathcal{P}$ and $\mathcal{R}$ are not replaced by the same proposition; such instances are not instances of $\neg\mathcal{P} \vee \mathcal{Q} \wedge \mathcal{P}$.

Example 3.8

Find the maximal form with the fewest letters for

‘*Jupiter is not a giant planet or both Jupiter and Saturn are giant planets.*’

Solution

The main connective is a disjunction, and so we can write

‘*Jupiter is not a giant planet*’ $\vee$ ‘*Both Jupiter and Saturn are giant planets*’

The second of these propositions is a conjunction:

‘*Jupiter is a giant planet*’ $\wedge$ ‘*Saturn is a giant planet*’

Thus our original proposition can be written as

‘*Jupiter is not a giant planet*’ $\vee$
‘*Jupiter is a giant planet*’ $\wedge$ ‘*Saturn is a giant planet*’

The first proposition is the negation of a later one: $\neg$‘*Jupiter is a giant planet*’ Hence the complete proposition is

$\neg$‘*Jupiter is a giant planet*’ $\vee$
‘*Jupiter is a giant planet*’ $\wedge$ ‘*Saturn is a giant planet*’

This is an instance of the propositional form $\neg\mathcal{P} \vee \mathcal{Q} \wedge \mathcal{P}$. □

In this last example, the maximal form had repeated occurrences of the same schematic letter to indicate that one atomic proposition was used more than once in the compound proposition. Such a propositional form can be regarded as displaying the essential structure of the compound proposition.

Definition 3.2

For any given proposition, a **characteristic form** is a maximal form having the fewest possible letters. The proposition is an instance of the characteristic form in which different schematic letters are instantiated to different atomic propositions. □

Example 3.9

Find the characteristic form for

$$\neg\,(\,2 \times 3 = 6\, \wedge 2 \geq 3\,) \vee \neg\,(\,2 \geq 3 \wedge\, 3 \geq 3\,) \vee\, 2 \times 3 = 6\, \wedge \neg\, 6 \geq 3$$

Solution

The atomic propositions are: $2 \times 3 = 6$, $2 \geq 3$, $3 \geq 3$ and $6 \geq 3$. Taking these as instances of $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ respectively gives the characteristic form $\neg(\mathcal{P}_1 \wedge \mathcal{P}_2) \vee \neg(\mathcal{P}_2 \wedge \mathcal{P}_3) \vee \mathcal{P}_1 \wedge \mathcal{P}_4$. □

Exercise 14: Propositional forms for compound propositions

Find a characteristic form for each of the following compound propositions. Hence derive a list of propositional forms for each of which the given proposition is an instance.

1. '*The sun is not made of gold and the moon is not made of blue cheese*'.
2. '*Either the moon orbits the earth and the earth orbits the sun or Copernicus made a mistake*'.
3. $2 \neq 3 \ \vee \ 3 = 4$.
4. '*Vienna and Islamabad are capital cities, but Bergen and Rio are not*'.
5. '*Rex has four legs*' $\vee$ $\neg$'*Rex has four legs*'
6. $2 > 0 \ \wedge \ 4 \neq 1 + 3 \ \vee \ 2 \not> 0 \ \wedge \ 4 = 1 + 3$

3.3 Truth tables for propositional forms

In section 2.13 we saw how the truth value of a compound proposition could be derived from the truth values of the atomic propositions. An essential feature of this process is that the answer is determined solely by the propositional form and the truth values associated with its schematic letters.

Example 3.10

Suppose a certain compound proposition is an instance of the propositional form $\neg\mathcal{P} \vee \mathcal{Q}$; further suppose that the corresponding instances of $\mathcal{P}$ and $\mathcal{Q}$ have truth values of T and F respectively. What is the truth value of the compound proposition?

Solution

The propositional form parses to $(\neg\mathcal{P}) \vee \mathcal{Q}$. Hence the truth value of the compound proposition is obtained by replacing $\mathcal{P}$ by T and $\mathcal{Q}$ by F in $(\neg\mathcal{P}) \vee \mathcal{Q}$ to give

$$(\neg T) \vee F =_T F \vee F =_T F$$

Thus we can say that the compound proposition referred to has truth value of F, even though we do not know what the proposition is! □

Example 3.11

In a second instance of the propositional form $\neg\mathcal{P} \vee \mathcal{Q}$, suppose that $\mathcal{P}$ and $\mathcal{Q}$ have truth values of F and T respectively. What is the truth value of this second instance?

Solution

$(\neg F) \vee T =_T T \vee T =_T T$. This instance has truth value of T. □

Example 3.12

List all possible combinations of truth values associated with $\mathcal{P}$ and $\mathcal{Q}$ and the corresponding truth values associated with $\neg\mathcal{P} \vee \mathcal{Q}$.

Solution

When $\mathcal{P}$ has truth value T and $\mathcal{Q}$ has truth value T then $(\neg\mathcal{P}) \vee \mathcal{Q}$ has truth value $(\neg T) \vee T =_T F \vee T =_T T$. When $\mathcal{P}$ has truth value F and $\mathcal{Q}$ has truth value F then $(\neg\mathcal{P}) \vee \mathcal{Q}$ has truth value $(\neg F) \vee F =_T T \vee F =_T T$. We can summarize these calculations, together with those in the preceding examples, in the form of a table.

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{P} \vee \mathcal{Q}$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

□

In this last example, we can summarize the truth values of $\neg\mathcal{P} \vee \mathcal{Q}$ for each possible combination of truth values associated with the schematic letters.

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P} \vee \mathcal{Q}$
T	T	T
T	F	F
F	T	T
F	F	T

This looks like the **truth tables** we used for defining connective schemas. However, we now have a propositional form, and the truth values have been calculated. In general, we think of truth tables as expressing truth values for propositional forms rather than simply serving to define connective schemas. (Connective schemas are, of course, a special kind of propositional form.) We can then use truth tables to find the truth values of particular instances of propositional forms.

Example 3.13

Suppose '*Fido has three legs*' $=_T F$ and '*Rex has four legs*' $=_T T$. What is the truth value $\neg$'*Fido has three legs*' $\vee$ '*Rex has four legs*'?

Solution

This proposition is an instance of the propositional form $\neg\mathcal{P} \vee \mathcal{Q}$ with '*Fido has three legs*' as the instance of $\mathcal{P}$, and '*Rex has four legs*' as the instance of $\mathcal{Q}$. Hence we can look up the truth value from the truth table above for $\neg\mathcal{P} \vee \mathcal{Q}$ by referring to the third row in which $\mathcal{P}$ has truth value F and $\mathcal{Q}$ has truth value T. From this we see that the truth value must be T. □

Example 3.14

Find the truth table for $\neg\mathcal{P} \vee \neg\mathcal{Q}$.

Solution

The first step is to parse $\neg\mathcal{P} \vee \neg\mathcal{Q}$ as $(\neg\mathcal{P}) \vee (\neg\mathcal{Q})$. We see that in deriving the truth values for $\neg\mathcal{P} \vee \neg\mathcal{Q}$ we must first obtain those for $\neg\mathcal{P}$ and $\neg\mathcal{Q}$.

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{P} \vee \neg\mathcal{Q}$

Next we need to identify all the possible combinations of truth values associated with the schematic letters $\mathcal{P}$ and $\mathcal{Q}$. The conventional way of writing these down is to start with T, T and finish with F, F.

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{P} \vee \neg\mathcal{Q}$
T	T			
T	F			
F	T			
F	F			

Notice how the truth values under $\mathcal{Q}$ alternate every row, while those under $\mathcal{P}$ alternate every two rows. We can now obtain the truth table for $\neg\mathcal{P} \vee \neg\mathcal{Q}$:

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{P} \vee \neg\mathcal{Q}$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

A more compact way of setting down the working is to set out truth values for $\neg\mathcal{P}$ and $\neg\mathcal{Q}$ under the corresponding subexpressions of $\neg\mathcal{P} \vee \neg\mathcal{Q}$, rather than to use separate columns.

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\vee$	$\neg\mathcal{Q}$
T	T	F	$\boldsymbol{F}$	F
T	F	F	$\boldsymbol{T}$	T
F	T	T	$\boldsymbol{T}$	F
F	F	T	$\boldsymbol{T}$	T

It is a matter of choice whether to use this more compact layout. The reader is advised to use extra columns at first, but as confidence grows to try using the more compact layout. □

So far we have restricted ourselves to finding truth values of propositional forms with two schematic letters, $\mathcal{P}$ and $\mathcal{Q}$. In practice we may have any number of letters, possibly with subscripts. Conventionally we list the letters alphabetically; if there are subscripts, then we list the corresponding letters in ascending numerical order. The possible combinations of truth values are usually listed such that those in the rightmost column alternate every row; those in the previous column alternate every two rows; those in the column before that alternate every four rows; and so on.

Example 3.15

Find the truth table for $\neg\mathcal{R} \wedge \mathcal{R}$.

Solution

$\mathcal{R}$	$\neg\mathcal{R}$	$\neg\mathcal{R} \wedge \mathcal{R}$
T	F	F
F	T	F
T	F	F
F	T	F

□

Example 3.16

What is the truth table of $\mathcal{Q}_2 \vee \mathcal{Q}_1 \vee \neg\mathcal{Q}_2$?

Solution

$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_2 \vee \mathcal{Q}_1$	$\neg\mathcal{Q}_2$	$\mathcal{Q}_2 \vee \mathcal{Q}_1 \vee \neg\mathcal{Q}_2$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

□

Example 3.17

What is the truth table for $\neg(\mathcal{R} \wedge \neg(\mathcal{P} \wedge \mathcal{Q}))$

Solution

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{P} \wedge \mathcal{Q}$	$\neg(\mathcal{P} \wedge \mathcal{Q})$	$R \wedge \neg(\mathcal{P} \wedge \mathcal{Q})$	$\neg(R \wedge \neg(\mathcal{P} \wedge \mathcal{Q}))$
T	T	T	T	F	F	T
T	T	F	T	F	F	T
T	F	T	F	T	T	F
T	F	F	F	T	F	T
F	T	T	F	T	T	F
F	T	F	F	T	F	T
F	F	T	F	T	T	F
F	F	F	F	T	F	T

□

Exercise 15: Truth tables

Obtain the truth table for each of the following propositional forms.

1. $\neg\neg\mathcal{P}$
2. $\mathcal{P} \vee \neg\mathcal{P}$
3. $\neg\neg\neg\mathcal{Q}$
4. $\neg(\mathcal{P} \vee \neg\mathcal{Q})$
5. $\mathcal{P} \wedge \neg\mathcal{Q}$
6. $\mathcal{P}_1 \wedge \mathcal{P}_2 \wedge \mathcal{P}_3$
7. $\mathcal{Q} \wedge (\mathcal{P} \wedge \mathcal{R})$
8. $\mathcal{P} \vee \mathcal{Q} \vee \mathcal{R}$
9. $\mathcal{P} \vee (\mathcal{Q} \vee \mathcal{R})$
10. $(\mathcal{P} \vee \mathcal{Q}) \wedge \mathcal{R}$
11. $(\mathcal{P} \wedge \mathcal{R}) \vee (\mathcal{Q} \wedge \mathcal{R})$
12. $\mathcal{Q}_1 \vee (\mathcal{Q}_2 \wedge \mathcal{Q}_3)$
13. $(\mathcal{P} \vee \mathcal{Q}) \wedge (\mathcal{P} \vee \mathcal{R})$
14. $\neg(\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R}))$
15. $\neg\mathcal{P} \wedge (\neg\mathcal{Q} \vee \neg\mathcal{R})$
16. $\neg(\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}))$
17. $\neg\mathcal{P}_3 \vee (\neg\mathcal{P}_2 \wedge \neg\mathcal{P}_1)$
18. $\neg(\neg(\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R}) \vee (\mathcal{P} \wedge \neg(\mathcal{Q} \vee \mathcal{R})))$
19. $\neg\mathcal{P} \vee (\neg\mathcal{Q} \wedge \mathcal{R})$
20. $\neg((\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{R} \wedge \mathcal{S}))$

3.4 Language and metalanguage

In the preceding sections, we have seen how a compound proposition can be viewed as an instance of a propositional form; furthermore, the truth value of a compound proposition can be determined by reference to the truth table for the corresponding propositional form. For these reasons, the *theory* of logic is concerned with propositional forms rather than particular propositions. It is in the *application* of logic that we consider compound propositions as instances of propositional forms.

Propositional forms as a language

The notation of propositional forms includes a variety of symbols:

- schematic letters such as $\mathcal{P}$ and $\mathcal{R}_1$;
- connectives such as $\wedge$ and $\neg$;
- parentheses (and).

These symbols can be combined according to certain rules to yield propositional forms: for example, $\neg\neg(\mathcal{P}_1 \wedge \mathcal{P}_3)$ is a propositional form whereas $\wedge(\wedge\mathcal{P}$ is not. Although these rules have not been stated explicitly, they are implicit in what has been said in the earlier sections of this book. The combination of notation and rules constitutes a **formal language**, in this case the language of propositional forms. The rules are said to comprise the **syntax** of the language: thus $\neg\neg(\mathcal{P}_1 \wedge \mathcal{P}_3)$ is syntactically correct, while $\wedge(\wedge\mathcal{P}$ is not. The collection of symbols is known as the **alphabet** of the language, while syntactically correct expressions such as $\neg\neg(\mathcal{P}_1 \wedge \mathcal{P}_3)$ are said to be **well formed**. Propositional forms may be regarded as being **words** in the formal language.

Note that the concept of formal language is very important in computer science. A programming language is indeed a formal language. Although it is not the intention in this book to discuss formal languages in any depth, the reader should gain some insight into their nature.

Exercise 16: Well formed words

Which of the following words are well formed?

1. $(\mathcal{P} \wedge \mathcal{Q}) \vee \mathcal{R}$
2. $\vee(\wedge\mathcal{P}\mathcal{Q})\mathcal{R}$
3. $(\mathcal{P} \wedge \mathcal{Q})$
4. $\mathcal{P}\,\&\,\mathcal{Q}$
5. $((\neg\mathcal{P})) \vee \mathcal{Q}$
6. $(((\mathcal{P} \wedge \mathcal{Q}) \vee \mathcal{R})$

Application

In talking generally about the application of propositional forms, we also need symbols to represent propositions. For this purpose we use lower case letters such as p and q_2 to represent unspecified propositions. These symbols form part of the **application language**. They are not, however, part of the formal language of propositional forms.

Metalanguage

In this book, as part of a wider theory of logic, we shall be developing a theory about propositional forms known as **propositional logic**. At the same time, we shall be extending the formal language of propositional forms to include new symbols; we shall refer to this extended formal language as the **language of propositional logic**. The theory of propositional logic is concerned with the properties of propositional forms and the relationships that exist between them. In order to present this theory, however, we need to develop a suitable language and notation. But this language is *not* the formal language of propositional logic; instead, it is a language for talking *about* the formal language. We refer to such a language as a **metalanguage**. In this section, we introduce some of the notation of the metalanguage.

Now just as in our application language we use letters such as p and q to represent propositions, so in the metalanguage we use letters to represent propositional forms.

Notation: $\mathfrak{A}$,$\mathfrak{B}$,$\mathfrak{C}$

Capital letters from near the beginning of the alphabet, such as $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$, refer to propositional forms (possibly unspecified). □

In this book, a blackletter font is used to provide a visual reminder that letters such as $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are part of the metalanguage of propositional forms. If you need to write these letters, then it is recommended that you use normal style capital letters.

Frequently we shall need to talk about a propositional form with a specific main connective, but without specifying the form in full. We can accomplish this in our metalanguage by adopting the following convention.

Notation: $\neg\mathfrak{A}$, $\mathfrak{A} \vee \mathfrak{B}$, $\mathfrak{A} \wedge \mathfrak{B}$

The notation $\neg\mathfrak{A}$ refers to a propositional form in which the main connective is negation; likewise, $\mathfrak{A} \vee \mathfrak{B}$ refers to a propositional form in which the main connective is disjunction; finally, $\mathfrak{A} \wedge \mathfrak{B}$ refers to a propositional form in which the main connective is conjunction. □

Truth values

We use the symbols T and F to represent truth values. However, we have a problem in deciding what kind of entity is represented by an expression such as $T \wedge F$. This expression cannot be a proposition. We can however regard it as a propositional form in which the letters T and F can be instantiated to true and false propositions respectively.

Notation: T, F

The symbols T, F are **restricted schematic letters**. The letter T may only be instantiated to a true proposition; the letter F may only be instantiated to a false proposition. □

Thus the letters T and F are each associated with a set of propositions. In **classical logic** every proposition is in just one of these two sets; that is, every proposition is either true of false but not be both. Strange as it might seem it is possible to conceive of different kinds of logic in which this fact does not hold; for example, three-valued logic. The reader may be pleased to hear that in this book, we shall confine our studies to classical logic. As far as propositional logic is concerned, we have added a further two symbols to the formal language, namely T and F.

It may be useful at this point to recap what we already know about this formal language. The alphabet includes:

- schematic letters – $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}, \ldots$;
- truth values – T and F;
- connectives – $\neg$, $\vee$, $\wedge$;
- parentheses – (,) .

Well formed expressions include propositional forms constructed according to the principles given earlier in the current chapter.

Exercise 17: Well formed expressions

Which of the following expressions are well formed in the formal language of propositional logic?

1. $\mathcal{P} \wedge \mathcal{Q}$
2. $\mathcal{P}\mathcal{R}$
3. $\neg\mathcal{P}\,\mathcal{Q} \wedge \mathcal{R}$
4. $\mathcal{P} \wedge \mathcal{Q} \vee \mathcal{R}$
5. $(\neg\mathcal{P} \wedge T) \vee F$
6. $\mathfrak{A} \vee F$
7. $\mathcal{P} \vee F$

3.5 Properties of propositional forms

We are now ready to begin building a theory of propositional logic. In this section we shall consider propositional forms which have special properties, while in the next section we shall look at relationships between propositional forms. In so doing we shall develop the metalanguage of propositional logic, and hence the means to develop the semantics.

Tautologies

Some propositional forms have the special property that all instances are true. To see how this can be, consider the propositional form $\mathcal{P} \vee \neg\mathcal{P}$. Its truth table is:

$\mathcal{P}$	$\neg\mathcal{P}$	$\mathcal{P} \vee \neg\mathcal{P}$
T	F	T
F	T	T

This form has the special property that irrespective of the truth value associated with $\mathcal{P}$, the truth value of $\mathcal{P} \vee \neg\mathcal{P}$ is always T: all instances of $\mathcal{P} \vee \neg\mathcal{P}$ are true. For example, we know that '*Fido has three legs*' $\vee$ $\neg$'*Fido has three legs*' is true even if we do not know the truth value of '*Fido has three legs*'. We have a special name for a form such as $\mathcal{P} \vee \neg\mathcal{P}$.

Definition 3.3
A **tautology** is a propositional form whose truth table is always T. □

Example 3.18
Show that $\neg\mathcal{P} \vee \neg\mathcal{Q} \vee \mathcal{P} \wedge \mathcal{Q}$ is a tautology.

Solution
The truth table for $\neg\mathcal{P} \vee \neg\mathcal{Q} \vee \mathcal{P} \wedge \mathcal{Q}$ is

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{P} \vee \neg\mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$	$\neg\mathcal{P} \vee \neg\mathcal{Q} \vee \mathcal{P} \wedge \mathcal{Q}$
T	T	F	F	F	T	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

From this truth table we see that $\neg\mathcal{P} \vee \neg\mathcal{Q} \vee \mathcal{P} \wedge \mathcal{Q}$ is a tautology. □

Example 3.19
Show that the following proposition is true.

$$29 \times 123 \neq 3667 \ \vee\ 236 \times 34 \neq 8024 \ \vee\ 29 \times 123 = 3667 \ \wedge\ 236 \times 34 = 8024$$

Solution
One approach to answering this question might be to calculate the multiplications and then to use the truth values of the component propositions. However, a simpler approach is to observe that the compound proposition is an instance of $\neg\mathcal{P} \vee \neg\mathcal{Q} \vee \mathcal{P} \wedge \mathcal{Q}$ with $29 \times 123 = 3667$ and $236 \times 34 = 8024$ as the respective instances of $\mathcal{P}$ and $\mathcal{Q}$. Since it is an instance of a tautology (as shown in the previous example), the compound proposition *must* be true. □

We can always determine whether a propositional form is a tautology by finding its truth table. This is not always desirable, however, as calculating truth tables can be time consuming and tedious. For example, if we have ten different schematic letters, then the number of rows needed in the truth table to cover all possible combinations of truth value is $2^{10} = 1024$. One possible way around this problem is demonstrated in the next example.

Example 3.20

Show that $(\mathcal{P}_1 \vee (\neg\mathcal{P}_2 \wedge \neg\neg\mathcal{P}_3) \wedge \mathcal{P}_4) \vee \neg(\mathcal{P}_1 \vee (\neg\mathcal{P}_2 \wedge \neg\neg\mathcal{P}_3) \wedge \mathcal{P}_4)$ is a tautology.

Solution

We could tackle this problem by finding the full truth table. This would require 16 rows of working, with each row spread over something like 12 columns. A simpler approach is possible, however, because the propositional form is the disjunction of a propositional form $\mathcal{P}_1 \vee (\neg\mathcal{P}_2 \wedge \neg\neg\mathcal{P}_3) \wedge \mathcal{P}_4$ and the negation of that propositional form. A **reduced truth table** can be built in terms of the two possible truth values of $\mathcal{P}_1 \vee (\neg\mathcal{P}_2 \wedge \neg\neg\mathcal{P}_3) \wedge \mathcal{P}_4$ rather than the 16 possible combinations of truth values for $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$. To simplify the presentation of this reduced truth table, we shall use $\mathfrak{A}$ to represent the propositional form $\mathcal{P}_1 \vee (\neg\mathcal{P}_2 \wedge \neg\neg\mathcal{P}_3) \wedge \mathcal{P}_4$; the propositional form $(\mathcal{P}_1 \vee (\neg\mathcal{P}_2 \wedge \neg\neg\mathcal{P}_3) \wedge \mathcal{P}_4) \vee \neg(\mathcal{P}_1 \vee (\neg\mathcal{P}_2 \wedge \neg\neg\mathcal{P}_3) \wedge \mathcal{P}_4)$ can then be represented more simply as $\mathfrak{A} \vee \neg\mathfrak{A}$. Thus we can write the reduced truth table as:

$\mathfrak{A}$	$\neg\mathfrak{A}$	$\mathfrak{A} \vee \neg\mathfrak{A}$
T	F	T
F	T	T

From this we see that any instance of the propositional form $\mathfrak{A}$ which has a truth value of T must correspond to an instance of the propositional form $\mathfrak{A} \vee \neg\mathfrak{A}$ which has a truth value of T. Furthermore, any instance of the propositional form $\mathfrak{A}$ which has a truth value of F must again correspond to an instance of the propositional form $\mathfrak{A} \vee \neg\mathfrak{A}$ which has truth value of T. Thus any instance of $\mathfrak{A} \vee \neg\mathfrak{A}$ must have a truth value which is T. Hence the propositional form $(\mathcal{P}_1 \vee (\neg\mathcal{P}_2 \wedge \neg\neg\mathcal{P}_3) \wedge \mathcal{P}_4) \vee \neg(\mathcal{P}_1 \vee (\neg\mathcal{P}_2 \wedge \neg\neg\mathcal{P}_3) \wedge \mathcal{P}_4)$ is a tautology. □

An alternative approach to avoiding large truth tables is to make use of facts from the theory of propositional logic. For ease of reference we shall label this theory of propositional logic as *Prop*. We have already encountered one such fact in the previous example.

Prop1

If $\mathfrak{A}$ refers to any propositional form then the propositional form which corresponds to $\mathfrak{A} \vee \neg\mathfrak{A}$ is a tautology. □

Justification

In example 3.20 we argued that any instance of a propositional form $\mathfrak{A} \vee \neg\mathfrak{A}$ has a truth value of T, no matter what the truth value of the corresponding instance of $\mathfrak{A}$. Although in that example we used $\mathfrak{A}$ to refer to a specific propositional form, the argument would apply whatever form $\mathfrak{A}$ refers to. □

Example 3.21

Use *Prop1* to show that $(\mathcal{Q} \vee \neg\mathcal{R} \wedge \mathcal{S}) \vee \neg(\mathcal{Q} \vee \neg\mathcal{R} \wedge \mathcal{S})$ is a tautology.

Solution

If $\mathfrak{A}$ represents the propositional form $(\mathcal{Q} \vee \neg\mathcal{R} \wedge \mathcal{S})$, then $\mathfrak{A} \vee \neg\mathfrak{A}$ represents the tautology $(\mathcal{Q} \vee \neg\mathcal{R} \wedge \mathcal{S}) \vee \neg(\mathcal{Q} \vee \neg\mathcal{R} \wedge \mathcal{S})$. □

Unfortunately, not all tautologies are covered by *Prop1*. To overcome this we could introduce other facts about propositional logic, such as the fact that $\mathfrak{A} \vee \mathfrak{B} \vee \neg(\mathfrak{A} \wedge \mathfrak{B})$ is a tautology for any propositional forms $\mathfrak{A}$ and $\mathfrak{B}$. Although this approach might seem reasonable, the difficulty is that we could never cover all possibilities. Later we shall see how we can circumvent this problem; for the time being, however, we shall have to use truth tables where necessary.

Exercise 18: Tautologies

Use *Prop1* to show that some of the following propositional forms are tautologies. For each of the remaining forms, construct a truth table to decide if it is a tautology.

1. $\neg(\mathcal{P} \wedge \neg\mathcal{P})$
2. $\mathcal{Q} \vee \neg\mathcal{Q}$
3. $\neg\mathcal{P} \vee \mathcal{P}$
4. $\mathcal{P} \vee \neg\mathcal{P} \vee \mathcal{Q}$
5. $\neg\mathcal{P} \wedge \neg\mathcal{Q} \wedge \neg(\neg\mathcal{P} \wedge \neg\mathcal{Q})$
6. $\mathcal{P} \wedge \neg\mathcal{Q} \vee \neg(\mathcal{P} \wedge \neg\mathcal{Q})$
7. $\mathcal{P} \wedge (\neg\mathcal{P} \vee \mathcal{Q})$

Contradictions

Some propositional forms always have the special property that all instances are false. This leads us to the following definition.

Definition 3.4

A **contradiction** is a propositional form whose truth table is always F. □

Example 3.22

Show that $\mathcal{P} \wedge \neg\mathcal{P}$ is a contradiction.

Solution

The truth table for $\mathcal{P} \wedge \neg\mathcal{P}$ is

$\mathcal{P}$	$\neg\mathcal{P}$	$\mathcal{P} \wedge \neg\mathcal{P}$
T	F	F
F	T	F

□

Example 3.23

Show that $\neg\mathcal{P} \wedge \neg\mathcal{Q} \wedge (\mathcal{P} \vee \mathcal{Q})$ is a contradiction.

Solution

The truth table for $\neg\mathcal{P} \vee \neg\mathcal{Q} \vee \mathcal{P} \wedge \mathcal{Q}$ is

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{P} \wedge \neg\mathcal{Q}$	$\mathcal{P} \vee \mathcal{Q}$	$\neg\mathcal{P} \wedge \neg\mathcal{Q} \wedge (\mathcal{P} \vee \mathcal{Q})$
T	T	F	F	F	T	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	T	F	F

□

We also have the following fact about propositional forms.

Prop2

If $\mathfrak{A}$ refers to any propositional form then the propositional form which corresponds to $\mathfrak{A} \wedge \neg\mathfrak{A}$ is a contradiction. □

Justification

We can relate truth values for instances of $\mathfrak{A}$ to the truth values for instances of $\mathfrak{A} \wedge \neg\mathfrak{A}$ as follows.

$\mathfrak{A}$	$\neg\mathfrak{A}$	$\mathfrak{A} \wedge \neg\mathfrak{A}$
T	F	F
F	T	F

From this we see that all instances of $\mathfrak{A} \wedge \neg\mathfrak{A}$ have a truth value of F. □

Example 3.24

Show $(\mathcal{Q}_1 \wedge \mathcal{Q}_2 \wedge \neg\neg\mathcal{Q}_3 \vee \mathcal{Q}_4) \wedge \neg(\mathcal{Q}_1 \wedge \mathcal{Q}_2 \wedge \neg\neg\mathcal{Q}_3 \vee \mathcal{Q}_4)$ is a contradiction.

Solution

If $\mathfrak{A}$ represents $\mathcal{Q}_1 \wedge \mathcal{Q}_2 \wedge \neg\neg\mathcal{Q}_3 \vee \mathcal{Q}_4$ then $\mathfrak{A} \wedge \neg\mathfrak{A}$ represents the contradiction $(\mathcal{Q}_1 \wedge \mathcal{Q}_2 \wedge \neg\neg\mathcal{Q}_3 \vee \mathcal{Q}_4) \wedge \neg(\mathcal{Q}_1 \wedge \mathcal{Q}_2 \wedge \neg\neg\mathcal{Q}_3 \vee \mathcal{Q}_4)$. □

Exercise 19: Contradictions

Use *Prop2* to show that some of the following propositional forms are contradictions. For each of the remaining forms, construct a truth table to decide if it is a contradiction.

1. $\neg(\neg\mathcal{P} \vee \mathcal{P})$
2. $\neg(\neg\mathcal{P} \wedge \mathcal{P})$
3. $\mathcal{P} \wedge \neg\mathcal{P} \vee \mathcal{Q}$

4. $\mathcal{P} \wedge \neg\mathcal{P} \wedge \mathcal{Q}$
5. $(\mathcal{P} \wedge \neg\mathcal{P} \vee \mathcal{Q}) \wedge \neg(\mathcal{P} \wedge \neg\mathcal{P} \vee \mathcal{Q})$
6. $\mathcal{P} \wedge \neg\mathcal{P} \wedge \mathcal{Q} \wedge \neg(\mathcal{P} \wedge \neg\mathcal{P} \wedge \mathcal{Q})$

3.6 Equivalent propositional forms

Consider the two propositions '*Rex has four legs*' and '*It is not the case that Rex does not have four legs*'. Although the two sentences have different connotations,[1] the denotations are the same: that is, both propositions convey the same information. We say that the two propositions are **equivalent.** One of the functions of logic is to enable us to decide whether any two propositions are equivalent.

First, we define **semantic equivalence** of propositional forms; then we define semantic equivalence of propositions in terms of **corresponding instances** of equivalent propositional forms.

Definition 3.5

Two propositional forms $\mathfrak{A}$ and $\mathfrak{B}$ are semantically equivalent if they have the same truth tables. Usually, we shall simply say that $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent. □

Definition 3.6

Corresponding instances of two propositional forms $\mathfrak{A}$ and $\mathfrak{B}$ are such that any schematic letter common to $\mathfrak{A}$ and $\mathfrak{B}$ is instantiated to the same proposition. □

Definition 3.7

Two propositions p and q are equivalent if they are corresponding instances of equivalent propositional forms $\mathfrak{A}$ and $\mathfrak{B}$. □

Example 3.25

Show that the proposition '*Rex has four legs*' is equivalent to the proposition '*It is not the case that Rex does not have four legs*'.

Solution

The two propositions are instances of $\mathcal{P}$ and $\neg\neg\mathcal{P}$ with $\mathcal{P}$ instantiated to the same proposition, namely '*Rex has four legs*', in each case; the instances correspond. Now the truth table for $\neg\neg\mathcal{P}$ is

$\mathcal{P}$	$\neg\neg\mathcal{P}$
T	T
F	F

from which we see that $\mathcal{P}$ and $\neg\neg\mathcal{P}$ are equivalent propositional forms. Hence the two propositions are equivalent. □

[1] The second sentence carries some sense of denial of a fact that may be held by some people.

Example 3.26

Show that $\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R}$ and $\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})$ are equivalent propositional forms.

Solution

Note that $\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R}$ parses to $(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}$, and is therefore different to $\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})$. We can construct truth tables for these forms as follows.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{P} \wedge \mathcal{Q}$	$(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}$	$\mathcal{Q} \wedge \mathcal{R}$	$\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})$
T	*T*	*T*	*T*	***T***	*T*	***T***
T	*T*	*F*	*T*	***F***	*F*	***F***
T	*F*	*T*	*F*	***F***	*F*	***F***
T	*F*	*F*	*F*	***F***	*F*	***F***
F	*T*	*T*	*F*	***F***	*T*	***F***
F	*T*	*F*	*F*	***F***	*F*	***F***
F	*F*	*T*	*F*	***F***	*F*	***F***
F	*F*	*F*	*F*	***F***	*F*	***F***

The truth tables for $\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R}$ and $\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})$ are shown in bold, and are clearly the same. The two propositional forms are equivalent. □

Example 3.27

What is a suitable symbolic representation of '*Venus, Earth and Mars are planets*'?

Solution

Two possible symbolic representations of this are

('*Venus is a planet*' ∧ '*Earth is a planet*') ∧ '*Mars is a planet*'

and

'*Venus is a planet*' ∧ ('*Earth is a planet*' ∧ '*Mars is a planet*').

Now these are corresponding instances of the equivalent forms $(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}$ and $\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})$, and hence must also be equivalent. For this reason, it does not matter too much which version we write. Usually we choose the first of the two forms,simply because we can drop the parentheses to give

'*Venus is a planet*' ∧ '*Earth is a planet*' ∧ '*Mars is a planet*'

Strictly speaking, neither of the two versions we have considered truly conveys the sense of the original, which may be regarded as a simultaneous conjunction of three atomic propositions. It *is* possible to define connectives of more than two operands; in particular we could define an extended version of conjunction which applies to any number of atomic propositions. If we were to do this, however, then the resulting expression would again prove equivalent to the two we have already considered. For this reason, the symbolic representation using ordinary conjunction is satisfactory for the purposes of logic. □

Exercise 20: Semantic equivalence

1. For which of the following pairs of propositional forms are the two forms equivalent?

   (a) $\mathcal{P},\ \neg\mathcal{P}$
   (b) $\neg\neg\neg\mathcal{Q},\ \neg\mathcal{Q}$
   (c) $\mathcal{P} \vee \mathcal{P},\ \mathcal{P}$
   (d) $\mathcal{P} \wedge \mathcal{P},\ \mathcal{P}$
   (e) $\mathcal{P} \vee \mathcal{Q},\ \mathcal{P}$
   (f) $\mathcal{P} \vee \mathcal{Q},\ \mathcal{P} \wedge \mathcal{Q}$
   (g) $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}),\ (\mathcal{P} \wedge \mathcal{Q}) \vee \mathcal{R}$
   (h) $\mathcal{P} \wedge \mathcal{Q},\ \mathcal{Q} \wedge \mathcal{P}$
   (i) $\mathcal{P} \vee \mathcal{Q}\ \ \mathcal{Q} \vee \mathcal{P}$
   (j) $\neg(\mathcal{P} \wedge \mathcal{Q}),\ \neg\mathcal{P} \wedge \neg\mathcal{Q}$
   (k) $\neg(\mathcal{P} \vee \mathcal{Q}),\ \neg\mathcal{P} \wedge \neg\mathcal{Q}$
   (l) $\neg(\mathcal{P} \wedge \mathcal{Q}),\ \neg\mathcal{P} \vee \neg\mathcal{Q}$
   (m) $\neg(\mathcal{P} \vee \mathcal{Q}),\ \neg\mathcal{P} \vee \neg\mathcal{Q}$
   (n) $\mathcal{P} \vee \mathcal{Q} \vee \mathcal{R},\ \mathcal{P} \vee (\mathcal{Q} \vee \mathcal{R})$
   (o) $\mathcal{P} \vee \mathcal{Q} \wedge \mathcal{R},\ (\mathcal{P} \vee \mathcal{Q}) \wedge \mathcal{R}$
   (p) $\mathcal{P} \vee \mathcal{Q} \wedge \mathcal{R},\ (\mathcal{P} \vee \mathcal{Q}) \wedge (\mathcal{P} \vee \mathcal{R})$
   (q) $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}),\ \mathcal{P} \wedge \mathcal{Q} \vee \mathcal{P} \wedge \mathcal{R}$
   (r) $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}),\ \mathcal{P} \wedge \mathcal{Q} \vee \mathcal{R}$

2. For which of the following pairs of propositions are the two propositions equivalent?

   (a) $3 > 0\ \wedge\ 7 > 3$,
   $7 > 3\ \wedge\ 3 > 0$
   (b) $3 > 0\ \wedge\ 7 > 3$,
   $7 > 3\ \wedge\ 3 > 2\ \wedge\ 2 > 0$
   (c) $\neg$(‘*Rex has four legs*’ $\wedge$ ‘*Fido has three legs*’),
   $\neg$‘*Rex has four legs*’ $\vee$ $\neg$‘*Fido has three legs*’
   (d) $\neg\neg$‘*The sky is blue*’,
   ‘*The sky is blue*’ $\vee$ ‘*The sun is golden*’

Notation for semantic equivalence

We have already met the symbol $=_T$ to represent equality of truth values. For example, we can indicate that $17 + 32 = 49$ and $3^2 = 9$ have the same truth values by writing $17 + 32 = 49\ =_T\ 3^2 = 9$.

But suppose we write $\mathcal{P} \wedge \mathcal{Q} =_T \mathcal{Q} \wedge \mathcal{P}$. What is signified by this expression? To answer this question, consider what happens when we take any instance of $\mathcal{P}$ and $\mathcal{Q}$. For example, if we take ‘*Rex has four legs*’ as the instance of $\mathcal{P}$ and ‘*Fido has three legs*’ as the instance of $\mathcal{Q}$, we get an equality of truth values of two compound propositions:

‘*Rex has four legs*’ $\wedge$ ‘*Fido has three legs*’
$=_T$ ‘*Fido has three legs*’ $\wedge$ ‘*Rex has four legs*’

But whatever instance we take for $\mathcal{P}$ and $\mathcal{Q}$, we would get an equality of truth values for the corresponding instances of $\mathcal{P} \wedge \mathcal{Q}$ and $\mathcal{Q} \wedge \mathcal{P}$. Thus the expression

$$\mathcal{P} \wedge \mathcal{Q} =_T \mathcal{Q} \wedge \mathcal{P}$$

denotes that $\mathcal{P} \wedge \mathcal{Q}$ and $\mathcal{Q} \wedge \mathcal{P}$ are equivalent propositional forms.

Notation: $\mathfrak{A} =_T \mathfrak{B}$

$\mathfrak{A} =_T \mathfrak{B}$ denotes that two propositional forms $\mathfrak{A}$ and $\mathfrak{B}$ are semantically equivalent. □

Example 3.28

The equivalence of $\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R}$ and $\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})$ can be represented as $\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R} =_T \mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})$ □

Note that $=_T$ is used to indicate either that two propositions have the same truth value, or that that two propositional forms are equivalent; it cannot, however, be used to indicate that two propositions are equivalent. For example, we write $17+32 = 49 =_T 3^2 = 9$, even though $17+32 = 49$ and $3^2 = 9$ are *not* equivalent. A different notation for expressing the equivalence of propositions is needed; this will be introduced in Chapter 4.

Example 3.29

What does the equivalence $\mathcal{P} \vee \neg\mathcal{P} =_T T$ signify?

Solution

By definition, corresponding instances of the propositional forms either side of the $=_T$ symbol have the same truth values. If we regard T as a special kind of schematic letter, then we read the equivalence as asserting that any instance of $\mathcal{P} \vee \neg\mathcal{P}$ must have the same truth value as any instance of T. But only true instances of T are allowed. Hence any instance of $\mathcal{P} \vee \neg\mathcal{P}$ is true: the equivalence asserts that $\mathcal{P} \vee \neg\mathcal{P}$ is a tautology.

Normally we think of T as a truth value, rather than as a special kind of schematic letter. We then read the equivalence as simply stating that any instance of $\mathcal{P} \vee \neg\mathcal{P}$ has a truth value of T. □

3.7 Some laws of equivalence

There are infinitely many equivalences between propositional forms. Some of the more important of these are summarized in Table 3.1. They represent useful properties of propositional forms. Logicians and mathematicians often have special names for these properties, and for completeness they are given in the table. But the reader is advised not to try to memorize this terminology as little use will be made of it in this book.

Consider the first of these equivalences. For any propositional form $\mathfrak{A}$, the propositional form $\mathfrak{A} \vee \neg\mathfrak{A}$ is equivalent T. Thus, any instance of $\mathfrak{A} \vee \neg\mathfrak{A}$ must have a truth value of T. Therefore $\mathfrak{A} \vee \neg\mathfrak{A}$ is a tautology. We have already

Tautology

Prop1 $\mathfrak{A} \vee \neg\mathfrak{A} =_T T$

Contradiction

Prop2 $\mathfrak{A} \wedge \neg\mathfrak{A} =_T F$

Unit

Prop3 $\mathfrak{A} \vee F =_T \mathfrak{A}$

Prop4 $\mathfrak{A} \wedge T =_T \mathfrak{A}$

Zero

Prop5 $\mathfrak{A} \vee T =_T T$

Prop6 $\mathfrak{A} \wedge F =_T F$

Idempotent

Prop7 $\mathfrak{A} \vee \mathfrak{A} =_T \mathfrak{A}$

Prop8 $\mathfrak{A} \wedge \mathfrak{A} =_T \mathfrak{A}$

Double negation

Prop9 $\neg\neg\mathfrak{A} =_T \mathfrak{A}$

Commutative

Prop10 $\mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}$

Prop11 $\mathfrak{A} \wedge \mathfrak{B} =_T \mathfrak{B} \wedge \mathfrak{A}$

Associative

Prop12 $\mathfrak{A} \vee (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \vee \mathfrak{C}$

Prop13 $\mathfrak{A} \wedge (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \wedge \mathfrak{C}$

Distributive

Prop14 $\mathfrak{A} \vee (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \wedge (\mathfrak{A} \vee \mathfrak{C})$

Prop15 $\mathfrak{A} \wedge (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \vee (\mathfrak{A} \wedge \mathfrak{C})$

Absorption

Prop16 $\mathfrak{A} \vee (\mathfrak{A} \wedge \mathfrak{B}) =_T \mathfrak{A}$

Prop17 $\mathfrak{A} \wedge (\mathfrak{A} \vee \mathfrak{B}) =_T \mathfrak{A}$

de Morgan

Prop18 $\neg(\mathfrak{A} \vee \mathfrak{B}) =_T \neg\mathfrak{A} \wedge \neg\mathfrak{B}$

Prop19 $\neg(\mathfrak{A} \wedge \mathfrak{B}) =_T \neg\mathfrak{A} \vee \neg\mathfrak{B}$

Table 3.1: Equivalences involving T, F, $\neg$, $\wedge$ and $\vee$

met this fact, and given it the label *Prop1*. The second equivalence in the list asserts that for any propositional form $\mathfrak{A}$, the propositional form $\mathfrak{A} \wedge \neg\mathfrak{A}$ is a contradiction. This is a restatement of the fact *Prop2* we met earlier.

Note that we have made reference to the 'propositional form $\mathfrak{A} \vee \neg\mathfrak{A}$'. Care must be taken in understanding what this means. The expression $\mathfrak{A} \vee \neg\mathfrak{A}$ is not itself a propositional form, but is a way of referring to any propositional form whose main connective is disjunction, and for which the right operand is the negation of the left; any such propositional form is a tautology. Likewise, the expression $\mathfrak{A} \wedge \neg\mathfrak{A}$ refers to any propositional form whose main connective is conjunction, and for which the right operand is the negation of the left; any such propositional form is a contradiction. Furthermore, an expression such as $\mathfrak{A} \vee \neg\mathfrak{A} =_T T$ is not, strictly speaking, an equivalence. Instead it represents an infinite collection of equivalences. To avoid confusion, we shall refer to the results listed in Table 3.1 as **laws.**

Example 3.30

Show that $\neg\mathcal{P}_2 \vee \mathcal{P}_1 \vee F$ and $(\neg\mathcal{P}_2 \vee \mathcal{P}_1) \wedge T$ are both equivalent to $\neg\mathcal{P}_2 \vee \mathcal{P}_1$ using the laws of equivalence. Confirm these results with truth tables.

Solution

Suppose $\mathfrak{A}$ represents the propositional form $\neg\mathcal{P}_2 \vee \mathcal{P}_1$, then $\mathfrak{A} \vee F$ represents the propositional form $\neg\mathcal{P}_2 \vee \mathcal{P}_1 \vee F$, and $\mathfrak{A} \wedge T$ represents the propositional form $(\neg\mathcal{P}_2 \vee \mathcal{P}_1) \wedge T$. From *Prop3* we see that

$$\neg\mathcal{P}_2 \vee \mathcal{P}_1 =_T \neg\mathcal{P}_2 \vee \mathcal{P}_1 \vee F$$

while from *Prop4* we see that

$$\neg\mathcal{P}_2 \vee \mathcal{P}_1 =_T (\neg\mathcal{P}_2 \vee \mathcal{P}_1) \wedge T$$

These results are conformed by the truth tables as shown.

$\mathcal{P}_1$	$\mathcal{P}_2$	$\neg\mathcal{P}_2$	$\neg\mathcal{P}_2 \vee \mathcal{P}_1$	$\neg\mathcal{P}_2 \vee \mathcal{P}_1 \vee F$	$(\neg\mathcal{P}_2 \vee \mathcal{P}_1) \wedge T$
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	F	F	F
F	F	F	T	T	T

We say that F is a **unit** for disjunction and T is a unit for conjunction. □

The derivation of some equivalences necessitate the use of more than one law listed in Table 3.1. To achieve this, we combine laws in a way that is familiar from the use of equality in ordinary arithmetic. In fact we shall be using special properties of truth equality, even though we may not be consciously aware of doing so. For example we shall make use of a property of truth equality $=_T$ known as **transitivity.**

Transitivity of $=_T$

If $\mathfrak{A} =_T \mathfrak{B}$ and $\mathfrak{B} =_T \mathfrak{C}$ then $\mathfrak{A} =_T \mathfrak{C}$. □

Justification

Suppose we choose any pair of corresponding instances of $\mathfrak{A}$ and $\mathfrak{C}$. We need to show that for any such pair of instances, the truth values are equal.

Since the instances of $\mathfrak{A}$ and $\mathfrak{C}$ correspond then, by definition, all the schematic letters common to both $\mathfrak{A}$ and $\mathfrak{C}$ are instantiated to the same propositions. Hence it is possible to choose an instance of $\mathfrak{B}$ such that all schematic letters in common with $\mathfrak{A}$ are instantiated to the same propositions; and all schematic letters in common with $\mathfrak{C}$ are instantiated to the same propositions. Thus our chosen instance of $\mathfrak{B}$ corresponds to both $\mathfrak{A}$ and $\mathfrak{C}$. But since $\mathfrak{A} =_T \mathfrak{B}$ the truth values of the instances of $\mathfrak{A}$ and $\mathfrak{B}$ are equal; likewise, since $\mathfrak{B} =_T \mathfrak{C}$, the truth values of the instances of $\mathfrak{B}$ and $\mathfrak{C}$ are equal. Thus the truth values of $\mathfrak{A}$ and $\mathfrak{C}$ are equal, as required. □

Although this may seem somewhat confusing to may readers, the application of this property is straightforward enough as the following example shows.

Example 3.31

Show that $\neg\mathcal{P} \vee \mathcal{P}$ is a tautology.

Solution

It would be wrong to use *Prop1* directly, since that law applies only to disjunctions where the right operand is the negation of the left. In $\neg\mathcal{P} \vee \mathcal{P}$ the left operand is the negation of the right. Fortunately, the law *Prop10* enables us to swap the two operands around:

$$\neg\mathcal{P} \vee \mathcal{P} =_T \mathcal{P} \vee \neg\mathcal{P}$$

This equivalence is an application of the **commutative** property of disjunction. We can now apply the tautology law *Prop1* to the righthand side:

$$\mathcal{P} \vee \neg\mathcal{P} =_T T$$

Combining these two equivalences we get the result

$$\neg\mathcal{P} \vee \mathcal{P} =_T T$$

from which we see that $\neg\mathcal{P} \vee \mathcal{P}$ is a a tautology, as required. This argument can be summarized as follows, with the law used at each stage being indicated.

$$\begin{array}{lll} & \neg\mathcal{P} \vee \mathcal{P} & \\ =_T & \mathcal{P} \vee \neg\mathcal{P} & \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle \\ =_T & T & \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle \end{array}$$

Notice that the transitivity property of $=_T$ is not stated explicitly. It is used naturally in linking the truth equalities together. □

Symmetry
If $\mathfrak{A} =_T \mathfrak{B}$ then $\mathfrak{B} =_T \mathfrak{A}$.

Transitivity
If $\mathfrak{A} =_T \mathfrak{B}$ and $\mathfrak{B} =_T \mathfrak{C}$ then $\mathfrak{A} =_T \mathfrak{C}$

Substitution of equivalents
Suppose the propositional form $\mathfrak{B}_1$ is a subexpression of a *fully parsed* propositional form $\mathfrak{A}_1$. Further suppose that $\mathfrak{A}_2$ is the propositional form obtained by replacing $\mathfrak{B}_1$ with an equivalent form $\mathfrak{B}_2$. Then $\mathfrak{A}_1 =_T \mathfrak{A}_2$.

Table 3.2: Properties of $=_T$

Other properties of $=_T$ are summarized in Table 3.2. The symmetry property allows us to use a law either way round, as illustrated in the following example.

Example 3.32
Show that $(\neg\mathcal{P} \vee \mathcal{Q}) \wedge (\neg\mathcal{P} \vee \mathcal{R}) =_T \neg\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$.

Solution

$$\begin{array}{lll} & (\neg\mathcal{P} \vee \mathcal{Q}) \wedge (\neg\mathcal{P} \vee \mathcal{R}) & \\ =_T & \neg\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}) & \langle\ \mathfrak{A} \vee (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \wedge (\mathfrak{A} \vee \mathfrak{C})\ \rangle \end{array}$$

□

The substitution property enables us to replace a subexpression in a propositional form by an equivalent expression. However, it is very important to parse the propositional form first.

Example 3.33
Show that $\mathcal{P} \vee \mathcal{Q} \vee \neg\mathcal{Q}$ is a tautology.

Solution
Without parsing the propositional form, it might seem that we could simply replace $\mathcal{Q} \vee \neg\mathcal{Q}$ by T. However, this would be incorrect reasoning even though it would lead to the right answer! The fully parsed expression is $(\mathcal{P} \vee \mathcal{Q}) \vee (\neg\mathcal{Q})$, from which we can see that $\mathcal{Q} \vee \neg\mathcal{Q}$ is not a subexpression; furthermore, the subexpression $\mathcal{Q}) \vee (\neg\mathcal{Q})$ is not a well formed propositional form.

$$\begin{array}{lll} & (\mathcal{P} \vee \mathcal{Q}) \vee \neg\mathcal{Q} & \\ =_T & \mathcal{P} \vee (\mathcal{Q} \vee \neg\mathcal{Q}) & \langle\ \mathfrak{A} \vee (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \vee \mathfrak{C}\ \rangle \\ =_T & \mathcal{P} \vee T & \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle \\ =_T & T & \langle\ \mathfrak{A} \vee T =_T T\ \rangle \end{array}$$

□

Always parse logical expressions even though you may not show this parsing explicitly.

Table 3.3: A Golden Rule for Doing Logic

Note that the presentation in this last example has been made easier to read by dropping the parentheses around the negation. Although it is indeed necessary to parse a propositional form fully before using the substitution property, this parsing may be done 'in the head' without writing it down explicitly. To avoid overloading the reader with too many parentheses, most logicians adopt this principle. Unfortunately, this can be confusing to the beginner and is possibly the underlying reason for many errors made by students. The reader would do well to remember the *Golden Rule for Doing Logic* presented in Table 3.3.

Example 3.34

Simplify $\mathcal{P} \vee \neg\mathcal{P} \wedge \mathcal{Q}$.

Solution

To simplify a propositional form means to find an equivalent form with fewer connectives or fewer occurrences of schematic letters. The maximum simplification in this example would normally be written as follows.

$$
\begin{array}{lll}
 & \mathcal{P} \vee \neg\mathcal{P} \wedge \mathcal{Q} & \\
=_T & (\mathcal{P} \vee \neg\mathcal{P}) \wedge (\mathcal{P} \vee \mathcal{Q}) & \langle\ \mathfrak{A} \vee (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \wedge (\mathfrak{A} \vee \mathfrak{C})\ \rangle \\
=_T & T \wedge (\mathcal{P} \vee \mathcal{Q}) & \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle \\
=_T & (\mathcal{P} \vee \mathcal{Q}) \wedge T & \langle\ \mathfrak{A} \wedge \mathfrak{B} =_T \mathfrak{B} \wedge \mathfrak{A}\ \rangle \\
=_T & \mathcal{P} \vee \mathcal{Q} & \langle\ \mathfrak{A} \wedge T =_T \mathfrak{A}\ \rangle
\end{array}
$$

When reading such a presentation, however, always bear in mind the parsed expressions. Although this requires greater effort, the result will reflect more closely the mental reasoning of the writer. In this case, the reader may find it very helpful to write out the working with extra parentheses added in order to emphasize the parsing.

$$
\begin{array}{lll}
 & \mathcal{P} \vee (\neg\mathcal{P} \wedge \mathcal{Q}) & \\
=_T & (\mathcal{P} \vee \neg\mathcal{P}) \wedge (\mathcal{P} \vee \mathcal{Q}) & \langle\ \mathfrak{A} \vee (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \wedge (\mathfrak{A} \vee \mathfrak{C})\ \rangle \\
=_T & T \wedge (\mathcal{P} \vee \mathcal{Q}) & \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle \\
=_T & (\mathcal{P} \vee \mathcal{Q}) \wedge T & \langle\ \mathfrak{A} \wedge \mathfrak{B} =_T \mathfrak{B} \wedge \mathfrak{A}\ \rangle \\
=_T & \mathcal{P} \vee \mathcal{Q} & \langle\ \mathfrak{A} \wedge T =_T \mathfrak{A}\ \rangle
\end{array}
$$

□

Example 3.35

Show that $\neg\neg\mathcal{P} \wedge \mathcal{Q} \vee \mathcal{R} =_T \neg(\neg\mathcal{R} \wedge (\neg\mathcal{P} \vee \neg\mathcal{Q}))$

Solution

$$
\begin{array}{lll}
 & (\neg\neg\mathcal{P} \wedge \mathcal{Q}) \vee \mathcal{R} & \\
=_T & (\neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q}) \vee \mathcal{R} & \langle\ \neg\neg\mathfrak{A} =_T \mathfrak{A}\ \rangle \\
=_T & \neg(\neg\mathcal{P} \vee \neg\mathcal{Q}) \vee \mathcal{R} & \langle\ \neg(\mathfrak{A} \vee \mathfrak{B} =_T \neg\mathfrak{A} \wedge \neg\mathfrak{B})\ \rangle \\
=_T & \mathcal{R} \vee \neg(\neg\mathcal{P} \vee \neg\mathcal{Q}) & \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle \\
=_T & \neg\neg\mathcal{R} \vee \neg(\neg\mathcal{P} \vee \neg\mathcal{Q}) & \langle\ \neg\neg\mathfrak{A} =_T \mathfrak{A}\ \rangle \\
=_T & \neg(\neg\mathcal{R} \wedge (\neg\mathcal{P} \vee \neg\mathcal{Q})) & \langle\ \neg(\mathfrak{A} \wedge \mathfrak{B} =_T \neg\mathfrak{A} \vee \neg\mathfrak{B})\ \rangle
\end{array}
$$

□

Exercise 21: Using laws of equivalence

Prove the following using the laws of equivalence.

1. $\neg\mathcal{P} \wedge \mathcal{P}$ is a contradiction.
2. $\neg(\mathcal{P} \vee \neg\mathcal{Q}) \vee (\mathcal{P} \vee \neg\mathcal{Q})$ is a tautology.
3. $\mathcal{P} \wedge \mathcal{Q} \vee \mathcal{P} =_T \mathcal{P}$.
4. $(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{P} =_T \mathcal{P} \wedge \mathcal{Q}$.
5. $(\mathcal{P} \vee \neg\mathcal{P}) \wedge \neg(\mathcal{Q} \wedge \neg\mathcal{Q})$ is a tautology.
6. $\neg T =_T F$.
7. $\neg F =_T T$.

3.8 Semantic entailment

In earlier sections we have looked at the properties of individual propositional forms; in particular we have defined the concepts of tautology and contradiction. We have also looked at the properties of pairs of propositional forms; in particular we have defined the concept of equivalence. In this section we shall look at properties involving sets of propositional forms.

Sets of propositional forms

In logic, we frequently need to refer to a set of propositional forms. The concept of set was introduced in section 1.8. In a set the order in which items are listed is immaterial, as are repetitions of the same item. To indicate a set, the usual mathematical convention is to enclose the list of items between curly braces $\{\ldots\}$. For example, suppose we have

$$\{\mathcal{P} \wedge \mathcal{Q},\ \neg\mathcal{R}\} = \{\neg\mathcal{R},\ \mathcal{P} \wedge \mathcal{Q}\} = \{\neg\mathcal{R},\ \mathcal{P} \wedge \mathcal{Q},\ \neg\mathcal{R}\}$$

All three expressions refer to the same set of two items. It might seem, therefore, that whenever we refer to a set of forms in logic we should enclose the corresponding list between braces. However, in practice it is not necessary to do this. When we need to represent lists in logic, they are usually presented as more than simple lists: for example, as enumerated lists: for example,

1. $\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})$
2. $\mathcal{P}$
3. $\mathcal{Q} \wedge \mathcal{R}$
4. $\mathcal{R}$
5. $\mathcal{R} \wedge \mathcal{P}$

We adopt the convention that a simple list always refers to the corresponding set. For example, $\mathcal{P} \wedge \mathcal{Q}, \neg\mathcal{R}$ refers to the set $\{\mathcal{P} \wedge \mathcal{Q},\ \neg\mathcal{R}\}$; we could equally well write $\neg\mathcal{R},\ \mathcal{P} \wedge \mathcal{Q}$ or even $\neg\mathcal{R},\ \mathcal{P} \wedge \mathcal{Q}$.

In section 3.4 we introduced the ideas of a formal language of propositional forms, and a metalanguage for talking about propositional forms. The curly braces $\{\ldots\}$ are part of the metalanguage. The concept of set union $\cup$ (defined in section 1.8) may also be used as part of the metalanguage. For example, we may write

$$\{\mathcal{P} \wedge \mathcal{Q},\ \neg\mathcal{R}\} \cup \{\mathcal{P} \vee \neg\mathcal{R},\ \mathcal{S} \vee \mathcal{P},\ \neg\mathcal{Q},\ \neg\mathcal{R}\} = \{\mathcal{P} \wedge \mathcal{Q},\ \neg\mathcal{R},\ \mathcal{P} \vee \neg\mathcal{R},\ \mathcal{S} \vee \mathcal{P},\ \neg\mathcal{Q}\}$$

Note however, that these three sets would normally be written more simply as

$$\mathcal{P} \wedge \mathcal{Q},\ \neg\mathcal{R}, \quad \mathcal{P} \vee \neg\mathcal{R},\ \mathcal{S} \vee \mathcal{P},\ \neg\mathcal{Q},\ \neg\mathcal{R} \quad \text{and} \quad \mathcal{P} \wedge \mathcal{Q},\ \neg\mathcal{R},\ \mathcal{P} \vee \neg\mathcal{R},\ \mathcal{S} \vee \mathcal{P},\ \neg\mathcal{Q}\,.$$

In addition, we also need symbols in the metalanguage to represent sets of propositional forms.

Notation: Γ

The symbol Γ – the capital Greek letter 'gamma' –is used to denote a set of propositional forms. When we need to refer to more than one sets, we shall use subscripts: Γ_1, Γ_2 for example. □

Often we need to refer to the union of two sets. This can be represented as $\Gamma_1 \cup \Gamma_2$, or simply as Γ_1, Γ_2 when there is no possibility of confusion.

We also need to be able to indicate a set having a specific number of propositional forms. For example we might write $\{\mathfrak{A}\}$ to indicate a set with just one form; or $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\}$ to indicate a set with n distinct forms. A rather special case arises when we write $\{\ \}$; this indicates a set having *no* propositional forms. Although this may seem a rather absurd idea, it turns out to be a very important one. The set is known as the **empty set**. In the formal language, it is represented simply by leaving a blank.

Exercise 22: Sets of propositional forms

What might each of the following expressions represent, if anything?

1. $\{\mathcal{P} \wedge \mathcal{Q},\ \neg\mathcal{Q}\}$
2. $\mathcal{P} \wedge \mathcal{Q},\ \neg\mathcal{Q}$
3. $\{\neg\mathcal{Q}\}$
4. $\neg\mathcal{Q}$

5. $\{F\}$
6. F
7. $\{\mathfrak{A} \wedge \mathfrak{B}, \neg\mathfrak{B}\}$
8. $\mathfrak{A} \wedge \mathfrak{B}, \neg\mathfrak{C}$
9. $\mathfrak{A}_1 \vee F, T \wedge \mathfrak{A}_2$
10. $\mathfrak{A} \wedge \mathfrak{B}$
11. $\{\mathcal{Q} \wedge \mathcal{R}, \mathcal{Q} \wedge \mathcal{R}\}$
12. $(\neg\mathcal{P}, \mathcal{Q} \wedge \mathcal{R})$
13. $\Gamma \cup \{\mathfrak{A}\}$
14. $\Gamma, \mathfrak{A}, \mathfrak{B}$
15. $\Gamma \cup \mathfrak{A}$
16. $\{\ \} \cup \{\mathcal{P}_1 \vee \neg\mathcal{P}_2\}$

Semantic entailment

Semantic entailment is a relationship between a set Γ of propositional forms and a single propositional form $\mathfrak{A}$. The concept can be illustrated by a simple example.

Example 3.36

Suppose we have a set of propositional forms $\{\mathcal{P} \wedge \mathcal{Q}, \mathcal{Q} \wedge \mathcal{R}\}$ and a further single form $\mathcal{P} \wedge \mathcal{R}$. Now the truth tables for all three forms are:

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{P} \wedge \mathcal{Q}$	$\mathcal{Q} \wedge \mathcal{R}$	$\mathcal{P} \wedge \mathcal{R}$
T	*T*	*T*	*T*	*T*	*T*
T	*T*	*F*	*T*	***F***	***F***
T	*F*	*T*	***F***	***F***	***F***
T	*F*	*F*	***F***	***F***	*T*
F	*T*	*T*	***F***	*T*	***F***
F	*T*	*F*	***F***	***F***	***F***
F	*F*	*T*	***F***	***F***	***F***
F	*F*	*F*	***F***	***F***	***F***

From these truth tables we see that whenever an instance of $\mathcal{P} \wedge \mathcal{R}$ has a truth value of F, at least one of the corresponding instances in the set$\{\mathcal{P} \wedge \mathcal{Q}, \mathcal{Q} \wedge \mathcal{R}\}$ also has a truth value of F. We say that the set $\{\mathcal{P} \wedge \mathcal{Q}, \mathcal{Q} \wedge \mathcal{R}\}$ semantically entails $\mathcal{P} \wedge \mathcal{R}$. □

Definition 3.8

Suppose that for every false instance of $\mathfrak{A}$ there is a corresponding false instance of a propositional form in Γ. We say that the set Γ **semantically entails** the propositional form $\mathfrak{A}$. □

Example 3.37

Find an entailment involving all the three propositional forms $\mathcal{P} \vee \mathcal{Q}$, $\mathcal{P}$ and $\mathcal{Q}$.

Solution

If an instance of $\mathcal{P} \vee \mathcal{Q}$ is false the corresponding instances of both $\mathcal{P}$ and $\mathcal{Q}$ are false. There are in fact three possible entailments:

- $\{\mathcal{P}, \mathcal{Q}\}$ entails $\mathcal{P} \vee \mathcal{Q}$;
- $\{\mathcal{P}\}$ entails $\mathcal{P} \vee \mathcal{Q}$;
- $\{\mathcal{Q}\}$ entails $\mathcal{P} \vee \mathcal{Q}$.

Of these, only the first involves all three forms: $\mathcal{P}, \mathcal{Q} \vDash \mathcal{P} \vee \mathcal{Q}$. □

Notation: $\vDash$

If Γ semantically entails $\mathfrak{A}$ then we write

$$\Gamma \vDash \mathfrak{A}$$

The symbol, $\vDash$, is called the **semantic turnstile** and is read as 'entails'. □

Example 3.38

Use the semantic turnstile $\vDash$ to express the fact that $\{\mathcal{P} \wedge \mathcal{Q}, \mathcal{Q} \wedge \mathcal{R}\}$ semantically entails $\mathcal{P} \wedge \mathcal{R}$.

Solution

Using the full set notation we might write $\{\mathcal{P} \wedge \mathcal{Q}, \mathcal{Q} \wedge \mathcal{R}\} \vDash \mathcal{P} \wedge \mathcal{R}$. However, it is customary to miss out the braces and simply write

$$\mathcal{P} \wedge \mathcal{Q}, \mathcal{Q} \wedge \mathcal{R} \vDash \mathcal{P} \wedge \mathcal{R}$$

□

Prop20

If $\Gamma \vDash \mathfrak{A}$ then whenever all the corresponding instances of Γ are true, then the corresponding instance of $\mathfrak{A}$ is also true. □

Justification

Suppose that the corresponding instance of $\mathfrak{A}$ *were* false. Then since $\Gamma \vDash \mathfrak{A}$, at least one of the corresponding instances of Γ would also have to be false. But we know that this is not the case. Therefore, the instance of $\mathfrak{A}$ cannot be false; it must be true. □

Prop21

Suppose that whenever all the corresponding instances of Γ are true then the corresponding instance of $\mathfrak{A}$ is also true. Then $\Gamma \vDash \mathfrak{A}$. This is often used as the definition of semantic entailment. □

Justification

Suppose that Γ did *not* entail $\mathfrak{A}$. This would mean that for at least one false instance of $\mathfrak{A}$ all the corresponding instances of Γ would be true. But we know that this is not the case. Therefore Γ must entail $\mathfrak{A}$: $\Gamma \vDash \mathfrak{A}$. □

Example 3.39

We know from the truth table for ∧ that whenever both $\mathcal{P}_1$ and $\mathcal{P}_2$ have truth values T, then so does $\mathcal{P}_1 \wedge \mathcal{P}_2$. Hence $\mathcal{P}_1, \mathcal{P}_2 \vDash \mathcal{P}_1 \wedge \mathcal{P}_2$. □

Example 3.40

Find an entailment involving just the four propositional forms $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$.

Solution

From the definitions of ∧ and ∨ we know that whenever $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are all true then so is $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$. Hence

$$\mathcal{P}, \mathcal{Q}, \mathcal{R} \vDash \mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$$

Is this the only entailment involving the four propositional forms? To answer this question we need to construct the truth table for $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{Q} \vee \mathcal{R}$	$\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$
T	*T*	*T*	*T*	*T*
T	*T*	***F***	*T*	*T*
T	***F***	*T*	*T*	*T*
T	*F*	*F*	*F*	*F*
F	*T*	*T*	*T*	***F***
F	*T*	*F*	*T*	***F***
F	*F*	*T*	*T*	***F***
F	*F*	*F*	*F*	***F***

We see from the second row that it is possible to have a false instance of $\mathcal{R}$ with the corresponding instances of $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$ all true. From the definition of semantic entailment it cannot be the case that $\mathcal{R}$ is entailed by the other three forms. Likewise from the third row of the truth table, we can also see that $\mathcal{Q}$ cannot be entailed by the other three forms. However, from the last four rows of the truth table we see that for all false instances of $\mathcal{P}$ the corresponding instance of $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$ is also false. Hence any set containing $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$ will entail $\mathcal{P}$. In particular $\mathcal{Q}, \mathcal{R}, \mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}) \vDash \mathcal{P}$. □

The set of propositional forms Γ may contain just one element or may even be the empty set. If Γ contains just one propositional form, $\mathfrak{B}$ say, and entails the propositional form $\mathfrak{A}$ then we normally write $\mathfrak{B} \vDash \mathfrak{A}$; the fact that $\mathfrak{A}$ represents the set $\{\mathfrak{A}\}$ is obvious form the context. Such an entailment would mean that whenever $\mathfrak{A}$ is false, then so is $\mathfrak{B}$; and that whenever $\mathfrak{B}$ is true so is $\mathfrak{A}$.

Example 3.41

Whenever $\mathcal{P} \wedge \mathcal{Q}$ is true both $\mathcal{P}$ and $\mathcal{Q}$ are true. How might this be expressed using entailments?

Solution

Two entailments are required: $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{P}$ and $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{Q}$. □

Prop22

Two propositional forms $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent, that is $\mathfrak{A} =_T \mathfrak{B}$, if and only if both $\mathfrak{A} \vDash \mathfrak{B}$ and $\mathfrak{B} \vDash \mathfrak{A}$. □

Justification

Suppose first that we are given $\mathfrak{A} =_T \mathfrak{B}$. Then corresponding instances of $\mathfrak{A}$ and $\mathfrak{B}$ always have the same truth values. Thus, whenever, $\mathfrak{A}$ is true then $\mathfrak{B}$ is also true: $\mathfrak{A} \vDash \mathfrak{B}$. Also, whenever, $\mathfrak{A}$ is false then $\mathfrak{B}$ is also false: $\mathfrak{B} \vDash \mathfrak{A}$. Hence if $\mathfrak{A} =_T \mathfrak{B}$, then it follows that both $\mathfrak{A} \vDash \mathfrak{B}$ and $\mathfrak{B} \vDash \mathfrak{A}$.

Now suppose instead that we are given that both $\mathfrak{A} \vDash \mathfrak{B}$ and $\mathfrak{B} \vDash \mathfrak{A}$. From the first entailment it follows that whenever $\mathfrak{A}$ is true then $\mathfrak{B}$ is also true, while form the second entailment it follows that whenever $\mathfrak{A}$ is false then $\mathfrak{B}$ is also false. We see that $\mathfrak{A}$ and $\mathfrak{B}$ always the same truth values: that is, $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent. Hence, if both $\mathfrak{A} \vDash \mathfrak{B}$ and $\mathfrak{B} \vDash \mathfrak{A}$, then it follows that $\mathfrak{A} =_T \mathfrak{B}$. □

If Γ is the empty set, then the entailment $\{\ \} \vDash \mathfrak{A}$ is normally written $\vDash \mathfrak{A}$. Such entailments have a rather special property.

Prop23

If $\vDash \mathfrak{A}$ then $\mathfrak{A}$ is a tautology. □

Justification

Suppose there was an instance of $\mathfrak{A}$ with truth value F. Then from the definition of entailment there must be at least one corresponding instance in the set $\{\ \}$ with truth value F. But this is impossible because $\{\ \}$ contains no elements. Hence all instances of $\mathfrak{A}$ must have truth value T. □

Thus if $\mathfrak{A}$ is a tautology, we usually indicate this by writing $\vDash \mathfrak{A}$.

Example 3.42

Express using an entailment the fact that $\mathcal{P} \vee \neg\mathcal{P}$ is a tautology.

Solution

$\vDash \mathcal{P} \vee \neg\mathcal{P}$ □

Exercise 23: Semantic entailment

1. Derive each of the following entailments.

   (a) $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{P} \vee \mathcal{Q}$

   (b) $\neg(\mathcal{P}_1 \vee \mathcal{P}_2) \vDash \neg(\mathcal{P}_2 \wedge \mathcal{P}_1)$

   (c) $\mathcal{P} \vee \neg\mathcal{Q}, \neg\mathcal{P} \vDash \neg\mathcal{Q}$

(d) $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$
(e) $\neg \mathcal{Q}_1 \vee \mathcal{Q}_2, \neg \mathcal{Q}_2 \vee \mathcal{Q}_3 \vDash \neg \mathcal{Q}_1 \vee \mathcal{Q}_3$

2. Explain the property of entailment that if $\mathfrak{A} \vDash \mathfrak{B}$ then $\neg\mathfrak{B} \vDash \neg\mathfrak{A}$.

3.9 Uniform replacement

Definition 3.9

In any context, the **uniform replacement** of a schematic letter by a propositional form means that *every* occurrence of that schematic letter is replaced by the same form *enclosed in parentheses.* □

Example 3.43

What are the new propositional forms obtained when $\mathcal{Q}$ is uniformly replaced by $\mathcal{P} \wedge \mathcal{Q}$ in the following pair of forms?

$\mathcal{Q} \wedge \mathcal{R}, \neg \mathcal{Q}$.

Solution

Every occurrence of $\mathcal{Q}$ in $\mathcal{Q} \wedge \mathcal{R}, \neg \mathcal{Q}$ is to be replaced by $(\mathcal{P} \wedge \mathcal{Q})$. This yields a new pair of forms:

$(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}, \neg(\mathcal{P} \wedge \mathcal{Q})$.

Note, however, that since $\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R}$ parses to $(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}$, the parentheses may be omitted in the first of this new pair. The final answer can therefore be written:

$\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R}, \neg(\mathcal{P} \wedge \mathcal{Q})$. □

Example 3.44

What is the result of uniformly replacing $\mathcal{P}$ by $\mathcal{Q}$ and $\mathcal{Q}$ by $\mathcal{P}$ in the propositional form $\neg \mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{P}) \vee \mathcal{Q}$?

Solution

Care needs to be taken to ensure that both replacements are done simultaneously. That is, we work from left to right across the propositional form, replace each schematic letter appropriately and then immediately move on to the next schematic letter. Thus the first letter we encounter is $\mathcal{P}$ in $\neg \mathcal{P} \ldots$; this letter is replaced by $\mathcal{Q}$ to give $\neg \mathcal{Q} \vee \ldots$. We immediately move to the next schematic letter, which is the $\mathcal{Q}$ in $\ldots(\mathcal{Q} \wedge \mathcal{P})\ldots$; this is replaced by $\mathcal{P}$ to give $\neg \mathcal{Q} \vee (\mathcal{P} \wedge \ldots$. Again we move on immediately. Repeating the process we obtain the final answer: $\neg \mathcal{Q} \vee (\mathcal{P} \wedge \mathcal{Q}) \vee \mathcal{P}$. □

Example 3.45

Show that $\neg \mathcal{P} \vee \neg \mathcal{Q}, \mathcal{Q} \vDash \neg \mathcal{P}$. Suppose now that $\mathcal{P}$ and $\mathcal{Q}$ are uniformly replaced throughout all three propositional forms by $\mathcal{P} \wedge \mathcal{R}$ and $\neg \mathcal{P} \wedge \neg \mathcal{R}$ respectively. What are the new propositional forms? Is there a corresponding entailment between these new forms?

Solution

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{P}\vee\neg\mathcal{Q}$
T	*T*	*F*	*F*	*F*
T	*F*	*F*	*T*	*T*
F	***T***	***T***	*F*	***T***
F	*F*	*T*	*T*	*T*

From the third row of the truth table we see that when both $\neg\mathcal{P}\vee\neg\mathcal{Q}$ and $\mathcal{Q}$ are true, then $\neg\mathcal{P}$ is also true. Thus $\neg\mathcal{P}\vee\neg\mathcal{Q},\mathcal{Q}\vDash\neg\mathcal{P}$. Now after uniform replacement the propositional forms become $\neg(\mathcal{P}\wedge\mathcal{R})\vee\neg(\neg\mathcal{P}\wedge\neg\mathcal{R})$, $(\neg\mathcal{P}\wedge\neg\mathcal{R})$ and $\neg(\mathcal{P}\wedge\mathcal{R})$. To check whether the first two forms still entail the third, we could draw up a complete truth table for all possible combinations of truth values of $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. However, it is easier to draw a reduced truth table for the possible combinations of truth values for $\mathcal{P}\wedge\mathcal{R}$ and $\neg\mathcal{P}\wedge\neg\mathcal{R}$.

$(\mathcal{P}\wedge\mathcal{R})$	$(\neg\mathcal{P}\wedge\neg\mathcal{R})$	$\neg(\mathcal{P}\wedge\mathcal{R})$	$\neg(\neg\mathcal{P}\wedge\neg\mathcal{R})$	$\neg(\mathcal{P}\wedge\mathcal{R})\vee\neg(\neg\mathcal{P}\wedge\neg\mathcal{R})$
T	*T*	*F*	*F*	*F*
T	*F*	*F*	*T*	*T*
F	***T***	***T***	*F*	***T***
F	*F*	*T*	*T*	*T*

This truth table is the same as the first one, except that $\mathcal{P}$ and $\mathcal{Q}$ have been uniformly replaced by $\mathcal{P}\wedge\mathcal{R}$ and $\neg\mathcal{P}\wedge\neg\mathcal{R}$. Thus the semantic entailment still holds:

$$\neg(\mathcal{P}\wedge\mathcal{R})\vee\neg(\neg\mathcal{P}\wedge\neg\mathcal{R}),(\neg\mathcal{P}\wedge\neg\mathcal{R})\vDash\neg(\mathcal{P}\wedge\mathcal{R})$$

□

Prop24

Semantic entailment and equivalence are preserved under uniform replacement. That is if $\Gamma\vDash\mathfrak{B}$, then we can apply any uniform replacement to obtain $\Gamma^*\vDash\mathfrak{B}^*$. Similarly, if $\mathfrak{A}=_T\mathfrak{B}$, then we can use any uniform replacement to give $\mathfrak{A}^*=_T\mathfrak{B}^*$. □

Justification

Suppose we have constructed a truth table to show that $\Gamma\vDash\mathfrak{B}$; then $\mathfrak{B}$ must be true in every row in which all the propositional forms of Γ are true. Now apply uniform replacement to this truth table to yield a reduced truth table for Γ^* and $\mathfrak{B}^*$. Since the truth values remain unchanged by the uniform replacement, the rows of the new truth table are the same as the original; in particular, it must be the case that $\mathfrak{B}^*$ is true in every row in which all the propositional forms of Γ^* are true. Hence $\Gamma^*\vDash\mathfrak{B}^*$.

Now if $\mathfrak{A}=_T\mathfrak{B}$, then we know from *Prop13* that both $\mathfrak{A}\vDash\mathfrak{B}$ and $\mathfrak{B}\vDash\mathfrak{A}$. Applying uniform replacement to these two entailments will give $\mathfrak{A}^*\vDash\mathfrak{B}^*$ and $\mathfrak{B}^*\vDash\mathfrak{A}^*$. From *Prop13* we conclude that $\mathfrak{A}^*=_T\mathfrak{B}^*$. □

Note that we made implicit use of *Prop14* in Table 3.2. Note also that, as a consequence of *Prop14*, if uniform replacement is applied to a tautology, then a tautology results; likewise, if uniform replacement is applied to a contradiction, then a contradiction results.

Example 3.46

Given that $\neg(\mathcal{P}_1 \wedge \neg\mathcal{P}_2), \neg\mathcal{P}_2 \models \neg\mathcal{P}_1$ show that $\neg((\mathcal{P} \vee \mathcal{Q}) \wedge \neg(\mathcal{R} \vee \mathcal{S})), \neg(\mathcal{R} \vee \mathcal{S}) \models \neg(\mathcal{P} \vee \mathcal{Q})$.

Solution

$\neg((\mathcal{P} \vee \mathcal{Q}) \wedge \neg(\mathcal{R} \vee \mathcal{S})), \neg(\mathcal{R} \vee \mathcal{S}) \models \neg(\mathcal{P} \vee \mathcal{Q})$ is obtained by replacing $\mathcal{P}_1$ with $(\mathcal{P} \vee \mathcal{Q})$ and $\mathcal{P}_2$ with $(\mathcal{R} \vee \mathcal{S})$ throughout $\neg(\mathcal{P}_1 \wedge \neg\mathcal{P}_2), \neg\mathcal{P}_2 \models \neg\mathcal{P}_1$. □

Example 3.47

Show that $(\mathcal{P} \wedge (\neg\mathcal{Q} \vee \mathcal{R})) \vee \neg(\mathcal{P} \wedge (\neg\mathcal{Q} \vee \mathcal{R}))$ is a tautology.

Solution

$(\mathcal{P} \wedge (\neg\mathcal{Q} \vee \mathcal{R})) \vee \neg(\mathcal{P} \wedge (\neg\mathcal{Q} \vee \mathcal{R}))$ can be obtained as a result of replacing $\mathcal{P}$ by $(\mathcal{P} \wedge (\neg\mathcal{Q} \vee \mathcal{R}))$ throughout the tautology $\mathcal{P} \vee \neg\mathcal{P}$. □

Exercise 24: Uniform replacement

In each of the following questions, derive the second entailment from the first.

1. $\mathcal{P}, \mathcal{Q} \models \mathcal{P} \wedge \mathcal{Q}$, $\neg\mathcal{P}, \neg\mathcal{Q} \models \neg\mathcal{P} \wedge \neg\mathcal{Q}$
2. $\mathcal{P}_1, \neg\mathcal{P}_1 \vee \mathcal{P}_2 \models \mathcal{P}_2$, $(\mathcal{P} \wedge \mathcal{Q}), \neg(\mathcal{P} \wedge \mathcal{Q}) \vee \mathcal{Q} \models \mathcal{Q}$
3. $\neg(\mathcal{R} \wedge \mathcal{S}) \models \neg\mathcal{R}$, $\neg(\mathcal{S} \wedge \mathcal{S}) \models \neg\mathcal{S}$

4 Natural Deduction

4.1 Arguments and validity

Arguments and argument forms

In the previous chapter, we have been concerned with truth values of propositions, and also with relationships between propositional forms, namely equivalence and semantic entailment, which can be defined in terms of truth values. In this chapter we shall be looking at how we can argue from a given set of propositions known as **premisses** to obtain a further proposition known as the **conclusion**. There are many such ways in which we can justify an argument, not all of which are acceptable in logic. Consider the following example.

Example 4.1

What conclusion might you draw from the following set of premisses?

{ ‘ $-2^2 + -2 + 41$ *is prime*’
‘ $-1^2 + -1 + 41$ *is prime*’
‘ $0^2 + 0 + 41$ *is prime*’
‘ $1^2 + 1 + 41$ *is prime*’
‘ $2^2 + 2 + 41$ *is prime*’
‘ $3^2 + 3 + 41$ *is prime*’
‘ $4^2 + 4 + 41$ *is prime*’
‘ $5^2 + 5 + 41$ *is prime*’
‘ $6^2 + 6 + 41$ *is prime*’ }

Solution

All the numerical expressions fit the pattern $n^2 + n + 41$. In this example we find that for all values of n between -2 and 6 the resulting value is a prime number. We might therefore argue that this is sufficient evidence to conclude ‘*all numbers of the form* $n^2 + n + 41$, *where* n *is an integer, are prime*’. □

This particular kind of argument is called **induction.** Induction is based upon finding a pattern that fits all the given facts. It is very common in everyday life; indeed it is essential for normal living. Suppose that you have become ill each time after you have eaten a certain food ingredient; then you would conclude that the ingredient makes you ill.

The difficulty with inductive arguments is that that the observed pattern might have arisen by coincidence; further similar observations may not fit the general pattern. One approach to dealing with this difficulty is a method of argument known as **scientific method.** The scientific method is a development of inductive argument: a general pattern obtained from observations is known as a **hypothesis**; further observations are made in order to test this hypothesis; if all these further observations agree with the hypothesis, then the hypothesis is verified and becomes a conclusion. A similar process is often used in program development. Suppose you have written a program. The hypothesis is that the program fulfils the requirements of the specification. To test the hypothesis, the program is run under various conditions and with various inputs. If the program executes as required under each of the test conditions, then the hypothesis is verified and the program passed as accepted.

Example 4.2

How might we use the scientific method to test the following hypothesis? '*All numbers of the form* $n^2 + n + 41$, *where* n *is an integer, are prime*'.

Solution

The hypothesis has been based upon observations for n between -2 and 6. To test the hypothesis, we need to choose some other values of n and assess whether $n^2 + n + 41$ is prime. For example, we might choose all the remaining values of n between -36 and $+36$; if we were to do this, then we would find that all the resulting values were prime numbers. The hypothesis is verified.

□

In spite of the popularity of the scientific method, the method is not foolproof. The history of science is full of theories which have been overturned by later observations; many, if not most, computer programs of any complexity contain residual 'bugs' and will fail under certain conditions not anticipated in the tests. Part of the skill of applying the method is to choose the test values carefully to reduce the chance of residual bugs.

Example 4.3

What might be a good test value of n in assessing whether $n^2 + n + 41$ is prime for any value of n?

Solution

The expression contains the number 41, which may therefore be considered to be of some special significance in this case. This suggests that we try a test value for n of 41. The result is $41^2 + 41 + 41$, which is equal to $41 \times (41 + 1 + 1) = 41 \times 43$; this value is clearly *not* a prime number. The hypothesis is disproved. □

Arguing by induction is just one method of argument. Unfortunately, it can lead to wrong conclusions. By contrast, in traditional logic we seek methods of argument which cannot lead to a wrong conclusion (unless one or more of the premisses are wrong). A program, or a computer circuit, which has been proved by logic to satisfy the specification requirements should not, in principle, have any bugs!

Validity of argument forms

In order to assess whether a particular method of argument is logically acceptable, we need to define the concept of a **logically valid** argument. Before we do that however, it will be useful to introduce some more notation and terminology.

Notation: $\therefore$

Suppose that we argue from a set of premisses $\{p_1, p_2, \ldots, p_n\}$ to yield a conclusion q. Then we write this argument as $p_1, p_2, \ldots, p_n \therefore q$. The symbol $\therefore$ is pronounced 'therefore'. □

Example 4.4

Suppose we argue from the set of premisses

{'*Rex has four legs*', '*Rover has four legs*'}

to get the conclusion '*all dogs have four legs*'. Represent this argument using the $\therefore$ symbol.

Solution

The set of premisses is written, without braces, to the left of the $\therefore$ symbol, and the conclusion to the right.

'*Rex has four legs*', '*Rover has four legs*' $\therefore$ '*all dogs have four legs*' □

The premisses $p_1, p_2, \ldots, p_n$, some of which may be compound, can be regarded as instances of propositional forms $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$. Naturally, if a particular premiss p_i is an atomic proposition, then the corresponding propositional form $\mathfrak{A}_i$ can only be a single schematic letter. Likewise the conclusion q can be regarded as an instance of a propositional form $\mathfrak{B}$. The argument $p_1, p_2, \ldots, p_n \therefore q$ can be then be regarded as an instance of $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n \therefore \mathfrak{B}$

Definition 4.1

If $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n, \mathfrak{B}$ are propositional forms, then $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n \therefore \mathfrak{B}$ is an **argument form.** □

Definition 4.2

An argument form $\Gamma \therefore \mathfrak{B}$ is **valid** if, and only if, $\Gamma \vDash \mathfrak{B}$; that is if $\mathfrak{B}$ is semantically entailed by the set of premiss forms Γ. An argument form that is not valid is said to be **invalid.** □

Justification

A desirable feature of an argument form is that whenever all the premisses of an instance are true, the corresponding conclusion must also be true. We label this property as 'validity'. But this property is precisely that which is conveyed by semantic entailment. □

Example 4.5

Which of the following argument forms are valid?

- $\mathcal{P} \wedge \mathcal{Q} \therefore \mathcal{P} \vee \mathcal{Q}$
- $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \therefore \mathcal{Q}$

Solution

From truth tables, we find that $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{P} \vee \mathcal{Q}$ but that $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \nvDash \mathcal{Q}$. Hence, $\mathcal{P} \wedge \mathcal{Q} \therefore \mathcal{P} \vee \mathcal{Q}$ is a valid argument form but $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \therefore \mathcal{Q}$ is not. □

Exercise 25: Validity of argument forms

Which of the following argument forms are valid?

1. $\mathcal{P} \vee \mathcal{Q} \therefore \mathcal{P} \wedge \mathcal{Q}$
2. $\mathcal{P}, \mathcal{Q} \therefore \mathcal{P} \wedge \mathcal{Q}$
3. $\mathcal{P} \wedge \mathcal{Q} \therefore \mathcal{P}$
4. $\mathcal{P} \vee \mathcal{Q} \therefore \mathcal{P}$
5. $\mathcal{P} \wedge \mathcal{Q} \therefore \mathcal{Q}$
6. $\mathcal{P} \vee \mathcal{Q} \therefore \mathcal{Q}$
7. $\mathcal{P} \therefore \mathcal{P} \vee \mathcal{Q}$
8. $\mathcal{P} \therefore \mathcal{P} \wedge \mathcal{Q}$
9. $\neg\mathcal{P} \therefore \mathcal{P}$
10. $\neg\neg\mathcal{P} \therefore \mathcal{P}$
11. $\mathcal{P} \therefore \neg\mathcal{P}$
12. $\mathcal{P} \therefore \neg\neg\mathcal{P}$
13. $\neg\mathcal{P} \vee \mathcal{Q}, \mathcal{P} \therefore \mathcal{Q}$
14. $\neg\mathcal{P} \vee \mathcal{Q}, \neg\mathcal{P} \therefore \neg\mathcal{Q}$
15. $\neg\mathcal{P} \vee \mathcal{Q}, \neg\mathcal{Q} \therefore \neg\mathcal{P}$
16. $\mathcal{P} \vee \mathcal{Q}, \mathcal{Q} \therefore \neg\mathcal{P}$
17. $\mathcal{P} \vee \mathcal{Q}, \neg\mathcal{P} \therefore \mathcal{Q}$

Validity of arguments

Definition 4.3

An argument is **valid** if it is an instance of a valid argument form. □

Example 4.6

Is the following argument valid?

'*Fido has four legs and a wet nose*'
$\therefore$
'*Fido either has four legs or has a wet nose*'

Solution

The given argument is an instance of the valid argument form

$$\mathcal{P} \wedge \mathcal{Q} \ \therefore\ \mathcal{P} \vee \mathcal{Q}$$

and so must be valid. □

Care must be taken when an argument is an instance of an *invalid* argument form; it does not necessarily follow that the argument itself is invalid.

Example 4.7

The argument form

'*Fido has four legs and a wet nose*'
$\therefore$
'*Fido either has four legs or has a wet nose*'

is an instance of $\mathcal{P} \ \therefore\ \mathcal{Q}$ and also of $\mathcal{P} \wedge \mathcal{Q} \ \therefore\ \mathcal{R} \vee \mathcal{S}$, neither of which is a valid argument form. Is the argument itself invalid?

Solution

No, the argument itself is *not* invalid since it is also an instance of the valid argument form $\mathcal{P} \wedge \mathcal{Q} \ \therefore\ \mathcal{P} \vee \mathcal{Q}$. □

In order to show that an argument is not valid, we need to be sure that all possible forms of which the argument is an instance are invalid. Fortunately it is not necessary to check all possible argument forms; only one argument form is needed provided it is a **characteristic argument form.** Characteristic argument forms are an extension of the idea of characteristic propositional forms met in section 3.2 and enable us to write down an alternative definition for validity.

Definition 4.4

The characteristic form of an argument is such that the argument can be obtained by instantiating different schematic letters to different atomic propositions. □

Definition 4.5

An argument is valid if and only if its characteristic form is valid. □

Justification

Suppose an argument

$$p_1, p_2, \ldots, p_n \therefore q$$

has a characteristic form

$$\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n \therefore \mathfrak{B}$$

with p_1, p_2,..., p_n instances of the propositional forms $\mathfrak{A}_1$, $\mathfrak{A}_2$,..., $\mathfrak{A}_n$ respectively, and q an instance of the propositional form $\mathfrak{B}$. If this argument form is valid, then from the earlier definition we know that the argument must also be valid.

If, however, the characteristic form is invalid, then we need to show that any other form of which the argument is an instance is also invalid. First we note that the propositional forms $\mathfrak{A}_1$, $\mathfrak{A}_2$,..., $\mathfrak{A}_n$ and $\mathfrak{B}$ must be characteristic of the propositions p_1, p_2,..., p_n and q, otherwise the argument form would not itself be characteristic. Now in section 3.2 we saw that if p_1 is an instance of a characteristic form $\mathfrak{A}_1$ and also of another form $\mathfrak{A}_1^*$, then all instances of $\mathfrak{A}_1$ are also instances of $\mathfrak{A}_1^*$. Similar arguments apply to the other propositions. Thus all instances of the characteristic form $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n \therefore \mathfrak{B}$ must also be instances of any other form $\mathfrak{A}_1^*, \mathfrak{A}_2^*, \ldots, \mathfrak{A}_n^* \therefore \mathfrak{B}^*$ of which our argument is an instance. But if the characteristic form is invalid, this means that that there is at least one instance for which all of $\mathfrak{A}_1$, $\mathfrak{A}_2$,..., $\mathfrak{A}_n$ are true but $\mathfrak{B}$ is false. Since this instance is also an instance of $\mathfrak{A}_1^*, \mathfrak{A}_2^*, \ldots, \mathfrak{A}_n^* \therefore \mathfrak{B}^*$, then this other form must also be invalid. □

Example 4.8

Which of the argument forms $\mathcal{P} \therefore \mathcal{Q}$, $\mathcal{P} \land \mathcal{Q} \therefore \mathcal{P} \lor \mathcal{Q}$ and $\mathcal{P} \land \mathcal{Q} \therefore \mathcal{R} \lor \mathcal{S}$ is a characteristic form for the following?

'*Fido has four legs and a wet nose*'
∴
'*Fido either has four legs or has a wet nose*'

Solution

The given argument is an instance of $\mathcal{P} \therefore \mathcal{Q}$, but with $\mathcal{P}$ and $\mathcal{Q}$ instantiated to the compound propositions '*Fido has four legs*' ∧ '*Fido has a wet nose*' and '*Fido has four legs*' ∨ '*Fido has a wet nose*' respectively. The form $\mathcal{P} \therefore \mathcal{Q}$ is not characteristic.

The given argument is also an instance of $\mathcal{P} \land \mathcal{Q} \therefore \mathcal{R} \lor \mathcal{S}$, but with the schematic letters $\mathcal{P}$ and $\mathcal{R}$ both instantiated to the same atomic proposition '*Fido has four legs*'; and with $\mathcal{Q}$ and $\mathcal{S}$ both instantiated to the same atomic proposition '*Fido has a wet nose*'. The form $\mathcal{P} \land \mathcal{Q} \therefore \mathcal{R} \lor \mathcal{S}$ is not characteristic.

Only $\mathcal{P} \wedge \mathcal{Q} \therefore \mathcal{P} \vee \mathcal{Q}$ is characteristic; the argument can be obtained by taking the atomic proposition '*Fido has four legs*' as the instance of $\mathcal{P}$ and the different atomic proposition '*Fido has a wet nose*' as the instance of $\mathcal{Q}$

Note that $\mathcal{P} \wedge \mathcal{Q} \therefore \mathcal{P} \vee \mathcal{Q}$ satisfies the criterion that there should be a one to one correspondence between schematic letters and atomic propositions, whereas neither $\mathcal{P} \therefore \mathcal{Q}$ nor $\mathcal{P} \wedge \mathcal{Q} \therefore \mathcal{R} \vee \mathcal{S}$ satisfies this criterion. □

Example 4.9

Is the following argument valid?

$$\begin{array}{c} \text{`}2^2 + 2 + 41 \textit{ is prime}\text{'} \\ \text{`}-14^2 + -14 + 41 \textit{ is prime}\text{'} \\ \text{`}37^2 + 37 + 41 \textit{ is prime}\text{'} \\ \therefore \\ \text{`}111^2 + 111 + 41 \textit{ is prime}\text{'} \end{array}$$

Solution

A characteristic form is $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \therefore \mathcal{Q}$. This is not, however, a valid argument form. Hence the given argument is invalid. □

Note that in this last example, the argument may well have resulted from the application of induction. In general, inductive arguments are not logically valid. Not that there is anything inherently wrong with induction; indeed some philosophers (notably John Stuart Mill) have proposed an *inductive logic*. However, in the *deductive logic* with which we are concerned, induction is not an acceptable form of argument. There will henceforth be little reference to inductive arguments.

Example 4.10

Is the following argument valid?

'*Fido does not have four legs*'
'*Fido either has four legs or has a wet nose*'
∴
'*Fido has a wet nose*'

Solution

The atomic propositions are '*Fido has four legs*' and '*Fido has a wet nose*'. Take these to be instances of the schematic letters $\mathcal{P}$ and $\mathcal{Q}$ respectively. The given argument is thus an instance of

$$\neg\mathcal{P},\ \mathcal{P} \vee \mathcal{Q} \therefore \mathcal{Q}$$

Now from the truth tables for $\neg\mathcal{P}$ and $\mathcal{P} \vee \mathcal{Q}$ we find that

$$\neg\mathcal{P},\ \mathcal{P} \vee \mathcal{Q} \models \mathcal{Q}$$

Hence the argument form is valid, and so must be the given argument. □

Note that although the argument form in this case happens to be a characteristic form, we do not need to use this in order to determine that the argument is valid; it is only in proving an argument is *invalid* that we *must* use a characteristic form.

Example 4.11

Show that $\neg\neg(7 > 0 \wedge 2 + 3 = 5) \therefore 7 > 0 \wedge 2 + 3 = 5$ is a valid argument.

Solution

The argument is an instance of $\neg\neg\mathcal{P} \therefore \mathcal{P}$. But from truth tables we find that $\neg\neg\mathcal{P} \vDash \mathcal{P}$. Hence we conclude that the argument is valid even though the argument form is not characteristic. □

Just because an argument is valid, it does not necessarily follow that the conclusion is true. Recall that if $\Gamma \vDash \mathfrak{B}$ then if $\mathfrak{B}$ is false, then at least one of the premisses in Γ is false. Hence a valid argument guarantees that the conclusion is true only if *all* the premisses are true.

Example 4.12

Is the following argument valid?

$$1 + 1 \neq 2, 1 + 1 = 2 \vee 6 \times 9 = 42 \therefore 6 \times 9 = 42$$

Solution

The argument is an instance of $\neg\mathcal{P}, \mathcal{P} \vee \mathcal{Q} \therefore \mathcal{Q}$. But from truth tables we can ascertain that $\neg\mathcal{P}, \mathcal{P} \vee \mathcal{Q} \vDash \mathcal{Q}$. Hence the argument form is valid, and so is the argument. But the conclusion is false. The reason for this is that the the premiss $1 + 1 \neq 2$ is also false. □

Example 4.13

Is the following argument valid?

$$1 + 1 \neq 3, 1 + 1 = 3 \vee 2 \times 3 = 6 \therefore 3^2 = 9$$

Solution

The argument is an instance of the form $\neg\mathcal{P}, \mathcal{P} \vee \mathcal{Q} \therefore \mathcal{R}$.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\neg\mathcal{P}$	$\mathcal{P} \vee \mathcal{Q}$
T	*T*	*T*	*F*	*T*
T	*T*	*F*	*F*	*T*
T	*F*	*T*	*F*	*T*
T	*F*	*F*	*F*	*T*
F	*T*	*T*	*T*	*T*
F	*T*	***F***	***T***	***T***
F	*F*	*T*	*T*	*F*
F	*F*	*F*	*T*	*F*

From the sixth row of the truth tables for $\neg\mathcal{P}$, $\mathcal{P} \vee \mathcal{Q}$ and $\mathcal{R}$, we see that $\{\neg\mathcal{P}, \mathcal{P} \vee \mathcal{Q}\}$ does not entail R. Hence the argument form is not valid. Since the argument form is a characteristic form, then it also follows that the given argument is also invalid. The fact that the conclusion and both premisses are all true does not make the argument valid in the logical sense. □

Example 4.14

Show that the following argument is invalid.

$$1 + 1 = 2, 2 \times 3 = 6 \therefore 9 \times 6 = 42$$

Although we cannot generally use the truth values of premisses and conclusion to determine the validity of an argument, there is one special circumstance in which this is indeed possible.

Solution

Suppose the argument is an instance of the form $\mathfrak{A}_1, \mathfrak{A}_2 \therefore \mathfrak{B}$. Now the conclusion is false while both the premisses are true. Thus we have an instance of $\mathfrak{B}$ which is false but for which the corresponding instances of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are both true. Hence from the definition of entailment, $\{\mathfrak{A}_1, \mathfrak{A}_2\}$ cannot entail $\mathfrak{B}$; $\mathfrak{A}_1, \mathfrak{A}_2 \therefore \mathfrak{B}$ is not a valid argument form; the original argument is not valid. □

Exercise 26: Validity of arguments

Which of the following arguments are valid?

1. '*Rex has four legs*' $\vee$ '*Fido has three legs*'
   $\therefore$
   '*Rex has four legs*' $\wedge$ '*Fido has three legs*'
2. '*Rex has four legs*' $\wedge$ '*Fido has three legs*'
   $\therefore$
   '*Rex has four legs*' $\vee$ '*Fido has three legs*'
3. '*Rex has four legs*' $\wedge$ '*Fido has three legs*'
   $\therefore$
   '*Rex has four legs*' $\vee$ '*Rover has four legs*'
4. $1 + 1 = 2 \wedge 2 \times 3 = 6 \therefore 1 + 1 = 2$
5. $1 + 1 = 2, 2 \times 3 = 6 \therefore 1 + 1 = 2 \wedge 2 \times 3 = 6$
6. $1 + 1 = 2, 2 \times 3 = 6 \therefore 1 + 1 = 2 \vee 2 \times 3 = 6$
7. $1 + 1 = 2 \therefore 1 + 1 = 2 \vee 2 \times 3 = 6$
8. $1 + 1 = 2 \therefore$ '*Rex has four legs*' $\vee\ 1 + 1 = 2$
9. $1 + 1 = 2 \therefore 3 > 0 \vee$ '7 *is prime*'
10. $\neg\neg(6 \times 9 = 42 \wedge 4 \not> 5) \therefore 6 \times 9 = 42 \wedge 4 \not> 5$

4.2 Natural deduction

We have *defined* the concept of validity in terms of semantic entailment, and have used this definition in order to assess whether particular arguments and argument forms are valid. Unfortunately, this definition is not always a practical method for finding and checking valid arguments. Various alternative approaches are possible. One approach is to use **deductive reasoning** to draw a conclusion from a set of premisses. Several examples of such reasoning have already been given in the various justifications of properties earlier in this book. A further, simple example of deductive reasoning follows.

Example 4.15

Suppose we are given $1 + 1 = 2 \vee 6 \times 9 = 42$ and $1 + 1 \neq 2$ as premisses. Explain why we can conclude that $6 \times 9 = 42$.

Solution

We are given that either $1 + 1 = 2$ or $6 \times 9 = 42$; but we are also given that $1 + 1 \neq 2$; therefore the only remaining possibility is that $6 \times 9 = 42$. □

Note that the *reasoning* in this last example is correct: it is only because we have started with one or more false premisses that we have obtained a false conclusion.

Deductive reasoning such as this is often less cumbersome than using truth tables, especially where there are many schematic letters and hence very large truth tables. Nevertheless, we need to be able to represent such reasoning symbolically rather than have to rely upon the use of natural language. In particular we need to be able to represent the fact that a conclusion is **deducible** from a set of premisses. The approach we shall look at is called **natural deduction**.

Definition 4.6

Suppose that from the set of premisses $\{p_1, p_2, \ldots, p_n\}$ we can deduce the conclusion q. Then we write the **inference**

$$p_1, p_2, \ldots, p_n \vdash q$$

The inference can be read as 'q is deducible from $p_1, p_2, \ldots, p_n$'. We also say that the argument $p_1, p_2, \ldots, p_n \therefore q$ is deducible. □

This symbol $\vdash$, known as the **syntactic turnstile.** Note that, as usual, the set of premisses is written without braces.

Example 4.16

In an earlier example we saw that from the set of premisses

$$\{1 + 1 = 2 \vee 6 \times 9 = 42, 1 + 1 \neq 2\}$$

we could reason deductively to obtain $6 \times 9 = 42$. How can this be expressed symbolically?

Solution

The set of premisses can be represented as a list before the turnstile $\vdash$. Thus we write the inference as

$$1 + 1 = 2 \lor 6 \times 9 = 42, 1 + 1 \neq 2 \vdash 6 \times 9 = 42$$ □

Example 4.17

Given premisses '2 *is prime*' and '2 *is even*' we can deduce the conjunction '2 *is prime*' $\land$ '2 *is even*'. How can this inference be represented using the $\vdash$ symbol? Is the corresponding argument valid?

Solution

The deduction can be represented as

'2 *is prime*', '2 *is even*' $\vdash$ '2 *is prime*' $\land$ '2 *is even*'

The corresponding argument is

'2 *is prime*', '2 *is even*' $\therefore$ '2 *is prime*' $\land$ '2 *is even*'

Now this argument is an instance of the argument form $\mathcal{P}, \mathcal{Q} \therefore \mathcal{P} \land \mathcal{Q}$. But from truth tables we know that $\mathcal{P}, \mathcal{Q} \vDash \mathcal{P} \land \mathcal{Q}$. Hence the argument is not only deducible but also valid. □

Care needs to be taken not to confuse the semantic turnstile $\vDash$ with the syntactic turnstile $\vdash$. They represent very different concepts even though they look similar. The notation $\Gamma \vDash \mathfrak{A}$ is used to show a relationship between the truth table for the propositional form $\mathfrak{A}$ and the truth tables for the propositional forms in Γ. The notation $p_1, p_2, \ldots, p_n \vdash q$ is used to denote that the proposition q is deducible from the propositions $p_1, p_2, \ldots, p_n$.

Inference forms and deduction rules

Definition 4.7

Suppose that all instances of an argument form $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \therefore \mathcal{Q}$ are deducible, then we can write the **inference form**

$$\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \vdash \mathcal{Q}$$

Inferences can be found as instances of such an inference form. □

Example 4.18

Suppose $\mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \land \mathcal{Q}$ is an inference form. What inference is obtained by instantiating $\mathcal{P}$ to '2 *is prime*' and $\mathcal{Q}$ to '2 *is even*'?

Solution

'2 *is prime*', '2 *is even*' $\vdash$ '2 *is prime*' $\land$ '2 *is even*' □

Example 4.19

Show that $p_1 \lor \neg p_3, \neg p_2 \land p_3 \vdash (p_1 \lor \neg p_3) \land (\neg p_2 \land p_3)$. is an instance of $\mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \land \mathcal{Q}$.

Solution

Taking $p_1 \vee \neg p_3$ as the instance of $\mathcal{P}$ and $\neg p_2 \wedge p_3$ as the instance of $\mathcal{Q}$ in the inference form $\mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$ gives the required inference. □

Natural deduction consists of **rules of deduction** which enable us to find or prove inferences. Several deduction rules are simply inference forms; for example, the inference form $\mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$ is one such deduction rule. This rule is denoted as $\wedge$I, pronounced 'AND introduction', because we regard the conclusion as having been formed by *introducing* the $\wedge$ connective between the two premisses.

Example 4.20

What inference is obtained from the $\wedge$I rule by taking $\mathcal{P}$ and $\mathcal{Q}$ to be $\neg p$ and $\neg q$ respectively?

Solution

The inference obtained is $\neg p, \neg q \vdash \neg p \wedge \neg q$. □

Soundness of natural deduction

We have already met the concept of logical validity and the importance of ensuring that any argument is valid. Hence, we choose the rules of deduction so that whenever natural deduction yields the inference $p_1, p_2, \ldots, p_n \vdash q$ then the corresponding argument $p_1, p_2, \ldots, p_n \therefore q$ is valid. This property of natural deduction is called **soundness**; we say that natural deduction is **sound**.

In order to ensure that natural deduction is sound, we shall only accept an inference form $\Gamma \vdash \mathfrak{A}$ as a rule of deduction if the corresponding argument form is valid, that is if $\Gamma \vDash \mathfrak{A}$. Such a deduction rule is said to be sound.

Example 4.21

Show that the $\wedge$I rule is sound.

Solution

The $\wedge$I rule is $\mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$. From truth tables we know that $\mathcal{P}, \mathcal{Q} \vDash \mathcal{P} \wedge \mathcal{Q}$. Hence the $\wedge$I rule is sound. □

We can now list some rules of deduction involving each of the three connectives: conjunction; disjunction; and negation.

Conjunction

We have already met a rule for *introducing* conjunction and shown this rule to be sound.

Rule: $\wedge I$

$\mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$ □

There are also two **elimination rules** for conjunction. In each of these rules there is a single premiss having conjunction as the main connective, while the conclusion is one of the conjuncts. We imagine that in going from the premiss to the conclusion, the conjunction $\wedge$ has been *eliminated*.

Rule: $\wedge E_1$

$\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P}$ □

Rule: $\wedge E_2$

$\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{Q}$ □

Justification

From the definition of $\mathcal{P} \wedge \mathcal{Q}$ we now that both $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{P}$ and $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{Q}$. Hence the two elimination rules are sound. □

Example 4.22

What can be deduced from $0 < 3 \wedge 3 \leq 5$. using the $\wedge\mathrm{E}_1$ rule?

Solution

In order for the premiss $0 < 3 \wedge 3 \leq 5$ to be an instance of $\mathcal{P} \wedge \mathcal{Q}$, the corresponding instance of $\mathcal{P}$ must be $0 < 3$, and the corresponding instance of the $\wedge$I rule must be $0 < 3 \wedge 3 \leq 5 \vdash 0 < 3$. The conclusion is $0 < 3$. □

Example 4.23

What can be deduced from

'*Mars has an atmosphere*' $\wedge$ '*Venus has an atmosphere*'

using the $\wedge\mathrm{E}_2$ rule?

Solution

If '*Mars has an atmosphere*' $\wedge$ '*Venus has an atmosphere*' is the instance of $\mathcal{P} \wedge \mathcal{Q}$ in $\wedge\mathrm{E}_2$, then the corresponding instance of $\mathcal{Q}$ must be '*Venus has an atmosphere*'. □

Example 4.24

If p and q are two propositions show that $p, q \vdash q \wedge p$.

Solution

We need to find a deduction rule such that the required deduction is an instance. Since there are two premisses, neither $\wedge\mathrm{E}_1$ nor $\wedge\mathrm{E}_2$ will be suitable as these rules only have one premiss. The $\wedge$I rule has two premisses as required, but seemingly in the wrong order. However, recall that the notation p, q refers to the *set* of premisses $\{p, q\}$; likewise, the notation $\mathcal{P}, \mathcal{Q}$ refers to the set of premiss forms $\{\mathcal{P}, \mathcal{Q}\}$. Now for the conclusion $q \wedge p$ to be an instance of $\mathcal{P} \wedge \mathcal{Q}$, we need to take q as the instance of $\mathcal{P}$ and p as the instance of $\mathcal{Q}$. It follows that $\{p, q\}$ would then be the corresponding instance of $\{\mathcal{P}, \mathcal{Q}\}$ as required in the $\wedge$I rule. □

Example 4.25
If p is a proposition, show that $p \vdash p \wedge p$ is an inference of natural deduction.

Solution
Clearly the only rule applicable here is the $\wedge$I rule, $\mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$. If we take p as the instance of both $\mathcal{P}$ and $\mathcal{Q}$, then the conclusion is $p \wedge p$ while the premiss set is $\{p, p\} = \{p\}$; hence the inference is $p \vdash p \wedge p$. □

Disjunction

There are two introduction rules for disjunction.

Rule: $\vee I_1$

$$\mathcal{P} \vdash \mathcal{P} \vee \mathcal{Q}$$

□

Rule: $\vee I_2$

$$\mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$$

□

Justification
From the definition for $\mathcal{P} \vee \mathcal{Q}$ we now that both $\mathcal{P} \vDash \mathcal{P} \vee \mathcal{Q}$ and $\mathcal{Q} \vDash \mathcal{P} \vee \mathcal{Q}$. Hence the two deduction rules are sound. □

Example 4.26
Show that $0 < 3 \vdash 0 < 3 \vee 3 \leq 5$.

Solution From arithmetic we know that $0 < 3$. Using rule $\vee I_1$ with $\mathcal{P}$ replaced by $0 < 3$ and $\mathcal{Q}$ replaced by $3 \leq 5$ gives the required inference. □

Example 4.27
Show that '*The moon is made of green cheese*' $\vee$ $1+1=2$ follows from $1+1=2$.

Solution
Instantiating $\mathcal{P}$ to '*The moon is made of green cheese*' and $\mathcal{Q}$ to $1+1=2$ in $\wedge E_2$ yields the inference

'*The moon is made of green cheese*' $\vee$ $1+1=2 \vdash 1+1=2$

The conclusion may seem rather ridiculous, yet it is perfectly legitimate in natural deduction. In practice, however, we would not normally have the need to make such unusual inferences. □

There is also an **elimination rule** for disjunction; unfortunately this rule cannot be expressed as an inference form, and will be considered later.

Negation

There are no simple rules for introducing or eliminating negation that can be expressed as inference forms. There are, however, very simple rules for introducing or eliminating **double negation**. Consider the proposition '*I have not eaten nothing*'. In colloquial English the use of the two negatives '*not*' and '*nothing*' represents an emphasis of the negation; it is a more emphatic way of saying '*I have not eaten anything*'. More formally, however, the two negatives are considered to cancel out to give '*I have eaten something*'. This leads us to a deduction rule for eliminating double negation.

Rule: $\neg\neg E$

$\neg\neg\mathcal{P} \vdash \mathcal{P}$ □

We also have a rule for introducing double negation.

Rule: $\neg\neg I$

$\mathcal{P} \vdash \neg\neg\mathcal{P}$ □

Justification

Both $\neg\neg\mathcal{P} \vDash \mathcal{P}$ and $\mathcal{P} \vDash \neg\neg\mathcal{P}$ so both rules are sound. □

Example 4.28

Show that $\neg(2 \neq 3) \vdash 2 = 3$.

Solution

The inequality $2 \neq 3$ can be regarded as a shorthand for $\neg(2 = 3)$. Thus $\neg(2 \neq 3)$ can be represented as $\neg\neg(2 = 3)$. Hence taking $\mathcal{P}$ to be $2 = 3$ in ¬¬E we have

$$\neg(2 \neq 3) \vdash (2 = 3).$$

Note that both the premiss and conclusion are false; the argument however is valid. □

Example 4.29

Show that '*It is not the case that Mercury is not smaller than Mars*' can be deduced from '*Mercury is smaller than Mars*'.

Solution

If $\mathcal{P}$ is instantiated to '*Mercury is smaller than Mars*' in ¬¬I, then we obtain

'*Mercury is smaller than Mars*' ⊢ ¬¬'*Mercury is smaller than Mars*'

in which '*It is not the case that Mercury is not smaller than Mars*' is represented by the conclusion. □

Further examples

So far we have been looking at examples in which the schematic letters $\mathcal{P}$ and $\mathcal{Q}$ have been instantiated to atomic propositions. More usually, however, they will be instantiated to compound propositions. In deciding what deduction rule applies, it is important to consider the main connectives of the premisses and of the conclusion

Example 4.30

Show that $2 + 3 = 5 \lor 5 < 0, \neg 5 < 0 \vdash (2 + 3 = 5 \lor 5 < 0) \land \neg 5 < 0$.

Solution

The conclusion of the required inference parses to $(2 + 3 = 5 \lor 5 < 0) \land (\neg 5 < 0))$. We see the main connective in the conclusion is conjunction. Hence we make use of the $\land$I rule. Taking $\mathcal{P}$ to be $2 + 3 = 5 \lor 5 < 0$ and $\mathcal{Q}$ to be $\neg 5 < 0$ in $\land$I we get the required inference. □

Example 4.31

Show that $2 + 3 = 5 \land 1 + 1 = 6 \vdash \neg\neg(2 + 3 = 5 \land 1 + 1 = 6)$.

Solution The main connective in the conclusion is negation. At the moment the only possible rule we have for dealing with this is $\neg\neg$I. Thus we use the $\neg\neg$I rule with $\mathcal{P}$ instantiated to $2 + 3 = 5 \land 1 + 1 = 6$. □

Example 4.32

Show that $\neg\neg p \land (q \lor r) \vdash q \lor r$.

Solution

The main connective in the conclusion is disjunction; this suggests the possibility of either the $\lor I_1$ rule or the $\lor I_1$ rule. The main connective in the premiss is conjunction; this suggests the possibility of either the $\land E_1$ rule or the $\land E_1$ rule. Now neither of the two disjuncts of the conclusion is the same as the premiss. Hence neither $\lor I_1$ nor $\lor I_1$ can be applicable. However, the conclusion is just the second conjunct of the premiss. Thus we see that the $\land E_2$ rule must apply, with $\mathcal{P}$ taken to be $\neg\neg p$ and $\mathcal{Q}$ taken to be $q \lor r$. □

Example 4.33

Can we obtain the inference $p \lor q \land r \vdash p \lor q$ as an instance of the $\land E_1$ rule?

Solution

The main connective in the premiss is disjunction. It is therefore not possible to use the $\land E_1$, which can only apply to a premiss whose *main* connective is conjunction. In fact the deduction can be obtained by using the disjunction elimination rule, but this will not be considered until later. □

Exercise 27: Inference forms as rules of deduction

1. What instances of $\mathcal{P}$ and $\mathcal{Q}$ are needed in order to obtain each of the following inferences from the given deduction rule? (As usual, p, q, r and s refer to propositions.)

   (a) $p, (q \vee r) \vdash p \wedge (q \vee r)$ from $\wedge$I.
   (b) $p \wedge (q \vee r) \vdash p$ from $\wedge E_1$.
   (c) $p \wedge (q \vee r) \vdash q \vee r$ from $\wedge E_2$.
   (d) $\neg\neg\neg q \wedge (p \vee \neg\neg q) \vdash p \vee \neg\neg q$ from $\wedge E_2$.
   (e) $\neg\neg\neg q \wedge (p \vee \neg\neg q) \vdash \neg\neg\neg q$ from $\wedge E_1$.
   (f) $p \vdash p \vee (q \wedge r)$ from $\vee I_1$.
   (g) $(\neg p \vee s) \vdash (\neg q \vee r) \vee (\neg p \vee s)$ from $\vee I_2$.
   (h) $\neg\neg\neg p \vdash \neg p$ from $\neg\neg$E.
   (i) $\neg\neg\neg\neg p \vdash p$ from $\neg\neg$E.
   (j) $\neg\neg(p \wedge (r \vee q)) \vdash p \wedge (r \vee q)$ from $\neg\neg$E.
   (k) $\neg\neg p \vdash \neg\neg\neg\neg p$ from $\neg\neg$I.
   (l) $\neg p \vdash \neg\neg\neg p$ from $\neg\neg$I.
   (m) $p \vee (r \wedge q) \vdash \neg\neg(p \vee (r \wedge q))$ from $\neg\neg$I.

2. Obtain each of the following from an appropriate rule of deduction.

   (a) '*The sky is blue*', '*Grass is green*' $\vdash$ '*The sky is blue*' $\wedge$ '*Grass is green*'
   (b) $p_1 \vdash p_1 \vee p_2$
   (c) $\neg\neg$'*Fido has three legs*' $\vdash$ '*Fido has three legs*'
   (d) '73 *is prime*' $\wedge$ '73 *is odd*' $\vdash$ '73 *is odd*'
   (e) '*The sky is blue*' $\vdash \neg\neg$'*The sky is blue*'
   (f) '71 *is prime*' $\vdash$ '71 *is prime*' $\vee$ '26 *is prime*'
   (g) '*Rex has four legs*' $\vdash$ '*Rex has a wet nose*' $\vee$ '*Rex has four legs*'
   (h) $r \wedge s \vdash r$
   (i) $1 + 1 = 2 \wedge 2 \times 3 = 6 \vdash 3^2 = 9 \vee 1 + 1 = 2 \wedge 2 \times 3 = 6$
   (j) $\neg\neg$('*Roses are red*' $\wedge$ '*Violets are blue*') $\vdash$ '*Roses are red*' $\wedge$ '*Violets are blue*'

4.3 New inference forms

In natural deduction, all inferences can be obtained as instances of inference forms. Indeed, the study of natural deduction is concerned primarily with inference forms and proving properties about them, rather than with particular inferences.

Some inference forms we have already met as rules of deduction, but not all inferences follow directly from these rules. For example, we can show that $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{P} \vee \mathcal{Q}$; this suggests the inference form $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$. One approach to a deductive system of logic might be to list all such inference forms as rules of deduction.

Example 4.34

Show that $\neg\mathcal{P}, \mathcal{P} \vee \mathcal{Q} \vDash \mathcal{Q}$ What inference form corresponds to this entailment?

Solution

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\mathcal{P} \vee \mathcal{Q}$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

The forms $\neg\mathcal{P}$ and $\mathcal{P} \vee \mathcal{Q}$ are both T only in the third row, for which $\mathcal{Q}$ is also T. Hence $\neg\mathcal{P}, \mathcal{P} \vee \mathcal{Q} \vDash \mathcal{Q}$. Thus we could, if we so wish, add the inference form $\neg\mathcal{P}, \mathcal{P} \vee \mathcal{Q} \vdash \mathcal{Q}$ to the other rules of deduction. Some systems of deduction do indeed include this form as a rule of deduction; it is known as **disjunctive syllogism.** □

Unfortunately there are infinitely many inference forms possible, and it is not possible to list them all. Instead, in natural deduction we introduce a new type of deduction rule which is not an inference form but which enables us instead to write down new inference forms.

Definition 4.8

A **method of deduction** is a rule by means of which we can write down a new inference form given one or more other inference forms. Often we shall refer to methods of deduction simply as **deduction methods**. □

Starting with basic inference forms, such as the deduction rules from the previous section, we can construct further inference forms using these methods of deduction. Many deduction methods are intuitively obvious and reflect natural ways of reasoning.

Example 4.35

Show that $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$.

Solution

Recall that the $\wedge$E rule is $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P}$, while the $\vee I_1$ rule is $\mathcal{P} \vdash \mathcal{P} \vee \mathcal{Q}$. Thus the $\wedge E_1$ rule enables us to infer the conclusion form $\mathcal{P}$ from $\mathcal{P} \wedge \mathcal{Q}$. But this conclusion form is itself the premiss form of the $\vee I_1$ rule, from which we can infer $\mathcal{P} \vee \mathcal{Q}$. Hence we see that from $\mathcal{P} \wedge \mathcal{Q}$ we can infer $\mathcal{P} \vee \mathcal{Q}$. □

Re-ordering and repetition of premisses

Strictly speaking, this is not a deduction method but a clarification of notation. Suppose we have an inference form $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n \vdash \mathfrak{B}$, then the list of premiss forms $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$ represents the set $\{\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n\}$. By definition, the same set is indicated no matter in what order the premiss forms are listed or how many times each form is listed. We can therefore reorder or repeat premiss forms at will. For example, the inference form $\mathcal{P} \wedge \neg \mathcal{Q}, \mathcal{Q} \wedge \mathcal{R} \vdash \mathcal{S}$ could be equally well written as, say, $\mathcal{Q} \wedge \mathcal{R}, \mathcal{P} \wedge \neg \mathcal{Q}, \mathcal{P} \wedge \neg \mathcal{Q} \vdash \mathcal{S}$.

Example 4.36
Show that $\mathcal{Q}, \mathcal{P} \vdash \mathcal{P} \wedge \mathcal{Q}$.

Solution
The premiss forms are a reordering of those in the $\wedge$I. □

Uniform replacement

In section 3.9 we defined the concept of uniform replacement as the replacement of all occurrences of a schematic letter by a propositional form. We saw that uniform replacement could be applied to truth tables, semantic entailments and equivalences.

Example 4.37
What semantic entailment is obtained if $\mathcal{P}$ is uniformly replaced by $\mathcal{P}_1 \vee \mathcal{P}_2$ in $\mathcal{P} \wedge \mathcal{Q} \vDash \mathcal{P} \vee \mathcal{Q}$?

Solution
$(\mathcal{P}_1 \vee \mathcal{P}_2) \wedge \mathcal{Q} \vDash (\mathcal{P}_1 \vee \mathcal{P}_2) \vee \mathcal{Q}$. □

Method of Deduction: Uniform Replacement
Given any inference form $\Gamma \vdash \mathfrak{B}$, we can always write down a new inference form $\Gamma^* \vDash \mathfrak{B}^*$ by uniformly replacing each schematic letter with any propositional form. □

Justification
Suppose that we have $\Gamma \vdash \mathfrak{B}$. Then we know that $\Gamma \therefore \mathfrak{B}$ is valid, and hence that $\Gamma \vDash \mathfrak{B}$. But from *Prop*14 we know that if $\Gamma \vDash \mathfrak{B}$ then $\Gamma^* \vDash \mathfrak{B}^*$ where Γ^* and $\mathfrak{B}^*$ are obtained from Γ and $\mathfrak{B}$ by uniform replacement. Thus $\Gamma^* \therefore \mathfrak{B}^*$ is also valid. □

Example 4.38
Show that $\mathcal{P}, \mathcal{Q}_1 \wedge \mathcal{Q}_2 \vdash \mathcal{P} \wedge (\mathcal{Q}_1 \wedge \mathcal{Q}_2)$.

Solution
Replacing $\mathcal{Q}$ by $\mathcal{Q}_1 \wedge \mathcal{Q}_2$ throughout $\wedge$I gives $\mathcal{P}, \mathcal{Q}_1 \wedge \mathcal{Q}_2 \vdash \mathcal{P} \wedge (\mathcal{Q}_1 \wedge \mathcal{Q}_2)$.
□

Example 4.39
Show that $\mathcal{P} \vdash \mathcal{P} \wedge \mathcal{P}$.

Solution
Replacing $\mathcal{Q}$ by $\mathcal{P}$ throughout $\wedge$I gives $\mathcal{P}, \mathcal{P} \vdash \mathcal{P} \wedge \mathcal{P}$. But the repeated premiss form $\mathcal{P}$ can be simplified to a single occurrence: $\mathcal{P} \vdash \mathcal{P} \wedge \mathcal{P}$. □

The uniform replacement rule can be incorporated into natural deduction by rewriting the introduction and elimination rules in terms of unspecified propositional forms $\mathfrak{A}$ and $\mathfrak{B}$ rather than schematic letters $\mathcal{P}$ and $\mathcal{Q}$:

- $\wedge E_1$: $\mathfrak{A} \wedge \mathfrak{B} \vdash \mathfrak{A}$
- $\wedge E_2$: $\mathfrak{A} \wedge \mathfrak{B} \vdash \mathfrak{B}$
- $\wedge I$: $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{A} \wedge \mathfrak{B}$
- $\vee I_1$: $\mathfrak{A} \vdash \mathfrak{A} \vee \mathfrak{B}$
- $\vee I_2$: $\mathfrak{A} \vdash \mathfrak{B} \vee \mathfrak{A}$
- $\neg\neg I$: $\mathfrak{A} \vdash \neg\neg\mathfrak{A}$
- $\neg\neg E$: $\neg\neg\mathfrak{A} \vdash \mathfrak{A}$

Example 4.40
Show that $\mathcal{R} \vee \neg\mathcal{S} \vdash \mathcal{R} \vee \neg\mathcal{S} \vee \mathcal{Q}$

Solution
Taking $\mathfrak{A}$ as $\mathcal{R} \vee \neg\mathcal{S}$ and $\mathfrak{B}$ as $\mathcal{Q}$ in $\vee I_1$ gives the required inference form.
□

Chain rule

We have already seen how the inference form $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$ can be formed by taking the conclusion form $\mathcal{P}$ of the $\wedge E_1$ rule to be the premiss form $\mathcal{P}$ to the $\vee I_1$ rule. That is, we can chain together

- $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P}$ and
- $\mathcal{P} \vdash \mathcal{P} \vee \mathcal{Q}$

to form a new inference form, $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$. Taking this idea a little further suggest that we should be able to chain together inference forms

- $\Gamma \vdash \mathfrak{A}$ and
- $\mathfrak{A} \vdash \mathfrak{B}$

to create $\Gamma \vdash \mathfrak{B}$

Example 4.41
Show that $\mathcal{P}, \mathcal{Q} \vdash \neg\neg(\mathcal{P} \wedge \mathcal{Q})$.

Solution
From the $\wedge I$ rule we can write down $\mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$, while from the $\neg\neg I$ rule we can write down $\mathcal{P} \wedge \mathcal{Q} \vdash \neg\neg(\mathcal{P} \wedge \mathcal{Q})$. Chaining these two inference forms together gives $\mathcal{P}, \mathcal{Q} \vdash \neg\neg(\mathcal{P} \wedge \mathcal{Q})$. □

But the chaining idea can be generalized still further.

Method of Deduction: Chain

If $\Gamma_1 \vdash \mathfrak{A}$ and $\Gamma_2, \mathfrak{A} \vdash \mathfrak{B}$ then $\Gamma_1, \Gamma_2 \vdash \mathfrak{B}$. □

Justification

If $\Gamma_1 \vdash \mathfrak{A}$ and $\Gamma_2, \mathfrak{A} \vdash \mathfrak{B}$ then $\Gamma_1 \vDash \mathfrak{A}$ and $\Gamma_2, \mathfrak{A} \vDash \mathfrak{B}$. Now consider an instance in which all of Γ_1 and Γ_2 are true. From $\Gamma_1 \vDash \mathfrak{A}$, we see that $\mathfrak{A}$ must be true; hence all of $\Gamma_2, \mathfrak{A}$ must be true. But from $\Gamma_2, \mathfrak{A} \vdash \mathfrak{B}$, we see that $\mathfrak{B}$ must be true. Hence we have shown that whenever Γ_1, Γ_2 are all true, $\mathfrak{B}$ must be true. Hence $\Gamma_1, \Gamma_2 \vDash \mathfrak{B}$, so $\Gamma_1, \Gamma_2 \therefore \mathfrak{B}$ is a valid argument form. □

Example 4.42

Show that $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{Q} \wedge \mathcal{P}$.

Solution

The deduction rules $\wedge E_1$, $\wedge E_2$ and $\wedge I$ give the inference forms:

$$
\begin{array}{lrcl}
1 & \mathcal{P} \wedge \mathcal{Q} & \vdash & \mathcal{P} \\
2 & \mathcal{P} \wedge \mathcal{Q} & \vdash & \mathcal{Q} \\
3 & \mathcal{Q}, \mathcal{P} & \vdash & \mathcal{Q} \wedge \mathcal{P}
\end{array}
$$

Applying the chain rule to inference forms 2 and 3 gives

$$
\begin{array}{lrcl}
4 & \mathcal{P} \wedge \mathcal{Q}, \mathcal{P} & \vdash & \mathcal{Q} \wedge \mathcal{P}
\end{array}
$$

Applying the chain rule to inference forms 1 and 4 gives

$$\mathcal{P} \wedge \mathcal{Q}, \mathcal{P} \wedge \mathcal{Q} \quad \vdash \quad \mathcal{Q} \wedge \mathcal{P}$$

However, note that we do not normally repeat $\mathcal{P} \wedge \mathcal{Q}$ but simply write

$$
\begin{array}{lrcl}
5 & \mathcal{P} \wedge \mathcal{Q} & \vdash & \mathcal{Q} \wedge \mathcal{P}
\end{array}
$$

This proves the result. The complete working could be shown as below.

$$
\begin{array}{lrcll}
1 & \mathcal{P} \wedge \mathcal{Q} & \vdash & \mathcal{P} & \wedge E_1 \\
2 & \mathcal{P} \wedge \mathcal{Q} & \vdash & \mathcal{Q} & \wedge E_1 \\
3 & \mathcal{Q}, \mathcal{P} & \vdash & \mathcal{Q} \wedge \mathcal{P} & \wedge I \\
4 & \mathcal{P} \wedge \mathcal{Q}, \mathcal{P} & \vdash & \mathcal{Q} \wedge \mathcal{P} & 2, 3 \\
5 & \mathcal{P} \wedge \mathcal{Q} & \vdash & \mathcal{Q} \wedge \mathcal{P} & 1, 4
\end{array}
$$

□

Derived rules

We could generalize the result from the last example to write down a **derived rule**:

$$\mathfrak{A} \wedge \mathfrak{B} \vdash \mathfrak{B} \wedge \mathfrak{A}$$

where $\mathfrak{A}$ and $\mathfrak{B}$ could be any propositional forms. Having proved such a rule, we can use it in subsequent derivations of inference forms in the same way as the original introduction and elimination rules. In order to do this, however, we need to give the derived rule a name: for example, an appropriate name for this rule might be 'AND commutation', which we could denote as $\wedge$comm.

Example 4.43

Show that $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{Q} \wedge \mathcal{P} \vee \neg\mathcal{P}$

Solution

Although we could derive the required inference form by starting with only introduction and elimination rules, the first five lines would be the same as we used in the previous example to show $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{Q} \wedge \mathcal{P}$. Rather than repeat these five lines we can instead invoke the $\wedge$comm rule as the first line.

1	$\mathcal{P} \wedge \mathcal{Q}$	$\vdash$	$\mathcal{Q} \wedge \mathcal{P}$	$\wedge$ comm
2	$\mathcal{P} \wedge \mathcal{Q}$	$\vdash$	$\mathcal{Q} \wedge \mathcal{P} \vee \neg\mathcal{P}$	$\vee$ I

□

We obtained the $\wedge$comm derived rule by replacing schematic letters $\mathcal{P}$ and $\mathcal{Q}$ with $\mathfrak{A}$ and $\mathfrak{B}$ in the inference form $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{Q} \wedge \mathcal{P}$. Usually, however, we obtain a derived rule directly by working with $\mathfrak{A}, \mathfrak{B}, \ldots$ rather than $\mathcal{P}, \mathcal{Q}, \ldots$, as the following example shows.

Example 4.44

Prove the identity rule, denoted ident: $\mathfrak{A} \vdash \mathfrak{A}$

Solution

1	$\mathfrak{A}$	$\vdash$	$\neg\neg\mathfrak{A}$	$\neg\neg$I
2	$\neg\neg\mathfrak{A}$	$\vdash$	$\mathfrak{A}$	$\neg\neg$E
3	$\mathfrak{A}$	$\vdash$	$\mathfrak{A}$	1, 2

□

Exercise 28: Tabular derivations

Derive each of the following, and present your working in a table.

1. $\mathcal{P} \wedge \mathcal{Q} \vdash \neg\neg\mathcal{P}$
2. $\neg\neg\mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$
3. $\neg\neg(\mathcal{P} \vee \mathcal{Q}) \vdash \mathcal{P} \vee \mathcal{Q} \vee \neg\neg(\mathcal{P} \wedge \mathcal{Q})$
4. $\neg\neg\mathfrak{A}, \neg\neg\mathfrak{B} \vdash \mathfrak{A} \wedge \mathfrak{B}$

4.4 Deduction trees

The presentation of working in tabular form, as shown in the previous section, can become difficult to read because of the need to keep referring back to earlier lines. In this section, we shall look at an alternative approach which displays the working in diagrammatic form known as a **deduction tree.**

Notation: Vertical presentation of inference forms

An inference form $\Gamma \vdash \mathfrak{B}$ used as a rule of deduction can be written in vertical form with the name of the rule placed at the end of a horizontal line.

$$\frac{\Gamma}{\mathfrak{B}}\ \textit{rule-name}$$

□

The introduction and elimination rules we have met so far can be written in this form:

- $$\frac{\mathfrak{A} \wedge \mathfrak{B}}{\mathfrak{A}} \wedge E_1$$
- $$\frac{\mathfrak{A} \wedge \mathfrak{B}}{\mathfrak{B}} \wedge E_2$$
- $$\frac{\mathfrak{A} \qquad \mathfrak{B}}{\mathfrak{A} \wedge \mathfrak{B}} \wedge I$$
- $$\frac{\mathfrak{A}}{\mathfrak{A} \vee \mathfrak{B}} \vee I_1$$
- $$\frac{\mathfrak{B}}{\mathfrak{A} \vee \mathfrak{B}} \vee I_2$$
- $$\frac{\mathfrak{A}}{\neg\neg\mathfrak{A}} \neg\neg I$$
- $$\frac{\neg\neg\mathfrak{A}}{\mathfrak{A}} \neg\neg I$$

Example 4.45

Show that $\mathcal{P} \wedge \mathcal{P} \vdash \mathcal{P}$.

Solution

$$\frac{\mathcal{P} \wedge \mathcal{P}}{\mathcal{P}} \wedge E_1$$

□

Example 4.46

Show that $\neg\neg(\mathcal{P} \vee \neg\mathcal{Q}) \vdash \mathcal{P} \vee \neg\mathcal{Q}$.

Solution

$$\dfrac{\neg\neg(\mathcal{P} \vee \neg\mathcal{Q})}{\mathcal{P} \vee \neg\mathcal{Q}}\ \neg\neg\mathrm{E}$$

□

The chain rule is a method of deduction, not an inference form. It cannot be expressed vertically in the same manner as the basic introduction and elimination rules above.

Notation: Vertical presentation of the chain rule

We can chain the inference forms $\Gamma_1 \vdash \mathfrak{A}$ and $\Gamma_2, \mathfrak{A} \vdash \mathfrak{B}$ vertically with $\mathfrak{A}$ providing the common link:

$$\dfrac{\dfrac{\Gamma_1}{\mathfrak{A}} \qquad \Gamma_2}{\mathfrak{B}}$$

□

Thus the chain rule provides the basic method of building deduction trees. Note that there is no explicit mention of the chain rule since it is implicit in the construction of deduction trees. In order to describe how we can read deduction trees, it will be useful to introduce two further terms.

Definition 4.9

The **root** of a deduction tree is the conclusion form at the lowest point of the tree; the root is the only form which does not have a horizontal line *underneath* it. □

Definition 4.10

A **leaf** of a deduction tree is a premiss form which does not have anything above it; that is, a leaf does not have a horizontal line *above* it. □

Example 4.47

What are the leaves and root in the following deduction tree? (Note that the derivation of this tree involves repeated application of the chain rule as explained below.)

$$\dfrac{\dfrac{\neg\neg\mathcal{P}}{\mathcal{P}}\ \neg\neg\mathrm{E} \qquad \dfrac{\neg\neg\mathcal{Q}}{\mathcal{Q}}\ \neg\neg\mathrm{E}}{\mathcal{P} \wedge \mathcal{Q}}\ \wedge\mathrm{I}$$

Solution

The root is $\mathcal{P} \wedge \mathcal{Q}$ and the leaves are $\neg\neg\mathcal{P}$ and $\neg\neg\mathcal{Q}$. □

For any deduction tree there is a corresponding inference form with conclusion form given by the root of the tree, and premiss forms given by the leaves.

Example 4.48

What inference form can be read off from the following deduction tree? From what inference forms has it been derived using the chain rule?

$$\dfrac{\dfrac{\mathcal{P}_1 \qquad \mathcal{P}_2}{\mathcal{P}_1 \wedge \mathcal{P}_2}\,\wedge\mathrm{I} \qquad\qquad \mathcal{P}_3}{\mathcal{P}_1 \wedge \mathcal{P}_2 \wedge \mathcal{P}_3}\,\wedge\mathrm{I}$$

Solution

The conclusion can be read off from the root of the tree as $\mathcal{P}_1 \wedge \mathcal{P}_2 \wedge \mathcal{P}_3$. Likewise, the premisses can be read off from the leaves of the tree as $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$. Hence the tree represents the inference $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \vdash \mathcal{P}_1 \wedge \mathcal{P}_2 \wedge \mathcal{P}_3$. This has been obtained by applying the chain rule to the inferences $\mathcal{P}_1, \mathcal{P}_2 \vdash \mathcal{P}_1 \wedge \mathcal{P}_2$ and $\mathcal{P}_1 \wedge \mathcal{P}_2, \mathcal{P}_3 \vdash \mathcal{P}_1 \wedge \mathcal{P}_2 \wedge \mathcal{P}_3$. □

Note that in this last example we spoke of the 'conclusion', 'premisses' and 'inference' rather than 'conclusion form', 'premiss forms' and 'inference form'. Although the latter terms would have technically been more correct, the meaning intended should by now be clear to the reader from the context.

Example 4.49

Show using a deduction tree that $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$.

Solution

We have seen previously that this inference can be obtained by applying the chain rule to $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P}$ (from $\wedge\mathrm{E}_1$) and $\mathcal{P} \vdash \mathcal{P} \vee \mathcal{Q}$ (from $\vee\mathrm{I}_1$). Writing these inferences vertically and making $\mathcal{P}$ the common link we get:

$$\dfrac{\dfrac{\mathcal{P} \wedge \mathcal{Q}}{\mathcal{P}}\,\wedge\mathrm{E}_1}{\mathcal{P} \vee \mathcal{R}}\,\vee\mathrm{I}_1$$

□

Repetition of the chain rule

The chain rule is frequently needed more than once in the derivation of an inference. For example, suppose we have derived the following deduction tree.

$$\frac{\dfrac{\Gamma_1}{\mathfrak{A}} \qquad \Gamma_2}{\mathfrak{B}}$$

Now suppose that the conclusion of this tree is a premiss in a further inference.

$$\frac{\mathfrak{B} \qquad \Gamma_3}{\mathfrak{C}}$$

Then we can chain this additional inference to the bottom of the deduction tree.

$$\frac{\dfrac{\dfrac{\Gamma_1}{\mathfrak{A}} \qquad \Gamma_2}{\mathfrak{B}} \qquad \Gamma_3}{\mathfrak{C}}$$

Example 4.50

Demonstrate as a deduction tree that $\neg\neg(\mathcal{P} \wedge \mathcal{Q}) \vdash \neg\neg\mathcal{P}$.

Solution

In tabular from we can write

1	$\neg\neg(\mathcal{P} \wedge \mathcal{Q})$	$\vdash$	$\mathcal{P} \wedge \mathcal{Q}$	$\neg\neg$E
2	$\mathcal{P} \wedge \mathcal{Q}$	$\vdash$	$\mathcal{P}$	$\wedge\text{E}_1$
3	$\neg\neg(\mathcal{P} \wedge \mathcal{Q})$	$\vdash$	$\mathcal{P}$	1, 2
4	$\mathcal{P}$	$\vdash$	$\neg\neg\mathcal{P}$	$\neg\neg$I
5	$\neg\neg(\mathcal{P} \wedge \mathcal{Q})$	$\vdash$	$\neg\neg\mathcal{P}$	3, 4

The derivation of inference 3 can be represented in a deduction tree as

$$\frac{\dfrac{\neg\neg(\mathcal{P} \wedge \mathcal{Q})}{\mathcal{P} \wedge \mathcal{Q}}\,\neg\neg\text{E}}{\mathcal{P}}\,\wedge\text{E}_1$$

while inference 4 in vertical form is

$$\frac{\mathcal{P}}{\neg\neg\mathcal{P}}\,\neg\neg\text{I}$$

We can chain these two trees together with $\mathcal{P}$ as the common link to give the required deduction tree.

$$\cfrac{\cfrac{\cfrac{\neg\neg(\mathcal{P} \wedge \mathcal{Q})}{\mathcal{P} \wedge \mathcal{Q}}\ \neg\neg\mathrm{E}}{\mathcal{P}}\ \wedge\mathrm{E}_1}{\neg\neg\mathcal{P}}\ \neg\neg\mathrm{I}$$

□

In the last example, the chain rule was used to extend the deduction tree downwards. Sometimes, however, it results in **branching**. Suppose that we have inferences $\Gamma_1 \vdash \mathfrak{A}$, $\Gamma_2 \vdash \mathfrak{B}$ and $\Gamma_3, \mathfrak{A}, \mathfrak{B} \vdash \mathfrak{C}$. The chain rule can be applied twice – once with $\mathfrak{A}$ as the link, and once with $\mathfrak{B}$ as the link – to yield $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \mathfrak{C}$.

$$\cfrac{\cfrac{\Gamma_1}{\mathfrak{A}} \qquad\qquad \cfrac{\Gamma_2}{\mathfrak{B}}}{\mathfrak{C}}$$

Example 4.51

Show that $\neg\neg\mathcal{P}, \neg\neg\mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$.

Solution

Each premiss is the starting point for a branch of the deduction tree.

$$\cfrac{\neg\neg\mathcal{P}}{\mathcal{P}}\ \neg\neg\mathrm{E} \qquad\qquad \cfrac{\neg\neg\mathcal{Q}}{\mathcal{Q}}\ \neg\neg\mathrm{E}$$

Each of these can be chained in turn with

$$\cfrac{\mathcal{P} \qquad \mathcal{Q}}{\mathcal{P} \wedge \mathcal{Q}}\ \wedge\mathrm{I}$$

firstly to give

$$\cfrac{\cfrac{\neg\neg\mathcal{P}}{\mathcal{P}}\ \neg\neg\mathrm{E} \qquad\qquad \mathcal{Q}}{\mathcal{P} \wedge \mathcal{Q}}\ \wedge\mathrm{I}$$

and then to give

$$\cfrac{\cfrac{\neg\neg\mathcal{P}}{\mathcal{P}}\ \neg\neg\mathrm{E} \qquad \cfrac{\neg\neg\mathcal{Q}}{\mathcal{Q}}\ \neg\neg\mathrm{E}}{\mathcal{P} \wedge \mathcal{Q}}\ \wedge\mathrm{I}$$

□

Example 4.52

Show that $\mathcal{P}, \neg\neg\mathcal{R} \vdash (\mathcal{P} \vee \mathcal{Q}) \wedge \mathcal{R}$.

Solution

$$\dfrac{\dfrac{\mathcal{P}}{\mathcal{P} \vee \mathcal{Q}}\,\vee\mathrm{I}_1 \qquad \dfrac{\neg\neg\mathcal{R}}{\mathcal{R}}\,\neg\neg\mathrm{E}}{(\mathcal{P} \vee \mathcal{Q}) \wedge \mathcal{R}}\,\wedge\mathrm{I}$$

□

In the previous examples of branching, each branch corresponded to a different premiss. Sometimes, however, the same premiss may be repeated in different branches. The following example illustrates this.

Example 4.53

Show that $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{Q} \wedge \mathcal{P}$.

Solution

$$\dfrac{\dfrac{\mathcal{P} \wedge \mathcal{Q}}{\mathcal{Q}}\,\wedge\mathrm{E}_2 \qquad \dfrac{\mathcal{P} \wedge \mathcal{Q}}{\mathcal{P}}\,\wedge\mathrm{E}_1}{\mathcal{Q} \wedge \mathcal{P}}\,\wedge\mathrm{I}$$

□

Sometimes we may chain an inference to the top of a previously derived tree.

Example 4.54

Show that $\mathcal{P} \wedge \mathcal{Q}, \neg\neg\mathcal{R} \vdash (\mathcal{P} \vee \mathcal{Q}) \wedge \mathcal{R}$.

Solution

We have already derived the following tree in an earlier example.

$$\dfrac{\dfrac{\mathcal{P}}{\mathcal{P} \vee \mathcal{Q}}\,\vee\mathrm{I}_1 \qquad \dfrac{\neg\neg\mathcal{R}}{\mathcal{R}}\,\neg\neg\mathrm{E}}{(\mathcal{P} \vee \mathcal{Q}) \wedge \mathcal{R}}\,\wedge\mathrm{I}$$

Furthermore we know

$$\dfrac{\mathcal{P} \wedge \mathcal{Q}}{\mathcal{P}}\,\wedge\mathrm{E}_1$$

We can chain this last inference to the top of the first to obtain

$$\dfrac{\dfrac{\dfrac{\mathcal{P} \wedge \mathcal{Q}}{\mathcal{P}}\,\wedge\mathrm{E}_1}{\mathcal{P} \vee \mathcal{Q}}\,\vee\mathrm{I}_1 \qquad \dfrac{\neg\neg\mathcal{R}}{\mathcal{R}}\,\neg\neg\mathrm{E}}{(\mathcal{P} \vee \mathcal{Q}) \wedge \mathcal{R}}\,\wedge\mathrm{I}$$

□

Clearly by repeated application of chaining we can build successively more complex trees. In particular, the branches of a tree may themselves have branches.

Example 4.55

Show that $\mathcal{P}, \mathcal{Q} \vdash \neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q} \wedge (\mathcal{P} \wedge \mathcal{Q})$.

Solution

$$\dfrac{\dfrac{\dfrac{\mathcal{P}}{\neg\neg\mathcal{P}}\,\neg\neg\mathrm{I} \qquad \dfrac{\mathcal{Q}}{\neg\neg\mathcal{Q}}\,\neg\neg\mathrm{I}}{\neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q}}\,\wedge\mathrm{I} \qquad \dfrac{\mathcal{P} \qquad \mathcal{Q}}{\mathcal{P} \wedge \mathcal{Q}}\,\wedge\mathrm{I}}{\neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q} \wedge (\mathcal{P} \wedge \mathcal{Q})}\,\wedge\mathrm{I}$$

□

Care must be taken with the parsing of propositional forms. Remember that introduction and elimination rules only ever apply to the main connective of the conclusion or a premiss.

Example 4.56

Show that $\mathcal{P}, \mathcal{Q} \vdash \neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q} \wedge \mathcal{P} \wedge \mathcal{Q}$.

Solution

The conclusion of this inference is not the same as the previous example. In $\neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q} \wedge (\mathcal{P} \wedge \mathcal{Q})$ the main connective is the last but one conjunction, whereas in $\neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q} \wedge \mathcal{P} \wedge \mathcal{Q}$, the main connective is the last conjunction. This difference is reflected in the deduction trees.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{P}}{\neg\neg\mathcal{P}}\,\neg\neg\mathrm{I} \qquad \dfrac{\mathcal{Q}}{\neg\neg\mathcal{Q}}\,\neg\neg\mathrm{I}}{\neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q}}\,\wedge\mathrm{I} \qquad \mathcal{P}}{\neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q} \wedge \mathcal{P}}\,\wedge\mathrm{I} \qquad \mathcal{Q}}{\neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q} \wedge \mathcal{P} \wedge \mathcal{Q}}\,\wedge\mathrm{I}$$

□

Example 4.57

Prove the following by deriving its characteristic inference form.

$$\neg\, 2 + 3 \neq 5, 5 > 0 \vdash (2 + 3 = 5 \vee 6 - 2 = 3) \wedge (3^2 = 8 \vee 5 > 0)$$

Solution

The proposition $\neg\, 2 + 3 \neq 5$ is the same as $2 + 3 = 5$. Hence, the atomic propositions are $2 + 3 = 5$, $6 - 2 = 3$, $5 > 0$ and $3^2 = 8$. Taking these as instances of $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ respectively, we find the characteristic inference form to be

$$\neg\neg\mathcal{P}, \mathcal{Q} \vdash (\mathcal{P} \vee \mathcal{R}) \wedge (\mathcal{S} \vee \mathcal{Q})$$

This inference form can be derived as follows:

$$\dfrac{\dfrac{\dfrac{\neg\neg\mathcal{P}}{\mathcal{P}}\,\neg\neg\mathrm{E}}{\mathcal{P} \vee \mathcal{Q}}\,\vee\mathrm{I}_1 \qquad \dfrac{\mathcal{Q}}{\mathcal{S} \vee \mathcal{Q}}\,\vee\mathrm{I}_2}{(\mathcal{P} \vee \mathcal{R}) \wedge (\mathcal{S} \vee \mathcal{Q})}\,\wedge\mathrm{I}$$

□

Exercise 29: Deduction trees

1. Write down a deduction tree for each of the following. To help you, the required deduction rules are given.
   (a) $\mathcal{P} \wedge \mathcal{Q} \vdash \neg\neg\mathcal{P}$ using $\wedge\mathrm{E}_1$ then $\neg\neg\mathrm{I}$.
   (b) $\neg\neg\neg\mathcal{Q} \wedge (\mathcal{P} \vee \neg\neg\mathcal{Q}) \vdash \neg\mathcal{Q}$ using $\wedge\mathrm{E}_1$ then $\neg\neg\mathrm{E}$.
   (c) $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$ using $\wedge\mathrm{E}_1$ then $\vee\mathrm{I}_1$.
   (d) $\neg\neg(\mathcal{P} \vee \mathcal{Q}) \wedge (\neg\mathcal{Q} \wedge \mathcal{R}) \vdash (\mathcal{P} \vee \mathcal{Q}) \vee (\mathcal{R} \wedge \mathcal{S})$ using $\wedge\mathrm{E}_1$, $\neg\neg\mathrm{E}$ and then $\vee\mathrm{I}_1$.
   (e) $\mathcal{P}, (\mathcal{Q} \vee \mathcal{R}) \vdash \neg\neg\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$ using $\neg\neg\mathrm{I}$ then $\wedge\mathrm{I}$.
   (f) $\mathcal{Q} \wedge \mathcal{R}, \mathcal{S} \vdash \mathcal{Q} \wedge \mathcal{S}$ using $\wedge\mathrm{E}_1$ then $\wedge\mathrm{I}$.
   (g) $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{Q} \wedge \mathcal{P}$ using $\wedge\mathrm{E}_1$ and $\wedge\mathrm{E}_2$ followed by $\wedge\mathrm{I}$.
   (h) $\mathcal{P} \vee \mathcal{Q}, \mathcal{R} \vdash (\mathcal{R} \vee (\mathcal{Q} \wedge \neg\mathcal{P})) \wedge (\mathcal{P} \vee \mathcal{Q})$ using $\vee\mathrm{I}_1$ then $\wedge\mathrm{I}$.
2. Construct a deduction tree for each of the following using some or all of the deduction rules $\wedge\mathrm{I}$, $\wedge\mathrm{E}_1$, $\wedge\mathrm{E}_2$, $\vee\mathrm{I}_1$, $\vee\mathrm{I}_2$ $\neg\neg\mathrm{E}$, $\neg\neg\mathrm{I}$.
   (a) $\mathcal{Q}, \mathcal{R} \vdash \mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R})$
   (b) $(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R} \vdash \mathcal{R} \wedge (\mathcal{Q} \wedge \mathcal{P})$
   (c) $\mathcal{P} \wedge \mathcal{Q} \vdash \neg\mathcal{P} \vee Q$
   (d) $\neg\neg\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$
   (e) $\neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$
   (f) $\neg\neg(\mathcal{P} \wedge \mathcal{Q}) \vdash \neg\neg(\mathcal{P} \vee \mathcal{Q})$
   (g) $\neg\neg(\mathcal{P} \wedge \mathcal{Q}) \vdash \neg\neg\mathcal{P} \vee \neg\neg\mathcal{Q}$
   (h) $\neg\neg(\mathcal{P} \wedge \mathcal{Q}) \vdash \neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q}$
   (i) $\neg\neg\mathcal{P} \wedge \neg\neg\mathcal{Q} \vdash \neg\neg(\mathcal{P} \wedge \mathcal{Q})$
3. Prove each of the following inferences by deriving its characteristic inference form.
   (a) '*Rex has four legs*' $\wedge$ '*Fido has three legs*' $\vdash$ '*Fido has three legs*' $\vee$ '*Rex has four legs*'
   (b) '*Rex has four legs*' $\wedge$ $\neg$'*Fido has three legs*' $\vdash$ $\neg$'*Fido has three legs*' $\wedge$ '*Rex has four legs*'
   (c) $2 + 3 = 5,\ 7 < 8 \vdash 1 + 1 = 6 \ \vee\ 7 < 8 \ \wedge\ 2 + 3 = 5$

4.5 Other methods of deduction

We have seen how deduction trees can be constructed from basic inference forms by the chain rule, which is an example of a deduction method. Another deduction method we have met is the rule of uniform replacement, although we do not usually make explicit use of this rule since it is implicit in the way we have written the introduction and elimination rules. The question remains of whether we can derive an inference $\Gamma \vdash \mathfrak{B}$ corresponding to any valid argument $\Gamma \therefore \mathfrak{B}$, that is to any semantic entailment. Using only the methods of deduction we have encountered so far, the answer is 'no'.

Justification

Suppose $\mathfrak{A}$ is a tautology, then we can write $\vDash \mathfrak{A}$. Thus we can have a semantic entailment with the empty set as the set of premisses. However, we cannot derive an inference with an empty set of premisses.

- The introduction and elimination rules all have at least one premiss.
- The chain rule yields an inference $\Gamma_1, \Gamma_2 \vdash \mathfrak{B}$ whose premiss set could only be empty if Γ_1 were empty; however, this can only be the case if we already have inferences with empty premiss sets.

□

We say that the system of deduction we have so far developed is **incomplete**. In order to achieve **completeness** we need further deduction methods. It is these further methods that we consider in this section.

Thinning

The first deduction method allows us to add premisses to the premiss set of an existing inference. Paradoxically, it is referred to as **thinning**.

Method of Deduction: Thinning

Given the inference form $\Gamma_1 \vdash \mathfrak{A}$ we can deduce the inference form $\Gamma_1, \Gamma_2 \vdash \mathfrak{A}$, where Γ_2 is any set of additional premisses. □

Justification

If $\Gamma_1 \vdash \mathfrak{A}$ then, since our deductive system is sound, it follows that $\Gamma_1 \vDash \mathfrak{A}$. Now, if all of Γ_1, Γ_2 are true, then necessarily all of Γ_1 are also true; but since $\Gamma_1 \vDash \mathfrak{A}$, it follows that $\mathfrak{A}$ is true; hence $\Gamma_1, \Gamma_2 \vDash \mathfrak{A}$. □

Example 4.58

Show that $P, Q \vdash \mathcal{P} \vee \mathcal{Q}$.

Solution

$\mathcal{P} \vdash \mathcal{P} \vee \mathcal{Q}$ from the $\vee I_1$ rule. Hence $P, \mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$ by thinning. □

Alternative solution

$\mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$ from the $\vee I_2$ rule. Hence $P, \mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$ by thinning. □

Proof by contradiction

The next deduction method is known as **proof by contradiction**, although in older books it is often referred to as ***reductio ad absurdum***. It has been used since ancient times to prove many important results in mathematics.

Example 4.59

A number which can be written as one integer divided by another is known as a **rational number**. Two examples of rational numbers are $0.6 = 6/10 = 3/5$ and $-2.72 = -272/100 = 68/25$. Note that we have simplified each fraction to its lowest terms by dividing out common factors between numerator and denominator.

Theorem: $\sqrt{2}$, the square root of 2, is not a rational number.

Proof:
First assume $\sqrt{2}$ to be a rational number. Now if it were a rational number, then we could simplify the fraction to its lowest terms, and hence write

$$k/l = \sqrt{2}$$

where k and l are integers with no factor in common (other than 1). Squaring this equation and multiplying by l^2 gives

$$k^2 = 2l^2$$

Hence k^2 would be even. But this would imply that k was also even, so that we could write $k = 2m$ for some integer m. Substituting for k in the previous equation gives

$$4m^2 = 2l^2$$

Dividing this new equation by 2 gives

$$2m^2 = l^2$$

from which we see that l^2 would be even. Hence l would also be even. Thus we see that both k and l would be divisible by 2. But we have already stipulated that k and l have no common factor greater than 1. Thus from our original assumption that $\sqrt{2}$ is a rational number, we have deduced a contradiction. We must therefore reject our assumption and conclude that $\sqrt{2}$ is not a rational number. □

Method of Deduction: Proof by contradiction

If $\Gamma, \mathfrak{A} \vdash \mathfrak{B} \wedge \neg\mathfrak{B}$ then $\Gamma \vdash \neg\mathfrak{A}$ □

Justification

The propositional form represented by $B \wedge \neg\mathfrak{B}$ is a contradiction; hence all instances must be false. It follows from the definition of semantic entailment, that every instance of $\Gamma, \mathfrak{A}$ must also contain at least one false proposition.

Thus whenever all the propositions in an instance of Γ are true, the corresponding instance of $\mathfrak{A}$ must be false; that is, the corresponding instance of $\neg\mathfrak{A}$ must be true. Hence we can write $\Gamma \vDash \neg\mathfrak{A}$. □

Suppose our goal is to prove an inference $\Gamma \vdash \neg\mathfrak{A}$. One possibility is to set ourselves the **subgoal** of deriving $\Gamma, \mathfrak{A} \vdash \mathfrak{B} \wedge \neg\mathfrak{B}$ for some propositional form $\mathfrak{B}$, and then apply proof by contradiction.

Example 4.60

Prove $\mathcal{P} \vdash \neg\neg\mathcal{P}$ *without* using the $\neg\neg$I rule.

Solution

The goal is to prove $\mathcal{P} \vdash \neg\neg\mathcal{P}$. To achieve this, we set ourselves the subgoal of deriving $\mathcal{P}, \neg\mathcal{P} \vdash \mathfrak{B} \wedge \neg\mathfrak{B}$ for some propositional form $\mathfrak{B}$. Hence we build a deduction tree with premisses $\mathcal{P}, \neg\mathcal{P}$.

$$\dfrac{\mathcal{P} \qquad \neg\mathcal{P}}{\mathcal{P} \wedge \neg\mathcal{P}} \wedge\mathrm{I}$$

We have thus achieved our subgoal. Hence we have proved by contradiction that $\mathcal{P} \vdash \neg\neg\mathcal{P}$ □

Example 4.61

Show that $\neg\mathcal{P}, \mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{R}$.

Solution

In order to prove $\neg\mathcal{P}, \mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{R}$ by contradiction we prove the subgoal $\neg\mathcal{P}, \mathcal{P} \wedge \mathcal{Q}, \neg\mathcal{R} \vdash \mathfrak{B} \wedge \neg\mathfrak{B}$. Now from the deduction tree

$$\dfrac{\dfrac{\mathcal{P} \wedge \mathcal{Q}}{\mathcal{P}} \wedge\mathrm{E}_1 \qquad \neg\mathcal{P}}{\mathcal{P} \wedge \neg\mathcal{P}} \wedge\mathrm{I}$$

we see that $\neg\mathcal{P}, \mathcal{P} \wedge \mathcal{Q} \vdash \mathfrak{B} \wedge \neg\mathfrak{B}$. Hence by the thinning rule we have also shown that $\neg\mathcal{P}, \mathcal{P} \wedge \mathcal{Q}, \neg\mathcal{R} \vdash \mathfrak{B} \wedge \neg\mathfrak{B}$. □

This last example is a little absurd! The schematic letter $\mathcal{R}$ does not occur anywhere in the premisses and hence can be instantiated to any proposition we like. For example, we could instantiate $\mathcal{P}$ to $3 > 0$, $\mathcal{Q}$ to $1 + 1 = 2$ and $\mathcal{R}$ to '*the moon is made of green cheese*'; thus from premisses $3 \not> 0$ and $3 > 0 \wedge 1 + 1 = 2$ we can conclude '*the moon is made of green cheese*'. This problem arises because the premisses are **inconsistent**; that is, we can infer a contradiction from the premisses.

Earlier in this chapter we have seen how we may obtain derived rules such as the identity rule, $\mathfrak{A} \vdash \mathfrak{A}$. It is now possible to obtain other derived rules using proof by contradiction. In particular, we can regard the ¬¬I rule as being derived from the other rules: from

$$\frac{\mathfrak{A} \qquad \neg\mathfrak{A}}{\mathfrak{A} \wedge \neg\mathfrak{A}} \wedge\text{I}$$

we prove by contradiction that $\mathfrak{A} \vdash \neg\neg\mathfrak{A}$.

Example 4.62

Show that if $\Gamma, \neg\mathfrak{A} \vdash \mathfrak{B} \wedge \neg\mathfrak{B}$ then $\Gamma \vdash \mathfrak{A}$. This derived rule is sometimes used as an alternative form of proof by contradiction.

Solution

From $\Gamma, \neg\mathfrak{A} \vdash \mathfrak{B} \wedge \neg\mathfrak{B}$ we prove by contradiction that $\Gamma \vdash \neg\neg\mathfrak{A}$. Now the ¬¬E rule gives $\neg\neg\mathfrak{A} \vdash \mathfrak{A}$. Chaining these last two inferences together, with $\neg\neg\mathfrak{A}$ as the common link, gives $\Gamma \vdash \mathfrak{A}$, as required. □

Reasoning by cases

The next method of deduction we shall look at is known as **reasoning by cases**. Here is an everyday example of this kind of reasoning.

Example 4.63

A shopkeeper who needs to shut up shop while away on holiday might reason as follows:

- Suppose I take a holiday. Then I will need to close the shop while I am on holiday. So I will lose money.
- Suppose I do not take a holiday. I shall become ill through overwork. Then I will need to close the shop while I am ill. So I will lose money.

But clearly either I do take a holiday or I do not take a holiday. Either way I shall lose money. □

In this last example the shopkeeper has considered two alternative cases, and in each case has arrived at the same conclusion; the conclusion '*I will lose money*' is inevitable.

An old name for this type of argument is **dilemma**, which literally means *double premiss*. Thus in a dilemma, the same conclusion is drawn from two different premisses (the dilemma), one of which must apply. This form of argument has been used extensively through the centuries, particularly when it was necessary to prove something unpopular! This has given rise to the current everyday meaning of the word *dilemma*.

Method of Deduction: Reasoning by cases

If $\Gamma_1, \mathfrak{A} \vdash \mathfrak{C}$ and $\Gamma_2, \mathfrak{B} \vdash \mathfrak{C}$ then $\Gamma_1, \Gamma_2, \mathfrak{A} \vee \mathfrak{B} \vdash \mathfrak{C}$. □

Justification

Suppose $\Gamma_1, \mathfrak{A} \vDash \mathfrak{C}$ and $\Gamma_2, \mathfrak{B} \vDash \mathfrak{C}$. Now consider an instance of $\Gamma_1, \Gamma_2, \mathfrak{A} \vee \mathfrak{B}$ in which all the propositions are true. Since $\mathfrak{A} \vee \mathfrak{B}$ is true in this instance, at least one of $\mathfrak{A}$ and $\mathfrak{B}$ is true. There are two cases to consider.

1. If $\mathfrak{A}$ is true, then all of $\Gamma_1, \mathfrak{A}$ are true. Hence from $\Gamma_1, \mathfrak{A} \vDash \mathfrak{C}$ it follows that C is true.

2. If $\mathfrak{A}$ is false, then $\mathfrak{B}$ must be true, so all of $\Gamma_2, \mathfrak{B}$ are true. Hence from $\Gamma_2, \mathfrak{B} \vDash \mathfrak{C}$ it follows that $\mathfrak{C}$ is true.

Thus we find that $\mathfrak{C}$ is true whenever all of $\Gamma_1, \Gamma_2, \mathfrak{A} \vee \mathfrak{B}$ are true; that is $\Gamma_1, \Gamma_2, \mathfrak{A} \vee \mathfrak{B} \vDash \mathfrak{C}$. □

Example 4.64

Show that $\mathcal{P} \vee \mathcal{Q} \vdash \mathcal{Q} \vee \mathcal{P}$.

Solution

We achieve the goal of proving $\mathcal{P} \vee \mathcal{Q} \vdash \mathcal{Q} \vee \mathcal{P}$ by proving the two subgoals:

1. $\mathcal{P} \vdash \mathcal{Q} \vee \mathcal{P}$

2. $\mathcal{Q} \vdash \mathcal{Q} \vee \mathcal{P}$

This we do in the following deduction trees:

$$\frac{\mathcal{P}}{\mathcal{Q} \vee \mathcal{P}} \vee I_2$$

$$\frac{\mathcal{Q}}{\mathcal{Q} \vee \mathcal{P}} \vee I_1$$

□

Example 4.65 Show that $\mathcal{P} \vee \mathcal{Q} \vdash \neg(\neg\mathcal{P} \wedge \neg\mathcal{Q})$.

Solution

We can achieve the goal $\mathcal{P} \vee \mathcal{Q} \vdash \neg(\neg\mathcal{P} \wedge \neg\mathcal{Q})$ by proving the two subgoals:

1. $\mathcal{P} \vdash \neg(\neg\mathcal{P} \wedge \neg\mathcal{Q})$

2. $\mathcal{Q} \vdash \neg(\neg\mathcal{P} \wedge \neg\mathcal{Q})$

Now $\mathcal{P} \vdash \neg(\neg\mathcal{P} \wedge \neg\mathcal{Q})$ can be proved by contradiction from $\mathcal{P}, \neg\mathcal{P} \wedge \neg\mathcal{Q} \vdash \mathfrak{B} \wedge \neg\mathfrak{B}$

$$\frac{\dfrac{\neg\mathcal{P} \wedge \neg\mathcal{Q}}{\neg\mathcal{P}} \wedge E_1 \qquad \mathcal{P}}{\mathcal{P} \wedge \neg\mathcal{P}} \wedge I$$

and $\mathcal{Q} \vdash \neg(\neg\mathcal{P} \wedge \neg\mathcal{Q})$ can be proved by contradiction from $\mathcal{Q}, \neg\mathcal{P} \wedge \neg\mathcal{Q} \vdash \mathfrak{B} \wedge \neg\mathfrak{B}$

$$\dfrac{\dfrac{\neg\mathcal{P} \wedge \neg\mathcal{Q}}{\neg\mathcal{Q}}\,\wedge\text{E}_2 \qquad \mathcal{Q}}{\mathcal{Q} \wedge \neg\mathcal{Q}}\,\wedge\text{I}$$

□

Representation in deduction trees

So far we have not been able to represent proof by contradiction or reasoning by cases in deduction trees. This we shall do shortly, but first we need to introduce some new terminology and notation.

Notation: ¬I

An alternative name for *proof by contradiction* is the *NOT introduction* rule; it is denoted by ¬I. □

Justification

The method of proof by contradiction enables us to write down an inference in which the conclusion has negation as its main connective. In a deduction tree this negation will appear *below* a horizontal line; compare this with other introduction rules.

$$\dfrac{\mathfrak{A} \quad \mathfrak{B}}{\mathfrak{A} \wedge \mathfrak{B}}\,\wedge\text{I} \qquad \dfrac{\mathfrak{B}}{\mathfrak{A} \vee \mathfrak{B}}\,\vee\text{I}_2 \qquad \dfrac{\vdots}{\neg\mathfrak{A}}\,\neg\text{I}$$

□

Notation: ∨E

An alternative name for *reasoning by cases* is the *OR elimination* rule; it is denoted by ∨E. □

Justification

The method of reasoning by cases enables us to write down an inference in which one of the premisses is a disjunction. In a deduction tree, this disjunction will appear *above* a horizontal line; compare this with other elimination rules.

$$\dfrac{\mathfrak{A} \wedge \mathfrak{B}}{\mathfrak{A}}\,\wedge\text{E}_1 \qquad \dfrac{\neg\neg\mathfrak{A}}{\mathfrak{A}}\,\neg\neg\text{E} \qquad \dfrac{\mathfrak{A} \vee \mathfrak{B} \quad \vdots}{\mathfrak{C}}\,\vee\text{E}$$

□

This leaves the question of how we incorporate these notations into a deduction tree. The difficulty is that the ¬I and ∨E rules are methods of deduction, not inference forms. In order to remind ourselves of this fact we shall place a marker such as $*$ or † at the end of the horizontal line just before the ¬I or ∨E notation.

$$\frac{\vdots}{\neg\mathfrak{A}}\,*\neg\mathrm{I} \qquad \frac{\vdots}{\neg\mathfrak{A}}\,\dagger\neg\mathrm{I} \qquad \frac{\mathfrak{A}\vee\mathfrak{B} \quad \vdots}{\mathfrak{C}}\,*\vee\mathrm{E} \qquad \frac{\mathfrak{A}\vee\mathfrak{B} \quad \vdots}{\mathfrak{C}}\,\dagger\vee\mathrm{E}$$

Now methods of deduction such as $\neg$I and $\vee$E depend upon deriving other inferences first. The deduction trees for these inferences can be placed above the horizontal line in the position indicated by vertical dots in the above examples. Suppose we wish to apply the $\neg$I rule to the inference

$$\cfrac{\cfrac{\Gamma \qquad \mathfrak{A}}{\vdots}}{\mathfrak{B}\wedge\neg\mathfrak{B}}$$

then we can incorporate this into the complete deduction tree as follows:

$$\cfrac{\cfrac{\cfrac{\Gamma \qquad \cfrac{}{\mathfrak{A}}\,*}{\vdots}}{\mathfrak{B}\wedge\neg\mathfrak{B}}}{\neg\mathfrak{A}}\,*\neg\mathrm{I}$$

A line has been placed above the $\mathfrak{A}$ so that it will not be included among the premisses to the complete tree; the complete tree thus represents the derivation of $\Gamma \vdash \neg\mathfrak{A}$. The same marker as that used in the application of the $\neg$I rule is placed to the right of this line to indicate that $\mathfrak{A}$ has not been deduced from an empty set of premisses, but is a **temporary premiss.** Below the second marker, and after the application of the $\neg$I rule, the temporary premiss is no longer used in the deduction tree; the temporary premiss is said to be **discharged** at this point.

Example 4.66

Derive $\mathcal{P} \vdash \neg\neg\mathcal{P}$.

Solution

First show that $\mathcal{P}, \neg\mathcal{P} \vdash \mathcal{P}\wedge\neg\mathcal{P}$:

$$\frac{\mathcal{P} \qquad \neg\mathcal{P}}{\mathcal{P}\wedge\neg\mathcal{P}}\,\wedge\mathrm{I}$$

The $\neg$I rule can be applied to this inference to give

$$\cfrac{\cfrac{\mathcal{P} \qquad \cfrac{}{\neg\mathcal{P}}\,*}{\mathcal{P}\wedge\neg\mathcal{P}}\,\wedge\mathrm{I}}{\neg\neg\mathcal{P}}\,*\neg\mathrm{I}$$

□

Usually the application of the $\neg$I rule will be part of a more complex tree.

Example 4.67

Show that $\neg\mathcal{P} \vdash (\neg\mathcal{P} \lor \mathcal{Q}) \land \neg(\mathcal{P} \land \mathcal{Q})$.

Solution

The main connective in the conclusion is a conjunction. This suggests that the final step in the deduction tree is an application of the $\land$I rule to $\neg\mathcal{P} \lor \mathcal{Q}$ and $\neg(\mathcal{P} \land \mathcal{Q})$.

$$\dfrac{\begin{array}{c}\vdots\\ \neg\mathcal{P}\lor\mathcal{Q}\end{array}\qquad\begin{array}{c}\vdots\\ \neg(\mathcal{P}\land\mathcal{Q})\end{array}}{(\neg\mathcal{P}\lor\mathcal{Q})\land\neg(\mathcal{P}\land\mathcal{Q})}\,\land\mathrm{I}$$

Thus we have two subgoals:

- $\neg\mathcal{P} \vdash \neg\mathcal{P} \lor \mathcal{Q}$
- $\neg\mathcal{P} \vdash \neg(\mathcal{P} \land \mathcal{Q})$

The first of these is a simple case of the $\lor\mathrm{I}_1$ rule:

$$\dfrac{\dfrac{\neg\mathcal{P}}{\neg\mathcal{P}\lor\mathcal{Q}}\,\lor\mathrm{I}_1\qquad\begin{array}{c}\vdots\\ \neg(\mathcal{P}\land\mathcal{Q})\end{array}}{(\neg\mathcal{P}\lor\mathcal{Q})\land\neg(\mathcal{P}\land\mathcal{Q})}\,\land\mathrm{I}$$

In the second inference, the main connective of the conclusion is a negation occurring singly. This suggests that this inference results from an application of the $\neg$I rule to

$$\neg\mathcal{P}, \mathcal{P} \land \mathcal{Q} \vdash \mathfrak{B} \land \neg\mathfrak{B}$$

for some $\mathfrak{B}$. Since the premiss set of this latest inference contains both $\mathcal{P}$ and $\neg\mathcal{P}$, an obvious choice for $\mathfrak{B}$ is simply $\mathcal{P}$ itself.

$$\dfrac{\neg\mathcal{P}\qquad\dfrac{\mathcal{P}\land\mathcal{Q}}{\mathcal{P}}\,\land\mathrm{E}_1}{\mathcal{P}\land\neg\mathcal{P}}\,\land\mathrm{I}$$

Thus the deduction tree for $\neg\mathcal{P} \vdash \neg(\mathcal{P} \land \mathcal{Q})$ is

$$\dfrac{\dfrac{\neg\mathcal{P}\qquad\dfrac{\dfrac{}{\mathcal{P}\land\mathcal{Q}}\,*}{\mathcal{P}}\,\land\mathrm{E}_1}{\mathcal{P}\land\neg\mathcal{P}}\,\land\mathrm{I}}{\neg(\mathcal{P}\land\mathcal{Q})}\,*\,\neg\mathrm{I}$$

This can be incorporated into the complete tree for $\neg\mathcal{P} \vdash (\neg\mathcal{P}\vee\mathcal{Q})\wedge\neg(\mathcal{P}\wedge\mathcal{Q})$:

$$\dfrac{\dfrac{\neg\mathcal{P}}{\neg\mathcal{P}\vee\mathcal{Q}}\vee\mathrm{I}_1 \qquad \dfrac{\dfrac{\neg\mathcal{P} \qquad \dfrac{\dfrac{}{\mathcal{P}\wedge\mathcal{Q}}*}{\mathcal{P}}\wedge\mathrm{E}_1}{\mathcal{P}\wedge\neg\mathcal{P}}\wedge\mathrm{I}}{\neg(\mathcal{P}\wedge\mathcal{Q})}*\ \neg\mathrm{I}}{(\neg\mathcal{P}\vee\mathcal{Q})\wedge\neg(\mathcal{P}\wedge\mathcal{Q})}\wedge\mathrm{I}$$

□

Representing the $\vee$E rule in a deduction tree requires two temporary premisses.

$$\dfrac{\mathfrak{A}\vee\mathfrak{B} \qquad \dfrac{\dfrac{\Gamma_1 \quad \dfrac{}{\mathfrak{A}}*}{\vdots}}{\mathfrak{C}} \qquad \dfrac{\dfrac{\Gamma_2 \quad \dfrac{}{\mathfrak{B}}*}{\vdots}}{\mathfrak{C}}}{\mathfrak{C}}*\ \vee\mathrm{E}$$

Example 4.68

Show that $\mathcal{P}\vee\mathcal{Q}\vdash\mathcal{Q}\vee\mathcal{P}$ using a deduction tree.

Solution

$$\dfrac{\mathcal{P}\vee\mathcal{Q} \qquad \dfrac{\dfrac{}{\mathcal{P}}*}{\mathcal{Q}\vee\mathcal{P}}\vee\mathrm{I}_2 \qquad \dfrac{\dfrac{}{\mathcal{Q}}*}{\mathcal{Q}\vee\mathcal{P}}\vee\mathrm{I}_1}{\mathcal{Q}\vee\mathcal{P}}\vee\mathrm{E}$$

□

Example 4.69

Show that $(\mathcal{P}\wedge\mathcal{Q})\vee(\mathcal{P}\wedge\mathcal{R})\vdash\mathcal{P}\wedge(\mathcal{Q}\vee\mathcal{R})$.

Solution

See Figure 4.1 □

Sometimes, it is necessary both the $\neg$I rule and the $\vee$E rule in a deduction tree, or even to repeat the same rule. When this happens, different markers will be necessary.

Example 4.70

Show that $\neg\mathcal{P}\vee\neg\mathcal{Q}\vdash\neg(\mathcal{P}\wedge\mathcal{Q})$

$$
\dfrac{(\mathcal{P}\wedge\mathcal{Q})\vee(\mathcal{P}\wedge\mathcal{R})
\qquad
\dfrac{\dfrac{\dfrac{}{\mathcal{P}\wedge\mathcal{Q}}\,*}{\mathcal{P}}\,\wedge\mathrm{E}_1
\quad
\dfrac{\dfrac{\dfrac{}{\mathcal{P}\wedge\mathcal{Q}}\,*}{\mathcal{Q}}\,\wedge\mathrm{E}_2}{\mathcal{Q}\vee\mathcal{R}}\,\vee\mathrm{I}_1}{\mathcal{P}\wedge(\mathcal{Q}\vee\mathcal{R})}\,\wedge\mathrm{I}
\qquad
\dfrac{\dfrac{\dfrac{}{\mathcal{P}\wedge\mathcal{R}}\,*}{\mathcal{P}}\,\wedge\mathrm{E}_1
\quad
\dfrac{\dfrac{\dfrac{}{\mathcal{P}\wedge\mathcal{R}}\,*}{\mathcal{R}}\,\wedge\mathrm{E}_2}{\mathcal{Q}\vee\mathcal{R}}\,\vee\mathrm{I}_2}{\mathcal{P}\wedge(\mathcal{Q}\vee\mathcal{R})}\,\wedge\mathrm{I}}{\mathcal{P}\wedge(\mathcal{Q}\vee\mathcal{R})}\,*\vee\mathrm{E}
$$

Figure 4.1: Deduction tree for $(\mathcal{P}\wedge\mathcal{Q})\vee(\mathcal{P}\wedge\mathcal{R})\vdash\mathcal{P}\wedge(\mathcal{Q}\vee\mathcal{R})$

Solution

$$\dfrac{\neg\mathcal{P}\vee\neg\mathcal{Q} \qquad \dfrac{\dfrac{\dfrac{}{\neg\mathcal{P}}\,* \quad \dfrac{\dfrac{}{\mathcal{P}\wedge\mathcal{Q}}\,\dagger}{\mathcal{P}}\,\wedge\mathrm{E}_1}{\mathcal{P}\wedge\neg\mathcal{P}}\,\wedge\mathrm{I}}{\neg(\mathcal{P}\wedge\mathcal{Q})}\,\dagger\,\neg\mathrm{I} \qquad \dfrac{\dfrac{\dfrac{}{\neg\mathcal{Q}}\,* \quad \dfrac{\dfrac{}{\mathcal{P}\wedge\mathcal{Q}}\,\ddagger}{\mathcal{Q}}\,\wedge\mathrm{E}_2}{\mathcal{Q}\wedge\neg\mathcal{Q}}\,\wedge\mathrm{I}}{\neg(\mathcal{P}\wedge\mathcal{Q})}\,\ddagger\,\neg\mathrm{I}}{\neg(\mathcal{P}\wedge\mathcal{Q})}\,*\,\neg\mathrm{I}$$

□

Exercise 30: Using $\neg$I and $\vee$E

Prove each of the following by constructing deduction trees.

1. $\neg\mathcal{Q} \vdash \neg(\mathcal{P}\wedge\mathcal{Q})$
2. $\mathcal{P}\wedge\mathcal{Q} \vdash \neg(\neg\mathcal{P}\wedge\mathcal{R})$
3. $\mathcal{P}\wedge\neg\mathcal{P} \vdash \mathcal{Q}$
4. $\mathcal{P}\vee\mathcal{Q} \vdash \mathcal{P}\vee(\mathcal{Q}\vee\mathcal{R})$
5. $\neg\mathcal{P}\vee\mathcal{Q}, \neg\mathcal{Q} \vdash \neg\mathcal{P}$
6. $\neg\mathcal{P}\wedge\neg\mathcal{Q} \vdash \neg(\mathcal{P}\vee\mathcal{Q})$

Soundness and completeness

With the addition of the $\neg$I and $\vee$E, we now have a complete and sound logical system. Using this system we can create all the valid argument forms for propositions built up from the connectives $\neg$, $\wedge$ and $\vee$. That is, for the propositional forms which we can construct using these three connectives:

- whenever $\Gamma \vDash \mathfrak{A}$ then $\Gamma \vdash \mathfrak{A}$;
- and whenever $\Gamma \vdash \mathfrak{A}$ then $\Gamma \vDash \mathfrak{A}$.

4.6 Theorems of natural deduction

Definition 4.11

Suppose we have an inference form whose premiss set is empty, $\vdash \mathfrak{A}$. Then the conclusion, $\mathfrak{A}$, is said to be a **theorem.** □

Example 4.71

Show that $\neg(\mathcal{P}\wedge\neg\mathcal{P})$ is a theorem.

Solution

Perhaps the simplest way to prove that $\neg(\mathcal{P} \wedge \neg\mathcal{P})$ is a theorem is to use the identity rule to write down $\mathcal{P} \wedge \neg\mathcal{P} \vdash \mathcal{P} \wedge \neg\mathcal{P}$, and then apply the $\neg$I rule to give $\vdash \neg(\mathcal{P} \wedge \neg\mathcal{P})$. A deduction tree with no derived rules is as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{P} \wedge \neg\mathcal{P}}\,*}{\mathcal{P}}\,\wedge\mathrm{E}_1 \qquad \dfrac{\dfrac{}{\mathcal{P} \wedge \neg\mathcal{P}}\,*}{\neg\mathcal{P}}\,\wedge\mathrm{E}_2}{\mathcal{P} \wedge \neg\mathcal{P}}\,\wedge\mathrm{I}}{\neg(\mathcal{P} \wedge \neg\mathcal{P})}\,*\,\neg\mathrm{I}$$

□

This last result can be generalized to give the following:

Thm.1

$\vdash \neg(\mathfrak{A} \wedge \neg\mathfrak{A})$ □

It is often convenient to use *Thm.1* as a derived rule of natural deduction.

Example 4.72

Prove by constructing a deduction tree that $\neg\neg\neg((\mathcal{P} \vee \mathcal{Q}) \wedge \neg(\mathcal{P} \vee \mathcal{Q}))$ is a theorem.

Solution

From *Thm.1* we know that $\vdash \neg((\mathcal{P}\vee\mathcal{Q})\wedge\neg(\mathcal{P}\vee\mathcal{Q}))$. Hence we can construct the following deduction tree.

$$\dfrac{\dfrac{}{\neg((\mathcal{P} \vee \mathcal{Q}) \wedge \neg(\mathcal{P} \vee \mathcal{Q}))}\,\mathit{Thm.}1}{\neg\neg\neg((\mathcal{P} \vee \mathcal{Q}) \wedge \neg(\mathcal{P} \vee \mathcal{Q}))}\,\neg\neg\mathrm{I}$$

□

Exercise 31: Theorems

By constructing deduction trees, prove that each of the following propositional forms is a theorem.

1. $\neg(\neg\mathcal{P} \wedge \mathcal{P}) \vee \mathcal{Q}$
2. $\neg(\neg\mathcal{P} \wedge \mathcal{P}) \wedge \neg(\mathcal{Q} \wedge \neg\mathcal{Q})$
3. $\neg(\mathcal{P} \wedge (\mathcal{Q} \wedge \neg\mathcal{P}))$

Theorems and tautologies

A propositional form is a theorem if, and only if, it is a tautology.

Justification

Since our deductive system is sound then whenever $\vdash \mathfrak{A}$, that is whenever $\mathfrak{A}$ is a theorem, $\vDash \mathfrak{A}$, that is $\mathfrak{A}$ is a tautology. And since our deductive system is complete then whenever $\vDash \mathfrak{A}$, that is whenever $\mathfrak{A}$ is a tautology, $\vdash \mathfrak{A}$, that is $\mathfrak{A}$ is a theorem. □

4.7 Syntactic equivalence

We have already met the concept of **semantic equivalence** between propositional forms in Chapter 3, although in that chapter it was referred to simply as 'equivalence': two forms $\mathfrak{A}$ and $\mathfrak{B}$ are semantically equivalent if both $\mathfrak{A} \vDash \mathfrak{B}$ and $\mathfrak{B} \vDash \mathfrak{A}$. We denote the semantic equivalence of $\mathfrak{A}$ and $\mathfrak{B}$ by $\mathfrak{A} =_T \mathfrak{B}$. There is, however, another way in which we can define equivalence.

Definition 4.12

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two propositional forms for which

- $\mathfrak{A} \vdash \mathfrak{B}$
- $\mathfrak{B} \vdash \mathfrak{A}$

then we say that $\mathfrak{A}$ and $\mathfrak{B}$ are **syntactically equivalent.** □

Notation: ≡

If $\mathfrak{A}$ and $\mathfrak{B}$ are syntactically equivalent, we write $\mathfrak{A} \equiv \mathfrak{B}$. □

Example 4.73

Show that $(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R} \equiv \mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})$.

Solution

First show that $(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R} \vdash \mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})$.

$$\dfrac{\dfrac{\dfrac{(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}}{\mathcal{P} \wedge \mathcal{Q}}\wedge\mathrm{E}_1}{\mathcal{P}}\wedge\mathrm{E}_1 \qquad \dfrac{\dfrac{\dfrac{(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}}{\mathcal{P} \wedge \mathcal{Q}}\wedge\mathrm{E}_1}{\mathcal{Q}}\wedge\mathrm{E}_2 \qquad \dfrac{(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}}{\mathcal{R}}\wedge\mathrm{E}_2}{\mathcal{Q} \wedge \mathcal{R}}\wedge\mathrm{I}}{\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})}\wedge\mathrm{I}$$

Then show that $\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R}) \vdash (\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}$.

$$\dfrac{\dfrac{\dfrac{\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})}{\mathcal{P}}\wedge\mathrm{E}_1 \qquad \dfrac{\dfrac{\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})}{\mathcal{Q} \wedge \mathcal{R}}\wedge\mathrm{E}_2}{\mathcal{Q}}\wedge\mathrm{E}_1}{\mathcal{P} \wedge \mathcal{Q}}\wedge\mathrm{I} \qquad \dfrac{\dfrac{\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})}{\mathcal{Q} \wedge \mathcal{R}}\wedge\mathrm{E}_2}{\mathcal{R}}\wedge\mathrm{E}_2}{(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}}\wedge\mathrm{I}$$

□

Now since our deductive system is both sound and complete, it follows that:

- whenever $\mathfrak{A} =_T \mathfrak{B}$ then $\mathfrak{A} \equiv \mathfrak{B}$;
- and whenever $\mathfrak{A} \equiv \mathfrak{B}$ then $\mathfrak{A} =_T \mathfrak{B}$.

Semantically equivalent forms are also syntactically equivalent, and *vice versa*. For this reason we often talk about 'equivalent' forms without stipulating what type of equivalence we mean.

Exercise 32: Syntactic equivalence

Prove each of the equivalences by constructing deduction trees.

1. $\mathcal{P} \equiv \neg\neg\mathcal{P}$
2. $\mathcal{P} \wedge \mathcal{Q} \equiv \mathcal{Q} \wedge \mathcal{P}$
3. $\mathcal{P} \vee \mathcal{Q} \equiv \mathcal{Q} \vee \mathcal{P}$
4. $(\mathcal{P} \vee \mathcal{Q}) \vee \mathcal{R} \equiv \mathcal{P} \vee (\mathcal{Q} \vee \mathcal{R})$

Conditional Connective 5

5.1 Symbolic representation of information

This book is about logic and language, and in particular it is about the representation of information and reasoning in symbolic form. So far we have made use of three connectives, namely: negation $\neg$; conjunction $\wedge$; and disjunction $\vee$. There are, however, limitations in using only these three connectives.

For example we use conjunction $\wedge$ to represent both '*and*' and '*but*', even though '*but*' usually has some additional nuance of meaning that '*and*' does not have; in representing '*but*' by $\wedge$ we have lost some of the meaning. For example the statement '*Rex has four legs and Fido has three legs*' is a simple assertion of the two facts

- '*Rex has four legs*'
- '*Fido has three legs*'

and is fully represented by '*Rex has four legs*' $\wedge$ '*Fido has three legs*' . However, the statement '*Rex has four legs but Fido has three legs*' also suggests that we could reasonably have expected Fido to have the same number of legs as Rex; its sense is not completely captured by the use of $\wedge$. In practice this does not represent a problem since whenever the statement with '*but*' is true, then so is the statement with '*and*'. If we obtain a conjunction as the result of symbolic reasoning, then we use additional understanding of the situation to decide whether '*but*' might be appropriate.

Example 5.1

What conjunction can be deduced from '2 *is prime but* 2 *is even*'?

Solution

The two atomic propositions are '2 *is prime*' and '2 *is even*'. The conjunction of these two is '2 *is prime*' $\wedge$ '2 *is even*'. This is certainly true given the original statement, but the suggestion that prime numbers are not usually even has been lost. (In fact all prime numbers except 2 are odd, which is why '*but*' was used in the original.) □

Example 5.2

How might '57 *is prime*' $\wedge$ '57 *is odd*' be interpreted in plain language?

Solution

The basic interpretation is '57 *is prime and* 57 *is odd*'. Now we know that prime numbers are usually odd, so it is not surprising that 57 is odd given that it is prime. In this case it would be misleading to use '*but*'. □

Example 5.3

How might the following be interpreted in plain language?

'*Sydney is the best known city in Australia*'
$\wedge$ $\neg$'*Sydney is the capital of Australia*'

Solution

The basic interpretation is '*Sydney is the best known city in Australia and Sydney is not the capital of Australia*'. Now we might reasonably expect that the best known city of any country is its capital; thus the fact that Sydney is not the capital of Australia is unexpected. In this case it would be entirely appropriate to use '*but*' and write

'*Sydney is the best known city in Australia but is not the capital.*' □

Now it might be thought that it would be a simple matter to introduce a new symbol to represent '*but*': $\barwedge$ say. Thus we would be able to represent the proposition '2 *is prime but* 2 *is even*' as '2 *is prime*' $\barwedge$ '2 *is even*'. Such a statement would be seen as an instance of the propositional form $\mathcal{P} \barwedge \mathcal{Q}$. This leads to questions of what the elimination and introduction rules would be for $\barwedge$; and what the truth table would be for $\mathcal{P} \barwedge \mathcal{Q}$. From our understanding of the nature of $\barwedge$ we could write down the inference form

$$\mathcal{P} \barwedge \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q} \tag{5.1}$$

from which we could derive elimination rules $\mathcal{P} \barwedge \mathcal{Q} \vdash \mathcal{P}$ and $\mathcal{P} \barwedge \mathcal{Q} \vdash \mathcal{Q}$. Unfortunately, we would not be able to deduce $\mathcal{P} \barwedge \mathcal{Q}$ from $\mathcal{P}$ and $\mathcal{Q}$ since for some instances of $\mathcal{P}$ and $\mathcal{Q}$ the use of '*but*' would not be appropriate; for example '57 *is prime*' and '57 *is odd*'. Hence there would be no introduction rule for $\barwedge$. The only rule needed would be the inference form 5.1; in effect we apply this rule whenever we write a compound proposition with '*but*' as a conjunction $p \wedge q$. Thus there is nothing to be gained by introducing a special symbol such as $\barwedge$ for '*but*'.

But what about a truth table for $\mathcal{P} \barwedge \mathcal{Q}$? From the inference form 5.1, it follows that $\mathcal{P} \barwedge \mathcal{Q} \therefore \mathcal{P} \wedge \mathcal{Q}$ would be a valid argument form. Hence whenever $\mathcal{P} \wedge \mathcal{Q}$ is false, that is whenever at least one of $\mathcal{P}$ and $\mathcal{Q}$ is false, $\mathcal{P} \barwedge \mathcal{Q}$ would be false. This leaves us with the problem of deciding the truth value of $\mathcal{P} \barwedge \mathcal{Q}$ when $\mathcal{P}$ and $\mathcal{Q}$ are both true. The answer is that the truth value of $\mathcal{P} \barwedge \mathcal{Q}$ would depend upon the *particular* instances of $\mathcal{P}$ and $\mathcal{Q}$, as we have seen in the earlier examples. Hence there would be no single truth value for $\mathcal{P} \barwedge \mathcal{Q}$ when $\mathcal{P}$ and $\mathcal{Q}$ both true. Now the concept of connective used in this book is that of an operator such that the truth value of a compound proposition depends only upon the truth values of the operands. The symbol $\barwedge$ does not fit this definition, and therefore cannot be regarded as a connective. The best we can do is to use the connective $\wedge$, since whenever '$\mathcal{P}$ *but* $\mathcal{Q}$' is true then $\mathcal{P} \wedge \mathcal{Q}$ is also true. However, the reverse does not hold: the conjunction in $\mathcal{P} \wedge \mathcal{Q}$ does not always represent '*but*'. For these reasons we conclude that the meaning of '*but*' cannot be completely captured by a connective, and hence we do not use a symbolic representation for '*but*'.

The important point to realize is that logic, or at least the logic that we are considering, does not do everything that we might want to do; but nevertheless it is still a very useful tool.

5.2 Causality, conditional statements and implication

This section is concerned with the representation as compound propositions of causality statements, conditional statements and implication. It will be seen that, like '*but*'-statements, none of these can be completely represented by a connective. And, just as we can capture some of the sense of '*but*'-statements by using the connective $\wedge$, we can capture something of the sense of these statements by using a new connective $\Rightarrow$.

Causality

Sometimes we want to represent the fact that one thing *causes* another. Consider the statement:

'*If Siobhan is sick tomorrow, she will stay at home*'

The intended meaning of this is that Siobhan's sickness tomorrow would *cause* her to stay at home. Now if we analyse this statement, we can identify two atomic propositions:

- p : '*Siobhan will be sick tomorrow*'
- q : '*Siobhan will stay at home tomorrow*'

where the letters p and q will be used for convenience instead of the actual propositions. Thus our original statement asserts that p *causes* q. This suggests that we could use some symbol, $\rightsquigarrow$ say, to represent causality and write $p \rightsquigarrow q$.

By the very nature of causality, if p and $p \rightsquigarrow q$ were true then q must also be true; $p, p \rightsquigarrow q \therefore q$ would be a valid argument. Thus we could adopt the inference form $\mathcal{P}, \mathcal{P} \rightsquigarrow \mathcal{Q} \vdash \mathcal{Q}$ as a rule of deduction. Furthermore we see that whenever $\mathcal{Q}$ is false, at least one of $\mathcal{P}$ and $\mathcal{P} \rightsquigarrow \mathcal{Q}$ would have to be false; thus if $\mathcal{P}$ is true then $\mathcal{P} \rightsquigarrow \mathcal{Q}$ would be false. Unfortunately, it is not possible to uniquely determine truth values of $\mathcal{P} \rightsquigarrow \mathcal{Q}$ for other combinations of truth values for $\mathcal{P}$ and $\mathcal{Q}$. Suppose we consider instances of $\mathcal{P}$ and $\mathcal{Q}$ which are both true: for some instances there will be a genuine causal relationship so that $\mathcal{P} \rightsquigarrow \mathcal{Q}$ would be true; for other instances, however, there will not be a causal relationship so that $\mathcal{P} \rightsquigarrow \mathcal{Q}$ would be false. (As an example of the latter case, consider taking $1 + 1 = 2$ and '*Archimedes was Greek*' as instances of $\mathcal{P}$ and $\mathcal{Q}$.) Thus we see that the symbol $\rightsquigarrow$ would not be a connective. Hence we cannot represent causality completely by a connective.

Conditional statements

Sometimes we want to express the fact that something will happen *only if* some condition is true. For example, consider the statement

'*You will succeed only if you work hard*'.

We can identify two atomic propositions:

- p = '*You will succeed*'
- q = '*You work hard*'

Thus the original statement asserts that p *only if* q. Suppose we introduce a symbol, $\succ$ say, to represent conditionality. Then we could write $p \succ q$.

Suppose $p \succ q$ to be true. This means that p could be true only if q were true; that is, if p is true, then so is q. Thus we would require $p, p \succ q \therefore q$ to be a valid argument form, and the inference form $\mathcal{P}, \mathcal{P} \succ \mathcal{Q} \vdash \mathcal{Q}$ to be a rule of deduction. Using arguments similar to those given for causality, we find that for true instances of $\mathcal{P}$ with false instances of $\mathcal{Q}$ the corresponding instances of $\mathcal{P} \succ \mathcal{Q}$ would be false; otherwise, however, we are not be able to determine uniquely what the truth value for the instance of $\mathcal{P} \succ \mathcal{Q}$ would be. Thus the symbol $\succ$ is not a connective. We cannot capture the notion of conditionality completely with a connective.

Valid arguments from one premiss

We have made several statements to the effect that certain arguments are valid, but these statements themselves may be regarded as having a truth value! Like all books, there will be errors in this one, and it is not impossible that I might claim some arguments to be valid which are in fact false. (Reader beware!) In this section we consider arguments in which there is just one premiss, and explore whether statements about the validity of such arguments might be expressed as propositions using a connective.

Consider the following statement:

'$3 > 1 \ \wedge\ 1 + 3 = 4 \ \therefore\ 1 + 3 = 4$ *is a valid argument*'.

Can we express this symbolically as a compound proposition? If we let p represent the compound proposition $3 > 1 \ \wedge \ 1+3=4$ and q the proposition $1+3=4$, then the statement can be written as

'$p \ \therefore \ q$ *is a valid argument*'.

However, it may not be immediately apparent how this can be regarded as a compound proposition. To overcome this difficulty, we introduce the notion of implication; implication is the assertion, possibly false, that an argument is valid. Thus we can write down the proposition

'p *implies* q'.

Now if we had a special symbol, $\rhd$ say, to represent implication, we could write the proposition as

$$p \rhd q \,.$$

The intention is that $\rhd$ captures the notion of implication completely; that is to say, *both* of the following deductions can always be made:

- 'p *implies* q' $\vdash p \rhd q$
- $p \rhd q \vdash$ 'p *implies* q'

The properties of this symbol are similar to those we found for $\rightsquigarrow$ (causality) and $\succ$ (conditionality). In particular we find that we would need a similar deduction rule:

$$\mathcal{P}, \mathcal{P} \rhd \mathcal{Q} \vdash \mathcal{Q}$$

Also, the truth value of $\mathcal{P} \rhd \mathcal{Q}$ would be F when that of $\mathcal{P}$ is T and that of $\mathcal{Q}$ is F; otherwise the truth value is not uniquely determined by the truth values of $\mathcal{P}$ and $\mathcal{Q}$. Hence the symbol $\rhd$ is not be a connective; there is no connective which completely captures the notion of a valid argument.

5.3 A new connective

In the previous section, we saw that it was not possible to capture completely the notions of causality, conditionality or implication by connectives. Although we could introduce symbols such as $\rightsquigarrow$, $\succ$ and $\rhd$, when we try to obtain the corresponding truth tables the best we can achieve may be summarized as follows:

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \rightsquigarrow \mathcal{Q}$	$\mathcal{P} \succ \mathcal{Q}$	$\mathcal{P} \rhd \mathcal{Q}$
T	T	T, F	T, F	T, F
T	F	F	F	F
F	T	T, F	T, F	T, F
F	F	T, F	T, F	T, F

Now consider the connective $\Rightarrow$[1] defined by the truth table

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{Q}$
T	T	T
T	F	F
F	T	T
F	F	T

From the above truth tables we see that whenever any one of $\mathcal{P} \rightsquigarrow \mathcal{Q}$, $\mathcal{P} \succ \mathcal{Q}$ or $\mathcal{P} \rhd \mathcal{Q}$ is true, then so is $\mathcal{P} \Rightarrow \mathcal{Q}$. However, the reverse is not necessarily the case. For example,

$1 + 1 = 2 \Rightarrow$ '*Jupiter is a giant planet*'

is true, even though the fact that $1 + 1 = 2$ is not the cause of Jupiter being a giant planet; nor is Jupiter being a giant planet a necessary condition for $1 + 1$ to equal 2; nor can we deduce that Jupiter is a giant planet from the fact that $1 + 1 = 2$.

Thus the connective $\Rightarrow$ *partially* captures the notion of causality, *partially* captures the notion of conditionality and *partially* captures the notion of valid argument in much the same way as the connective $\wedge$ only *partially* captures the notion of '*but*'. Unlike $\wedge$, however, the connective $\Rightarrow$ does not represent a clearly defined relationship between its operands; it is a somewhat abstract connective. Nevertheless it is necessary to give it a name and to decide upon a pronunciation for the symbol, simply in order to enable us to talk about it. In those contexts where it derives from a logical argument, it would be reasonable to refer to it as the **implication** connective and read as '*implies*'; in those contexts where it derives from a conditional relationship, it would be reasonable to refer to it as the **conditional** connective and read as '*only if*'; in those contexts where it derives from a causal relationship, it would be reasonable to refer to it as the **causal** connective and read as '*if ... then ...*'. In this book, it will generally be referred to as the conditional connective; the pronunciation is largely a matter of choice for the reader, but '*only if*' is recommended for general use.

We sometimes use special names for the propositions that occur before and after the conditional connective, especially in the context of implication. In the compound proposition $p \Rightarrow q$, the proposition p is referred to as the **antecedent** and q as the **consequent**. The propositional forms $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathfrak{A} \Rightarrow \mathfrak{B}$ are also referred to as the antecedent and consequent respectively.

Example 5.4

Write a compound proposition using $\Rightarrow$ which follows from

'*If Siobhan is sick tomorrow, she will stay at home.*'

[1] For the time being, we shall still pronounce this connective as '*links to*'.

Solution

We can identify two propositions, the first of which causes the second: '*Siobhan will be sick tomorrow*' causes '*Siobhan will stay at home tomorrow*'. From this we can deduce

'*Siobhan will be sick tomorrow*' ⇒ '*Siobhan will stay at home tomorrow*' .

□

Example 5.5

Write a compound proposition using ⇒ which follows from

'*You will succeed only if you work hard*'.

Solution

We can identify two propositions, the first of which is true only if the second is true: '*You will succeed*' only if '*You work hard*'. From this we can deduce

'*You will succeed*' ⇒ '*You work hard*' . □

Where compound propositions include more than one connective, care must be taken to include parentheses appropriately. In section 5.5, however, we shall consider the priority of the conditional connective, and how parentheses may be removed.

Example 5.6

Write a compound proposition using ⇒ which follows from

'$3 > 1 \;\wedge\; 1 + 3 = 4 \;\therefore\; 1 + 3 = 4$ *is a valid argument*'.

Solution

We can reword the statement as '$3 > 1 \;\wedge\; 1 + 3 = 4$ *implies* $1 + 3 = 4$' from which we can deduce

$(3 > 1 \;\wedge\; 1 + 3 = 4) \Rightarrow 1 + 3 = 4$.

Note that we have included parentheses to ensure that the conditional is the main connective in this compound proposition. □

Propositional forms may also include occurrences of the conditional connective. When interpreting instances of such forms, however, it may be necessary to use the context to obtain a meaningful statement in English.

Example 5.7

Write down a meaningful statement in English which might correspond to the instance of $(\neg\mathcal{P} \wedge \mathcal{Q}) \Rightarrow \mathcal{R}$ in which

- $\mathcal{P}$ is instantiated to '*I take an umbrella*'
- $\mathcal{Q}$ is instantiated to '*It will rain today*'
- $\mathcal{R}$ is instantiated to '*I shall get wet*'

Solution

The propositional form parses to $((\neg \mathcal{P}) \wedge \mathcal{Q}) \Rightarrow \mathcal{R}$. Now the instance of $\neg \mathcal{P}$ may be interpreted as '*I do not take an umbrella*', and so the instance of $((\neg \mathcal{P}) \wedge \mathcal{Q})$ may be interpreted as '*I do not take an umbrella and it rains today*'. The instance of $(\neg \mathcal{P} \wedge \mathcal{Q}) \Rightarrow \mathcal{R}$ is therefore

'*I do not take an umbrella and it rains today*' ⇒ '*I shall get wet*'

Although correct, this is not a meaningful statement in English. We have to use our judgment to decide what might be appropriate. In this case, it seems likely that we have causality:

'*If I do not take an umbrella and it rains today then I shall get wet*'. □

Example 5.8

Write down a meaningful statement in English which might correspond to the instance of $\neg \mathcal{P} \Rightarrow \mathcal{Q}$ in which

- $\mathcal{P}$ is instantiated to '*I will visit next month*'
- $\mathcal{Q}$ is instantiated to '*You tell me not to visit next month*'

Solution

In this instance we have

'*I will not visit next month*' ⇒ '*You tell me not to visit next month*'

It would seem that a conditional statement might be appropriate:

'*I will not visit next month only if you tell me not to*'.

Unfortunately the meaning of his statement is not clear. A more understandable reading is

'*I will visit next month unless you tell me not to*'.

In this instance we have read $\neg \mathcal{P} \Rightarrow \mathcal{Q}$ as '$\mathcal{P}$ *unless* $\mathcal{Q}$'. □

Example 5.9

Is there a meaningful statement in English which might correspond to the following instance of $\mathcal{P} \Rightarrow \mathcal{Q}$?

- $\mathcal{P}$ is instantiated to $1 + 1 = 2$
- $\mathcal{Q}$ is instantiated to '*Rex has four legs*'

Solution

Clearly $1 + 1 = 2 \therefore$ '*Rex has four legs*' is not a valid argument form. Furthermore, there seems no possibility of any real connection between the the two facts: to say '$1+1=2$ *only if Rex has four legs*' sounds somewhat bizarre. There does not appear to be any meaningful statement in English in this instance. □

Exercise 33: Compound propositions with the conditional

1. Write down a compound proposition using ⇒ which follows from each of the following.
   (a) '*If it rains today then I shall get wet.*'
   (b) '$a = 2$ *implies* $a^2 = 4$.'
   (c) '*If I understand logic then I shall become a good programmer.*'
   (d) '*John will only go to the party if Mary goes.*'
   (e) '*John will go to the party if Mary goes.*'
   (f) '*John will go to the party if, and only if, Mary goes.*'
   (g) '*Unless my car breaks down, we shall go to the seaside tomorrow.*'
2. For each of the following proposition forms, use the given instances of $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ to obtain a a meaningful statement in English.
   (a) $\mathcal{P} \Rightarrow \mathcal{Q}$ where
   - $\mathcal{P}$: '*You are nice to me*'
   - $\mathcal{Q}$: '*I will be your friend*'

   (b) $\mathcal{Q} \Rightarrow \mathcal{P}$ where
   - $\mathcal{P}$: '*You are nice to me*'
   - $\mathcal{Q}$: '*I will be your friend*'

   (c) $\neg\mathcal{P} \Rightarrow \neg\mathcal{Q}$ where
   - $\mathcal{P}$: '*You are a good boy*'
   - $\mathcal{Q}$: '*Father Christmas will leave you presents*'

   (d) $\mathcal{Q} \Rightarrow \mathcal{P}$ where
   - $\mathcal{P}$: '*You are a good boy*'
   - $\mathcal{Q}$: '*Father Christmas will leave you presents*'

   (e) $(\mathcal{Q} \Rightarrow (\mathcal{R} \wedge \neg\mathcal{S})) \wedge (\neg\mathcal{Q} \Rightarrow (\mathcal{S} \wedge \neg\mathcal{R}))$ where
   - $\mathcal{Q}$: '*The driver has a parking permit*'
   - $\mathcal{R}$: '*The driver will be admitted to the car park*'
   - $\mathcal{S}$: '*The driver will be sent a reminder about parking regulations*'

5.4 Properties of the conditional connective

Newcomers to logic are frequently troubled by the need to know what the connective ⇒ 'really means'. As indicated in the previous sections, there is no simple meaning that can be attached to the connective. Nevertheless, we can list some properties which, in a sense, give some sort of meaning.

Equivalence to a disjunction

Perhaps the most useful property for giving some sort of meaning is the following equivalence.

Prop25

$$\mathcal{P} \Rightarrow \mathcal{Q} =_T \neg\mathcal{P} \vee \mathcal{Q}$$ □

Justification

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{P} \vee \mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{Q}$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

□

Thus we can regard the conditional as meaning either the consequent is true or the antecedent is false (or possibly both).

Modus ponens

Prop26

$$\mathcal{P}, \mathcal{P} \Rightarrow \mathcal{Q} \vDash \mathcal{Q}$$ □

Justification
From the first row of the following truth table we see that $\mathcal{Q}$ is true whenever both $\mathcal{P}$ and $\mathcal{P} \Rightarrow \mathcal{Q}$ are true.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{Q}$
$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$
T	F	F
F	T	T
F	F	T

□

(In traditional logic this relationship is known as *modus ponens ponendis* or as *modus ponens* for short.) Thus $\mathcal{P} \Rightarrow \mathcal{Q}$ provides a link that enables us to conclude $\mathcal{Q}$ from $\mathcal{P}$. A sensible name for the connective might thus seem to be be **linkage**, with the symbol pronounced as '*links to*'; such usage would, however, be highly unconventional.

Validity of argument forms

Although $p \Rightarrow q$ does not necessarily represent the fact that $p \therefore q$ is a valid argument for propositions p and q, there is a very important result concerning propositional *forms* $\mathfrak{A} \Rightarrow B$ in which the conditional connective is the main connective.

Prop27

$\mathfrak{A} \therefore \mathfrak{B}$ is a valid argument form if, and only if, $\mathfrak{A} \Rightarrow B$ is a tautology. That is, if $\mathfrak{A} \vDash \mathfrak{B}$ if, and only if, $\vDash \mathfrak{A} \Rightarrow \mathfrak{B}$. □

Justification

There are two parts to this justification. Firstly we must show that $\mathfrak{A} \vDash \mathfrak{B}$ *only if* $\vDash \mathfrak{A} \Rightarrow \mathfrak{B}$. We shall do this by supposing that $\mathfrak{A} \vDash \mathfrak{B}$, then showing that $\vDash \mathfrak{A} \Rightarrow \mathfrak{B}$. Secondly we must show that $\mathfrak{A} \vDash \mathfrak{B}$ *if* $\vDash \mathfrak{A} \Rightarrow \mathfrak{B}$. We shall do this by supposing that $\vDash \mathfrak{A} \Rightarrow \mathfrak{B}$, then showing that $\mathfrak{A} \vDash \mathfrak{B}$. Combining these two facts gives us the '*if, and only if*' relationship between $\mathfrak{A} \vDash \mathfrak{B}$ and $\vDash \mathfrak{A} \Rightarrow \mathfrak{B}$.

Suppose firstly, therefore, that $\mathfrak{A} \vDash \mathfrak{B}$. This means that whenever $\mathfrak{A}$ is true $\mathfrak{B}$ must also be true, that is $\mathfrak{B}$ cannot be false. Thus the only combinations of truth values for $\mathfrak{A}, \mathfrak{B}$ are T, T, F, T and F, F. From the truth table for $\mathfrak{A} \Rightarrow \mathfrak{B}$

$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{A} \Rightarrow \mathfrak{B}$
T	T	T
T	F	F
F	T	T
F	F	T

we see that $\mathfrak{A} \Rightarrow \mathfrak{B}$ is true in all these instances. That is $\mathfrak{A} \Rightarrow \mathfrak{B}$ is a tautology: $\vDash \mathfrak{A} \Rightarrow \mathfrak{B}$. Thus we have shown that $\mathfrak{A} \vDash \mathfrak{B}$ *only if* $\vDash \mathfrak{A} \Rightarrow \mathfrak{B}$.

Now suppose that $\vDash \mathfrak{A} \Rightarrow \mathfrak{B}$. From the truth table we see that whenever $\mathfrak{A} \Rightarrow \mathfrak{B}$ is true, then either both $\mathfrak{A}$ and $\mathfrak{B}$ are true, or $\mathfrak{A}$ is false. Thus in all instances for which the truth value of $\mathfrak{A}$ is T, the truth value of $\mathfrak{B}$ must also be T; that is, $\mathfrak{A} \vDash \mathfrak{B}$. Thus we have shown that $\mathfrak{A} \vDash \mathfrak{B}$ *if* $\vDash \mathfrak{A} \Rightarrow \mathfrak{B}$. □

5.5 Priority of the conditional connective

In Section 2.11 an order of priority is given such that in replacing missing parentheses $\neg$ is considered before $\wedge$, which in turn is considered before $\vee$. Following this order of priority, the conditional connective $\Rightarrow$ is considered last. Thus $\mathcal{P} \wedge \mathcal{Q} \Rightarrow \mathcal{R}$ is interpreted as $(\mathcal{P} \wedge \mathcal{Q}) \Rightarrow \mathcal{R}$.

Example 5.10

Parse $\mathcal{P} \wedge \mathcal{Q} \Rightarrow \mathcal{R}$.

Solution

There are two possibilities: $(\mathcal{P} \wedge \mathcal{Q}) \Rightarrow \mathcal{R}$ and $\mathcal{P} \wedge (\mathcal{Q} \Rightarrow \mathcal{R})$. Since conjunction has the higher priority, the correct parsing is $(\mathcal{P} \wedge \mathcal{Q}) \Rightarrow \mathcal{R}$. □

Example 5.11

Parse $\neg\mathcal{P} \vee \mathcal{Q} \Rightarrow \mathcal{P} \wedge \mathcal{Q}$

Solution

Negation has the highest priority:

$$(\neg\mathcal{P}) \vee \mathcal{Q} \Rightarrow \mathcal{P} \wedge \mathcal{Q}$$

Next comes conjunction:

$$(\neg\mathcal{P}) \vee \mathcal{Q} \Rightarrow (\mathcal{P} \wedge \mathcal{Q})$$

Then comes disjunction:

$$((\neg\mathcal{P}) \vee \mathcal{Q}) \Rightarrow (\mathcal{P} \wedge \mathcal{Q})$$

□

Example 5.12

What is the truth table for $\neg\mathcal{P} \vee \mathcal{Q} \Rightarrow \mathcal{P} \wedge \mathcal{Q}$? What is a simpler propositional form equivalent to this?

Solution

The first step in answering a question like this is to replace the missing parentheses. This has already been done in Example 5.5: $((\neg\mathcal{P}) \vee \mathcal{Q}) \Rightarrow (\mathcal{P} \wedge \mathcal{Q})$.

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{P} \vee \mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$	$\neg\mathcal{P} \vee \mathcal{Q} \Rightarrow \mathcal{P} \wedge \mathcal{Q}$
T	*T*	*F*	*T*	*T*	*T*
T	*F*	*F*	*F*	*F*	*T*
F	*T*	*T*	*T*	*F*	*F*
F	*F*	*T*	*T*	*F*	*F*

Clearly $\neg\mathcal{P} \vee \mathcal{Q} \Rightarrow \mathcal{P} \wedge \mathcal{Q} =_T \mathcal{P}$ □

Example 5.13

Are $(\mathcal{P} \Rightarrow \mathcal{Q}) \Rightarrow \mathcal{R}$ and $\mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{R})$ equivalent forms?

Solution

From the truth tables we see that the two forms are *not* equivalent, although it is the case that $(\mathcal{P} \Rightarrow \mathcal{Q}) \Rightarrow \mathcal{R} \vDash \mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{R})$

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{P} \Rightarrow \mathcal{Q}$	$(\mathcal{P} \Rightarrow \mathcal{Q}) \Rightarrow \mathcal{R}$	$\mathcal{Q} \Rightarrow \mathcal{R}$	$\mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{R})$
T	*T*	*T*	*T*	*T*	*T*	*T*
T	*T*	*F*	*T*	*F*	*F*	*F*
T	*F*	*T*	*F*	*T*	*T*	*T*
T	*F*	*F*	*F*	*T*	*T*	*T*
F	*T*	*T*	*T*	*T*	*T*	*T*
F	*T*	*F*	*T*	***F***	*F*	***T***
F	*F*	*T*	*T*	*T*	*T*	*T*
F	*F*	*F*	*T*	***F***	*T*	***T***

□

There is still the issue of how priorities are decided between two or more occurrences of the conditional connective. For example, should $\mathcal{P} \Rightarrow \mathcal{Q} \Rightarrow \mathcal{R}$ be parsed as $(\mathcal{P} \Rightarrow \mathcal{Q}) \Rightarrow \mathcal{R}$ or as $\mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{R})$? (From the previous example we know that these two forms are not equivalent.) Following the procedure for conjunction and disjunction, we shall agree that conditionals to the left have higher priority. Thus, $(\mathcal{P} \Rightarrow \mathcal{Q}) \Rightarrow \mathcal{R}$ may be written without parentheses as $\mathcal{P} \Rightarrow \mathcal{Q} \Rightarrow \mathcal{R}$.

Exercise 34: Truth tables and the conditional connective

1. Obtain the truth table for each of the following propositional forms,

   (a) $\neg\mathcal{P} \Rightarrow \neg\mathcal{Q}$
   (b) $\neg\mathcal{P} \Rightarrow \neg\mathcal{Q} \vee \mathcal{R}$
   (c) $\neg\mathcal{P} \Rightarrow \mathcal{Q} \wedge \mathcal{P} \Rightarrow \mathcal{Q}$

2. Construct truth tables and hence decide whether each of the following propositional schemas is a contradiction, a tautology or neither.

   (a) $\mathcal{P} \Rightarrow \neg\mathcal{P}$
   (b) $(\mathcal{P} \Rightarrow \neg\mathcal{P}) \wedge (\neg\mathcal{P} \Rightarrow \mathcal{P})$
   (c) $\mathcal{P} \wedge \mathcal{Q} \Rightarrow \mathcal{P}$
   (d) $\mathcal{P} \Rightarrow \mathcal{P} \wedge \mathcal{Q}$
   (e) $\mathcal{P} \vee \mathcal{Q} \Rightarrow \mathcal{P}$
   (f) $\mathcal{P} \Rightarrow \mathcal{P} \vee \mathcal{Q}$
   (g) $\mathcal{P} \vee \neg\mathcal{P} \Rightarrow \mathcal{P} \wedge \neg\mathcal{P}$
   (h) $(\mathcal{P} \Rightarrow \mathcal{Q}) \vee \mathcal{P}$
   (i) $\neg\mathcal{P} \wedge \neg\mathcal{Q} \Rightarrow \mathcal{P} \vee \mathcal{Q}$
   (j) $\mathcal{P} \vee \mathcal{Q} \Rightarrow \neg\mathcal{P} \wedge \neg\mathcal{Q}$
   (k) $(\mathcal{P} \Rightarrow \mathcal{P}) \Rightarrow (\mathcal{P} \Rightarrow \neg\mathcal{P})$
   (l) $(\mathcal{P} \wedge \mathcal{Q} \Rightarrow \mathcal{Q} \wedge \mathcal{P}) \wedge (\mathcal{Q} \wedge \mathcal{P} \Rightarrow \mathcal{P} \wedge \mathcal{Q})$

3. Use truth tables to prove each of the following equivalences.

   (a) $\mathcal{P} \Rightarrow \neg\mathcal{P} =_T \neg\mathcal{P}$
   (b) $\mathcal{P} \Rightarrow \mathcal{Q} =_T \neg(\mathcal{P} \wedge \neg\mathcal{Q})$
   (c) $\neg(\mathcal{P} \Rightarrow \mathcal{Q}) =_T \mathcal{P} \wedge \neg\mathcal{Q}$
   (d) $\mathcal{P} \Rightarrow \mathcal{Q} \wedge \mathcal{R} =_T (\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{P} \Rightarrow \mathcal{R})$
   (e) $\mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{R}) =_T (\mathcal{P} \wedge \mathcal{Q}) \Rightarrow \mathcal{R}$

4. Use truth tables to prove each of the following entailments.

   (a) $\models \mathcal{P} \Rightarrow \mathcal{P}$
   (b) $\mathcal{P} \Rightarrow \mathcal{Q}, \neg\mathcal{Q} \models \neg\mathcal{P}$
   (c) $\mathcal{P} \models (\mathcal{Q} \Rightarrow \mathcal{P})$
   (d) $\mathcal{P} \wedge \mathcal{Q} \models \mathcal{P} \Rightarrow \mathcal{Q}$
   (e) $(\mathcal{P} \Rightarrow \mathcal{Q}) \Rightarrow \mathcal{R} \models \mathcal{Q} \Rightarrow \mathcal{R}$
   (f) $\mathcal{P}_1 \Rightarrow \mathcal{P}_2, \mathcal{P}_2 \Rightarrow \mathcal{P}_3 \models \mathcal{P}_1 \Rightarrow \mathcal{P}_3$
   (g) $\models \mathcal{P} \wedge (\neg\mathcal{Q} \Rightarrow (\mathcal{R} \vee \neg\mathcal{S})) \Rightarrow \mathcal{P}$

5.6 Equational logic

Table 3.1 listed properties of propositional forms involving $\neg$, $\wedge$ and $\vee$. We can now add to these some further properties involving the conditional connective; see Table 5.1. Note that the equivalence expressed in *Prop25* is often taken as the definition of the conditional connective. It can be used to prove other equivalences.

Example 5.14
Show that $\mathcal{P} \Rightarrow (\mathcal{Q} \wedge \mathcal{R}) =_T (\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{P} \Rightarrow \mathcal{R})$.

Solution

$$
\begin{array}{lll}
 & \mathcal{P} \Rightarrow (\mathcal{Q} \wedge \mathcal{R}) & \\
=_T & \neg\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R}) & \langle\ \mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{A} \vee \mathfrak{B}\ \rangle \\
=_T & (\neg\mathcal{P} \vee \mathcal{Q}) \wedge (\neg\mathcal{P} \vee \mathcal{R}) & \langle\ \mathfrak{A} \vee (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \wedge (\mathfrak{A} \vee \mathfrak{C})\ \rangle \\
=_T & (\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{P} \Rightarrow \mathcal{R}) & \langle\ \mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{A} \vee \mathfrak{B}\ \rangle
\end{array}
$$

□

Example 5.15
Show that $(\mathcal{P} \vee \mathcal{Q}) \Rightarrow \mathcal{R} =_T (\mathcal{P} \Rightarrow \mathcal{R}) \wedge (\mathcal{Q} \Rightarrow \mathcal{R})$.

Solution

$$
\begin{array}{lll}
 & (\mathcal{P} \vee \mathcal{Q}) \Rightarrow \mathcal{R} & \\
=_T & \neg(\mathcal{P} \vee \mathcal{Q}) \vee \mathcal{R} & \langle\ \mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{A} \vee \mathfrak{B}\ \rangle \\
=_T & (\neg\mathcal{P} \wedge \neg\mathcal{Q}) \vee \mathcal{R} & \langle\ \neg(\mathfrak{A} \vee \mathfrak{B}) =_T \neg\mathfrak{A} \wedge \neg\mathfrak{B}\ \rangle \\
=_T & \mathcal{R} \vee (\neg\mathcal{P} \wedge \neg\mathcal{Q}) & \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle \\
=_T & (\mathcal{R} \vee \neg\mathcal{P}) \wedge (\mathcal{R} \vee \neg\mathcal{Q}) & \langle\ \mathfrak{A} \vee (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \wedge (\mathfrak{A} \vee \mathfrak{C})\ \rangle \\
=_T & (\neg\mathcal{P} \vee \mathcal{R}) \wedge (\neg\mathcal{Q} \vee \mathcal{R}) & \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle \\
=_T & (\mathcal{P} \Rightarrow \mathcal{R}) \wedge (\mathcal{Q} \Rightarrow \mathcal{R}) & \langle\ \mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{A} \vee \mathfrak{B}\ \rangle
\end{array}
$$

□

The law expressed in *Prop27* enables us to rewrite semantic entailments in terms of the conditional connective, and hence to prove such entailments by using equivalences. Thus in order to prove the entailment

$$\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n \models \mathfrak{B}$$

we can prove instead

$$\mathfrak{A}_1 \wedge \mathfrak{A}_2 \wedge \ldots \wedge \mathfrak{A}_n \Rightarrow \mathfrak{B} =_T T$$

Example 5.16
Show that $\neg\neg\mathcal{P} \vee \neg\neg\mathcal{Q}\ \therefore\ \mathcal{P} \vee \mathcal{Q}$ is a valid argument form.

Definition

Prop25 $\mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{A} \vee \mathfrak{B}$

Modus ponens

Prop26 $\mathfrak{A}, \mathfrak{A} \Rightarrow \mathfrak{B} \vDash \mathfrak{B}$

Valid argument forms

Prop27 $\mathfrak{A} \vDash \mathfrak{B}$ if and only if $\mathfrak{A} \Rightarrow \mathfrak{B} =_T T$

Tautology

Prop 28 $\mathfrak{A} \Rightarrow \mathfrak{A} =_T T$

......

Prop 29 $\mathfrak{A} \Rightarrow \neg\mathfrak{A} =_T \neg\mathfrak{A}$

Prop 30 $\mathfrak{A} \Rightarrow T =_T T$

Prop 31 $T \Rightarrow \mathfrak{A} =_T \mathfrak{A}$

Prop 32 $\mathfrak{A} \Rightarrow F =_T \neg\mathfrak{A}$

Prop 33 $F \Rightarrow \mathfrak{A} =_T T$

Negation

Prop 34 $\neg(\mathfrak{A} \Rightarrow \mathfrak{B}) =_T \mathfrak{A} \wedge \neg\mathfrak{B}$

Contrapositive

Prop 35 $\neg\mathfrak{B} \Rightarrow \neg\mathfrak{A} =_T \mathfrak{A} \Rightarrow \mathfrak{B}$

Exportation

Prop 36 $\mathfrak{A} \Rightarrow (\mathfrak{B} \Rightarrow \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \Rightarrow \mathfrak{C}$

Distribution to the right

Prop 37 $\mathfrak{A} \Rightarrow (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \Rightarrow \mathfrak{B}) \wedge (\mathfrak{A} \Rightarrow \mathfrak{C})$

Prop 38 $\mathfrak{A} \Rightarrow (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \Rightarrow \mathfrak{B}) \vee (\mathfrak{A} \Rightarrow \mathfrak{C})$

......

Prop 39 $(\mathfrak{A} \vee \mathfrak{B}) \Rightarrow \mathfrak{C} =_T (\mathfrak{A} \Rightarrow \mathfrak{C}) \wedge (\mathfrak{B} \Rightarrow \mathfrak{C})$

Hypothetical syllogism

Prop 40 $\mathfrak{A} \Rightarrow \mathfrak{B}, \mathfrak{B} \Rightarrow \mathfrak{C} \vDash \mathfrak{A} \Rightarrow \mathfrak{C}$

......

Prop 41 $(\mathfrak{A} \Rightarrow \mathfrak{B}) \Rightarrow \mathfrak{A} =_T \mathfrak{A}$

Prop 42 $(\mathfrak{A} \Rightarrow \mathfrak{B}) \Rightarrow \mathfrak{B} =_T T$

Table 5.1: Special properties of $\Rightarrow$

Solution

$$\begin{array}{lll} & (\neg\neg\mathcal{P} \vee \neg\neg\mathcal{Q}) \Rightarrow (\mathcal{P} \vee \mathcal{Q}) & \\ =_T & (\mathcal{P} \vee \mathcal{Q}) \Rightarrow (\mathcal{P} \vee \mathcal{Q}) & \langle\ \neg\neg\mathfrak{A} =_T \mathfrak{A}\ \rangle \\ =_T & T & \langle\ \mathfrak{A} \Rightarrow \mathfrak{A} =_T T\ \rangle \end{array}$$

Since $(\neg\neg\mathcal{P} \vee \neg\neg\mathcal{Q}) \Rightarrow (\mathcal{P} \vee \mathcal{Q}) =_T T$ then $\neg\neg\mathcal{P} \vee \neg\neg\mathcal{Q} \vDash \mathcal{P} \vee \mathcal{Q}$. Hence $\neg\neg\mathcal{P} \vee \neg\neg\mathcal{Q} \therefore \mathcal{P} \vee \mathcal{Q}$ is a valid argument form. □

Example 5.17

Prove the *modus ponens* law. That is, show that $\mathcal{P}, \mathcal{P} \Rightarrow \mathcal{Q} \therefore \mathcal{Q}$ is a valid argument form.

Solution

This problem is essentially to prove *Prop26*, and so we cannot use *Prop26* itself.

$$\begin{array}{lll} & (\mathcal{P} \wedge (\mathcal{P} \Rightarrow \mathcal{Q})) \Rightarrow \mathcal{Q} & \\ =_T & \neg(\mathcal{P} \wedge (\neg\mathcal{P} \vee \mathcal{Q})) \vee \mathcal{Q} & \langle\ \mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{A} \vee \mathfrak{B}\ \rangle \\ =_T & (\neg\mathcal{P} \vee \neg(\neg\mathcal{P} \vee \mathcal{Q})) \vee \mathcal{Q} & \langle\ \neg(\mathfrak{A} \wedge \mathfrak{B}) =_T \neg\mathfrak{A} \vee \neg\mathfrak{B}\ \rangle \\ =_T & (\neg(\neg\mathcal{P} \vee \mathcal{Q}) \vee \neg\mathcal{P}) \vee \mathcal{Q} & \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle \\ =_T & \neg(\neg\mathcal{P} \vee \mathcal{Q}) \vee (\neg\mathcal{P} \vee \mathcal{Q}) & \langle\ \mathfrak{A} \vee (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \vee \mathfrak{C}\ \rangle \\ =_T & (\neg\mathcal{P} \vee \mathcal{Q}) \vee \neg(\neg\mathcal{P} \vee \mathcal{Q}) & \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle \\ =_T & T & \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle \end{array}$$

Since $(\mathcal{P} \wedge (\mathcal{P} \Rightarrow \mathcal{Q})) \Rightarrow \mathcal{Q} =_T T$ then $\mathcal{P} \wedge \mathcal{P} \Rightarrow \mathcal{Q} \vDash \mathcal{Q}$, and so $\mathcal{P}, \mathcal{P} \Rightarrow \mathcal{Q} \vDash \mathcal{Q}$. Hence $\mathcal{P}, \mathcal{P} \Rightarrow \mathcal{Q} \therefore \mathcal{Q}$ is a valid argument form. □

Proving equivalences, perhaps as a step to proving valid argument forms, is known as **equational logic**. Although equational logic is a legitimate method for proving the validity of arguments, in this book we shall mainly use the method of natural deduction. The next section considers how natural deduction can be extended to include the conditional connective.

Exercise 35: Equational logic

In this exercise, use the laws of propositional logic as given in Tables 3.1 and 5.1; do not construct truth tables or use natural deduction.

1. Derive each of the following equivalences.

   (a) $\mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \neg\mathcal{P}) =_T \neg\mathcal{P} \vee \neg\mathcal{Q}$
   (b) $(\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{P} \Rightarrow \neg\mathcal{Q}) =_T \neg\mathcal{P}$
   (c) $\mathcal{P} \Rightarrow (\mathcal{P} \Rightarrow \mathcal{Q}) =_T \mathcal{P} \Rightarrow \mathcal{Q}$
   (d) $\mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{P}) =_T T$
   (e) $(\mathcal{P} \Rightarrow \mathcal{Q}) \vee (\neg\mathcal{P} \Rightarrow \mathcal{Q}) =_T T$
   (f) $\mathcal{P} \Rightarrow \mathcal{Q} \vee \neg\mathcal{P} \Rightarrow \mathcal{Q} =_T T$
   (g) $\mathcal{P} \Rightarrow \mathcal{Q} \wedge \mathcal{P} \Rightarrow \neg\mathcal{Q} =_T \neg\mathcal{Q}$

2. Prove each of the following semantic entailments.

   (a) $\mathcal{P} \models \mathcal{P} \vee \mathcal{Q}$
   (b) $\mathcal{P} \models (\mathcal{Q} \Rightarrow \mathcal{P})$
   (c) $\mathcal{P} \wedge \mathcal{Q} \models \mathcal{P} \Rightarrow \mathcal{Q}$
   (d) $\neg(\neg\mathcal{P} \vee \mathcal{Q}) \models \mathcal{P}$
   (e) $(\mathcal{P} \vee \mathcal{Q}) \Rightarrow \mathcal{Q} \models \mathcal{P} \Rightarrow \mathcal{Q}$
   (f) $\mathcal{P} \Rightarrow \mathcal{Q}, \neg\mathcal{Q} \models \neg\mathcal{P}$

5.7 Natural deduction with the conditional connective

In Chapter 4 we saw how we could derive inference forms $\Gamma \vdash \mathfrak{A}$ using rules natural deduction. Some basic inference forms are taken as rules of deduction, for example: the $\wedge$I rule, $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{A} \wedge \mathfrak{B}$. Other inference forms can be deduced from these by using methods of deduction; these methods of deduction are represented as deduction trees and rules of deduction, such as $\neg$I and $\vee$E. Natural deduction using the existing set of rules and methods can still be applied to propositional forms containing the conditional connective. However, there is also an introduction rule and an elimination rule for the conditional connective itself.

Existing rules of deduction

Example 5.18

Deduce $(\mathcal{P} \vee \mathcal{Q}) \wedge (\mathcal{R} \Rightarrow \neg\mathcal{S}) \vdash \neg\neg(\mathcal{P} \vee \mathcal{Q})$

Solution

$$\dfrac{\dfrac{(\mathcal{P} \vee \mathcal{Q}) \wedge (\mathcal{R} \Rightarrow \neg\mathcal{S})}{\mathcal{P} \vee \mathcal{Q}}\ \wedge\mathrm{E}_1}{\neg\neg(\mathcal{P} \vee \mathcal{Q})}\ \neg\neg\mathrm{I}$$

□

Exercise 36: Deductions using ¬, ∧ and ∨ rules

Deduce each of the following inference forms.

1. $\neg\neg(\mathcal{P} \Rightarrow (\mathcal{R} \vee \mathcal{Q})) \vdash \mathcal{P} \Rightarrow (\mathcal{R} \vee \mathcal{Q})$
2. $\neg\neg\neg\mathcal{Q} \wedge (\mathcal{P} \Rightarrow \neg\neg\mathcal{Q}) \vdash (\mathcal{P} \Rightarrow \neg\neg\mathcal{Q})$
3. $\mathcal{P} \vdash \mathcal{P} \vee (\mathcal{Q} \Rightarrow \mathcal{R})$
4. $(\neg\mathcal{P} \vee \mathcal{S}) \vdash (\mathcal{Q} \Rightarrow \mathcal{R}) \vee (\neg\mathcal{P} \vee \mathcal{S})$
5. $\neg\neg\neg\mathcal{Q} \wedge (\mathcal{P} \Rightarrow \neg\neg\mathcal{Q}) \vdash \neg\mathcal{Q}$
6. $\neg\neg(\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\neg\mathcal{Q} \Rightarrow \mathcal{R}) \vdash (\mathcal{P} \Rightarrow \mathcal{Q}) \vee (\mathcal{R} \wedge \mathcal{S})$

The ⇒ E rule

The ⇒ E rule (*ONLY IF elimination*) is an inference form.

Rule: ⇒ *E*

$\mathfrak{A}, \mathfrak{A} \Rightarrow \mathfrak{B} \vdash \mathfrak{C}$ □

Justification

In earlier sections we have proved the semantic entailment $\mathcal{P}, \mathcal{P} \Rightarrow \mathcal{Q} \vDash \mathcal{Q}$, often known as *modus ponens*. This can be generalized to $\mathfrak{A}, \mathfrak{A} \Rightarrow \mathfrak{B} \vDash \mathfrak{B}$ for any propositional forms $\mathfrak{A}$ and $\mathfrak{B}$. □

Example 5.19

Show that $\mathcal{P} \wedge \mathcal{Q}$, $\mathcal{P} \Rightarrow \mathcal{R} \vdash \mathcal{R}$.

Solution

$$\dfrac{\dfrac{\mathcal{P} \wedge \mathcal{Q}}{\mathcal{P}}\wedge E_1 \qquad \mathcal{P} \Rightarrow \mathcal{R}}{\mathcal{R}} \Rightarrow E$$

□

Example 5.20

Show that $\mathcal{Q}$, $\mathcal{P} \wedge (\mathcal{Q} \Rightarrow \mathcal{R}) \vdash \mathcal{R}$.

Solution

$$\dfrac{\mathcal{Q} \qquad \dfrac{\mathcal{P} \wedge (\mathcal{Q} \Rightarrow \mathcal{R})}{\mathcal{Q} \Rightarrow \mathcal{R}}\wedge E_2}{\mathcal{R}} \Rightarrow E$$

□

Exercise 37: Deductions using the ⇒ E rule

Deduce each of the following inference forms.

1. $\neg\mathcal{P}, \neg\mathcal{P} \Rightarrow \mathcal{Q} \vdash \mathcal{Q}$
2. $\neg\neg\mathcal{P}, \mathcal{P} \Rightarrow \mathcal{Q} \vdash \mathcal{Q}$
3. $\mathcal{P}, \mathcal{P} \Rightarrow \neg\neg\mathcal{Q} \vdash \mathcal{Q}$
4. $\mathcal{P}, \mathcal{P} \Rightarrow (\mathcal{P} \wedge \mathcal{Q}) \vdash \mathcal{Q}$
5. $\mathcal{P}, \mathcal{P} \Rightarrow (\mathcal{P} \Rightarrow \mathcal{Q}) \vdash \mathcal{Q}$
6. $\mathcal{P} \wedge \mathcal{Q}, \mathcal{Q} \Rightarrow \mathcal{R} \vdash \mathcal{R}$
7. $\mathcal{P} \Rightarrow \mathcal{Q} \vdash \neg(\mathcal{P} \wedge \neg\mathcal{Q})$
8. $\mathcal{P} \wedge \neg\mathcal{Q} \vdash \neg(\mathcal{P} \Rightarrow \mathcal{Q})$
9. $\mathcal{P} \vee \mathcal{Q}, \mathcal{P} \Rightarrow \mathcal{R}, \mathcal{Q} \Rightarrow \mathcal{R} \vdash \mathcal{R}$
10. $(\mathcal{P} \Rightarrow \mathcal{Q}) \vee (\neg\mathcal{Q} \Rightarrow \mathcal{R}), \mathcal{P} \wedge \neg\mathcal{R} \vdash \mathcal{Q}$

Conditional proof – the $\Rightarrow$ I rule

The $\Rightarrow$ I rule (*ONLY IF introduction*) is not an inference form but a method of deduction; that is we can use it to deduce one inference form from another. It is also known as the method of **conditional proof.**

Rule: $\Rightarrow$ *I*

If $\Gamma, \mathfrak{A} \vdash \mathfrak{B}$ then $\Gamma \vdash \mathfrak{A} \Rightarrow \mathfrak{B}$. □

Justification

From *Prop*27 we know that $\mathfrak{A} \vDash \mathfrak{B}$ if and only if $\mathfrak{A} \Rightarrow \mathfrak{B} =_T T$. We can generalize this to the case when

$$\Gamma, \mathfrak{A} \vDash \mathfrak{B} \tag{5.2}$$

Now consider an instance of Γ in which all the propositions are true. If the corresponding instance of $\mathfrak{A}$ is also true, then from the entailment 5.2 it follows that $\mathfrak{B}$ is also true; thus from the truth table for the conditional connective, $\mathfrak{A} \Rightarrow \mathfrak{B}$ is also true. If, however, the corresponding instance of $\mathfrak{A}$ is false, then from the truth table for the conditional connective, $\mathfrak{A} \Rightarrow \mathfrak{B}$ is again true. Thus we have shown that if $\Gamma, \mathfrak{A} \vDash \mathfrak{B}$ then $\Gamma \vDash \mathfrak{A} \Rightarrow \mathfrak{B}$. □

Example 5.21

Show that $\mathcal{P} \Rightarrow \mathcal{Q} \vdash (\mathcal{P} \wedge \mathcal{R}) \Rightarrow \mathcal{Q}$.

Solution

In order to show that $\mathcal{P} \Rightarrow \mathcal{Q} \vdash (\mathcal{P} \wedge \mathcal{R}) \Rightarrow \mathcal{Q}$, we need to deduce the subgoal of $\mathcal{P} \Rightarrow \mathcal{Q}, \mathcal{P} \wedge \mathcal{R} \vdash \mathcal{Q}$. This we can do with a deduction tree as follows.

$$\dfrac{\dfrac{\mathcal{P} \wedge \mathcal{R}}{\mathcal{P}}\wedge\mathrm{E}_1 \qquad \mathcal{P} \Rightarrow \mathcal{Q}}{\mathcal{Q}} \Rightarrow \mathrm{E}$$

Applying the method conditional proof to this inference yields the required result. □

The $\Rightarrow$ I rule (that is, conditional proof) can be incorporated into a deduction tree in a manner similar to the $\neg$I rule.

$$\dfrac{\dfrac{\dfrac{\Gamma \quad \overline{\mathfrak{A}}^{\,*}}{\vdots}}{\mathfrak{B}}}{\mathfrak{A} \Rightarrow \mathfrak{B}} * \Rightarrow \mathrm{I}$$

Example 5.22

Show the derivation of $\mathcal{P} \Rightarrow \mathcal{Q} \vdash (\mathcal{P} \wedge \mathcal{R}) \Rightarrow \mathcal{Q}$ as a deduction tree.

Solution

We have already obtained the deduction tree for $\mathcal{P} \Rightarrow \mathcal{Q}, \mathcal{P} \wedge \mathcal{R} \vdash \mathcal{Q}$ in the preceding example. This subtree can now be incorporated into the complete tree as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{P} \wedge \mathcal{R}}\ *}{\mathcal{P}}\ \wedge\mathrm{E}_1 \qquad \mathcal{P} \Rightarrow \mathcal{Q}}{\mathcal{Q}}\ \Rightarrow \mathrm{E}}{(\mathcal{P} \wedge \mathcal{R}) \Rightarrow \mathcal{Q}}\ * \Rightarrow \mathrm{I}$$

□

Example 5.23

Show that $\vdash \mathcal{P} \wedge \mathcal{Q} \Rightarrow \mathcal{P} \vee \mathcal{Q}$.

Solution

If we can show that $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$, then we can deduce $\vdash \mathcal{P} \wedge \mathcal{Q} \Rightarrow \mathcal{P} \vee \mathcal{Q}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{P} \wedge \mathcal{Q}}\ *}{\mathcal{P}}\ \wedge\mathrm{E}_1}{\mathcal{P} \vee \mathcal{Q}}\ \vee\mathrm{I}_1}{\mathcal{P} \wedge \mathcal{Q} \Rightarrow \mathcal{P} \vee \mathcal{Q}}\ * \Rightarrow \mathrm{I}$$

□

In this last example, the set of premisses in the final inference form is the empty set. The conclusion of such an inference form is known as a **formal theorem**, or more simply as a **theorem**.

Example 5.24

Prove that $\mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{P})$ is a theorem .

Solution

We need to show that $\vdash \mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{P})$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{P}}\ * \qquad \dfrac{}{\mathcal{Q}}\ \dagger}{\mathcal{P} \wedge \mathcal{Q}}\ \wedge\mathrm{I}}{\mathcal{P}}\ \wedge\mathrm{E}_1}{\mathcal{Q} \Rightarrow \mathcal{P}}\ \dagger \Rightarrow \mathrm{I}}{\mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{P})}\ * \Rightarrow \mathrm{I}$$

□

Exercise 38: Using $\Rightarrow$ I

Prove each of the following by constructing deduction trees.

1. $\vdash \neg\neg\mathcal{P} \Rightarrow \mathcal{P}$
2. $\vdash (\mathcal{P} \wedge \mathcal{R}) \Rightarrow (\mathcal{R} \vee \mathcal{Q})$
3. $\mathcal{P} \vdash (P \Rightarrow \mathcal{Q}) \Rightarrow \mathcal{Q}$
4. $\mathcal{P} \Rightarrow \mathcal{Q}, \mathcal{Q} \Rightarrow \mathcal{R} \vdash \mathcal{P} \Rightarrow \mathcal{R}$

5.8 Derived rules

Our system for propositional logic is based upon introduction and elimination rules for each of the four basic connectives. Although it is not necessary to use any other rule, it is often convenient to make use of additional inference forms derived from the basic rules. Some pre-existing inference forms are so useful that they are given special names. Two important ones are known as ***modus tollens*** and **hypothetical syllogism.**

Modus tollens

Rule: MT

$\mathfrak{A} \Rightarrow \mathfrak{B},\ \neg\mathfrak{B} \vdash \neg\mathfrak{A}$ □

Justification

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathfrak{A}}\,* \qquad \mathfrak{A} \Rightarrow \mathfrak{B}}{\mathfrak{B}}\Rightarrow \mathrm{E} \qquad \neg\mathfrak{B}}{\mathfrak{B} \wedge \neg\mathfrak{B}}\wedge\mathrm{I}}{\neg\mathfrak{A}}\,*\neg\mathrm{I}$$

Note also that from truth tables we can show that $\mathfrak{A} \Rightarrow \mathfrak{B},\ \neg\mathfrak{B} \models \neg\mathfrak{A}$:

$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{A} \Rightarrow \mathfrak{B}$	$\neg\mathfrak{B}$	$\neg\mathfrak{A}$
T	*T*	*T*	*F*	*F*
T	*F*	*F*	*T*	*F*
F	*T*	*T*	*F*	*T*
F	*F*	***T***	***T***	***T***

□

Example 5.25

Prove that $\mathcal{P} \Rightarrow \neg(\mathcal{Q} \vee \mathcal{R}),\ \mathcal{Q} \vdash \neg\mathcal{P}$.

Solution

$$\dfrac{\dfrac{\dfrac{\mathcal{Q}}{\mathcal{Q}\vee\mathcal{R}}\,\vee\mathrm{I}_1}{\neg\neg(\mathcal{Q}\vee\mathcal{R})}\,\neg\neg\mathrm{I} \qquad \mathcal{P}\Rightarrow\neg(\mathcal{Q}\vee\mathcal{R})}{\neg\mathcal{P}}\,\mathrm{MT}$$

□

Hypothetical syllogism

Rule: HS

$\mathfrak{A}\Rightarrow\mathfrak{B}, \mathfrak{B}\Rightarrow\mathfrak{C} \vdash \mathfrak{A}\Rightarrow\mathfrak{C}$. □

Justification

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathfrak{A}}\,* \qquad \mathfrak{A}\Rightarrow\mathfrak{B}}{\mathfrak{B}}\,\Rightarrow\mathrm{E} \qquad \mathfrak{B}\Rightarrow\mathfrak{C}}{\mathfrak{C}}\,\Rightarrow\mathrm{E}}{\mathfrak{A}\Rightarrow\mathfrak{C}}\,\Rightarrow\mathrm{I}$$

Note also that from truth tables we can show that $\mathfrak{A}\Rightarrow\mathfrak{B}, \mathfrak{B}\Rightarrow\mathfrak{C} \vDash \mathfrak{A}\Rightarrow\mathfrak{C}$.

$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{A}\Rightarrow\mathfrak{B}$	$\mathfrak{B}\Rightarrow\mathfrak{C}$	$\mathfrak{A}\Rightarrow\mathfrak{C}$
T	*T*	*T*	*T*	*T*	*T*
T	*T*	*F*	*T*	*F*	*F*
T	*F*	*T*	*F*	*T*	*T*
T	*F*	*F*	*F*	*T*	*F*
F	*T*	*T*	*T*	*T*	*T*
F	*T*	*F*	*T*	*F*	*T*
F	*F*	*T*	***T***	***T***	***T***
F	*F*	*F*	***T***	***T***	***T***

□

Example 5.26

Prove that $\mathcal{P}\wedge\mathcal{Q}, \mathcal{P}\Rightarrow\mathcal{R}, \mathcal{R}\Rightarrow\mathcal{S} \vdash \mathcal{S}\wedge\mathcal{Q}$.

Solution

$$\dfrac{\dfrac{\dfrac{\mathcal{P}\wedge\mathcal{Q}}{\mathcal{P}}\,\wedge\mathrm{E}_1 \qquad \dfrac{\mathcal{P}\Rightarrow\mathcal{R} \qquad \mathcal{R}\Rightarrow\mathcal{S}}{\mathcal{P}\Rightarrow\mathcal{S}}\,\mathrm{HS}}{\mathcal{S}}\,\Rightarrow\mathrm{E} \qquad \dfrac{\mathcal{P}\wedge\mathcal{Q}}{\mathcal{Q}}\,\wedge\mathrm{E}_2}{\mathcal{S}\wedge\mathcal{Q}}\,\wedge\mathrm{I}$$

□

Exercise 39: Derived rules of deduction – MT and HS

1. Prove each of the following by constructing a deduction tree in which the MT rule is used.
   (a) $\mathcal{P} \Rightarrow \neg\mathcal{Q}, \neg\neg\mathcal{Q} \vdash \neg\mathcal{P}$
   (b) $\mathcal{P} \Rightarrow \neg\mathcal{Q}, \mathcal{Q} \vdash \neg\mathcal{P}$
   (c) $\neg\mathcal{P} \Rightarrow \mathcal{Q}, \neg\mathcal{Q} \vdash \mathcal{P}$
   (d) $\mathcal{P} \Rightarrow \mathcal{Q}, \neg\mathcal{Q} \vdash \neg\mathcal{P} \vee \mathcal{Q}$
   (e) $\mathcal{P} \Rightarrow (\mathcal{Q} \wedge \mathcal{R}), \neg(\mathcal{Q} \wedge \mathcal{R}) \vdash \neg\mathcal{P} \vee \mathcal{S}$
2. Prove each of the following by constructing a deduction tree in which the HS rule is used.
   (a) $\mathcal{P}, \mathcal{P} \Rightarrow \mathcal{Q}, \mathcal{Q} \Rightarrow \mathcal{R} \vDash \mathcal{R}$
   (b) $\neg\mathcal{P} \Rightarrow \neg\mathcal{Q}, \neg\mathcal{Q} \Rightarrow \mathcal{R}, \neg\mathcal{R} \vdash \mathcal{P}$

5.9 The biconditional connective

Consider the statement

'*Mohammed will go to the meeting if and only if Jonathan goes*'.

The phrase '*if and only if*' is used to assert that either both men go, or neither goes to the meeting. It is important to understand that this is very different to the statement

'*Mohammed will go to the meeting if Jonathan goes*',

in which the possibility exists that Jonathan will still go to the meeting even if Mohammed does not go. Thus if we let

- p represent '*Mohammed will go the meeting*'
- q represent '*Jonathan will go the meeting*'

then either p and q are both true or both are false. Thus we can represent the original statement symbolically as $(p \wedge q) \vee (\neg p \wedge \neg q)$. This is an instance of the propositional form $(\mathcal{P} \wedge \mathcal{Q}) \vee (\neg\mathcal{P} \wedge \neg\mathcal{Q})$ whose truth table is given by

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{P} \wedge \neg\mathcal{Q}$	$(\mathcal{P} \wedge \mathcal{Q}) \vee (\neg\mathcal{P} \wedge \neg\mathcal{Q})$
T	T	T	F	F	F	T
T	F	F	F	T	F	F
F	T	F	T	F	F	F
F	F	F	T	T	T	T

It is convenient to introduce a new connective to represent '*if and only if*'; this connective is called the **biconditional connective** and is represented by the symbol $\Leftrightarrow$.

Definition 5.1

The connective schema $\mathcal{P} \Leftrightarrow \mathcal{Q}$ for the biconditional is defined as being equivalent to the propositional form $(\mathcal{P} \wedge \mathcal{Q}) \vee (\neg\mathcal{P} \wedge \neg\mathcal{Q})$. □

The notation and terminology for the biconditional $\Leftrightarrow$ is clearly related to that of the conditional $\Rightarrow$. This is because of the following property, which indeed is often used to define the biconditional.

Prop43

$\mathcal{P} \Leftrightarrow \mathcal{Q} =_T (\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \Rightarrow \mathcal{P})$ □

Justification

This equivalence follows immediately from the truth table for $\mathcal{P} \Leftrightarrow \mathcal{Q} =_T (\mathcal{P} \Rightarrow \mathcal{Q})$ and $(\mathcal{Q} \Rightarrow \mathcal{P})$.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{Q}$	$\mathcal{Q} \Rightarrow \mathcal{P}$	$(\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \Rightarrow \mathcal{P})$	$\mathcal{P} \Leftrightarrow \mathcal{Q}$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

□

Parsing

When parsing compound propositions, the biconditional $\Leftrightarrow$ has a lower priority than $\neg$, $\wedge$, $\vee$ and $\Rightarrow$.

Example 5.27

Parse $\mathcal{P} \wedge \mathcal{Q} \Leftrightarrow \mathcal{P} \Rightarrow \mathcal{Q}$, and represent the parsing by introducing appropriate parentheses.

Solution

$\mathcal{P} \wedge \mathcal{Q} \Leftrightarrow \mathcal{P} \Rightarrow \mathcal{Q}$ is parsed as $((\mathcal{P} \wedge \mathcal{Q}) \Leftrightarrow (\mathcal{P} \Rightarrow \mathcal{Q}))$. □

Deductions with the biconditional

Note that the biconditional is regarded as being no more than a notational convenience; $\mathcal{P} \Leftrightarrow \mathcal{Q}$ is merely a shorthand for $(\mathcal{P} \wedge \mathcal{Q}) \vee (\neg\mathcal{P} \wedge \neg\mathcal{Q})$ or, more usually, $(\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \Rightarrow \mathcal{P})$. It is not regarded as an additional basic connective, and we do not specify any deduction rules for introducing or eliminating $\Leftrightarrow$ in propositional logic.

Example 5.28

Show that $\neg\neg(\mathcal{P} \Leftrightarrow \mathcal{Q}) \wedge \mathcal{R} \vdash (\mathcal{P} \Leftrightarrow \mathcal{Q}) \vee \mathcal{S}$.

Solution

$$\dfrac{\dfrac{\dfrac{\neg\neg(\mathcal{P} \Leftrightarrow \mathcal{Q}) \wedge \mathcal{R}}{\neg\neg(\mathcal{P} \Leftrightarrow \mathcal{Q})} \wedge E_1}{\mathcal{P} \Leftrightarrow \mathcal{Q}} \neg\neg E}{(\mathcal{P} \Leftrightarrow \mathcal{Q}) \vee \mathcal{S}} \vee I_1$$

□

Example 5.29

Show that $\neg\mathcal{P}, \mathcal{P} \Leftrightarrow \mathcal{Q} \vdash \neg\mathcal{Q}$.

Solution

Writing $\mathcal{P} \Leftrightarrow \mathcal{Q}$ as $(\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \Rightarrow \mathcal{P})$ we can construct the deduction tree as follows.

$$\dfrac{\dfrac{(\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \Rightarrow \mathcal{P})}{\mathcal{Q} \Rightarrow \mathcal{P}} \wedge E_2 \qquad \neg\mathcal{P}}{\neg\mathcal{Q}} \text{MT}$$

Note that it would also have been possible to have construct a deduction tree by writing $\mathcal{P} \Leftrightarrow \mathcal{Q}$ as $(\mathcal{P} \wedge \mathcal{Q}) \vee (\neg\mathcal{P} \wedge \neg\mathcal{Q})$, but this would have necessitated the $\vee$E rule (and also, in this case, the $\neg$I rule); thus the deduction tree would have been less straightforward. For this reason, it is usually more convenient to write $\mathcal{P} \Leftrightarrow \mathcal{Q}$ as $(\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \Rightarrow \mathcal{P})$. □

Negation of the biconditional

Prop44

$\neg(\mathcal{P} \Leftrightarrow \mathcal{Q}) =_T \neg\mathcal{P} \Leftrightarrow \mathcal{Q} =_T \mathcal{P} \Leftrightarrow \neg\mathcal{Q}$ □

Justification

$\mathcal{P}$	$\mathcal{Q}$	$\neg(\mathcal{P} \Leftrightarrow \mathcal{Q})$	$\neg\mathcal{P} \Leftrightarrow \mathcal{Q}$	$\mathcal{P} \Leftrightarrow \neg\mathcal{Q}$
T	T	F	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	F	F

□

In a biconditional statement of the form $\mathcal{P} \Leftrightarrow \mathcal{Q}$, either both $\mathcal{P}$ and $\mathcal{Q}$ are T or both are F; it is not possible to have one true without the other. By contrast, in the negation, $\neg(\mathcal{P} \Leftrightarrow \mathcal{Q})$, we assert that precisely one of $\mathcal{P}$ and $\mathcal{Q}$ is true. This relationship is sometimes referred to as the **exclusive 'or'**, and may be represented symbolically as $\not\Leftrightarrow$.

Example 5.30

The compound proposition '*Either John or Mary will be present*' can be written as

'*John will be present*' $\not\Leftrightarrow$ '*Mary will be present*'

if we wish to convey the sense that '*Both will not be present*', that is, if the exclusive 'or' is intended. However, it may be argued that it is better to make the intended meaning more explicit by writing

'*John will be present*' $\wedge$ $\neg$'*Mary will be present*'
$\vee$
'*Mary will be present*' $\wedge$ $\neg$'*John will be present*'

We shall not make use of this notation for the exclusive 'or'. □

Predicate Logic 6

6.1 Propositions and predicates

So far we have been looking at how we can reason with propositions. For example, suppose we have the following two propositions.

- '*If Rex is a dog then Rex has four legs.*'
- '*Rex is a dog.*'

From these we can conclude:

- '*Rex has four legs.*'

We know this argument is valid since it is an instance of the inference form: $\mathcal{P}, \mathcal{P} \Rightarrow \mathcal{Q} \vdash \mathcal{Q}$. Now in using our current set of deduction rules, we do not create any new atomic propositions; for example, in the above example, the conclusion is one of the atomic propositions in the premisses. The conclusion is always a combination of one or more existing atomic propositions.

Now consider the following premisses.

- '*Every dog has four legs.*'
- '*Rex is a dog.*'

From these we *should* be able to conclude:

- '*Rex has four legs.*'

However, the conclusion now is *not* based upon existing atomic propositions, but is itself a new atomic proposition. We cannot arrive at the conclusion using our current deduction rules, or even indeed with our current notation. Worse still, since both premisses and the conclusion are all atomic propositions it is not possible to overcome the problem by introducing a new connective.

To overcome these difficulties we need to introduce a new concept, namely that of a **predicate.** The clue to solving the problem is given by looking at the first premiss '*Every dog has four legs*' and the conclusion '*Rex has four legs*'. The common factor to these is the property,

> '... *has four legs*' :

this is an example of what is called a **predicate.** The three dots constitute a symbol called **ellipsis.** An ellipsis is frequently used in print to indicate that some text is missing. In this case the ellipsis ... indicates a gap which we can fill with the name of a particular object, such as '*Rex*', '*Rover*' or '*Buttercup*'. In general we shall use some kind of **label** that refers to an object or a value. A label can be a name, or **proper noun**, such as '*Rex*'; or it can be some other form of expression such as '*Mrs. Joel's dog*'. When we fill the gap in this way we obtain a proposition, for example:

- '*Rex has four legs.*'
- '*Rover has four legs.*'
- '*Buttercup has four legs.*'
- '*Mrs. Joel's dog has four legs.*'

Note that we normally need to restrict ourselves to what objects we can refer to. For example, it would be pointless to use the name of a city as statements such as '*Tokyo has four legs*' are meaningless. To this end, we need to specify a **universe of discourse**; this is a non-empty set of objects about which we are reasoning. In this case our universe of discourse might be ANIMALS.

Example 6.1

What is the predicate in '*Rex is a dog*'? What property does it denote? What might be a suitable universe of discourse?

Solution

The predicate '... *is a dog*' denotes the property of being a dog. The universe of discourse might be ANIMALS, or it might be LIVING THINGS. □

Exercise 40: Predicates from propositions

Identify the predicate in each of the following propositions and suggest one possibility for a suitable universe of discourse. What property does the predicate represent?

1. '*This flower is red*'
2. '*4 is a perfect square*'
3. '*Sadie has brown hair*'
4. '*Daffodils are yellow*'

6.2 Predicates with more than one gap

The predicates we have so far considered have just one gap in each case. It is also possible, however, to have more than one gap in a predicate.

Example 6.2

The predicate '… *has* … *legs*' has two gaps. The first gap could be filled, for example, by the name of an animal. It would be associated with a universe of discourse of ANIMALS. The second gap could be filled by a whole number greater than or equal to zero. It would be associated with the universe of discourse of NONNEGATIVE INTEGERS. Thus, if we fill the first gap with the label '*My goldfish*' and the second gap with '*zero*', we would obtain the proposition

'*My goldfish has zero legs*'.

A more natural way of writing this would be

'*My goldfish has no legs*'.

□

Similarly we can have predicates with three or more gaps.

6.3 Free variables

We have seen how we can fill the gap in a unary predicate with a label for a specific object or value. A simple ellipsis … is used to indicate the gap. We have also seen that we may have more than one gap in a predicate. In the examples we have looked at the gaps in a predicate may be filled with different labels; indeed in some cases (where there are different universes of discourse) they *must* be filled with different labels.

Sometimes, however, we may have a predicate in which two or more gaps must be filled with the *same* label. For example, we may want a predicate to state that a number is equal to itself. On way to do this would be to write

$$\ldots = \ldots$$

with the intention that both gaps are filled with the same labels to generate propositions such as $2 = 2$ or $-3.1 = -3.1$. Unfortunately, as it stands, the gaps can be filled with different labels to yield propositions such as $2 = -3.1$, which we do not want.

We need some way to indicate when gaps must be filled with the same label. There are several ways of doing this. One approach would be to use a letter as **tag** to each ellipsis; ellipses with the same tag must be filled with the same label. Thus if we use x as a tag in the example above, we could write

$$\overset{x}{\ldots} = \overset{x}{\ldots}$$

Thus now we can only fill both gaps with the same label.

Now since a tag is only ever written over an ellipsis, there is no longer any real need to write the ellipsis explicitly. Usually we shall leave out the ellipses, and simply write the tags.

Example 6.3
The following are predicates in which the ellipses have been omitted.

- '$x = x$'
- '*x has four legs*'
- '*x and y are the parents of z*'

□

Example 6.4
The predicate '... *is a dog*' can be written using the letter x instead of ... as '*x is a dog*'. The letter x can be replaced by a label which refers to an object from the universe of discourse. For example x could be replaced by '*Rex*', '*Rover*' or '*Buttercup*'. □

In accordance with convention, we shall refer to tags as **free variables**. We shall also use lower case letters near the end of the alphabet to represent free variables.

Example 6.5
List the free variables in each of the following predicates.

- '*x is a dog*'
- '*y has four legs*'
- '$x = x$'
- '*x and y are the parents of z*'

Solution

- In '*x is a dog*', there is one free variable, x.
- In '*y has four legs*', there is one free variable, y.
- In '$x = x$', there is one free variable, x.
- In '*x and y are the parents of z*', there are three free variables, x, y and z.

□

Definition 6.1
A predicate with one free variable is known as a **unary** predicate. A unary predicate represents a property. □

Definition 6.2
A predicate with two free variables is known as a **binary** predicate. A binary predicate represents a relation between two objects. □

Example 6.6

Which of the following predicates are unary and which are binary?

- '*x is a cow*'
- '*y has a wet nose*'
- '$x = x$'
- '$x = y$'
- '*x and y are the parents of z*'

Solution

- '*x is a cow*' is a unary predicate.
- '*y has a wet nose*' is a unary predicate.
- '$x = x$' is a unary predicate.
- '$x = y$' is a binary predicate.
- '*x and y are the parents of z*' is neither unary nor binary. (It is in fact a **tertiary** predicate.)

□

Exercise 41: Unary predicates

1. Replace the free variable in each of the following predicates by the given value to obtain a proposition.
   (a) Replace x by 3 in $x > 2$.
   (b) Replace y by 1 in $y > 2$.
   (c) Replace z by 6 in $z = z$.
   (d) Replace y by '*Sydney*' in '*y is the capital of Australia*'.
   (e) Replace y by '*Canberra*' in '*y is the capital of Australia*'.
   (f) Replace z by '*Queen Elizabeth II*' in '*z is the Duke of Normandy*'.
2. Identify an appropriate unary predicate for each of the following propositions, and express this predicate using a free variable. In each case suggest an appropriate universe of discourse for the free variable.
   (a) '3 *is a prime number*'.
   (b) '*Antarctica is very cold*'.
   (c) '*The door to my office is blue*'.
   (d) '*The sky is blue*'.
   (e) $7 \geq 7$

Exercise 42: Predicates with more than one free variable

1. Replace the free variable indicated in each of the following binary predicates by the given value to obtain a predicate in the remaining free variable.
   (a) Replace x by 3 in $x > y$.
   (b) Replace y by 7 in $x > y$.
   (c) Replace x by '*Canberra*' in '*x is the capital of y*'.
   (d) Replace y by '*the United Kingdom*' in '*x is the capital of y*'.

2. Replace the remaining free variable in each predicate in question 1 with the given value to obtain a proposition.

   (a) Replace y by 2.
   (b) Replace x by 5.
   (c) Replace y by '*Australia*'.
   (d) Replace x by '*Westminster*'.

3. Identify an appropriate 'two place' predicate in each of the following statements, and express this using free variables. Suggest an appropriate universe of discourse for *each* free variable.

   (a) '*Mount Everest is higher than Mount Snowdon*'.
   (b) '*The Danube flows through Austria*'.
   (c) $2 + 3 = 6$

6.4 Compound predicates

So far we have seen how an atomic proposition can be obtained from an **atomic predicate.** In a similar manner we can obtain a compound proposition from a **compound predicate.**

Example 6.7

From what compound predicate might the compound proposition

'*Rex is a dog and has four legs*'

be obtained?

Solution

'*Rex is a dog and has four legs*' can be represented as a compound proposition as

'*Rex is a dog*' $\wedge$ '*Rex has four legs*'.

The atomic proposition '*Rex is a dog*' is obtained from the predicate '*x is a dog*' and the atomic proposition '*Rex has four legs*' is obtained from the predicate '*x has four legs*'. Thus the compound proposition '*Rex is a dog and has four legs*' is obtained from the **compound predicate**

'*x is a dog*' $\wedge$ '*x has four legs*'.

□

We often refer to a proposition by a single letter such as p, q and r. Likewise we can refer to a predicate by a single letter followed by the list of its free variables in parentheses, for example: $p(x)$ or $q(x, y, z)$.

Example 6.8

Represent symbolically the compound predicate from which '*Rex is a dog and Rex has four legs*' might be obtained.

Solution

The predicate '*x is a dog*' can be represented by $p(x)$ and the predicate '*x has four legs*' by $q(x)$. Hence the proposition '*Rex is a dog*' is $p($'*Rex*'$)$ and the proposition '*Rex has four legs*' is $q($'*Rex*'$)$. The compound proposition $p($'*Rex*'$) \wedge q($'*Rex*'$)$ is obtained by replacing the free variable x in $p(x) \wedge q(x)$ by the constant '*Rex*'. □

In formulating compound predicates, care needs to be taken with choosing the free variables. The predicates $p(x) \wedge q(x)$ and $p(x) \wedge q(y)$ are different: $p($'*Rex*'$) \wedge q($'*Rex*'$)$ can be obtained from both predicates, but $p($'*Rex*'$) \wedge q($'*Fido*'$)$ can be obtained from $p(x) \wedge q(y)$ only.

Exercise 43: Compound predicates

For each of the following propositions, identify an appropriate compound predicate and free variables.

1. '*Either you are late or my watch is fast*'
2. '*Canberra and Rabat are capital cities*'
3. $3>2 \Rightarrow 3^2>2^2$
4. $7>3 \wedge 1+1=2$

6.5 Constants and functions

We have met two ways of labelling an object or value. Sometimes we use a proper name such as '*Rex*'. Such a label we shall refer to as a **proper constant** or simply as a **constant.** In talking about constants in general terms we shall find it helpful to use letters such as k, l, m and n to stand for proper constants; in any particular case, each such letter will stand for a specific name.

Another way we can refer to an object or a value is to use an expression such as '*Mrs. Joel's dog*'. In this kind of label, the object referred to is not named, but instead is associated with another object. In this case a dog is associated with the person whose name is '*Mrs. Joel*'. Note that in an expression such as '*Mrs. Joel's dog*', there is the implication that Mrs. Joel has just one dog. If Mrs. Joel had no dog, or had more than one dog, then the label '*Mrs. Joel's dog*' would not make sense.

Example 6.9

Suppose we have the predicate '*x was born in 1930*' where the universe of discourse for the free variable x is PEOPLE. Then we can replace x by a label such as '*John's father*'. □

In this second approach to identifying objects (or values), the object being identified is associated to another, named object by means of a **function.**

Definition 6.3

A function associates a single object or value called the **result** with another object or value called the **argument.** We say that the function is **applied to** the argument to give the result. □

Unfortunately that the word '*argument*' is used in two different senses: we can talk of a '*logical argument*'; and also of the '*argument of a function*'. In practice, the context will determine which of these two meanings is intended.

Example 6.10

Suppose we have a universe of discourse PEOPLE, then we can define a function '*father_ of*'. The result of applying this function to a person will be another person who is the father of the first person. Thus '*John's father*' will be the result of applying the function '*father_ of*' to '*John*'. □

We need a notation to indicate the application of a function to its argument. The notation we shall use is to enclose the argument in parentheses after the function name. Now in describing functions in general we shall use letters such as f, g and h. Thus the result of applying a function f to a constant k will be written as $f(k)$.

Example 6.11

How might '*John's father*' be written using the notation of functional application?

Solution The label '*John's father*' results from the application of the function '*father_ of*' to '*John*' and can be written as '*father_of*'('*John*'). □

Care must be taken to ensure that any function we define is indeed a function, as the following example shows.

Example 6.12

Suppose we have a universe of discourse PEOPLE and another universe of discourse DOGS. Then we might consider setting up a function '*owned_ by*' which has an argument taken from PEOPLE and result in the set DOGS. Thus the label '*Mrs. Joel's dog*' would be the result of applying the function '*owned_ by*' to '*Mrs. Joel*' to give as result the dog which is owned by Mrs. Joel. The problem with this is that although Mrs. Joel may indeed only own one dog, other people may own more than one dog. Suppose that Ali owns two dogs, Rex and Fido say. Then what would be the result of applying '*owned_ by*' to '*Ali*'? If there were an answer it could be either '*Rex*' or '*Fido*'. The expression '*Ali's dog*' would be meaningless; it would not refer to a single dog, and is not a label. □

Thus we have two ways of labelling an object:

- by a constant, such as k;
- by a function applied to a constant, such as $f(k)$.

Now a function can be applied to an object labelled in any way. Thus instead of applying a function f to a constant k, we could instead apply f to $g(l)$, which itself is a function g applied to a constant l. Thus we would get $f(g(l))$. This nesting process can be repeated; for example we could have $f(g(h(m)))$. We talk of such identifiers as **closed terms.**

Definition 6.4

A closed term is either

- a constant, or
- a function applied to a closed term.

□

Predicates with functions

We can regard a proposition as $p(f(k))$ as being formed from a predicate $p(\ldots)$ by filling the gap $\ldots$ with $f(k)$. There is however an alternative way of analysing $p(f(k))$: we can regard $p(f(k))$ as being formed from the expression $p(f(\ldots))$ by filling the gap with k. Now we have already seen how we can use a free variable to tag $\ldots$. Thus, using the free variable notation, then we can regard $p(f(k))$ as being obtained in one of two ways:

- the free variable x in the predicate $p(x)$ is replaced by $f(k)$;
- the free variable x in the expression $p(f(x))$ is replaced by k.

The subexpression $f(x)$ is often said to be a function of the variable x. This terminology, however, may be a little misleading. Perhaps a better terminology is to describe $f(x)$ as a **function expression**, and to describe f as a **function letter**.

Just as predicates can have more than one free variable we can also have function expressions which have more than one free variable; for example, $f((x, y, z))$.

Exercise 44: Functions

1. Express each of the following in English, or as a mathematical statement

   (a) $p(f(m))$ where $p(\ldots)$ is the predicate '$\ldots$ *was a grocer*', $f(\ldots)$ is the function '$\ldots$*'s father*' and m is the constant '*Margaret Thatcher*'.

   (b) $q(g(k), g(l))$ where $q(x, y)$ is the predicate 'x *is further east than* y', $g(\ldots)$ is the function '*the capital of* $\ldots$', k is the constant '*Scotland*' and l is the constant '*England*'.

   (c) $p_1(f_1((k_1, k_2)), f_2((k_1, k_2)))$ where

   - $p_1(x, y)$ is the predicate $x \geq y$;
   - $f_1((x, y))$ is the function $x + y$;
   - $f_2((x, y))$ is the function $\sqrt{x \times y}$;
   - k_1 is the constant 2;
   - k_2 is the constant 3.

2. Express each of the following propositions symbolically by using appropriate letters to stand for predicates, functions and constants.
   (a) '*Fido's mother has four legs*'
   (b) '*Fido's mother has four legs but Fido only has three legs*'
   (c) '*Fido's mother is older than Fido*'
   (d) '*Everyone's mother was born before them*'

Arbitrary constants

Suppose we have the following premisses:

- '*at least one capital city has rivers*'
- '*capital cities which have rivers are beautiful*'

We can argue as follows.

> From the first premiss we know that it is possible to select a capital city which has a river. Suppose we make such a selection. From the second premiss we know that the capital city which we have just selected must be beautiful. Thus we can can conclude that at least one capital city is be beautiful.

This argument is a little cumbersome. It relies upon the idea of making a selection, but does not specify what that selection is. If there are several capital cities which have rivers, then any one of these would do. We do not care what the selection is, as long as such a selection is possible. Thus the description '*the capital city which we have just selected*' is *not* a label since it does not necessarily refer to a unique capital city. Nevertheless we can present the argument by referring to the selected city by something that resembles a constant; for example we could refer to it as '*river city*'. It is important to remember that an expression such as '*river city*' is not a name, and is not a label for a specific object. The argument might then run along the following lines.

> From the first premiss we know that it is possible to select a capital city which has a river. Suppose we make such a selection, '*river city*'. From the second premiss we know that '*river city*' must be beautiful. Thus we can can conclude that at least one capital city is be beautiful.

The expression '*river city*' is not a proper constant; it is an example of what we shall call an **arbitrary constant**. An arbitrary constant is treated as a constant but it does not necessarily refer to a unique object. Conventionally, we normally use small letters from the beginning of the alphabet as arbitrary constants. Free variables in predicates and function expressions can be replaced by arbitrary constants as well as proper constants.

Example 6.13

In the predicate 'x *is a dog*', we could replace the free variable x by an arbitrary constant a to give the proposition 'a *is a dog*'. □

Example 6.14

In the function expression $\sqrt{x+y}$, we could replace the free variable x by the arbitrary constant b and the free variable y by the proper constant 2 to give $\sqrt{b+2}$. □

Terms

We see that we can replace a free variable by

- a proper constant;
- an arbitrary constant;
- a function application.

This leads us to define the concept of a **term.**

Definition 6.5

Each of the following is a term:

- a proper constant;
- an arbitrary constant;
- a function applied to a term.

□

Example 6.15

Suppose f and g are functions, and a is an arbitrary constant. Is the expression $f(g(a))$ a term?

Example 6.16

The expression $f(g(a))$ represents the application of the function f to $g(a)$. But $g(a)$ is itself the application of the function g to a. Now since a is an arbitrary constant, by definition it must be a term. Hence $g(a)$ is a function applied to a term, and so by definition must is also be a term. Similarly we find that $f(g(a))$ is a function applied to a term, and so must also be a term. □

A free variable in a predicate or a function expression can always be replaced by a term.

Example 6.17

Suppose we have a predicate $p(x)$, '*x was born in 1930*'; a function f where $f(x)$ stands for '*the father of x*'; and an arbitrary constant c. Then we can apply f to c to give the term $f(c)$ which can then replace the free variable x in the predicate to give the proposition $p(f(c))$. □

In talking about terms in general, it is useful to introduce letters $\mathfrak{t}_1$, $\mathfrak{t}_2$, $\mathfrak{t}_3$ and so on to represent unspecified terms. These are part of our metalanguage of predicate logic.

6.6 Predicate forms

Just as we had propositional forms or propositional schemas, so we can have **predicate forms**.

Forms (schemas) involving schematic letters $\mathcal{P}, \mathcal{Q}, \mathcal{R} \ldots$ can be used to represent compound predicates as well as compound propositions. Each schematic letter can be instantiated to either a proposition, or a predicate with one or more free variables.

Example 6.18

Instantiating $\mathcal{P}$ to $2+1=3$, $\mathcal{Q}$ to $x+1=7$ and $\mathcal{R}$ to $x>y$ in $\mathcal{P} \wedge \mathcal{R} \Rightarrow \mathcal{Q}$ gives the binary predicate

$$(2+1=3) \wedge (x+1=7) \Rightarrow (x>y)$$

□

Often we wish to indicate that a schematic letter is to be instantiated to a predicate and to indicate some or all of the free variables. This can be done by including a list of free variables in brackets after the schematic letter. In this book we adopt the convention that whenever the list of variables is enclosed within parentheses, then the list is complete: there are no other free variables. Whenever the list is enclosed in square brackets, however, then the list *may* be incomplete: the possibility exists that an instance may contain other free variables.

Example 6.19

The predicate $x > y$ is an instance of $\mathcal{P}(x,y)$, $\mathcal{P}[x,y]$, $\mathcal{P}[x]$, $\mathcal{P}[y]$ and $\mathcal{P}$ but not of $\mathcal{P}(x)$, $\mathcal{P}(y)$, $\mathcal{P}[x,y,z]$ or $\mathcal{P}(z,w)$. □

We saw above that a free variable can be replaced by a term. To indicate that a free variable has been replaced by a term, we place a backslash then the term after the free variable. Alternatively, the term replace the free variable in the same position.

Example 6.20

$\mathcal{P}(x, y\backslash k)$ or simply $\mathcal{P}(x,k)$ is the result of replacing y by the proper constant k in the predicate form $\mathcal{P}(x,y)$; it is another predicate form with just one free variable, namely x. Thus, for example, if $\mathcal{P}(x,y)$ is instantiated to $x>y$ then $\mathcal{P}(x,y\backslash 0)$, or $\mathcal{P}(x,0)$, is instantiated to $x>0$.

Replacing the free variable x by $f(a)$ in this new predicate form, yields $\mathcal{P}(x\backslash f(a), y\backslash k)$ which is a propositional form; alternative ways of writing this are:

- $\mathcal{P}(x\backslash f(a), k)$
- $\mathcal{P}(f(a), y\backslash k)$
- $\mathcal{P}(f(a), k)$

Note that $\mathcal{P}(k, f(a))$ is *not* the same as $\mathcal{P}(f(a), k)$; $\mathcal{P}(k, f(a))$ written in full would be $\mathcal{P}(x\backslash k, y\backslash f(a))$. □

Example 6.21

$\mathcal{Q}[x] \vee \mathcal{R}[x]$ is a predicate form with at least one free variable, x. Replacing x by the arbitrary constant b gives $\mathcal{Q}[x \backslash b] \vee \mathcal{R}[x \backslash b]$ or $\mathcal{Q}[b] \vee \mathcal{R}[b]$, which may be either a propositional form or a predicate form according to whether or not there are still free variables. □

We can extend this idea to replacing one free variable by another.

Example 6.22

Replacing y by z in $\mathcal{P}(x, y)$ gives $\mathcal{P}(x, y \backslash z)$ or simply $\mathcal{P}(x, z)$, which is a predicate form with two free variables x and z. For example if $\mathcal{P}(x, y)$ is instantiated to $x > y$, then $\mathcal{P}(x, z)$ is instantiated to $x > z$. Now recall that the free variables are no more than tags for the gaps in a predicate. Then both $x > y$ and $x > z$ are essentially the same predicate, $\ldots > \ldots$, but with a different choice of tags. □

Example 6.23

Replacing y by x in $\mathcal{P}(x, y)$ gives $\mathcal{P}(x, y \backslash x)$ or simply $\mathcal{P}(x, x)$, which is a predicate form with only one free variable, namely x. For example if $\mathcal{P}(x, y)$ is instantiated to $x > y$, then $\mathcal{P}(x, x)$ is instantiated to $x > x$. In this case $x > y$ and $x > x$ are not the same predicate since in $x > x$ both gaps must be filled with the same value, whereas in $x > y$ they need not be. □

In general, suppose that we have a predicate form $\mathfrak{A}$ with a free variable x, and that we replace x by y in $\mathfrak{A}$. Then the resulting predicate form will be essentially the same as the original, provided that y does not occur in the original.

Exercise 45: Predicate forms

1. The predicate $x^2 = y$ could be an instance of which of the following predicate forms.

   (a) $\mathcal{Q}$
   (b) $\mathcal{P}(x)$
   (c) $\mathcal{P}(x, y)$
   (d) $\mathcal{P}(x, y, z \backslash 2)$
   (e) $\mathcal{P}[x]$
   (f) $\mathcal{Q}[y, x]$
   (g) $\mathcal{P}[x, y, z]$
   (h) $\mathcal{P}[x, y, z \backslash 2]$
   (i) $\mathcal{P}[x, y \backslash 2, z]$

2. The compound predicate $x + 1 = y \Rightarrow x + 2 = z$ could be an instance of which of the following predicate forms.

   (a) $\mathcal{P}(x, y, z)$
   (b) $\mathcal{P}(v \backslash 1, w \backslash 2, x, y, z)$
   (c) $\mathcal{Q}(x, y) \Rightarrow \mathcal{P}(x, z)$
   (d) $\mathcal{Q}(x, z) \Rightarrow \mathcal{P}(x, y)$
   (e) $\mathcal{P}[x] \Rightarrow \mathcal{Q}[x]$
   (f) $\mathcal{P}[x, y, z] \Rightarrow \mathcal{Q}[x, y, z]$
   (g) $\mathcal{P}[x, y, z \backslash 1] \Rightarrow \mathcal{Q}[x, y \backslash 2, z]$
   (h) $\mathcal{P}[x, y, z \backslash 1] \Rightarrow \mathcal{P}[x, y \backslash 2, z]$

6.7 Quantifiers

Universal quantifier

We are still left with the problem of how to deal with '*Every dog has four legs*'. At first sight it might seem straightforward to simply replace x by '*every dog*' in the predicate 'x *has four legs*'. But we cannot do this, because '*every dog*' does not refer to an individual object, not even an arbitrary one.

In order to illustrate how this problem can be overcome, consider the following simple model. Suppose the free variable x in the predicate 'x *has four legs*' has DOGS, the set of dogs, as its universe of discourse; and suppose there are just three dogs in this set {'*Rex*', '*Fido*', '*Rover*'}. Now we can list all the possible propositions that arise from this predicate:

- '*Rex has four legs*'
- '*Fido has four legs*'
- '*Rover has four legs*'

Now if all of these propositions are true, then clearly the proposition '*Every dog has four legs*' has truth value T, while if one or more propositions is false, then '*Every dog has four legs*' has truth value F. Thus the proposition

'*Every dog has four legs*'

is equivalent to

'*The proposition* '$\mathfrak{t}$ *has four legs*' *has truth value* T *for any arbitrary term* $\mathfrak{t}$.'

This is indicated symbolically by using the symbol $\forall$; this symbol is called the **universal quantifier**.

$\forall x$ ('x *has four legs*')

In general, for any predicate $\mathcal{P}(x)$ we can form a proposition $\forall x\,(\mathcal{P}(x))$ which asserts that any proposition $\mathcal{P}(x\backslash\mathfrak{t})$ is true. In general the universe of discourse may be very large so that listing all the possibilities would be impractical, or it may be infinite so that listing all the possibilities would be impossible.

Existential quantifier

Suppose we want to represent '*Some dogs have four legs*' using symbols. We can illustrate this with the same model that we used in the previous section. Suppose we start with the predicate 'x *has four legs*' and take DOGS as the universe of discourse. If we were to consider all the possible names of dogs to replace x, then there must be at least one for which the resulting proposition is true. This can be represented using the **existential quantifier**, $\exists$.

$\exists x$ ('x *has four legs*')

In general, for any predicate $\mathcal{P}(x)$ we can form a proposition $\exists x\,(\mathcal{P}(x))$ which asserts that there is at least one term $\mathtt{t}$ for which $\mathcal{P}(\mathtt{t})$ is true.

Scope of quantifiers

Definition 6.6

The parentheses associated with a quantifier indicate the predicate or predicate form to which the quantifier applies; this predicate (form) is called the **scope** of the quantifier. Note that when the scope includes just the predicate or form immediately following the quantifier, it is possible to leave out the parentheses. □

Example 6.24

The expression $\forall x\,(p(x))$ can be more simply written as $\forall x p(x)$. However, $\forall x\,(p(x) \Rightarrow q(x))$ cannot be written as $\forall x p(x) \Rightarrow q(x)$; the expression $\forall x p(x) \Rightarrow q(x)$ is in fact a simpler way of writing $\forall x\,(p(x)) \Rightarrow q(x)$. □

This needs to be remembered in parsing expressions in which there are quantifiers.

Example 6.25

Parse $\forall x \mathcal{P}(x) \Rightarrow \mathcal{Q}(y) \wedge \neg\mathcal{R} \Leftrightarrow \exists y \mathcal{P}(y)$.

Solution

First we indicate the scope of each quantification:

$$\forall x\,(\mathcal{P}(x)) \Rightarrow \mathcal{Q}(y) \wedge \neg\mathcal{R} \Leftrightarrow \exists y\,(\mathcal{P}(y))$$

We then parse for the connectives to yield:

$$((\forall x\,(\mathcal{P}(x)) \Rightarrow (\mathcal{Q}(y) \wedge (\neg\mathcal{R}))) \Leftrightarrow \exists y\,(\mathcal{P}(y)))$$

□

6.8 Semantics

The word **semantics** is concerned with meaning. For example, the semantics of a connective such as $\Rightarrow$ is concerned with the meaning of that connective. In logic we prescribe the semantics of each connective in terms of truth values. Thus the connective $\Rightarrow$ is defined by its truth table. Likewise the semantics of a propositional form is given by its truth table. We also define what a valid deduction is in terms of truth values, that is in terms of semantic entailment.

We now extend this approach to the quantifiers $\forall$ and $\exists$. Indeed we have already given definitions for the quantifiers in terms of truth values. The proposition $\exists x\,\mathcal{P}(x)$ is true if we can find a value a in the universe of discourse for which $\mathcal{P}(a) =_T T$; the proposition $\forall x\,\mathcal{P}(x)$ is true if $\mathcal{P}(a) =_T T$ for any value a in the universe of discourse. There are, however, difficulties with these definitions.

For example, if the universe of discourse is infinite we cannot check every possible proposition $\mathcal{P}(a)$. Nevertheless, we can still reason (using natural language) about quantified expressions to arrive at equivalences such as

$$\exists x\, \mathcal{P} =_T \neg \forall x \neg \mathcal{P}$$

and semantic entailments such as

$$\forall x\, \mathcal{P} \vDash \exists x\, \mathcal{P}$$

Such results are very useful in predicate logic.

If a universe of discourse could be the empty set, then there would be no propositions to check. Determining the truth values of $\exists x\, \mathcal{P}(x)$ and $\forall x\, \mathcal{P}(x)$ in such case would not be straightforward. At first it may seem that both $\forall x\, \mathcal{P}(x)$ and $\exists x\, \mathcal{P}(x)$ must be false since there are no true propositions. But by the same reasoning, $\forall x \neg \mathcal{P}(x)$ would also be false, so $\neg \forall x \neg \mathcal{P}(x)$ would be true. Thus if the equivalence

$$\exists x\, \mathcal{P} =_T \neg \forall x \neg \mathcal{P}$$

were still to hold, then this would mean that $\exists x\, (\mathcal{P}(x))$ must be true. This would clearly contradict our earlier conclusion that $\exists x\, \mathcal{P}(x)$ must be false. The solution to this difficulty that is adopted in the traditional approach to logic is that empty universes of discourse are not allowed; when we make a statement such as $\forall x\, \mathcal{P}(x)$ or $\exists x\, \mathcal{P}(x)$, then implicit to this statement is the fact that there exists at least one value we can substitute for x.

We can now begin to reason about quantified expressions.

Example 6.26

Show that $\forall x\, \mathcal{P}(x) \vDash \exists x\, \mathcal{P}(x)$.

Solution

If $\forall x\, \mathcal{P}(x) =_T T$ then by definition all propositions are true. Now the universe of discourse must contain at least one value. Thus we can choose a value a from the universe of discourse. Furthermore from the definition of universal quantifiers, since $\forall x\, \mathcal{P}(x) =_T T$ then the proposition $\mathcal{P}(a)$ is true. We therefore conclude that whenever $\forall x\, \mathcal{P}(x)$ is true then so is $\exists x\, \mathcal{P}(x)$. We can write this using semantic entailment:

$$\forall x\, \mathcal{P}(x) \vDash \exists x\, \mathcal{P}(x)$$

Note that this argument depends upon the fact that the universe of discourse is not empty. □

Example 6.27

If the universe of discourse is DOGS, how can the proposition '*Every dog has four legs*' be represented symbolically? What is the negation of this proposition.

Solution

Suppose we denote the predicate '*x has four legs*' by $p(x)$. Then since the universe of discourse is DOGS the proposition '*Every dog has four legs*' can be represented as $\forall x\,(p(x))$.

Thus to negate '*Every dog has four legs*', we need to find the negation of $\forall x\,(p(x))$. The negation is that not all propositions arising from $p(x)$ are true; there is at least one value a for which $p(a) =_T F$. Thus there is at least one value a for which $\neg p(a) =_T T$. We can express this using existential quantification as $\exists x\,(\neg p(x))$. This can be interpreted as

'*Some dogs do not have four legs*'

or as

'*There exist dogs who do not have four legs*'.

Note that this is an example of the more general result that

$$\neg \forall x\, p(x) =_T \exists x\,(\neg p(x))$$

□

In the previous example, we were given that the universe of discourse was DOGS. But the problem can also be tackled if have a larger set for the universe of discourse; for example we might have ANIMALS, LIVING_THINGS or even THINGS. In such a case, we need to introduce a predicate $q(x)$ to represent the property of being a dog.

Example 6.28

Suppose we have a universe of discourse of LIVING_THINGS, with $p(x)$ as the predicate '*x has four legs*', and $q(x)$ as the predicate '*x is a dog*'. Then what is the interpretation of $\forall x\,(q(x) \Rightarrow p(x))$? What is the negation of this proposition?

Solution

Consider the predicate $q(x) \Rightarrow p(x)$. Suppose we were to choose a living thing a which is not a dog. Then $q(a)$ would be false. Hence from the truth table for $\Rightarrow$, the proposition $q(a) \Rightarrow p(a)$ must be true irrespective of the truth value of $p(a)$; if a is not a dog, then the proposition '*a has four legs*' may be either true or false.

Now suppose we consider a value of a which represents a dog. Then, since $q(a)$ would be true, $q(a) \Rightarrow p(a)$ would only be true if $p(a)$ were true. But we know from $\forall x\,(q(x) \Rightarrow p(x))$ that $q(a) \Rightarrow p(a)$ cannot be false. Hence we conclude that $p(a) =_T T$ whenever we choose a which is a dog. The interpretation of $\forall x\,(q(x) \Rightarrow p(x))$ is

'*Every dog has four legs*'.

The negation of this proposition is '$\forall x\,(q(x) \Rightarrow p(x))$ *is false*'. That is, there is at least one value, b say, for which $q(b) \Rightarrow p(b)$ is false. From the truth table for $\Rightarrow$ we know that this can only happen if $q(b) =_T T$ and $p(b) =_T F$: that is, b must represent a dog which does not have four legs. The negation is thus

'*There exists at least one dog who does not have four legs*'.

Now since $p(b) =_T F$, then $\neg p(b) =_T T$. Hence $q(b) \wedge \neg p(b) =_T T$. From this we can deduce that

$$\exists x\, (q(x) \wedge \neg p(x))$$

is true. This must therefore be the symbolic representation of the negation of $\forall x\, (q(x) \Rightarrow p(x))$. □

Exercise 46: Universal quantifier

1. What is the interpretation of $\forall x\, \mathcal{P}(x)$ for each of the following instances?
   (a) $\mathcal{P}(x)$: '$x^2 \geq x$.'
   Universe of discourse: INTEGERS.

   (b) $\mathcal{P}(x)$: 'x *has three legs.*'
   Universe of discourse: DOGS.

   (c) $\mathcal{P}(x)$: 'x *has hosted the Olympic Games*'
   Universe of discourse: CAPITAL CITIES.

2. What is the interpretation of $\forall x\, (\mathcal{P}(x) \Rightarrow \mathcal{Q}(x))$ for each of the following instances?
   (a) $\mathcal{P}(x)$: '$x > 0$.'
   $\mathcal{Q}(x)$: '$x = x^2$.'
   Universe of discourse: INTEGERS.

   (b) $\mathcal{P}(x)$: 'x *is a dog.*'
   $\mathcal{Q}(x)$: 'x *has three legs.*'
   Universe of discourse: ANIMALS.

   (c) $\mathcal{P}(x)$: 'x *is a capital city.*'
   $\mathcal{Q}(x)$: 'x *has hosted the Olympic Games*'
   Universe of discourse: TOWNS AND CITIES.

3. What is the interpretation of $\forall x\, (\mathcal{P}(x) \wedge \mathcal{Q}(x))$ for each of the following instances?
   (a) $\mathcal{P}(x)$: '$x > 0$.'
   $\mathcal{Q}(x)$: '$x = x^2$.'
   Universe of discourse: INTEGERS.

(b) $\mathcal{P}(x)$: 'x *is a dog.*'
$\mathcal{Q}(x)$: 'x *has three legs.*'
Universe of discourse: ANIMALS.

(c) $\mathcal{P}(x)$: 'x *is a capital city.*'
$\mathcal{Q}(x)$: 'x *has hosted the Olympic Games*'
Universe of discourse: TOWNS AND CITIES.

4. What is the interpretation of $\forall x\,(\mathcal{P}(x)) \wedge \forall x\,(\mathcal{Q}(x))$ for each of the following instances?

(a) $\mathcal{P}(x)$: '$x > 0$.'
$\mathcal{Q}(x)$: '$x = x^2$.'
Universe of discourse: INTEGERS.

(b) $\mathcal{P}(x)$: 'x *is a dog.*'
$\mathcal{Q}(x)$: 'x *has three legs.*'
Universe of discourse: ANIMALS.

(c) $\mathcal{P}(x)$: 'x *is a capital city.*'
$\mathcal{Q}(x)$: 'x *has hosted the Olympic Games*'
Universe of discourse: TOWNS AND CITIES.

5. What is the interpretation of $\forall x\,(\mathcal{P}(x) \vee \mathcal{Q}(x))$ for each of the following instances?

(a) $\mathcal{P}(x)$: '$x > 0$.'
$\mathcal{Q}(x)$: '$x = x^2$.'
Universe of discourse: INTEGERS.

(b) $\mathcal{P}(x)$: 'x *is a dog.*'
$\mathcal{Q}(x)$: 'x *has three legs.*'
Universe of discourse: ANIMALS.

(c) $\mathcal{P}(x)$: 'x *is a capital city.*'
$\mathcal{Q}(x)$: 'x *has hosted the Olympic Games*'
Universe of discourse: TOWNS AND CITIES.

6. What is the interpretation of $\forall x\,(\mathcal{P}(x)) \vee \forall x\,(\mathcal{Q}(x))$ for each of the following instances?

(a) $\mathcal{P}(x)$: '$x > 0$.'
$\mathcal{Q}(x)$: '$x = x^2$.'
Universe of discourse: INTEGERS.

(b) $\mathcal{P}(x)$: '*x is a dog.*'
$\mathcal{Q}(x)$: '*x has three legs.*'
Universe of discourse: ANIMALS.

(c) $\mathcal{P}(x)$: '*x is a capital city.*'
$\mathcal{Q}(x)$: '*x has hosted the Olympic Games*'
Universe of discourse: TOWNS AND CITIES.

Exercise 47: Existential quantifier

1. What is the interpretation of $\exists x\,(\mathcal{P}(x))$ for each of the following instances?
   (a) $\mathcal{P}(x)$: '$x^2 \geq x$.'
   Universe of discourse: INTEGERS.

   (b) $\mathcal{P}(x)$: '*x has three legs.*'
   Universe of discourse: DOGS.

   (c) $\mathcal{P}(x)$: '*x has hosted the Olympic Games*'
   Universe of discourse: CAPITAL CITIES.

2. What is the interpretation of $\exists x\,(\mathcal{P}(x) \wedge \mathcal{Q}(x))$ for each of the following instances?
   (a) $\mathcal{P}(x)$: '$x > 0$.'
   $\mathcal{Q}(x)$: '$x = x^2$.'
   Universe of discourse: INTEGERS.

   (b) $\mathcal{P}(x)$: '*x is a dog.*'
   $\mathcal{Q}(x)$: '*x has three legs.*'
   Universe of discourse: ANIMALS.

   (c) $\mathcal{P}(x)$: '*x is a capital city.*'
   $\mathcal{Q}(x)$: '*x has hosted the Olympic Games*'
   Universe of discourse: TOWNS AND CITIES.

3. What is the interpretation of $\exists x\,(\mathcal{P}(x)) \wedge \exists x\,(\mathcal{Q}(x))$ for each of the following instances?
   (a) $\mathcal{P}(x)$: '$x > 0$.'
   $\mathcal{Q}(x)$: '$x = x^2$.'
   Universe of discourse: INTEGERS.

   (b) $\mathcal{P}(x)$: '*x is a dog.*'
   $\mathcal{Q}(x)$: '*x has three legs.*'
   Universe of discourse: ANIMALS.

(c) $\mathcal{P}(x)$: '*x is a capital city.*'
$\mathcal{Q}(x)$: '*x has hosted the Olympic Games*'
Universe of discourse: TOWNS AND CITIES.

4. What is the interpretation of $\exists x\,(\mathcal{P}(x) \vee \mathcal{Q}(x))$ for each of the following instances?

   (a) $\mathcal{P}(x)$: '$x > 0$.'
   $\mathcal{Q}(x)$: '$x = x^2$.'
   Universe of discourse: INTEGERS.

   (b) $\mathcal{P}(x)$: '*x is a dog.*'
   $\mathcal{Q}(x)$: '*x has three legs.*'
   Universe of discourse: ANIMALS.

   (c) $\mathcal{P}(x)$: '*x is a capital city.*'
   $\mathcal{Q}(x)$: '*x has hosted the Olympic Games*'
   Universe of discourse: TOWNS AND CITIES.

5. What is the interpretation of $\exists x\,(\mathcal{P}(x)) \vee \exists x\,(\mathcal{Q}(x))$ for each of the following instances?

   (a) $\mathcal{P}(x)$: '$x > 0$.'
   $\mathcal{Q}(x)$: '$x = x^2$.'
   Universe of discourse: INTEGERS.

   (b) $\mathcal{P}(x)$: '*x is a dog.*'
   $\mathcal{Q}(x)$: '*x has three legs.*'
   Universe of discourse: ANIMALS.

   (c) $\mathcal{P}(x)$: '*x is a capital city.*'
   $\mathcal{Q}(x)$: '*x has hosted the Olympic Games*'
   Universe of discourse: TOWNS AND CITIES.

6. What is the interpretation of $\exists x\,(\mathcal{P}(x) \Rightarrow \mathcal{Q}(x))$ for each of the following instances?

   (a) $\mathcal{P}(x)$: '$x > 0$.'
   $\mathcal{Q}(x)$: '$x = x^2$.'
   Universe of discourse: INTEGERS.

   (b) $\mathcal{P}(x)$: '*x is a dog.*'
   $\mathcal{Q}(x)$: '*x has three legs.*'
   Universe of discourse: ANIMALS.

(c) $\mathcal{P}(x)$: '*x is a capital city.*'
$\mathcal{Q}(x)$: '*x has hosted the Olympic Games*'
Universe of discourse: TOWNS AND CITIES.

6.9 Deduction with quantified predicates

Rules of propositional logic

We can apply the deduction rules from propositional logic to any propositions or propositional forms; these include propositions or forms which have quantified predicates. This is illustrated in the following examples.

Example 6.29
Show that $\forall x\, \mathcal{P}(x),\ \forall x\, \mathcal{P}(x) \Rightarrow \exists x\, \mathcal{P}(x) \vdash \exists x\, \mathcal{P}(x)$

Solution

$$\dfrac{\forall x\, \mathcal{P}(x) \qquad \forall x\, \mathcal{P}(x) \Rightarrow \exists x\, \mathcal{P}(x)}{\exists x\, \mathcal{P}(x)} \Rightarrow \mathrm{E}$$

□

Example 6.30
Show that $\vdash (\forall x\, \mathcal{P}(x) \wedge \forall y\, \mathcal{Q}(y)) \Rightarrow (\forall x\, \mathcal{P}(x) \wedge \forall y\, \mathcal{Q}(y))$

Solution

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x\, \mathcal{P}(x) \wedge \forall y\, \mathcal{Q}(y)} *}{\forall x\, \mathcal{P}(x)} \wedge \mathrm{E}_1}{\forall x\, \mathcal{P}(x) \wedge \forall y\, \mathcal{Q}(y)} \vee \mathrm{I}_1}{(\forall x\, \mathcal{P}(x) \wedge \forall y\, \mathcal{Q}(y)) \Rightarrow (\forall x\, \mathcal{P}(x) \wedge \forall y\, \mathcal{Q}(y))} * \Rightarrow \mathrm{I}$$

□

Deduction Rules: ∀E and ∃I

In addition to the rules for introducing and eliminating connectives, there are also rules for introducing and eliminating the quantifiers. In this subsection we shall look at the rule for eliminating the universal quantifier ∀ and the rule for introducing the existential quantifier ∃. Both these rules are inference forms. As before we shall show justify each new rule of deduction by showing that it corresponds to a semantic entailment, that is to a valid argument.

Rule: ∀E
$\forall x\, \mathcal{P}(x) \vdash \mathcal{P}(\mathfrak{t})$ □

Justification
First suppose that $\forall x\, \mathcal{P}(x)$ is a proposition with truth value T. Then for any term $\mathfrak{t}$ which refers to an object in the universe of discourse, it must be the case that $\mathcal{P}(\mathfrak{t}) =_T T$. Hence we can write:

$$\forall x\, \mathcal{P}(x) \vDash \mathcal{P}(\mathfrak{t})$$

□

Rule: $\exists I$

$\mathcal{P}(\mathfrak{t}) \vdash \exists x\, \mathcal{P}(x)$ □

Justification

Suppose that for some term $\mathfrak{t}$ referring to an object in the universe of discourse, $\mathcal{P}(\mathfrak{t}) =_T T$, then clearly it follows that $\exists x\, \mathcal{P}(x) =_T T$. Hence we can write:

$$\mathcal{P}(\mathfrak{t}) \vDash \exists x\, \mathcal{P}(x)$$

□

As we did with deduction rules for connectives, we can generalize these rules for general predicate forms.

Rule: $\forall E$

For any predicate form $\mathfrak{A}(x)$, $\forall x\, \mathfrak{A}(x) \vdash \mathfrak{A}(\mathfrak{t})$ □

Rule: $\exists I$

For any predicate form $\mathfrak{A}(x)$, $\mathfrak{A}(\mathfrak{t}) \vdash \exists x\, \mathfrak{A}(x)$ □

Example 6.31

Show that '*All dogs have four legs*' ⊢ '*Some dogs have four legs*'.

Solution

Suppose we represent the predicate '*x has four legs*' as $p(x)$ and choose the universe of discourse to be DOGS. Naturally we are making the assumption that dogs do indeed exist! Then the proposition

'*all dogs have four legs*'

can be represented by $\forall x\, p(x)$. Furthermore the proposition

'*Some dogs have four legs*'

can be represented by $\exists x\, p(x)$. We therefore need to show that

$$\forall x\, p(x) \vdash \exists x\, p(x)$$

This we can do by deducing the inference form

$$\forall x\, \mathcal{P}(x) \vdash \exists x\, \mathcal{P}(x)$$

using rules of natural deduction.

$$\cfrac{\cfrac{\forall x\, \mathcal{P}(x)}{\mathcal{P}(\text{'}Rex\text{'})}\ \forall\text{E}}{\exists x\, \mathcal{P}(x)}\ \exists\text{I}$$

Note that the term chosen, in this case the proper constant '*Rex*', is largely immaterial, so long as the term refers to a dog in the universe of discourse. With the same proviso, we could equally have chosen '*Fido*', or '*Sadie*', or '*Mrs. Joel's dog*', or indeed an arbitrary constant such as a. In general we do not need to know what the term is; we represent the term by a letter such as t. Typically the deduction tree would be presented as

$$\dfrac{\dfrac{\forall x\, \mathcal{P}(x)}{\mathcal{P}(\mathsf{t})}\ \forall\mathrm{E}}{\exists x\, \mathcal{P}(x)}\ \exists\mathrm{I}$$

Note that other instances will yield different valid arguments from the same inference form, for example:

'*Everyone can sing*' $\vdash$ '*Some people can sing*'

□

Alternative solution

Suppose that the universe of discourse is a **superset** of DOGS; that is DOGS is a subset of this universe of discourse. We shall need to introduce a further predicate to represent the property of being a dog: let $q(x)$ be the predicate '*x is a dog*'. Now the proposition '*All dogs have four legs*' can then be thought of as

'*For all x, if x is a dog then x has four legs*'

and hence can be represented as $\forall x\, (q(x) \Rightarrow p(x))$. Similarly, the proposition '*Some dogs have four legs*' can be thought of

'*There exists x such that x is a dog and x has four legs*'

and hence can be represented as $\exists x\, (q(x) \wedge p(x))$. Unfortunately the argument

$$\forall x\, (q(x) \Rightarrow p(x)) \;\therefore\; \exists x\, (q(x) \wedge p(x))$$

is not valid! This is because there may not be any dogs; in that case $q(x)$ would be false for all values of x so $q(x) \Rightarrow p(x)$ would always be true while $q(x) \wedge p(x)$ would always be false. To overcome this problem, we need to introduce a premiss to the effect that dogs do indeed exist: $\exists x\, q(x)$. The problem is now that of showing

$$\forall x\, (q(x) \Rightarrow p(x)), \exists x\, q(x) \vdash \exists x\, (q(x) \wedge p(x))$$

This we can do by deducing the inference form

$$\forall x\, (\mathcal{Q}(x) \Rightarrow \mathcal{P}(x)), \exists x\, \mathcal{Q}(x) \vdash \exists x\, (\mathcal{Q}(x) \wedge \mathcal{P}(x))$$

using the rules of natural deduction. The deduction tree for this inference form, however, requires the use of the $\exists$E rule, which is considered below. □

Example 6.32

Show that $\forall x\, \mathcal{P}(x), \forall x\, \mathcal{Q}(x) \vdash \exists x\, (\mathcal{P}(x) \wedge \mathcal{Q}(x))$.

Solution

The derivation tree is:

$$\dfrac{\dfrac{\forall x\, \mathcal{P}(x)}{\mathcal{P}(\mathsf{t})}\,\forall\text{E} \qquad \dfrac{\forall x\, \mathcal{Q}(x)}{\mathcal{Q}(\mathsf{t})}\,\forall\text{E}}{\dfrac{\mathcal{P}(\mathsf{t}) \wedge \mathcal{Q}(\mathsf{t})}{\exists x\, (\mathcal{P}(x) \wedge \mathcal{Q}(x))}\,\exists\text{I}}\,\wedge\text{I}$$

Hence $\forall x\, \mathcal{P}(x), \forall x\, \mathcal{Q}(x) \vdash \exists x\, (\mathcal{P}(x) \wedge \mathcal{Q}(x))$. □

Example 6.33

Show that '*Rex is a dog*', '*All dogs have four legs*' ⊢ '*Rex has four legs*'.

Solution

First we must represent the propositions symbolically using appropriate predicates, a universe of discourse and quantifiers. Since the proposition '*Rex is a dog*' asserts the property of being a dog for '*Rex*', we can represent this proposition as p('*Rex*') where $p(x)$ is the predicate 'x *is a dog*'. Clearly it would be senseless to take the universe of discourse to be DOGS for the predicate 'x *is a dog*'; we need a larger universe such as ANIMALS. In order to represent the proposition '*Rex has four legs*', we introduce the predicate $q(x)$ = 'x *has four legs*' with the same universe of discourse. We can now represent '*All dogs have four legs*' as $\forall x\, (p(x) \Rightarrow q(x))$. Thus we have to prove that

$$p(\text{'}Rex\text{'}),\ \forall x\, (p(x) \Rightarrow q(x)) \vdash q(\text{'}Rex\text{'})$$

Note that it is not necessary to introduce a premiss to express that dogs exist since this is implicit in the premiss p('*Rex*'). The deduction tree for the corresponding inference form is:

$$\dfrac{\mathcal{P}(\text{'}Rex\text{'}) \qquad \dfrac{\forall x\, (\mathcal{P}(x) \Rightarrow \mathcal{Q}(x))}{\mathcal{P}(\text{'}Rex\text{'}) \Rightarrow \mathcal{Q}(\text{'}Rex\text{'})}\,\forall\text{E}}{\mathcal{Q}(\text{'}Rex\text{'})}\,\Rightarrow\text{E}$$

Notice that in using the ∀E rule, *any* term may be chosen; in this case the proper constant '*Rex*' was chosen. The deduction tree can be generalized to use an arbitrary term

$$\dfrac{\mathcal{P}(\mathsf{t}) \qquad \dfrac{\forall x\, (\mathcal{P}(x) \Rightarrow \mathcal{Q}(x))}{\mathcal{P}(\mathsf{t}) \Rightarrow \mathcal{Q}(\mathsf{t})}\,\forall\text{E}}{\mathcal{Q}(\mathsf{t})}\,\Rightarrow\text{E}$$

to give the general inference form

$$\mathcal{P}(\mathsf{t}),\ \forall x\, (\mathcal{P}(x) \Rightarrow \mathcal{Q}(x)) \vdash \mathcal{Q}(\mathsf{t})$$

The required inference is an instance of this form in which $\mathcal{P}(x)$ has been instantiated to 'x *is a dog*', $\mathcal{Q}(x)$ to 'x *has four legs*' and t to '*Rex*'. □

Exercise 48: Deduction rules: $\forall$E and $\exists$I

1. Prove each of the following inference forms.
   (a) $\forall x\,(\mathcal{P}(x) \wedge \mathcal{Q}(x)) \vdash \exists x\,(\mathcal{P}(x) \vee \mathcal{R}(x))$
   (b) $\neg\neg\forall x\,(\mathcal{P}(x) \wedge \mathcal{Q}(x)) \vdash \exists x\,(\mathcal{P}(x) \wedge \mathcal{Q}(x))$
   (c) $\neg\mathcal{Q}(a), \forall x\,(\mathcal{P}(x) \Rightarrow \mathcal{Q}(x)) \vdash \exists x\,(\neg\mathcal{P}(x))$
2. Prove each of the following inferences by deducing an appropriate inference form.
   (a) $\forall x\,(\neg p(x)) \vdash \exists x\,(\neg p(x))$
   (b) $\forall x\,(\neg\neg p(x)) \vdash \exists x\,(p(x))$
   (c) '*Canberra is the capital city of Australia*' $\vdash$ '*Australia has a capital city*'

6.10 Methods of deduction

The two rules of deduction for predicate logic we have met so far are inference forms. They can be used together with the inference forms and methods of deduction introduced in earlier chapters to give further inference forms. However, there are also methods of deduction for predicate logic; that is rules of deduction that enable us to write down one inference form from another.

Substitution

The simplest way in which we can create a new inference from an existing one is by substitution.

Rule: Term Subst.
Suppose we have a inference form $\Gamma \vdash \mathfrak{A}$, and suppose that some of Γ and $\mathfrak{A}$ contain the term $\mathfrak{t}_1$. Replacing every occurrence of the term $\mathfrak{t}_1$ by another term $\mathfrak{t}_2$ will yield another inference form. □

However, *great* care must be taken with how the new inference form is written, as the following example shows.

Example 6.34
Suppose we have predicate forms $\mathcal{P}(x)$, $\mathcal{P}(x) \Rightarrow \mathcal{Q}(x)$ and $\mathcal{Q}(x)$; suppose further that we replace the free variable x throughout these forms by the proper constant k to yield $\mathcal{P}(k)$, $\mathcal{P}(k) \Rightarrow \mathcal{Q}(k)$. Now from the $\Rightarrow$ E rule we know that

$$\mathcal{P}(k), \mathcal{P}(k) \Rightarrow \mathcal{Q}(k) \vdash \mathcal{Q}(k)$$

Substituting *all* occurrences k throughout this inference form by another term such as the proper constant l will yield a new inference form, but we cannot simply write this new inference form as

$$\mathcal{P}(l), \mathcal{P}(l) \Rightarrow \mathcal{Q}(l) \vdash \mathcal{Q}(l)$$

The reason for this is that some instances of $\mathcal{P}(x)$ and $\mathcal{Q}(x)$ will contain the constant k *before* substituting a term for the free variable x. In such instances the expressions $\mathcal{P}(l)$, $\mathcal{P}(l) \Rightarrow \mathcal{Q}(l)$ and $\mathcal{Q}(l)$ will still contain the constant k.

To illustrate this, consider the instance in which the universe of discourse is INTEGERS, $\mathcal{P}(x)$ is instantiated to the predicate $x > 2$ and $\mathcal{Q}(x)$ to the predicate $x > 1$. Now suppose that we take a value for x of 2. This gives the inference

$$2 > 2, (2 > 2) \Rightarrow (2 > 1) \vdash 2 > 1$$

Now if we substitute *all* occurrences of 2 by 1 in this inference we get

$$1 > 1, (1 > 1) \Rightarrow (1 > 1) \vdash 1 > 1$$

However, taking x to have the value 1 in the predicate $x > 2$ gives $1 > 2$ and *not* $1 > 1$. □

Universal quantifier introduction: $\forall$I

Rule: $\forall I$

If $\Gamma \vdash \mathfrak{A}(\mathfrak{t})$ then $\Gamma \vdash \forall x\, \mathfrak{A}(x)$ provided $\mathfrak{t}$ does not occur in the predicate form $\mathfrak{A}(x)$ nor in any proposition of Γ. □

Justification

Suppose that we have been able to show $\Gamma \vdash \mathfrak{A}(\mathfrak{t})$ for some term $\mathfrak{t}$. Then we can use term substitution of $\mathfrak{t}$ by any arbitrary constant b to obtain a new inference form. Provided we consider only instances of Γ and $\mathfrak{A}(x)$ which do *not* contain the term $\mathfrak{t}$ then this new inference form can be written as

$$\Gamma \vdash \mathfrak{A}(b)$$

Now from soundness it follows that

$$\Gamma \vDash \mathfrak{A}(b)$$

Thus if all of Γ are true then $\mathcal{P}(b)$ is true whatever the choice for b. Hence we conclude that $\forall x\, \mathcal{P}(x)$ is true:

$$\Gamma \vDash \forall x\, \mathcal{P}(x)$$

□

Example 6.35

Show that $\forall x\, \mathcal{P}(x), \forall x\, \mathcal{Q}(x) \vdash \forall x\, (\mathcal{P}(x) \wedge \mathcal{Q}(x))$.

Solution

The derivation tree begins like that of example (6.9) except that we must stipulate that a is chosen such that it does not occur in either $\mathcal{P}(x)$ or $\mathcal{Q}(x)$.

$$\dfrac{\dfrac{\forall x\, \mathcal{P}(x)}{\mathcal{P}(a)}\,\forall\text{E} \qquad \dfrac{\forall x\, \mathcal{Q}(x)}{\mathcal{Q}(a)}\,\forall\text{E}}{\mathcal{P}(a) \wedge \mathcal{Q}(a)}\,\wedge\text{I}$$

From this tree we can see that

$$\forall x\, \mathcal{P}(x), \forall x\, \mathcal{Q}(x) \vdash \mathcal{P}(a) \wedge \mathcal{Q}(a)$$

But neither of the premisses contain a; hence we can use the $\forall$I rule to deduce $\forall x\,(\mathcal{P}(x) \wedge \mathcal{Q}(x))$. we can represent this by simply adding another line to the deduction tree.

$$\cfrac{\cfrac{\cfrac{\forall x\, \mathcal{P}(x)}{\mathcal{P}(a)}\ \forall E \qquad \cfrac{\forall x\, \mathcal{Q}(x)}{\mathcal{Q}(a)}\ \forall E}{\mathcal{P}(a) \wedge \mathcal{Q}(a)}\ \wedge I}{\forall x\,(\mathcal{P}(x) \wedge \mathcal{Q}(x))}\ \forall I$$

□

Existential quantifier elimination: $\exists$E

The final rule of deduction, the $\exists$E rule is much more difficult to justify. It is perhaps best to simply state the rule, and demonstrate some examples of its use.

Rule: $\exists E$

If $\Gamma, \mathcal{P}(\mathfrak{t}) \vdash \mathcal{Q}$ then provided we consider only instances in which $\mathfrak{t}$ does not occur in $\mathcal{P}(x)$, $\mathcal{Q}(x)$ nor in any expression of Γ, then $\Gamma, \exists x\, \mathcal{P}(x) \vdash \mathcal{Q}$. □

Example 6.36

Show that $\exists x\, \forall y\, \mathcal{P}(x,y) \vdash \forall y\, \exists x\, \mathcal{P}(x,y)$

Solution

We approach the solution to this problem by first showing that

$$\forall y\, \mathcal{P}(a,y) \vdash \forall y\, \exists x\, \mathcal{P}(x,y)$$

for some constant a which does not appear in $\mathcal{P}(x,y)$; once we have done this we can invoke the $\exists$E rule to obtain our desired result. In order to achieve this first stage, however, we must use the $\forall$E rule by introducing another constant b which does not occur in $\mathcal{P}(x,y)$.

Thus we choose a and b such that neither occurs in $\mathcal{P}(x,y)$ and proceed as follows.

$$\cfrac{\cfrac{\cfrac{\forall y\, \mathcal{P}(a,y)}{\mathcal{P}(a,b)}\ \forall E}{\exists x\, \mathcal{P}(x,b)}\ \exists I}{\forall y\, \exists x\, \mathcal{P}(x,y)}\ \forall I$$

Since we have established

$$\forall y\, \mathcal{P}(a,y) \vdash \forall y\, \exists x\, \mathcal{P}(x,y)$$

we can invoke the $\exists$E rule to give

$$\exists x\, \forall y\, \mathcal{P}(x,y) \vdash \forall y\, \exists x\, \mathcal{P}(x,y)$$

This argument can be presented as a single tree in which the assumption $\forall y\, \mathcal{P}(a, y)$ is discharged by the $\exists$E rule

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall y\, \mathcal{P}(a,y)}\,*}{\mathcal{P}(a,b)}\,\forall\text{E}}{\exists x\, \mathcal{P}(x,b)}\,\exists\text{I}}{\forall y\, \exists x\, \mathcal{P}(x,y)}\,\forall\text{I} \qquad \exists x\, \forall y\, \mathcal{P}(x,y)}{\forall y\, \exists x\, \mathcal{P}(x,y)}\,*\exists\text{E}$$

□

Example 6.37

Prove that the conclusion '*Some furry creatures have whiskers*' can be deduced from the premisses '*All rabbits have whiskers*' and '*Some furry creatures are rabbits*'.

Solution

Choose the universe of discourse to be CREATURES, and introduce predicates $p(x)$ = 'x *is a rabbit*', $q(x)$ = 'x *has whiskers*' and $r(x)$ = 'x *is furry*'. The inference we have to prove can be written as

$$\forall x\, (p(x) \Rightarrow q(x)),\ \exists x\, (p(x) \wedge r(x)) \vdash \exists x\, (q(x) \wedge r(x))$$

This we can do by proving the inference form:

$$\forall x\, (\mathcal{P}(x) \Rightarrow \mathcal{Q}(x)),\ \exists x\, (\mathcal{P}(x) \wedge \mathcal{R}(x)) \vdash \exists x\, (\mathcal{Q}(x) \wedge \mathcal{R}(x))$$

To do this we first prove the subgoal:

$$\forall x\, (\mathcal{P}(x) \Rightarrow \mathcal{Q}(x)),\ \mathcal{P}(a) \wedge \mathcal{R}(a) \vdash \exists x\, (\mathcal{Q}(x) \wedge \mathcal{R}(x))$$

where in any instance the value of the arbitrary constant a is chosen such that it does not occur in any of the instances of $\mathcal{P}(x)$, $\mathcal{Q}(x)$ or $\mathcal{R}(x)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{P}(a) \wedge \mathcal{R}(a)}{\mathcal{P}(a)}\,\wedge\text{E}_1 \qquad \dfrac{\forall x\, (\mathcal{P}(x) \Rightarrow \mathcal{Q}(x))}{\mathcal{P}(a) \Rightarrow \mathcal{Q}(a)}\,\forall\text{E}}{\mathcal{Q}(a)}\,\Rightarrow\text{E} \qquad \dfrac{\mathcal{P}(a) \wedge \mathcal{R}(a)}{\mathcal{R}(a)}\,\wedge\text{E}_2}{\mathcal{Q}(a) \wedge \mathcal{R}(a)}\,\wedge\text{I}}{\exists x\, (\mathcal{Q}(x) \wedge \mathcal{R}(x))}\,\exists\text{I}$$

We can now apply the $\exists$E rule to write down

$$\forall x\, (\mathcal{P}(x) \Rightarrow \mathcal{Q}(x)),\ \exists x\, (\mathcal{P}(x) \wedge \mathcal{R}(x)) \vdash \exists x\, (\mathcal{Q}(x) \wedge \mathcal{R}(x))$$

The complete deduction tree is shown in Figure 6.1. □

$$
\dfrac{\exists x\,(\mathcal{P}(x) \wedge \mathcal{R}(x)) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\mathcal{P}(a) \wedge \mathcal{R}(a)}\,*}{\mathcal{P}(a)}\,\wedge\mathrm{E}_1 \qquad \dfrac{\forall x\,(\mathcal{P}(x) \Rightarrow \mathcal{Q}(x))}{\mathcal{P}(a) \Rightarrow \mathcal{Q}(a)}\,\forall\mathrm{E}}{\mathcal{Q}(a)}\,\Rightarrow\mathrm{E} \qquad \dfrac{\overline{\mathcal{P}(a) \wedge \mathcal{R}(a)}\,*}{\mathcal{R}(a)}\,\wedge\mathrm{E}_2}{\mathcal{Q}(a) \wedge \mathcal{R}(a)}\,\wedge\mathrm{I}}{\exists x\,(\mathcal{Q}(x) \wedge \mathcal{R}(x))}\,\exists\mathrm{I}}{\exists x\,(\mathcal{Q}(x) \wedge \mathcal{R}(x))}\,*\exists\mathrm{E}
$$

Figure 6.1: Solution to Example 6.38

Example 6.38

Earlier we saw that in one approach to proving

‘*All dogs have four legs*’ ⊢ ‘*Some dogs have four legs*’

was to show that

$$\forall x\,(\mathcal{Q}(x) \Rightarrow \mathcal{P}(x)), \exists x\,\mathcal{Q}(x) \vdash \exists x\,(\mathcal{Q}(x) \wedge \mathcal{P}(x))$$

Prove this inference form.

Solution

$$\dfrac{\exists x\,(\mathcal{Q}(x)) \qquad \dfrac{\dfrac{\dfrac{}{\mathcal{Q}(a)}\,* \qquad \dfrac{\dfrac{}{\mathcal{Q}(a)}\,* \qquad \dfrac{\forall x\,(\mathcal{Q}(x) \Rightarrow \mathcal{P}(x))}{\mathcal{Q}(a) \Rightarrow \mathcal{P}(a)}\,\forall\text{E}}{\mathcal{P}(a)}\,\Rightarrow\text{E}}{\mathcal{Q}(a) \wedge \mathcal{P}(a)}\,\wedge\text{I}}{\exists x\,(\mathcal{Q}(x) \wedge \mathcal{P}(x))}\,\exists\text{I}}{\exists x\,(\mathcal{Q}(x) \wedge \mathcal{P}(x))}\,*\exists\text{E}$$

□

Exercise 49: Deduction rules: ∀I and ∃E

1. Prove each of the following inference forms, making use of the ∀I rule.
   (a) $\forall x\,\mathcal{Q}(x) \vdash \forall x\,\neg\mathcal{P}(x) \vee \mathcal{Q}(x)$
   (b) $\forall x\,\forall y\,\mathcal{P}(x,y) \vdash \forall y\,\forall x\,\mathcal{P}(x,y)$
2. Prove each of the following inference forms, making use of the ∃E rule.
   (a) $\forall x\,\mathcal{P}(x), \exists x\,(\mathcal{P}(x) \Rightarrow \mathcal{Q}(x)) \vdash \exists x\,\mathcal{Q}(x)$
   (b) $\exists x\,\exists y\,\mathcal{P}(x,y) \vdash \exists y\,\exists x\,\mathcal{P}(x,y)$

7 First Order Theories

7.1 First order logic with identity

The logic we have been considering in this book is commonly known as **first-order predicate logic**, or just simply **first-order logic**. We have developed a system of natural deduction in order that we may draw valid conclusions from a set of premisses in first order logic. One notion we have not looked at so far, however, is that of **identity**, also known as **equality**. In this section we shall introduce this notion and define it by extending our rules of deduction to include introduction and elimination rules for identity. The resulting logic is thus known as **first order logic with identity** or as **first order logic with equality**.

We have seen that an object can be referred to by a term. Thus it is possible that two different terms t_1 and t_2 can refer to the same object. We can express this by the **identity relation** $\ldots = \_\_\_$

$$t_1 = t_2$$

For example, if a and $f(b)$ both refer to the same object, we would write $a = f(b)$.

Example 7.1

Superstitious actors will not refer to Shakespeare's play '*Macbeth*' by that name but instead prefer to call it '*The Scottish play*'. We can express this using the

'*Macbeth*' = '*The Scottish play*'

Another example of identity, this time involving a function, is:

author('*Macbeth*') = '*William Shakespeare*'

though there are some scholars who would maintain that this proposition is false. □

So far we have dealt briefly with the semantics of identity. Much has been written on the semantics of identity, but in this book we shall largely accept the basic intuitive concept outlined above. Rather we shall concentrate more on the deductive properties of identity. In addition to the deduction rules of first order predicate logic, we have the following rules for identity.

Rule: =I

$\vdash \mathsf{t} = \mathsf{t}$ □

Justification

The introduction rule for identity is simply a statement that any term is equal to itself. □

Rule: =E

$\mathsf{t}_1 = \mathsf{t}_2,\ \mathcal{P}(\mathsf{t}_1) \vdash \mathcal{P}(\mathsf{t}_2)$ □

Justification

This deduction rule is valid, since if t_1 and t_2 are equal then they refer to the same object or value; hence $\mathcal{P}(\mathsf{t}_1)$ and $\mathcal{P}(\mathsf{t}_2)$ will have the same truth values, and in particular

$$\mathcal{P}(\mathsf{t}_1) \vDash \mathcal{P}(\mathsf{t}_2)$$

□

Example 7.2

Prove that $\vdash 2 = 2$.

Solution

Instantiating t to 2 in =E we get $\vdash 2 = 2$. □

Example 7.3

We can show that $\forall x\,(\mathcal{P}(f(x))), \exists x\,(f(x) = b) \vdash \mathcal{P}(b)$

Solution

$$\cfrac{\cfrac{\cfrac{\forall x\,\mathcal{P}(f(x))}{\mathcal{P}(f(a))}\,\forall E \qquad \cfrac{}{f(a) = b}\,*}{\mathcal{P}(b)}\,=E \qquad \exists x\,(f(x) = b)}{\mathcal{P}(b)}\,*\exists E$$

□

Exercise 50: Reasoning about identity

Prove each of the following:

1. $\forall x\,\mathcal{P}(x, f(x)),\ \forall x\,\exists y\,(x = g(y)) \vdash \forall x\,\exists y\,\mathcal{P}(g(y), f(x))$
2. $\exists x\,\exists y\,(x = f(y)),\ \forall x\,\mathcal{P}(x) \vdash \exists y\,\mathcal{P}(f(y))$
3. $\forall x\,(x = f(f(x)) \vee x = a),\ \forall x\,\mathcal{P}(f(f(f(x)))) \vdash \forall x\,(\mathcal{P}(f(x)) \vee x = a)$

7.2 Theories

Before developing some examples of **first order theories** it is useful to summarize our understanding of what a theory is.

A theory is a body of knowledge. Thus we can talk about the theory of music, network theory or the theory of relativity. A body of knowledge consists of all the true statements that can be made about the area concerned. For example, the theory of integer arithmetic consists of all the true statements that can made about whole numbers. Such statements include:

$$
\begin{aligned}
1 + 1 &= 2 \\
45^2 - 2 \times (-78) &= 2181 \\
\forall x \, \exists y \; y &> x \\
736/32 &= 23 \\
\sqrt{64} &= 8
\end{aligned}
$$

A theory can be about a specific entity such as a network. Alternatively it can be about a collection of entities, in which case the theory describes the general properties shared by all such entities; thus network theory is the body of knowledge which applies to all networks.

All theories are infinite, that is there are infinitely many true statements we can make about anything. The reason for this is simple. Any statement which is an instantiation of a tautology must be true by definition. But there are infinitely many tautologies. Hence there must be infinitely many true statements. The outcome of this is that it is not possible in practice to store a complete theory in a computer system, or in a book. Now, we have seen how we can use logic to obtain new items of information from existing ones by means of deduction. We can use this to reduce the number of items of information we need to hold, and to regenerate the missing items by logic. A common approach is to choose a set of **axiom**s, which are basic truths of the theory, and to prove other results using first order logic. We say that the axioms define a **first order theory**. Ideally the first order theory will not give rise to any statement which is false, that is the first order theory will be sound. Furthermore, we also want to be able to prove all true results using the first order theory, that is we want the first order theory to be complete. Unfortunately, first order theories are not always sound and complete; in particular, it can be shown that it is impossible for a first order theory of theory of arithmetic to be both sound and complete.

In spite of the limitation of first order theories, they are fundamental to much of mathematics and computing. In this chapter we shall first look at ways in which we can build a theory of digital circuits, and then at a formal definition of a first order theories.

7.3 Digital circuits

Digital circuits, as used in computer hardware for example, depend upon the idea that any point in a circuit can be at one of two voltages: a high voltage (typically 5V), or a low voltage (typically 0V). The voltage at certain points in a circuit (the outputs) are determined by the voltages at other points (the inputs). Thus an output voltage can be expressed as a function of one or more input voltages. Now, more complex circuits can be built up from simpler circuits, which in turn can be built up from even simpler circuits; for most modern circuits this process can be repeated until we end up with a collection of just one basic type circuit known as the ***NAND*-gate**.

Sheffer's stroke

It is possible to develop a theory of digital circuits based around *NAND*-gates. A *NAND*-gate has two inputs and one output. Thus the output is a function of two inputs. One way of writing this function is to use a special symbol | known as **Sheffer's stroke**. The Sheffer's stroke is placed between its two arguments: $a|b$ is the result of applying the Sheffer function to (a, b). In addition to the Sheffer's stroke, we also need symbols to denote the high voltage and the low voltage. One possibility would be to use $\top$ for the high voltage and $\bot$ for the low voltage. We can define a theory for the Sheffer function by using three axioms.

S1 $\forall x\,(x = \bot \vee x = \top)$ This axiom is simply a statement of the fact that only two values are possible namely $\bot$ and $\top$.

S2 $\forall x\,(x|\bot = \top \wedge \bot|x = \top)$ This axiom states that whenever either input is a low voltage ($\bot$) the output will be a high voltage ($\top$).

S3 $\top|\top = \bot$ This axiom states that when both inputs are high ($\top$), the output will be low ($\bot$).

Thus the axioms define the behaviour of the Sheffer stroke under all possible conditions. In combination with the rules of deduction for first order logic with equality, we can deduce the behaviour of any circuit built up from *NAND*-gates.

Example 7.4

Suppose we have a circuit of two *NAND*-gates such that the output from the first gate is applied to the first input of the second gate, while the second input of each gate is connected to a high voltage $\top$. The only input not fixed is the first input to the first gate; refer to this input as x. Prove that the output will be the same as the input.

Solution

The output of the whole circuit will be equal to $(x|\top)|\top$. We can prove from our first order theory of the Sheffer's stroke that

$$(x|\top)|\top = x$$

or more formally that

$$\Gamma_| \vdash \forall x\,((x|\top)|\top = x)$$

where $\Gamma_|$ denotes the set of axioms for the Sheffer function. The deduction tree is shown in Figure 7.1. □

From this last example we see that the proof of a simple result can be complicated. Yet in this proof, all that we are essentially doing is to evaluate $(x|\top)|\top$ for all possible values of x and show that in each case the result is equal to x. This can be done more conveniently in the form of a table in which we make use of the following facts.

- $\bot|\bot = \top$, $\bot|\top = \top$ and $\top|\bot = \top$ from the axiom S2.
- $\top|\top = \bot$ which is axiom S3.

Example 7.5

Calculate the value of the output for each of the two possible input values and hence show that the output is always equal to the input.

Solution

x	$x\|\top$	$(x\|\top)\|\top$
$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\top$

From this we see that in both cases $(x|\top)|\top = x$; that is, $\Gamma_| \vdash \forall x\,((x|\top)|\top = x)$ □

It also makes for easier reading to use 0 to represent the low voltage and 1 to represent the high voltage. This is a common convention; no problem should arise as long as it is understood that these new symbols represent voltage states and are not numbers.

Example 7.6

Calculate the values of $x|y$ and $y|x$ for every possible combination of values for x and y. Hence show that $\Gamma_| \vdash \forall x\,(x|y = y|x)$.

Solution

x	y	$x\|y$	$y\|x$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

From this example we see that $x|y = y|x$ for all possible combinations of values of x and y, that is $\Gamma_| \vdash \forall x\,(x|y = y|x)$ □

$$
\dfrac{
\dfrac{
\dfrac{\perp|\top = \top \qquad \top|\top = \perp}{(\perp|\top)|\top = \perp}\,{=}\mathrm{E}
\qquad
\dfrac{}{a = \perp}\,*
}{(a|\top)|\top = a}\,{=}\mathrm{E}
\qquad
\dfrac{
\dfrac{\top|\top = \perp \qquad \dfrac{\dfrac{\forall x\,(x|\perp = \top \wedge \perp|x = \top)}{\top|\perp = \top = \top \wedge \perp|\top}\,\forall\mathrm{E}}{\perp|\top = \top}\,\wedge\mathrm{E}_2}{(\top|\top)|\top = \top}\,{=}\mathrm{E}
\qquad
\dfrac{}{a = \top}\,*
}{(a|\top)|\top = a}\,{=}\mathrm{E}
\qquad
\dfrac{\forall x\,(x = \perp \vee x = \top)}{a = \perp \vee a = \top}\,\forall\mathrm{E}
}{
\dfrac{(a|\top)|\top = a}{\forall x\,((x|\top)|\top = x)}\,\forall\mathrm{I}
}\,*\vee\mathrm{E}
$$

Figure 7.1: Solution to Example 7.4

We can also make use of previously proven results. For example we know that $x|y = y|x$ for all values of x and y. This result is called a theorem of the first order theorem. In order to refer to this theorem in subsequent work, we need to give it a name: we call it the **commutative law** for $|$ and use the abbreviation $Cm|$ in proofs.

Example 7.7

Prove that $\Gamma_| \vdash \forall x\, \forall y\, ((x|y)|1 = (y|x)|1)$.

Solution

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(a|b)|1 = (a|b)|1}\,{=}\mathrm{I} \qquad \dfrac{\dfrac{\dfrac{\Gamma_|}{\forall x\, \forall y\, x|y = y|x}\,Cm|}{\forall y\, a|y = y|a}\,\forall\mathrm{E}}{a|b = b|a}\,\forall\mathrm{E}}{(a|b)|1 = (b|a)|1}\,{=}\mathrm{E}}{\forall x\, (x|b)|1 = (b|x)|1}\,\forall\mathrm{I}}{\forall x\, \forall y\, (x|y)|1 = (y|x)|1}\,\forall\mathrm{I}$$

□

From the last example, we see we have yet another theorem relating to a circuit in which the output from a *NAND*-gate becomes the first input to a second *NAND*-gate. If the second input to the second *NAND*-gate is fixed at 1, then the complete circuit has two variable inputs and one output. This new circuit is common in circuit design; it is known as an ***AND*-gate.** Its behaviour corresponds to a function. The conventional way of denoting this function is to place a $\cdot$ symbol between the two input values:

$$x \cdot y = (x|y)|1$$

Thus the theorem proved in the last example can be written as

$$\Gamma_| \vdash \forall x\, \forall y\, (x \cdot y = y \cdot x)$$

This is the commutative law for $\cdot$, and is denoted by $Cm\cdot$.

Other circuits can be devised and theorems proved to express properties of these circuits. One of the simplest circuits is the ***NOT*-gate** which is a single *NAND*-gate with the second input fixed at 1; the function corresponding to a *NOT*-gate is represented by placing a bar over the argument.

$$\overline{x} = x|1$$

From example 7.3 we see that

$$\forall x\, (\overline{\overline{x}} = x)$$

Another circuit involving three *NAND*-gates is the *OR*-gate; if the input values are x and y then the output is represented by $x + y$ and is defined as

$$x + y = (x|1)|(y|1)$$

Simplifying the presentation of deductions

In our deductions we use universal quantifier elimination on the axioms at the start and universal quantifier introduction at the end. Essentially the deduction depends upon the middle part, which has no quantifiers but only terms (such as arbitrary constants). We can simplify the first order theory by stating the axioms without quantifiers, and using the term $\mathfrak{t}$ instead of the variables x

S1 $\mathfrak{t} = 0 \vee \mathfrak{t} = 1$

S2 $\mathfrak{t}|0 = 1 \wedge 0|\mathfrak{t} = 1$

S3 $1|1 = 0$.

The term $\mathfrak{t}$ can be replaced by any arbitrary constant or by any function of terms and arbitrary constants.

Example 7.8

What is the result of replacing $\mathfrak{t}$ by a in axioms S1 and S2?

Solution

S1 gives $a = 0 \vee a = 1$ and S2 gives $a|0 = 1 \wedge 0|a = 1$. □

Example 7.9

What is the result of replacing $\mathfrak{t}$ by $(\mathfrak{t}_1|\mathfrak{t}_2)$ in axioms S1 and S2?

Solution

S1 gives $(\mathfrak{t}_1|\mathfrak{t}_2) = 0 \vee (\mathfrak{t}_1|\mathfrak{t}_2) = 1$ and S2 gives $(\mathfrak{t}_1|\mathfrak{t}_2)|0 = 1 \wedge 0|(\mathfrak{t}_1|\mathfrak{t}_2) = 1$. □

Furthermore, since we are always using the same premisses throughout, we can drop the $\Gamma_| \vdash$ from each theorem we prove. Thus we can write

$$\mathfrak{t}_1|\mathfrak{t}_2 = \mathfrak{t}_2|\mathfrak{t}_1$$

for the commutative law *Cm*| instead of the more cumbersome

$$\Gamma_| \vdash \forall x\,(\forall y\,(x|y = y|x))$$

Example 7.10

Prove the commutative law $\Gamma_| \vdash \forall x\,\forall y\,(x \cdot y = y \cdot x)$ by deducing

$$\mathfrak{t}_1 \cdot \mathfrak{t}_2 = \mathfrak{t}_2 \cdot \mathfrak{t}_1$$

Solution

Since by definition $t_1 \cdot t_2 = (t_1|t_2)|1$, we can solve this problem by first deducing $(t_1|t_2)|1 = (t_2|t_1)|1$.

$$\dfrac{\dfrac{\dfrac{}{(t_1|t_2)|1 = (t_1|t_2)|1}\ {=}\mathrm{I} \qquad \dfrac{}{t_1|t_2 = t_2|t_1}\ Cm|}{(t_1|t_2)|1 = (t_2|t_1)|1}\ {=}\mathrm{E}}{t_1 \cdot t_2 = t_2 \cdot t_1}\ Df\cdot$$

□

Hence we can rewrite the axioms for our first order theory in the following form:

S1 $t = 0 \vee t = 1$

S2 $t|0 = 1 \wedge 0|t = 1$

S3 $1|1 = 0$

In addition, we can define three new operations in terms of the Sheffer's stroke:

Df $\overline{}$ $\overline{t} = t|1$. This is the definition of *NOT* in terms of the Sheffer's stroke. It can be used in definitions for *AND* and *OR*.

Df $\cdot$ $t_1 \cdot t_2 = \overline{t_1|t_2}$. to $(t_1|t_2)|1$.

Df $+$ $t_1 + t_2 = \overline{t_1}\,|\,\overline{t_2}$.

Note that the definition of *NOT* also enables us to rewrite axiom S3 as

S3 $\overline{1} = 0$

In addition to these axioms, we also have the commutative law for Sheffer's stroke and the cancellation law for *NOT* :

Cm $|$ $t_1|t_2 = t_2|t_1$

DN $\overline{\overline{t}} = t$

Example 7.11

Prove the following law due to de Morgan: $\overline{t_1 + t_2} = \overline{t}_1 \cdot \overline{t}_2$.

Solution

$$\dfrac{\dfrac{}{t_1 + t_2 = \overline{t_1}|\overline{t_2}}\ Df+ \qquad \dfrac{}{\overline{t_1} \cdot \overline{t_2} = \overline{\overline{\overline{t_1}}|\overline{\overline{t_2}}}}\ Df\cdot}{\overline{t_1} \cdot \overline{t_2} = \overline{t_1 + t_2}}\ {=}\mathrm{E}$$

□

Exercise 51: First Order Theory of Sheffer's Stroke

1. Use the revised version of the axioms and definitions (just given) together with the commutative law for | to prove each of the following laws:
   (a) *Cm*$\cdot$ $\quad t_1 \cdot t_2 = t_2 \cdot t_1$, the commutative law for *AND*.
   (b) *Cm*$+$ $\quad t_1 + t_2 = t_2 + t_1$, the commutative law for *OR*.
2. Prove following theorem due to de Morgan using the revised version of the axioms and definitions together with the commutative laws for |, $\cdot$ and +: $\overline{t_1 \cdot t_2} = \overline{t_1} + \overline{t_2}$. (Note that not all the axioms and laws may be needed.)

7.4 Equational theories

In this section we shall look at a simple style of presenting deductions which does not involve deduction trees. The approach depends upon the **symmetry** of equality:

$$t_1 = t_2 \vdash t_2 = t_1$$

We shall refer to this theorem of first order logic as *Sym* =. In particular, the =E and *Sym* = rules enable us to make successive substitutions in a formula.

Example 7.12
Show that $\mathcal{P}(t_1), t_1 = t_2, t_3 = t_2 \vdash \mathcal{P}(t_3)$.

Solution

$$\dfrac{\dfrac{\mathcal{P}(t_1) \qquad t_1 = t_2}{\mathcal{P}(t_2)}\,{=}\mathrm{E} \qquad \dfrac{t_3 = t_2}{t_2 = t_3}\,\mathit{Sym}{=}}{\mathcal{P}(t_3)}\,{=}\mathrm{E}$$

We see that the premiss $t_1 = t_2$ enables us to substitute t_2 for t_1 in $\mathcal{P}(t_1)$ to give $\mathcal{P}(t_2)$; subsequently, the premiss $t_3 = t_2$ enables us to substitute t_3 for t_2 in $\mathcal{P}(t_2)$ to give $\mathcal{P}(t_3)$. These substitutions can be summarized in the following form

$$\begin{array}{lll} & \mathcal{P}(t_1) & \\ = & \mathcal{P}(t_2) & \langle\ t_1 = t_2\ \rangle \\ = & \mathcal{P}(t_3) & \langle\ t_3 = t_2\ \rangle \end{array}$$

□

Example 7.13
Prove that for arbitrary constants a_1, a_2, b_1, b_2 and c

$$f((a_1, a_2)) = g(b_1), g(b_2) = h(c), y_2 = d, y_1 = d \vdash f((a_1, a_2)) = h(c)$$

Solution

$$\dfrac{\dfrac{\dfrac{f((a_1,a_2)) = g(b_1) \quad b_1 = c}{f((a_1,a_2)) = g(d)}\,{=}\mathrm{E} \qquad \dfrac{b_1 = d}{d = b_1}\,Sym{=}}{f((a_1,a_2)) = g(b_2)}\,{=}\mathrm{E} \qquad g(b_2) = h(c)}{f((a_1,a_2)) = h(c)}\,{=}\mathrm{E}$$

This deduction can be summarized as follows:

$$\begin{aligned} f((a_1,a_2)) &= g(b_1) \\ &= g(d) \quad \langle\ b_1 = d\ \rangle \\ &= g(b_2) \quad \langle\ b_2 = d\ \rangle \\ &= h(z) \quad \langle\ g(y_2) = h(z)\ \rangle \end{aligned}$$

□

Thus we can see how a deduction tree can be summarized more simply using a succession of equalities. This style of presentation is often to as **equational logic**. Given a deduction in equational form we can always reconstruct the full deduction tree.

Equational logic for Sheffer's stroke

In order to build an equational theory for Sheffer's stroke, we need to rewrite the axioms of the theory as equations. Axiom S3 is already an equation: $\overline{1} = 0$. Axiom S2 is not a simple equation but the conjunction of two equations; thus we can split S2 into two separate axioms, each of which is an equation.

S2a $t|0 = 1$

S2b $0|t = 1$

Unfortunately, axiom S1 is the disjunction of two equations and cannot be split into separate equalities like S2. We must therefore omit axiom S1 from the equational theory.

Example 7.14

Prove the following law due to de Morgan.

$$\overline{t_1 + t_2} = \overline{t}_1 \cdot \overline{t}_2$$

Solution

$$\begin{aligned} &\overline{t_1 + t_2} \\ = \ &\overline{\overline{\overline{t_1}\,|\,\overline{t_2}}} \quad \langle\ Df{+}\ \rangle \\ = \ &\overline{t_1} \cdot \overline{t_2} \quad \langle\ Df{\cdot}\ \rangle \end{aligned}$$

□

Unfortunately the equational theory does not enable us to deduce everything in the first order theory of Sheffer's stroke. This is because we have left out axiom S1 of the first order theory; the axiom which states that there are only two possible values, labelled 0 and 1. Now we know that from axioms S1, S2 and S3 we can check any theorem of the non-equational theory by writing down a table for all possible combinations of values. Thus we could write, for example,

x	y	$x\|y$	$\overline{x}$	$\overline{y}$	$x \cdot y$	$x + y$	$\overline{x} \cdot \overline{y}$	$\overline{x + y}$
0	0	1	1	1	0	0	1	1
0	1	1	1	0	0	1	0	0
1	0	1	0	1	0	1	0	0
1	1	0	0	0	1	1	0	0

From this table we can see that $\forall x\, \forall y\, (\overline{x} \cdot \overline{y} = \overline{x + y})$, that is that

$$\overline{t_1} \cdot \overline{t_2} = \overline{t_1 + t_2}$$

Without the restriction of S1 in our equational theory, however, we can no longer assume that there are just two possible values. For example, we could imagine a theory of circuits in which there are *three* possible voltage states labelled 0, 1 and 2. As before, $x|y$ could be used to represent the output of a circuit with inputs x and y, but its behaviour would be different to that described above. There are many different possible behaviours we could choose for $x|y$ in this three value theory. For example, $x|y$ might behave as given in the following table:

x	y	$x\|y$
0	0	1
0	1	1
0	2	1
1	0	1
1	1	0
1	2	0
2	0	1
2	1	0
2	2	2

Thus we have a different **interpretation** for our equational logic. Choosing different sets of possible values, or different tables of behaviour for | will lead to other interpretations; in some of these interpretations, we will find that all the axioms S2a, S2b and S3 will be true. In particular for the interpretation just given, we can show that all the axioms are true. For example we can show that

$$\overline{t_1} \cdot \overline{t_2} = \overline{t_1 + t_2}$$

In other words, we can show that $\forall x\ \forall y\ (\overline{x} \cdot \overline{y} = \overline{x + y})$.

x	y	$x \mid y$	$\overline{x}$	$\overline{y}$	$x \cdot y$	$x + y$	$\overline{x} \cdot \overline{y}$	$\overline{x + y}$
0	0	1	1	1	0	0	1	1
0	1	1	1	0	0	1	0	0
0	2	1	1	0	0	1	0	0
1	0	1	0	1	0	1	0	0
1	1	0	0	0	1	1	0	0
1	2	0	0	0	1	1	0	0
2	0	1	0	1	0	1	0	0
2	1	0	0	0	1	1	0	0
2	2	2	0	0	0	1	0	0

We say that an interpretation for which all the axioms are true is a **model** for that set of axioms.

Thus we have found two different models for set of equational axioms. This in itself would not be a problem if every property of either model was shared by the other; but this is not the case. In one model, there are two possible values, while in the other model there are three possible values. We also know that for our two value model

$$\forall x\ (x \cdot x = x)$$

that is that $\mathtt{t} \cdot \mathtt{t} = \mathtt{t}$. This property is not true, however, for our three value model:

x	$x \cdot x$
0	0
1	1
2	0

Now if this equality could be deduced from the axioms S2a, S2b and S3 using equational reasoning, then it would be true for *any* model. Thus we conclude that we cannot deduce $\mathtt{t} \cdot \mathtt{t} = \mathtt{t}$ from our set of equational axioms: the set of equational axioms does not give us a complete theory for our (two value) digital circuits. Fortunately, it is possible to extend the equational theory by adding extra equalities to the existing axioms. For example we could add the equality $\mathtt{t} \cdot \mathtt{t} = \mathtt{t}$, known as the **idempotent rule** for $\cdot$. Our three value interpretation is not a model of this extended theory. By extending our theory with appropriate equalities, we can arrive at a set of equalities from which we can deduce any equality which is true for our (two value) digital circuits. This is most conveniently done in terms of the operators $\cdot$, $+$ and $\overline{}$ rather than the operator $\mid$. If required, Sheffer's stroke can be defined by

$$x \mid y = \overline{x + y}$$

We shall look at this theory in the next section.

7.5 Boolean algebras

It is possible to describe digital circuits using the following set of equational axioms.

B1 $t_1 \cdot t_2 = t_2 \cdot t_1$

B2 $t_1 + t_2 = t_2 + t_1$

B3 $t_1 + (t_2 + t_3) = (t_1 + t_2) + t_3$

B4 $t_1 \cdot (t_2 \cdot t_3) = (t_1 \cdot t_2) \cdot t_3$

B5 $t_1 \cdot (t_2 + t_3) = (t_1 \cdot t_2) + (t_1 \cdot t_3)$

B6 $t_1 \cdot (t_1 + t_2) = t_1$

B7 $t_1 + (t_1 \cdot t_2) = t_1$

B8 $t + \bar{t} = 1$

B9 $t \cdot \bar{t} = 0$

B10 $t + 1 = 1$

B11 $t \cdot 0 = 0$

Any model for this set of axioms is called a **boolean algebra.** One such boolean algebra is provided by the theory of digital circuits. Other boolean algebras are possible, but it can be shown that the number possible different values is always a power of two: 2, 4, 8, 16, ... We can now prove some important results about boolean algebras.

Example 7.15
Show that $a \cdot 1 = a$.

Solution

$$
\begin{array}{lll}
 & a \cdot 1 & \\
= & a \cdot (a + 1) & \langle\ B10\ \rangle \\
= & a & \langle\ B6\ \rangle
\end{array}
$$

□

Example 7.16
Show that $a \cdot a = a$.

Solution

$$
\begin{array}{lll}
 & a \cdot a & \\
= & a \cdot (a + (a \cdot 1)) & \langle\ B7\ \rangle \\
= & a & \langle\ B6\ \rangle
\end{array}
$$

□

Example 7.17
Show that $a = a + 0$.

Solution

$$\begin{array}{lll} & a & \\ = & a \cdot 1 & \langle\ Example\ 7.15\ \rangle \\ = & a \cdot (a + \overline{a}) & \langle\ B8\ \rangle \\ = & a \cdot a + a \cdot \overline{a} & \langle\ B5\ \rangle \\ = & a + 0 & \langle\ B9\ \rangle \end{array}$$

□

The results in these examples, and some other similar results, suggest the following additional axioms for boolean algebras.

B12 $t \cdot 1 = t$

B13 $t + 0 = t$

B14 $t \cdot t = t$

B15 $t + t = t$

B16 $\overline{\overline{t}} = t$

B17 $t_1 + (t_2 \cdot t_3) = (t_1 + t_2) \cdot (t_1 + t_3)$

B18 $\overline{t_1 + t_2} = \overline{t_1} \cdot \overline{t_2}$

B19 $\overline{t_1 \cdot t_2} = \overline{t_1} + \overline{t_2}$

Experience shows that these nineteen axioms are more convenient for developing the theory of boolean algebras, than simply restricting ourselves to just B1–B11.

Exercise 52: Boolean algebras

1. Prove each of the following using only axioms taken from B1–B14.

   (a) $a + a = a$

   (b) $(a + b) \cdot (a + c) = a + b \cdot c$

2. Prove, using only axioms taken from B1–B18, that $\overline{a \cdot b} = \overline{a} + \overline{b}$

3. Prove, using only axioms taken from B1–B15, that $\overline{a} \cdot \overline{b} = \overline{a + b}$

7.6 Equational theory of logic

In this book we have been developing a body of knowledge, namely logic. In the broadest sense, we have a theory of logic. In this section we explore the extent to which logic itself can be presented as a first order theory. Essentially we shall develop a first order theory of logic derived from the theory of boolean algebras.

Boolean logic

We have already defined an equational theory of boolean algebras. We saw that one interpretation of these algebras is the theory of digital circuits. Historically, however, the first interpretation was propositional logic itself. In this earlier interpretation, the letters a, b and c represent propositions, the symbols $\bar{\ }$, $\cdot$ and $+$ represent negation, conjunction and disjunction respectively, and the constants 0 and 1 represent falsehood and truth respectively. As with the circuit interpretation, there are only two possible (truth) values. By convention, a boolean algebra with just two possible values is called **boolean logic**. Although propositional logic is not the only possible interpretation of a boolean logic, we still refer to 'truth tables'.

Example 7.18

Find the truth table for the boolean expression $a \cdot b + a \cdot c$.

Solution

The truth table shows the value of $a \cdot b + a \cdot c$ for every possible combination of values of a, b and c.

a	b	c	$a \cdot b$	$a \cdot c$	$a \cdot b + a \cdot c$
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	1

□

One limitation of boolean logic is that there are no symbols for the conditional connective or biconditional. Although it *would* be possible to define appropriate symbols in terms of , $\cdot$ and $+$, the simplicity of boolean logic would be lost; we may just as well use the notation introduced in earlier chapters, and indeed this is just what we shall do.

Exercise 53: Boolean logic

1. Calculate using Boolean arithmetic:

   (a) $1 \cdot 0$

   (b) $1 + 1$

   (c) $1 + 1 + 1$

   (d) $(1 + 1) \cdot 0$

   (e) $1 + 1 \cdot 0$

   (f) $1 \cdot 1 + 0 \cdot 0$

2. Find the truth tables for the following Boolean expressions:
   (a) $a + (b \cdot a)$
   (b) $a \cdot (b + a)$
   (c) $\overline{a + (b \cdot a)}$
   (d) $\overline{a \cdot (b + a)}$
   (e) $\overline{(a \cdot b)} + (\overline{a} \cdot \overline{b})$
   (f) $\overline{a} + (\overline{b} \cdot \overline{a})$
   (g) $\overline{a} \cdot (\overline{b} + \overline{a})$
   (h) $a + (b \cdot c)$
   (i) $a \cdot (b + c)$
   (j) $\overline{a + (b \cdot c)}$
   (k) $\overline{a \cdot (b + c)}$
   (l) $\overline{a} + (\overline{b} \cdot \overline{c})$
   (m) $\overline{a} \cdot (\overline{b} + \overline{c})$
   (n) $a \cdot b \cdot \overline{c} + \overline{a} \cdot b \cdot c + a \cdot \overline{b} \cdot c$
   (o) $(\overline{a} + b + c) \cdot (a + \overline{b} + c) \cdot (a + b + \overline{c})$
   (p) $\overline{a \cdot b + c \cdot d} + a \cdot b \cdot c \cdot \overline{d + a}$
   (q) $\overline{\overline{a + b \cdot c} + a \cdot \overline{b + c}}$
3. Which of the above expressions are equal to each other?
4. Where possible, simplify each of the above expressions.

Equational theory of propositional forms

We can now develop an equational theory of propositional forms that is based upon boolean logic but which uses our familiar notation.

- schematic letters such as $\mathcal{P}$, $\mathcal{Q}$ instead of letters a and b;
- constants T and F instead of 1 and 0;
- connectives $\wedge$, $\vee$ and $\neg$ instead of $\cdot$, $+$ and $\overline{}$ – the connectives are now treated as functions;
- the symbol $=_T$ instead of the equals sign $=$. Like equality, semantic equivalence is an identity.

Exercise 54: Converting boolean logic to propositional logic

Rewrite each of the following Boolean expressions using the conventional notation of propositional logic, for example:

- $\mathcal{P} \wedge \mathcal{Q}$ instead of $a \cdot b$
- $\mathcal{R} \vee \mathcal{S}$ instead of $c + d$

1. $a + (b + c)$
2. $(a + b) + c$
3. $a \cdot (b \cdot c)$
4. $(a \cdot b) \cdot c$
5. $(a + b) \cdot c$
6. $(a \cdot c) + (b \cdot c)$
7. $a + (b \cdot c)$
8. $(a + b) \cdot (a + c)$
9. $\overline{a + (b \cdot c)}$
10. $\overline{a} \cdot (\overline{b} + \overline{c})$
11. $\overline{a \cdot (b + c)}$
12. $\overline{a} + (\overline{b} \cdot \overline{c})$
13. $\overline{(a \cdot b) + (c \cdot d)}$
14. $\overline{\overline{a + b \cdot c} + a \cdot \overline{b + c}}$

In addition to $+$ and $\cdot$, we can define two further functions:

- the conditional $\Rightarrow$, defined by

$$\mathcal{P} \Rightarrow \mathcal{Q} =_T \neg\mathcal{P} \vee \mathcal{Q}$$

- the biconditional $\Leftrightarrow$, defined by

$$\mathcal{P} \Leftrightarrow \mathcal{Q} =_T (\mathcal{P} \wedge \mathcal{Q}) \vee (\neg\mathcal{P} \wedge \neg\mathcal{Q})$$

Using our familiar notation the axioms of boolean logic now become:

Commutative

L1 $\mathcal{P} \wedge \mathcal{Q} =_T \mathcal{Q} \wedge \mathcal{P}$

L2 $\mathcal{P} \vee \mathcal{Q} =_T \mathcal{Q} \vee \mathcal{P}$

Associative

L3 $\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R}) =_T (\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}$

L4 $\mathcal{P} \vee (\mathcal{Q} \vee \mathcal{R}) =_T (\mathcal{P} \vee \mathcal{Q}) \vee \mathcal{R}$

Distributive

L5 $\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}) =_T (\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \mathcal{R})$

L6 $\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R}) =_T (\mathcal{P} \vee \mathcal{Q}) \wedge (\mathcal{P} \vee \mathcal{R})$

Absorption

L7 $\mathcal{P} \wedge (\mathcal{P} \vee \mathcal{Q}) =_T \mathcal{P}$

L8 $\mathcal{P} \vee (\mathcal{P} \wedge \mathcal{Q}) =_T \mathcal{P}$

Unit

L9 $\mathcal{P} \wedge T =_T \mathcal{P}$

L10 $\mathcal{P} \vee F =_T \mathcal{P}$

Zero

L11 $\mathcal{P} \wedge F =_T F$

L12 $\mathcal{P} \vee T =_T T$

Complement

L13 $\mathcal{P} \wedge \neg\mathcal{P} =_T F$

L14 $\mathcal{P} \vee \neg\mathcal{P} =_T T$

Idempotent

L15 $\mathcal{P} \wedge \mathcal{P} =_T \mathcal{P}$

L16 $\mathcal{P} \vee \mathcal{P} =_T \mathcal{P}$

Double negation

L17 $\neg\neg\mathcal{P} =_T \mathcal{P}$

de Morgan

L18 $\neg(\mathcal{P} \vee \mathcal{Q}) =_T \neg\mathcal{P} \wedge \neg\mathcal{Q}$

L19 $\neg(\mathcal{P} \wedge \mathcal{Q}) =_T \neg\mathcal{P} \vee \neg\mathcal{Q}$

Note we have changed the axiom labelling: in particular the two distributive axioms have been numbered sequentially (L5 and L6). The axioms involving constants have also been reordered. These axioms are in fact no more than a restatement of those in Table 3.1.

We can now begin building our theory of propositional forms; in particular we can prove some basic results for the connectives $\Rightarrow$ and $\Leftrightarrow$. For convenience, axioms will be referred to as far as possible in terms of the properties they represent: L1 will be referred to as $Cm\wedge$, L2 as $Cm\vee$, L3 as $Ass\wedge$ and so on. Apart from this, the working is similar to that presented in earlier chapters.

Example 7.19

Show that that the biconditional is commutative: $\mathcal{P} \Leftrightarrow \mathcal{Q} =_T \mathcal{Q} \Leftrightarrow \mathcal{P}$.

Solution

$$
\begin{array}{lll}
 & \mathcal{P} \Leftrightarrow \mathcal{Q} & \\
=_T & (\mathcal{P} \wedge \mathcal{Q}) \vee (\neg\mathcal{P} \wedge \neg\mathcal{Q}) & \langle\ Df \Leftrightarrow\ \rangle \\
=_T & (\mathcal{Q} \wedge \mathcal{P}) \vee (\neg\mathcal{Q} \wedge \neg\mathcal{P}) & \langle\ Cm \wedge\ \rangle \\
=_T & \mathcal{Q} \Leftrightarrow \mathcal{P} & \langle\ Df \Leftrightarrow\ \rangle
\end{array}
$$

□

This theorem can be used in later proofs, where it can be referred to as $Cm \Leftrightarrow$.

Example 7.20
Show that $\mathcal{P} \Rightarrow \mathcal{Q} =_T \neg\mathcal{Q} \Rightarrow \neg\mathcal{P}$

Solution

$$
\begin{array}{lll}
 & \mathcal{P} \Rightarrow \mathcal{Q} & \\
=_T & \neg\mathcal{P} \vee \mathcal{Q} & \langle\ Df \Rightarrow\ \rangle \\
=_T & \neg\mathcal{P} \vee \neg\neg\mathcal{Q} & \langle\ \neg\neg\ \rangle \\
=_T & \neg\neg\mathcal{Q} \vee \neg\mathcal{P} & \langle\ Cm \vee\ \rangle \\
=_T & \neg\mathcal{Q} \Rightarrow \neg\mathcal{P} & \langle\ Df \Rightarrow\ \rangle
\end{array}
$$

□

Thus the $\Rightarrow$ connective is not commutative. We say that $\neg\mathcal{Q} \Rightarrow \neg\mathcal{P}$ is the **contrapositive** of $\mathcal{P} \Rightarrow \mathcal{Q}$. We have shown that a conditional is equivalent to its contrapositive; we can refer to this theorem as the contrapositive theorem and give it the label *Contra*.

Tautologies and contradictions

We can define the concepts of tautology and contradiction in the equational theory of propositional logic.

Definition 7.1
If $\mathfrak{A} =_T T$ for some propositional form $\mathfrak{A}$, then this form is said to be a tautology. □

Definition 7.2
If $\mathfrak{A} =_T F$ for some propositional form $\mathfrak{A}$, then this form is said to be a contradiction. □

Example 7.21
Show that $\mathcal{P} \Rightarrow \mathcal{P}$ is a tautology.

Solution

$$
\begin{array}{lll}
 & \mathcal{P} \Rightarrow \mathcal{P} & \\
=_T & \neg\mathcal{P} \vee \mathcal{P} & \langle\ Df \Rightarrow\ \rangle \\
=_T & T & \langle\ Cpt \vee\ \rangle
\end{array}
$$

□

Example 7.22

Show that $(\mathcal{P} \vee \neg\mathcal{P}) \Rightarrow (\mathcal{Q} \wedge \neg\mathcal{Q})$ is a contradiction.

Solution

$$\begin{array}{lll} & (\mathcal{P} \vee \neg\mathcal{P}) \Rightarrow (\mathcal{Q} \wedge \neg\mathcal{Q}) & \\ =_T & T \Rightarrow (\mathcal{Q} \wedge \neg\mathcal{Q}) & \langle\ Cpt\vee\ \rangle \\ =_T & T \Rightarrow F & \langle\ Cpt\wedge\ \rangle \\ =_T & \neg T \vee F & \langle\ Df \Rightarrow\ \rangle \\ =_T & F \vee F & \langle\ prop.\ of\ T\ proved\ above\ \rangle \\ =_T & F & \langle\ Idempt\vee\ \rangle \end{array}$$

□

Valid arguments and equivalences

It is also possible to define the concepts of valid argument and equivalence $\equiv$ in the equational theory of propositional logic.

Definition 7.3

$\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n \therefore \mathfrak{B}$ is a valid argument form if and only if

$$\mathfrak{A}_1 \wedge \mathfrak{A}_2 \wedge \ldots \wedge \mathfrak{A}_n \Rightarrow \mathfrak{B} =_T T$$

□

Definition 7.4

$\mathcal{P} \equiv \mathcal{Q}$ if and only if $\mathcal{P} \Leftrightarrow \mathcal{Q} =_T T$. □

Example 7.23

Show that the argument form $\mathcal{P}, \mathcal{P} \Rightarrow \mathcal{Q} \therefore \mathcal{Q}$ is valid.

Solution

$$\begin{array}{lll} & (\mathcal{P} \wedge (\mathcal{P} \Rightarrow \mathcal{Q})) \Rightarrow \mathcal{Q} & \\ =_T & \neg(\mathcal{P} \wedge (\neg\mathcal{P} \vee \mathcal{Q})) \vee \mathcal{Q} & \langle\ Df \Rightarrow\ \rangle \\ =_T & (\neg\mathcal{P} \vee \neg(\neg\mathcal{P} \vee \mathcal{Q})) \vee \mathcal{Q} & \langle\ de\ Morgan\ \rangle \\ =_T & (\neg\mathcal{P} \vee (\neg\neg\mathcal{P} \wedge \neg\mathcal{Q})) \vee \mathcal{Q} & \langle\ de\ Morgan\ \rangle \\ =_T & (\neg\mathcal{P} \vee (\mathcal{P} \wedge \neg\mathcal{Q})) \vee \mathcal{Q} & \langle\ DN\ \rangle \\ =_T & (\neg\mathcal{P} \vee \mathcal{P}) \wedge (\neg\mathcal{P} \vee \neg\mathcal{Q})) \vee \mathcal{Q} & \langle\ Dist\ \rangle \\ =_T & (T \wedge (\neg\mathcal{P} \vee \neg\mathcal{Q})) \vee \mathcal{Q} & \langle\ Cpt\vee\ \rangle \\ =_T & (\neg\mathcal{P} \vee \neg\mathcal{Q}) \vee \mathcal{Q} & \langle\ Unit\wedge\ \rangle \\ =_T & \neg\mathcal{P} \vee (\neg\mathcal{Q} \vee \mathcal{Q}) & \langle\ Ass\vee\ \rangle \\ =_T & \neg\mathcal{P} \vee T & \langle\ Cpt\vee\ \rangle \\ =_T & T & \langle\ Zero\vee\ \rangle \end{array}$$

□

Decidability

Given any argument form $\Gamma \therefore \mathfrak{A}$ of propositional logic we can use the equational theory of propositional logic to determine whether or not the argument form is valid. As a result of this, it can be argued that it must be possible to write an algorithm – a computer program – to determine whether or not any given argument form is valid. Indeed, one approach to writing such an algorithm rests is based testing for semantic entailment using truth tables. We say that propositional logic is **decidable**.

Exercise 55: Equational theory of propositional forms

1. Prove each of the following using equational logic.
   (a) $\neg T =_T F$
   (b) $\neg F =_T T$
   (c) $(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{P} =_T \mathcal{P} \wedge \mathcal{Q}$
   (d) $\mathcal{P} \Rightarrow (\mathcal{P} \wedge \mathcal{Q}) =_T \mathcal{P} \Rightarrow \mathcal{Q}$
   (e) $\mathcal{P} \Rightarrow \mathcal{Q} =_T \neg(\mathcal{P} \wedge \neg\mathcal{Q})$
   (f) $\mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{R}) =_T (\mathcal{P} \wedge \mathcal{Q}) \Rightarrow \mathcal{R}$
   (g) $\mathcal{P} \Leftrightarrow \neg\mathcal{Q} =_T (\mathcal{P} \vee \mathcal{Q}) \wedge (\neg\mathcal{P} \vee \neg\mathcal{Q})$
   (h) $\neg(\mathcal{P} \Leftrightarrow \mathcal{Q}) =_T (\mathcal{P} \vee \mathcal{Q}) \wedge (\neg\mathcal{P} \vee \neg\mathcal{Q})$
   (i) $\neg(\mathcal{P} \Leftrightarrow \mathcal{Q}) =_T \mathcal{P} \Leftrightarrow \neg\mathcal{Q}$
2. Prove that $\mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{P})$ is a tautology. What deduction does this tautology correspond to?

7.7 First order logic

In the first order theory of propositional forms we can identify the variables, constants and functions of the theory:

- one binary predicate namely $=_T$.;
- two constants namely T and F;
- variables $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and so;
- three functions namely $\neg$, $\wedge$ and $\vee$.

Unfortunately, it is not possible to devise a first order theory of first order logic because it is not possible to relate the quantifiers to functions, variable and constants. Besides, such a theory would be tantamount to a first order theory of first order logic; this would seem rather strange! Furthermore, because we cannot write down a first order theory of first order logic, we cannot therefore devise an effective algorithm to check deducibility as described above. Indeed, it can be shown that no such algorithm exists for deciding whether or not *any* given argument form of first order logic is valid.

Although we cannot build a complete equational theory of logic with quantifiers, it is, however, possible to write down useful equivalences and to reason with these.

Commutative

L20 $\forall x\, \forall y\, (\mathcal{P}) =_T \forall y\, \forall x\, (\mathcal{P})$

Distributive

L22 $\forall x\, \mathcal{P} \wedge \mathcal{Q} =_T \forall x\, \mathcal{P} \wedge \forall x\, \mathcal{Q}$

Constants

L24 $\forall x\, T =_T T$

L25 $\forall x\, F =_T F$

Double quantification

L28 $\forall x\, \forall x\, \mathcal{P} =_T \forall x\, \mathcal{P}$

In addition we can define existential quantification in terms of universal quantification.

***Df*$\exists$** $\exists x\, \mathcal{P} =_T \neg \forall x\, \neg \mathcal{P}$

Using this definition we can prove laws for existential quantification corresponding to those for universal quantification.

Commutative

L21 $\exists x\, \exists y\, (\mathcal{P}) =_T \exists y\, \exists x\, (\mathcal{P})$

Distributive

L23 $\exists x\, (\mathcal{P} \vee \mathcal{Q}) =_T \exists x\, \mathcal{P} \vee \exists x\, \mathcal{Q}$

Constants

L26 $\exists x\, T =_T T$

L27 $\exists x\, F =_T F$

Double quantification

L29 $\exists x\, \exists x\, \mathcal{P} =_T \exists x\, \mathcal{P}$

In addition we can add some extra laws which involve both universal and existential quantification.

Example 7.24

Show that $\forall x\, \mathcal{P} \Rightarrow \exists x\, \mathcal{P} =_T T$; in other words, that $\forall x\, \mathcal{P} \Rightarrow \exists x\, \mathcal{P}$ is a tautology.

Solution

$$
\begin{array}{lll}
 & \forall x\, \mathcal{P} \Rightarrow \exists x\, (\mathcal{P}) & \\
=_T & \neg \forall x\, \mathcal{P} \vee \exists x\, \mathcal{P} & \langle\ Df \Rightarrow\ \rangle \\
=_T & \neg \forall x\, (\neg\neg \mathcal{P}) \vee \exists x\, \mathcal{P} & \langle\ \text{double negation}\ \rangle \\
=_T & \exists x\, (\neg \mathcal{P}) \vee \exists x\, \mathcal{P} & \langle\ Df\exists\ \rangle \\
=_T & \exists x\, (\neg \mathcal{P} \vee \mathcal{P}) & \langle\ Dist\exists\ \rangle \\
=_T & \exists x\, T & \langle\ L14\ \rangle \\
=_T & T & \langle\ L26\ \rangle
\end{array}
$$

□

L30 $\forall x\, \mathcal{P} \Rightarrow \exists x\, \mathcal{P} =_T T$

L31 $\forall x\, \mathcal{P} \wedge \exists x\, \mathcal{P} =_T \forall x\, \mathcal{P}$

Exercise 56: Properties of quantifiers

Prove each of the following using L1–L28 and $Df\exists$.

1. $\forall x\, (\neg \mathcal{P}) =_T \neg \exists x\, \mathcal{P}$
2. $\exists x\, (\mathcal{P} \vee \mathcal{Q}) =_T \exists x\, \mathcal{P} \vee \exists x\, \mathcal{Q}$
3. $\exists x\, \exists y\, (\mathcal{P}) =_T \exists y\, \exists x\, (\mathcal{P})$
4. $\exists x\, T =_T T$
5. $\exists x\, F =_T F$

8 An Introduction to Logic Programming

8.1 Limitations of natural deduction

In this book we have been largely concerned with using the system of natural deduction to prove the validity of arguments. Generally speaking, natural deduction is relatively straightforward to use and understand. Unfortunately, it is not always easy to construct deduction trees. The reader might be wondering why no method for constructing deduction trees has been given: the answer is simply that no such method is possible to cover *all* argument forms. Thus, although software tools now exist which can help in the application of natural deduction, it is impossible to automate the process completely; if a person is unable to construct a deduction tree then this *may* mean that the argument form under consideration is not valid, but it may simply mean that the person has not been able to spot the deduction. This leads us to consider whether there is some other automatic method of deciding validity. For propositional logic, one such method is based upon truth tables.

Example 8.1

Verify that $p \wedge q, p \Rightarrow r \therefore q \Rightarrow r$ is a valid argument.

Solution

The corresponding argument form is $\mathcal{P} \wedge \mathcal{Q}, \mathcal{P} \Rightarrow \mathcal{R} \therefore \mathcal{Q} \Rightarrow \mathcal{R}$; determining the validity or otherwise of this argument form is a problem of propositional logic and can be automated by constructing and analysing truth tables. The truth tables in this case are given in Figure 8.1. □

Unfortunately, as discussed briefly in Chapter 7, first order logic is not decidable. Although it is possible for a human to reason using truth values about the validity of an argument form in first order logic, this reasoning cannot be automated.

Example 8.2

Argue that $\forall x\, \mathcal{P} \therefore \exists x\, \mathcal{P}$ is a valid argument form using truth values.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{P} \wedge \mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{R}$	$\mathcal{Q} \Rightarrow \mathcal{R}$
T	*T*	*T*	*T*	*T*	*T*
T	*T*	*F*	*T*	***F***	***F***
T	*F*	*T*	*F*	*T*	*T*
T	*F*	*F*	*F*	*F*	*T*
F	*T*	*T*	*F*	*T*	*T*
F	*T*	*F*	***F***	*T*	***F***
F	*F*	*T*	*F*	*T*	*T*
F	*F*	*F*	*F*	*T*	*T*

Figure 8.1: Truth tables for Example 8.1

Solution

We cannot show that $\forall x\, \mathcal{P} \models \exists x\, \mathcal{P}$ by constructing tables of truth values: to do so would require us to consider all the possible predicates for $\mathcal{P}$ (and associated universes of discourse). Furthermore, for each predicate we would need to consider the truth values of all the propositions (possibly infinitely many) that could arise. Clearly this is impossible. The best we can do is to argue intuitively. Thus we could argue intuitively that if $\mathcal{P}$ is T for *all* values of x, it must necessarily be T for *some* values of x. Such reasoning cannot, however, be automated.

We can *test* the argument form by considering some particular cases. For example, consider the simple case in which the universe of discourse is the set {'*Rex*', '*Fido*'} and the predicate is 'x *has four legs*'. Then there are two propositions:

- '*Rex has four legs*'
- '*Fido has four legs*'

Now if $\exists x$ ('x *has four legs*') is false, then at least one of these two propositions would be false. For sake of argument, consider the case in which

- '*Rex has four legs*' $=_T F$
- '*Fido has four legs*' $=_T T$

In this case, $\forall x$ ('x *has four legs*') would be false, a result which is consistent with the argument form being valid. It does not however *prove* that the argument form is valid since there is still the possibility that for some other case the conclusion is false even though the premiss is false; such a case would constitute a **counterexample**, and would be sufficient evidence to show that the argument form is invalid. However, there is no way of proving the argument form valid by such testing. □

In this chapter we shall look at a different approach to logic and how this can be automated in **logic programming.**

8.2 Consistency and refutation

Definition 8.1

A set of propositional forms is **inconsistent** if there is no instance of this set in which all the propositions are true. A set of propositions is inconsistent if it is an instance of an inconsistent set of propositional forms. □

Now suppose we have a semantic entailment $\mathfrak{A}_1, \ldots, \mathfrak{A}_n \vDash \mathfrak{B}$. For any instance of $\mathfrak{B}$ which is false, at least one of the corresponding instances of $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ must be false. But the corresponding instance of $\neg\mathfrak{B}$ must be true. Hence $\mathfrak{A}_1, \ldots, \mathfrak{A}_n, \neg\mathfrak{B}$ must be an inconsistent set of propositional forms. This is the basis of the **method of refutation**: we can prove that $\mathfrak{A}_1, \ldots, \mathfrak{A}_n \therefore \mathfrak{B}$ is a valid argument by showing that $\mathfrak{A}_1, \ldots, \mathfrak{A}_n, \neg\mathfrak{B}$ is inconsistent.

Note that the definition of inconsistency given above is for propositional logic only. There *is* a more general definition which applies to first order logic, but the need for this can be circumvented as illustrated below.

Consider the theory defined the following axioms:

'*All dogs have four legs.*'

'*Rex is a dog.*'

'*Rajah is a dog.*'

'*Fido does not have four legs.*'

We could represent these symbolically as

$\forall x\,(q(x) \Rightarrow p(x))$

$q(Rex)$

$q(Rajah)$

$\neg p(Fido)$

where

- $p(x)$ is the predicate '*x has four legs*';
- $q(x)$ is the predicate '*x is a dog*';

Suppose we want to determine whether '*Rex has four legs*' is a valid conclusion of this theory. We can attempt to do this by adding the proposition $\neg p(Rex)$ to the list of axioms to give the following set (Set 1).

Set 1

$$\forall x\,(q(x) \Rightarrow p(x))$$
$$q(Rex)$$
$$q(Rajah)$$
$$\neg p(Fido)$$
$$\neg p(Rex)$$

Unfortunately, because of the quantified predicate in the first axiom, it is not possible to consider the inconsistency of this set of propositions directly. What we can do however is to replace the first axiom $\forall x\,(q(x) \Rightarrow p(x))$ by the related axiom $q(\mathfrak{t}) \Rightarrow p(\mathfrak{t})$, so that our set of propositions is now

Set 2

$$q(\mathfrak{t}) \Rightarrow p(\mathfrak{t})$$
$$q(Rex)$$
$$q(Rajah)$$
$$\neg p(Fido)$$
$$\neg p(Rex)$$

It is important to realize that the replacement axiom is *not* equivalent to the original. Nevertheless there is a relation between them: if $\forall x\,(q(x) \Rightarrow p(x))$ is true, then $q(\mathfrak{t}) \Rightarrow p(\mathfrak{t})$ must be true for every possible value of the term $\mathfrak{t}$; and conversely, if $q(\mathfrak{t}) \Rightarrow p(\mathfrak{t})$ is true for every possible value of the term $\mathfrak{t}$, then $\forall x\,(q(x) \Rightarrow p(x))$ must be true. Furthermore, if we can find a value of $\mathfrak{t}$ for which Set 2 is inconsistent, then the original Set 1 must also be inconsistent.

Now there are only three proper constants in the theory: *Rajah*, *Fido* and *Rex*. Each of these gives rise to a different case for Set 2.

Case 1

$$q(Rajah) \Rightarrow p(Rajah)$$
$$q(Rex)$$
$$q(Rajah)$$
$$\neg p(Fido)$$
$$\neg p(Rex)$$

This case is an instance of the set of propositional forms

$$\mathcal{Q}_1 \Rightarrow \mathcal{P}_1,\ \mathcal{Q}_3,\ \mathcal{Q}_1,\ \neg\mathcal{P}_2,\ \neg\mathcal{P}_3$$

This set is not inconsistent. For example, take true instances of $\mathcal{P}_1$, $\mathcal{Q}_1$ and $\mathcal{Q}_3$ and false instances of $\mathcal{P}_2$ and $\mathcal{P}_3$. □

Case 2

$q(Fido) \Rightarrow p(Fido)$

$q(Rex)$

$q(Rajah)$

$\neg p(Fido)$

$\neg p(Rex)$

This case is an instance of the set of propositional forms

$$\mathcal{Q}_2 \Rightarrow \mathcal{P}_2,\ \mathcal{Q}_3,\ \mathcal{Q}_1,\ \neg\mathcal{P}_2,\ \neg\mathcal{P}_3$$

This set is not inconsistent. For example, take true instances of $\mathcal{Q}_1$ and $\mathcal{Q}_3$ and false instances of $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{Q}_2$. □

Case 3

$q(Rex) \Rightarrow p(Rex)$

$q(Rex)$

$q(Rajah)$

$\neg p(Fido)$

$\neg p(Rex)$

This case is an instance of the set of propositional forms

$$\mathcal{Q}_3 \Rightarrow \mathcal{P}_3,\ \mathcal{Q}_3,\ \mathcal{Q}_1,\ \neg\mathcal{P}_2,\ \neg\mathcal{P}_3$$

This set *is* inconsistent as may be seen, for example, from truth tables. □

Since we have found an inconsistent case for Set 2, the original Set 1 must also be inconsistent. Hence we can conclude $p(Rex)$.

8.3 Clauses

Definition 8.2

A **positive literal** is a letter, such as p, which represents an atomic proposition. A **negative literal** is the negation of a positive literal; for example $\neg p$. A literal is either a positive literal or a negative literal. □

Definition 8.3

A **clause** is either a literal on its own, or the disjunction of two or more literals. □

Example 8.3

Examples of clauses include $p, \neg p, p \vee q$ and $p_1 \vee \neg p_2 \vee \neg p_3 \vee \neg p_5 \vee p_7$. But the following are *not* clauses: $p \wedge q, \neg(p \vee q)$ and $\neg\neg p \vee q$ □

In **clausal logic** we represent facts in a theory using **clauses**; a theory can be defined by a list of clauses.

Metanotation

In discussing the logic of clauses, we often wish to refer to unspecified literals. We shall use $L_1, L_2, \ldots$ for literals which may be either positive or negative, $P_1, P_2, \ldots$ for positive literals and $N_1, N_2, \ldots$ for negative literals. Note that if P is a positive literal, then the corresponding negative literal can be written as $\neg P$. We also want to refer to unspecified clauses; we shall use $C_1, C_2, \ldots$ for this purpose. Note that if C_1 is a clause, then so are $C_1 \vee C_2$, $C_1 \vee \mathcal{P}$ and $C_1 \vee \neg\mathcal{P}$.

Resolution

Although we could use the rules of natural deduction from propositional logic to reason about clauses, it can be shown that in clausal logic only *one* rule of deduction is needed, known as **resolution**. This rule can be thought of as cancelling a positive literal P in one clause by the corresponding negative literal $\neg P$ in another clause to yield a new clause from what remains. Using our current notation form propositional logic, the resolution rule can be written as follows.

Rule: resolution

$C_1 \vee P$, $C_2 \vee \neg P \therefore C_1 \vee C_2$ is a valid argument for any clauses C_1 and C_2 and positive literal P. □

Justification

If we consider the possible truth values for C_1, C_2 and P, then we see that $C_1 \vee P$, $C_2 \vee \neg P \vDash C_1 \vee C_2$.

C_1	C_2	P	$\neg P$	$C_1 \vee P$	$C_2 \vee \neg P$	$C_1 \vee C_2$
T	*T*	*T*	*F*	*T*	*T*	*T*
T	*T*	*F*	*T*	*T*	*T*	*T*
T	*F*	*T*	*F*	*T*	*F*	*T*
T	*F*	*F*	*T*	*T*	*T*	*T*
F	*T*	*T*	*F*	*T*	*T*	*T*
F	*T*	*F*	*T*	*F*	*T*	*T*
F	*F*	*T*	*F*	*T*	***F***	***F***
F	*F*	*F*	*T*	*F*	*T*	*F*

□

The clause $C_1 \vee C_2$ is said to be the **resolvent** of $C_1 \vee P$ and $C_2 \vee \neg P$. In fact, when applying resolution, we do not worry about the order of letters in a disjunct. Thus, for example, it is also the case that

$$P \vee C_1, C_2 \vee \neg P \vdash C_2 \vee C_1$$

Example 8.4

Show that $\neg p \vee \neg q \vee r, q \vee \vee s \vdash \neg p \vee r \vee s$.

Solution

Resolving $\neg p \vee \neg q \vee r$ and $q \vee \neg r \vee s$ for q gives a resolvent of $\neg p \vee r \vee s$.

□

It might seem that there are two flaws in this approach to logic.

- We can resolve a positive literal at *any* position in one clause with a corresponding negative literal at *any* position in another clause. Using the standard notation of propositional logic, it is not possible to express this idea simply.
- The fact that only clauses can be considered in this approach may seem too restrictive.

Fortunately, we can overcome these problems by introducing an alternative notation for clauses based upon sets, and by showing how any problem in deductive logic has a corresponding problem in clausal logic.

Set notation for clauses

A clause is built up from literals where each literal corresponds either to an atomic proposition p (in which the case the literal is said to be positive) or to the negation of an atomic proposition $\neg q$ (in which the case the literal is said to be negative). Now we know by definition that any clause is obtained by forming the disjunction of its constituent literals, and so we do not need to write down explicitly the disjunction symbol $\vee$. It is sufficient to write down the set of literals. The set notation emphasizes that the order in which the literals are given is immaterial.

Example 8.5

How can the clause $p_1 \vee \neg p_2 \vee p_3$ be represented as a set of literals?

Solution

The clause is a disjunction of the literals p_1, $\neg p_2$ and p_3. Thus the clause can be represented by the set $\{p_1, \neg p_2, p_3\}$.

Note, however, that other representations of the same set are possible, such as $\{\neg p_2, p_1, p_3\}$ and $\{p_3, p_1, \neg p_2\}$. □

A theory can thus be defined by a set of clauses, each of which is itself a set of literals. For example:

$$\{\{p_1, p_2, \neg p_7\}, \{\neg p_5\}, \{p_2, p_7, \neg p_5\}, \{p_3, \neg p_1, \neg p_4\}, \{p_6\}\}$$

To simplify the presentation, we shall usually represent a set of clauses as a list, for example:

$\{p_1, p_2, \neg p_7\}$

$\{\neg p_5\}$

$\{p_2, p_7, \neg p_5\}$

$\{p_3, \neg p_1, \neg p_4\}$

$\{p_6\}$

Indeed we could take this one stage further. Writing the set of clauses as a vertical list, we can represent each clause as a horizontal list.

$p_1, p_2, \neg p_7$

$\neg p_5$

$p_2, p_7, \neg p_5$

$p_3, \neg p_1, \neg p_4$

p_6

Note that it is possible to conceive of a set which has no members, and which we can write as { }. In clausal logic we can therefore represent a clause with *no* literals at all. This special clause is called the **empty clause.**

Using this set notation for clauses, we can now write down the resolution rule.

Rule: Resolution

$\{\mathsf{L}_1, \mathsf{L}_2, \ldots, \mathsf{L}_m, \mathsf{P}\}, \{\mathsf{L}_{m+1}, \mathsf{L}_{m+2}, \ldots, \mathsf{L}_{m+n}, \neg \mathsf{P}\} \vdash \{\mathsf{L}_1, \mathsf{L}_2, \ldots, \mathsf{L}_{m+n}\}$ □

Normal forms

Given any propositional form we can find other propositional forms which are equivalent. Two types of propositional form which are particularly useful are known as the **disjunctive normal form** and the **conjunctive normal form**. Conjunctive normal forms are especially important for converting a compound proposition into a corresponding a set of clauses.

Disjunctive normal form

We have seen above that several different propositional forms may be equivalent. Now suppose we have the situation where we know the truth table for a propositional form, and we wish to find what that propositional form is. The short answer is that there is no one unique answer; there will be infinitely may propositional forms, all of which are equivalent. It is, however, very easy to find what is known as the **disjunctive normal form** or **DNF.**

Definition 8.4

For any propositional form $\mathfrak{A}$, the disjunctive normal form is an equivalent form $\mathfrak{B}_1 \vee \mathfrak{B}_2 \vee \ldots \vee \mathfrak{B}_n$, where each of the disjuncts $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n$ is a conjunction of propositional forms. □

The method of finding the DNF corresponding to a truth table is perhaps best explained by means of an example.

Example 8.6

Find the DNF corresponding to the following truth table.

$\mathcal{P}$	$\mathcal{Q}$	
T	T	T
T	F	F
F	T	F
F	F	T

Solution

From the truth table we see that the result has a truth value of T either when $\mathcal{P}$ and $\mathcal{Q}$ are both true, that is when $\mathcal{P} \wedge \mathcal{Q} =_T T$, or when $\mathcal{P}$ and $\mathcal{Q}$ are both false, that is when $\neg\mathcal{P} \wedge \neg\mathcal{Q} =_T T$. Thus one possible propositional form corresponding to the truth table is given by $\mathcal{P} \wedge \mathcal{Q} \vee \neg\mathcal{P} \wedge \neg\mathcal{Q}$. This is the disjunctive normal form. □

Exercise 57: Disjunctive normal forms

1. For each of the following truth tables, determine the corresponding disjunctive normal form, and simplify the answer.

(a)

$\mathcal{P}$	$\mathcal{Q}$	
T	T	T
T	F	T
F	T	F
F	F	F

(b)

$\mathcal{P}$	$\mathcal{Q}$	
T	T	T
T	F	T
F	T	F
F	F	T

(c)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

(d)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

(e)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{S}$	
T	T	T	T	F
T	T	T	F	F
T	T	F	T	T
T	T	F	F	F
T	F	T	T	T
T	F	T	F	F
T	F	F	T	T
T	F	F	F	F
F	T	T	T	F
F	T	T	F	T
F	T	F	T	F
F	T	F	F	F
F	F	T	T	F
F	F	T	F	F
F	F	F	T	F
F	F	F	F	F

Conjunctive normal form

Definition 8.5

For any propositional form $\mathfrak{A}$, the conjunctive normal form is an equivalent form $\mathfrak{B}_1 \wedge \mathfrak{B}_2 \wedge \ldots \wedge \mathfrak{B}_n$, where each of the conjuncts $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n$ is a disjunction of propositional forms. □

Example 8.7

Express $(\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \Leftrightarrow \mathcal{R})$ in conjunctive normal form.

Solution

Since $\mathcal{P} \Rightarrow \mathcal{Q} =_T \neg\mathcal{P} \vee \mathcal{Q}$ and $\mathcal{Q} \Leftrightarrow \mathcal{R} =_T (\neg\mathcal{Q} \vee \mathcal{R}) \wedge (\mathcal{Q} \vee \neg\mathcal{R})$ then it follows that $(\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \Leftrightarrow \mathcal{R}) =_T (\neg\mathcal{P} \vee \mathcal{Q}) \wedge (\neg\mathcal{Q} \vee \mathcal{R}) \wedge (\mathcal{Q} \vee \neg\mathcal{R})$, which is the conjunctive normal form. □

In this last example, we found the conjunctive normal form by using equational logic. Another approach is to find the *disjunctive* normal form for the *negation* of the proposition, and then negate this to obtain the *conjunctive* normal form using de Morgan's laws:

- $\neg(\mathfrak{A} \wedge \mathfrak{B}) \equiv \neg\mathfrak{A} \vee \neg\mathfrak{B}$
- $\neg(\mathfrak{A} \vee \mathfrak{B}) \equiv \neg\mathfrak{A} \wedge \neg\mathfrak{B}$

Example 8.8

Convert $\mathcal{P} \vee (\mathcal{P} \Leftrightarrow \mathcal{Q})$ into conjunctive normal form.

Solution

First we find the truth table for $\neg(\mathcal{P} \vee (\mathcal{P} \Leftrightarrow \mathcal{Q}))$.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \Leftrightarrow \mathcal{Q}$	$\mathcal{P} \vee (\mathcal{P} \Leftrightarrow \mathcal{Q})$	$\neg(\mathcal{P} \vee (\mathcal{P} \Leftrightarrow \mathcal{Q}))$
T	T	T	T	F
T	F	F	T	F
F	T	F	F	T
F	F	T	T	F

Hence $\neg(\mathcal{P} \vee (\mathcal{P} \Leftrightarrow \mathcal{Q})) \equiv \neg\mathcal{P} \wedge \mathcal{Q}$. Thus we can write down

$$
\begin{array}{lll}
 & \mathcal{P} \vee (\mathcal{P} \Leftrightarrow \mathcal{Q}) & \\
=_T & \neg\neg(\mathcal{P} \vee (\mathcal{P} \Leftrightarrow \mathcal{Q})) & \langle\ \neg\neg\mathfrak{A} =_T \mathfrak{A}\ \rangle \\
=_T & \neg(\neg\mathcal{P} \wedge \mathcal{Q}) & \langle\ \text{from truth table}\ \rangle \\
=_T & \neg\neg\mathcal{P} \vee \neg\mathcal{Q} & \langle\ \neg(\mathfrak{A} \wedge \mathfrak{B}) =_T \neg\mathfrak{A} \vee \neg\mathfrak{B}\ \rangle \\
=_T & \mathcal{P} \vee \neg\mathcal{Q} & \langle\ \neg\neg\mathfrak{A} =_T \mathfrak{A}\ \rangle
\end{array}
$$

□

Example 8.9

What set of clauses corresponds to $(p \Rightarrow q) \wedge (q \Leftrightarrow r)$?

Solution

The compound proposition in conjunctive normal form that is equivalent to $(p \Rightarrow q) \wedge (q \Leftrightarrow r)$ is $(\neg p \vee q) \wedge (\neg q \vee r) \wedge (q \vee \neg r)$. There are three clauses:

1. $\neg p \vee q$ with positive literal q and negative literal $\neg p$;
2. $\neg q \vee r$ with positive literal r and negative literal $\neg q$;
3. $q \vee \neg r$ with positive literal q and negative literal $\neg r$.

Thus $(p \Rightarrow q) \wedge (q \Leftrightarrow r)$ corresponds to the following set of clauses:

$q, \neg p$

$r, \neg q$

$q, \neg r$

□

Example 8.10

What set of clauses corresponds to $p \vee (p \Leftrightarrow q)$?

Solution

The proposition in conjunctive normal form equivalent to $p \vee (p \Leftrightarrow q)$ is $p \vee \neg q$, which is a single clause. Thus $p \vee (p \Leftrightarrow q)$ corresponds to the following set of clauses:

$p, \neg q$

□

In general, for any set of premisses of propositional logic we can find a corresponding set of clauses in clausal logic.

Example 8.11

Convert the following list of propositions into a list of clauses.

$$(p_1 \vee p_2) \Rightarrow \neg p_3,\ p_1 \wedge (p_4 \Rightarrow p_5),\ p_2 \vee p_5$$

Solution

- $p_2 \vee p_5$ is a clause in its own right; it is already in conjunctive normal form. This clause can be written as $\{p_2, p_5\}$.
- Since $p_4 \Rightarrow p_5$ is equivalent to $\neg p_4 \vee p_5$, $p_1 \wedge (p_4 \Rightarrow p_5)$ corresponds to the two clauses p_1, a single literal, and $\neg p_4 \vee p_5$. These clauses can be written as $\{p_1\}$ and $\{p_5, \neg p_4\}$.
- From the laws of the conditional connective (see Table 5.1) we know that $(p_1 \vee p_2) \Rightarrow \neg p_3$ is equivalent to $(p_1 \Rightarrow \neg p_3) \wedge (p_2 \Rightarrow \neg p_3)$; furthermore the first of these conjuncts $p_1 \Rightarrow \neg p_3$ is equivalent to the clause $\neg p_1 \vee \neg p_3$, and the second conjunct $p_2 \Rightarrow \neg p_3$ is equivalent to the clause $\neg p_2 \vee \neg p_3$. These clauses can be written as $\{\neg p_1, \neg p_3\}$ and $\{\neg p_1, \neg p_3\}$.

Combining all these clauses results in the following set of clauses:

p_2, p_5

p_1

$p_5, \neg p_4$

$\neg p_1, \neg p_3$

$\neg p_1, \neg p_3$

□

Example 8.12

Prove that $p \Rightarrow q, q \Rightarrow r \therefore p \Rightarrow r$ is a valid argument.

Solution

We know that $p \Rightarrow q$ is equivalent to $\neg p \vee q$ and that $q \Rightarrow r$ is equivalent to $\neg q \vee r$. Hence we have the following clauses:

$q, \neg p$

$r, \neg q$

Resolving these two clauses for q gives a resolvent clause of $r, \neg p$. Now this resolvent corresponds to $\neg p \vee r$, which is equivalent to the required conclusion $p \Rightarrow r$. □

Exercise 58: Converting propositions into clausal form

1. Convert each of the following propositions into conjunctive normal form, and hence obtain a corresponding set of clauses.

   (a) $p \Rightarrow (q \Rightarrow r)$
   (b) $p \Rightarrow (q \wedge r)$
   (c) $(p \Rightarrow q) \vee (\neg q \Rightarrow r)$
   (d) $(p \Rightarrow q) \Rightarrow r$

2. Prove each of the following arguments are valid by using resolution on appropriate sets of clauses:

   (a) $p, p \Rightarrow (q \Rightarrow r) \therefore q \Rightarrow r$
   (b) $p \vee q, p \Rightarrow r, q \Rightarrow r \therefore r$
   (c) $(p \Rightarrow q) \vee (\neg q \Rightarrow r), p \wedge \neg r \therefore q$

8.4 Refutation in clausal logic

Refutation is a very important tool in clausal logic. If a set of clauses can be reduced to the empty clause, then the original set of clauses is inconsistent.

Suppose we have a number (possibly zero) of premiss clauses and wish to determine whether another clause (the **query**) can be deduced from these. We can do this by adding to the negation of the query to the premisses and to see if the empty clause can be obtained. If an empty clause is obtained, then the negation of the query is inconsistent with the original premisses; the query can be deduced from the original premisses.

Example 8.13

Show using refutation that $\{\} \therefore p \Rightarrow (q \Rightarrow p)$ is a valid argument.

Solution

From properties of the conditional connective, we know that $p \Rightarrow (q \Rightarrow p)$ is equivalent to $(p \wedge q) \Rightarrow p)$ which in turn is equivalent to $\neg((p \wedge q) \wedge \neg p)$. Hence $\neg(p \Rightarrow (q \Rightarrow p))$ is equivalent to $p \wedge q \wedge \neg p$. This gives the set of clauses

p

q

$\neg p$

to p and $\neg p$ yields a clause the empty clause, $\{\}$. Thus we conclude that $\{\} \therefore p \Rightarrow (q \Rightarrow p)$ is a valid argument. □

Example 8.14

Show using refutation that $\neg p \vee \neg q \vee r, q \vee s \therefore \neg p \vee r \vee s$ is a valid argument.

Solution

The two premisses correspond to the two clauses

$r,\ \neg p,\ \neg q$

$q,\ s$

Now the negation of $\neg p \vee r \vee s$ is $p \wedge \neg r \wedge \neg s$, which gives three further clauses:

p

$\neg r$

$\neg s$

Thus we arrive at the following set of clauses:

1	$r,\ \neg p,\ \neg q$	
2	$q,\ s$	
3	p	
4	$\neg r$	
5	$\neg s$	

Now we can use resolution on these premisses to obtain further clauses:

1	$r,\ \neg p,\ \neg q$	
2	$q,\ s$	
3	p	
4	$\neg r$	
5	$\neg s$	
6	$\neg p,\ \neg q$	$res\ 1,4$
7	q	$res\ 2,5$
8	$\neg p$	$res\ 6,7$
9	$\{\}$	$res\ 3,8$

□

Exercise 59: Refutation in clausal logic

Prove each of the following arguments are valid by using refutation and resolution.

1. $p, p \Rightarrow (q \wedge r) \vdash q \Rightarrow r$
2. $p \vee q, p \Rightarrow r, q \Rightarrow r \vdash r$
3. $(p \Rightarrow q) \vee (\neg q \Rightarrow r), p \wedge \neg r \vdash q$

8.5 Horn clauses

Clearly we have a well defined procedure using resolution and refutation for assessing whether any given argument is valid. Although we have only considered the application of this procedure for problems in propositional logic, it is possible to adapt the method of Section 8.2 such that problems in predicate logic may also be tackled. Unfortunately, the procedure as described is fairly inefficient. However, it is possible to improve efficiency by restricting clauses to those in which there is no more than one positive literal.

Definition 8.6

A clause in which there is at most one positive literal is called a **Horn clause.** □

Definition 8.7

A **logic program** is a set of Horn clauses. □

Example 8.15

Horn clauses include $\neg q$, $p \vee \neg q$, t and $\neg p_1 \vee \neg p_2 \vee \neg p_3 \vee \neg p_4$. □

Exercise 60: Horn clauses

1. Which of the following clauses are Horn clauses?
   (a) s
   (b) $\neg p_1$
   (c) $s, \neg q$
   (d) $t, s, \neg q$
   (e) $t, s, \neg q, \neg r$
   (f) $s, \neg q, \neg r$
   (g) $\neg q, \neg r$
   (h) $\neg t, s, \neg q, \neg r$
2. Which of the following correspond to a set Horn clauses?
   (a) $p \Rightarrow (q \Rightarrow r)$
   (b) $p \Rightarrow (q \wedge r)$
   (c) $(p \Rightarrow q) \vee (\neg q \Rightarrow r)$
   (d) $(p \Rightarrow q) \Rightarrow r$

Alternative notation for clauses

The set notation for clauses can be modified to a form which is particularly useful for writing Horn clauses, and which forms the basis of the programming language prolog.

The basic idea behind the notation is that the positive literal, if there is one, is placed before the symbol :-; the list of negative literals follows the :-.

If there are no negative literals, then the :- symbol may be omitted. The empty clause can be represented simply as :-.

Example 8.16

What is the logic program that corresponds to $(p \wedge q) \Leftrightarrow r$.

Solution

From the definition of $\Leftrightarrow$ we know that $(p \wedge q) \Leftrightarrow r$ is equivalent to the conjunction $((p \wedge q) \Rightarrow r) \wedge (r \Rightarrow (p \wedge q))$. The first of the two conjuncts $(p \wedge q) \Rightarrow r$ can be shown to be equivalent to $\neg p \vee \neg q \vee r$. This gives the single clause $\neg p \vee \neg q \vee r$; in the new notation we can write this as r :- p, q. The second of the two conjuncts can be shown to be equivalent to $(\neg r \vee p) \wedge (\neg r \vee q)$. This gives two clauses, $\neg r \vee p$ and $\neg r \vee q$; in the new notation we can write these as p :- r and q :- r. We have therefore converted the original expression, $(p \wedge q) \Leftrightarrow r$, into three Horn clauses:

r :- p, q
p :- r
q :- r □

With this new notation, the resolution rule takes on a very simple form: a query clause

:-$p, q_1, q_2, \ldots, q_n$

resolves with the program clause

p :- $r_1, r_2, \ldots, r_n$

to give a new query clause

:-$q_1, q_2, \ldots, q_n, r_1, r_2, \ldots, r_n$

This can be imagined as the ps either side of the :- symbol 'cancelling' out in a combined clause.

If we write down the program clauses of a logic program as a numbered list, we can add a query clause as further numbered item to this list. We then use resolution between query clauses and program clauses to generate further query clauses as numbered items. The clauses used to generate each new clause are indicated by reference to their numbers in the list.

Example 8.17

Show that s can be deduced from the following logic program.

t :- p,q s :- q,t p q

Solution

The logic program as a numbered list becomes

1 t :- p,q
2 s :- q,t
3 p
4 q

Next we negate the query s and add it to the program clauses as :-s.

1	t :- p, q
2	s :- q, t
3	p
4	q
5	:-s

Finally we use resolution to obtain new clauses until the empty clause, :-, is obtained.

1	t :- p, q	
2	s :- q, t	
3	p	
4	q	
5	:-s	
6	:-q, t	*Res.* 2, 5
7	:-t	*Res.* 4, 6
8	:-p, q	*Res.* 1, 7
9	:-q	*Res.* 3, 8
10	:-	*Res.* 4, 9

Since we obtain the empty clause, we have shown that s follows from the program. Note that clause 4 has been used twice; it has not been 'used up' in creating clause 7. □

Example 8.18

Use logic programming to show that $(p \wedge q) \Leftrightarrow r, q \wedge r \vdash p \vee s$.

Solution

We have already seen from Example 8.5 that $(p \wedge q) \Leftrightarrow r$ can be converted into the three program clauses r :- p, q, p :- r and q :- r. The second premiss $q \wedge r$ can quite simply be converted into the two program clauses q and r. Finally the negation $\neg(p \vee s)$ of the conclusion is equivalent to $\neg p \wedge \neg s$ which gives two query clauses, namely: :-p and :-s. The logic program together with the query clauses begins a sequence of resolutions which leads to an empty clause.

1	r :- p, q	
2	p :- r	
3	q :- r	
4	q	
5	r	
6	:-p	
7	:-s	
8	:-r	*Res.* 2, 6
9	:-	*Res.*5, 8

□

Exercise 61: Logic programming with propositional clauses

1. Decide whether t follows from the following logic program.

   q :- p
   r :- s, t, u
   s :- p, v
   t :- q
   w :- r, s
   p
   s

2. Use logic programming to show that $p \land q,\ p \Rightarrow r,\ q \Rightarrow s \vdash r \land s$

Solutions to Exercises A

Solution 1: Symbolic connectives

1. (a) $\frac{6}{8} \neq \frac{3}{4}$
   (b) $\frac{12}{16} = \frac{6}{8} = \frac{3}{4}$, that is '$\frac{12}{16} = \frac{6}{8}$ *and* $\frac{6}{8} = \frac{3}{4}$'.
   Only (b) is true.
2. $r \wedge s$
3. p is '*Two is a prime number*'
   q is '*Two is an even number*'
   $p \vee q$

Solution 2: Arithmetic operators

1. $3 + 1 + 4$ or, more accurately, $3 + (1 + 4)$. Left operand = 3 and right operand = 5 so result = $3 + 5 = 8$.
2. $(2 + 3) - (7 - 5) = 5 - 2 = 3$ but $(7 - 5) - (2 + 3) = 2 - 5 = -3$, which is different.

Solution 3: Conjunction

1. '*My cat is black but your cat is white*'
   '*My cat is black*' $\wedge$ '*Your cat is white*' $=_T F \wedge F =_T F$
2. '*Shakespeare wrote both Hamlet and MacBeth*'
   '*Shakespeare wrote Hamlet*' $\wedge$ '*Shakespeare wrote MacBeth*' $=_T T \wedge T =_T T$
3. $q \wedge p =_T T \wedge F =_T F$
4. $2 \times 7 = 27 \ \wedge \ 3^2 = 9$
   $=_T F \wedge T =_T F$
5. $3 > 2 \ \wedge \ 3 > 2 =_T T \wedge T =_T T$

Solution 4: Negation

1. '*The moon is not made of blue cheese*' $=_T \neg F =_T T$.
2. '1234 *is an not even number*' $=_T \neg T =_T F$.
   Note that '1234 *is an odd number*' would be incorrect answer to this question as it depends upon the *additional* fact that an odd number is one which is not even.

Solution 5: Disjunction

1. '*Either you broke the window or else I'm a Martian*' $=_T F \vee F =_T F$.
   Note that in general use this is just a more emphatic way of saying 'you broke the window'.
2. '*Either Shakespeare or Francis Bacon wrote Hamlet*' $=_T T \vee F =_T T$.
3. $r_1 \vee r_2 =_T F \vee T =_T T$
4. '*Either* 2 *is even or* 3 *is odd*' $=_T T \vee T =_T T$

Solution 6: Connective schemas

1. (a) '*Rex does not have a wet nose*'
   (b) $2 + 3 \neq 8$
   (c) $\neg p$
   (d) $\neg q$
   (e) $\neg r_2$
   (f) '*Rex is black but Rover is white*'
   (g) '*Either* $2 + 3 = 8$ *or* $3^2 = 9$'
   (h) $p \wedge q$
   (i) $p_1 \wedge p_2$
   (j) $r_2 \vee q$
2. The following are possible answers. Note how some of the meaning of the original may be lost.
   (a) $\mathcal{P} \wedge \mathcal{Q}$ where
   - $\mathcal{P}$ is interpreted as '*This book is long*';
   - $\mathcal{Q}$ is interpreted as '*I read it quickly*'.

   (b) $\mathcal{P} \vee \mathcal{Q}$ where
   - $\mathcal{P}$ is '*There is a hole in the exhaust*'
   - $\mathcal{Q}$ is '*A bracket has worked loose*'.

   (c) $\mathcal{P} \vee \mathcal{Q}$ where
   - $\mathcal{P}$ is '*Lunch will be served during the flight*'.
   - $\mathcal{Q}$ is '*Dinner will be served during the flight*'.

   (d) $\neg\mathcal{P}$ where $\mathcal{P}$ is $3 \times 6 = 7$

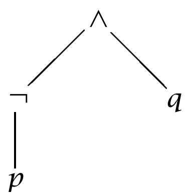
Solution 7: Instantiation to compound propositions

1. $\neg\neg$‘*Rover is a brave dog*’
2. $2 \leq 3 \wedge 2^2 \leq 3^2 \vee 2 \geq 3 \wedge 2^2 \geq 3^2$
3. $p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge p_4$, though strictly speaking this should be written as $(p_1 \wedge \neg p_2) \wedge (\neg p_3 \wedge p_4)$ – see sections 2.9–2.12.

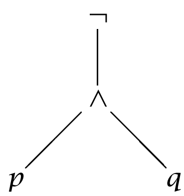
Solution 8: Constructing parse trees

Main connectives are shown in bold.

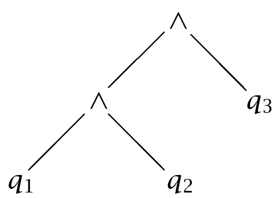
1. $(\neg p) \boldsymbol{\wedge} q$



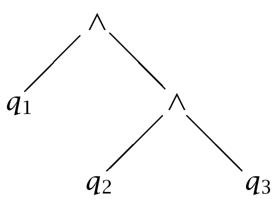
2. $\boldsymbol{\neg}(p \wedge q)$



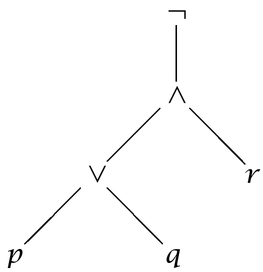
3. $(q_1 \wedge q_2) \boldsymbol{\wedge} q_3$



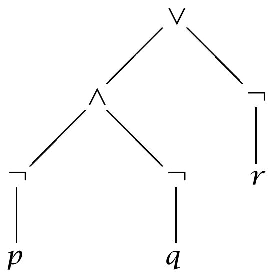
4. $q_1 \boldsymbol{\wedge} (q_2 \wedge q_3)$



5. $\neg((p \vee q) \wedge r)$



6. $((\neg p) \wedge (\neg q)) \vee (\neg r)$



Solution 9: Compound propositions from parse trees

1. $q \wedge (p \wedge r)$
2. $\neg\neg(p \wedge q)$

Solution 10: Connective priorities

1. (a) $\neg(\neg p)$
   (b) $(\neg p) \wedge q$
   (c) $q \wedge (\neg r)$
   (d) $(\neg p) \vee q$
   (e) $(p \wedge q) \wedge (\neg r)$
   (f) $(p \vee q) \vee (\neg r)$
   (g) $(q \wedge (\neg p)) \vee q$
   (h) $(p_1 \vee p_2) \vee (\neg p_3)$
   (i) $3 > 0 \vee ((\neg 1 + 1 = 2) \wedge 2 + 3 = 5)$
   (j) ($\neg$'*Fido has three legs*') $\vee$ '*Rex has four legs*'
   (k) $(p \wedge q) \vee ((\neg r) \wedge p)$
   (l) $(\neg p) \wedge ((\neg q) \vee (p \wedge r))$
   (m) $\neg(\neg((\neg p_2) \wedge (\neg p_1)))$

2. The main connective is shown in bold.
   (a) $\neg(p_1 \wedge p_2) \boldsymbol{\vee} p_3$
   (b) '*Rex has four legs*' $\vee$ '*Fido has three legs*' $\boldsymbol{\vee}$ $\neg$'*Rover has a wet nose*'
   (c) $\boldsymbol{\neg}\neg q$
   (d) $\neg\, 1 + 1 = 2 \;\boldsymbol{\vee}\; (1+1)^2 = 2^2$
   (e) $\boldsymbol{\neg}(p \vee q \wedge r \vee \neg s)$
   (f) $\neg(\neg(\neg p_1 \vee p_2 \wedge p_3) \vee \neg p_4) \boldsymbol{\wedge} \neg(p_5 \wedge p_6 \vee \neg\neg p_7)$

Solution 11: Removing parentheses

1. $\neg p \vee q$
2. $p_1 \vee \neg p_2 \vee p_3$
3. $p \wedge \neg q \vee \neg r$
4. $\neg q_1 \wedge (q_1 \vee q_2) \vee q_1 \wedge q_2$,
   though perhaps $\neg q_1 \wedge (q_1 \vee q_2) \vee (q_1 \wedge q_2)$ is easier to read
5. $r \wedge (\neg p \vee q)$

Solution 12: Truth values of compound propositions

1. $\neg(\neg T) =_T \neg F =_T T$
2. $\neg T \vee F =_T F \vee F =_T F$
3. $(F \vee F) \vee F =_T F \vee F =_T F$
4. $\neg\neg(T \vee \neg F) \wedge \neg(F \vee F)$
   $=_T \neg\neg(T \vee T) \wedge \neg F$
   $=_T \neg\neg T \wedge T$
   $=_T \neg F \wedge T$
   $=_T T \wedge T$
   $=_T T$
5. $\neg F \vee T =_T T \vee T =_T T$
6. $\neg F \wedge (T \vee \neg T) =_T T \wedge (T \vee F) =_T T \wedge T =_T T$

Solution 13: Compound propositions from propositional forms

1. '*Rex has four legs and either Fido has three legs or Rover does not have a wet nose.*'
2. (a) '*It is not cold but it is snowing.*'
   (b) '*Either the water pump is not working or there is no anti-freeze in the radiator and last night was very cold.*'

Solution 14: Propositional forms for compound propositions

Note that the choice of schematic letters is not unique.

1. One possible answer is: $\neg\mathcal{P} \wedge \neg\mathcal{Q}$, $\mathcal{P} \wedge \neg\mathcal{Q}$, $\neg\mathcal{P} \wedge \mathcal{Q}$, $\mathcal{P} \wedge \mathcal{Q}$, $\mathcal{P}$
   Another possibility is: $\neg\mathcal{P}_1 \wedge \neg\mathcal{P}_2$, $\mathcal{Q}_1 \wedge \neg\mathcal{P}_2$, $\neg\mathcal{P}_1 \wedge \mathcal{Q}_2$, $\mathcal{Q}_1 \wedge \mathcal{Q}_2$, $\mathcal{R}$

   In the first answer, occurrences of a schematic letter in one propositional form would not always be instantiated to the same proposition as occurrences of the same letter in another propositional form in the list: thus the schematic letter $\mathcal{P}$ in the propositional form $\neg\mathcal{P} \wedge \neg\mathcal{Q}$ would need to be instantiated to '*The sun is made of gold*' whereas in $\mathcal{P} \wedge \neg\mathcal{Q}$ it would need to be instantiated to '*The sun is not made of gold*'. In the second answer, however, all occurrences of a schematic letter would need to be instantiated to the same proposition wherever it occurred in the list: thus $\mathcal{P}_1$ would always need to be instantiated to '*The sun is made of gold*'.

2. One possible answer is: $(\mathcal{P} \wedge \mathcal{Q}) \vee \mathcal{R}$, $\mathcal{P} \vee \mathcal{R}$, $\mathcal{P}$
   Another possibility is: $(\mathcal{P}_1 \wedge \mathcal{P}_2) \vee \mathcal{P}_3$, $\mathcal{Q}_1 \vee \mathcal{P}_3$, $\mathcal{R}_1$

3. One possible answer is: $\neg\mathcal{P} \vee \mathcal{Q}$, $\mathcal{P} \vee \mathcal{Q}$, $\mathcal{P}$
   Another possibility is: $\neg\mathcal{P}_1 \vee \mathcal{P}_2$, $\mathcal{Q}_1 \vee \mathcal{P}_2$, $\mathcal{R}_1$

4. $(\mathcal{P}_1 \wedge \mathcal{P}_2) \wedge (\mathcal{P}_3 \wedge \mathcal{P}_4)$, $\mathcal{Q}_1 \wedge (\mathcal{P}_3 \wedge \mathcal{P}_4)$, $(\mathcal{P}_1 \wedge \mathcal{P}_2) \wedge \mathcal{Q}_2$, $\mathcal{Q}_1 \wedge \mathcal{Q}_2$, $\mathcal{R}_1$ or
   $(\mathcal{P} \wedge \mathcal{Q}) \wedge (\mathcal{R} \wedge \mathcal{S})$, $\mathcal{P} \wedge (\mathcal{R} \wedge \mathcal{S})$, $(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R}$, $\mathcal{P} \wedge \mathcal{R}$, $\mathcal{P}$

5. $\mathcal{P} \vee \neg\mathcal{P}$ $\mathcal{P} \vee \mathcal{Q}$ $\mathcal{R}$ or
   $\mathcal{P} \vee \neg\mathcal{P}$ $\mathcal{P} \vee \mathcal{Q}$ $\mathcal{P}$

6. $(\mathcal{P}_1 \wedge \neg\mathcal{P}_2) \vee (\neg\mathcal{P}_1 \wedge \mathcal{P}_2)$, $(\mathcal{P}_1 \wedge \mathcal{Q}_2) \vee (\neg\mathcal{P}_1 \wedge \mathcal{P}_2)$, $(\mathcal{P}_1 \wedge \neg\mathcal{P}_2) \vee (\mathcal{Q}_1 \wedge \mathcal{P}_2)$, $(\mathcal{P}_1 \wedge \mathcal{Q}_2) \vee (\mathcal{Q}_1 \wedge \mathcal{P}_2)$, $\mathcal{R}_1 \vee (\mathcal{Q}_1 \wedge \mathcal{P}_2)$, $(\mathcal{P}_1 \wedge \mathcal{Q}_2) \vee \mathcal{R}_2$, $\mathcal{R}_1 \vee \mathcal{R}_2$, $\mathcal{S}_1$

   or

   $(\mathcal{P} \wedge \neg\mathcal{Q}) \vee (\neg\mathcal{P} \wedge \mathcal{Q})$ $(\mathcal{P} \wedge \mathcal{R}) \vee (\neg\mathcal{P} \wedge \mathcal{Q})$ $(\mathcal{P} \wedge \neg\mathcal{Q}) \vee (\mathcal{S} \wedge \mathcal{Q})$ $(\mathcal{P} \wedge \mathcal{R}) \vee (\mathcal{S} \wedge \mathcal{Q})$
   $\mathcal{P} \vee (\mathcal{S} \wedge \mathcal{Q})$ $(\mathcal{P} \wedge \mathcal{R}) \vee \mathcal{Q}$ $\mathcal{P} \vee \mathcal{Q}$ $\mathcal{P}$

   Notice care must be taken over the choice of schematic letters, since the characteristic form has repeated occurrences of the same letter. For example: in the second answer, the propositional form $(\mathcal{P} \wedge \mathcal{Q}) \vee (\neg\mathcal{P} \wedge \mathcal{Q})$ would *not* be correct since the letter $\mathcal{Q}$ would need to be instantiated to different propositions in that one form – this is not acceptable.

Solution 15: Truth tables

1.

$\mathcal{P}$	$\neg\mathcal{P}$	$\neg\neg\mathcal{P}$
T	F	T
F	T	F

2.

$\mathcal{P}$	$\neg\mathcal{P}$	$\mathcal{P} \vee \neg\mathcal{P}$
T	F	T
F	T	T

3.

$\mathcal{Q}$	$\neg\mathcal{Q}$	$\neg\neg\mathcal{Q}$	$\neg\neg\neg\mathcal{Q}$
T	F	T	F
F	T	F	T

4.

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{Q}$	$\mathcal{P} \vee \neg\mathcal{Q}$	$\neg(\mathcal{P} \vee \neg\mathcal{Q})$
T	T	F	T	F
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

5.

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{Q}$	$\mathcal{P} \wedge \neg\mathcal{Q}$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

6.

$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_1 \wedge \mathcal{P}_2$	$(\mathcal{P}_1 \wedge \mathcal{P}_2) \wedge \mathcal{P}_3$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

7.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{P} \wedge \mathcal{R}$	$\mathcal{Q} \wedge (\mathcal{P} \wedge \mathcal{R})$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

8.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{P} \vee \mathcal{Q}$	$(\mathcal{P} \vee \mathcal{Q}) \vee \mathcal{R}$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

9.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{Q} \vee \mathcal{R}$	$\mathcal{P} \vee (\mathcal{Q} \vee \mathcal{R})$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

10.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{P} \vee \mathcal{Q}$	$(\mathcal{P} \vee \mathcal{Q}) \wedge \mathcal{R}$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

11.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{P} \wedge \mathcal{R}$	$\mathcal{Q} \wedge \mathcal{R}$	$(\mathcal{P} \wedge \mathcal{R}) \vee (\mathcal{Q} \wedge \mathcal{R}))$
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	F	T
T	F	F	F	F	F
F	T	T	F	T	T
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

12.

$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_2 \wedge \mathcal{Q}_3$	$\mathcal{Q}_1 \vee (\mathcal{Q}_2 \wedge \mathcal{Q}_3)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

13.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{P} \vee \mathcal{Q}$	$\mathcal{P} \vee \mathcal{Q}$	$(\mathcal{P} \vee \mathcal{Q}) \wedge (\mathcal{P} \vee \mathcal{R})$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	F	F

14.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{Q} \wedge \mathcal{R}$	$\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R})$	$\neg(\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R}))$
T	T	T	T	T	F
T	T	F	F	T	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	F
F	T	F	F	F	T
F	F	T	F	F	T
F	F	F	F	F	T

15.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\neg\mathcal{Q}$	$\neg\mathcal{R}$	$\neg\mathcal{Q} \vee \neg\mathcal{R}$	$\neg\mathcal{P}$	$\neg\mathcal{P} \wedge (\neg\mathcal{Q} \vee \neg\mathcal{R})$
T	T	T	F	F	F	F	F
T	T	F	F	T	T	F	F
T	F	T	T	F	T	F	F
T	F	F	T	T	T	F	F
F	T	T	F	F	F	T	F
F	T	F	F	T	T	T	T
F	F	T	T	F	T	T	T
F	F	F	T	T	T	T	T

16.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{Q} \vee \mathcal{R}$	$\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$	$\neg(\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}))$
T	*T*	*T*	*T*	*T*	*F*
T	*T*	*F*	*T*	*T*	*F*
T	*F*	*T*	*T*	*T*	*F*
T	*F*	*F*	*F*	*F*	*T*
F	*T*	*T*	*T*	*F*	*T*
F	*T*	*F*	*T*	*F*	*T*
F	*F*	*T*	*T*	*F*	*T*
F	*F*	*F*	*F*	*F*	*T*

17.

$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\neg\mathcal{P}_2$	$\neg\mathcal{P}_1$	$\neg\mathcal{P}_2 \wedge \neg\mathcal{P}_1$	$\neg\mathcal{P}_3$	$\neg\mathcal{P}_3 \vee (\neg\mathcal{P}_2 \wedge \neg\mathcal{P}_1)$
T	*T*	*T*	*F*	*F*	*F*	*F*	*F*
T	*T*	*F*	*F*	*F*	*F*	*T*	*T*
T	*F*	*T*	*T*	*F*	*F*	*F*	*F*
T	*F*	*F*	*T*	*F*	*F*	*T*	*T*
F	*T*	*T*	*F*	*T*	*F*	*F*	*F*
F	*T*	*F*	*F*	*T*	*F*	*T*	*T*
F	*F*	*T*	*T*	*T*	*T*	*F*	*T*
F	*F*	*F*	*T*	*T*	*T*	*T*	*T*

18. See Figure A.1

19.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{Q} \wedge \mathcal{R}$	$\neg\mathcal{P} \vee (\neg\mathcal{Q} \wedge \mathcal{R})$
T	*T*	*T*	*F*	*F*	*F*	*F*
T	*T*	*F*	*F*	*F*	*F*	*F*
T	*F*	*T*	*F*	*T*	*T*	*T*
T	*F*	*F*	*F*	*T*	*F*	*F*
F	*T*	*T*	*T*	*F*	*F*	*T*
F	*T*	*F*	*T*	*F*	*F*	*T*
F	*F*	*T*	*T*	*T*	*T*	*T*
F	*F*	*F*	*T*	*T*	*F*	*T*

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{Q} \wedge \mathcal{R}$	$\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R})$	$\neg(\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R}))$	$\mathcal{Q} \vee \mathcal{R}$	$\neg(\mathcal{Q} \vee \mathcal{R})$	$\mathcal{P} \wedge \neg(\mathcal{Q} \vee \mathcal{R})$	$\neg(\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R})) \vee (\mathcal{P} \wedge \neg(\mathcal{Q} \vee \mathcal{R}))$	$\neg(\neg(\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R})) \vee (\mathcal{P} \wedge \neg(\mathcal{Q} \vee \mathcal{R})))$
T	T	T	T	T	F	T	F	F	F	T
T	T	F	F	T	F	T	F	F	F	T
T	F	T	F	T	F	T	F	F	F	T
T	F	F	F	T	F	F	T	T	T	F
F	T	T	T	T	F	T	F	F	F	T
F	T	F	F	F	T	T	F	F	T	F
F	F	T	F	F	T	T	F	F	T	F
F	F	F	F	F	T	F	T	F	T	F

Figure A.1: Solution to Exercise 15, Question 18

20.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{S}$	$\mathcal{P} \wedge \mathcal{Q}$	$\mathcal{R} \wedge \mathcal{S}$	$(\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{R} \wedge \mathcal{S})$	$\neg((\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{R} \wedge \mathcal{S}))$
T	T	T	T	T	T	T	F
T	T	T	F	T	F	T	F
T	T	F	T	T	F	T	F
T	T	F	F	T	F	T	F
T	F	T	T	F	T	T	F
T	F	T	F	F	F	F	T
T	F	F	T	F	F	F	T
T	F	F	F	F	F	F	T
F	T	T	T	F	T	T	F
F	T	T	F	F	F	F	T
F	T	F	T	F	F	F	T
F	T	F	F	F	F	F	T
F	F	T	T	F	T	T	F
F	F	T	F	F	F	F	T
F	F	F	T	F	F	F	T
F	F	F	F	F	F	F	T

Solution 16: Well formed words

Only 1, 3 and 5 are well formed. Note that 6 has three left parentheses '(' but only two right parentheses ')'.

Solution 17: Well formed expressions

Only 1, 4, 5 and 7 are well formed.

Solution 18: Tautologies

From *Prop*1 it follows that propositional forms 2 and 6 are tautologies. Propositional forms 1 and 3 are also tautologies, but these do not follow directly from *Prop*1. Note that in 3 the first disjunct is negation whereas in *Prop*1 the second disjunct is negation.

Solution 19: Contradictions

From *Prop*2 it follows that propositional forms 5 and 6 are contradictions. Propositional forms 1 and 4 are also contradictions, but these do not follow directly from *Prop*2. Note that propositional form 2 is the negation of a contradiction and so must be a tautology.

Solution 20: Equivalence

1. b, c, d, h, i, k, l, n, p, q
2. a, c

Solution 21: Using laws of equivalence

1. $\neg\mathcal{P} \wedge \mathcal{P}$
   $=_T \mathcal{P} \wedge \neg\mathcal{P} \quad \langle\ \mathfrak{A} \wedge \mathfrak{B} =_T \mathfrak{B} \wedge \mathfrak{A}\ \rangle$
   $=_T F \quad \langle\ \mathfrak{A} \wedge \neg\mathfrak{A} =_T F\ \rangle$

2. $\neg(\mathcal{P} \vee \neg\mathcal{Q}) \vee (\mathcal{P} \vee \neg\mathcal{Q})$
   $=_T (\mathcal{P} \vee \neg\mathcal{Q}) \vee \neg(\mathcal{P} \vee \neg\mathcal{Q}) \quad \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle$
   $=_T T \quad \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle$

3. $(\mathcal{P} \wedge \mathcal{Q}) \vee \mathcal{P}$
   $=_T \mathcal{P} \vee (\mathcal{P} \wedge \mathcal{Q}) \quad \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle$
   $=_T \mathcal{P} \quad \langle\ \mathfrak{A} \vee \mathfrak{A} =_T \mathfrak{A}\ \rangle$

4. $(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{P}$
   $=_T \mathcal{P} \wedge (\mathcal{P} \wedge \mathcal{Q}) \quad \langle\ \mathfrak{A} \wedge \mathfrak{B} =_T \mathfrak{B} \wedge \mathfrak{A}\ \rangle$
   $=_T (\mathcal{P} \wedge \mathcal{P}) \wedge \mathcal{Q} \quad \langle\ \mathfrak{A} \wedge (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \wedge \mathfrak{C}\ \rangle$
   $=_T \mathcal{P} \wedge \mathcal{Q} \quad \langle\ \mathfrak{A} \wedge \mathfrak{A} =_T \mathfrak{A}\ \rangle$

5. $(\mathcal{P} \vee \neg\mathcal{P}) \wedge \neg(\mathcal{Q} \wedge \neg\mathcal{Q})$
   $=_T T \wedge \neg(\mathcal{Q} \wedge \neg\mathcal{Q}) \quad \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle$
   $=_T \neg(\mathcal{Q} \wedge \neg\mathcal{Q}) \wedge T \quad \langle\ \mathfrak{A} \wedge \mathfrak{B} =_T \mathfrak{B} \wedge \mathfrak{A}\ \rangle$
   $=_T \neg(\mathcal{Q} \wedge \neg\mathcal{Q}) \quad \langle\ \mathfrak{A} \wedge T =_T \mathfrak{A}\ \rangle$
   $=_T \neg\mathcal{Q} \vee \neg\neg\mathcal{Q} \quad \langle\ \neg(\mathfrak{A} \wedge \mathfrak{B}) =_T \neg\mathfrak{A} \vee \neg\mathfrak{B}\ \rangle$
   $=_T \neg\mathcal{Q} \vee \mathcal{Q} \quad \langle\ \neg\neg\mathfrak{A} =_T \mathfrak{A}\ \rangle$
   $=_T \mathcal{Q} \vee \neg\mathcal{Q} \quad \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle$
   $=_T T \quad \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle$

6. $\neg T$
   $=_T \neg(\mathcal{P} \vee \neg\mathcal{P}) \quad \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle$
   $=_T \neg\mathcal{P} \wedge \neg\neg\mathcal{P} \quad \langle\ \neg(\mathfrak{A} \vee \mathfrak{B}) =_T \neg\mathfrak{A} \wedge \mathfrak{B}\ \rangle$
   $=_T \neg\mathcal{P} \wedge \mathcal{P} \quad \langle\ \neg\neg\mathfrak{A} =_T \mathfrak{A}\ \rangle$
   $=_T \mathcal{P} \wedge \neg\mathcal{P} \quad \langle\ \mathfrak{A} \wedge \mathfrak{B} =_T \mathfrak{B} \wedge \mathfrak{A}\ \rangle$
   $=_T F \quad \langle\ \mathfrak{A} \wedge \neg\mathfrak{A} =_T F\ \rangle$

7. F
   $=_T \neg\neg T \quad \langle\ \mathit{Qu.\ 6}\ \rangle$
   $=_T T \quad \langle\ \neg\neg\mathfrak{A} =_T \mathfrak{A}\ \rangle$

   Note that an earlier proof has been quoted in order to avoid repeating the steps used in that proof – the earlier proof can be viewed as a 'subprogram' called by the later proof.

Solution 22: Sets of propositional forms

1. Set of two propositional forms, namely $\mathcal{P} \wedge \mathcal{Q}$ and $\neg\mathcal{Q}$.
2. Set of two propositional forms, namely $\mathcal{P} \wedge \mathcal{Q}$ and $\neg\mathcal{Q}$.
3. Set of one propositional form, namely $\neg\mathcal{Q}$.
4. The propositional form $\neg\mathcal{Q}$.
5. Set of one truth value, namely F.
6. The truth value F.
7. General expression for a set of two propositional forms, one of which has $\wedge$ as the main connective and the other of which has $\neg$ as the main connective.
8. General expression for a set of two propositional forms, one of which has $\wedge$ as the main connective and the other of which has $\neg$ as the main connective.
9. General expression for a set of two propositional forms, one of which has $\vee$ as the main connective with F as the right disjunct and the other of which has $\wedge$ as the main connective with left conjunct T.
10. A propositional form with $\wedge$ as the main connective.
11. Set of one propositional form, namely $\mathcal{Q} \wedge \mathcal{R}$.
12. Does not represent anything meaningful in our notation – it is not a well formed expression.
13. Set containing *at least* one propositional form.
14. Set containing at least one propositional form; if $\mathfrak{A}$ and $\mathfrak{B}$ are different, then the set contains at least two propositional forms.
15. Does not represent anything meaningful as $\mathfrak{A}$ is not a set – it is not a well formed expression.
16. A set with one propositional form, namely $\mathcal{P}_1 \vee \neg\mathcal{P}_2$

Solution 23: Semantic entailment

1. (a)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$	$\mathcal{P} \vee \mathcal{Q}$
T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
T	F	F	T
F	T	F	T
F	F	F	F

(b)

$\mathcal{P}_1$	$\mathcal{P}_2$	$\neg(\mathcal{P}_1 \vee \mathcal{P}_2)$	$\neg(\mathcal{P}_2 \wedge \mathcal{P}_1)$
T	T	F	F
T	F	F	T
F	T	F	T
F	F	$\boldsymbol{T}$	$\boldsymbol{T}$

(c)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \vee \neg\mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{Q}$
T	T	T	F	F
T	F	T	F	T
F	T	F	T	F
F	F	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$

(d)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{P} \wedge \mathcal{Q}$	$\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R})$
T	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
T	T	F	$\boldsymbol{T}$	$\boldsymbol{T}$
T	F	T	F	T
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(e)

$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\neg\mathcal{Q}_1 \vee \mathcal{Q}_2$	$\neg\mathcal{Q}_2 \vee \mathcal{Q}_3$	$\neg\mathcal{Q}_1 \vee \mathcal{Q}_3$
T	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$
F	T	F	T	F	T
F	F	T	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$
F	F	F	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$

2. If $\mathfrak{A} \vDash \mathfrak{B}$, then whenever $\mathfrak{B}$ is F then $\mathfrak{A}$ is F; so whenever $\neg\mathfrak{B}$ is T then $\neg\mathfrak{A}$ is also T; that is, $\neg\mathfrak{B} \vDash \neg\mathfrak{A}$.

Solution 24: Uniform replacement

1. Replace $\mathcal{P}$ by $\neg\mathcal{P}$ and $\mathcal{Q}$ by $\neg\mathcal{Q}$.
2. Replace $\mathcal{P}_1$ by $\mathcal{P} \wedge \mathcal{Q}$ and $\mathcal{P}_2$ by $\mathcal{Q}$.
3. Replace $\mathcal{R}$ by $\mathcal{S}$.

Solution 25: Validity of argument forms

From the truth tables it can be seen that 2, 3, 5, 7, 10, 12, 13, 15 and 17 are valid argument forms. Note that *modus ponendo tollens* is not a valid argument form.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \vee \mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$	$\neg\mathcal{P}$	$\neg\neg\mathcal{P}$	$\neg\mathcal{P} \vee \mathcal{Q}$	$\neg\mathcal{Q}$
T	T	T	T	F	T	T	F
T	F	T	F	F	T	F	T
F	T	T	F	T	F	T	F
F	F	F	F	T	F	T	T

Solution 26: Validity of arguments

The characteristic argument form is given for each argument, from which it is possible to decide whether or not the argument is valid.

1. $\mathcal{P} \vee \mathcal{Q} \therefore \mathcal{P} \wedge \mathcal{Q}$ (not valid).
2. $\mathcal{P} \wedge \mathcal{Q} \therefore \mathcal{P} \vee \mathcal{Q}$ (valid).
3. $\mathcal{P} \wedge \mathcal{Q} \therefore \mathcal{P} \vee \mathcal{R}$ (valid).
4. $\mathcal{P} \wedge \mathcal{Q} \therefore \mathcal{P}$ (valid).
5. $\mathcal{P}, \mathcal{Q} \therefore \mathcal{P} \wedge \mathcal{Q}$ (valid).
6. $\mathcal{P}, \mathcal{Q} \therefore \mathcal{P} \vee \mathcal{Q}$ (valid).
7. $\mathcal{P} \therefore \mathcal{P} \vee \mathcal{Q}$ (valid).
8. $\mathcal{P} \therefore \mathcal{Q} \vee \mathcal{P}$ (valid).
9. $\mathcal{P} \therefore \mathcal{Q} \vee \mathcal{R}$ (not valid).
10. $\neg\neg(\mathcal{P} \wedge \neg\mathcal{Q}) \therefore \mathcal{P} \wedge \neg\mathcal{Q}$ (valid). Note that it is also possible to use the fact that $\neg\neg\mathcal{P} \therefore \mathcal{P}$ is a valid argument form to show that the argument itself is valid.

Solution 27: Deduction rules

1. In each case the instances of $\mathcal{P}$ and $\mathcal{Q}$ are given together with the corresponding inference form.
   (a) $\mathcal{P} : p, \quad \mathcal{Q} : q \vee r, \quad \mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$.
   (b) $\mathcal{P} : p, \quad \mathcal{Q} : q \vee r, \quad \mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P}$.
   (c) $\mathcal{P} : p, \quad \mathcal{Q} : q \vee r, \quad \mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{Q}$.
   (d) $\mathcal{P} : \neg\neg\neg q, \quad \mathcal{Q} : p \vee \neg\neg q, \quad \mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{Q}$.
   (e) $\mathcal{P} : \neg\neg\neg q, \quad \mathcal{Q} : p \vee \neg\neg q, \quad \mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P}$.
   (f) $\mathcal{P} : p, \quad \mathcal{Q} : q \wedge r, \quad \mathcal{P} \vdash \mathcal{P} \vee \mathcal{Q}$.
   (g) $\mathcal{P} : \neg q \vee r, \quad \mathcal{Q} : \neg p \vee s, \quad \mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$.
   (h) $\mathcal{P} : \neg p, \quad \neg\neg\mathcal{P} \vdash \mathcal{P}$.
   (i) $\mathcal{P} : \neg\neg p, \quad \neg\neg\mathcal{P} \vdash \mathcal{P}$.
   (j) $\mathcal{P} : (p \wedge (r \vee q)), \quad \neg\neg\mathcal{P} \vdash \mathcal{P}$.
   (k) $\mathcal{P} : \neg\neg p, \quad \mathcal{P} \vdash \neg\neg\mathcal{P}$.
   (l) $\mathcal{P} : \neg p, \quad \mathcal{P} \vdash \neg\neg\mathcal{P}$.
   (m) $\mathcal{P} : (p \vee (r \wedge q)), \quad \mathcal{P} \vdash \neg\neg\mathcal{P}$.
2. For each question, the appropriate rule of deduction is given followed by the necessary instantiations and then the inference form which constitutes that rule of deduction.
   (a) $\wedge$I, $\mathcal{P}$: '*The sky is blue*', $\mathcal{Q}$: '*Grass is green*', $\mathcal{P}, \mathcal{Q} \vdash \mathcal{P} \wedge \mathcal{Q}$.
   (b) $\vee\mathrm{I}_1$, $\mathcal{P} : p_1$, $\mathcal{Q} : p_2$, $\mathcal{P} \vdash \mathcal{P} \vee \mathcal{Q}$.
   (c) $\neg\neg$E, $\mathcal{P}$: '*Fido has three legs*', $\neg\neg\mathcal{P} \vdash \mathcal{P}$.
   (d) $\wedge\mathrm{E}_2$, $\mathcal{P}$: '73 *is prime*', $\mathcal{Q}$: '73 *is odd*', $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{Q}$.

(e) $\neg\neg\mathrm{I}$, $\mathcal{P}$: '*The sky is blue*', $\mathcal{P} \vdash \neg\neg\mathcal{P}$.

(f) $\vee\mathrm{I}_1$, $\mathcal{P}$: '71 *is prime*', $\mathcal{Q}$: '26 *is prime*', $\mathcal{P} \vdash \mathcal{P} \vee \mathcal{Q}$.

(g) $\vee\mathrm{I}_2$, $\mathcal{P}$: '*Rex has four legs*', $\mathcal{Q}$: '*Rex has a wet nose*', $\mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$.

(h) $\wedge\mathrm{E}_1$, $\mathcal{P} : r$, $\mathcal{Q} : s$, $\mathcal{P} \wedge \mathcal{Q} \vdash \mathcal{P}$.

(i) $\vee\mathrm{I}_2$, $\mathcal{P} : 3^2 = 9$, $\mathcal{Q} : 1 + 1 = 2 \wedge 2 \times 3 = 6$, $\mathcal{Q} \vdash \mathcal{P} \vee \mathcal{Q}$.

(j) $\neg\neg\mathrm{E}$, $\mathcal{P}$: '*Roses are red*', $\mathcal{Q}$: '*Violets are blue*', $\neg\neg\mathcal{P} \vdash \mathcal{P}$.

Solution 28: Tabular derivations

1.
$$\begin{array}{llcll} 1 & \mathcal{P} \wedge \mathcal{Q} & \vdash & \mathcal{P} & \wedge\mathrm{E}_1 \\ 2 & \mathcal{P} & \vdash & \neg\neg\mathcal{P} & \neg\neg\mathrm{I} \\ 3 & \mathcal{P} \wedge \mathcal{Q} & \vdash & \neg\neg\mathcal{P} & 1,2 \end{array}$$

2.
$$\begin{array}{lrcll} 1 & \neg\neg\mathcal{P} & \vdash & \mathcal{P} & \neg\neg\mathrm{E} \\ 2 & \mathcal{P},\mathcal{Q} & \vdash & \mathcal{P} \wedge \mathcal{Q} & \wedge\mathrm{I} \\ 3 & \neg\neg\mathcal{P},\mathcal{Q} & \vdash & \mathcal{P} \wedge \mathcal{Q} & 1,2 \end{array}$$

3.
$$\begin{array}{lrcll} 1 & \neg\neg(\mathcal{P} \vee \mathcal{Q}) & \vdash & \mathcal{P} \vee \mathcal{Q} & \neg\neg\mathrm{E} \\ 2 & \mathcal{P} \vee \mathcal{Q} & \vdash & (\mathcal{P} \vee \mathcal{Q}) \vee \neg\neg(\mathcal{P} \wedge \mathcal{Q}) & \vee\mathrm{I}_1 \\ 3 & \neg\neg(\mathcal{P} \vee \mathcal{Q}) & \vdash & (\mathcal{P} \vee \mathcal{Q}) \vee \neg\neg(\mathcal{P} \wedge \mathcal{Q}) & 1,2 \end{array}$$

4.
$$\begin{array}{lrcll} 1 & \neg\neg\mathfrak{A} & \vdash & \mathfrak{A} & \neg\neg\mathrm{E} \\ 2 & \mathfrak{A},\mathfrak{B} & \vdash & \mathfrak{A} \wedge \mathfrak{B} & \wedge\mathrm{I} \\ 3 & \neg\neg\mathfrak{A},\mathfrak{B} & \vdash & \mathfrak{A} \wedge \mathfrak{B} & 1,2 \\ 4 & \neg\neg\mathfrak{B} & \vdash & \mathfrak{B} & \neg\neg\mathrm{E} \\ 5 & \neg\neg\mathfrak{A},\neg\neg\mathfrak{B} & \vdash & \mathfrak{A} \wedge \mathfrak{B} & 4,3 \end{array}$$

Solution 29: Deduction trees

1. (a)
$$\dfrac{\dfrac{\mathcal{P} \wedge \mathcal{Q}}{\mathcal{P}}\,\wedge\mathrm{E}_1}{\neg\neg\mathcal{P}}\,\neg\neg\mathrm{I}$$

(b)
$$\dfrac{\dfrac{\neg\neg\neg\mathcal{Q} \wedge (\mathcal{P} \vee \neg\neg\mathcal{Q})}{\neg\neg\neg\mathcal{Q}}\,\wedge\mathrm{E}_1}{\neg\mathcal{Q}}\,\neg\neg\mathrm{E}$$

(c)
$$\dfrac{\dfrac{\mathcal{P} \wedge \mathcal{Q}}{\mathcal{P}}\,\wedge\mathrm{E}_1}{\mathcal{P} \vee \mathcal{Q}}\,\vee\mathrm{I}_1$$

(d)

$$\dfrac{\dfrac{\dfrac{\neg\neg(\mathcal{P}\lor\mathcal{Q})\land(\neg\mathcal{Q}\land\mathcal{R})}{\neg\neg(\mathcal{P}\lor\mathcal{Q})}\,{\land}\mathrm{E}_1}{\mathcal{P}\lor\mathcal{Q}}\,\neg\neg\mathrm{E}}{(\mathcal{P}\lor\mathcal{Q})\lor(\mathcal{R}\land\mathcal{S})}\,{\lor}\mathrm{I}_1$$

(e)

$$\dfrac{\dfrac{\mathcal{P}}{\neg\neg\mathcal{P}}\,\neg\neg\mathrm{I} \qquad \mathcal{Q}\lor\mathcal{R}}{\neg\neg\mathcal{P}\land(\mathcal{Q}\lor\mathcal{R})}\,{\land}\mathrm{I}$$

(f)

$$\dfrac{\dfrac{\mathcal{Q}\land\mathcal{R}}{\mathcal{Q}}\,{\land}\mathrm{E}_1 \qquad \mathcal{S}}{\mathcal{Q}\land\mathcal{S}}\,{\land}\mathrm{I}$$

(g)

$$\dfrac{\dfrac{\mathcal{P}\land\mathcal{Q}}{\mathcal{Q}}\,{\land}\mathrm{E}_2 \qquad \dfrac{\mathcal{P}\land\mathcal{Q}}{\mathcal{P}}\,{\land}\mathrm{E}_1}{\mathcal{Q}\land\mathcal{P}}\,{\land}\mathrm{I}$$

(h)

$$\dfrac{\dfrac{\mathcal{R}}{\mathcal{R}\lor(\mathcal{Q}\land\neg\mathcal{P})}\,{\lor}\mathrm{I}_1 \qquad \mathcal{P}\lor\mathcal{Q}}{(\mathcal{R}\lor(\mathcal{Q}\land\neg\mathcal{P}))\land(\mathcal{P}\lor\mathcal{Q})}\,{\land}\mathrm{I}$$

2. (a)

$$\dfrac{\dfrac{\mathcal{Q} \qquad \mathcal{R}}{\mathcal{Q}\land\mathcal{R}}\,{\land}\mathrm{I}}{\mathcal{P}\lor(\mathcal{Q}\land\mathcal{R})}\,{\lor}\mathrm{I}_2$$

(b)

$$\dfrac{\dfrac{(\mathcal{P}\land\mathcal{Q})\land\mathcal{R}}{\mathcal{R}}\,{\land}\mathrm{E}_2 \qquad \dfrac{\dfrac{\dfrac{(\mathcal{P}\land\mathcal{Q})\land\mathcal{R}}{\mathcal{P}\land\mathcal{Q}}\,{\land}\mathrm{E}_1}{\mathcal{Q}}\,{\land}\mathrm{E}_2 \qquad \dfrac{\dfrac{(\mathcal{P}\land\mathcal{Q})\land\mathcal{R}}{\mathcal{P}\land\mathcal{Q}}\,{\land}\mathrm{E}_1}{\mathcal{P}}\,{\land}\mathrm{E}_1}{\mathcal{Q}\land\mathcal{P}}\,{\land}\mathrm{I}}{\mathcal{R}\land(\mathcal{Q}\land\mathcal{P})}\,{\land}\mathrm{I}$$

(c)

$$\dfrac{\dfrac{\mathcal{P}\land\mathcal{Q}}{\mathcal{Q}}\,{\land}\mathrm{E}_2}{\neg\mathcal{P}\lor\mathcal{Q}}\,{\lor}\mathrm{I}_2$$

(d)

$$\dfrac{\dfrac{\dfrac{\neg\neg\mathcal{P}\wedge\mathcal{Q}}{\neg\neg\mathcal{P}}\,\wedge\mathrm{E}_1}{\mathcal{P}}\,\neg\neg\mathrm{E} \qquad \dfrac{\neg\neg\mathcal{P}\wedge\mathcal{Q}}{\mathcal{Q}}\,\wedge\mathrm{E}_2}{\mathcal{P}\wedge\mathcal{Q}}\,\wedge\mathrm{I}$$

(e)

$$\dfrac{\dfrac{\dfrac{\neg\neg\mathcal{P}\wedge\neg\neg\mathcal{Q}}{\neg\neg\mathcal{P}}\,\wedge\mathrm{E}_1}{\mathcal{P}}\,\neg\neg\mathrm{E} \qquad \dfrac{\dfrac{\neg\neg\mathcal{P}\wedge\neg\neg\mathcal{Q}}{\neg\neg\mathcal{Q}}\,\wedge\mathrm{E}_2}{\mathcal{Q}}\,\neg\neg\mathrm{E}}{\mathcal{P}\wedge\mathcal{Q}}\,\wedge\mathrm{I}$$

(f)

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\neg(\mathcal{P}\wedge\mathcal{Q})}{\mathcal{P}\wedge\mathcal{Q}}\,\neg\neg\mathrm{E}}{\mathcal{Q}}\,\wedge\mathrm{E}_2}{\mathcal{P}\vee\mathcal{Q}}\,\vee\mathrm{I}_2}{\neg\neg(\mathcal{P}\vee\mathcal{Q})}\,\neg\neg\mathrm{I}$$

or, alternatively,

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\neg(\mathcal{P}\wedge\mathcal{Q})}{\mathcal{P}\wedge\mathcal{Q}}\,\neg\neg\mathrm{E}}{\mathcal{P}}\,\wedge\mathrm{E}_1}{\mathcal{P}\vee\mathcal{Q}}\,\vee\mathrm{I}_1}{\neg\neg(\mathcal{P}\vee\mathcal{Q})}\,\neg\neg\mathrm{I}$$

(g)

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\neg(\mathcal{P}\wedge\mathcal{Q})}{\mathcal{P}\wedge\mathcal{Q}}\,\neg\neg\mathrm{E}}{\mathcal{P}}\,\wedge\mathrm{E}_1}{\neg\neg\mathcal{P}}\,\neg\neg\mathrm{I}}{\neg\neg\mathcal{P}\vee\neg\neg\mathcal{Q}}\,\vee\mathrm{I}_1$$

or, alternatively,

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\neg(\mathcal{P}\wedge\mathcal{Q})}{\mathcal{P}\wedge\mathcal{Q}}\,\neg\neg\mathrm{E}}{\mathcal{Q}}\,\wedge\mathrm{E}_2}{\neg\neg\mathcal{Q}}\,\neg\neg\mathrm{I}}{\neg\neg\mathcal{P}\vee\neg\neg\mathcal{Q}}\,\vee\mathrm{I}_2$$

(h)

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\neg(\mathcal{P}\wedge\mathcal{Q})}{\mathcal{P}\wedge\mathcal{Q}}\,\neg\neg E}{\mathcal{P}}\,\wedge E_1}{\neg\neg\mathcal{P}}\,\neg\neg I \qquad \dfrac{\dfrac{\dfrac{\neg\neg(\mathcal{P}\wedge\mathcal{Q})}{\mathcal{P}\wedge\mathcal{Q}}\,\neg\neg E}{\mathcal{Q}}\,\wedge E_2}{\neg\neg\mathcal{Q}}\,\neg\neg I}{\neg\neg\mathcal{P}\wedge\neg\neg\mathcal{Q}}\,\wedge I$$

(i)

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\neg\mathcal{P}\wedge\neg\neg\mathcal{Q}}{\neg\neg\mathcal{P}}\,\wedge E_1}{\mathcal{P}}\,\neg\neg E \qquad \dfrac{\dfrac{\neg\neg\mathcal{P}\wedge\neg\neg\mathcal{Q}}{\neg\neg\mathcal{Q}}\,\wedge E_2}{\mathcal{Q}}\,\neg\neg E}{\mathcal{P}\wedge\mathcal{Q}}\,\wedge I}{\neg\neg(\mathcal{P}\wedge\mathcal{Q})}\,\neg\neg I$$

3. (a)

$$\dfrac{\dfrac{\mathcal{P}\wedge\mathcal{Q}}{\mathcal{P}}\,\wedge E_1}{\mathcal{Q}\wedge\mathcal{P}}\,\vee I_2$$

(b)

$$\dfrac{\dfrac{\mathcal{P}\wedge\neg\mathcal{Q}}{\neg\mathcal{Q}}\,\wedge E_2 \qquad \dfrac{\mathcal{P}\wedge\neg\mathcal{Q}}{\mathcal{P}}\,\wedge E_1}{\neg\mathcal{Q}\wedge\mathcal{P}}\,\wedge I$$

(c)

$$\dfrac{\dfrac{\mathcal{P}\qquad\mathcal{Q}}{\mathcal{P}\wedge\mathcal{Q}}\,\wedge I}{\mathcal{R}\vee\mathcal{P}\wedge\mathcal{Q}}\,\vee I_2$$

Solution 30: Using ¬I and ∨E

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{P}\wedge\mathcal{Q}}\,*}{\mathcal{Q}}\,\wedge E_2 \qquad \neg\mathcal{Q}}{\mathcal{Q}\wedge\neg\mathcal{Q}}\,\wedge I}{\neg(\mathcal{P}\wedge\mathcal{Q})}\,*\neg I$$

2.

$$\dfrac{\dfrac{\dfrac{\mathcal{P}\wedge\mathcal{Q}}{\mathcal{P}}\,\wedge E_1 \qquad \dfrac{\dfrac{}{\neg\mathcal{P}\wedge\mathcal{R}}\,*}{\neg\mathcal{P}}\,\wedge E_1}{\mathcal{P}\wedge\neg\mathcal{P}}\,\wedge I}{\neg(\neg\mathcal{P}\wedge\mathcal{R})}\,*\neg I$$

3.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{P}\wedge\neg\mathcal{P}}{\mathcal{P}}\,{\wedge}\mathrm{E}_1 \quad \dfrac{}{\neg\mathcal{Q}}\,*}{\mathcal{P}\wedge\neg\mathcal{Q}}\,{\wedge}\mathrm{I}}{\mathcal{P}}\,{\wedge}\mathrm{E}_1 \qquad \dfrac{\dfrac{\dfrac{\mathcal{P}\wedge\neg\mathcal{P}}{\neg\mathcal{P}}\,{\wedge}\mathrm{E}_2 \quad \dfrac{}{\neg\mathcal{Q}}\,*}{\neg\mathcal{P}\wedge\neg\mathcal{Q}}\,{\wedge}\mathrm{I}}{\neg\mathcal{P}}\,{\wedge}\mathrm{E}_1}{\mathcal{P}\wedge\neg\mathcal{P}}\,{\wedge}\mathrm{I}}{\neg\neg\mathcal{Q}}\,{*}\neg\mathrm{I}}{\mathcal{Q}}\,\neg\neg\mathrm{E}$$

4.

$$\dfrac{\mathcal{P}\vee\mathcal{Q} \qquad \dfrac{\dfrac{}{\mathcal{P}}\,*}{\mathcal{P}\vee(\mathcal{Q}\vee\mathcal{R})}\,{\vee}\mathrm{I}_1 \qquad \dfrac{\dfrac{\dfrac{}{\mathcal{Q}}\,*}{\mathcal{Q}\vee\mathcal{R}}\,{\vee}\mathrm{I}_1}{\mathcal{P}\vee(\mathcal{Q}\vee\mathcal{R})}\,{\vee}\mathrm{I}_2}{\mathcal{P}\vee(\mathcal{Q}\vee\mathcal{R})}\,{*}\vee\mathrm{E}$$

5. See Figure A.2

6. See Figure A.3

Solution 31: Theorems

1.

$$\dfrac{\dfrac{}{\neg(\neg\mathcal{P}\wedge\mathcal{P})}}{\neg(\neg\mathcal{P}\wedge\mathcal{P})\vee\mathcal{Q}}\,{\vee}\mathrm{I}_1$$

2. This follows from the theorem proved in Example 4.71, both directly, and also from its generalization: $\vdash \neg(\neg\mathfrak{A}\wedge\mathfrak{A})$.

$$\dfrac{\dfrac{}{\neg(\neg\mathcal{P}\wedge\mathcal{P})}\,\text{Ex. 4.71} \qquad \dfrac{}{\neg(\neg\mathcal{Q}\wedge\mathcal{Q})}\,\text{Gen. Ex. 4.71}}{\neg(\neg\mathcal{P}\wedge\mathcal{P})\wedge\neg(\neg\mathcal{Q}\wedge\mathcal{Q})}\,{\wedge}\mathrm{I}$$

3.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{P}\wedge(\mathcal{Q}\wedge\neg\mathcal{P})}\,*}{\mathcal{P}}\,{\wedge}\mathrm{E}_1 \qquad \dfrac{\dfrac{\dfrac{}{\mathcal{P}\wedge(\mathcal{Q}\wedge\neg\mathcal{P})}\,*}{\mathcal{Q}\wedge\neg\mathcal{P}}\,{\wedge}\mathrm{E}_2}{\neg\mathcal{P}}\,{\wedge}\mathrm{E}_2}{\mathcal{P}\wedge\neg\mathcal{P}}\,{\wedge}\mathrm{I}}{\neg(\mathcal{P}\wedge(\mathcal{Q}\wedge\neg\mathcal{P}))}\,{*}\neg\mathrm{I}$$

$$\dfrac{\neg\mathcal{P}\vee\mathcal{Q} \qquad \dfrac{\dfrac{\dfrac{}{\neg\mathcal{P}}\,\dagger \quad \neg\mathcal{Q}}{\neg\mathcal{P}\wedge\neg\mathcal{Q}}\,\wedge\mathrm{I}}{\neg\mathcal{P}}\,\wedge\mathrm{E}_1 \qquad \dfrac{\dfrac{\dfrac{}{\mathcal{Q}}\,\dagger \quad \dfrac{\dfrac{\neg\mathcal{Q} \quad \dfrac{}{\mathcal{P}}\,*}{\neg\mathcal{Q}\wedge\mathcal{P}}\,\wedge\mathrm{I}}{\neg\mathcal{Q}}\,\wedge\mathrm{E}_1}{\mathcal{Q}\wedge\neg\mathcal{Q}}\,\wedge\mathrm{I}}{\neg\mathcal{P}}\,*\neg\mathrm{I}}{\neg\mathcal{P}}\,\dagger\vee\mathrm{I}$$

Figure A.2: Solution to Exercise 30, Question 5

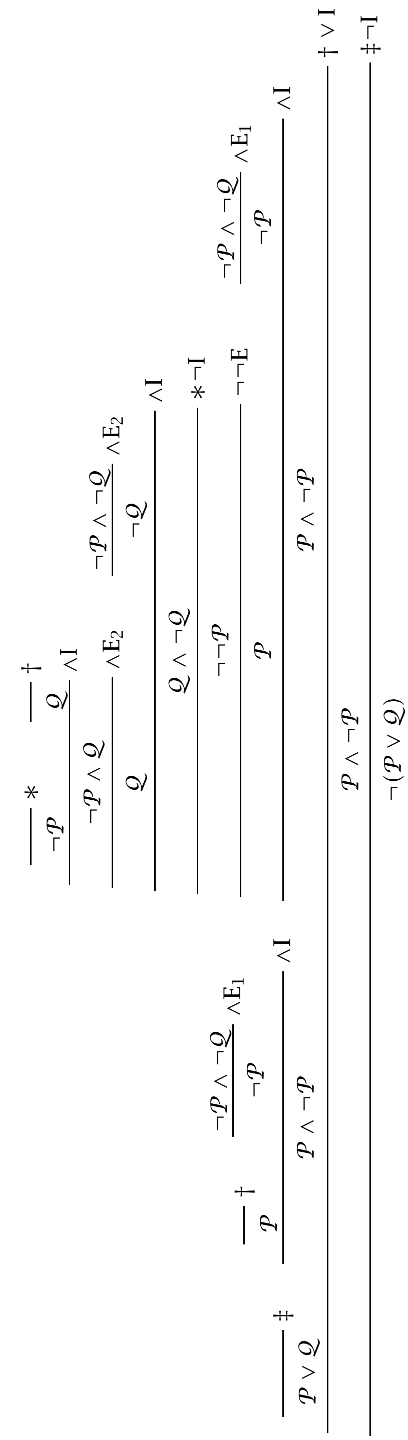


Figure A.3: Solution to Exercise 30, Question 6

Solution 32: Syntactic equivalence

1. $$\dfrac{\mathcal{P}}{\neg\neg\mathcal{P}}\,\neg\neg\mathrm{I} \qquad\qquad \dfrac{\neg\neg\mathcal{P}}{\mathcal{P}}\,\neg\neg\mathrm{E}$$

2. $$\dfrac{\dfrac{\mathcal{P}\wedge\mathcal{Q}}{\mathcal{Q}}\,\wedge\mathrm{E}_2 \qquad \dfrac{\mathcal{P}\wedge\mathcal{Q}}{\mathcal{P}}\,\wedge\mathrm{E}_1}{\mathcal{Q}\wedge\mathcal{P}}\,\wedge\mathrm{I}$$

$$\dfrac{\dfrac{\mathcal{Q}\wedge\mathcal{P}}{\mathcal{P}}\,\wedge\mathrm{E}_2 \qquad \dfrac{\mathcal{Q}\wedge\mathcal{P}}{\mathcal{Q}}\,\wedge\mathrm{E}_1}{\mathcal{P}\wedge\mathcal{Q}}\,\wedge\mathrm{I}$$

3. $$\dfrac{\mathcal{P}\vee\mathcal{Q} \qquad \dfrac{\dfrac{}{\mathcal{P}}\,*}{\mathcal{Q}\vee\mathcal{P}}\,\vee\mathrm{I}_2 \qquad \dfrac{\dfrac{}{\mathcal{Q}}\,*}{\mathcal{Q}\vee\mathcal{P}}\,\vee\mathrm{I}_1}{\mathcal{Q}\vee\mathcal{P}}\,*\vee\mathrm{E}$$

$$\dfrac{\mathcal{Q}\vee\mathcal{P} \qquad \dfrac{\dfrac{}{\mathcal{Q}}\,*}{\mathcal{P}\vee\mathcal{Q}}\,\vee\mathrm{I}_2 \qquad \dfrac{\dfrac{}{\mathcal{P}}\,*}{\mathcal{P}\vee\mathcal{Q}}\,\vee\mathrm{I}_1}{\mathcal{P}\vee\mathcal{Q}}\,*\vee\mathrm{E}$$

4. See Figure A.4

Solution 33: Compound propositions with the conditional connective

1. (a) ‘*It will rain today*’ ⇒ ‘*I shall get wet*’
   (b) $a = 2 \Rightarrow a^2 = 4$
   (c) ‘*I understand logic*’ ⇒ ‘*I shall become a good programmer*’
   (d) ‘*John will go to the party*’ ⇒ ‘*Mary will go to the party*’
   (e) ‘*Mary will go to the party*’ ⇒ ‘*John will go to the party*’
   (f) (‘*John will go to the party*’ ⇒ ‘*Mary will go to the party*’)
   ∧
   (‘*Mary will go to the party*’ ⇒ ‘*John will go to the party*’)
   (g) ¬‘*We shall go to the seaside tomorrow*’ ⇒ ‘*My car will break down*’
2. (a) ‘*If you are nice to me then I will be your friend.*’
   (b) ‘*I will be your friend only if you are nice to me.*’
   (c) ‘*If you are not a good boy, then Father Christmas will not leave you any presents.*’
   (d) ‘*Father Christmas will only leave you presents if you are a good boy.*’
   (e) ‘*If the driver displays a parking permit then he or she will be admitted to the car park and will not be sent a reminder about parking regulations, otherwise she or he will not be admitted and will be sent a reminder.*’

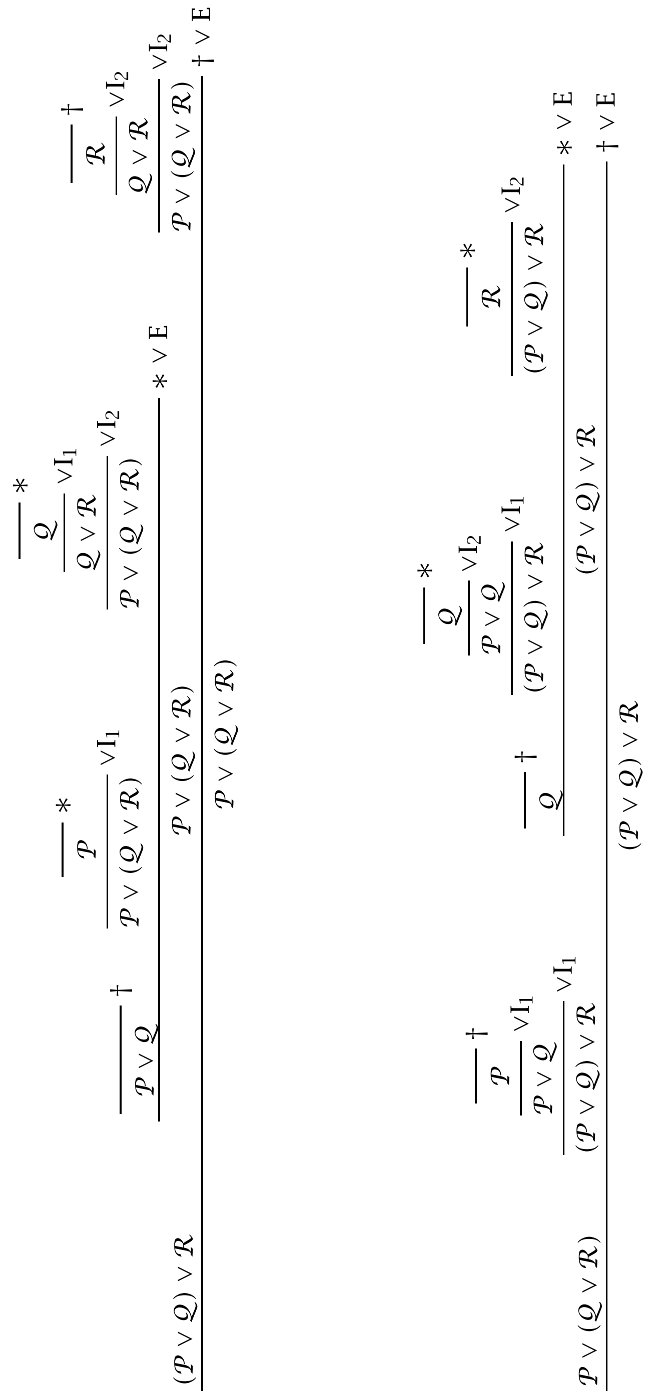


Figure A.4: Solution to Exercise 32, Question 4

Solution 34: Truth tables and the conditional connective

1. (a)

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{P} \Rightarrow \neg\mathcal{Q}$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

(b)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{Q} \vee \mathcal{R}$	$\mathcal{P} \Rightarrow (\neg\mathcal{Q} \vee \mathcal{R})$
T	T	T	F	F	T	T
T	T	F	F	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	F	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

(c)

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\mathcal{Q} \wedge \mathcal{P}$	$\neg\mathcal{P} \Rightarrow (\mathcal{Q} \wedge \mathcal{P})$	$(\neg\mathcal{P} \Rightarrow (\mathcal{Q} \wedge \mathcal{P})) \Rightarrow \mathcal{Q}$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	F	T
F	F	T	F	F	T

2. (a) Neither
   (b) Contradiction

$\mathcal{P}$	$\neg\mathcal{P}$	$\neg\mathcal{P} \Rightarrow \mathcal{P}$	$\mathcal{P} \Rightarrow \neg\mathcal{P}$	$(\mathcal{P} \Rightarrow \neg\mathcal{P}) \wedge (\neg\mathcal{P} \Rightarrow \mathcal{P})$
T	F	T	F	F
F	T	F	T	F

   (c) Tautology
   (d) Neither

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$	$(\mathcal{P} \wedge \mathcal{Q}) \Rightarrow \mathcal{P}$	$\mathcal{P} \Rightarrow (\mathcal{P} \wedge \mathcal{Q})$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	T	T

   (e) Neither
   (f) Tautology

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \vee \mathcal{Q}$	$(\mathcal{P} \vee \mathcal{Q}) \Rightarrow \mathcal{P}$	$\mathcal{P} \Rightarrow (\mathcal{P} \vee \mathcal{Q})$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

(g) Contradiction

$\mathcal{P}$	$\neg\mathcal{P}$	$\mathcal{P} \vee \neg\mathcal{P}$	$\mathcal{P} \wedge \neg\mathcal{P}$	$(\mathcal{P} \vee \neg\mathcal{P}) \Rightarrow (\mathcal{P} \wedge \neg\mathcal{P})$
T	F	T	F	F
F	T	T	F	F

(h) Tautology

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{Q}$	$(\mathcal{P} \Rightarrow \mathcal{Q}) \vee \mathcal{P}$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

(i) Tautology

(j) Neither (See Figure A.5)

(k) Neither

$\mathcal{P}$	$\mathcal{P} \Rightarrow \mathcal{P}$	$\neg\mathcal{P}$	$\mathcal{P} \Rightarrow \neg\mathcal{P}$	$(\mathcal{P} \Rightarrow \mathcal{P}) \Rightarrow (\mathcal{P} \Rightarrow \neg\mathcal{P})$
T	T	F	F	F
F	T	T	T	T

(l) Tautology (See Figure A.6)

3. (a)

$\mathcal{P}$	$\neg\mathcal{P}$	$\mathcal{P} \Rightarrow \neg\mathcal{P}$
T	F	F
F	T	T

(b)

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{Q}$	$\mathcal{P} \wedge \neg\mathcal{Q}$	$\neg(\mathcal{P} \wedge \neg\mathcal{Q})$	$\mathcal{P} \Rightarrow \mathcal{Q}$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

(c)

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{Q}$	$\mathcal{P} \wedge \neg\mathcal{Q}$	$\neg(\mathcal{P} \Rightarrow \mathcal{Q})$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

(d)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{Q} \wedge \mathcal{R}$	$\mathcal{P} \Rightarrow \mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{R}$	$\mathcal{P} \Rightarrow (\mathcal{Q} \wedge \mathcal{R})$	$(\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{P} \Rightarrow \mathcal{R})$
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F
T	F	T	F	F	T	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

$\mathcal{P}$	$\mathcal{Q}$	$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{P} \wedge \neg\mathcal{Q}$	$\mathcal{P} \vee \mathcal{Q}$	$(\neg\mathcal{P} \wedge \neg\mathcal{Q}) \Rightarrow (\mathcal{P} \vee \mathcal{Q})$	$(\mathcal{P} \vee \mathcal{Q}) \Rightarrow (\neg\mathcal{P} \wedge \neg\mathcal{Q})$
T	T	F	F	F	T	T	F
T	F	F	T	F	T	T	F
F	T	T	F	F	T	T	F
F	F	T	T	T	F	T	T

Figure A.5: Solution to Exercise 34, Question 2(j)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$	$\mathcal{Q} \wedge \mathcal{P}$	$(\mathcal{P} \wedge \mathcal{Q}) \Rightarrow (\mathcal{Q} \wedge \mathcal{P})$	$(\mathcal{Q} \wedge \mathcal{P}) \Rightarrow (\mathcal{P} \wedge \mathcal{Q})$	$((\mathcal{P} \wedge \mathcal{Q}) \Rightarrow (\mathcal{Q} \wedge \mathcal{P})) \wedge ((\mathcal{Q} \wedge \mathcal{P}) \Rightarrow (\mathcal{P} \wedge \mathcal{Q}))$
T	T	T	T	T	T	T
T	F	F	F	T	T	T
F	T	F	F	T	T	T
F	F	F	F	T	T	T

Figure A.6: Solution to Exercise 34, Question 2(l)

(e)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{Q} \Rightarrow \mathcal{R}$	$\mathcal{P} \wedge \mathcal{Q}$	$\mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{R})$	$(\mathcal{P} \wedge \mathcal{Q}) \Rightarrow \mathcal{R}$
T	T	T	T	T	T	T
T	T	F	F	T	F	F
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	F	T	T
F	T	F	F	F	T	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

4. Either the conclusion is true when all the premisses are true, or at least one premiss is false when the conclusion is false. If there are no premisses, then the conclusion form must be a tautology.

(a)

$\mathcal{P}$	$\mathcal{P} \Rightarrow \mathcal{P}$
T	$\boldsymbol{T}$
F	$\boldsymbol{T}$

(b)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{Q}$	$\neg\mathcal{Q}$	$\neg\mathcal{P}$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$

(c)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{Q} \Rightarrow \mathcal{P}$
$\boldsymbol{T}$	T	$\boldsymbol{T}$
$\boldsymbol{T}$	F	$\boldsymbol{T}$
F	T	F
F	F	T

(d)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{Q}$
T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
T	F	F	F
F	T	F	T
F	F	F	T

(e)

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{P} \Rightarrow \mathcal{Q}$	$(\mathcal{P} \Rightarrow \mathcal{Q}) \Rightarrow \mathcal{R}$	$\mathcal{Q} \Rightarrow \mathcal{R}$
T	T	T	T	T	T
T	T	F	T	$\boldsymbol{F}$	$\boldsymbol{F}$
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	$\boldsymbol{F}$	$\boldsymbol{F}$
F	F	T	T	T	T
F	F	F	T	F	T

(f)

$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_1 \Rightarrow \mathcal{P}_2$	$\mathcal{P}_2 \Rightarrow \mathcal{P}_3$	$\mathcal{P}_1 \Rightarrow \mathcal{P}_3$
T	T	T	T	T	T
T	T	F	T	$\boldsymbol{F}$	$\boldsymbol{F}$
T	F	T	F	T	T
T	F	F	$\boldsymbol{F}$	T	$\boldsymbol{F}$
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

(g) See Figure A.7

Solution 35: Equational logic

1. (a)

$$
\begin{array}{lll}
 & \mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \neg\mathcal{P}) & \\
=_T & \neg\mathcal{P} \vee (\neg\mathcal{Q} \vee \neg\mathcal{P}) & \langle\ \mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{A} \vee \mathfrak{B}\ \rangle \\
=_T & \neg\mathcal{P} \vee (\neg\mathcal{P} \vee \neg\mathcal{Q}) & \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle \\
=_T & (\neg\mathcal{P} \vee \neg\mathcal{P}) \vee \neg\mathcal{Q} & \langle\ \mathfrak{A} \vee (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \vee \mathfrak{C}\ \rangle \\
=_T & \neg\mathcal{P} \vee \neg\mathcal{Q} & \langle\ \mathfrak{A} \vee \mathfrak{A} =_T \mathfrak{A}\ \rangle
\end{array}
$$

(b)

$$
\begin{array}{lll}
 & (\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{P} \Rightarrow \neg\mathcal{Q}) & \\
=_T & \mathcal{P} \Rightarrow (\mathcal{Q} \wedge \neg\mathcal{Q}) & \langle\ \mathfrak{A} \Rightarrow (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \Rightarrow \mathfrak{B}) \wedge (\mathfrak{A} \Rightarrow \mathfrak{C})\ \rangle \\
=_T & \mathcal{P} \Rightarrow F & \langle\ \mathfrak{A} \wedge \neg\mathfrak{A} =_T F\ \rangle \\
=_T & \neg\mathcal{P} & \langle\ \mathfrak{A} \Rightarrow F =_T \neg\mathfrak{A}\ \rangle
\end{array}
$$

(c)

$$
\begin{array}{lll}
 & \mathcal{P} \Rightarrow (\mathcal{P} \Rightarrow \mathcal{Q}) & \\
=_T & (\mathcal{P} \wedge \mathcal{P}) \Rightarrow \mathcal{Q} & \langle\ \mathfrak{A} \Rightarrow (\mathfrak{B} \Rightarrow \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \Rightarrow \mathfrak{C}\ \rangle \\
=_T & \mathcal{P} \Rightarrow \mathcal{Q} & \langle\ \mathfrak{A} \wedge \mathfrak{A} =_T \mathfrak{A}\ \rangle
\end{array}
$$

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{S}$	$\neg\mathcal{Q}$	$\neg\mathcal{S}$	$\mathcal{R} \vee \neg\mathcal{S}$	$\neg\mathcal{Q} \Rightarrow (\mathcal{R} \vee \neg\mathcal{S})$	$\mathcal{P} \wedge (\neg\mathcal{Q} \Rightarrow (\mathcal{R} \vee \neg\mathcal{S}))$	$(\mathcal{P} \wedge (\neg\mathcal{Q} \Rightarrow (\mathcal{R} \vee \neg\mathcal{S}))) \Rightarrow \mathcal{P}$
T	T	T	T	F	F	T	T	T	$\boldsymbol{T}$
T	T	T	F	F	T	T	T	T	$\boldsymbol{T}$
T	T	F	T	F	F	F	T	T	$\boldsymbol{T}$
T	T	F	F	F	T	T	T	T	$\boldsymbol{T}$
T	F	T	T	T	F	T	T	T	$\boldsymbol{T}$
T	F	T	F	T	T	T	T	T	$\boldsymbol{T}$
T	F	F	T	T	F	F	F	F	$\boldsymbol{T}$
T	F	F	F	T	T	T	T	T	$\boldsymbol{T}$
F	T	T	T	F	F	T	T	F	$\boldsymbol{T}$
F	T	T	F	F	T	T	T	F	$\boldsymbol{T}$
F	T	F	T	F	F	F	T	F	$\boldsymbol{T}$
F	T	F	F	F	T	T	T	F	$\boldsymbol{T}$
F	F	T	T	T	F	T	T	F	$\boldsymbol{T}$
F	F	T	F	T	T	T	T	F	$\boldsymbol{T}$
F	F	F	T	T	F	F	F	F	$\boldsymbol{T}$
F	F	F	F	T	T	T	T	F	$\boldsymbol{T}$

Figure A.7: Solution to Exercise 34, Question 4(g)

(d)
$$\begin{array}{lll} & \mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{P}) & \\ =_T & \mathcal{P} \Rightarrow (\neg\mathcal{P} \Rightarrow \neg\mathcal{Q}) & \langle\ \mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{B} \Rightarrow \neg\mathfrak{A}\ \rangle \\ =_T & (\mathcal{P} \wedge \neg\mathcal{P}) \Rightarrow \mathcal{Q} & \langle\ \mathfrak{A} \Rightarrow (\mathfrak{B} \Rightarrow \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \Rightarrow \mathfrak{C}\ \rangle \\ =_T & F \Rightarrow \neg\mathcal{Q} & \langle\ \mathfrak{A} \wedge \neg\mathfrak{A} =_T F\ \rangle \\ =_T & T & \langle\ F \Rightarrow \mathfrak{A} =_T T\ \rangle \end{array}$$

(e)
$$\begin{array}{lll} & (\mathcal{P} \Rightarrow \mathcal{Q}) \vee (\neg\mathcal{P} \Rightarrow \mathcal{Q}) & \\ =_T & (\neg\mathcal{Q} \Rightarrow \neg\mathcal{P}) \vee (\neg\mathcal{Q} \Rightarrow \neg\neg\mathcal{P}) & \langle\ \mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{B} \Rightarrow \neg\mathfrak{A}\ \rangle \\ =_T & \neg\mathcal{Q} \Rightarrow (\neg\mathcal{P} \vee \neg\neg\mathcal{P}) & \\ & \langle\ \mathfrak{A} \Rightarrow (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \Rightarrow \mathfrak{B}) \vee (\mathfrak{A} \Rightarrow \mathfrak{C})\ \rangle & \\ =_T & \neg\mathcal{Q} \Rightarrow T & \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle \\ =_T & T & \langle\ \mathfrak{A} \Rightarrow T =_T T\ \rangle \end{array}$$

(f)
$$\begin{array}{lll} & (\mathcal{P} \Rightarrow (\mathcal{Q} \vee \neg\mathcal{P})) \Rightarrow \mathcal{Q} & \\ =_T & (\mathcal{P} \Rightarrow (\neg\mathcal{P} \vee \mathcal{Q})) \Rightarrow \mathcal{Q} & \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle \\ =_T & (\mathcal{P} \Rightarrow (\mathcal{P} \Rightarrow \mathcal{Q})) \Rightarrow \mathcal{Q} & \langle\ \mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{A} \vee \mathfrak{B}\ \rangle \\ =_T & ((\mathcal{P} \wedge \mathcal{P}) \Rightarrow \mathcal{Q}) \Rightarrow \mathcal{Q} & \langle\ \mathfrak{A} \Rightarrow (\mathfrak{B} \Rightarrow \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \Rightarrow \mathfrak{C}\ \rangle \\ =_T & (\mathcal{P} \Rightarrow \mathcal{Q}) \Rightarrow \mathcal{Q} & \langle\ \mathfrak{A} \wedge \mathfrak{A} =_T \mathfrak{A}\ \rangle \\ =_T & T & \langle\ (\mathfrak{A} \Rightarrow \mathfrak{B}) \Rightarrow \mathfrak{B} =_T T\ \rangle \end{array}$$

(g)
$$\begin{array}{lll} & (\mathcal{P} \Rightarrow (\mathcal{Q} \wedge \mathcal{P})) \Rightarrow \neg\mathcal{Q} & \\ =_T & ((\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{P} \Rightarrow \mathcal{P})) \Rightarrow \neg\mathcal{Q} & \\ & \langle\ \mathfrak{A} \Rightarrow (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \Rightarrow \mathfrak{B}) \wedge (\mathfrak{A} \Rightarrow \mathfrak{C})\ \rangle & \\ =_T & ((\mathcal{P} \Rightarrow \mathcal{Q}) \wedge T) \Rightarrow \neg\mathcal{Q} & \langle\ \mathfrak{A} \Rightarrow \mathfrak{A} =_T T\ \rangle \\ =_T & (\mathcal{P} \Rightarrow \mathcal{Q}) \Rightarrow \neg\mathcal{Q} & \langle\ \mathfrak{A} \wedge T =_T \mathfrak{A}\ \rangle \\ =_T & (\neg\mathcal{Q} \Rightarrow \neg\mathcal{P}) \Rightarrow \neg\mathcal{Q} & \langle\ \mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{B} \Rightarrow \neg\mathfrak{A}\ \rangle \\ =_T & \neg\mathcal{Q} & \langle\ (\mathfrak{A} \Rightarrow \mathfrak{B}) \Rightarrow \mathfrak{A} =_T \mathfrak{A}\ \rangle \end{array}$$

2. (a)
$$\begin{array}{lll} & \mathcal{P} \Rightarrow (\mathcal{P} \vee \mathcal{Q}) & \\ =_T & (\mathcal{P} \Rightarrow \mathcal{P}) \vee (\mathcal{P} \Rightarrow \mathcal{Q}) & \langle\ \mathfrak{A} \Rightarrow (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \Rightarrow \mathfrak{B}) \vee (\mathfrak{A} \Rightarrow \mathfrak{C})\ \rangle \\ =_T & T \vee (\mathcal{P} \Rightarrow \mathcal{Q}) & \langle\ \mathfrak{A} \Rightarrow \mathfrak{A} =_T T\ \rangle \\ =_T & (\mathcal{P} \Rightarrow \mathcal{Q}) \vee T & \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle \\ =_T & T & \langle\ \mathfrak{A} \vee T =_T T\ \rangle \end{array}$$

(b) From Qu.1(d) we know that $\mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{P}) =_T T$.

(c)
$$\begin{array}{lll} & (\mathcal{P} \wedge \mathcal{Q}) \Rightarrow (\mathcal{P} \Rightarrow \mathcal{Q}) & \\ =_T & ((\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{P}) \Rightarrow \mathcal{Q} & \langle\ \mathfrak{A} \Rightarrow (\mathfrak{B} \Rightarrow \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \Rightarrow \mathfrak{C}\ \rangle \\ =_T & (\mathcal{P} \wedge (\mathcal{P} \wedge \mathcal{Q})) \Rightarrow \mathcal{Q} & \langle\ \mathfrak{A} \wedge \mathfrak{B} =_T \mathfrak{B} \wedge \mathfrak{A}\ \rangle \\ =_T & ((\mathcal{P} \wedge \mathcal{P}) \wedge \mathcal{Q}) \Rightarrow \mathcal{Q} & \langle\ \mathfrak{A} \wedge (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \wedge \mathfrak{C}\ \rangle \\ =_T & (\mathcal{P} \wedge \mathcal{Q}) \Rightarrow \mathcal{Q} & \langle\ \mathfrak{A} \wedge \mathfrak{A} =_T \mathfrak{A}\ \rangle \\ =_T & \mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{Q}) & \langle\ \mathfrak{A} \Rightarrow (\mathfrak{B} \Rightarrow \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \Rightarrow \mathfrak{C}\ \rangle \\ =_T & \mathcal{P} \Rightarrow T & \langle\ \mathfrak{A} \Rightarrow \mathfrak{A} =_T T\ \rangle \\ =_T & T & \langle\ \mathfrak{A} \Rightarrow T =_T T\ \rangle \end{array}$$

(d)

$$\begin{array}{lll}
 & \neg(\neg\mathcal{P} \vee \mathcal{Q}) \Rightarrow \mathcal{P} & \\
=_T & \neg\neg(\neg\mathcal{P} \vee \mathcal{Q}) \vee \mathcal{P} & \langle\ \mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{A} \vee \mathfrak{B}\ \rangle \\
=_T & (\neg\mathcal{P} \vee \mathcal{Q}) \vee \mathcal{P} & \langle\ \neg\neg\mathfrak{A} =_T \mathfrak{A}\ \rangle \\
=_T & \mathcal{P} \vee (\neg\mathcal{P} \vee \mathcal{Q}) & \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle \\
=_T & (\mathcal{P} \vee \neg\mathcal{P}) \vee \mathcal{Q} & \langle\ \mathfrak{A} \vee (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \vee \mathfrak{C}\ \rangle \\
=_T & T \vee \mathcal{Q} & \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle \\
=_T & \mathcal{Q} \vee T & \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle \\
=_T & T & \langle\ \mathfrak{A} \vee T =_T T\ \rangle
\end{array}$$

(e)

$$\begin{array}{lll}
 & ((\mathcal{P} \vee \mathcal{Q}) \Rightarrow \mathcal{Q}) \Rightarrow (\mathcal{P} \Rightarrow \mathcal{Q}) & \\
=_T & ((\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \Rightarrow \mathcal{Q})) \Rightarrow (\mathcal{P} \Rightarrow \mathcal{Q}) & \\
 & & \langle\ (\mathfrak{A} \vee \mathfrak{B}) \Rightarrow \mathfrak{C} =_T (\mathfrak{A} \Rightarrow \mathfrak{C}) \wedge (\mathfrak{B} \Rightarrow \mathfrak{C})\ \rangle \\
=_T & ((\mathcal{P} \Rightarrow \mathcal{Q}) \wedge T) \Rightarrow (\mathcal{P} \Rightarrow \mathcal{Q}) & \langle\ \mathfrak{A} \Rightarrow \mathfrak{A} =_T T\ \rangle \\
=_T & (\mathcal{P} \Rightarrow \mathcal{Q}) \Rightarrow (\mathcal{P} \Rightarrow \mathcal{Q}) & \langle\ \mathfrak{A} \wedge T =_T \mathfrak{A}\ \rangle \\
=_T & T & \langle\ \mathfrak{A} \Rightarrow \mathfrak{A} =_T T\ \rangle
\end{array}$$

(f)

$$\begin{array}{lll}
 & ((\mathcal{P} \Rightarrow \mathcal{Q}) \wedge \neg\mathcal{Q}) \Rightarrow \neg\mathcal{P} & \\
=_T & \neg((\neg\mathcal{P} \vee \mathcal{Q}) \wedge \neg\mathcal{Q}) \vee \neg\mathcal{P} & \langle\ \mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{A} \vee \mathfrak{B}\ \rangle \\
=_T & (\neg(\neg\mathcal{P} \vee \mathcal{Q}) \vee \neg\neg\mathcal{Q}) \vee \neg\mathcal{P} & \langle\ \neg(\mathfrak{A} \wedge \mathfrak{B}) =_T \neg\mathfrak{A} \vee \neg\mathfrak{B}\ \rangle \\
=_T & ((\neg\neg\mathcal{P} \wedge \neg\mathcal{Q}) \vee \neg\neg\mathcal{Q}) \vee \neg\mathcal{P} & \langle\ \neg(\mathfrak{A} \vee \mathfrak{B}) =_T \neg\mathfrak{A} \wedge \neg\mathfrak{B}\ \rangle \\
=_T & ((\mathcal{P} \wedge \neg\mathcal{Q}) \vee \mathcal{Q}) \vee \neg\mathcal{P} & \langle\ \neg\neg\mathfrak{A} =_T \mathfrak{A}\ \rangle \\
=_T & (\mathcal{Q} \vee (\mathcal{P} \wedge \neg\mathcal{Q})) \vee \neg\mathcal{P} & \langle\ \mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}\ \rangle \\
=_T & ((\mathcal{Q} \vee \mathcal{P}) \wedge (\mathcal{Q} \vee \neg\mathcal{Q})) \vee \neg\mathcal{P} & \\
 & & \langle\ \mathfrak{A} \vee (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \vee \mathfrak{C}\ \rangle \\
=_T & ((\mathcal{Q} \vee \mathcal{P}) \wedge T) \vee \neg\mathcal{P} & \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle \\
=_T & (\mathcal{Q} \vee \mathcal{P}) \vee \neg\mathcal{P} & \langle\ \mathfrak{A} \wedge T =_T \mathfrak{A}\ \rangle \\
=_T & \mathcal{Q} \vee (\mathcal{P} \vee \neg\mathcal{P}) & \\
 & & \langle\ \mathfrak{A} \vee (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \vee \mathfrak{C}\ \rangle \\
=_T & \mathcal{Q} \vee T & \langle\ \mathfrak{A} \vee \neg\mathfrak{A} =_T T\ \rangle \\
=_T & T & \langle\ \mathfrak{A} \vee T =_T T\ \rangle
\end{array}$$

Solution 36: Deductions using ¬, ∧ and ∨ rules

1. $$\frac{\neg\neg(\mathcal{P} \Rightarrow (\mathcal{R} \vee \mathcal{Q}))}{\mathcal{P} \Rightarrow (\mathcal{R} \vee \mathcal{Q})}\ \neg\neg\mathrm{E}$$

2. $$\frac{\neg\neg\neg\mathcal{Q} \wedge (\mathcal{P} \Rightarrow \neg\neg\mathcal{Q})}{\mathcal{P} \Rightarrow \neg\neg\mathcal{Q}}\ \wedge\mathrm{E}_2$$

3. $$\frac{\mathcal{P}}{\mathcal{P} \vee (\mathcal{Q} \Rightarrow \mathcal{R})}\ \vee\mathrm{I}_1$$

4. $$\frac{\neg\mathcal{P} \vee \mathcal{S}}{(\mathcal{Q} \Rightarrow \mathcal{R}) \vee (\neg\mathcal{P} \vee \mathcal{S})}\ \vee\mathrm{I}_2$$

5. $$\dfrac{\dfrac{\neg\neg\neg\mathcal{Q} \wedge (\mathcal{P} \Rightarrow \neg\neg\mathcal{Q})}{\neg\neg\neg\mathcal{Q}}\,\wedge E_1}{\neg\mathcal{Q}}\,\neg\neg E$$

6. $$\dfrac{\dfrac{\dfrac{\neg\neg(\mathcal{P} \Rightarrow \mathcal{Q}) \wedge (\neg\mathcal{Q} \Rightarrow \mathcal{R})}{\neg\neg(\mathcal{P} \Rightarrow \mathcal{Q})}\,\wedge E_1}{\mathcal{P} \Rightarrow \mathcal{Q}}\,\neg\neg E}{(\mathcal{P} \Rightarrow \mathcal{Q}) \vee (\mathcal{R} \wedge \mathcal{S})}\,\vee I_1$$

Solution 37: Deductions using the $\Rightarrow$ E rule

1. $$\dfrac{\neg\mathcal{P} \qquad \neg\mathcal{P} \Rightarrow \mathcal{Q}}{\mathcal{Q}}\,\Rightarrow E$$

2. $$\dfrac{\dfrac{\neg\neg\mathcal{P}}{\mathcal{P}}\,\neg\neg E \qquad \mathcal{P} \Rightarrow \mathcal{Q}}{\mathcal{Q}}\,\Rightarrow E$$

3. $$\dfrac{\dfrac{\mathcal{P} \qquad \mathcal{P} \Rightarrow (\mathcal{P} \wedge \mathcal{Q})}{\mathcal{P} \wedge \mathcal{Q}}\,\Rightarrow E}{\mathcal{Q}}\,\wedge E_2$$

4. $$\dfrac{\dfrac{\mathcal{P} \qquad \mathcal{P} \Rightarrow (\mathcal{P} \Rightarrow \mathcal{Q})}{\mathcal{P} \Rightarrow \mathcal{Q}}\,\Rightarrow E \qquad \mathcal{P}}{\mathcal{Q}}\,\Rightarrow E$$

5. $$\dfrac{\dfrac{\mathcal{P} \wedge \mathcal{Q}}{\mathcal{Q}}\,\wedge E_2 \qquad \mathcal{Q} \Rightarrow \mathcal{R}}{\mathcal{R}}\,\Rightarrow E$$

6. $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{P} \wedge \neg\mathcal{Q}}\,*}{\mathcal{P}}\,\wedge E_1 \qquad \mathcal{P} \Rightarrow \mathcal{Q}}{\mathcal{Q}}\,\Rightarrow E \qquad \dfrac{\dfrac{}{\mathcal{P} \wedge \neg\mathcal{Q}}\,*}{\neg\mathcal{Q}}\,\wedge E_2}{\mathcal{Q} \wedge \neg\mathcal{Q}}\,\wedge I}{\neg(\mathcal{P} \wedge \neg\mathcal{Q})}\,*\neg I$$

7. The deduction tree is very similar to that of the previous example except that the ¬I rule is applied to yield $\neg(\mathcal{P} \Rightarrow \mathcal{Q})$ rather than $\neg(\mathcal{P} \wedge \neg\mathcal{Q})$

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{P}\wedge\neg\mathcal{Q}}{\mathcal{P}}\,\wedge\mathrm{E}_1 \qquad \dfrac{}{\mathcal{P}\Rightarrow\mathcal{Q}}\,*}{\mathcal{Q}}\,\Rightarrow\mathrm{E} \qquad \dfrac{\mathcal{P}\wedge\neg\mathcal{Q}}{\neg\mathcal{Q}}\,\wedge\mathrm{E}_2}{\mathcal{Q}\wedge\neg\mathcal{Q}}\,\wedge\mathrm{I}}{\neg(\mathcal{P}\Rightarrow\mathcal{Q})}\,*\neg\mathrm{I}$$

8.

$$\dfrac{\mathcal{P}\vee\mathcal{Q} \qquad \dfrac{\dfrac{}{\mathcal{P}}\,* \qquad \mathcal{P}\Rightarrow\mathcal{R}}{\mathcal{R}}\,\Rightarrow\mathrm{E} \qquad \dfrac{\dfrac{}{\mathcal{Q}}\,* \qquad \mathcal{Q}\Rightarrow\mathcal{R}}{\mathcal{R}}\,\Rightarrow\mathrm{E}}{\mathcal{R}}\,*\vee\mathrm{E}$$

9. See Figure A.8

Solution 38: Using ⇒ I

1.

$$\dfrac{\dfrac{\dfrac{}{\neg\neg\mathcal{P}}\,*}{\mathcal{P}}\,\neg\neg\mathrm{E}}{\neg\neg\mathcal{P}\Rightarrow\mathcal{P}}\,*\Rightarrow\mathrm{I}$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{P}\wedge\mathcal{R}}\,*}{\mathcal{R}}\,\wedge\mathrm{E}_2}{\mathcal{R}\vee\mathcal{Q}}\,\vee\mathrm{I}_1}{(\mathcal{P}\wedge\mathcal{R})\Rightarrow(\mathcal{R}\vee\mathcal{Q})}\,*\Rightarrow\mathrm{I}$$

3.

$$\dfrac{\dfrac{\mathcal{P} \qquad \dfrac{}{\mathcal{P}\Rightarrow\mathcal{Q}}\,*}{\mathcal{Q}}\,\Rightarrow\mathrm{E}}{(\mathcal{P}\Rightarrow\mathcal{Q})\Rightarrow\mathcal{Q}}\,*\Rightarrow\mathrm{I}$$

4.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{P}}\,* \qquad \mathcal{P}\Rightarrow\mathcal{Q}}{\mathcal{Q}}\,\Rightarrow\mathrm{E} \qquad \mathcal{Q}\Rightarrow\mathcal{R}}{\mathcal{R}}\,\Rightarrow\mathrm{E}}{\mathcal{P}\Rightarrow\mathcal{R}}\,*\Rightarrow\mathrm{I}$$

$$
\dfrac{
(\mathcal{P} \Rightarrow \mathcal{Q}) \vee (\neg\mathcal{Q} \Rightarrow \mathcal{R})
\qquad
\dfrac{
\dfrac{\mathcal{P} \wedge \neg\mathcal{R}}{\mathcal{P}}\,{\scriptstyle \wedge \mathrm{E}_1}
\qquad
\dfrac{}{\mathcal{P} \Rightarrow \mathcal{Q}}\,{\scriptstyle \dagger}
}{\mathcal{Q}}\,{\scriptstyle \Rightarrow \mathrm{E}}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\mathcal{P} \wedge \neg\mathcal{R}}{\neg\mathcal{R}}\,{\scriptstyle \wedge \mathrm{E}_2}
\qquad
\dfrac{
\dfrac{}{\neg\mathcal{Q}}\,{\scriptstyle *}
\qquad
\dfrac{}{\neg\mathcal{Q} \Rightarrow \mathcal{R}}\,{\scriptstyle \dagger}
}{\mathcal{R}}\,{\scriptstyle \Rightarrow \mathrm{E}}
}{\mathcal{R} \wedge \neg\mathcal{R}}\,{\scriptstyle \wedge \mathrm{I}}
}{\neg\neg\mathcal{Q}}\,{\scriptstyle * \neg \mathrm{I}}
}{\mathcal{Q}}\,{\scriptstyle \neg\neg \mathrm{E}}
}{\mathcal{Q}}\,{\scriptstyle \dagger \vee \mathrm{E}}
$$

Figure A.8: Solution to Exercise 37, Question 9

Solution 39: Derived rules of deduction – MT and HS

1. (a)
$$\dfrac{\mathcal{P} \Rightarrow \neg\mathcal{Q} \qquad \neg\neg\mathcal{Q}}{\neg\mathcal{P}}\,\text{MT}$$

   (b)
$$\dfrac{\mathcal{P} \Rightarrow \neg\mathcal{Q} \qquad \dfrac{\mathcal{Q}}{\neg\neg\mathcal{Q}}\,\neg\neg\text{I}}{\neg\mathcal{P}}\,\text{MT}$$

   (c)
$$\dfrac{\dfrac{\neg\mathcal{P} \Rightarrow \mathcal{Q} \qquad \neg\mathcal{Q}}{\neg\neg\mathcal{P}}\,\text{MT}}{\mathcal{P}}\,\neg\neg\text{E}$$

   (d)
$$\dfrac{\dfrac{\mathcal{P} \Rightarrow \mathcal{Q} \qquad \neg\mathcal{Q}}{\neg\mathcal{P}}\,\text{MT}}{\neg\mathcal{P} \vee \mathcal{Q}}\,\vee\text{I}_1$$

   (e)
$$\dfrac{\dfrac{\mathcal{P} \Rightarrow (\mathcal{Q} \wedge \mathcal{R}) \qquad \neg(\mathcal{Q} \wedge \mathcal{R})}{\neg\mathcal{P}}\,\text{MT}}{\neg\mathcal{P} \vee \mathcal{S}}\,\vee\text{I}_2$$

2. (a)
$$\dfrac{\mathcal{P} \qquad \dfrac{\mathcal{P} \Rightarrow \mathcal{Q} \qquad \mathcal{Q} \Rightarrow \mathcal{R}}{\mathcal{P} \Rightarrow \mathcal{R}}\,\text{HS}}{\mathcal{R}}\,\Rightarrow\text{E}$$

   (b)
$$\dfrac{\dfrac{\dfrac{\neg\mathcal{P} \Rightarrow \neg\mathcal{Q} \qquad \neg\mathcal{Q} \Rightarrow \mathcal{R}}{\neg\mathcal{P} \Rightarrow \mathcal{R}}\,\text{HS} \qquad \neg\mathcal{R}}{\neg\neg\mathcal{P}}\,\text{MT}}{\mathcal{P}}\,\neg\neg\text{E}$$

Solution 40: Predicates from propositions

1. '…*is red*', FLOWERS. The predicate represents the property that the subject is red.
2. '…*is a perfect square*', INTEGERS ('integer' is another name for 'whole number'). The property represents the property that the subject is a perfect square, that is it is equal to the square of an integer.Note that although NUMBERS would also be a possibility, the concept of a perfect square usually arises when we are are talking about INTEGERS, or, even more specifically, NON-NEGATIVE INTEGERS.

3. '*...has brown hair*', PEOPLE. Again, there are other possibilities; DOGS for example. In a textbook question such as this, we are working in the abstract somewhat; in a real situation, an appropriate universe of discourse would be much clearer to identify.
4. '*...are yellow*', SPECIES OF FLOWERS. Care needs to be taken in choosing an appropriate universe of discourse. The subject '*Daffodils*' refers to the collection of all daffodils, not just one particular daffodil. The predicate represents the property that *each member* of the subject is yellow.

Solution 41: Unary predicates

1. (a) $3 > 2$.
   (b) $1 > 2$.
   (c) $6 = 6$.
   (d) '*Sydney is the capital of Australia*'.
   (e) '*Canberra is the capital of Australia*'.
   (f) '*Queen Elizabeth II is the Duke of Normandy*'.
2. The universes of discourse are suggestions only. Other appropriate letters may be used for the free variables.
   (a) '*x is a prime number.*' POSITIVE INTEGERS
   (b) '*y is very cold.*' CONTINENTS
   (c) '*z is blue.*' DOORS
   (d) It might seem that this is derived from the predicate '*w is blue*', but it is difficult to decide upon an appropriate universe of discourse! Perhaps NATURAL SIGHTS is appropriate, though this is not entirely unambiguous.
   (e) Possibilities include $x \geq 7$, $7 \geq x$ or even $x \geq x$. REAL NUMBERS

Solution 42: Predicates with more than one free variable

1. (a) $3 > y$.
   (b) $x > 7$.
   (c) '*Canberra is the capital of y*'.
   (d) '*x is the capital of the United Kingdom*'.
2. (a) $3 > 2$.
   (b) $5 > 7$.
   (c) '*Canberra is the capital of Australia*'.
   (d) '*Westminster is the capital of the United Kingdom*'.
3. (a) '*x is higher than y*'. For both free variables an appropriate universe of discourse is MOUNTAINS.
   (b) '*x flows through y*'. An appropriate universe of discourse for x is RIVERS and for y is COUNTRIES.
   (c) $x + y = 6$. An appropriate universe of discourse for both x and y is INTEGERS.

Solution 43: Compound predicates

1. $p(x) \vee q(y)$ where $p(x)$ is '*x is late*' and $q(y)$ is '*y is fast*'. Note that in English, the adjective '*fast*' has other meanings. However, the predicate $q(y)$ refers to one specific property, namely that of showing a time ahead of the correct time; thus TIMEPIECES would be an appropriate universe of discourse, while ANIMALS would not.
2. $p(x) \wedge p(y)$ where $p(x)$ is '*x is a capital city*'. In this case the same predicate is used twice but with different free variables.
3. One possibility is $x{>}y \Rightarrow x^2{>}y^2$. The atomic predicates are $x{>}y$ and $x^2{>}y^2$.
4. One possibility is $x_1{>}x_2 \wedge x_3{+}x_4{=}x_5$. The atomic predicates are $x_1{>}x_2$ and $x_3{+}x_4{=}x_5$.

Solution 44: Functions

1. (a) Margaret Thatcher's father was a grocer.
   (b) The capital of England is further east than the capital of Scotland.
   (c) $2 + 3 \geq \sqrt{2 \times 3}$
2. There may be more than one way of symbolizing each of these sentences. The following are *possible* solutions.

   (a) $p_1(f(a))$
   (b) $p_1(f(a)) \wedge p_2(a)$
   (c) $p_3(f(a), a)$
   (d) $\forall x\,(p_4(f(x), x))$

   where

   - $p_1(x)$ is the predicate '*x has four legs*';
   - $p_2(x)$ is the predicate '*x has three legs*';
   - $p_3(x, y)$ is the predicate '*x is older than y*';
   - $p_4(x, y)$ is the predicate '*x was born before y*';
   - $f(x)$ is the function '*the mother of x*';
   - a is the constant '*Fido*'.

Solution 45: Predicate forms

1. (a) $\mathcal{Q}$ Yes; a schematic letter on its own can represent any proposition.
   (b) $\mathcal{P}(x)$ No; only predicates with the single free variable x are possible.
   (c) $\mathcal{P}(x, y)$ Yes.
   (d) $\mathcal{P}(x, y, z\backslash 2)$ Yes; $x^z = y$ could be an instance of $\mathcal{P}(x, y, z)$ and substituting 2 for z would then give the required predicate. The predicate form $\mathcal{P}(x, y, z\backslash 2)$ has two free variables, x and y.
   (e) $\mathcal{P}[x]$ Yes.

(f) $\mathcal{Q}[y, x]$ Yes. The order in which free variables are listed does not have to be the same as the order in which they occur in the predicate.
(g) $\mathcal{P}[x, y, z]$ No. The free variable z does not occur in $x^2 = y$.
(h) $\mathcal{P}[x, y, z\backslash 2]$ Yes.
(i) $\mathcal{P}[x, y\backslash 2, z]$ No. The free variable z does not occur in $x^2 = y$.

2. (a) $\mathcal{P}(x, y, z)$ Yes; $\mathcal{P}$ does not necessarily refer to an atomic predicate.
(b) $\mathcal{P}(v\backslash 1, w\backslash 2, x, y, z)$ Yes; the free variables are x, y and z, as required.
(c) $\mathcal{Q}(x, y) \Rightarrow \mathcal{P}(x, z)$ Yes.
(d) $\mathcal{Q}(x, z) \Rightarrow \mathcal{P}(x, y)$ No.
(e) $\mathcal{P}[x] \Rightarrow \mathcal{Q}[x]$ Yes.
(f) $\mathcal{P}[x, y, z] \Rightarrow \mathcal{Q}[x, y, z]$ No.
(g) $\mathcal{P}[x, y, z\backslash 1] \Rightarrow \mathcal{Q}[x, y\backslash 2, z]$ Yes.
(h) $\mathcal{P}[x, y, z\backslash 1] \Rightarrow \mathcal{P}[x, y\backslash 2, z]$ No.

Solution 46: Universal quantifier

1. (a) ‘*Every integer is equal to its square*’
(b) ‘*All dogs have three legs*’
(c) ‘*Every capital city has hosted the Olympic games*’

2. (a) ‘*Every positive integer is equal to its square*’
(b) ‘*All dogs have three legs*’
(c) ‘*Every capital city has hosted the Olympic games*’
Note the similarities of interpretation in these first two questions.

3. (a) ‘*Every integer is positive and equal to its square*’
(b) ‘*All animals are dogs and have three legs*’
(c) ‘*Every town and city is a capital city has hosted the Olympic games*’

4. (a) ‘*Every integer is positive and every integer is equal to its square*’
(b) ‘*All animals are dogs and all animals have three legs*’
(c) ‘*Every town and city is a capital city and every town and city has hosted the Olympic games*’
Notice the fact that although interpretations in these two questions are different, the truth values are the same.

5. (a) ‘*Every integer is either positive or equal to its square*’
(b) ‘*Every animal is either a dog or has three legs*’
(c) ‘*Every town and city is either a capital city or has hosted the Olympic games*’

6. (a) ‘*Either every integer is positive or every integer is equal to its square*’
(b) ‘*Either all animals are dogs or every animal has three legs*’
(c) ‘*Either every town and city is a capital city or every town and city has hosted the Olympic games*’

Notice that in these last two questions, not only are the interpretations different but also the truth values may be different.

Solution 47: Existential quantifier

1. (a) '*At least one integer is equal to its square*'
   (b) '*Some dogs have three legs*'
   (c) '*Some capital cities have hosted the Olympic games*'

2. (a) '*At least one integer is positive and equal to its square*' More succinctly we can write, '*There exists a positive integer which is equal to its square*'
   (b) '*Some animals are three legged dogs*'
   (c) '*Some capital cities have hosted the Olympic games*'

3. (a) '*There exists a positive integer and there exists an integer which is equal to its square*'
   (b) '*Some animals are dogs and some animals have three legs*'
   (c) '*Some towns and cities are capital cities and some towns and cities have hosted the Olympic games*'

4. (a) '*At least one integer is either positive or equal to its square*'
   (b) '*Some animals are either dogs or have three legs*'
   (c) '*Some towns and cities are either capital cities or have hosted the Olympic games*'

5. (a) '*Either some integers are positive or some are equal to their squares*'
   (b) '*Either some animals are dogs or some animals have three legs*'
   (c) '*Either some towns and cities are either capital cities or some town and cities have hosted the Olympic games*'

   Notice that the expressions in these last two questions have different interpretations and may even have different truth values, yet the English sentences sound very similar. Frequently in English, a statement like '*Either some animals are dogs or some animals have three legs*' is shortened to '*Some animals are dogs or have three legs*', which is not an equivalent sentence.

6. The interpretation in these cases is a little problematic. Perhaps the best approach is to appeal to the truth value properties of $p(x) \Rightarrow q(x)$: the conditional is T either when the antecedent $p(x)$ is F or when the consequent $q(x)$ is T.

   (a) '*Some integers are either non-positive or equal to their squares*'
   (b) '*Some animals have three legs or are not dogs*'
   (c) '*Some towns and cities have hosted the Olympic games or are not capital cities*'

Solution 48: Deduction rules: $\forall$E and $\exists$I

1. (a) $\forall x\,(\mathcal{P}(x) \wedge \mathcal{Q}(x)) \vdash \exists x\,(\mathcal{P}(x) \vee \mathcal{R}(x))$

$$\dfrac{\dfrac{\dfrac{\forall x\,(\mathcal{P}(x) \wedge \mathcal{Q}(x))}{\mathcal{P}(a) \wedge \mathcal{Q}(a)}\,\forall\text{E}}{\mathcal{P}(a)}\,\wedge\text{E}_1}{\exists x\,(\mathcal{P}(x) \vee \mathcal{R}(x))}\,\exists\text{I}$$

   (b) $\neg\neg\forall x\,(\mathcal{P}(x) \wedge \mathcal{Q}x) \vdash \exists x\,(\mathcal{P}(x) \wedge \mathcal{Q}x)$

$$\dfrac{\dfrac{\dfrac{\neg\neg\forall x\,(\mathcal{P}(x) \wedge \mathcal{Q}x)}{\forall x\,(\mathcal{P}(x) \wedge \mathcal{Q}x)}\,\neg\neg\text{E}}{\mathcal{P}(a) \wedge \mathcal{Q}a}\,\forall\text{E}}{\exists x\,(\mathcal{P}(x) \wedge \mathcal{Q}x)}\,\exists\text{I}$$

   (c) $\neg\mathcal{Q}(a), \forall x\,(\mathcal{P}(x) \Rightarrow \mathcal{Q}(x)) \vdash \exists x\,(\neg\mathcal{P}(x))$

$$\dfrac{\dfrac{\dfrac{\forall x\,(\mathcal{P}(x) \Rightarrow \mathcal{Q}(x))}{\mathcal{P}(a) \Rightarrow \mathcal{Q}(a)}\,\forall\text{E} \qquad \neg\mathcal{Q}(a)}{\neg\mathcal{P}(a)}\,\text{MT}}{\exists x\,(\neg\mathcal{P}(x))}\,\exists\text{I}$$

2. (a) $\forall x\,(\neg\mathcal{P}(x)) \vdash \exists x\,(\neg\mathcal{P}(x))$

$$\dfrac{\dfrac{\forall x\,(\neg\mathcal{P}(x))}{\neg\mathcal{P}(a)}\,\forall\text{E}}{\exists x\,(\neg\mathcal{P}(x))}\,\exists\text{I}$$

   (b) $\forall x\,(\neg\neg\mathcal{P}(x)) \vdash \exists x\,(\mathcal{P}(x))$

$$\dfrac{\dfrac{\dfrac{\forall x\,(\neg\neg\mathcal{P}(x))}{\neg\neg\mathcal{P}(a)}\,\forall\text{E}}{\mathcal{P}(a)}\,\neg\neg\text{E}}{\exists x\,(\mathcal{P}(x))}\,\exists\text{I}$$

(c) '*Canberra is the capital city of Australia*' ⊢'*Australia has a capital city*'
Take TOWNS AND CITIES as the universe of discourse, and let $p(x)$ be the predicate '*x is a capital city of Australia*'. Then we can write the premiss as p('*Canberra*') and the conclusion as $\exists x\,(px)$. The proof follows immediately as an instance of the $\exists$I rule.

$$\dfrac{\mathcal{P}(\text{`}\textit{Canberra}\text{'})}{\exists x\,\mathcal{P}(x)}\,\exists I$$

Solution 49: Deduction rules: ∀I and ∃E

1. (a) $\forall x\,\mathcal{Q}(x) \vdash \forall x\,\neg\mathcal{P}(x) \vee \mathcal{Q}(x)$

$$\dfrac{\dfrac{\dfrac{\forall x\,\mathcal{Q}(x)}{\mathcal{Q}(a)}\,\forall E}{\neg\mathcal{P}(a) \vee \mathcal{Q}(a)}\,\vee I_2}{\forall x\,\neg\mathcal{P}(x) \vee \mathcal{Q}(x)}\,\forall I$$

(b) $\forall x\,\forall y\,\mathcal{P}(x,y) \vdash \forall y\,\forall x\,\mathcal{P}(x,y)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x\,\forall y\,\mathcal{P}(x,y)}{\forall y\,\mathcal{P}(a,y)}\,\forall E}{\mathcal{P}(a,b)}\,\forall E}{\forall x\,\mathcal{P}(x,b)}\,\forall I}{\forall y\,\forall x\,\mathcal{P}(x,y)}\,\forall I$$

2. (a) $\forall x\,\mathcal{P}(x),\ \exists x\,(\mathcal{P}(x) \Rightarrow \mathcal{Q}(x)) \vdash \exists x\,\mathcal{Q}(x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x\,\mathcal{P}(x)}{\mathcal{P}(a)}\,\forall E \qquad \dfrac{}{\mathcal{P}(a) \Rightarrow \mathcal{Q}(a)}\,*}{\mathcal{Q}(a)}\,\Rightarrow E}{\exists x\,\mathcal{Q}(x)}\,\exists I \qquad \exists x\,(\mathcal{P}(x) \Rightarrow \mathcal{Q}(x))}{\exists x\,\mathcal{Q}(x)}\,*\exists E$$

(b) $\exists x\,\exists y\,\mathcal{P}(x,y) \vdash \exists y\,\exists x\,\mathcal{P}(x,y)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{P}(a,b)}\,*}{\exists x\,\mathcal{P}(x,b)}\,\exists\mathrm{I}}{\exists y\,\exists x\,\mathcal{P}(x,y)}\,\exists\mathrm{I} \qquad \dfrac{}{\exists y\,\mathcal{P}(a,y)}\,\dagger}{\exists y\,\exists x\,\mathcal{P}(x,y)}\,*\exists\mathrm{E} \qquad \exists x\,\exists y\,\mathcal{P}(x,y)}{\exists y\,\exists x\,\mathcal{P}(x,y)}\,\dagger\exists\mathrm{E}$$

Solution 50: Reasoning about identity

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\,\mathcal{P}(x,f(x))}{\mathcal{P}(b,f(b))}\,\forall\mathrm{E} \qquad \dfrac{}{b=g(a)}\,*}{\mathcal{P}(g(a),f(b))}\,{=}\mathrm{E}}{\exists y\,\mathcal{P}(g(y),f(b))}\,\exists\mathrm{I} \qquad \dfrac{\forall x\,\exists y\,(x=g(y))}{\exists y\,(b=g(y))}\,\forall\mathrm{E}}{\exists y\,\mathcal{P}(g(y),f(b))}\,\exists\mathrm{E}}{\forall x\,\exists y\,\mathcal{P}(g(y),f(x))}\,\forall\mathrm{I}$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\,\mathcal{P}(x)}{\mathcal{P}(a)}\,\forall\mathrm{E} \qquad \dfrac{}{a=f(b)}\,*}{\mathcal{P}(f(b))}\,{=}\mathrm{E}}{\exists y\,\mathcal{P}(f(y))}\,\exists\mathrm{I} \qquad \dfrac{}{\exists y\,(a=f(y))}\,\dagger}{\exists y\,\mathcal{P}(f(y))}\,*\exists\mathrm{E} \qquad \exists x\,\exists y\,(x=f(y))}{\exists y\,\mathcal{P}(f(y))}\,\dagger\exists\mathrm{E}$$

3. See Figure A.9

Solution 51: First order theory of Sheffer's stroke

1. (a)

$$\dfrac{\dfrac{}{\mathsf{t}_2\cdot\mathsf{t}_1=\overline{\mathsf{t}_2|\mathsf{t}_1}}\,Df\cdot \qquad \dfrac{\dfrac{}{\mathsf{t}_1\cdot\mathsf{t}_2=\overline{\mathsf{t}_1|\mathsf{t}_2}}\,Df\cdot \qquad \dfrac{}{\mathsf{t}_1|\mathsf{t}_2=\mathsf{t}_2|\mathsf{t}_1}\,Cm|}{\mathsf{t}_1\cdot\mathsf{t}_2=\overline{\mathsf{t}_2|\mathsf{t}_1}}\,{=}\mathrm{E}}{\mathsf{t}_1\cdot\mathsf{t}_2=\mathsf{t}_2\cdot\mathsf{t}_1}\,{=}\mathrm{E}$$

$$\dfrac{\dfrac{\dfrac{\forall x (x = f(f(x))) \lor x = a}{b = f(f(b)) \lor b = a}\,\forall E \qquad \dfrac{\dfrac{\dfrac{\forall x \mathcal{P}(f(f(f(x))))}{\mathcal{P}(f(f(f(b))))}\,\forall E \qquad \dfrac{}{b = f(f(b))}\,*}{\mathcal{P}(f(b))}\,{=}E}{\mathcal{P}(f(b)) \lor x = a}\,\lor I_1 \qquad \dfrac{\dfrac{}{b = a}\,*}{\mathcal{P}(f(b)) \lor b = a}\,\lor I_2}{\mathcal{P}(f(b)) \lor b = a}\,* \lor E}{\forall x (\mathcal{P}(f(x)) \lor x = a)}\,\forall I$$

Figure A.9: Solution to Exercise 50, Question 3

(b)

$$\dfrac{\dfrac{}{t_2 + t_1 = \overline{t_2}|\overline{t_1}}\,Df+ \qquad \dfrac{\dfrac{}{t_1 + t_2 = \overline{t_1}|\overline{t_2}}\,Df+ \qquad \dfrac{}{\overline{t_1}|\overline{t_2} = \overline{t_2}|\overline{t_1}}\,Cm|}{t_1 + t_2 = \overline{t_2}|\overline{t_1}}\,{=}\mathrm{E}}{t_1 + t_2 = t_2 + t_1}\,{=}\mathrm{E}$$

2. See Figure A.10

Solution 52: Boolean algebras

1. (a)

$$\begin{array}{lll} & a + a & \\ = & a \cdot 1 + a \cdot 1 & \langle\ B12\ \rangle \\ = & a \cdot (1 + 1) & \langle\ B5\ \rangle \\ = & a \cdot 1 & \langle\ B10\ \rangle \\ = & a & \langle\ B12\ \rangle \end{array}$$

(b)

$$\begin{array}{lll} & (a + b) \cdot (a + c) & \\ = & (a + b) \cdot a + (a + b) \cdot c & \langle\ B5\ \rangle \\ = & a \cdot (a + b) + c \cdot (a + b) & \langle\ B1\ \rangle \\ = & (a \cdot a + a \cdot b) + (c \cdot a + c \cdot b) & \langle\ B5\ \rangle \\ = & (a + a \cdot b) + (c \cdot a + c \cdot b) & \langle\ B14\ \rangle \\ = & (a + a \cdot b) + (a \cdot c + b \cdot c) & \langle\ B1\ \rangle \\ = & a + (a \cdot b + (a \cdot c + b \cdot c)) & \langle\ B3\ \rangle \\ = & a + ((a \cdot b + a \cdot c) + b \cdot c) & \langle\ B3\ \rangle \\ = & a + (a \cdot (b + c) + b \cdot c) & \langle\ B5\ \rangle \\ = & a \cdot 1 + (a \cdot (b + c) + b \cdot c) & \langle\ B12\ \rangle \\ = & (a \cdot 1 + a \cdot (b + c)) + b \cdot c & \langle\ B3\ \rangle \\ = & (a \cdot (1 + (b + c))) + b \cdot c & \langle\ B5\ \rangle \\ = & a \cdot 1 + b \cdot c & \langle\ B10\ \rangle \\ = & a + b \cdot c & \langle\ B12\ \rangle \end{array}$$

2.

$$\begin{array}{lll} & \overline{a \cdot b} & \\ = & \overline{\overline{\overline{a}} \cdot \overline{\overline{b}}} & \langle\ B16\ \rangle \\ = & \overline{\overline{\overline{a} + \overline{b}}} & \langle\ B18\ \rangle \\ = & \overline{a} + \overline{b} & \langle\ B16\ \rangle \end{array}$$

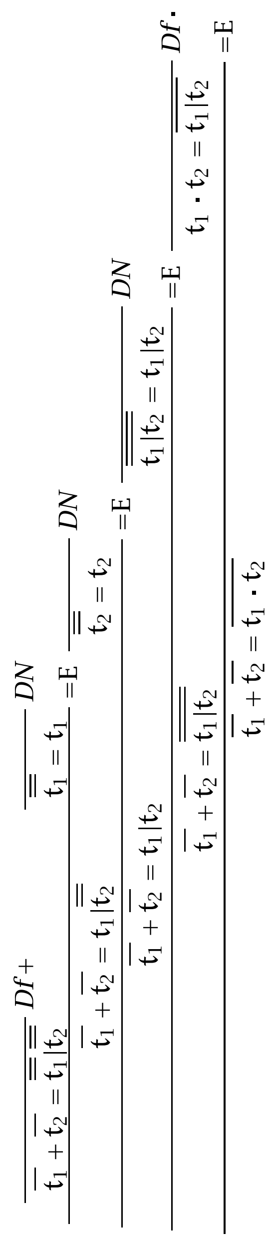


Figure A.10: Solution to Exercise 51, Question 2

3. Using only axioms B1–B15, the proof is very long and complicated; this is why it is useful to treat B18 as an additional axiom.

$$
\begin{aligned}
& \overline{a} \cdot \overline{b} \\
= \;& (\overline{a} \cdot \overline{b}) \cdot 1 && \langle\ B12\ \rangle \\
= \;& (\overline{a} \cdot \overline{b}) \cdot ((a + b) + \overline{a + b}) && \langle\ B8\ \rangle \\
= \;& (\overline{a} \cdot \overline{b}) \cdot (a + b) + (\overline{a} \cdot \overline{b}) \cdot \overline{a + b} && \langle\ B5\ \rangle \\
= \;& ((\overline{a} \cdot \overline{b}) \cdot a + (\overline{a} \cdot \overline{b}) \cdot b) + (\overline{a} \cdot \overline{b}) \cdot \overline{a + b} && \langle\ B5\ \rangle \\
= \;& (a \cdot (\overline{a} \cdot \overline{b}) + (\overline{a} \cdot \overline{b}) \cdot b) + (\overline{a} \cdot \overline{b}) \cdot \overline{a + b} && \langle\ B1\ \rangle \\
= \;& ((a \cdot \overline{a}) \cdot \overline{b} + \overline{a} \cdot (\overline{b} \cdot b)) + (\overline{a} \cdot \overline{b}) \cdot \overline{a + b} && \langle\ B3\ \rangle \\
= \;& ((a \cdot \overline{a}) \cdot \overline{b} + \overline{a} \cdot (b \cdot \overline{b})) + (\overline{a} \cdot \overline{b}) \cdot \overline{a + b} && \langle\ B1\ \rangle \\
= \;& (0 \cdot \overline{b} + \overline{a} \cdot 0) + (\overline{a} \cdot \overline{b}) \cdot \overline{a + b} && \langle\ B9\ \rangle \\
= \;& (\overline{b} \cdot 0 + \overline{a} \cdot 0) + (\overline{a} \cdot \overline{b}) \cdot \overline{a + b} && \langle\ B1\ \rangle \\
= \;& (0 + 0) + (\overline{a} \cdot \overline{b}) \cdot \overline{a + b} && \langle\ B11\ \rangle \\
= \;& 0 + (\overline{a} \cdot \overline{b}) \cdot \overline{a + b} && \langle\ B14\ \rangle \\
= \;& (a + b) \cdot \overline{a + b} + (\overline{a} \cdot \overline{b}) \cdot \overline{a + b} && \langle\ B9\ \rangle \\
= \;& \overline{a + b} \cdot (a + b) + \overline{a + b} \cdot (\overline{a} \cdot \overline{b}) && \langle\ B1\ \rangle \\
= \;& \overline{a + b} \cdot ((a + b) + \overline{a} \cdot \overline{b}) && \langle\ B5\ \rangle \\
= \;& \overline{a + b} \cdot ((a \cdot 1 + b \cdot 1) + \overline{a} \cdot \overline{b}) && \langle\ B12\ \rangle \\
= \;& \overline{a + b} \cdot ((a \cdot (b + \overline{b}) + b \cdot (a + \overline{a})) + \overline{a} \cdot \overline{b}) && \langle\ B8\ \rangle \\
= \;& \overline{a + b} \cdot (((a \cdot b + a \cdot \overline{b}) + (b \cdot a + b \cdot \overline{a})) + \overline{a} \cdot \overline{b}) && \langle\ B5\ \rangle \\
= \;& \overline{a + b} \cdot (((a \cdot b + a \cdot \overline{b}) + (a \cdot b + \overline{a} \cdot b)) + \overline{a} \cdot \overline{b}) && \langle\ B1\ \rangle \\
= \;& \overline{a + b} \cdot ((a \cdot b + (a \cdot \overline{b} + (a \cdot b + \overline{a} \cdot b))) + \overline{a} \cdot \overline{b}) && \langle\ B3\ \rangle \\
= \;& \overline{a + b} \cdot ((a \cdot b + ((a \cdot b + \overline{a} \cdot b) + a \cdot \overline{b})) + \overline{a} \cdot \overline{b}) && \langle\ B2\ \rangle \\
= \;& \overline{a + b} \cdot (((a \cdot b + (a \cdot b + \overline{a} \cdot b)) + a \cdot \overline{b}) + \overline{a} \cdot \overline{b}) && \langle\ B3\ \rangle \\
= \;& \overline{a + b} \cdot ((((a \cdot b + a \cdot b) + \overline{a} \cdot b) + a \cdot \overline{b}) + \overline{a} \cdot \overline{b}) && \langle\ B3\ \rangle \\
= \;& \overline{a + b} \cdot (((a \cdot b + \overline{a} \cdot b) + a \cdot \overline{b}) + \overline{a} \cdot \overline{b}) && \langle\ B15\ \rangle \\
= \;& \overline{a + b} \cdot ((a \cdot b + \overline{a} \cdot b) + (a \cdot \overline{b} + \overline{a} \cdot \overline{b})) && \langle\ B3\ \rangle \\
= \;& \overline{a + b} \cdot ((b \cdot a + b \cdot \overline{a}) + (\overline{b} \cdot a + \overline{b} \cdot \overline{a})) && \langle\ B1\ \rangle \\
= \;& \overline{a + b} \cdot (b \cdot (a + \overline{a}) + \overline{b} \cdot (a + \overline{a})) && \langle\ B5\ \rangle \\
= \;& \overline{a + b} \cdot (b \cdot 1 + \overline{b} \cdot 1) && \langle\ B8\ \rangle \\
= \;& \overline{a + b} \cdot (b + \overline{b}) && \langle\ B12\ \rangle \\
= \;& \overline{a + b} \cdot 1 && \langle\ B8\ \rangle \\
= \;& \overline{a + b} && \langle\ B12\ \rangle
\end{aligned}
$$

Solution 53: Boolean logic

1. (a) 0 (b) 1 (c) 1 (d) 0 (e) 1 (f) 1

2. (a)

a	b	$a + (b \cdot a)$
0	0	0
0	1	0
1	0	1
1	1	1

(b)

a	b	$a \cdot (b + a)$
0	0	0
0	1	0
1	0	1
1	1	1

(c)

a	b	$\overline{a + (b \cdot a)}$
0	0	1
0	1	1
1	0	0
1	1	0

(d)

a	b	$\overline{a \cdot (b + a)}$
0	0	1
0	1	1
1	0	0
1	1	0

(e)

a	b	$\overline{(a \cdot b)} + (\overline{a} \cdot \overline{b})$
0	0	1
0	1	1
1	0	1
1	1	0

(f)

a	b	$\overline{a} + (\overline{b} \cdot \overline{a})$
0	0	1
0	1	1
1	0	0
1	1	0

(g)

a	b	$\overline{a} \cdot (\overline{b} + \overline{a})$
0	0	1
0	1	1
1	0	0
1	1	0

(h)

a	b	c	$a + (b \cdot c)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

(i)

a	b	c	$a \cdot (b + c)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

(j)

a	b	c	$\overline{a + (b \cdot c)}$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

(k)

a	b	c	$\overline{a \cdot (b + c)}$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

(l)

a	b	c	$\overline{a} + (\overline{b} \cdot \overline{c})$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

(m)

a	b	c	$\overline{a} \cdot (\overline{b} + \overline{c})$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

(n)

a	b	c	$a \cdot b \cdot \overline{c} + \overline{a} \cdot b \cdot c + a \cdot \overline{b} \cdot c$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

(o)

a	b	c	$(\overline{a}+b+c)\cdot(a+\overline{b}+c)\cdot(a+b+\overline{c})$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

(p)

a	b	c	d	$\overline{a\cdot b+c\cdot d}+a\cdot b\cdot c\cdot\overline{d+a}$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	1
0	1	1	1	0
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

(q)

a	b	c	$\overline{\overline{a+b\cdot c}+a\cdot\overline{b+c}}$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

3.
    - a & b;
    - c, d, f & g;
    - k & l;
    - j & m.
4.
    (a) $a + (b \cdot a) = a$
    (b) $a \cdot (b + a) = a$
    (c) $\overline{a + (b \cdot a)} = \overline{a}$
    (d) $\overline{a \cdot (b + a)} = \overline{a}$
    (e) $\overline{(a \cdot b)} + (\overline{a} \cdot \overline{b}) = \overline{(a \cdot b)}$
    (f) $\overline{a} + (\overline{b} \cdot \overline{a}) = \overline{a}$
    (g) $\overline{a} \cdot (\overline{b} + \overline{a}) = \overline{a}$
    (h) $a + (b \cdot c)$ is already in simplest form
    (i) $a \cdot (b + c)$ is already in simplest form
    (j) $\overline{a + (b \cdot c)}$ is already in simplest form
    (k) $\overline{a \cdot (b + c)}$ is already in simplest form
    (l) $\overline{a} + (\overline{b} \cdot \overline{c}) = \overline{a \cdot (b + c)}$
    (m) $\overline{a} \cdot (\overline{b} + \overline{c}) = \overline{a + (b \cdot c)}$
    (n) $a \cdot b \cdot \overline{c} + \overline{a} \cdot b \cdot c + a \cdot \overline{b} \cdot c$ is already in simplest form
    (o) $(\overline{a} + b + c) \cdot (a + \overline{b} + c) \cdot (a + b + \overline{c}) = a \cdot b + a \cdot c + b \cdot c + \overline{a + b + c}$
    (p) $\overline{a \cdot b + c \cdot d} + a \cdot b \cdot c \cdot \overline{d + a} = \overline{a \cdot b + c \cdot d}$
    (q) $\overline{\overline{\overline{a + b \cdot c} + a \cdot \overline{b + c}}} = a \cdot b + a \cdot c + b \cdot c$

Solution 54: Converting boolean logic to propositional logic

1. $\mathcal{P} \vee (\mathcal{Q} \vee \mathcal{R})$
2. $(\mathcal{P} \vee \mathcal{Q}) \vee \mathcal{R})$
3. $\mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{R})$
4. $(\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{R})$
5. $(\mathcal{P} \vee \mathcal{Q}) \wedge \mathcal{R})$
6. $(\mathcal{P} \wedge \mathcal{R}) \vee (\mathcal{Q} \wedge \mathcal{R}))$
7. $\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R})$
8. $(\mathcal{P} \vee \mathcal{Q}) \wedge (\mathcal{P} \vee \mathcal{R})$
9. $\neg(\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R}))$
10. $\neg\mathcal{P} \wedge (\neg\mathcal{Q} \vee \neg\mathcal{R})$
11. $\neg(\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}))$
12. $\neg\mathcal{P} \vee (\neg\mathcal{Q} \wedge \neg\mathcal{R})$
13. $\neg((\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{R} \wedge s))$
14. $\neg(\neg(\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R})) \vee (\mathcal{P} \wedge \neg(\mathcal{Q} \vee \mathcal{R})))$

Solution 55: Equational theory of propositional forms

1. (a)

$$
\begin{array}{lll}
 & F & \\
=_T & T \wedge \neg T & \langle\ Cpt\wedge\ \rangle \\
=_T & \neg T \wedge T & \langle\ Cm\wedge\ \rangle \\
=_T & \neg T & \langle\ Unit\wedge\ \rangle
\end{array}
$$

(b)

$$
\begin{array}{lll}
 & \neg F & \\
=_T & \neg\neg T & \langle\ Qu.1a\ \rangle \\
=_T & T & \langle\ DN\ \rangle
\end{array}
$$

(c)

$$
\begin{array}{lll}
 & (\mathcal{P} \wedge \mathcal{Q}) \wedge \mathcal{P} & \\
=_T & \mathcal{P} \wedge (\mathcal{Q} \wedge \mathcal{P}) & \langle\ Ass\wedge\ \rangle \\
=_T & \mathcal{P} \wedge (\mathcal{P} \wedge \mathcal{Q}) & \langle\ Cm\wedge\ \rangle \\
=_T & (\mathcal{P} \wedge \mathcal{P}) \wedge \mathcal{Q} & \langle\ Ass\wedge\ \rangle \\
=_T & \mathcal{P} \wedge \mathcal{Q} & \langle\ Idempt\wedge\ \rangle
\end{array}
$$

(d)

$$
\begin{array}{lll}
 & \mathcal{P} \Rightarrow (\mathcal{P} \wedge \mathcal{Q}) & \\
=_T & \neg\mathcal{P} \vee (\mathcal{P} \wedge \mathcal{Q}) & \langle\ Df\Rightarrow\ \rangle \\
=_T & (\neg\mathcal{P} \vee \mathcal{P}) \wedge (\neg\mathcal{P} \vee \mathcal{Q}) & \langle\ Dist\ \rangle \\
=_T & T \wedge (\neg\mathcal{P} \vee \mathcal{Q}) & \langle\ Cpt\vee\ \rangle \\
=_T & \neg\mathcal{P} \vee \mathcal{Q} & \langle\ Unit\wedge\ \rangle \\
=_T & \mathcal{P} \Rightarrow \mathcal{Q} & \langle\ Df\Rightarrow\ \rangle
\end{array}
$$

(e)

$$
\begin{array}{lll}
 & \mathcal{P} \Rightarrow \mathcal{Q} & \\
=_T & \neg\mathcal{P} \vee \mathcal{Q} & \langle\ Df\Rightarrow\ \rangle \\
=_T & \neg\neg\neg\mathcal{P} \vee \neg\neg\mathcal{Q} & \langle\ DN\ \rangle \\
=_T & \neg(\neg\neg\mathcal{P} \wedge \neg\mathcal{Q}) & \langle\ de\ Morgan\ \rangle \\
=_T & \neg(\mathcal{P} \wedge \neg\mathcal{Q}) & \langle\ DN\ \rangle
\end{array}
$$

(f)

$$
\begin{array}{lll}
 & \mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{R}) & \\
=_T & \neg\mathcal{P} \vee (\neg\mathcal{Q} \vee \mathcal{R}) & \langle\ Df\Rightarrow\ \rangle \\
=_T & (\neg\mathcal{P} \vee \neg\mathcal{Q}) \vee \mathcal{R} & \langle\ Ass\vee\ \rangle \\
=_T & \neg(\mathcal{P} \wedge \mathcal{Q}) \vee \mathcal{R} & \langle\ de\ Morgan\ \rangle \\
=_T & (\mathcal{P} \wedge \mathcal{Q}) \Rightarrow \mathcal{R} & \langle\ Df\Rightarrow\ \rangle
\end{array}
$$

(g)

$$
\begin{array}{lll}
 & \mathcal{P} \Leftrightarrow \neg\mathcal{Q} & \\
=_T & (\mathcal{P} \wedge \neg\mathcal{Q}) \vee (\neg\mathcal{P} \wedge \neg\neg\mathcal{Q}) & \langle\ Df\Leftrightarrow\ \rangle \\
=_T & (\mathcal{P} \wedge \neg\mathcal{Q}) \vee (\neg\mathcal{P} \wedge \mathcal{Q}) & \langle\ DN\ \rangle \\
=_T & ((\mathcal{P} \wedge \neg\mathcal{Q}) \vee \neg\mathcal{P}) \wedge ((\mathcal{P} \wedge \neg\mathcal{Q}) \vee \mathcal{Q}) & \langle\ Dist\ \rangle \\
=_T & (\neg\mathcal{P} \vee (\mathcal{P} \wedge \neg\mathcal{Q})) \wedge (\mathcal{Q} \vee (\mathcal{P} \wedge \neg\mathcal{Q})) & \langle\ Cm\vee\ \rangle \\
=_T & ((\neg\mathcal{P} \vee \mathcal{P}) \wedge (\neg\mathcal{P} \vee \neg\mathcal{Q})) \wedge ((\mathcal{Q} \vee \mathcal{P}) \wedge (\mathcal{Q} \vee \neg\mathcal{Q})) & \langle\ Dist\ \rangle \\
=_T & ((\mathcal{P} \vee \neg\mathcal{P}) \wedge (\neg\mathcal{P} \vee \neg\mathcal{Q})) \wedge ((\mathcal{P} \vee \mathcal{Q}) \wedge (\mathcal{Q} \vee \neg\mathcal{Q})) & \langle\ Cm\vee\ \rangle \\
=_T & (T \wedge (\neg\mathcal{P} \vee \neg\mathcal{Q})) \wedge ((\mathcal{P} \vee \mathcal{Q}) \wedge T) & \langle\ Cpt\vee\ \rangle \\
=_T & (\neg\mathcal{P} \vee \neg\mathcal{Q}) \wedge (\mathcal{P} \vee \mathcal{Q}) & \langle\ Unit\wedge\ \rangle \\
=_T & (\mathcal{P} \vee \mathcal{Q}) \wedge (\neg\mathcal{P} \vee \neg\mathcal{Q}) & \langle\ Cm\wedge\ \rangle
\end{array}
$$

(h)
$$
\begin{array}{lll}
 & \neg(\mathcal{P} \Leftrightarrow \mathcal{Q}) & \\
=_T & \neg((\mathcal{P} \wedge \mathcal{Q}) \vee (\neg\mathcal{P} \wedge \neg\mathcal{Q})) & \langle\ \mathit{Df} \Leftrightarrow\ \rangle \\
=_T & \neg(\mathcal{P} \wedge \mathcal{Q}) \wedge \neg(\neg\mathcal{P} \wedge \neg\mathcal{Q}) & \langle\ \textit{de Morgan}\ \rangle \\
=_T & (\neg\mathcal{P} \vee \neg\mathcal{Q}) \wedge (\neg\neg\mathcal{P} \vee \neg\neg\mathcal{Q}) & \langle\ \textit{de Morgan}\ \rangle \\
=_T & (\neg\mathcal{P} \vee \neg\mathcal{Q}) \wedge (\mathcal{P} \vee \mathcal{Q}) & \langle\ \mathit{DN}\ \rangle \\
=_T & (\mathcal{P} \vee \mathcal{Q}) \wedge (\neg\mathcal{P} \vee \neg\mathcal{Q}) & \langle\ \mathit{Cm}\wedge\ \rangle
\end{array}
$$

(i)
$$
\begin{array}{lll}
 & \neg(\mathcal{P} \Leftrightarrow \mathcal{Q}) & \\
=_T & (\mathcal{P} \vee \mathcal{Q}) \wedge (\neg\mathcal{P} \vee \neg\mathcal{Q}) & \langle\ \mathit{Qu.1h}\ \rangle \\
=_T & \mathcal{P} \Leftrightarrow \neg\mathcal{Q} & \langle\ \mathit{Qu.1g}\ \rangle
\end{array}
$$

2.
$$
\begin{array}{lll}
 & \mathcal{P} \Rightarrow (\mathcal{Q} \Rightarrow \mathcal{P}) & \\
=_T & \neg\mathcal{P} \vee (\neg\mathcal{Q} \vee \mathcal{P}) & \langle\ \mathit{Df} \Rightarrow\ \rangle \\
=_T & \neg\mathcal{P} \vee (\mathcal{P} \vee \neg\mathcal{Q}) & \langle\ \mathit{Cm}\vee\ \rangle \\
=_T & (\neg\mathcal{P} \vee \mathcal{P}) \vee \neg\mathcal{Q} & \langle\ \mathit{Ass}\vee\ \rangle \\
=_T & T \vee \neg\mathcal{Q} & \langle\ \mathit{Cpt}\vee\ \rangle \\
=_T & T & \langle\ \mathit{Zero}\vee\ \rangle
\end{array}
$$

Solution 56: Properties of quantifiers

1.
$$
\begin{array}{lll}
 & \forall x\,(\neg\mathcal{P}) & \\
=_T & \neg\neg\forall x\,(\neg\mathcal{P}) & \langle\ \mathit{DN}\ \rangle \\
=_T & \neg\exists x\,\mathcal{P} & \langle\ \mathit{Df}\exists\ \rangle
\end{array}
$$

2.
$$
\begin{array}{lll}
 & \exists x\,(\mathcal{P} \vee \mathcal{Q}) & \\
=_T & \neg\neg\exists x\,(\mathcal{P} \vee \mathcal{Q}) & \langle\ \mathit{DN}\ \rangle \\
=_T & \neg\forall x\,\neg(\mathcal{P} \vee \mathcal{Q}) & \langle\ \mathit{Df}\exists\ \rangle \\
=_T & \neg\forall x\,(\neg\mathcal{P} \wedge \neg\mathcal{Q})) & \langle\ \textit{de Morgan}\ \rangle \\
=_T & \neg\forall x\,\neg\mathcal{P} \wedge \neg\forall x\,\neg\mathcal{Q} & \langle\ \mathit{Dist}\ \rangle \\
=_T & \neg(\forall x\,\neg\mathcal{P} \vee \forall x\,\neg\mathcal{Q}) & \langle\ \textit{de Morgan}\ \rangle \\
=_T & \exists x\,\mathcal{P} \vee \exists x\,\mathcal{Q} & \langle\ \mathit{Df}\exists\ \rangle
\end{array}
$$

3.
$$
\begin{array}{lll}
 & \exists x\,\exists y\,\mathcal{P} & \\
=_T & \neg\forall x\,(\neg\exists y\,\mathcal{P}) & \langle\ \mathit{Df}\exists\ \rangle \\
=_T & \neg\forall x\,(\neg\neg\forall y\,(\neg\mathcal{P})) & \langle\ \mathit{Df}\exists\ \rangle \\
=_T & \neg\forall x\,(\forall y\,(\neg\mathcal{P})) & \langle\ \textit{double negation}\ \rangle \\
=_T & \neg\forall y\,(\forall x\,(\neg\mathcal{P})) & \langle\ \mathit{Cm}\forall\ \rangle \\
=_T & \neg\forall y\,(\neg\neg\forall x\,(\neg\mathcal{P})) & \langle\ \textit{double negation}\ \rangle \\
=_T & \neg\forall y\,(\neg\exists x\,\mathcal{P}) & \langle\ \mathit{Df}\exists\ \rangle \\
=_T & \exists y\,\exists x\,\mathcal{P} & \langle\ \mathit{Df}\exists\ \rangle
\end{array}
$$

4.
$$
\begin{array}{lll}
 & \exists x\, T & \\
=_T & \neg\forall x \neg T & \langle\ Df\exists\ \rangle \\
=_T & \neg\forall x\, F & \langle\ Ex55\ Qu1a\ \rangle \\
=_T & \neg F & \langle\ \forall F\ \rangle \\
=_T & T & \langle\ Ex55\ Qu1b\ \rangle
\end{array}
$$

5.
$$
\begin{array}{lll}
 & \exists x\, F & \\
=_T & \neg\forall\neg F & \langle\ Df\exists\ \rangle \\
=_T & \neg\forall T & \langle\ Ex55\ Qu1b\ \rangle \\
=_T & \neg T & \langle\ \forall T\ \rangle \\
=_T & F & \langle\ Ex55\ Qu1a\ \rangle
\end{array}
$$

Solution 57: Disjunctive normal forms

1. (a) $(\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \neg\mathcal{Q}) =_T \mathcal{P}$
   (b) $(\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \neg\mathcal{Q}) \vee (\neg\mathcal{P} \wedge \neg\mathcal{Q}) =_T \mathcal{P} \vee \neg\mathcal{Q}$
   (c) $(\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R}) \vee (\mathcal{P} \wedge \mathcal{Q} \wedge \neg\mathcal{R}) \vee (\neg\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R}) =_T \mathcal{Q} \wedge (\mathcal{P} \vee \mathcal{R})$
   (d) $(\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R}) \vee (\mathcal{P} \wedge \mathcal{Q} \wedge \neg\mathcal{R}) \vee (\mathcal{P} \wedge \neg\mathcal{Q} \wedge \mathcal{R}) \vee (\neg\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R})$
   $=_T \mathcal{Q} \wedge (\mathcal{P} \vee \mathcal{R}) \vee (\mathcal{P} \wedge \mathcal{R})$
   (e) $(\mathcal{P} \wedge \mathcal{Q} \wedge \neg\mathcal{R} \wedge \mathcal{S}) \vee (\mathcal{P} \wedge \neg\mathcal{Q} \wedge \mathcal{R} \wedge \mathcal{S})$
   $\vee (\mathcal{P} \wedge \neg\mathcal{Q} \wedge \neg\mathcal{R} \wedge \mathcal{S}) \vee (\neg\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R} \wedge \neg\mathcal{S})$
   $=_T (\mathcal{P} \wedge (\neg\mathcal{Q} \vee \neg\mathcal{R}) \wedge \mathcal{S}) \vee (\neg\mathcal{P} \wedge \mathcal{Q} \wedge \mathcal{R} \wedge \neg\mathcal{S})$

Solution 58: Converting propositions into clausal form

1. (a) $p \Rightarrow (q \Rightarrow r) =_T \neg p \vee (\neg q \vee r) =_T \neg p \vee \neg q \vee r$ giving one clause: $\{\neg p, \neg q, r\}$.
   (b) $p \Rightarrow (q \wedge r) =_T \neg p \vee (q \wedge r) =_T (\neg p \vee q) \wedge (\neg p \vee r)$
   giving two clauses: $\{\neg p, q\}$ and $\{\neg p, r\}$.
   (c) $(p \Rightarrow q) \vee (\neg q \Rightarrow r) =_T \neg p \vee q \vee q \vee r =_T \neg p \vee q \vee r$. giving one clause: $\{\neg p, q, r\}$.
   (d) $(p \Rightarrow q) \Rightarrow r =_T \neg(\neg p \vee q) \vee r =_T (p \wedge \neg q) \vee r =_T (p \vee r) \wedge (\neg q \vee r)$
   giving two clauses: $\{p, r\}$ and $\{\neg q, r\}$.

2. (a)

1	p	
2	$\neg p, \neg q, r$	
3	$\neg q, r$	*Res.* 1, 2

   (b)

1	p, q	
2	$\neg p, r$	
3	$\neg q, r$	
4	q, r	*Res.* 1, 2
5	r	*Res.* 3, 4

(c)

1	$\neg p, q, r$	
2	p	
3	$\neg r$	
4	q, r	*Res.* 1, 2
5	q	*Res.* 3, 4

Solution 59: Refutation in clausal logic

1.

1	p	
2	$\neg p, \neg q, r$	
3	q	
4	$\neg r$	
5	$\neg q, r$	*Res.* 1, 2
6	r	*Res.* 3, 5
7	$\{\}$	*Res.* 4, 6

2.

1	p, q	
2	$\neg p, q$	
3	$\neg q, r$	
4	$\neg r$	
5	$\neg q$	*Res.* 3, 4
6	q	*Res.* 1, 2
7	$\{\}$	*Res.* 5, 6

3.

1	$\neg p, q, r$	
2	p	
3	$\neg r$	
4	$\neg q$	
5	$\neg p, q$	*Res.* 1, 3
6	q	*Res.* 2, 5
7	$\{\}$	*Res.* 4, 6

Solution 60: Horn clauses

1. All clauses are Horn clauses *except* (d) and (e).
2. (a) $p \Rightarrow (q \Rightarrow r) =_T \neg p \vee (\neg q \vee r) =_T \neg p \vee \neg q \vee r$, giving $\{\neg p, \neg q, r\}$ which is a Horn clause.
   (b) $p \Rightarrow (q \wedge r) =_T \neg p \vee (q \wedge r) =_T (\neg p \vee q) \wedge (\neg p \vee r)$, giving $\{\neg p, q\}$ and $\{\neg p, r\}$, both of which are Horn clauses.
   (c) $(p \Rightarrow q) \vee (\neg q \Rightarrow r) =_T \neg p \vee q \vee q \vee r =_T \neg p \vee q \vee r$, giving $\{\neg p, q, r\}$, which is *not* a Horn clause.
   (d) $(p \Rightarrow q) \Rightarrow r =_T \neg(\neg p \vee q) \vee r =_T (p \wedge \neg q) \vee r =_T (p \vee r) \wedge (\neg q \vee r)$ giving $\{p, r\}$ and $\{\neg q, r\}$, only the second of which is a Horn clause.

Solution 61: Logic programming with propositional clauses

1.

1	q :- p	
2	r :- s, t, u	
3	s :- p, v	
4	t :- q	
5	w :- r, s	
6	p	
7	s	
8	:-t	*negation of query*
9	:-q	*Res.* 4, 8
10	:-p	*Res.* 1, 9
11	:-	*Res.* 6, 10

Hence we conclude that t follows from the program.

2.

1	p	
2	q	
3	r :- p	
4	s :- q	
5	:-r, s	*negation of query*
6	:-p, s	*Res.* 3, 5
7	:-s	*Res.* 1, 6
8	:-q	*Res.* 4, 7
9	:-	*Res.* 2, 8

Summary of notation B

B.1 Letters

$p, q, r, s, p_1, p_2, \ldots, q_1, q_2, \ldots$ represent (constant) propositions.

$\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{Q}_1, \mathcal{Q}_2, \ldots$ represent schematic letters in propositional forms.

T, F are restricted schematic letters: the letter T may only be instantiated to a true proposition; the letter F may only be instantiated to a false proposition.

$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ and other capital letters from near the beginning of the alphabet refer to propositional forms (possibly unspecified); in print, a blackletter font is used.

$\Gamma, \Gamma_1, \Gamma_2, \ldots$ are used to denote a set of propositional forms. (Γ is the capital Greek letter 'gamma') The union of two sets Γ_1 and Γ_2 may be represented as Γ_1, Γ_2 when there is no possibility of confusion.

$x, x_1, x_2, \ldots, y, y_1, y_2, \ldots, z, z_1, z_2, \ldots$ and other lower case letters near the end of the alphabet are used to represent variables in predicate logic.

$a, b, c, a_1, a_2, \ldots, b_1, b_2, \ldots$ and other lower cases near the beginning of the alphabet represent arbitrary constants.

$f, f_1, f_2, \ldots, g, g_1, g_2, h, h_1, h_2, \ldots$ represent functions.

$\mathfrak{t}, \mathfrak{t}_1, \mathfrak{t}_2, \ldots$ represent terms.

B.2 Connectives

$\wedge$ represents conjunction ('*and*').

$\vee$ represents disjunction ('*or*').

$\neg$ represents disjunction ('*not*').

$\Rightarrow$ represents the conditional ('*only if*').

$\Leftrightarrow$ represents the biconditional ('*if and only if*').

B.3 Quantifiers

$\forall x\, \mathcal{P}$ represents the universal quantification of the predicate $\mathcal{P}$.

$\exists x\, \mathcal{P}$ represents the existential quantification of the predicate $\mathcal{P}$.

B.4 Propositional forms and truth values

$=_T$ indicates that two logical expressions have the same truth value.

$\neg\mathfrak{A}$, $\mathfrak{A} \vee \mathfrak{B}$, $\mathfrak{A} \wedge \mathfrak{B}$ refer to propositional forms in which the main connective is negation, disjunction and conjunction respectively.

$\mathfrak{A} =_T \mathfrak{B}$ denotes that two propositional forms $\mathfrak{A}$ and $\mathfrak{B}$ are semantically equivalent.

$\Gamma \vDash \mathfrak{A}$ represents the semantic entailment of $\mathfrak{A}$ from the set Γ.

B.5 Arguments and natural deduction

$p_1, p_2, \ldots, p_n \therefore q$ represents the argument that q follows from $\{p_1, p_2, \ldots, p_n\}$. (The symbol $\therefore$ is pronounced 'therefore'.)

$p_1, p_2, \ldots, p_n \vdash q$ represents the fact that q can be deduced from $\{p_1, p_2, \ldots, p_n\}$ using natural deduction.

$\mathfrak{A} \equiv \mathfrak{B}$ denotes that $\mathfrak{A}$ and $\mathfrak{B}$ are syntactically equivalent.

Vertical presentation of inference forms. An inference form $\Gamma \vdash \mathfrak{B}$ used as a rule of deduction can be written in vertical form with the name of the rule placed at the end of a horizontal line.

$$\frac{\Gamma}{\mathfrak{B}}\ \textit{rule-name}$$

Vertical presentation of the chain rule. The inference forms $\Gamma_1 \vdash \mathfrak{A}$ and $\Gamma_2, \mathfrak{A} \vdash \mathfrak{B}$ can be combined vertically:

$$\frac{\dfrac{\Gamma_1}{\mathfrak{A}} \qquad \Gamma_2}{\mathfrak{B}}$$

Glossary C

antecedent

See *conditional*

arbitrary constant

A label which refers to an unspecified item.

argument

If $p_1, p_2, \ldots, p_n, q$ are *propositions*, then $p_1, p_2, \ldots, p_n \therefore q$ is an argument. The propositions $p_1, p_2, \ldots, p_n$ are the premisses, and the proposition q is the conclusion.

argument form

If $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n, \mathfrak{B}$ are *propositional forms*, then $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n \therefore \mathfrak{B}$ is an argument form. The propositional forms $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$ are the premiss forms and $\mathfrak{B}$ is the conclusion form. Any instance of an argument form is an *argument*.

atomic proposition

A *proposition* which is not a *compound proposition*; that is, one which contains no *connective*.

biconditional

The connective *schema* $\mathcal{P} \Leftrightarrow \mathcal{Q}$ for the biconditional is defined as being equivalent to the *propositional form* $(\mathcal{P} \wedge \mathcal{Q}) \vee (\neg\mathcal{P} \wedge \neg\mathcal{Q})$.

binary predicate

A *predicate* with two free variables is known as a binary predicate. A binary predicate represents a relation between two items.

bound variable

See *quantifier*.

characteristic form of a proposition

For any given *proposition*, a characteristic form is a *maximal form* having the fewest possible *schematic letters*. The proposition is an *instance* of the characteristic form in which different schematic letters are instantiated to different *atomic propositions*.

characteristic form of an argument

The characteristic form of an *argument* is such that the argument can be obtained by *instantiating* different *schematic letters* to different *atomic propositions*.

clause

A clause is either a *literal* on its own, or the *disjunction* of two or more literals.

compound proposition

A *proposition* with one or more *connectives*.

conclusion

See *argument*.

conditional

The connective *schema* $P \Rightarrow Q$ for the conditional is defined by the truth table:

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{Q}$
T	T	T
T	F	F
F	T	T
F	F	T

$\mathcal{P}$ is known as the antecedent and $\mathcal{Q}$ as the consequent.

conjunct

See *conjunction*.

conjunction

$P \wedge \mathcal{Q}$ is the conjunction of $\mathcal{P}$ and $\mathcal{Q}$ and has the truth table:

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$
T	T	T
T	F	F
F	T	F
F	F	F

The two operands of conjunction are known as conjuncts.

conjunctive normal form

For any *propositional form* $\mathfrak{A}$, the conjunctive normal form is the propositional form $\mathfrak{B}_1 \wedge \mathfrak{B}_2 \wedge \ldots \wedge \mathfrak{B}_n$ equivalent to $\mathfrak{A}$ in which each of the *conjuncts* $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n$ is a *disjunction* of propositional forms.

connective

A symbol (or word) which combined with a *proposition* gives a more complex proposition.

connective priority

In a *propositional form*, connective priority determines the order in which different *connectives* apply:

$$\neg \quad \text{then} \quad \wedge \quad \text{then} \quad \vee \quad \text{then} \quad \Rightarrow$$

Parentheses are used to override this order.

consequent

See *conditional*.

constant

See *arbitrary constant* and *proper constant*.

contradiction

A contradiction is a *propositional form* for which the truth value is equal to F for all *instances*.

corresponding instances

Corresponding *instances* of two *propositional forms* $\mathfrak{A}$ and $\mathfrak{B}$ are such that any *schematic letter* common to $\mathfrak{A}$ and $\mathfrak{B}$ is instantiated to the same *proposition*.

disjunct

See *disjunction*.

disjunction

$P \vee Q$ is the disjunction of $\mathcal{P}$ and $\mathcal{Q}$ and has the truth table:

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \vee \mathcal{Q}$
T	T	T
T	F	T
F	T	T
F	F	F

The two operands of conjunction are known as disjuncts.

disjunctive normal form

For any *propositional form* $\mathfrak{A}$, the disjunctive normal form is the propositional form $\mathfrak{B}_1 \vee \mathfrak{B}_2 \vee \ldots \vee \mathfrak{B}_n$ equivalent to $\mathfrak{A}$ in which each of the *disjuncts* $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n$ is a *conjunction* of propositional forms.

elimination rule

An elimination rule for a symbol, such as a *connective*, *quantifier* or identity symbol, is a *rule of deduction* in which the *conclusion* of the resulting *inference* or *inference form* contains that symbol in all cases.

equivalence of propositions

Two *propositions* p and q are equivalent if they are *corresponding instances* of equivalent *propositional forms* $\mathfrak{A}$ and $\mathfrak{B}$. See *semantic equivalence* and *syntactic equivalence*.

existential quantifier

In $\exists x\, \mathcal{P}(x)$, the existential quantifier is $\exists$. The *proposition* $\exists x\, \mathcal{P}(x)$ is true if and only if there is at least one value of x for which $\mathcal{P}(x)$ is true.

free variable

See *predicate*.

function

A function associates a single value called the result with another value called the argument. We say that the function is applied to the argument to give the result.

Horn clause

A *clause* in which there is at most one positive *literal* is called a Horn clause.

inconsistency

A set of *propositional forms* is inconsistent if there is no *instance* of this set in which all the *propositions* are true. A set of propositions is inconsistent if it is an instance of an inconsistent set of propositional forms.

inference

If $p_1, p_2, \ldots, p_n, q$ are *propositions*, then $p_1, p_2, \ldots, p_n \vdash q$ is an inference. Inferences are obtained by application of *rules of deduction*. The propositions $p_1, p_2, \ldots, p_n$ are the premisses, and the proposition q is the conclusion.

inference form

If $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n, \mathfrak{B}$ are *propositional forms*, then $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n \vdash \mathfrak{B}$ is an inference form. Inference forms are obtained by application of *rules of deduction*. The propositional forms $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$ are the premiss forms and $\mathfrak{B}$ is the conclusion form. Any *instance* of an inference form is an *inference*.

instance

See *schema*.

instantiation

See *schema*.

introduction rule

An introduction rule for a symbol, such as a connective, quantifier or identity symbol, is a *rule of deduction* in which there is always a *premiss* containing that symbol in the resulting *inference* or *inference form*.

invalid argument

An *argument* is invalid if its *characteristic form* is invalid.

literal

A positive literal is a letter, such as p, which represents an *atomic proposition*. A negative literal is the *negation* of a positive literal; for example $\neg p$. A literal is either a positive literal or a negative literal.

logic program

A logic program is a set of *Horn clauses*.

maximal form

A maximal form for any *proposition* is a *propositional form* which has the greatest number of *connectives* and of which the given proposition is an *instance*. The given proposition can be obtained from the maximal form by *instantiating* each *schematic letter* to an *atomic proposition*.

methods of deduction

A method of deduction is a rule by means of which we can write down a new *inference form* given one or more other inference forms.

negation

$\neg\mathcal{P}$ is the negation of $\mathcal{P}$ and has the truth table:

$\mathcal{P}$	$\neg\mathcal{P}$
T	F
F	T

parse tree

A diagram to show how a *compound proposition* or *propositional form* is built up from simpler propositions or forms.

predicate

A predicate contains one or more *free variables* such that replacing each free variable with a *term* gives a *proposition*. For each free variable there is associated a set of possible values known as the universe of discourse.

predicate form

A *schema* whose *instances* are *predicates*.

premise

An alternative spelling for *premiss*.

premiss

See *argument*.

priority

See *connective priority*.

proper constant

A label which refers to a specific item; the name of an item.

proposition

A statement with which it is meaningful to associate a truth value.

propositional form

A *schema* whose *instances* are *propositions*.

quantifier

If $p(x_1, x_2, \ldots, x_n)$ is a *predicate* with free variables $x_1, x_2, \ldots, x_n$ and Q is a quantifier, then the quantified expression $\mathsf{Q}x_1\ p(x_1, x_2, \ldots, x_n)$ is a predicate with free variables $x_2, \ldots, x_n$. In the quantified expression, x_1 is a bound variable. See also *existential quantifier* and *universal quantifier*.

resolution

A *rule of deduction* for *clauses*: if C_1 and C_2 are clauses, and if P is a positive *literal*, then applying resolution to the clauses $\mathsf{C}_1 \vee \mathsf{P}$ and $\mathsf{C}_2 \vee \neg\mathsf{P}$ yields the resolvent clause $\mathsf{C}_1 \vee \mathsf{C}_2$.

resolvent

See *resolution*.

schema

A schema contains one or more schematic letters such as $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. A schematic letter can be replaced by a *proposition* or a *predicate*; this act of replacement is called instantiation. Replacing all schematic letters by propositions (or predicates) creates a proposition (or predicate) which is referred to as an instance of the schema.

schematic letter

See *schema*.

semantic entailment

Suppose that for every false *instance* of the *propositional form* $\mathfrak{A}$ there is at least one false *corresponding instance* of a propositional form in the set Γ. Then Γ is said to semantically entail $\mathfrak{A}$. Furthermore, whenever all the corresponding instances of Γ are true, then the corresponding instance of $\mathfrak{A}$ is also true

semantic equivalence

Two propositional forms $\mathfrak{A}$ and $\mathfrak{B}$ are semantically equivalent if *corresponding instances* always have the same truth values. If $\mathfrak{A} \models \mathfrak{B}$ and $\mathfrak{B} \models \mathfrak{A}$ then $\mathfrak{A}$ and $\mathfrak{B}$ are semantically equivalent.

syntactic equivalence

Two propositional forms $\mathfrak{A}$ and $\mathfrak{B}$ are syntactically equivalent if both $\mathfrak{A} \vdash \mathfrak{B}$ and $\mathfrak{B} \vdash \mathfrak{A}$.

tautology

A tautology is a *propositional form* for which the truth value is equal to T for all *instances*. If $\models \mathfrak{A}$ then $\mathfrak{A}$ is a tautology.

term

A term refers to an item. A term can either be a simple label, such as a *proper constant* or *arbitrary constant*, or a *function* applied to another term.

theorem

Suppose we have an *inference form* whose *premiss* set is empty, $\vdash \mathfrak{A}$. Then the *conclusion*, $\mathfrak{A}$, is said to be a theorem.

truth table

A table showing the truth value of a *propositional form* for each possible combination of truth values for the *schematic letters* of the form.

unary predicate

A *predicate* with one *free variable* is known as a unary predicate. A unary predicate represents a property.

uniform replacement

In any context, the uniform replacement of a *schematic letter* by a *propositional form* means that every occurrence of that schematic letter is replaced by the same form enclosed in parentheses.

universal quantifier

In $\forall x\, \mathcal{P}(x)$, the universal quantifier is $\forall$. The proposition $\forall x\, \mathcal{P}(x)$ is true if and only if $\mathcal{P}(x)$ is true for every value of x.

universe of discourse

See *predicate*.

valid argument

An *argument* is valid if it is an *instance* of a *valid argument form*.

valid argument form

An *argument form* $\Gamma \therefore \mathfrak{B}$ is valid if, and only if, $\Gamma \vDash \mathfrak{B}$.

An argument form that is not valid is said to be invalid.

variable

See *free variable* and *bound variable*.

Summary of deduction rules D

D.1 Inference forms

$\neg\neg$I : $\mathfrak{A} \vdash \neg\neg\mathfrak{A}$

$\neg\neg$E : $\neg\neg\mathfrak{A} \vdash \mathfrak{A}$

$\wedge$I : $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{A} \wedge \mathfrak{B}$

$\wedge E_1$: $\mathfrak{A} \wedge \mathfrak{B} \vdash \mathfrak{A}$

$\wedge E_2$: $\mathfrak{A} \wedge \mathfrak{B} \vdash \mathfrak{B}$

$\vee I_1$: $\mathfrak{A} \vdash \mathfrak{A} \vee \mathfrak{B}$

$\vee I_2$: $\mathfrak{B} \vdash \mathfrak{A} \vee \mathfrak{B}$

$\Rightarrow$ E : $\mathfrak{A}, \mathfrak{A} \Rightarrow \mathfrak{B} \vdash \mathfrak{B}$

$\forall$E : $\forall x\, \mathfrak{A}(x) \vdash \mathfrak{A}(\mathfrak{t})$

$\exists$I : $\mathfrak{A}(\mathfrak{t}) \vdash \exists x\, \mathfrak{A}(x)$

D.2 Methods of deduction

$\neg$I : If $\Gamma, \mathfrak{A} \vdash \mathfrak{B} \wedge \neg\mathfrak{B}$ then $\Gamma \vdash \neg\mathfrak{A}$.

$\vee$E : If $\Gamma, \mathfrak{A} \vdash \mathfrak{C}$ and $\Gamma, \mathfrak{B} \vdash \mathfrak{C}$ then $\Gamma, \mathfrak{A} \vee \mathfrak{B} \vdash \mathfrak{C}$.

$\Rightarrow$ I : If $\Gamma, \mathfrak{A} \vdash \mathfrak{B}$ then $\Gamma \vdash \mathfrak{A} \Rightarrow \mathfrak{B}$.

$\forall$I : If $\Gamma \vdash \mathfrak{A}(\mathfrak{t})$ then $\Gamma \vdash \forall x\, \mathfrak{A}(x)$
provided no constant in $\mathfrak{t}$ occurs in $\mathfrak{A}(x)$ nor in any expression of Γ.

$\exists$E : If $\Gamma, \mathfrak{A}(a) \vdash \mathfrak{B}$ then $\Gamma, \exists x\, \mathfrak{A}(x) \vdash \mathfrak{B}$
provided no constant in $\mathfrak{t}$ occurs in $\mathfrak{A}(x), \mathfrak{B}(x)$ nor in any expression of Γ.

D.3 Identity

=I : $\vdash \mathfrak{t} = \mathfrak{t}$

=E : $\mathfrak{t}_1 = \mathfrak{t}_2,\ \mathfrak{A}(\mathfrak{t}_1) \vdash \mathfrak{A}(\mathfrak{t}_2)$

D.4 Derived rules

MT : $\mathfrak{A} \Rightarrow \mathfrak{B}, \neg\mathfrak{B} \vdash \neg\mathfrak{A}$

HS : $\mathfrak{A} \Rightarrow \mathfrak{B}, \mathfrak{B} \Rightarrow \mathfrak{C} \vdash \mathfrak{A} \Rightarrow \mathfrak{C}$

Summary of equivalences E

E.1 Propositional forms

Tautology

Prop1 $\mathfrak{A} \vee \neg\mathfrak{A} =_T T$

Contradiction

Prop2 $\mathfrak{A} \wedge \neg\mathfrak{A} =_T F$

Unit

Prop3 $\mathfrak{A} \vee F =_T \mathfrak{A}$

Prop4 $\mathfrak{A} \wedge T =_T \mathfrak{A}$

Zero

Prop5 $\mathfrak{A} \vee T =_T T$

Prop6 $\mathfrak{A} \wedge F =_T F$

Idempotent

Prop7 $\mathfrak{A} \vee \mathfrak{A} =_T \mathfrak{A}$

Prop8 $\mathfrak{A} \wedge \mathfrak{A} =_T \mathfrak{A}$

Double negation

Prop9 $\neg\neg\mathfrak{A} =_T \mathfrak{A}$

Commutative

Prop10 $\mathfrak{A} \vee \mathfrak{B} =_T \mathfrak{B} \vee \mathfrak{A}$

Prop11 $\mathfrak{A} \wedge \mathfrak{B} =_T \mathfrak{B} \wedge \mathfrak{A}$

Associative

Prop12 $\mathfrak{A} \vee (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \vee \mathfrak{C}$

Prop13 $\mathfrak{A} \wedge (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \wedge \mathfrak{C}$

Distributive

Prop14 $\mathfrak{A} \vee (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \vee \mathfrak{B}) \wedge (\mathfrak{A} \vee \mathfrak{C})$

Prop15 $\mathfrak{A} \wedge (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \vee (\mathfrak{A} \wedge \mathfrak{C})$

Absorption

Prop16 $\mathfrak{A} \vee (\mathfrak{A} \wedge \mathfrak{B}) =_T \mathfrak{A}$

Prop17 $\mathfrak{A} \wedge (\mathfrak{A} \vee \mathfrak{B}) =_T \mathfrak{A}$

de Morgan

Prop18 $\neg(\mathfrak{A} \vee \mathfrak{B}) =_T \neg\mathfrak{A} \wedge \neg\mathfrak{B}$

Prop19 $\neg(\mathfrak{A} \wedge \mathfrak{B}) =_T \neg\mathfrak{A} \vee \neg\mathfrak{B}$

Definition

Prop25 $\mathfrak{A} \Rightarrow \mathfrak{B} =_T \neg\mathfrak{A} \vee \mathfrak{B}$

Modus ponens

Prop26 $\mathfrak{A}, \mathfrak{A} \Rightarrow \mathfrak{B} \models \mathfrak{B}$

Valid argument forms

Prop27 $\mathfrak{A} \models \mathfrak{B}$ if and only if $\mathfrak{A} \Rightarrow \mathfrak{B} =_T T$

Tautology

Prop 28 $\mathfrak{A} \Rightarrow \mathfrak{A} =_T T$

......

Prop 29 $\mathfrak{A} \Rightarrow \neg\mathfrak{A} =_T \neg\mathfrak{A}$

......

Prop 30 $\mathfrak{A} \Rightarrow T =_T T$

......

Prop 31 $T \Rightarrow \mathfrak{A} =_T \mathfrak{A}$

......

Prop 32 $\mathfrak{A} \Rightarrow F =_T \neg\mathfrak{A}$

......

Prop 33 $F \Rightarrow \mathfrak{A} =_T T$

Negation of the conditional

Prop 34 $\neg(\mathfrak{A} \Rightarrow \mathfrak{B}) =_T \mathfrak{A} \wedge \neg\mathfrak{B}$

Contrapositive

Prop 35 $\neg\mathfrak{B} \Rightarrow \neg\mathfrak{A} =_T \mathfrak{A} \Rightarrow \mathfrak{B}$

Exportation

Prop 36 $\mathfrak{A} \Rightarrow (\mathfrak{B} \Rightarrow \mathfrak{C}) =_T (\mathfrak{A} \wedge \mathfrak{B}) \Rightarrow \mathfrak{C}$

Distribution to the right

Prop 37 $\mathfrak{A} \Rightarrow (\mathfrak{B} \wedge \mathfrak{C}) =_T (\mathfrak{A} \Rightarrow \mathfrak{B}) \wedge (\mathfrak{A} \Rightarrow \mathfrak{C})$

Prop 38 $\mathfrak{A} \Rightarrow (\mathfrak{B} \vee \mathfrak{C}) =_T (\mathfrak{A} \Rightarrow \mathfrak{B}) \vee (\mathfrak{A} \Rightarrow \mathfrak{C})$

......

Prop 39 $(\mathfrak{A} \vee \mathfrak{B}) \Rightarrow \mathfrak{C} =_T (\mathfrak{A} \Rightarrow \mathfrak{C}) \wedge (\mathfrak{B} \Rightarrow \mathfrak{C})$

......

Prop 41 $(\mathfrak{A} \Rightarrow \mathfrak{B}) \Rightarrow \mathfrak{A} =_T \mathfrak{A}$

......

Prop 42 $(\mathfrak{A} \Rightarrow \mathfrak{B}) \Rightarrow \mathfrak{B} =_T T$

......

$\mathfrak{A} \Leftrightarrow \mathfrak{B} =_T (\mathfrak{A} \Rightarrow \mathfrak{B}) \wedge (\mathfrak{B} \Rightarrow \mathfrak{A})$

......

$\neg(\mathfrak{A} \Leftrightarrow \mathfrak{B}) =_T \neg\mathfrak{A} \Leftrightarrow \mathfrak{B} =_T \mathfrak{A} \Leftrightarrow \neg\mathfrak{B}$

E.2 Quantifiers

Definition of $\exists$

$\exists x\, \mathfrak{A} =_T \neg\forall x\, \neg\mathfrak{A}$

Commutative

$\forall x\, \forall y\, (\mathfrak{A}) =_T \forall y\, \forall x\, (\mathfrak{A})$

$\exists x\, \exists y\, (\mathfrak{A}) =_T \exists y\, \exists x\, (\mathfrak{A})$

Distributive

$\forall x\, \mathfrak{A} \wedge \mathfrak{B} =_T \forall x\, \mathfrak{A} \wedge \forall x\, \mathfrak{B}$

$\exists x\, (\mathfrak{A} \vee \mathfrak{B}) =_T \exists x\, \mathfrak{A} \vee \exists x\, \mathfrak{B}$

Constants

$\forall x\, T =_T T$

$\forall x\, F =_T F$

$\exists x\, T =_T T$

$\exists x\, F =_T F$

Double quantification

$\forall x\, \forall x\, \mathfrak{A} =_T \forall x\, \mathfrak{A}$

$\exists x\, \exists x\, \mathfrak{A} =_T \exists x\, \mathfrak{A}$

Miscellaneous

$\forall x\, \mathfrak{A} \vee \exists x\, \mathfrak{A} =_T \exists x\, \mathfrak{A}$

$\forall x\, \mathfrak{A} \wedge \exists x\, \mathfrak{A} =_T \forall x\, \mathfrak{A}$

$\forall x\, \mathfrak{A} \Rightarrow \exists x\, \mathfrak{A} =_T T$

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There is an abundance of books on logic covering many aspects of the subject, including the philosophical, linguistic and mathematical aspects; in addition there are books on the application of logic to computer science, software engineering and mathematics itself. Dean (1997) gives an elementary introduction to sets and logical notation, while Kelly (1997) gives a useful overview of a range of logical systems: both books are in the *Essence of Computing Series*. Burke & Foxley (1996) and Galton (1990) are specifically intended for the computer scientist, while Garnier & Taylor (1997) is intended for the mathematician: all three books consider the practical applications of logic and would be suitable for further reading at a slightly more advanced level than has been covered in the current text. Burris (1998) is somewhat more advanced and largely suitable for the more mathematically inclined reader; nevertheless, it also gives some very interesting insights into the history and development of mathematical logic. Enderton (2001) gives a good introduction to the theory of symbolic logic, while the two volumes by Cori & Lascar (2000, 2001) cover the theory of formal logic in somewhat greater depth; all three books are written from a point of view that should be useful both for the computer scientist and the mathematician. Mendelson (1997) has been a standard text in mathematical logic for many years and is an excellent and thorough book for mathematicians; it is perhaps less useful for computer scientists.

Lemmon (1971) covers natural deduction at a level similar to the current text but concentrates on a tabular approach to presenting deductions rather than deduction trees; some readers may find it useful as a supplementary text. Tennant (1990) also considers natural deduction, but at a somewhat more advanced level; it would be suitable for both supplementary and further reading. Barwise & Etchemendy (1990) provide useful insights into the relationship of symbolic logic to language; like the current book, but unlike other books, the conditional connective is delayed until after the basic ideas of logic have been introduced. Hodges (2001) is a very readable introduction to logic but yet manages to cover some fairly advanced and difficult concepts; it approaches the subject from the point of view of linguistic analysis, and develops the basic ideas of 'semantic tableaux' and models. Of the many books on classical logic, Luce (1973) provides a useful summary; sadly this book is now out of print. Quine (1986) gives a stimulating discussion of the philosophy of logic.

Note that not all the following books are currently in print, but should be available in libraries.

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