The Young Scientist

How Plants Grow in The 'Golden Mean' Ratio

by Ned Rosinsky, M.D.

Why do plants grow in the shapes that they do?

This question has fascinated scientists for thousands of years. Although the shapes of plants can become quite complicated, a great deal can be understood simply by considering what the plant needs in order to function. First, it needs to be exposed to sunlight for photosynthesis, so it will tend to grow in a shape with a large surface area exposed to the Sun. Second, it needs room to grow, so it will tend to grow in such a way that one part of the plant does not crowd another part.

These simple ideas, if examined carefully, lead to interesting conclusions about how the plant must be shaped and how the plant must grow into its proper shape. I will show here that the kind of shapes and growing forms that best allow the plant to do its work are all related to a particular mathematical ratio called the *golden mean*, which is approximately 1.62. The golden mean is the ratio in which the smaller part of a quantity is to the larger part of that same quantity as the larger part is to the whole (Figure 1). The quantity can be just a line you have drawn on paper and then divided, a container full of marbles that you divide, or something living, like a plant.

$a/b = b/(a + b)$

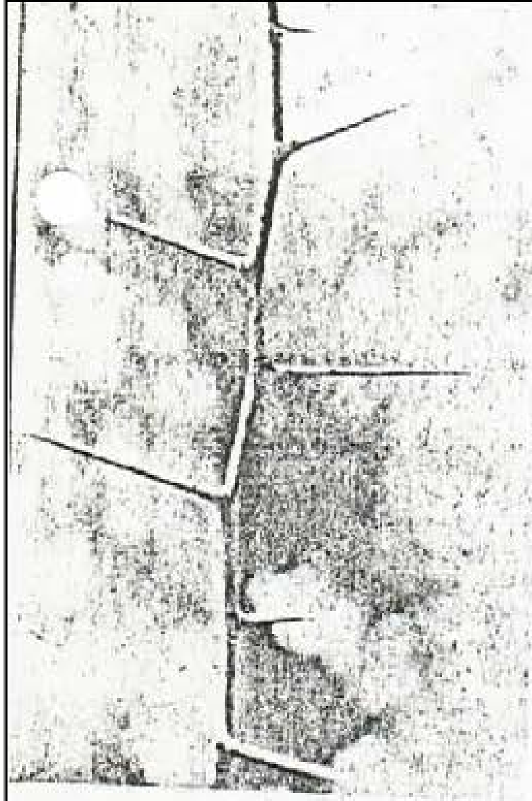
Figure 1
THE GOLDEN MEAN RATIO
The golden mean is the ratio of division in which the smaller part, a, is to the larger, b, as b is to (a + b).

To understand the importance of the golden mean, you must first notice that living things usually grow by multiplication, rather than addition. For example, if you start with 1 bacterium, after about 20 minutes it will have divided in half and produced 2 bacteria; after another 20 minutes, 4 bacteria, then 8, then 16, and so on. Every 20 minutes the number of bacteria doubles or, in other words, multiplies by 2.

A series of numbers that grows by multiplication is called a *geometric series*, such as 2, 4, 8, 16, and so on.

Figure 2
SELF-SIMILAR GROWTH
As the spruce tree grows, its overall shape does not change much, only its size. This gives it self-similarity in its shape as it grows.

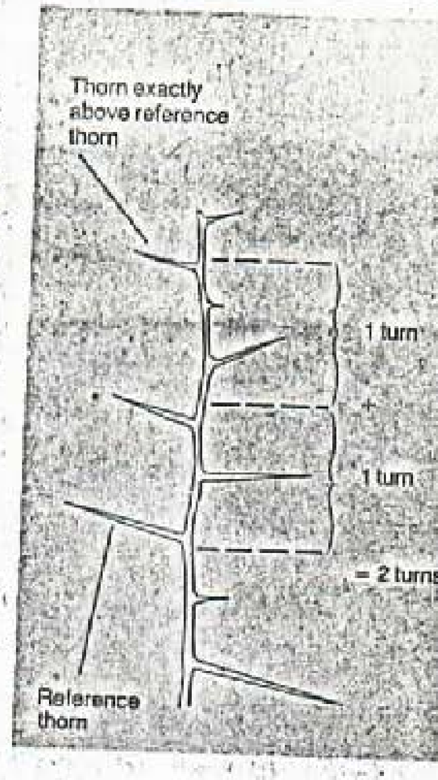
Figure 3
SNAIL SHELL
The snail grows at a rate proportional to its size at any particular time. Therefore, its rate of growth is always increasing. Since the shell is turning as it grows, it produces a spiral shape.



◄ Figure 4
SPIRAL PHYLLOTAXIS
In this plant the thorns are arranged around the stem in a spiral pattern. If you start with a particular thorn, can you see how many thorns you have to pass through above that thorn until you reach one exactly above the thorn you started with?

Source: *Patterns in Nature* by Peter S. Stevens, (Boston: Little, Brown and Company, 1974), p. 158.

Figure 5 ►
THE PHYLLOTAXIS RATIO
In Figure 4 you saw that two thorns that grow at the same position on the plant stem are separated along the stem by a certain number of thorns. The phyllotaxis ratio is the ratio of the number of turns in the spiral you had to make to get from your reference thorn to a thorn growing at the same position on the stem, divided by the number of thorns you passed through. Here it is two turns and five thorns, giving a ratio of 2/5.



In contrast, a series that grows by addition, such as 2, 4, 6, 8, 10, and so on (here, 2 is added each time), is called an *arithmetic series*. The main point is that plants tend to grow by geometric series, and that this kind of growth causes certain kinds of shapes in the plants (see Figure 2).

Self-Similar Growth

The reason that plants grow in geometric series (for example, doubling in size every six months) is that the entire plant is growing as one overall unit. This means that if it weighs 1 pound and it takes a month to grow another pound, then when the plant weighs 2 pounds it will take another month to grow another 2 pounds, and so on. The speed at which it grows increases, and is related to the plant's current size at any particular time. This causes the size to increase as shown in Figure 2. Since the plant is growing overall as a unit, it tends to keep the same shape even though it gets bigger. This is called *self-similar growth*.

If you look at the shape of a snail you can see the same pattern of geometric growth, only now since the shape of the snail is turning as it grows, it produces a spiral (Figure 3). This geometric spiral, also called a logarithmic spiral, is the main kind of shape you see in plants. If you can understand how this spiral works in plants you will begin to see why they are shaped the way they are.

A good example of a spiral in a plant is the way the leaves or thorns are located on a plant stem in many kinds of plants (Figure 4). Look at the overall shape of a spruce tree, which is a cone. The branches of the tree stick out in a pattern that spirals around the tree's cone shape (Figure 2). Now, let us see how this geometric growth connects to the specific ratio of the golden mean.

If you look at various types of plants, you find that the leaves around the stem are arranged in different kinds of spiral patterns. If you pick any leaf and call it a *reference leaf*, and then start counting leaves above it as they spiral around up the plant stem, sooner or later you will find a leaf that is directly above the reference leaf on the stem. You may have to go a number of times around the spiral before you find this leaf.

Now, if you count up the turns around the spiral you made, and divide it by the number of leaves you went through, you get a ratio that is characteristic for each plant species. For example if you made 5 turns and passed 13 leaves, the ratio is 5/13. This is called the *phyllotaxis ratio*. (Phyllotaxis comes from the Greek words for leaf, *phyllon*, and order, *taxis*; the plural of taxis is taxes.) See Figure 5.

The Phyllotaxis Ratio

These phyllotaxis ratios form an interesting series of fractions: 1/1, 1/2, 1/3, 2/5, 3/8, 5/13, 8/21, 13/34, and so on. What is interesting is that all the numbers come from a series called the *Fibonacci series*, named after the 12th-century mathematician who discovered it. The Fibonacci series is formed by starting with 1, adding another 1, and then getting each next new member of the series by adding the previous two members: $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, $3 + 5 = 8$, and so on. This forms the series 1, 1, 2, 3, 5, 8, 13, 21, 34, and so on. (See Professor von Puzzle, p. 59.)

The plant phyllotaxes are ratios formed by taking two numbers from this series that are separated by one number; for example 3 and 8 or 5 and 13.

This particular series has some interesting geometrical properties. Take a circle and divide it into a number of parts according to one of the Fibonacci numbers, say, 13. Then count off sections of the circle by the Fibonacci number that is two numbers behind 13, which is 5. This gives you the pattern shown in Figure 6.

Figure 6

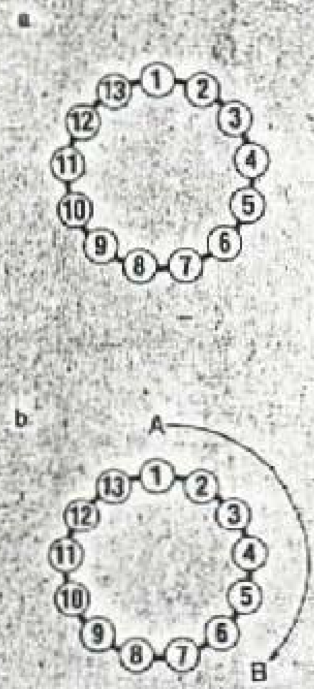
UNPEELING THE FIBONACCI SERIES

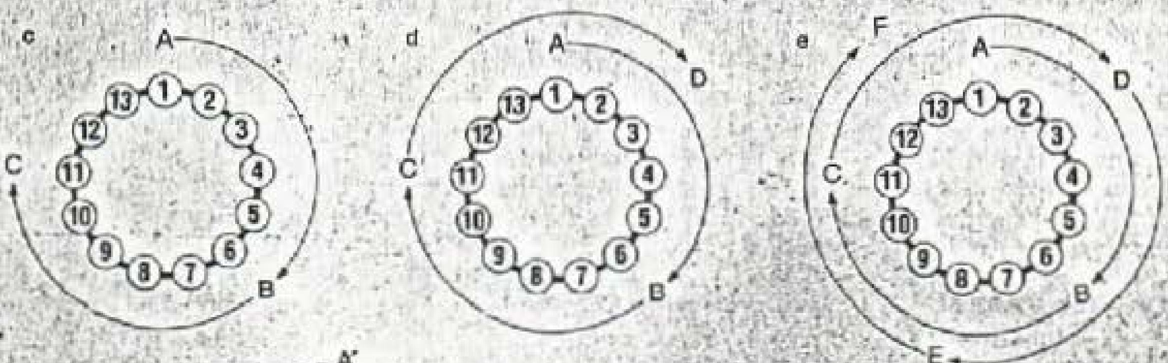
In this example we choose a Fibonacci number, 13, and divide a circle into that number of parts (a). We then begin with one of the sections and start counting off sections around the circle according to the number that is two numbers back on the Fibonacci series, 5 (b). Call the original point on the circle A and the next point B. Notice that B has divided its space of 13 sections into two parts of 5 and the remaining 8.

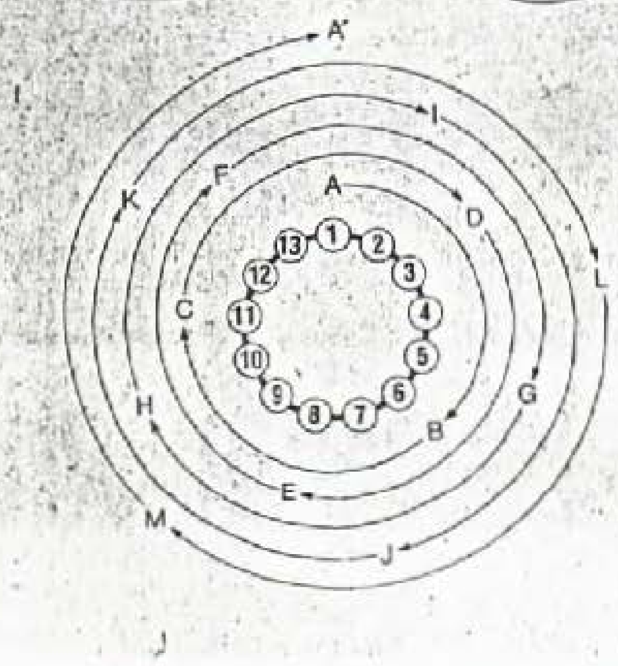
Next we continue the process by counting off another 5 sections to arrive at C (c). Point C divides its space, which is the 8 sections from B back around to A, into two parts of 5 and 3. Next, we go around another five sections, passing A and arriving at D (d). Note here that D divides its space, which consists of the 5 sections bounded by A and B, into the two parts of 2 and 3.

In e, two more points have been added: E divides its space into the parts 2 and 3, and F divides its space of 3 sections bounded by C and A into the parts of 1 and 2. In the final figure, f, the process is continued for the full 13 points, until the spiral returns to A.

Note several interesting things: First, all the divisions are Fibonacci ratios, including all of the remaining points after E. Second, each section of the original 13-section division of the circle has been used exactly once in this process. Third, notice that the counting-off process has revolved around the circle five times exactly, and has used up 13 spaces, which in a plant would give a phyllotaxis ratio of 5/13.







CALCULATIONS OF FIBONACCI RATIOS FOR THE FIRST FEW TERMS OF THE FIBONACCI SERIES

1/1	=	1.0000
2/1	=	2.0000
3/2	=	1.5000
5/3	=	1.6666
8/5	=	1.6000
13/8	=	1.6500
21/13	=	1.6154
34/21	=	1.6191

As you can see, each new place counted off on the circle divides its space into a Fibonacci-type ratio, either 1/1, 1/2, 2/3, 3/5, 5/8, 8/13, and so on. Notice that these ratios are the same as the ones you would have produced if you had counted off 8 spaces going around the circle in the opposite direction. The Fibonacci phyllotaxis can also be obtained by taking two consecutive Fibonacci numbers, 13 and 8 (that is, without skipping a number in the series), and going around your circle in the opposite direction (Figure 7b).

You will obtain a similar picture by starting out with a circle divided into equal parts by any other Fibonacci number, such as 21. (With the circle divided into 21 parts, you would then count off segments in groups of 13.)

Looking Down at the Plant

Now, consider the sectioned-off circle you have drawn as a diagram of a plant, looking down at the plant from the top, with each of the above countings in the circle representing a new leaf sprouting out. You can see that each leaf is dividing its space into a Fibonacci ratio. What can you tell about these ratios?

First, each place you have marked off on the circle divides the previous space marked off on the circle nearly in half. Thus, each new leaf is almost evenly placed between two previous leaves, giving it lots of *space in which to grow*. Second, each leaf also has lots of *space to get sunlight*. In these divisions, the space on one side of a leaf is never more than twice the space on the other side.

Now, calculate the ratios you have marked off as they get farther along in the series, dividing the numerator by the denominator. You can see that the ratios get closer and closer to the ratio 1.62, which is the golden mean (see table). Some plants, in fact, have the golden mean ratio as the angle of the separation of consecutive leaves (Figure 8).

What about other ratios? Let us try an experiment with a ratio that is not from the Fibonacci series. Keeping the 13-divided circle, if you count off by 7 sectors at a time (instead of 8 as in the Fibonacci series) you get within one section of the first leaf after just two leaves, causing unnecessary

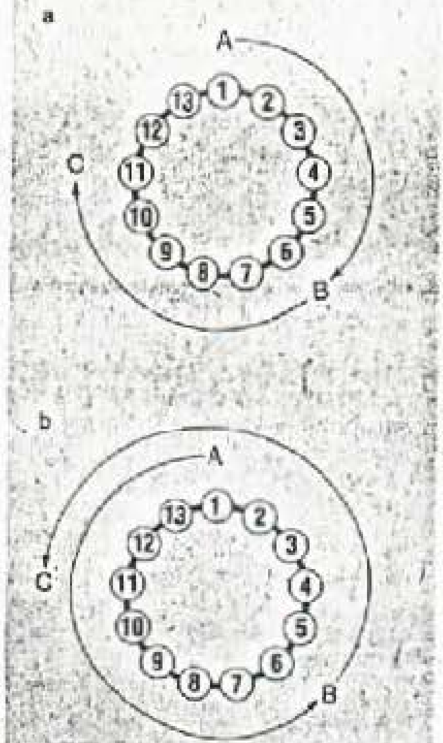


Figure 7
ALTERNATE FIBONACCI COUNTING

In a, the 13-sector circle is counted off in groups of five, giving the points A, B, and C. In b, the same circle is counted off in groups of 8, in the opposite direction, giving the exact same location of the points A, B, and C. We see, therefore, that the neighboring Fibonacci number to 13, which is 8, can be used to give the same results as 5.

If you work out this example all the way, as was done in Figure 6, you will see that all 13 sections are used up. Now, however, instead of 5 complete rotations around the circle, there are 8, giving a phyllotaxis ratio of 8/13.

crowding. If you use 4 to count off, or rotate, you get again within one section of the start within three leaves (Figure 9). You can try this with other numbers of section divisions and rotations.

To see why the Fibonacci numbers work so well, look more carefully at

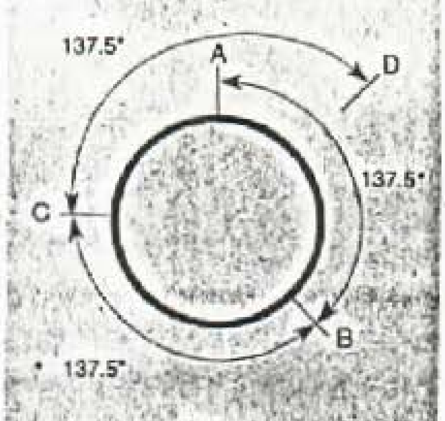


Figure 8
GOLDEN MEAN SEPARATION OF LEAF ATTACHMENT LOCATIONS

Here the circle is divided by the golden mean, which is approximately 1.62 or 137.5 degrees (out of 360 degrees). Notice that the points are located in positions that give them ample room as each is successively laid down.

what happens in the above example of a circle divided into 13 sections by counting off 5 (or the equivalent, 8, in the opposite direction). The first counting divides the circle into 8 and 5, which are the two previous terms in the Fibonacci series. The next counting, or rotation, of 5 sections divides the space of 8 sections between the first and second leaves into two portions of 5 and 3, which are the next numbers counting backwards in the Fibonacci series.

Now, the next counting of 5 goes past the first leaf and divides the space of 5 sections between the first two leaves into two portions of 2 and 3. Again, this is done by moving backwards in the Fibonacci series, continuing the process (Figure 6). This is really like an "unpeeling" of the Fibonacci series by subtraction!

Since the Fibonacci ratios get progressively closer to a constant value, the golden mean, the series becomes close to a geometric series in which the golden mean is the constant factor of multiplication. Plants frequently flip from one phyllotaxis to another in the course of early development, or in the evolution of new species of plants. Since the ratios are in a geometric series, this

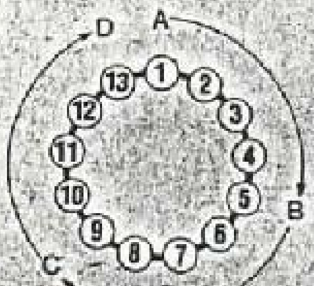


Figure 9
NON-FIBONACCI DIVISION OF THE CIRCLE
Using the circle divided into 13 sections, we try some counting number other than 5 (or the equivalent 8), such as 4. Here the first added leaf, B, divides its space up into 4 and 9, which is less symmetric than the 5 and 8 division. The second leaf added, C, divides its space between B and A of 9 sections into 4 and 5, a good division, but then its next leaf, D, divides its space of 5 sections into 4 and 1, a highly unequal division. Similar highly unequal divisions occur with other choices of the counting number, such as 3 or 7.

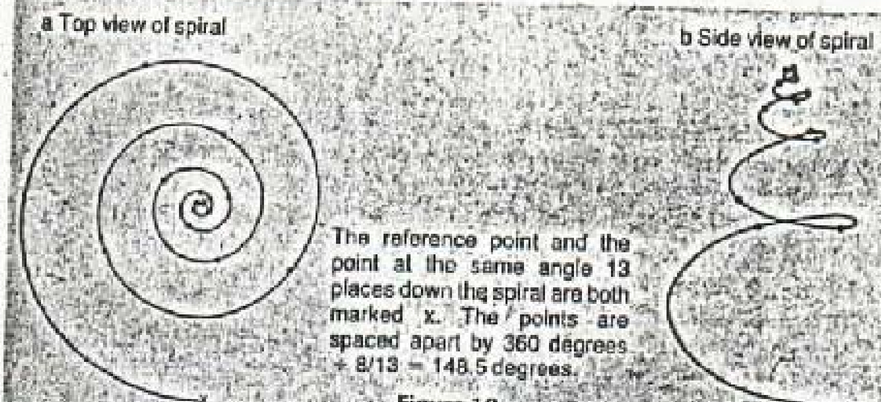


Figure 10
SPIRAL ARRANGEMENT OF LEAF POSITIONS
Here the circle representation is uncoiled to reveal the underlying spiral of leaf arrangement. In this example the 13-section circle becomes a 13-section spiral. After 5 turns above a given point in the spiral, there is another point at the same angle on the spiral. Again, another spiral could have been drawn through the same 13 points but in the opposite direction. This spiral would have had to turn 8 times between the matched points. The top view of this spiral is shown in a, and the side view of the spiral, looked at obliquely, is shown in b.

aspect of evolution is really a geometric jump, from one ratio to another similar ratio. This is another aspect of self-similar growth that is more easily seen in the overall growth of the snail or the overall shape of a tree as it grows.

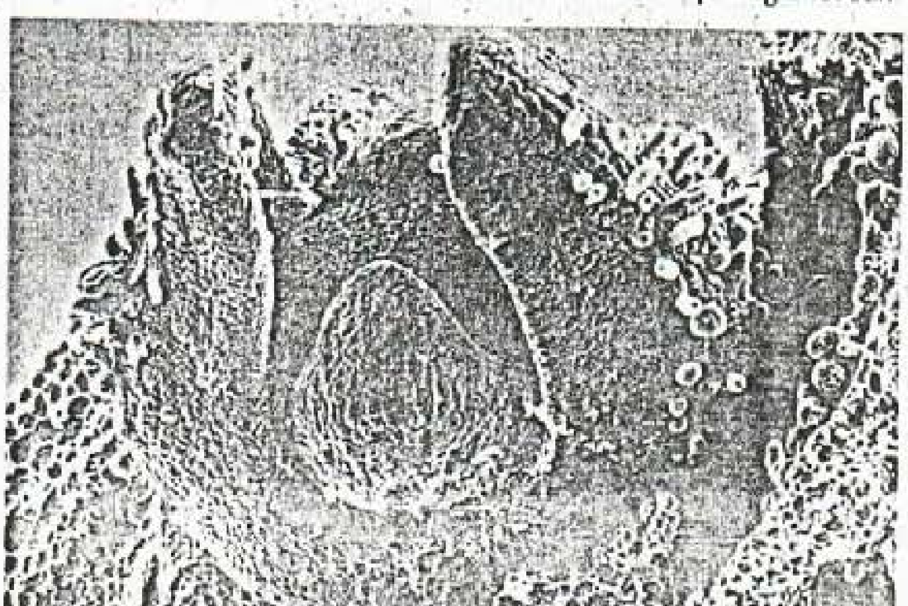
Next, take the circle on which you have drawn the leaves and re-draw it back to its original spiral shape (Figure 10). Here you can see that as the Fibonacci series is "unpeeled," the leaves are spread out vertically, so that neighboring leaves on the circle are far away from each other on the grown plant. The outward growth of elongation of the leaf stems makes sure that the new leaves do not shadow the old leaves, which may be directly under them or near them in the circle model.

This completes the overall picture, showing that the plant is generally shaped like a cone with a spiral of leaves coming out of a central stem. The plant always grows in such a way that it expands into a new size but keeps a similar conical shape.

The Question of Crowding

The same question of crowding applies to the small piece of plant tissue at the growing tip of the plant where the new leaves actually sprout. This tissue is called the *meristem*, and measures less than 1/32 of an inch across. The new leaf buds come out of this tissue at the same angles they will have on the stem as adult leaves. Therefore, the crowding question for growth and sunlight is exactly the same as it is in the case of starting the leaves out with room to grow on the meristem in the first place (Figure 11).

This shows that the question of what the best leaf spacing is for sun-



Photograph by Dr. R.L. Peterson, University of Guelph, Ontario, Canada

Figure 11
HOW BUDS GROW ON THE MERISTEM
The meristem shown here is magnified 250 times, and the outer leaves are cut away to expose the very youngest leaf buds as they begin to form on the surface of the meristem tissue. Notice how close the young leaf buds are to one another.

light, for growth, for growing out in the meristem, and for evolution as well, are all really the same. These problems require the same solutions, and the best solutions are all based on self-similar geometric shapes related to the golden mean.

Some Experiments

In order to look at these ideas more clearly you can do several kinds of experiments. First, do the example described above of dividing a circle into a Fibonacci number of sections and then counting off sequences of sections according to the next smaller Fibonacci number. Notice how the addition of each leaf divides the space for that leaf in a Fibonacci ratio.

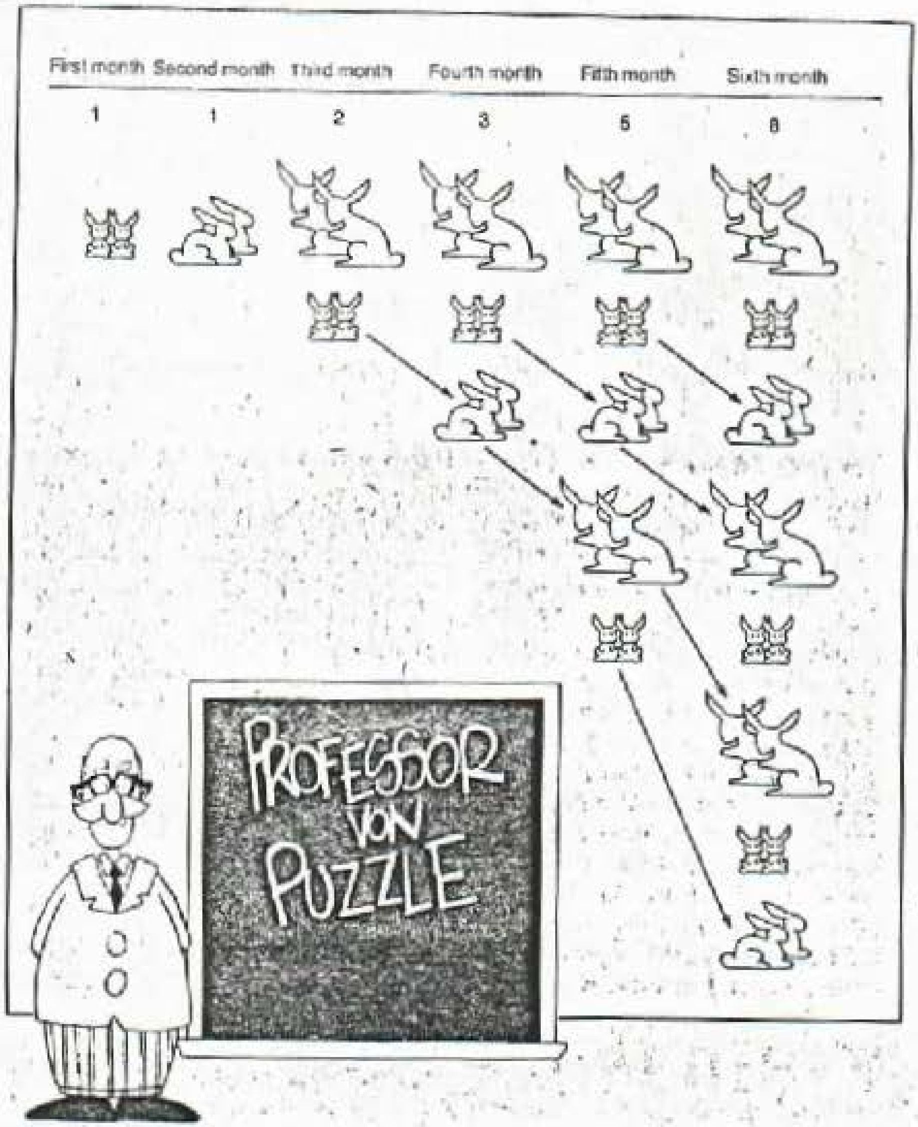
You can try this exercise with other numbers that are not from the Fibonacci series to see whether the leaf spacing is as good.

Second, collect plant samples that have the various Fibonacci ratios in their leaf or thorn patterns around the stem. Examine these specimens from various angles to see the effect of the spacing on sunlight exposure and growth crowding. A few common examples are the 1/3 ratio in eech trees, the 2/5 ratio in oak, the 3/8 ratio in poplar, and the 5/13 ratio in willows.

Third, dissect the meristem tip of a plant under a low power microscope or strong hand lens, to see the pattern of leaf buds and how they form a Fibonacci geometry. Notice the extremely close crowding of growth at the meristem tip.

Fourth, take snapshot pictures of a growing plant every day at the same time from the same place, in order to visualize the growth patterns that follow the self-similar patterns you have outlined.

Fifth, a more challenging experiment would be to investigate why some plants that may start out in a conical shape change to other shapes. I'll give you a hint: The increase in crowding of the plant (its population density) may affect its shape as it attempts to maximize exposure to the Sun or to ground water. For example a lone oak may be wider at its base, but an oak in a crowded forest may be wider on top.



Brother Bonacci's Rabbits

Filius (Brother) Bonacci, who was born in Pisa, Italy, in the year 1170 AD, discovered a unique series of numbers that have ever since borne his name—the Fibonacci series. Brother Bonacci is said to have discovered the series observing the reproduction of rabbits.

Suppose we want to raise rabbits, and we start with a newborn pair—a male and a female. Let's assume for simplicity that each rabbit pair takes one month to mature to the point that it can produce offspring, and that the female carries her young for one month. That means it will take two months from the birth of the first rabbit pair to produce the first offspring. Let's also assume for simplicity's sake that each litter consists of one male and one female rabbit.

Now let's look at the growth of the rabbit population by month. In the first month we have *one pair*, and so also in the second. At the beginning of the third month a new pair is born. Now we have *two rabbit pairs*. At the beginning of the fourth month, that new pair is maturing, and the original pair give birth to another pair. The total number of pairs now *equals three*.

What do you think happens in the fifth month? Let's see. The original pair gives birth to yet another pair, making four pairs. But the pair that was born at the beginning of the third month also has a pair. So the total number of pairs *equals five*.

The chart helps you to see how this growth continues, producing the series named after Filius Bonacci:

1, 1, 2, 3, 5, 8, 13, 21, . . .

This series has two very important

properties. First, any term in the series is equal to the sum of the two previous terms. For example:

$$2=1+1; \text{ or } 13=5+8$$

Knowing this, you can produce the series just beginning with 1 and 1.

Second, the ratio of any two neighboring terms of the series is almost equal to the number 1.618, which is the ratio of the golden section or golden mean (see "How Plants Grow," p. 54). The higher we get in the series, the more true this is. Try it yourself. Use the first property I have just shown you to write out the Fibonacci series a few terms beyond the highest number (21) that I have given you. Now take any two neighboring numbers and divide them, the higher by the lower. (For example $21 \div 13 = 1.615$). You may use a calculator to facilitate this long division.

You see that the higher you go in the series, the closer it gets to the ratio of the golden section, which is approximately 1.618. This is called *convergence*. The ratio of neighboring terms in the Fibonacci series converges on the golden section.

A Geometric Construction

You can also produce the Fibonacci series by a geometric construction. Start with a square whose sides we consider to be 1×1. Now, you need only follow a simple rule to generate the Fibonacci series. The rule is: Always add a square to the longest side of your figure. Since we begin with a square, no side is the longest; we can add a square to it on any side (Figure 1).

Figure 1

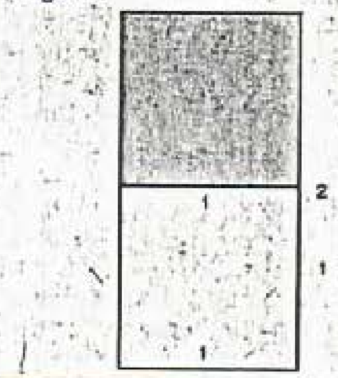
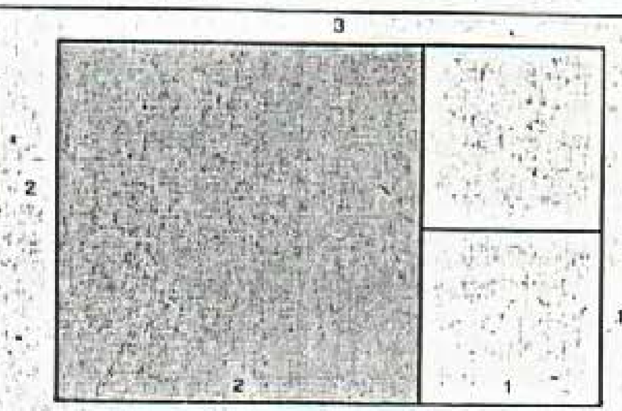


Figure 2



But now we have a figure twice as long as it is wide (2×1). So we add a square to its longest side (Figure 2).

Now our figure is 3×2. As we keep adding squares to the longest side, we get figures of 5×3, 8×5, 13×8, and so on. As the figure grows larger, the proportions of the rectangle hardly change. This is self-similar growth. The size increases, but not the shape. All living things grow this way.

Now for the puzzle: What will happen if you start with any rectangle and construct a square on its long side, making a new, larger rectangle? Repeat the process a number of times. Then make a series of numbers from the lengths of the rectangles you have constructed. What can you discover about these rectangles?

—*Laurence Hecht*