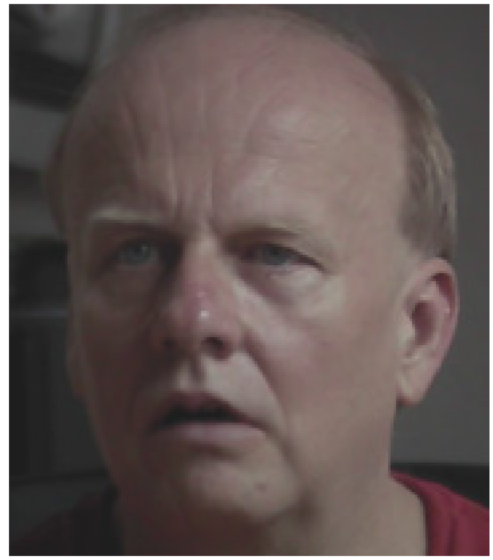
What is and what will be

Integrating spirituality and science

Paul P. Budnik Jr.



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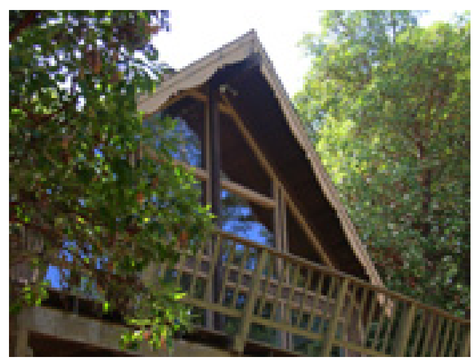

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This version of *What is and what will be* has images distributed throughout the text as visual poetry. The images were mostly taken at Esalen Institute near Big Sur California and at my home in the Santa Cruz Mountains near San Jose California. The dolphin pictures and a few others were taken in Hawaii.

The images in this version are formatted at 72 dots per inch. This is suitable for viewing on a display. Other formats and versions are available at: www.mtnmath.com/willbe.html.

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Chapter 1

Origins

In this age of extraordinarily powerful computers, the human brain remains the most complex object on earth. What led to the evolution of this costly labyrinthine structure? Why is it capable of such wondrous ecstatic experience? Why has it evolved conflicting instincts, that can lead to unbearable anguish and pain? Why do spiritual and religious instincts have such enormous power and influence? What is their evolutionary function?

Two ideas at the core of this book provide a useful perspective on the most fundamental questions of human existence. The first is the simple, even trivial, observation that controlling ones actions and predicting their consequences have enormous survival value. This is why nervous systems and brains evolved. By connecting this observation to the mathematical limitations of prediction proved by Kurt Gödel, we gain insight into the history and future potential of biological evolution.

The abilities to control ones actions and predict their consequences have no intrinsic value or meaning. Ultimately, only conscious experience does. Complex brains seem to be essential for the richness of human experience. What is the relationship between conscious experience and the physical structure of the brain? The answer developed here is an old idea invigorated by contemporary science and mathematics. Conscious experience in some from is not unique to human or animal existence. It is universal in all that exists. It is only its form that changes between individuals, between species and between animate and inanimate matter.

This is a philosophical assumption that is ultimately beyond proof, but it is developed in the spirit of science, by seeking the simplest assumption consistent with what we know. The case is made empirically by considering research that connects the structure of the living brain with reports of internal conscious experience. The case is made theoretically by showing that all of science and mathematics is built from simple structures that, at least in mathematics, are reducible to the empty set or nothing at all. It is only conscious experience that gives substance to the abstract structure of our intellectual understanding.

The remainder of this chapter tells three stories to explain the origin and motivation for these ideas. The first focuses on the abstract nature of mathematics. Human thought is not abstract. We think in terms of images, sounds, internal experience etc. Even when we think in terms of symbols, like the words on this page, we hear, see or experience something concrete. I hear the words I read. Abstraction was essential to the progress of mathematics and science but it did not come quickly or easily.

The other two stories are personal. One is about important results in mathematics and computer science that converged in my undergraduate mind to create a way of looking at reality. The third story is about the evolution of those ideas over decades.

1.1 The trouble with lines

It would seem that when we examine the structure of an object we must ultimately come to some irreducible components that have an essence or intrinsic nature. For example we might consider the lumber, nails, concrete foundation and roofing shingles as being among the fundamental components of a house. Contemporary mathematics has deliberately and systematically purged itself of any objects with an intrinsic nature. Mathematicians did this because starting with objects that had an intrinsic nature, like lines, led them to make false assumptions like the parallel postulate of Euclidean Geometry.

For centuries, the parallel postulate of Euclid was considered to be a self evident truth. Two lines are parallel if they are both perpendicular to a third line. For example the legs of a well made table are parallel because they are perpendicular to the table top. No matter how far one extends the legs they will never meet if one keeps them straight and parallel. This is the parallel postulate. It seems self evident.

Now consider the laws of geometry on the surface of the earth. Sailors determine their location in the ocean by latitude and longitude. These are imaginary lines that circle the earth. Lines of latitude are parallel to the equator. Lines of longitude are perpendicular to the lines of latitude. Thus all lines of longitude are parallel with each other. However, if you look at a globe with the major lines of latitude and longitude marked, you will see that all the lines of longitude intersect at the north and south poles.

The surface of a sphere does not conform to our intuitive notions about parallel lines. We call geometries that obey the parallel postulate Euclidean. Many important geometries are not Euclidean including the surface of our planet. General relativity defines the geometry of our universe in contemporary physics. It too is not Euclidean.

Mathematicians wanted to avoid making assumptions that are not universally true like the parallel postulate. To that end they removed any fundamental entities like lines, planes or points from the formulation of mathematics. They invented set theory. In set theory there is a single primitive entity, the empty set, and a single primitive relationship, set membership. The only objects are the empty set and things constructed from the empty set. For example the number one is the set containing the empty set. The number two is the set containing the number one and the empty set.

1.2 Reductionism

Physics and mathematics deal only with abstract structure. They have purged themselves of anything with an intrinsic nature. It is not natural to think this way. It is a conceptual leap. I will explain some of what led me to this conclusion as an undergraduate.

Computers were a comparative novelty in the late 60's and I was able to work with one of these extraordinary machines. I could program it to do complex tasks using simple instructions. The low level or "assembly language" for computers contains instructions like move the value stored in one place to a different place or add the values stored at two locations together and put the result in a third place.

The computer itself was constructed from simple operations. You could build all the machinery that controlled the computer from three components. These are devices that have input signals in the form of voltage levels and generate similar output levels. One range of voltages corresponds to the number one or switched on and another range corresponds to zero or switched off. Devices that have inputs and output signals like this are called logic circuits.

Two of these circuits had two inputs and one output. The other circuit has a single input and output. The first circuit is a logical AND. Its output is one if and only if both of its inputs are one. The next is an OR circuit. It outputs a one if either or both of its inputs are one. The third NOT circuit has an output that is the opposite of the input. The input/output function of these three circuits is shown in Table 5.1 on page 79 with 'true' and 'false' substituting for 1 and 0. Every logic circuit in any computer no matter how complex can be built out of these three simple circuits.

I was struck by what can be constructed from such simple building blocks. My interest and wonder was further aroused by the idea of a Universal Turing Machine. This is a very simple computer that can simulate any program that any computer could execute. One way to characterize computer programs is with the mathematical functions they can compute. A Universal Turing Machine can compute any mathematical function that *any* computer can compute.

A mathematical function is a formula or procedure for uniquely defining a function output number from a function input number. Many calculators have a square root function key. Enter any positive number and touch that key and the square root of the number is displayed. Square root is a function on the positive numbers. Any formula that defines a *unique* output number for every input number in some collection of possible inputs (like the positive numbers) defines a function. In the 1930's there were a number of proposals to characterize those functions that could be calculated by following exact mechanistic rules. All of the major contenders turned out to be equivalent. One of these was a Universal Turing Machine. At the time Alonzo Church proposed what has come to be called Church's Thesis. This states that the functions computable by a Universal Turing Machine are all the functions that can be computed by any precise mechanistic process. Church's thesis is almost universally accepted.

The AND, OR and NOT circuits of a computer are so simple that they have little content. The important thing is how they are put together to form complex circuits. The programs that control computers are made up of very simple instructions. The computer and its programs can be completely understood in reductionism terms. There is nothing to the simple instructions a program is built from or the simple circuits used to build a computer. It is only the structure of how these simple components are assembled that has significant content.

The sense that everything is structure was expanded when I studied set theory. All of mathematics was constructed with the single primitive entity of the empty set and the single relationship of set membership. This was much like the primitive circuits that made up a computer. All of this led me to ask what is it that is structured in reality?

1.3 From the empty set to God

Conscious experience has an intrinsic nature. The experience of the color blue has an irreducible reality that cannot be analyzed into constituent parts. In the technical language of philosophy it is a 'quale'. (The plural of quale is qualia.) It feels like something to experience blue.

What is the connection between the structure of physical reality and the intrinsic nature in immediate experience. There is no way to answer this question from science or mathematics. Through those disciplines we can understand a great deal about the *structure* of conscious experience. For example we know a great deal about optical illusions from studying the structure of the eye and the neural network that connects the eye to the brain. But this provides no clues as to why the experience of a blue sky is as it is.

One approach to such questions is to look for the simplest assumption consistent with what we experience. This is the approach used for fundamental laws of physics like Einstein's Special Theory of Relativity. The fundamental equations of this theory are known as the Lorentz Transformation because they were invented by Hendrik Lorentz not Albert Einstein. Einstein showed how these equations could be derived from very simple assumptions that had wider implications. The most famous of these is the equation $E = mc^2$.

The simplest assumption about conscious experience leads to a very old intuitive idea. Immediate experience in some form is universal in all that exists. This assumes there is nothing special about the matter in our brain that seems to embody conscious experience. In thinking about this many years ago I went further in the search for simplicity. I assumed that structured conscious experience is *all* that exists. There is no need for "physical stuff" of which external reality is constructed and "soul stuff" that embodies our consciousness. Everything is soul stuff.

The essence and totality of the existence of physical structure is immediate awareness in some form. Qualia are universal in all that exists. Simpler brains have simpler experiences. This can be extrapolated all the way down to inanimate matter. Immediate awareness in some form is *all* that exists. Once one can explain the structure of conscious experience, including the experience of the external world, there is nothing remaining that requires explanation. The physical world is the transformation of conscious experience and nothing but the transformation of conscious experience.

This assumption, which is part of the Totality Axiom of Section 4.1, came to me in an advanced philosophy course. My focus at the time was on mathematics and especially physics. If all that exists is the abstract structure of mathematics made real as immediate awareness, then physics at its core must be discrete and not continuous. For consciousness is finite. We seem to experience continuity in vision but that is an illusion created by the brain. Our eye detects light as discrete pixels just as a video camera does. It is only subsequent processing in the brain that groups these pixels and creates the illusion of continuous structures like lines.

This philosophical speculation and the observations that complex computers and all of mathematics are built up from simple discrete structures had a powerful impact on me. I became preoccupied with the idea that discrete models might be the solution to some of the paradoxical aspects of quantum mechanics. I entered graduate school in computer science because I felt the background research needed could qualify as computer science long before, and whether or not, it led to new physics. A thesis adviser agreed to sponsor me on this line of research, but I was not making enough progress and had to find a different topic.

I eventually completed a conventional thesis in computer science, but the pull of these other ideas was overwhelming. Since I was not able to make progress on creating new physics I started focusing on the mathematical implications of these ideas. If there were no infinite structures what was the immense and important body of mathematics based on these structures about?

Mathematics defines an extraordinary rich hierarchy of objects or structures. These are characterized by the mathematical language needed to define them. Some of these have obvious interpretations in a discrete universe of the sort I was considering. For example we have already discussed functions that can be defined by a precise set of mechanical instructions or a Universal Turing Machine. These are called recursive functions. Such functions can be though of as properties of the particular computer program that can generate the function.

I came to see how most sequences of integers, that can be defined mathematically, can be treated as properties of computer programs. To understand this one most know that one can assign a unique integer to every possible computer program. This is called Gödel numbering and it is explained in Section 5.8. One example of a set that is not recursive is the set of Gödel numbers of computers that halt. (Today computer programs do not literally *halt* the computer, but they did so in the early days of computing. Today programs return control to an operating system like Linux or Windows.)

If a computer program halts it does so in some finite time. To say that it never halts is a statement about an infinite sequence of events. But those events can all be generated by a computer. They are the sequence of instructions the program executes. Thus they have meaning in a universe they may exist forever but is finite in each moment of its existence. Such a universe is said to be potentially infinite.

Statements about infinite sequences of events enumerated by a computer are absolutely true or false. They are completely determined by the program or rules for generating the events. Yet there is no general way to determine if such a statement is true or false. This was proved by Kurt Gödel with his Incompleteness Theorem (see Section 5.8) in the 1930's.

One can generalize the question of whether a computer halts by asking if a computer will have an infinite number of outputs if it runs forever. By generalizing and iterating this property it was possible to treat most mathematically definable infinite sequences of integers as properties of computer programs that were meaningful in a finite but potentially infinite universe. I suspect that it is only mathematics that can be interpreted in this way that has an absolute meaning. There are mathematical questions like the Continuum Hypothesis discussed in Section 5.7 that cannot be interpreted in this way. Those questions are a little like the parallel postulate. They are not true or false in any absolute sense. Instead they may be true, false or undecidable in a particular formal mathematical system.

As far as I know this is a unique approach to mathematical truth. My insight about a computer program having an infinite number of outputs as a way to define many nonrecursive sets was not new. It had been anticipated in the U quantifier discussed in Section 6.1. I found this approach to mathematics compelling. It suggested that mathematics was in a sense an experimental science that dealt with properties that were not determined by a particular event but were determined by an infinite sequence of events that you can program a computer to generate.

These ideas totally engrossed me, but seem to have little interest for others. There seemed to be no place for me and the energy that moved me. Carl Jung's *Psychological Types*[32] helped me to gain some perspective on myself. In the language of Jungian theory I am an introverted thinking type. But my most powerful function is my intuition. It was my intuition that was always pushing me off on tangents or making connections that others did not see or did not think were important. These insights helped me make sense out of my chaotic life.

Jung's emphasis on the creative nature of psychic development was especially appealing to me. It connected with my mathematical understanding and my life experience. For me, the creative nature of the universe has always been its most astounding and appealing feature.

I was raised a Catholic but rejected that and every other conventional religion as a college sophomore. But the rituals of the Catholic Church instilled a profound sense of the spiritual that never left me. My idea that consciousness was the essence and totality of all the exists implied a unity between the spiritual and physical. I began to connect ideas from other religious traditions, especially Buddhism, with the sense of spirituality that was emerging from these ideas.

I was born on August 6, 1945, the day the atom bomb was first used to destroy human life. One might say I was born with a sense that our science could get out of hand and destroy us. The immense cruelty that permeated the twentieth century, as I lived through half of it and learned about it all, amplified my concern. I became aware of and concerned about the disparity between the steady growth of science and technology and the chaotic development of spirituality and values. To me the reason for the disparity was clear. The objective guidance of experiments allowed science to make steady progress. This contrasted with philosophy, religion, spirituality and values which remain permeated with prejudice and superstition. Scientists are able to reach agreement about any issues that can be investigated experimentally. Philosophers, theologians and ethicists can reach agreement about almost nothing.

The world faces a long list of potential dangers from human activity. At their core is this disconnect between the steady progress of science and the random walk that characterizes the development of spirituality and values. As the discrepancy between these grow the danger grows. Thus the most important motivation for and biggest ambition of this book is to point the way for starting to repair the split between science and meaning.

The unified view that emerges from this work is that of God as the unbounded evolutions of consciousness. This evolution is a physical process. We can measure it and characterize it mathematically. Its core and essence is experiential. We are the evolution of consciousness that is God. Why this is true is a mystery beyond explanation. But if we try to understand the reality we experience and to give the simplest possible description of what existence is, then this is the vision that emerges. That is the journey that this book is intended to take you on.

Chapter 2

Overview

The journey of this book begins with the distinction between structure and essence. The former is the world of mathematics, science and technology and the latter is the domain of conscious experience. I explain how complete this split is in contemporary scientific understanding. Next I ask what it means to exist. This leads to a connection between scientific understanding and values and meaning. With this philosophical base I, develop some mathematics and physics that is relevant to this connection. Some of the physics is speculative and labeled as such. I then develop this connection, first in philosophical terms and then in more concrete and practical ways. The book ends with philosophical and spiritual speculation that aims to develop something approaching an objective spirituality. This chapter is a tour guide for the journey. It outlines the ideas whose connections form the backbone of this book.

The overall goal of this journey is to begin the process of reconnection science an technology with meaning and value. Historically science could only progress by splitting off from philosophical, spiritual and religious traditions that were burdened with false assumptions about the physical world. Science succeeded by focusing on understanding what was observed and developing techniques to test that understanding. It is the power of this approach that has lead to the most fundamental problem we face today. Our power to manipulate the world is growing exponentially, but our values, that ultimately determine how we use that power, continue to develop haphazardly as they have throughout history.

The solution to this problem is not a return to orthodox religious traditions as so many are moving today albeit each to his own preferred tradition. The solution is the development of a new sense of spirituality, built equally on religious tradition and scientific understanding. From that base one can create a system of ethics and values that has some of the objectivity of the hard sciences and thus the ability to grow our our moral sense at a pace that can keep up with the enormous power that science and technology is creating.

2.1 Structure

As science has become more mathematical and abstract it has created an opening to reconnect with values and meaning. For the ultimate content of mathematics is the empty set or nothing at all. Mathematics describes only structure or how complex entities are constructed out of simpler ones. The empty set is the only primitive object not constructed as a set of simpler objects.

This high level of abstraction forces all assumptions about structure to be explicit. There are no inherent properties. There are only properties explicitly constructed with the relationship of set membership. Making all assumptions explicit avoids attributing logical necessity to implicit assumptions such as the parallel postulate in Euclidean Geometry. But reducing everything to structure makes it clear that something profoundly important is missing from our mathematically based scientific understanding. In fact everything with an intrinsic nature or ultimate content is missing.

For these abstractions are explicitly free of meaning and value. This contrasts with older metaphysics such as the one based on earth, air, fire and water. These each had an inherent nature and, through that nature, gave meaning and value to the objects that were constructed from them.

2.2 Connecting meaning to structure

The physical world is permeated with meaning through conscious experience. Making love or dying a painful death have intrinsic value. Every conscious moment of every sentient beings' existence has positive or negative intrinsic value.

Connecting physical structure with conscious experience is the starting point for integrating science and spirituality. In doing so we do not seek an explanation for consciousness. Science ultimately explains nothing. The fundamental laws of science are the simplest known *description* of a wide range of phenomena. Science makes no claims to explain why those descriptions are correct. The power of science comes from the ability to model and thus to manipulate a wide range of phenomena using relatively simple descriptive laws.

The nature of existence is a deep and profound mystery that cannot be solved. For any explanation depends on assumptions which themselves must be explained in an endless regress. At best one can develop the simplest possible assumptions consistent with what we know to be true as science has done with its assumptions about physical structure.

The simplest possible assumption about consciousness is that all physical structure exists as direct conscious experience. The essence and totality of the existence of physical structure is immediate conscious experience. Any other assumption must distinguish between conscious and unconscious structures in a way that adds unnecessary complexity. An embryo, fetus and then infant develop into a conscious being in a continuous process. An adult conscious mind gradually fades into an amorphous simplicity through Alzheimer's disease.

Once we recognize the continuity of consciousness from the human mind to the simplest physical structure there is no need to assume anything but structured conscious experience exists. One can raise many questions about and objections to this assumption. We lay the ground work for the assumption in Chapter 3. We develop the assumption and deal with some possible objections in Chapter 4.

This assumption connects the abstract structure of science with meaning. Physical structures that are painful have negative value and those that are joyful have positive value. (We are talking about intrinsic value. Any experience can be positive because of what it leads to or how it helps someone change.)

The connection becomes nontrivial when we focus on the evolution of structure and thus the evolution of consciousness. One can apply mathematics to questions about the evolution of physical structure in a way that leads to conclusions about the evolution of consciousness. This suggests that human consciousness is an an infinitesimal fragment what may be.

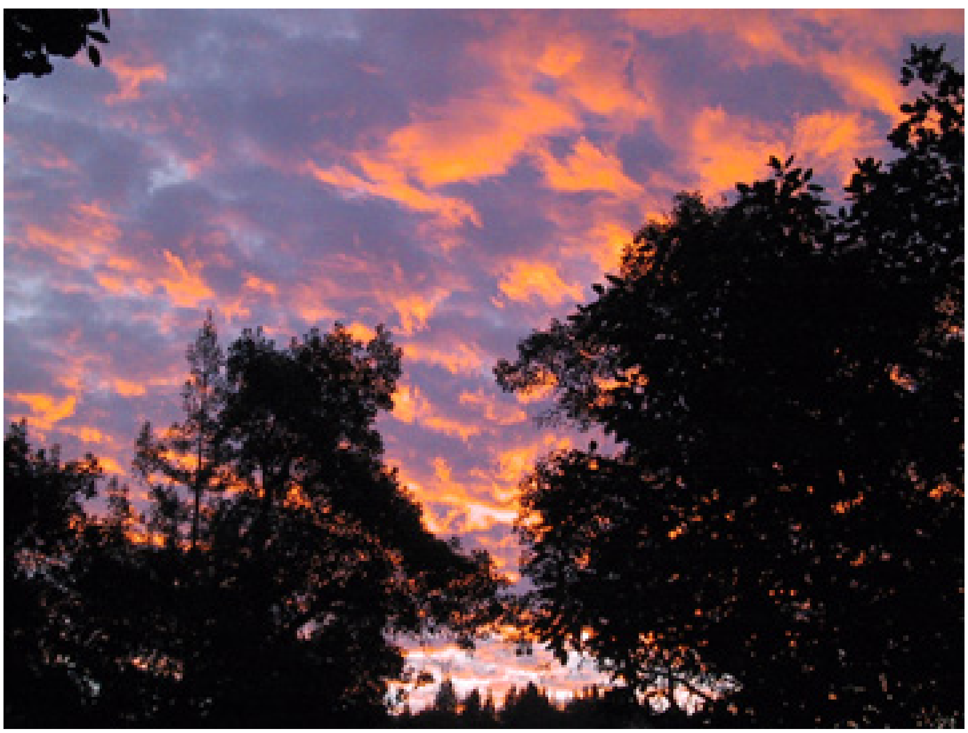
Historically we have seen ourselves as the center of the universe and the center of creation. Science has repeatedly dethroned us from such positions. We may be at the leading edge of evolution on one small planet, but we are not its end point or ultimate achievement. We. like the life we have evolved from, are a stepping stone to an ever expanding consciousness.

2.3 God evolving

We choose to call this creatively evolving consciousness God. We are the eyes of God with the power to create the world through our ability to understand and change biological evolution and our developing capacity to create artificial intelligence.

Evolution has become conscious of itself and is acquiring an understanding of its own structure. This is a dangerous but inevitable situation. We are encroaching on the power of God. But God is not an other worldly being. She is the creative process of evolving consciousness. We are the highest level of God's consciousness on this planet. We are not so much encroaching on God as we are God becoming conscious of her work and starting to consciously direct that work.

2.4 Consciousness is finite



In addition to assuming consciousness is all that exists I assume it is finite. This is the second fundamental assumption of this book. Consciousness is always specific and unique. You cannot add or remove something from a conscious experience without changing it. This is not true of an infinite set. For example you can take all the integers in an infinite set and double them. You can then add in all odd integers and you will be back to the original set of all integers. The universe may be potentially infinite. Consciousness may expand without any specific limit but it will always remain finite. There is no final state or ultimate destination. There is only an ever expanding deeper and richer conscious experience.

This vision of God as the evolution of physical structure and consciousness differs from the views of most religious traditions. Chapter 13 develops the idea that our spiritual instincts evolved biologically to connect us with the deepest creative forces in evolution.

Mathematics is crucial to understanding this creative process. Through Gödel's Incompleteness Theorem (see Section 5.8) we know that the creative evolution of structure can never be captured in finite form. It is an open ended ever expanding process. There is a hierarchy of mathematical truth that characterizes levels of abstraction or self reflection such as the self reflection that is a defining characteristic of human consciousness. Gödel proved that this hierarchy cannot be finitely described[22]. However it can be fully developed by exploring an every increasing number of paths without selecting a best or correct path. Biological evolution randomly explores possibilities through natural selection, becoming more diverse over time, if sufficient resources are available.

2.5 The shock of Gödel's proof

Gödel's result was and remains a shock to the mathematical community that sees mathematical truth as the one absolute certainty in a confusing world. Some mathematicians believe intuition about infinite sets borders on the mystical. By asserting the existence of complex infinite sets one can indirectly define levels in the hierarchy of mathematical truth that are difficult to approach in other ways. This suggests to some that mathematical intuition can transcend the limits Gödel's theorem imposes on any single path approach to extending mathematics. In Chapters 5 and 6 we describe the structure of the mathematical hierarchy of self reflecting structures and possible approaches to extending it.

Chapters 5 and 6 develop in an intuitive and semi-formal way the basics of formal set theory. This is done in terms of properties of logically determined sequence of events in a potentially infinite universe. Computer programs serve as an effective model of such processes. Developing mathematics in this way makes it more concrete and intuitive. Section 5.8 contains a sketch of a proof of a limited version of Gödel's Incompleteness Theorem called the Halting Problem. Chapter 6 speculates about extending mathematics in light of Gödel's result.

2.6 Digital space-time

The assumption that everything is finite has implications for physics. In Chapters 7 and 8 we explore the possibility that discrete as opposed to continuous models might be needed to explain physical reality as Einstein came to suspect near the end of his life[41]. Chapter 7 gives a brief overview of the two theories, relativity and quantum mechanics, that a discrete model must account for.

Chapter 8 discusses the problems of combining relativity and quantum mechanics. General relativity and quantum mechanics are incompatible when one attempts to combine them at small distances. This is a powerful suggestion that space and time are grainy or digital and that a radically different class of models is called for when we approach the scale of digital space-time.

2.7 Randomness and conservation laws

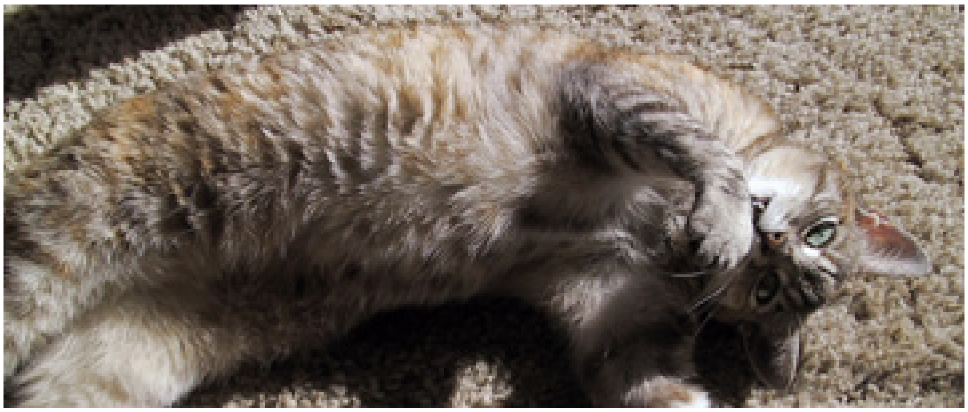
The path to a more complete theory may lie in reconciling quantum randomness, absolute conservation laws and locality. Einstein first called attention to these issues in 1935[21]. This has led through the work of Bell and others to practical experiments that could provide the data essential to developing a more complete and perhaps digital theory. Up to this point the experimental results remain consistent with quantum mechanics although experts agree that there are two loopholes, detection efficiency and timing, that have yet to be simultaneously addressed in a single experiment. We describe many of the relevant experiments and explain why we suspect a conclusive experiment will differ with the predictions of quantum mechanics in Section 8.6.

2.8 Creative evolution

Creativity is inherently risky with inevitable disappointment and failure. No matter where we are in the evolutionary process we can never assume we are on the right path. We must always recognize the need to give autonomy to other paths. Infectious disease is a lower form of life that has exercised its autonomy in a way that has created enormous evil. We are conquering this problem. We can repeatedly reach evolutionary levels that allow us to eliminate some lower level sources of evil. However we cannot eliminate the struggle between lower and higher levels without limiting the creative process. Understanding the mathematical structure of evolution can help us understand and deal with evil in a way that does not violate creativity. This is a subject of Chapter 10.

2.9 Spiritual instincts

Spiritual instincts connect our individual existence to a wider sense of self. This starts with our family and can expand to include our local community, our country, all of humanity, other species, all of life and ultimately the creative process itself. How we see ourselves is, to a significant degree, a matter of conscious choice. We are of course individuals with an individual destiny. But we are also part of a creative process. Our genes and our instincts are a product of, and designed to support, this creative process. Excessive focus on our individual existence does violence to this deeper sense of self. The development of a deeper sense of self is a subject of Chapter 12.



I assume there is no immaterial soul. On the contrary all matter is soul stuff. Consciousness is not connected to some essence that can be divorced from physical structure. We are the consciousness embodied in our bodies. There is no magic moment when a soul enters our body at the beginning of life nor leaves us in death. There is a continual transformation of consciousness. We are not separate from the physical world around us, but a part of the universal flow of consciousness. We are the universe becoming conscious of itself. We are the eyes of God with the power to create the world. Many spiritual traditions intuitively see this. By reintegrating essence to science we can move beyond intuition to understanding. The spiritual implications are explored in Chapter 13.

But understanding is not enough. One must feel a connection with the physical universe. One must feel that one is the universe becoming conscious of itself. Our values come from feeling. Values will determine if we continue the open ended exploration of consciousness that has created us.

2.10 An evolutionary crisis

Often, in spite of ourselves, our instincts push us in the right direction. They are doing so blindly and unconsciously and thus can take terrible detours. In contemporary Western culture we value some instincts and devalue others in a way that is very dangerous. But evolution is working with the wisdom of time. It can afford to take its time. The danger is that our technological prowess is advancing far beyond our wisdom to use it. That is why it is so important to understand the fundamental basis of the creative human instincts. It is one of many essential paths to the wisdom we need to use our rapidly expanding power.

The goals of this book are extraordinarily ambitious. Yet the assumptions are simple and conservative. Some of the ambition stems from the simplicity. The assumptions are *less* complex then the prevailing paradigms in mathematics and physics.

2.11 Toward an objective spirituality

The most ambitions goal is the development of an objective spirituality with a base in mathematics and physics and implications for those fields. Philosophical considerations have always guided science. When these were dogmatic views postulated by vested interests, they were obstacles to the growth of an objective science. When the philosophy stems in part from our scientific understanding, it has the potential to contribute to the scientific enterprise.

2.12 Intuition and intellect

This book is largely about connections between disparate existing ideas. It does not follow a logical deductive path. It starts in many places and leads many places to paint a picture of reality that is consistent with what we know, internally coherent and extraordinary in its implications. The two fundamental assumptions of this book (consciousness is all that exists and it is finite) are a crossroads where these paths meet and diverge. There are many reasons for adopting these assumptions and many implications of them. The journey across this terrain is more intuitive than intellectual but it is an intuition firmly rooted in intellect.

Western academia knows how to develop and foster intellect but not intuition. Those in the creative arts are more aware of what intuition is and how it can be developed. This is ironic since many of the greatest scientists such as the physicists Einstein, Bohr and Feynman were more intuitive geniuses than intellectual ones.

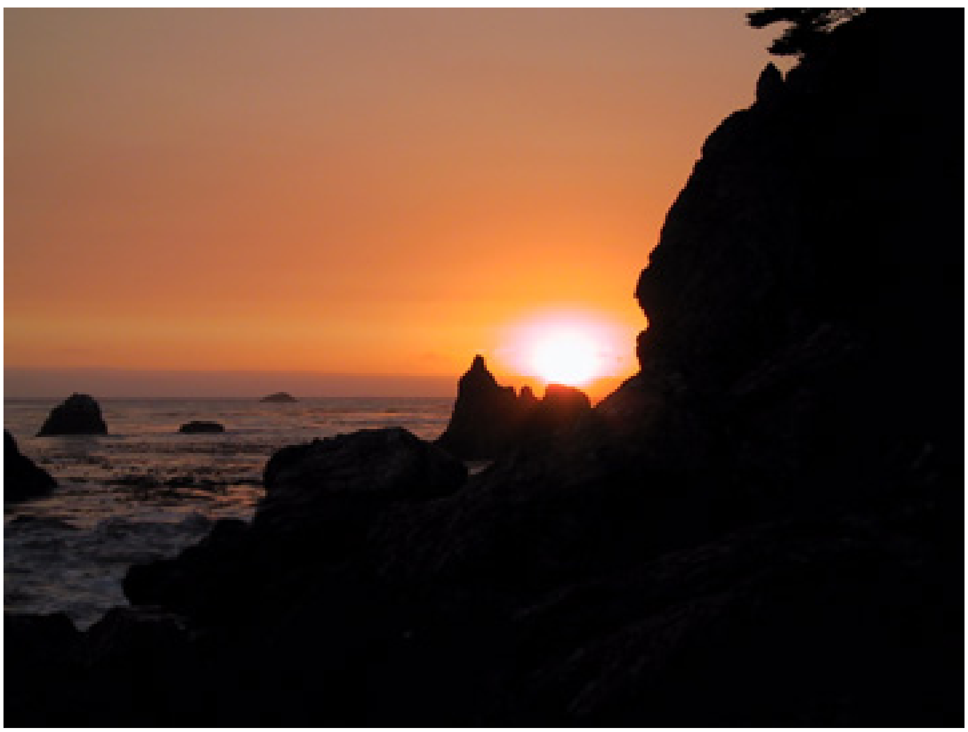
Intuition is a pattern recognition process. It senses when many pieces fit together to form a coherent whole not unlike facial recognition in which many features combine to form the face of someone familiar. Intuition was crucial in creating this book and is crucial in comprehending it. Intuitive talent is becoming increasingly important. Intuition has always led the way in creating the new idea or seeing the new possibility that intellect could develop. As we have mastered the territory that is well defined enough for intellect to deal with more of the major issues we confront fall outside of that domain. Intuition is the subject of Chapter 11.

2.13 An ever expanding consciousness

The unlimited potential for evolution in a potentially infinite universe does not exist on our planet. Its future is limited in time and resources and thus creative potential. If we avoid self destruction on a massive scale, we will almost certainly, within this millennium, begin to travel between the stars. We will do so on unmanned ships equipped with our knowledge and with biological material and machines that are capable of colonizing planets on which life could not develop spontaneously. We will reproduce and evolve not as individuals but as entire worlds. We will probably evolve as a combination of biological and manufactured components. Over time the manufacturing processes and biological processes may merge as the former become more subtle and efficient and the latter are more controlled and directed. The goal is the never ending expansion of conscious experience. We will grow more capable of pleasure, happiness, joy and ecstasy. The goal is the never ending journey of God becoming ever more deeply conscious of herself in her unbounded glory. That is the subject of Chapter 14.

Chapter 3

Structure and essence



Structure describes how components are put together to make a complex entity. Essence is the inherent fundamental nature of an entity. We naturally think of the essence of a complex structure as determined in part by the essence of its components. A steel bridge is strong because of how it is put together *and* the strength of steel used to build it.

3.1 Essence and chemistry

Chemistry defies this sort of analysis. Salt is made out of sodium, a highly reactive metal, and chlorine, a highly reactive gas. Yet salt has none of the properties of sodium or chlorine. It is not a metal nor a gas and it is relatively stable. The analysis that yields these properties says nothing about the fundamental nature of the elements, sodium and chlorine, that combine to form a molecule of salt. Instead it describes complex mathematical structures which model the way the electrons in the molecule are distributed in physical space. Even this description is a simplification. The chemical properties of elements can be understood most accurately by using quantum mechanics. That theory never describes the actual location of the electrons. It speaks only about the probability of observing electrons at a given location.

This may be true of salt, but one may protest that steel has essential properties that we can exploit to build a safe bridge. Craftsmen use intuitive understanding of the essential nature of materials. Enormous cathedrals have been constructed in this way.

But this approach is very limited compared to contemporary scientific understanding and engineering practice. Material properties directly connected to immediate experience provide far less knowledge than the abstractions of chemistry and physics. These abstractions have a basis in experience, but often in a very indirect and convoluted way. Scientists have succeeded in understanding the material world by completely replacing intuitive understanding from immediate experience with mathematical abstractions.

They did so only when they had no alternative. At the beginning of the 20th century many physicists thought they had a nearly complete understanding of the physical world. There was one small anomaly. A heated object emitted radiation in a way that could not be explained by Newtonian physics. This seemed an obscure problem, but its resolution led to quantum mechanics and a view of the world fundamentally and radically at odds with all previous physics.

Newtonian physics retained a direct connection with immediate experience. It was a physics of tiny billiard balls bouncing off each other. Quantum mechanics is a mathematical abstraction unlike anything we experience. The world of that experience is called macroscopic. It is at time and distance scales that are enormous compared to the scale of fundamental quantum effects such as the the creation or destruction of a particle. There is nothing in human experience remotely like effects at the quantum level.

Contemporary science has robbed physical objects of an essential nature. They only have properties defined by abstract mathematical relationships. There is no "essential nature" in this mathematics.

3.2 Essence as a Platonic ideal

Platonic philosophy attempted to capture essence in a different way. Plato thought what we see in the physical world is a dim reflection of the true ideal thing. For example circular objects are crude approximations to the ideal perfect circle. Platonic philosophy aims to understand reality in terms of the ideals that capture the real essence that is dimly reflected in physical existence.

Today mathematics comes close to capturing the ideal circle of Plato. It cannot be constructed, but its properties, like the area it covers, can be computed with any desired degree of accuracy. Mathematics can do this in a purely structural way building all objects including circles from the essence free empty set.

Essence free arguments are not the norm even in mathematics. Geometrical arguments are still phrased in terms of geometrical properties. Only they are done in such a way that it is clear how to convert them to arguments about sets. In normal discourse we take the merger of structure and essence as given. It is how we visualize the world and how we think. The problem is that the essence we attribute to external objects is from our own experience. It is not something that is part of the external objects. A soft touch, sharp slap, beautiful sunset or ugly wound, are things created in us when we have particular experiences.

We are not perceiving external reality as it truly is nor are we dimly perceiving some ideal platonic reality. We are creating the world in our conscious experience. There is a related external structure that our perception is causally connected to. But the perception of, for example, color is far more a construction of our sensory and nervous system than it is an effect from light of a particular frequency.

The idealization and abstraction of mathematics had a profound an unexpected impact. It lead to something called Gödelization. That is the ability to assign unique numbers to all mathematical expressions. Today Gödelization is a practical reality for all of us as we store documents, images and movies digitally as a series of numbers in a computer. But in the 1930's it was a deeply creative observation that led to a result that shook the world of mathematics and exploded the idea that there could be an absolute Platonic ideal even for mathematical truth.

3.3 Gödel and unfathomable complexity

Gödel's Incompleteness Theorem had a profound impact on mathematics when it was first established in the 1930's and a profound impact on me when I learned about it. At the beginning of the 20th century a famous mathematician, Hilbert, proposed the construction of a formula or mechanistic process for deciding all mathematical questions. Gödel proved this was impossible.

Mathematics then and now is based on formal systems. These are mechanistic processes (in effect computer programs) for enumerating theorems. Gödel used Gödelization to show how formal systems could talk about or model themselves. He showed that any sufficiently powerful formal system could not prove that the system itself never generated a contradiction. To be sufficiently powerful, the system had to support the definition of a Universal

Turing Machine.

One implication of this result is the Halting Problem. We can predict what a computer program will do at any time but we cannot predict if it will *ever* do something such as halt. Of course, if it does halt and we run it long enough, we will observe this. But, if it never halts, we can not know this from observation. There is no general way to determine when we will have waited long enough. For many programs we can decide the Halting Problem. Some programs have simple loops that continually repeat the same sequence in an obvious way. There are far more complex ways that a program can loop forever that we can understand. But we cannot do this in general for every possible computer program. There will always be programs that have some subtle way to loop or iterate that are beyond our current understanding. This suggests a creative aspect to mathematical truth. There will always be more interesting mathematics that we do not yet understand even if the human race, or at least the study of mathematics, is immortal.

Mathematical structure is devoid of any essential nature, but endowed with unfathomable complexity. Nowhere in the unbounded richness of mathematics and science is there anything that begins to touch on my own immediate experience. Science and mathematics can explain how aspects of experience are structured or related to each other, but the experience itself is completely beyond structural understanding.

This leads to the question: what is?

Chapter 4

What is

Who are we? Why are we here? What is this place? These questions cry our over the centuries. They cannot be approached as we address other issues because there is no context in which to answer them. They are questions about the ultimate context of existence.

A similar situation exists with the fundamental laws of physics. We can explain why a chemical reaction occurs using quantum mechanics and the properties of fundamental particles. Some day we may able to explain the properties of fundamental particles by a deeper, yet to be discovered theory, but for now we simply take them as given as we do the laws of quantum mechanics. We discover these laws by looking for the simplest explanation that accounts for as wide a range of experimental results as possible.

We do this because it works. Much of the world, including some extraordinarily complex things, can be explained by simple laws. Once we understand such laws we often gain power to control the phenomena the laws describe. There are also esthetic reasons. Simple laws can be profoundly beautiful. But in the end it is utility that carries the day. That which works is adopted. Those that refuse to do so are less effective and over time their influence and power diminish.

The same approach can be applied to the fundamental philosophical questions. We can search for the simplest description of what we know to be true. The starting point in this search for simplicity is unifying internal experience with external 'objective' reality.

4.1 Unifying external and internal reality

The world seems objective. A chair, a tree, a glass of water all seem to be physical things that we can feel, sit on, climb or drink. We translate our immediate experience into a sense of external reality automatically without thinking about it. We see a chair. We do not see a complex geometric shape and deduce that there must be a chair five feet in front of us.

There is an unconscious process of deciding a particular shape is a chair. The result enters consciousness when some part of our brain has decided that is a chair. We see the chair as a unity or gestalt and not a pattern of color. Only when that unconscious process is confused do we see a pattern that we cannot make out.

We construct a sense of objective reality for practical reasons. We interact with the external world to get what we want and need. We focus our conscious energy on novel or problematic events. We evolved ways to automatically deal with the routine and mundane. The external world of objective reality seems natural and necessary. We do not think about it.

The objective external world and experiential inner world seem radically different. Connecting the two has been a deep problem in philosophy for centuries. Is there some special soul stuff that translates the physical processes of our body into the inner experience of making love?

Des Carte speculated that the pineal gland, as the only part of the brain not part of a symmetrical pair, provided the connection between the body and soul. As we understand more of the brain we see no evidence for a special connection to soul stuff. All that we experience internally seems to be reflected in physical brain structures and dynamic neural processes. Experiments have shown that certain parts of the brain are active when we think about certain things without any external stimulus. Our internal states seem to have a measurable physical existence.

The brain is made of the same atoms and molecules as everything else. Our neurons are elegant but simple switches. They are more complex than the binary switches used to build computers, but fully comprehensible as physical and chemical processes. So where does the magic inner world that makes up the ultimate and only reality for each of us come from?

Physical brain structures seem to be capable of fully reflecting the structure of our internal experience. As the devices we use to observe the functioning brain improve in sensitivity we should be able to establish this as a scientific fact. For now all the evidence points in this direction. So I assumed that our conscious experience *is* the existence of structures in the brain. There is nothing special about these physical structures. Immediate conscious experience is not simply associated with physical structures. *Immediate experience in some form is the essence and totality of the existence of physical structure and structure is the only aspect of existence that can be communicated.* This is the Totality Axiom.

4.2 Panpsychism

The Totality Axiom is a form of panpsychism or the belief that consciousness is universal in all that exists. In its animistic form panpsychism is throughly discredited by contemporary science.

But it exists in more abstract forms as the Artificial Intelligence researcher and furturist Ray Kurzweil has suggested.

> So we could say that the universe —"all that is"— is indeed personal, is conscious in some way that we cannot fully comprehend. This is no more unreasonable an assumption or belief than believing that another person is conscious. Personally, I do feel this is the case. But this does not require me to go beyond the "mere" "material" world and its transcendent patterns. The world that is, is profound enough[36, p. 215].

Joseph Campbell, the former expert on the world's mythologies, had a similar sense of the universality of consciousness.

> It is part of the Cartesian mode to think of consciousness as being something peculiar to the head, that the head is the organ originating consciousness. It isn't. The head is an organ that inflects consciousness in a certain direction or to a certain set of purposes. But there is consciousness here in the body. The whole living world is informed by consciousness.
>
> I have a feeling that consciousness and energy are the same thing somehow. Where you really see life energy there is consciousness. Certainly the vegetable world is conscious. And when you live in the woods as I did as a kid, you can see all these different consciousnesses relating to themselves. There is a plant consciousness and there is an animal consciousness, and we share both these things. You eat certain foods, and the bile knows whether there's something to go to work on. The whole process is consciousness. Trying to interpret it in simply mechanistic terms won't work[10, p. 18].

Campbell was not saying that energy produces consciousness. He was saying that energy *is* consciousness.

The philosopher, David Chalmers, has proposed a tentative theory of consciousness based on information. In context Chalmers' information is almost a synonym for mathematical structure. Every structure contains information and any structure can be fully described using information. After outlining his ideas he observes that information is ubiquitous. He does not shrink from the conclusion that experience must also be ubiquitous.

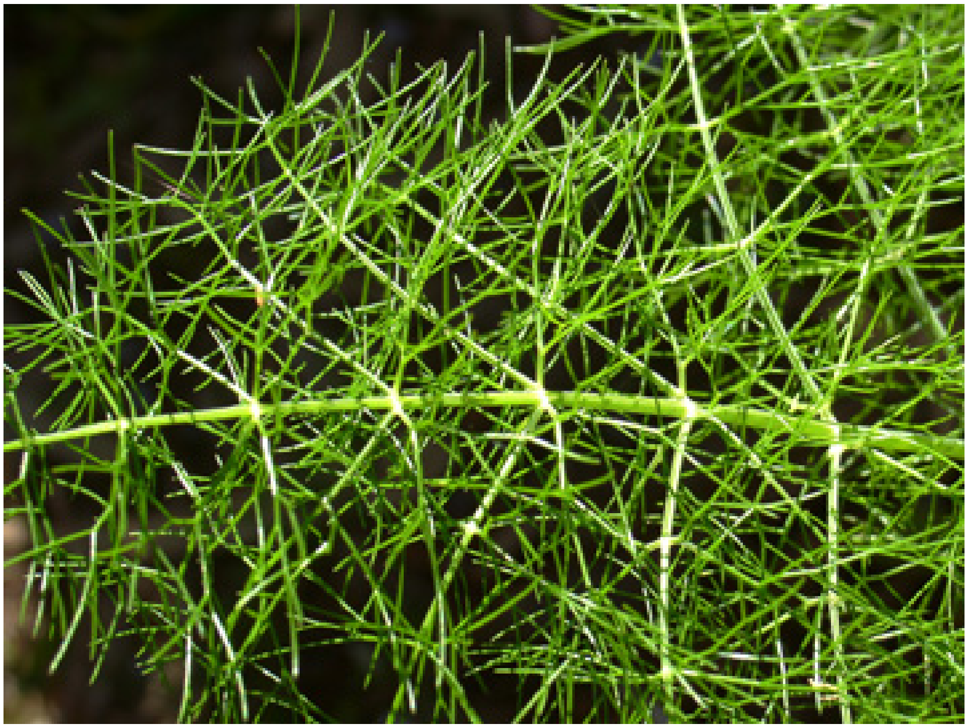
> If this [experience is ubiquitous] is correct then experience is associated with even very simple systems. This idea is often regarded as outrageous, or even crazy. But I think it deserves a close examination. It is not so *obvious* to me that the idea is misguided, and in some ways it has a certain appeal[12, p. 293].

Adult consciousness involves a limited set of brain structures. Much of the brain operates below consciousness. We are not conscious of most of our body most of the time. Experiences enter consciousness when something notable happens like stubbing a toe. Why not assume all the activity not entering our stream of consciousness is nonetheless conscious, but with a limited connection to stream of consciousness? What is left out is as important as what is present. The consciousness we experience is an executive control with a limited capacity to deal with information. So complex filters exist to insure only relevant experience gets through. There is nothing special about the neurons that make up this executive control. Why not assume *all* the structures in the brain correspond to a consciousness that *is* their structure.

Equating existence to immediate experience violates our sense of objective physical reality. That reality is a pragmatic creation of consciousness. In what sense could an objective reality beyond any conscious experience exist?

This is not denying our scientific understanding of physical structure. It is describing the context in which that structure has existence and meaning. The dynamic physical transformation of the universe over time is a transformation of consciousness and *nothing but* a transformation of consciousness.

4.3 Information theory and digitization



The second part of the Totality Axiom asserts that structure is the only aspect of essence that can be communicated. It can be regarded as a restatement of Shannon's definition of information as something that allows us to reduce the number of states a system may be in. For example, suppose we know that a flag must be red, blue, green or yellow. Then it can be in any of four states. Each state corresponds to a different color. If we are now told the flag is green we have reduced the four possible states to a single state. The amount of information transferred is that needed to reduce four states to one state. This requires a number between one and four. One can think of structure as defining the state of an object. In mathematics this is easily made explicit. One can assign a unique integer to every possible finite mathematical object. Defining a finite structure then requires nothing more than selecting the number that indexes it.

Shannon's definition of information applies equally to *everything* we can communicate. We can *always measure* the information communicated in terms of how much we have reduced the number of possible states. We can always communicate that information as a number that selects possible states as long as both the sender and receiver agree on which numbers correspond to which states.

Shannon's definition has had enormous consequences in the digitization of media. Everything we see and hear can be encoded as a sequence of numbers on a CD of DVD. These sequences preserve the structure of the sound or image. In the case of a CD the numbers represent sound pressure level at a given instant. By recreating the sequence of sound pressure level we recreate the original sound. That is what a audio system does with the numbers on a CD. In images the numbers represent the intensity of the three primary colors, at a given point in an image and instant in time. Recreate the intensity levels of these colors at the correct location and time and you recreate the image. That is what a DVD player connected to a television does.

Digitization has made Shannon's definition a practical reality of immense importance. Digitization is an example of the difference between structure and essence. The sequences of numbers on a CD contain all of the structure in a Bach Sonata, but none of the essence. The numbers on the CD only come to life when a conscious listener experiences the effect of the sound reproduced from the numbers on the CD.

4.4 Universal consciousness

The contrast between the numbers on a CD and the experience of a Bach Sonata could not be more dramatic. The numbers are ultimately meaningless. The experience is its own meaning. The assumption that consciousness is universal is the assumption that all of existence has intrinsic meaning and value. Instead of existing as isolated souls in a sea of empty matter we exist as focal points of intense consciousness in an ocean of universal consciousness.

My conscious experience is affected by the internal structure of my brain and by external events. Where do my senses begin and the external world end? We can see the issue most clearly in how we experience time. Each moment is unique and specific yet each flows into the next with no boundary. We can understand how this happens structurally. Our brain state changes in an almost continuous way because the firing of neurons that give rise to that change are numerous and weakly coordinated. They are like an unruly crowd and nothing like a marching band where everyone is in step. Just as there are no clear boundaries between experiences in time there is no clear boundary between a sense organ and the external world it senses. By doing away with a bit of soul stuff unique to each person we destroy any absolute boundary between the individual and the wider world. There are only the vague and shifting boundaries like those in our experience of time.

4.5 The essence that is

The idea that immediate experience is universal is similar to existing ideas such as that of Tao.

> In the beginning was the Tao.
> All things issue from it;
> All things return to it[48, v 52].

> The Tao gives birth to One.
> One gives birth to Two.

> Two gives birth to Three.
> Three gives birth to all things[48, v 42].

Immediate experience is always specific and finite. It is logical in the sense that what we experience is either something or not something. Language can be ambiguous or inadequate to express what we experience, but the experience itself is always a definite thing and not all the other possible definite things.

Immediate experience is finite. It may appear to be continuous as in our visual image, but this is an illusion constructed from discrete visual receptors in the eye.

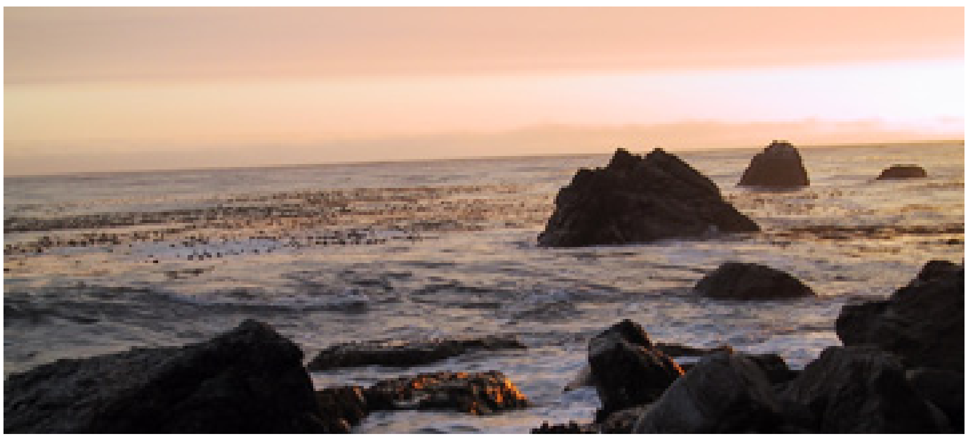
4.6 All that exists is finite

By giving essence to existence we give substance to the question of what it would mean for an infinite structure to exist. A defining property of infinite sets is that one can add something without changing them. A defining property of conscious experience is its indivisible wholeness. Any change in its structure changes its essence.

This suggests two classes of existence. The first is immediate gestalt experience. The second is the collection of all such experiences. This collection may be infinite, but is not an immediate gestalt experience. Mathematics already has such a distinction between sets and classes. This was necessary because of the contradictions that arise from assuming there can be a set that contains all sets. There is a class of all sets that cannot be a set. We are suggesting that this necessary boundary occurs between the finite and the infinite.

Mathematics studies all possible structures. The only constraint is logical consistency. The same constraint would seem to apply to a gestalt. When we have an immediate conscious experience it is a definite unique event. It may have many ambiguous interpretations, but the experience itself is exactly what it is. A patch of color cannot be red and also not red. As mathematics is the study of all possible logical structures it is also the study of the structure of all possible gestalts.

4.7 Boundaries



Where does one gestalt end and another begin? Gestalt experiences are the connecting glue that make the universe whole. Boundaries exist in the unity and irreducibility of each moment of experience. Boundaries fade away in the connectedness of individual gestalts. This is most obvious in the flow of time.

The boundary between me and the rest of the universe is like the boundary from one moment of time to the next. It is real and important, but also transitory. Without boundaries there can be no specifics and thus nothing at all. But boundaries are more pathways than limits. They lead to future moments of time, new experiences, other people, other times and other places.

4.8 Extensions of consciousness

There is no reason to think evolution is at or has a limit. Culture as well as biology generates extensions to consciousness. As culture expands awareness expands. This can be accomplished by passing on hard won knowledge and understanding. It can be accomplished by inventions that expand the senses. Scientists are designing direct interfaces to human neural circuits to help the handicapped. In time such devices will provide capacities that do not now exist. They have the potential to extend consciousness.

Telecommunications and the Internet have created a limited form of global consciousness. The Internet is a global nervous system that is becoming ever more important to commercial transactions. Eventually we will have direct neural connections to the Internet in a way that will vastly expand the information we have available through what appears to be direct thought and apprehension. Similarly we will be able to be in direct and immediate 'telepathic' contact with any other person who is open to such communication with us. These capabilities will transform both our capabilities and our consciousness. It will feel different to be a human being. At the same time we will be creating a global consciousness. This consciousness will reside in the Internet and all the devices and beings connected to it.

Chapter 5

Mathematical structure

It is helpful to have some understanding of the foundations of mathematics to fully grasp the structure and essence dichotomy at the core of this book. This chapter and the next develop the foundations of mathematics and connect these to philosophical issues.

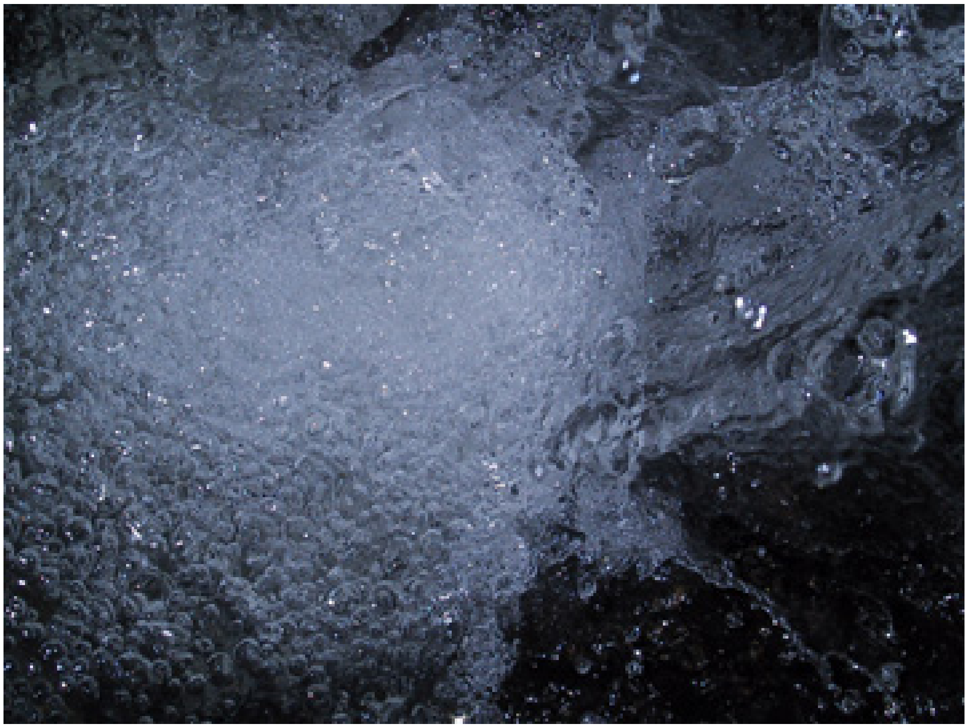
Many people are needlessly turned off by mathematics because it is often taught in a manner that is boring and unnecessarily difficult. Like most people, my mind goes blank when I face a page full of equations. Formal mathematics is important because it is connected with every day life, but it is often taught as if this were irrelevant or not true.

By using English language explanations and connecting mathematics to properties of computers, it is hoped that this chapter and the next will be intelligible and of interest to a wide audience. Experience with mathematical expressions and the idea of a function at the level of high school algebra is helpful. These are defined and explained, but may take thinking about and playing with to fully grasp.

Computer programs are used as a surrogate for logically determined or mechanistic processes. The focus on computers is not to suggest there is anything special about them. The universe allows us to predict future events to a limited degree using logic and mathematics. It is that aspect of mathematics that has practical importance and computers are an accurate metaphor for a logically determined sequence of events.

The computer metaphor suggests a deterministic universe where nothing truly novel happens. But this is not the case. Gödel proved in the 1930's that mathematics cannot be captured in any finite system. Gödel's result suggests that mathematics is an inherently creative endeavor albeit one that can create absolute truth. This result is derived and explored in the next two chapters. I will argue that it is central to understanding the creative evolution of consciousness.

5.1 Structure and consciousness



The capacity for subtle self reflection is a defining characteristic of human consciousness. The power of a mathematical system can be measured by the level of self reflection or iteration definable within the system. Gödel's proof shows there is no finite limit to the levels of subtle self reflection in finite systems.

Higher levels of self reflection seem to be associated with higher levels of conscious experience. Evolution creates such structures because they have survival value. They allow better prediction of the consequences of one's actions. Gödel's result suggests that unbounded creative evolution of self reflection and consciousness is possible only if diversity expands without limit.

5.2 Logically determined unsolvable problems

We do not think of mathematics as creative. It gives absolute truths like $2 + 2 = 4$. However, the search for logical absolutes uncovered a hierarchy of problems that are logically determined yet unsolvable.

Logically determined unsolvable problems exist because one can ask if a property is true for *any* or *all* integers. For example the Halting Problem asks if a computer program will *ever* halt[1] This is not a question about what the computer will do at some particular time. It is a question about what it will do over an unlimited time.

[1] Many computers have a specific instruction to stop processing instructions or halt. Today programmers never use such instructions unless they are writing operating systems, but, in the early days of computing, there were no operating systems and programmers had to halt the computer when the program completed. The Halting Problem need have nothing to do with halting. The question will a program *ever* do some specific action at *any* future time is all that is needed.

Real computers halt when the power goes off. They have a limited amount of memory. The Halting Problem is about an abstract computer that runs forever and has no fixed limit on memory.

Computers follow an exact set of rules. One always know what the next step is and thus what a computer will do at any time. But one cannot, in general, know if it will it ever do something like halt. If one waits long enough and it *does* halt, one knows that. Until and unless the program halts. one cannot know if one has waited long enough. To prove a computer *never* halts requires something more than following the steps the computer takes.

There is nothing special about halting. An equivalent problem comes from asking if the computer program will ever accept more input. No doubt you have experienced this problem while waiting for a response from your computer. You don't know if it requires rebooting or will eventually respond.

The Halting Problem is at the base of an unlimited hierarchy of unsolvable problems. A step up the hierarchy asks if a computer program will generate an infinite number of outputs. Instead of asking will it ever do something one asks will it keep doing something again and again with no limit.

One can go to higher levels by interpreting a program's output as a new computer program. This is possible because all computer programs can be numbered. They are stored in computer memory as very long sequences of digits.

One can interpret any number as a computer program. Most numbers will not correspond to a meaningful program, but some will.

Using this idea one can ask if a program has an infinite number of outputs an infinite subset of which encode a computer program that itself has an infinite number of outputs. This method of defining higher levels of unsolvable problems can be iterated and generalized in obvious and very complex non obvious ways.

There is a hierarchy of axioms of mathematics that solves these problems. A mathematical axiom is an assumption. It cannot be derived from more basic axioms. It must be assumed as true. The parallel postulate discussed in Section1.1 is an example. There is some axiom that can correctly solve every Halting Problem, but no finite set of axioms can solve all Halting Problems. In standard mathematics this hierarchy is extended by adding axioms that assert the existence of 'large' infinite sets. Since the unsolvable problems refer to mechanistic processes (what will a program do *eventually*?) it is possible to extend this hierarchy by adding axioms about such processes. The question of how best to extend this hierarchy is addressed in Section 6.9.

5.3 Formal logic

Mathematics starts with formal logic. This is a set of rules for making deductions that seem self evident. Syllogisms like the following occur in every day conversation.

All humans are mortal.
Socrates is a human.
Therefore Socrates is mortal.

Mathematical logic formalizes such deductions with rules precise enough to program a computer to decide if an argument is valid.

This is facilitated by representing objects and relationships symbolically. For example Use h for the set of humans, m for the set of mortal creatures and s for Socrates. Use the symbolic expression '$x \in y$' to indicate that object x is a member of set y. Thus 'Socrates is a human' is represented by $s \in h$. Use the symbol $\rightarrow$ in an expression '$a \rightarrow b$' to indicate that if statement a is true than statement b must be true. Use the 'quantifier' $\forall$ to indicate that all objects satisfy some condition. For example all men are mortal can be written as $\forall x \;\; x \in h \rightarrow x \in m$. $\forall$ is called a universal quantifier.

The syllogism can be formalized as follows.

$$\forall x \;\; x \in h \rightarrow x \in m$$

$$s \in h$$

therefore

$$s \in m$$

Logic assumes something cannot be both true and not true. It looks only at the truth value of a proposition. It involves simple relationships between these truth values. These can be represented by truth tables as shown in Table 5.1. The only logical operations required are the three in this figure. Others such as implication represented by '$\rightarrow$' can be constructed from these three. $A \rightarrow B$ is the same as $A \wedge B \vee \overline{A}$. A implies B requires that either both A and B are true or A is false.

Determining the truth of a logical expression that contains no quantifiers (like $\forall$) is a straightforward application of simple rules. One can use a truth table to evaluate each subexpression starting with those at the root of the expression tree as shown in Table 5.2. If a logical expression contains quantifiers, the logical relationship involved must be evaluated over a range of values to determine the

Proof that the sum of all integers less than or equal to k is $\frac{k\times(k+1)}{2}$.

1. Proof for 0: $\frac{0\times(0+1)}{2} = 0$.
2. Proof that if it is true for n it must be true for $n+1$.
   (a) Assume for any n that that the sum of all integers less than n is $\frac{n\times(n+1)}{2}$.
   (b) Then the sum of all integers less than $n+1$ must be $n+1+\frac{n\times(n+1)}{2}$.
   (c) Put the $n+1$ in the numerator of the fraction producing $\frac{2\times(n+1)+n\times(n+1)}{2}$.
   (d) Simplify using the common factor $n+1$ to get $\frac{(n+2)\times(n+1)}{2}$.
   (e) Substituting k for $n+1$ in the above equation yields $\frac{k\times(k+1)}{2}$.
3. This completes the proof that if the equation is true for n it must be true for $n+1$ and that completes the proof by induction.

Figure 5.1: Example of proof by induction on the integers

truth of the expression. If the range is infinite, there is no general way to evaluate the expression. Induction on the integers can solve some problems of this type.

To use induction to prove a property is true for all integers requires two steps. First prove the property holds for 0. Then prove that if the property is true for any number n it is also true for $n+1$. Having established these two results the principle of induction allows one to conclude the property is true for all integers. Figure 5.1 gives an example of such a proof. The principle of induction is an axiom of mathematics that seems self evident, but cannot be derived from more basic principles. It must be assumed.

AND $\wedge$		
A	B	A $\wedge$ B
false	false	false
false	true	false
true	false	false
true	true	true

OR $\vee$		
A	B	A $\vee$ B
false	false	false
false	true	true
true	false	true
true	true	true

NOT –	
A	$\overline{A}$
false	true
true	false

Table 5.1: Truth tables for AND, OR and NOT

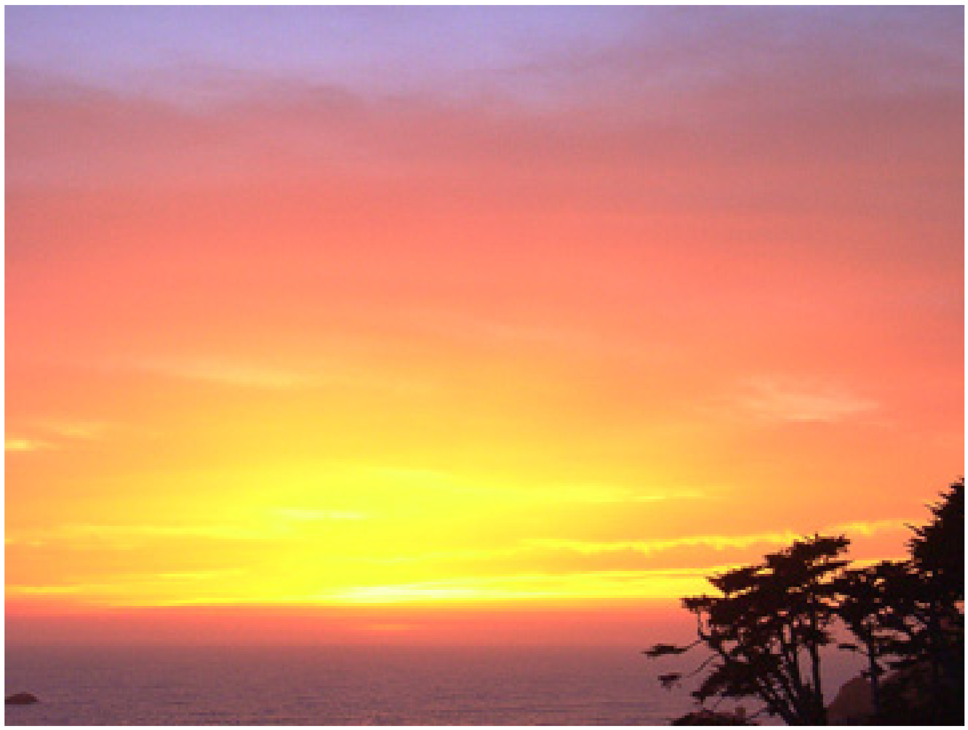
symbol	A	B	C	D	E	F
truth value	true	true	false	true	true	false

<table>
<tr><td colspan="5">expression evaluation</td></tr>
<tr><td colspan="5">$A \wedge \overline{B} \vee C \vee \overline{D \wedge E \wedge F}$</td></tr>
<tr><td></td><td>$\overline{B}$
false</td><td></td><td>$D \wedge E$
true</td><td></td></tr>
<tr><td colspan="2">$A \wedge \overline{B}$
false</td><td></td><td colspan="2">$D \wedge E \wedge F$
false</td></tr>
<tr><td colspan="2">$A \wedge \overline{B} \vee C$
false</td><td></td><td colspan="2">$\overline{D \wedge E \wedge F}$
true</td></tr>
<tr><td colspan="5">$A \wedge B \vee C \vee \overline{D \wedge E \wedge F}$
true</td></tr>
</table>

The order in which the logical operations are done is important. The rules for this are called precedence. First do NOT then AND finally OR. If the NOT sign extends over a subexpression, the subexpression is evaluated and the NOT operation is applied to that result. In this example at the first level evaluate $\overline{B}$ and $D \wedge E$. At subsequent levels the part of the expression evaluated at higher levels in in smaller type.

Table 5.2: Evaluating a logical expression

5.4 Formal mathematics



Formal mathematics builds on formal logic. Set theory has become the standard way to formalize the foundations of mathematics because of its simplicity and power. The only undefined primitive object in set theory is the empty. Its *only* property is that it contains no subsets. There is no clearer illustration of the absence of content or intrinsic nature in mathematics and physics than the construction of all of mathematics assuming the empty set as the only primitive entity.

The standard axioms of set theory are summarized in Figure 6.4. This figure references the sections where the axioms are explained. These axioms are adequate for all of conventional mathematics. Almost every mathematical abstraction that has ever been investigated can be derived as a set that these axioms imply exists. Almost every mathematical proof ever constructed can be made assuming nothing beyond these axioms. These axioms are less than a page long.

It is straightforward to program a computer to output all the theorems that can be deduced from these axioms. This is not a practical way to derive mathematics because most of the theorems are trivial and of no interest. Interesting theorems are extremely rare. It would take a long time before such theorems occur and it would be very difficult to select them out.

5.5 Axioms of Set Theory

To understand the formal version of these axioms you have to know in what order operations like $\wedge$ (AND) and $\vee$ (OR) are performed. This is determined by precedence as described in Table 5.2. Subexpressions involving $\in$ are evaluated before logical operations like $\wedge$ or $\rightarrow$. Parenthesis () and square brackets [] are used to override standard precedence and to make clearer how an expression is to be evaluated. The portion of an expression inside parenthesis is evaluated first.

The clearest example of how set theory builds on the empty set is the construction of the integers. The integer 1 is defined as the set containing the empty set. Two is defined as the set that contains 1 and the empty set (or 0). Not surprisingly 3 is the set that contains 0, 1 and 2. In general $n+1$ is defined as the union of all the elements of n plus n itself. Thus $n+1$ will contain $n+1$ elements as long as n contains n elements.

5.5.1 Axiom of extensionality

Without the axiom that defines $=$ there would be little point in defining the integers or anything else. The axiom of extensionality says sets are uniquely defined by their members.

$$\forall x \forall y \ \ (\forall z \ \ z \in x \equiv z \in y) \equiv (x = y)$$

$a \equiv b$ means a and b have the same truth value or are equivalent. They are either both true or both false. It is the same as $(a \rightarrow b) \wedge (b \rightarrow a)$. This axiom says a pair of sets x and y are equal if and only if they have exactly the same members.

5.5.2 Axiom of the empty set

The empty set must be defined before any other set can be defined. The axiom of the empty set uses the existential quantifier ($\exists$). $\exists x\, g(x)$ means there exists some set x for which $g(x)$ is true. Here $g(x)$ is any expression that includes x.

The notation $x \notin y$ indicates that x is not a member of set y.

The axiom of the empty set is as follows.

$$\exists x \forall y \ \ y \notin x$$

This says there exists an object x that no other set belongs to. x contains nothing. The empty set is denoted by the symbol $\emptyset$.

The definition of the integers requires two axioms for constructing finite sets.

5.5.3 Axiom of unordered pairs

From any two sets x and y one can construct a set that contains both x and y. The notation for that set is $\{x, y\}$.

This axiom constructs a new set from any two existing sets.

$$\forall x \forall y \ \exists z \ \forall w \ \ w \in z \equiv (w = x \vee w = y)$$

This says for every pair of sets x and y there exists a set w that contains x and y and no other members.

5.5.4 Axiom of union

A set is an arbitrary collection of objects. The axiom of union allows one to combine the objects in many different sets and make them members of a single new set. It says one can go down two levels taking not the members of a set, but the members of members of a set and combine them into a new set.

$$\forall x \exists y \ \forall z \ \ z \in y \equiv (\exists t \ z \in t \wedge t \in x)$$

This says for every set x there exists a set y that is the union of all the members of x. Specifically, for every z that belongs to the union set y there must be some set t such that t belongs to x and z belongs to t.

5.5.5 Axiom of infinity

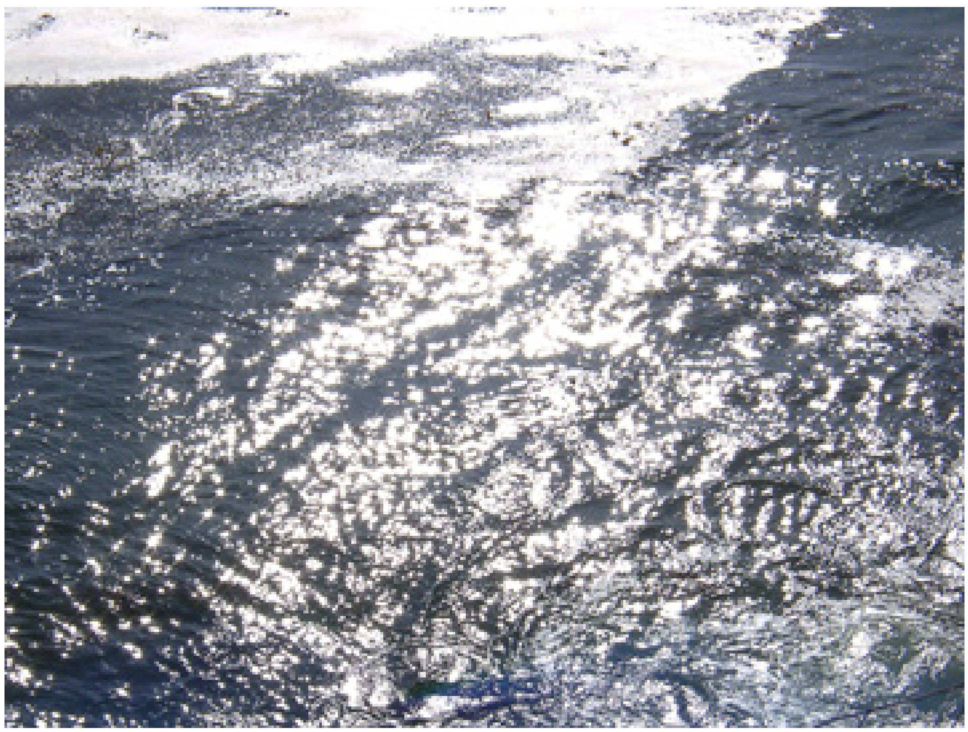
The integers are defined by an axiom that asserts the existence of a set ω that contains *all* the integers. ω is defined as the set containing 0 and having the property that if n is in ω then $n+1$ is in ω. Writing this compactly requires some notation. $\emptyset$ represents the empty set. From any set x then one can construct a set containing x. This set is written as $\{x\}$.

$$\exists x \, \emptyset \in x \wedge [\forall y \, (y \in x) \rightarrow (y \cup \{y\} \in x)]$$

This says there exists a set x that contains the empty set $\emptyset$ and for every set y that belongs to x the set $y + 1$ constructed as $y \cup \{y\}$ also belongs to x.

The remaining axioms are developed in the next chapter starting in Section 6.3. The discussion of the infinite at the end of this chapter and the start of the next lays the groundwork for those axioms.

5.6 Infinity



Infinite structures do not exist in the physical world as far as we know. So what do mathematicians mean when they assert the existence of ω? There is no universally accepted philosophy of mathematics, but the most common belief is that mathematics touches on an other worldly absolute truth. This idea has its origins in the Platonic concept of an ideal and perfect world of which the physical world is a dim reflection. Many mathematicians hold to this position in some form. They believe that mathematics involves a special perception of an idealized world of absolute truth. This comes in part from the recognition that all that exists in the physical world is imperfect and falls short of what we can apprehend with mathematical thinking.

For example all physical circles have imperfections, but the geometric circle is perfect. One can argue that this distinction no longer exists. It is impossible to construct a perfect circle, but there are computer programs that will describe the perfect circle to any desired degree of accuracy. A computer program is a physical device. Its instructions are usually carried out with absolute perfection. Computers can and do make errors, but the probability of an error is small and there are techniques that can make that probability arbitrarily small. Thus there are physical objects that comes close to being a Platonic ideal.

The infinite used to be thought of as a potential that is never fully realized. This perspective fell into disfavor as mathematicians constructed a hierarchy of infinite sets. that greatly extend mathematics. The practical consequences of these extensions led many mathematicians to feel that reasoning about complex infinite sets could provide an intuitive window into useful mathematics that could not be developed in other more mundane ways.

The hierarchy of infinite sets comes in two flavors, the ordinals and the cardinals. The ordinals generalize the construction of the integers. The successor of ω is constructed like the successor of a finite integer is constructed. It is the union of ω and all the members of ω. Ordinals are important because one can do induction on them in more powerful ways than one can do induction on the integers. It is induction up to particular ordinals that allow us to solve particular Halting Problems. For every Halting Problem there is some ordinal large enough to solve it. The ordinal numbers are developed starting in Section 6.2. Loosely speaking ordinals represent different ways of ordering operations. Cardinals represent objects of different absolute size. Every integer is a cardinal number.

5.7 Cardinal numbers

The cardinal numbers became problematic with Cantor's proof that there are 'more' real numbers than integers. This was the start of the modern concept of mathematical infinity.

George Cantor used a diagonalization argument to show there does not exist a function[2] that assigned a unique integer to every real number. He assumed such a function, $r(n)$, exists and used it to construct a real, d, not in the range of the function $r(n)$.

[2] A function has a domain or set of inputs and a range or set of outputs. For each possible input there is a unique output. For example $f(x) = x + 1$ is a function that adds one to its input x. limiting its domain to the integers greater than 0 forces its range to be the integers greater than 1. A more complex example is the function that gives the payments on a $100,000 mortgage from the interest rate. Such a function might have a domain of interest rates between 3% and 10% and a corresponding limited range of payments. Many functions like these two examples are computable. One can write a computer program to compute the output from the output. Mathematical functions need not be computable. Noncomputable functions can be defined using unsolvable problems like the Halting Problem.

This technique is proof by contradiction. If an assumption leads to a contradiction then the assumption must be false.

If there is a function that maps a unique integer onto each element of a set then that set is said to be countable. Cantor proved the reals are not countable.

In giving Cantor's proof one can limit the reals to those between 0 and 1 since, if these cannot be mapped uniquely to the integers, it is impossible to do so for a more inclusive set such as all reals. A real number in this range can be represented as an infinitely long decimal fraction with the first digit just to the right of the decimal point as shown in Table 5.3.

Cantor showed how to construct a real, d, whose nth digit differed from the nth digit of the the real that was mapped from n by r which is $r(n)$. The mth digit of the real d is written as d_m. The mth digit of the real $r(n)$ is written as $r(n)_m$. The real d Cantor defined satisfies the following:

$$d_m \neq r(m)_m$$

d differs in its mth digit from $r(m)_m$ for every integer m. Thus it differs from every real in the range of $r(n)$. An example is shown in Table 5.3. The assumption that $r(n)$ is a function that has the integers as its domain and *all* reals as its range is false.

Section 6.10 describes the counter point to Cantor's proof known as the Lowenheim Skolem Theorem. This shows that no matter how large the infinite sets a mathematical system claims to define, there is a countable model that satisfies the axioms of the system if there is any model that does so. A model is a collection of sets for which the axioms of the system hold.

The reals have cardinality greater than the integers by Cantor's argument. It is unknown if there are cardinals larger than the integers and smaller than the reals. The assertion that there are none is called the Continuum Hypothesis. We know today it cannot be proved or disproved from the standard axioms of mathematics[14]. I believe the Continuum Hypothesis is neither true nor false in an absolute sense. In a sense I think it is the modern version of the parallel postulate. It may be true false or undecidable in a particular formal system, just as the parallel postulate may be true or false in a particular geometry.

$r(n)$ tries to map integers to reals	
n	$r(n)$
0	$.\overline{0}0000000000000000...$
1	$.1\overline{1}111111111111111...$
2	$.22\overline{2}22222222222222...$
3	$.333\overline{3}3333333333333...$
4	$.4444\overline{4}444444444444...$
5	$.55555\overline{5}55555555555...$
6	$.666666\overline{6}6666666666...$
7	$.7777777\overline{7}777777777...$
8	$.88888888\overline{8}88888888...$
9	$.999999999\overline{9}9999999...$
10	$.1010101010\overline{1}010101...$
11	$.12121212121\overline{2}12121...$
12	$.131313131313\overline{1}3131...$
13	$.1414141414141\overline{4}141...$
14	$.15151515151515\overline{1}51...$
...	...

The above shows the start of a possible map from integers onto the reals. The digits along the diagonal have a line above them ($\overline{0}$). One can construct a real that is different from every real in the list by making its nth digit different from the nth digit of the nth real in the list. The digits that the constructed real has to differ from are .012345678912141... Any real that differs in every decimal position from this number cannot be in the list. Thus the real .101111111101010... constructed by putting 1 in every position that was not 1 on the diagonal and 0 in the remaining positions. This real will differ in at least one digit from every number in the list.

Table 5.3: Trying to map the integers onto the reals

The hierarchy of cardinal numbers is of practical importance because axioms asserting the existence of such sets solve problems that almost everyone agrees are meaningful. For example such axioms can solve instances of the Halting Problem. To see how this is possible we now turn our attention to Gödel's result. It was the starting point for developing the hierarchy of unsolvable problems. I think it is the key to revealing the fundamentally creative nature of mathematics.

5.8 Gödel's Incompleteness Theorem

This section lays the groundwork for a simplified version of Gödel's theorem that is proven in the next section. The proof is for the Halting Problem. Gödel proved that any formal system that defines the primitive recursive functions must be either incomplete or inconsistent. In particular one could not prove from within the system that the system itself was consistent even though the question could be formulated within the system.

The consistency of any finite formal system is equivalent to the Halting Problem for a particular computer program that is easily constructed from the axioms of the system. Gödel's result is more general than the Halting Problem because it is not limited to finite formal systems.

All formal systems that humans can write down are finite. However the idea of an arbitrary real number seems so obvious that mathematicians claim as formal systems a finite set of axioms plus an axiom for each real number that asserts the existence of that number.

Given a solution for the Halting Problem one could solve the consistency problem for finite formal systems. The idea of the proof is simple. A finite formal system is a mechanistic process for deducing theorems. Thus one can construct a computer program to generate all the theorems deducible from the axioms of the system. One can add to this program a check that tests each theorem as it is generated to see if it is inconsistent with any theorem previously generated. If an inconsistency is found the program halts.

Such a program will halt if and only if the original formal system is inconsistent. For the program will eventually generate and check every theorem that can be deduced from the system against every other theorem to insure no theorem is proved to be both true and false.

Crucial to the proof of the unsolvability of the Halting Problem is the ability to assign a unique integer to every computer program. Assigning a unique number to each element in a class of objects is known as Gödel numbering. It is trivial to see this is possible today when we Gödel number anything stored in computer memory including computer programs. In Gödel's time this aspect of his proof was very complex and its construction was a stroke of genius.

In a computer's memory programs are represented as a very long sequence of zeros and ones. Computer memory is almost always binary with two possible states at each storage element. These 'bits' (elements that can store 0 or 1) are organized in groups of eight known as bytes. When computer programs are stored in memory as bytes they are Gödel numbered. The numbering depends on the architecture of the computer running the program.

The proof of the Halting Problem depends on a Universal Turing Machine capable of simulating every possible program. Almost any computer is a Universal Turing Machine provided it has some way to reference a potentially infinite storage device. For example it might request that the next or previous disk be loaded from a set of disks that can grow without limit. The requirement in Gödel's proof that the formal system be powerful enough to embed the primitive recursive functions implies that the system must be able to model a Universal Turing Machine.

A Universal Truing Machine has two inputs. One is s computer program that can be executed on a particular computer. The other input is a complete description of the computer that can execute the program. The Universal Truing Machine uses the description of the computer to perform exactly the same operations that the real machine would do if it were executing the program. Of course this simulation will be much slower than executing the program directly on a machine designed for that programming language.

Computer programs often have parameters. Whenever a web site asks you to type in information you are entering a parameter to a computer program. Programs typically produce different responses to different input parameters. Using this capability of generating an output for each input one can create computer programs that define a function on the integers. For any integer input the program outputs a particular integer result.

Represent this function as $p(n)$ where p is the program n is the input and the value of $p(n)$ is the output the program produces with input n. There may be no output for some inputs.

5.9 The Halting Problem

Following is some needed notation.

1. A computer program with no inputs is represented by t_n where n is the Gödel number of the program.

2. A computer program with a single integer parameter is represented by $p_n(m)$. n is the Gödel number of the program. m is the parameter. If a program halts it may generate an output value. This is written as $V(t_n)$ and $V(p_n(m)$.

3. If t_n or $p_n(m)$ halts, write $H(t_n)$ or $H(p_n(m))$ respectively.

Following is a sketch of the proof of the unsolvability of the Halting Problem.

The Halting Problem asks does there exist a computer program $h(n)$ such that $V(h(n)) = 1$ if $H(t_n)$ and $V(h(n)) = 0$ otherwise.

Assume $h(n)$ exists. The following shows that this leads to a contradiction.

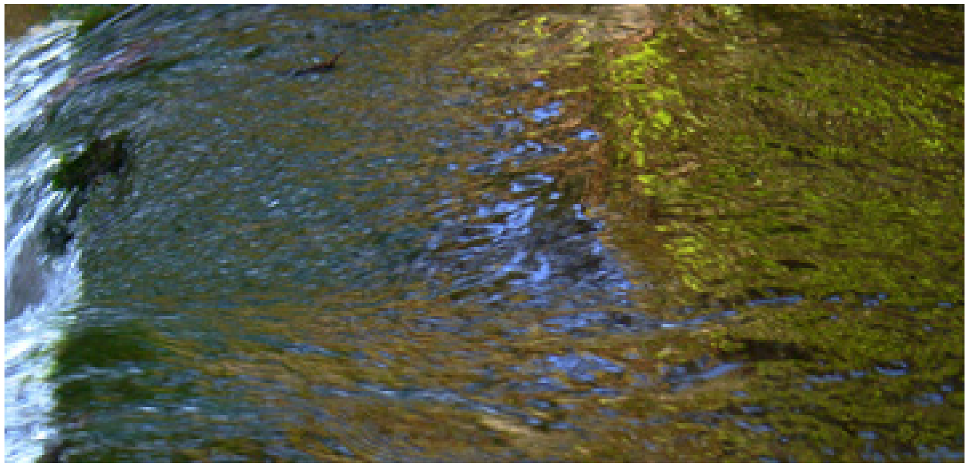
From $h(n)$ one can construct $s(n)$ (for the Self Halting Problem). $V(s(n)) = 1$ if $p_n(n)$ halts. otherwise it equals 0. In other words $s(n)$ decides whether any computer program that accepts a single integer parameter as input will halt when presented with its own Gödel number as input.

For any program p_n it is straightforward to compute k such that t_k behaves exactly as $p_n(n)$. To construct t_k from p_n expand the program memory to include the value n and read that data instead of reading an input parameter. Thus from a solution to the Halting Problem one can construct a solution to the Self Halting Problem.

Now s, which solves the Self Halting Problem, is a computer program that accepts an integer input, Thus it has a Gödel number r and $p_r(n)$ is identical to $s(n)$. From r one can construct a program with Gödel number q such that $p_q(n)$ halts if $V(s(n)) = 0$, otherwise it runs forever.

What does $p_q(q)$ do? If $p_q(q)$ halts then $V(s(q)) = 1$ and thus $p_q(q)$ will run forever. This is a contradiction and thus the original assumption that $h(n)$, a solution to the Halting Problem, exists is false.

This sketch of a proof is far simpler than Gödel's result. The program modifications described are straight forward for an experience programmer, but the details at the level of computer code are tedious. It is even more tedious to *prove* the modifications do what is intended. Gödel's proof in 1931 never mentioned computers. He went through the long and difficult exercise of showing that a formal system could be modeled as an arithmetic function definable within the formal system. He then proved the consistency of the formal system was equivalent to a question about this function. Using an argument similar to the above he showed one could not prove that property from within the system unless the system itself was inconsistent. One can prove anything in an inconsistent system.



Chapter 6

Creative mathematics

Gödel's result led to the discovery of a hierarchy of unsolvable problems. There is no general way to solve *all* problems even at the lowest level of this hierarchy. Yet there is some axiom of mathematics that can solve any individual problem at any level in the hierarchy. Biological evolution has created the human mind which is capable of developing a set of mathematical axioms that are very powerful and that seem intuitively obvious to most educated mathematical minds. These axioms are at the core of contemporary mathematics and science.

Gödel's result implies that any finite set of axioms is an infintessimal fragment of objectively true mathematics. This chapter develops the hierarchy of unsolvable problems and gives the remaining axioms of set theory that solve a significant, albeit infintessimal, fragment of these problems. The chapter ends with a philosophy of mathematical truth that connects biological evolution with the creative nature of mathematics. One crucial result is mathematical boundary conditions that would support unlimited creativity in future biological evolution and unlimited expansion of mathematics. This is vitally important because it is almost inevitable that future biological evolution will be, in large measure, consciously directed. Knowing the boundary conditions for unbounded creativity is essential to the wise use of the enormous power that science is providing us.

6.1 Arithmetical Hierarchy

The Arithmetical Hierarchy predated Gödel's result, but his work led to recognizing it as a hierarchy of unsolvable problems of increasing difficulty. Traditionally it was defined as the hierarchy that comes from logical expressions that contain quantifiers over the integers. This chapter develops the hierarchy by relating it to the Halting Problem for computers.

Consider the function $h(n)$ that is 1 if the computer program with Gödel number n halts and 0 otherwise. This function cannot be generated by a computer program. But one can write a program to output the Gödel numbers of any computer program n for which $h(n) = 1$. That is one can write a computer program that outputs the Gödel numbers of all computer programs that halt. What one cannot do is list the Gödel numbers of programs that do *not* halt.

To output the Gödel numbers of all computer programs that halt, program a single computer to execute the program corresponding to *every* Gödel number. This involves a sequence of steps. In the first step one instruction from the program with Gödel number 1 is executed. In the next step 2 instructions for programs 1 and 2 are executed. This is 4 instructions total. In the nth step n instructions are executed for programs 1 through n. This is n^2 instructions total. If any program halts during any step then the Gödel number of that program is output. Eventually, if any program halts, its Gödel number will be output. This is not a solution to the halting problem because it provides no way to know if a program does not halt. We have to wait an infinite time before we can be sure a program's Gödel number will *not* be output.

This simulation of many programs by a single program is called nondeterministic programming although there is nothing random or unpredictable about it. A computer running such a program is called a nondeterministic computer.

A set that can be listed using a computer program is said to be recursively enumerable. If one can also list by a computer the complement of the set (those integers not in the set) than it is said to be recursive. The set of Gödel numbers of computer programs that halt is recursively enumerable but not recursive. This is the first in a hierarchy of recursively unsolvable problems that form the Arithmetical Hierarchy.

One can speculate about 'more difficult' problems by assuming one had a solution for the halting problem and ask what new problems would remain unsolvable. This led to the idea of a computer with an oracle. An oracle is a magical device that solves some unsolvable problem like the Halting Problem. You input to it an integer n and in a finite time it outputs 1 or 0 to indicate if the program with Gödel number n will or will not halt.

Assuming a computer exists that has access to an oracle for the Halting Problem, are there functions it cannot compute? One can apply the original Halting Problem proof to this machine to prove it could not solve its own Halting Problem. One could give an oracle for this higher level Halting Problem and generate an even higher level problem. Thus was introduced the notion of degrees of unsolvability.

A related way to extend the hierarchy of unsolvable problems is to ask if a computer program will generate an infinite number of outputs. This property can be generalized by interpreting the output of a computer as the Gödel number of another computer. One can thin ask this question. Does a program have an infinite number of outputs an infinite subset of which, when interpreted as computer programs, have an infinite number of outputs? This can be iterated any finite number of times to create the Arithmetical Hierarchy.

This hierarchy is usually developed with the universal ($\forall$) and existential ($\exists$) quantifiers restricted to the integers rather than ranging over all possible sets.

An alternating pair of these quantifiers ($\forall\exists$) restricted to the integers has been shown to be equivalent to the Un quantifier. $Un\ g(n)$ is true if and only if $g(n)$ is true for an infinite subset of the integers. The Arithmetical Hierarchy can be defined using either the U quantifier or alternating pairs of existential and universal quantifiers.

Levels in the Arithmetical Hierarchy are labeled as Σ_n if they can be defined with an expression limited to $n - 1$ pairs of alternating quantifiers starting with Σ. Similarly statements that start with $\forall$ are labeled as Π_n. Σ_0 and Π_0 are defined as having no quantifiers and are equivalent. Σ_1 and Π_1 are defined as having a single quantifier. Table 6.1 summarizes these definitions.

Only alternating pairs of quantifiers are counted because two quantifiers of the same type occurring together are equivalent to a single quantifier. Table 6.2 shows a map from the integers onto all pairs of integers. Using this map one can convert a sequence like $\forall x \forall y\ g(x, y)$ to $\forall z\ g(x(z), y(z))$. The same technique applies to two consecutive existential ($\exists$) quantifiers. An expressions ending with $\forall x \exists w \forall y\ g(x, w, y)$ can be rewritten as an expression ending with $\forall z \exists w\ g(x(z), w, y(z))$. A similar reduction works with $\exists x \forall w \exists y\ g(x, w, y)$. So Table 6.1 gives all unique possibilities.

Level	Questions one can ask: will a computer program
Σ_0	halt in fixed time
Σ_1	*ever* halt
Π_1	*never* halt
Σ_2	have at *most* a finite number of outputs
Π_2	have an infinite number of outputs
Σ_3	have at *most* a finite number of Π_2 outputs
Π_3	have an infinite number of Π_2 outputs
Σ_n	have at *most* a finite number of Π_{n-1} outputs
Π_n	have an infinite number of Π_{n-1} outputs

Table 6.1: Arithmetical Hierarchy

x	z										
...	...	...	...	...	...	...	...	...	...	...	...
9	81	82	83	84	85	86	87	88	89	90	...
8	64	65	66	67	68	69	70	71	72	91	...
7	49	50	51	52	53	54	55	56	73	92	...
6	36	37	38	39	40	41	42	57	74	93	...
5	25	26	27	28	29	30	43	58	75	94	...
4	16	17	18	19	20	31	44	59	76	95	...
3	9	10	11	12	21	32	45	60	77	96	...
2	4	5	6	13	22	33	46	61	78	97	...
1	1	2	7	14	23	34	47	62	79	98	...
0	0	3	8	15	24	35	48	63	80	99	...
y	0	1	2	3	4	5	6	7	8	9	...

This table shows part of a mapping from the integers onto all pairs of integers. For every integer pair (x, y) there is a unique z. The table is generated by placing 0 at $(0, 0)$ and adding one to each successive entry. The sequence of entries go across to the diagonal and then down. The across down sequence is continually repeated starting just above the previous row.

Table 6.2: Mapping pairs of integers onto the integers

6.2 Ordinal induction

One can move beyond the Arithmetical Hierarchy by interpreting the output of a computer program in different ways. For example, one can interpret the output as an integer or as the Gödel number of another program. One could assume an even output is interpreted as an integer and an odd output is interpreted as a program's Gödel number. After deciding how an output is to be interpreted, it can be divided by 2 so every integer and every Gödel number is a possible result.

Using this convention one can define an n-tree. An output is called 0-tree if it is to be interpreted as an integer. An output is an n-tree if it is interpreted as the Gödel number of a computer program with the following property. When the program is executed it has an infinite number of outputs and an infinite subset of these are $(n-1)$-trees. A 1-tree must have an infinite number of integer outputs. A 2-tree must have an infinite number of outputs an infinite subset of these must represent computer programs that generate an infinite number of integer outputs.

With this property one can define sets that are not in the Arithmetical Hierarchy. Consider the question does a computer program have an output that is at least n-tree for all integers n. That is, for every n, is there at least one output that is an m-tree where $m \geq n$. I call this an ω-tree. This is no finite logical expression, limited to quantifiers over the integers, that can define this set. It is the self referencing structure of this definition that cannot be captured by the expressions that define the Arithmetical Hierarchy. This property can be defined in the Hyperarithmetical Hierarchy. New 'successor' levels in this new hierarchy can be created in the same way the Arithmetical Hierarchy is built up. To go to the next level one asks if a program has an infinite number of outputs of the previous level. One can also create new *limit* levels by asking if a computer program has outputs that span some infinite sequence of lower levels as we just did to move beyond the Arithmetical Hierarchy.

Solving problems in the Arithmetical and Hyperarithmetical hierarchies requires ordinal induction which is a generalization of induction on the integers.

Ordinals are defined by two properties. They are transitive and they are well ordered by the $\in$ relationship. A set is transitive if every *member of a member* of a set belongs to the set. A set S is well ordered if, for every two elements a and b of S, either $a > b$ or $a < b$ or $a = b$. Such a set is also said to be linearly ordered. The ordinals are will ordered by the $\in$ relationship. For any two ordinals a and b either $a \in b$ or $b \in a$ or $a = b$.

The integers have both of these properties and are the finite ordinals. The ordinals generalize the integers into the transfinite. ω is the smallest infinite ordinal. The next ordinal contains ω and all the elements of ω.

Induction on the integers and ordinal induction are compared in Figure 6.1. Ordinal induction is completely general. There is

Induction on the integers

$$(p(0) \wedge \forall n\,(n \in \omega)[(p(n) \rightarrow p(n+1))]) \rightarrow (\forall n\,(n \in \omega)p(n))$$

If $p(0)$ and, for all integers n, $p(n) \rightarrow p(n+1)$ then $p(n)$ is true for all integers n.

Ordinal induction

Ordinal induction generalizes induction on the integers.

$O(x)$ means x is an ordinal.

$$\forall \beta O(\beta)[p(0) \wedge [\forall \alpha(\alpha < \beta)[\forall x(x < \alpha)p(x)] \rightarrow p(\alpha)]] \rightarrow [\forall y(y < \beta)p(y)$$

This establishes conditions to prove a property p is true for all ordinals less than β. To prove this one must prove $p(0)$. One must also prove for every α less than $beta$ that if $p(x)$ is true for all x less than α then $p(\alpha)$ is true.

Figure 6.1: Induction on the integers and ordinals

no higher level scheme of induction. It obtains this generality by making ordinal number an open ended concept. The difficulty in increasing the power of a mathematical system to solve problems is the construction of ever larger ordinal numbers. The first tool for defining large ordinals in standard set theory is the axiom of replacement.

6.3 Axiom scheme of replacement

The general axiom scheme for building up complex sets like the ordinals is called replacement. It is an infinite list of axioms. These axioms could be defined by a single finite expression, but they are usually defined as an easily generated sequence.

The axiom of replacement scheme describes how new sets can be defined from exiting sets using any relationship $A(x, y)$ that defines y as a function of x. Recall that a function maps any element in its range (any input value) to a unique result or output value. The axiom of replacement scheme asserts that for any set x and any function f defined on all sets, one can construct a new set which consists of the sets obtained by applying f to each element of x.

The following notation simplifies the formal expression. $\exists!\, y\, g(y)$ says there exists one and only one set y such that $g(y)$ is true. The replacement axioms schema is as follows.

$$[\forall x \exists!\, y A_n(x,y)] \rightarrow \forall u \exists v (B(u,v))$$
$$B(u,v) \equiv [\forall r (r \in v \equiv \exists s [s \in u \wedge A_n(s,r)])]$$

This first part says if A_n defines y uniquely as a function of x then the for all u there exists v such that $B(u.v)$ is true. The second part defines $B(u,v)$ as equivalent to $r \in v$ if and only if there exists an $s \in u$ such that $A_n(s,r)$ is true. v is the set defined by applying the function defined by A_n to u. Since A_n is not defined in the form of a function one has to use this somewhat convoluted definition.

This axiom schema came about because previous attempts to formalize mathematics were too general and led to contradictions like the Barber Paradox[1]. By restricting new sets to those obtained by applying well defined functions to the elements of existing sets it was felt that one could avoid such contradictions. Sets are explicitly built up from sets defined in safe axioms. Sets cannot be defined as the *universe* of all objects satisfying some relationship. One cannot construct the set of all sets which inevitably leads to paradox.

We now turn our attention to developing the ordinals.

[1] The barber paradox concerns a barber who shaves everyone in the town except those who shave themselves. If the barber shaves himself then he must be among the exceptions and cannot shave himself. If he does shave himself that he does not shave himself. Such a barber cannot exist.

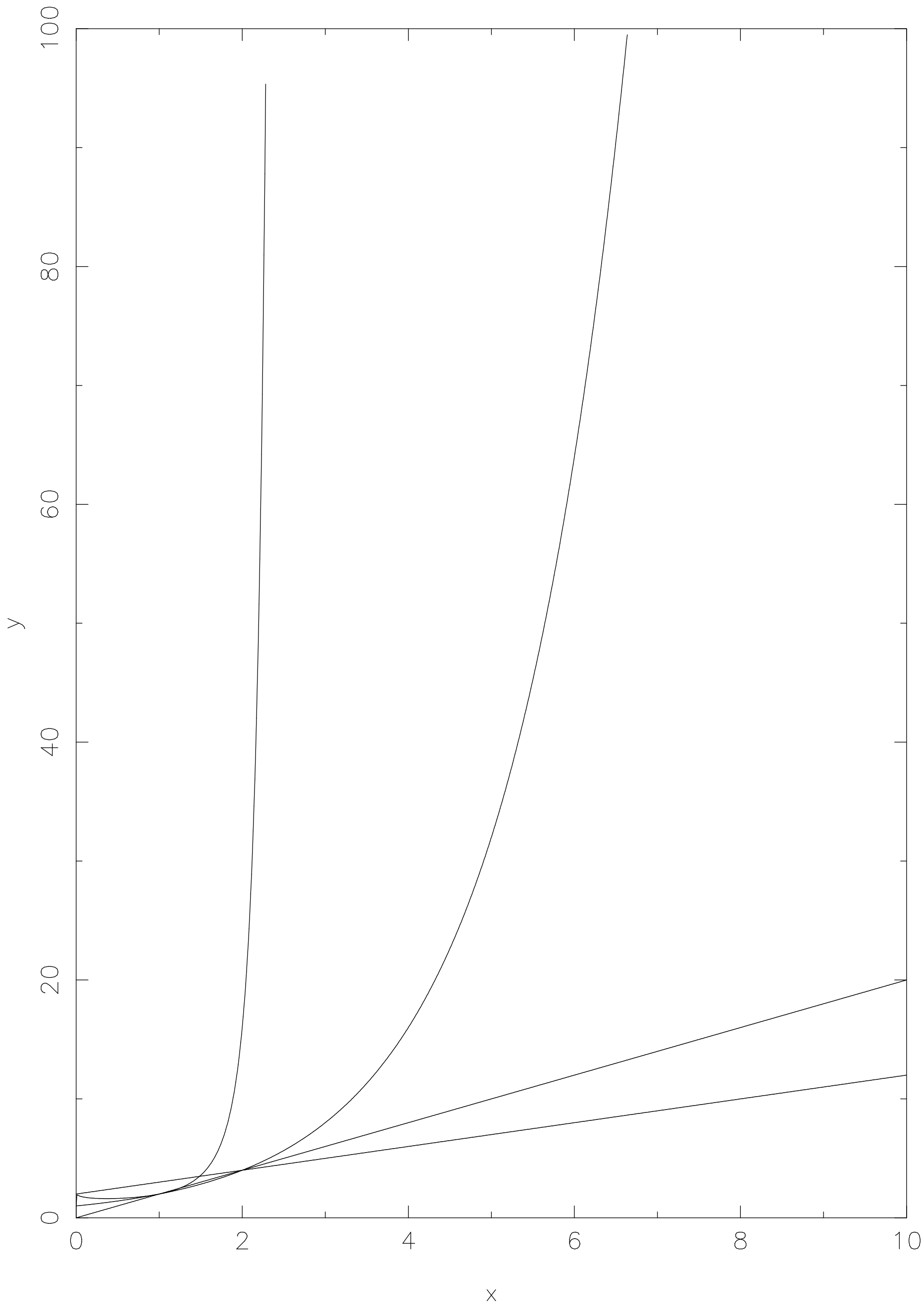
6.4 Ordinal numbers

One way to construct ordinals is through ordinal arithmetic. This is similar to arithmetic on the integers. $\omega + \omega$ or $2 \times \omega$ is the infinite successor to ω just as ω is the infinite successor to 0.

Consider the ordinal sequence ω, $\omega\times\omega$, $\omega\times\omega\times\omega$, $\ldots$. This leads to ω^ω. Then comes the series ω^ω, ω^{ω^ω}, $\omega^{\omega^{\omega^\omega}}$, $\omega^{\omega^{\omega^{\omega^\omega}}}$, $\ldots$. The limit of this sequence is the first ordinal that is not primitive recursive. Loosely speaking at cannot be obtained by simply iterating, iterating iteration etc. It contains all the iteration schemes one can generate in that way. To get this ordinal you need to introduce a higher level of abstraction. You need to view the process of generating iteration schemes as something that itself can be iterated.

There is a correspondence between the hierarchy of ordinal arithmetic and functions on the integers that increase more rapidly. Compare members in the following sequence.

$$z + n, z \times n, z^n, z^{n^n}, z^{n^{n^n}}, z^{n^{n^{n^n}}} \ldots$$



Plot of $y = 2 + x$, $y = 2 \times x$, $y = 2^x$ and $y = 2^{x^x}$.

Figure 6.2: Functions that grow rapidly

Figure 6.2 plots the first four of these functions with $z = 2$. More complex iteration generates functions that increase more rapidly. The complexity comes from abstracting and generalizing forms of iteration. Addition is iteration of successor. Multiplication is iteration of addition. With multiplication as a base, each level of the exponential hierarchy is iteration of the previous level. One can generalize from the entire exponential hierarchy to create a function that is not in the exponential hierarchy and grows faster than any exponential function. One such function is defined as follows,

$$B(0) = 1, B(1) = 1, B(2) = 2^{2^2}, B(3) = 3^{3^{3^3}} B(4) = 4^{4^{4^{4^4}}} \ldots$$

The Halting Problem could be solved with a function that increased rapidly enough. Assume a specific Gödel numbering of computer programs. For any integer n there is some integer m such that all programs with Gödel number less than n that halt will do so in m time steps. m can be the longest time to halt for programs with Gödel number less than n. If we had a function $m(n)$ that could tell us how long to wait we could solve the halting problem. Any function equal to or uniformly larger than this function will do. There is such a function that will solve problems at every level in the Arithmetical and Hyperarithmetical hierarchies.

The axiom of replacement supports complex iteration schemes and does not seem to lead to contradictions. But there are more general iteration schemes, that require concepts that seem to far removed from the universe of computer programs. By allowing quantification over the reals, one can ask questions about all possible paths in a branching tree. This is a very powerful concept that allows extremely complex iteration. The real numbers would seem to take us beyond the mathematics of finite computers. However it is the fate of computers following paths in a branching tree that are central to the mathematics of the reals.

6.5 Searching all possible paths

The ordinals and iteration schemes described in the previous section are recursive. Their structure can be modeled by a nondeterministic computer program like those used in defining an n-tree. By defining precisely what it means for the structure of an ordinal to be modeled by a computer, one can define the set of all recursive ordinals. This set is the smallest nonrecursive ordinal.

A computer program can model the structure of a recursive ordinal by outputting Gödel numbers that represent all the members of the ordinal in question. Every set except the empty set is represented by a program with at least one output. Use computer programs with integer outputs. 0 represents the empty set. All other integer outputs are interpreted as Gödel numbers of computer programs. These may or may not represent a set. A program represents a set only if all of its outputs represent sets.

Many computer programs (in fact an infinite number) will be representations of the same set. Sets are uniquely determined by their members but representations for sets are not. To deal with this require that only one Gödel number represents each each set.

A computer outputting representations for the members of an infinite set gives an ordering to the members that is not present in the original set. (The outputs must come in some sequence.) This can be confusing. For a set like $\omega + 1$ the order the representations are output must differ from the ordering of the set members defined by $\in$. Recall that ordinals are well ordered by $\in$. The representation for ω must be output before the representations for most of the elements of ω. Otherwise one would never get around to outputting the representation for ω.

Most computer programs will not represent sets. Most that do represent sets will not represent ordinals. To define which programs represent a set requires defining which are well founded. Sets constructed from the axioms of set theory are well founded because they are built up in a finite number of steps from the empty set. A set is well founded if it does not contain an infinite descending chain of members. If you take any member of a set then take any member of that set and keep repeating the process you will reach the empty set in a finite number of steps.

The outputs generated by the nondeterministic execution of a program are structured in a tree. The level of a node is the number of times the output of a program is interpreted as another program before getting to this node. The original program is level 0. Outputs of that program are level 1. Outputs of these programs are level 2 and they generate outputs at level 3, etc. Each output is a node in the tree that is either another program or the number 0 for the empty set. Each program node may have an infinite number of branches. To select a node or output at the first level requires a single integer. Selecting a second level node requires a pair of integers etc.

If a program is well founded, then repeating the process of taking any output at level n and using that output as the Gödel number of a program to take the output at level $n+1$. will reach the representation for the empty set in a finite number of steps.

If the program is well founded than every path will end in a finite sequence of integers. But for any integer N there can be a path of length N. With no *fixed* limit on the length of a path, one must search over paths that have no limit on their length. This requires an infinite sequence of integers.

To search all paths generated by an infinite sequence of integers requires the power set axiom defined in the next section. That axiom implies the set of all subsets of the integers exists. This is not the same as the set of all ordered sequences of integers needed to specify the position of every node in the tree. To get all ordered infinite sequences of integers from all subsets of the integers, let the integer size determine the ordering. Using such a sequence directly would result in strictly increasing sequences. That does not work. Instead use only the first (smallest) number in the subset as is. Each subsequent number is the difference between the previous number in the sequence and the current one. This allows an arbitrary ordered sequence of integers with 1 as a minimum.

With the power set axiom and the above construction One can write the following expression that is true if program with Gödel number p is well founded.

$$\forall s(s \in P(\omega))\exists n(n \in \omega)t(p, O(s), n)$$

O is the map between an unordered sequence of integers and the ordered sequence defined in the above paragraph. O also needs to extend any finite subset of the integers to make it infinite by, for example, repeating the last element in the finite set. $P(\omega)$ is the set of all subsets of ω. $t(p, O(s), n)$ is a recursive relationship between the program with Gödel number p, the ordered sequence of integers $O(s)$ and the integer n. It is true if there exists an output of 0 at level n from program p with path given by sequence $O(s)$. Thus for every possible path $O(s)$ in the tree of outputs from p the empty set will terminate the path at some finite level n. If the statement is true than p is well founded. .

Recall from Section 6.2 that an ordinal is defined as being well ordered by $\in$ and transitive. So in addition to being well founded an ordinal representation must meet these requirements. These properties require a search over all members of specific sets. Such an infinite search is not recursive. It requires an existential quantifier in an expression like $\exists nP(n)$. Recall Section 6.1 and Table 6.2

show how to replace multiple adjacent existential quantifiers with a single existential quantifier. Using this technique one can incorporate the transitive and well ordered properties as well as the requirement that each set has a unique representation into the $\exists n(n \in \omega)t(p, s, n)$ part of the above expression defining well founded programs.

6.6 Power set axiom

The power set axiom says the set of all subsets of any set exists. This is not needed for finite sets, but it is essential to define the set of all subsets of the integers.

$$\forall x \exists y \forall z[z \in y \equiv z \subseteq x]$$

This says for every set x there exists a set y that contains all the subsets of x. z is a subset of x ($z \subseteq x$) if every element of z is an element of x.

The axiom of the power set completes the axioms of ZF or Zermelo Frankel set theory. They are summarized in Figure 6.4 that also includes the axiom of choice described in the next section.

From the power set axiom one can conclude that the set of all subsets of the integers exists. From this set on can construct the real numbers. One approach is to use the natural order of integers in the subset. The smallest member defines the integer part of the real. Successive integers form successive digits in the fractional part. Each fractional digit is between 0 and 9. Take the remainder from dividing the integer by 10 to get a digit between 0 and 9. Applying this process to every subset of the integers generates every real number many times. This procedure is illustrated in Figure 6.3.

By Cantor's proof in Section 5.7 one cannot map the reals onto the integers and by the above construction the same applies to maps between the integers and all subsets of the integers.

Constructions like those outlined in this and the previous section show how natural it can be to talk about infinite processes operating on infinite and even uncountable sets. This is part of the reason mathematicians think of the infinite as if it were a physical reality. Such arguments are logically sound and can be important mathematically.

The problem is that taking this approach too literally leads to a false intuition about the nature of infinity. It seems as if the mind is grasping the infinite when it is actually arguing about recursive processes carried out in a potentially infinite universe. In the next section I touch on an alternative direction for extending mathematics that grows out of an acceptance of the infinite as a potential that can never be actualized. It is a philosophical view that sees mathematical creativity and all creativity as a divergent and not a convergent process.

As mentioned at the beginning of Chapter 5 Computers are an accurate metaphor for a logically determined sequence of events. There is nothing magical about computers. It is the laws of physics that allow us to build computers and imply a universe in which logic is important in predicting future events. Whether or not there is fundamental randomness in the laws of physics is an open ques-

Following are the first few elements or a subset of the integers listed in ascending order. The digits used in constructing a real are underlined.

$\{\underline{532}, 67\underline{5}, 95\underline{8}, 232\underline{1}, 332\underline{2}, 512\underline{1}, 798\underline{9}, \ldots\}$

To construct a real number from the above subset of the reals, do the following:

1. Put the first element to the left of the decimal place.
2. Take the least significant digit of each succeeding element.
3. Use that as the next position to the right of the decimal.

This generates $532.581219\ldots$

Figure 6.3: Constructing a real from a subset of the integers

tion that will be explored in the next two chapters.

6.7 Axiom of Choice

The Axiom of Choice is not part of ZF. It is however widely accepted and critical to some proofs. The combination of this axiom and the others in ZF is called ZFC.

The axiom states that for any collection of non empty sets C there exists a choice function f that can select an element from every member of C. In other words for every $e \in C$ $f(e) \in e$.

$$\forall C \exists f \forall e[(e \in C \wedge e \neq \emptyset) \rightarrow f(e) \in e]$$

A mathematically complete statement of the above requires a definition in the language of set theory of function. A function is a set of ordered pairs where the first element is in the domain of the function and the second element is in the range of the function. Each pair maps an element of the domain uniquely into an element of the range. Thus each first element must occur only once as in the set that defines the function.

Gödel proved that one could construct a model for the axioms of ZF using the constructible sets. Essentially these are the sets one can build up by applying the axioms of ZF. In this model the axiom of choice is true. However Paul Cohen constructed models of ZF in which the Axiom of Choice was false making it clear that this axiom cannot be derived from the other axioms.

It is a strange axiom since it would seem to be obvious. If one has a collection of sets then one should be able to choose one member from each set. But in general there is no way to do this using the axioms of ZF. It is one example of the strange nature of the infinite in formal mathematics. The real numbers derived from the power set allow one to search over all reals. This leads to many other strange questions and another postulate sometimes needed for theorems that is not derivable from the other axioms. This is the Continuum Hypothesis discussed in Section 5.7. Unlike the axiom of choice the Continuum Hypothesis is not generally accepted as true.

1. Axiom of extensionality (See Section 5.5.1).

$$\forall x \forall y \quad (\forall z \quad z \in x \equiv z \in y) \equiv (x = y)$$

2. Axiom of the empty set (See Section 5.5.2).

$$\exists x \forall y \quad y \notin x$$

3. Axiom of unordered pairs (See Section 5.5.3).

$$\forall x \forall y \; \exists z \; \forall w \quad w \in z \equiv (w = x \vee w = y)$$

4. Axiom of union (See Section 5.5.4).

$$\forall x \exists y \; \forall z \quad z \in y \equiv (\exists t \; z \in t \wedge t \in x)$$

5. Axiom of infinity (See Section 5.5.5).

$$\exists x \, \emptyset \in x \wedge [\forall y \, (y \in x) \rightarrow (y \cup \{y\} \in x)]$$

6. Axiom schema of replacement (See Section 6.3).

$$[\forall x \exists ! \, y A_n(x, y)] \rightarrow \forall u \exists v (B(u, v))$$

$$B(u, v) \equiv [\forall r (r \in v \equiv \exists s [s \in u \wedge A_n(s, r)])]$$

7. Axiom of the power set (See Section 6.6).

$$\forall x \exists y \forall z [z \in y \equiv z \subseteq x]$$

8. Axiom of choice (See Section 6.7).

$$\forall C \exists f \forall e [(e \in C \wedge e \neq \emptyset) \rightarrow f(e) \in e]$$

Figure 6.4: The axioms of ZFC set theory

6.8 Trees of trees

Searching all paths or quantifying over the reals can, in many cases, be interpreted as a property of computer programs. In such cases one is dealing with logically determined properties of computer programs. Everything a nondeterministic program does happens at some finite time and the collection of all those events determines if a given property like defining an ordinal representation is true. Far more complex problems can be defined that remain logically determined by the outputs generated from a computer.

One way to do this is by iterating the notion of a program being well founded for inputs of a given type. To do this have the program indicate how its output is to be interpreted. For example the program may have a label that says it requires an integer input. Alternatively it could say it requires an input which is the Gödel number of a program that accepts integers as input and is well founded for such inputs.

The question is a computer well founded for integer inputs fully captures the Hyperarithmetical Hierarchy. If one could decide that question for every possible computer program, one could decide any question in the Hyperarithmetical Hierarchy. Looking for computers that are well founded for various types of inputs is a natural way to extend the hierarchy of logically determined mathematical questions.

Beyond the Hyperarithmetical Hierarchy, mathematics becomes likes Swiss cheese. It is full of holes. One can list by a computer all meaningful mathematical questions at a given level of the Hyperarithmetical Hierarchy. But all levels of this hierarchy cannot be enumerate, even though structures that transcend the hierarchy can be defined by quantifying over the reals.

The idea of being well founded is a conceptual leap that allows us to do induction on more powerful structures. So what is the next conceptual leap and how do we find it? The axioms of set theory allow us to quantify over not just reals but functions from reals to reals (with cardinality $\aleph_2$) and much larger cardinals. The difficulty with using such abstractions to define higher level iterative structures is their disconnect from properties of computer programs. The total number of events that fully characterize the execution of a nondeterministic computer program is countable. Cardinals larger than the reals seem to have a limited relevance to such structures.

Perhaps it is possible to come up with more powerful structures by approaching the problem in a different way. Instead of looking to larger cardinals one can explicitly generalize the concept of trees of trees. A nondeterministic computer program can label its outputs in a variety of ways. For example they may be integers or the Gödel numbers of well founded computer programs that terminate with outputs that are integers. Once one has defined a hierarchy in this way one can use the hierarchy itself to label levels in a more complex hierarchy. What are the most powerful ways one can generalize and iterate such structures?

An advantage of this approach as that one can write and play with computer programs to develop intuition and understanding. Research in the foundation of mathematics is one of the few if not the only major scientific discipline in which computer simulations do not play a major role in developing the field. I believe that is unfortunate and significantly greater progress is possible by changing it.

6.9 Extending mathematics

If infinity is a potential and never a completed reality, then infinite sets do not exist. Mathematicians try to define the most general infinite structures imaginable because that seems to give the most bang for the buck. If no infinite sets exist this would be the construction of a fantasy.

Mathematics should be directly connected to properties of nondeterministic programs in a potentially infinite universe. This would limit extensions to a fragment of the countable ordinals and the sets that can be constructed from them. Of course the previous sentence only has meaning in the philosophical context of contemporary mathematics. In the philosophical approach I am advocating, it is illusion to think of the countable ordinals as having an objective existence.

The objects definable within a finite formal mathematical system, no matter what axioms of infinity it includes, are countable (they can be mapped onto the integers). This result is called the Lowenheim Skolem Theorem. The idea of the proof is that a formal system can be interpreted as a computer program for generating theorems. Such a program can output *all* of the names of the objects or sets definable with the system. These names and thus the collection of all objects they refer to are countable. They can be mapped onto the integers.

All real numbers and for that matter larger cardinals that can ever be defined in any mathematical system that finite creatures create will be countable. They will not necessarily be countable from within the system. Cantor's proof is correct as a proof about formal systems. If real numbers do not exist Cantor's proof is about the structure of formal systems and not some greater metaphysical reality.

This suggests that the theory of cardinals is an illusion. It is talking indirectly about ways of extending mathematics that are countable and reducible to properties of computer programs. The set of reals definable within a formal system is a countable set in a more powerful formal system. In the more powerful system there is a countable ordinal that characterizes this set.

There is no inconsistency in the illusion of the completed infinite. But perhaps this fantasy gets in the way of extending mathematics to the full degree that the human mind aided with computer simulations is capable of.

Of course there is a limit to what we are capable of understanding as a species. It is only by continuing evolution in a nondeterministic way following an ever increasing number of divergent paths that we can avoid creative stagnation in mathematics and everything else.

But we are far from understanding all that we are capable of. Using the enormous power of computers to leverage our intuition and intellect has led to great strides in science. It is ironic that research in the foundations of mathematics is still largely conducted with pencil and paper. As long as the focus is on the most abstract and powerful notions of the infinite, computer experiments seem irrelevant. That alone suggests this approach to mathematical truth is the contemporary version of historical failures. There is a tendency in mathematics to postulate axioms that are too strong.

Mathematicians have perhaps learned to avoid absolute contradictions but not the folly of Icarus. By attempting to fly too high foundations research in mathematics has stagnated.

6.10 Is the cardinality of the reals 0 or $\aleph_0$?

It is easy to talk about a formal system plus an uncountable number of axioms that state the existence of all reals. Each real number can be defined as an infinite sequence of digits. We cannot write the entire sequence but we have a sense of what an arbitrary real is. In this way mathematicians talk about the true set theory that includes all reals.

The human mathematical mind is the product of biological evolution. There is no evidence of a special facility that transcends the finite. On the contrary all the evidence suggests the opposite. The current 'theological' approach to mathematical truth flies in the face of the evidence. I believe the cardinality of the real numbers is 0. I do not think any real number as a completed infinite sequence exists. Of course the integers and rational numbers are also considered reals so one could argue that the cardinality of the reals is $\aleph_0$ or the same as ω.

It makes sense to consider all possible paths that a nondeterministic computer simulation (or for that matter biological evolution) can take. Mathematical theorems about searching all possible paths have a form of absolute meaning because their truth is determined by a recursively enumerable set of events. They are a complex statement about a well defined set of events all of which can occur in a potentially infinite universe. This does not require that the set of all reals or for that matter a single real exists.

6.11 A philosophy of mathematical truth

The mathematics described in this chapter is standard. The approach that focuses on properties of computer programs is unconventional but not controversial. In contrast the philosophy of mathematical truth is radical.

The real numbers have no physical existence. They are a human creation and thus a product of biological and cultural evolution. Denying the objective existence of completed infinite totalities, including the reals, leads to a very different notion of mathematical truth.

Mathematics is about the structure of immediate experience and the potentially infinite progression of sequences of such experiences. Mathematics involves the creation of truth which has an objective meaning. This is truth about what a computer does if it is allowed to run forever perhaps following an ever expanding number of paths. Mathematical statements that cannot be interpreted as questions about events, all of which can occur in a potentially infinite deterministic universe. are neither true nor false in any absolute sense. Of course, they may be useful properties that are either true or false relative to a particular formal system.

The essence of physical objects is immediate experience. This widens the implications of mathematics as the study of all possible structures. It implies that biological evolution is an evolution of consciousness and the structure of that evolution can be understood mathematically. Number is the mediator between the reality of here and now and our deeper existence as part of the unbounded creative process of evolving consciousness. Mathematics gives us some sense of what is possible and establishes some of the conditions necessary to realize those possibilities.

The subtlety and richness of human consciousness has evolved mainly to deal with other brains. If we survive as a species, the future, not just of our species, but of the entire evolutionary process will fall into our hands at least on planet earth. If we do survive, evolution will have passed the most remarkable turning point of its existence. Evolution will become conscious of what it is creating and will take conscious control of creation itself. Mathematics is crucial to understanding the conditions for unbounded creativity as we assume this extraordinary power and responsibility.

Chapter 7

Digital physics

The human eye and brain is designed to see continuous lines and other complex structures as a single entity. We do not see patterns of light. We see faces, bodies, chairs, tables. lines and circles. This helps us make sense of the complex patterns of light that strike our eyes.

The a continuous line is simpler than the ‘pixels’ that it is made of. We need to recognize and respond to complex structures, not in detail, but to their meaning. We need to recognize food, predators, escape routes and canyons that are potential traps. We recognize geometric shapes as a starting point for seeing what is relevant to our survival.

The reality that contemporary physics paints is radically different than the conceptual framework we develop naturally. What we see as continuous lines are complex structures made up of individual molecules and atoms. Light itself comes in discrete packets of energy called photons.

The mathematics of contemporary physics remains centered on continuous structures that are sophisticated versions of the concept of a continuous line that we are designed to recognize. The objects in physical space in contemporary physics have a discrete structure, but time and space are still modeled as continuous entities.

Part of the reason physicists focus on continuous space time models is that other models are far more complex and difficult to work with. Science must use the tools it has and these inevitably have limits and shortcomings. But it is important to distinguish between decisions made for sound scientific reasons and those made for pragmatic ones. For what is practical changes with our mathematical understanding and the tools of technology.

This chapter and the next explore the possibility that time and space are discrete or digital. Could there be a smallest possible distance and a smallest possible time just as atoms and molecules form the smallest possible unit of physical substances?

A continuous line would constitute a completed infinite totality. It would imply that infinite sets exist in some physical sense. We are a long way from being able to resolve this issue. It will take experimental techniques far beyond our existing capabilities to explore the time and distance scales where contemporary physics suggests we might observe the digitization of space-time. What we can do in this and the following chapter is paint a picture of contemporary physics that suggests why we might eventually discover that the universe is digital and finite. We can also point out some of the experimental approaches that might aid in the search.

7.1 Time and distance scales

> I consider it quite possible that physics cannot be based on the field concept, i. e., on continuous structures. In that case *nothing* remains of my entire castle in the air gravitation theory included, [and of] the rest of modern physics[41].

Einstein reached this conclusion near the end of his life in spite of the obvious pain it caused him because of its implications for his beloved relativity. I came to a similar conclusion as an undergraduate long before this quote was published. The implications for relativity were the biggest concern I had at the time. I do not know why Einstein came to this conclusion, but I can suggest some possibilities.

The universe has an absolute time and distance scale. Particles behave differently at the scale of single atoms then they do at distance scales we experience directly. At the much smaller scale of the Planck distance[1] the existing laws of physics cease to work. The continuum required for relativity is incompatible with the uncertainty that is fundamental to quantum mechanics. At larger scales this is only a theoretical problem but at the Planck distance the equations of physics no longer give meaningful answers. This suggests the Planck time and distance scales are approaching the point at which space and time are discrete or digital.[2]

7.2 Waves, particles and events

[1] The Planck distance is $\sqrt{G\hbar/c^3}$ or approximately 10^{-33} meters. Where G is the gravitational constant, $\hbar$ is Planck's constant divided by 2π and c is the speed of light.

[2] The most prominent attempt to reconcile relativity and quantum mechanics is string theory. This theory establishes minimum particle sizes to avoid the domain where the two fundamental theories of physics are incompatible. One cannot know if string theory is valid because its predictions are impossible to test with existing or foreseeable technology. String theory is not a branch of physics. It is mathematical philosophy. Science requires experiments.

Wave particle duality is another reason to suspect a discrete model might account for the seemingly paradoxical behavior at the quantum level, Quantum mechanics is a theory of waves not unlike the waves that appear on a still lake as a rock is dropped in. Detecting a particle has an effect like dropping the rock. The future expectation of where the particle can be found spreads out as a wave. The higher or lower the water level is relative to the still lake the more likely one will observe the particle at that location. The area where the particle might be found spreads out over time just as the ripples in the lake do.

Missing in contemporary physics is the connection between the wave and the particle. In quantum mechanics probabilities continually change but *nothing* ever happens. The connection between events (like seeing a particle) and probabilities is made through metaphysical interpretations. The first and most widely known of these is the called the Copenhagen Interpretation. This claims a conscious observation 'collapses' the wave function. We make one observation (drop a rock in the water) wait for a time and make a second observation. This second observation *instantly* causes the waves from the first observation to disappear making the lake perfectly still again. Then another rock is dropped in and a new wave spreads out from the new location where we observed the particle.

The waves that move out from an observation were not invented with quantum mechanics. Something similar happens in classical probability theory. The difference is that in the classical theory there is a fully deterministic model of how events proceed. However we have imperfect knowledge of the initial conditions. Thus, after making an observation of say the location of the ball on a roulette wheel, we can only make approximate predictions about where the ball will land. With more accurate information we can make better predictions and it is possible with enough computing power to beat the odds in Las Vegas until they kick you out of the casino.

In classical probability theory the collapse of a probability 'wave' function has no physical significance. It is only a matter of getting more accurate information. What collapses is our degree of ignorance about what is happening physically. The external world is unaffected. The quantum wave function is different. We can observe this function physically through interference effects. We can change its structure by, for example, having it pass through a polarizing filter as described in Section 8.3. Thus to think that a conscious observation collapses a physical wave function is to go beyond what most people think of as science. The Copenhagen Interpretation avoids this by asserting there are two domains of reality the quantum and the macroscopic. Consciousness observation connects the two domains. The process of quantum collapse becomes a metaphysical link between these domains rather than being a physical event.

7.3 Interpretations of quantum mechanics

The Copenhagen Interpretation is the basis for Schrödinger's famous cat.

> One can even set up quite ridiculous cases. A cat is penned up in a steel chamber, along with the following

> diabolical device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny bit of radioactive substance, so small that *perhaps* in the course of one hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives *if* meanwhile no atom has decayed. The first atomic decay would have poisoned it. The Psi function for the entire system would express this by having in it the living and the dead cat (pardon the expression) mixed or smeared out in equal parts.
>
> It is typical of these cases that an indeterminacy originally restricted to the atomic domain becomes transformed into macroscopic indeterminacy, which can then be *resolved* by direct observation. That prevents us from so naively accepting as valid a "blurred model" for representing reality. In itself it would not embody anything unclear or contradictory. There is a difference between a shaky or out-of-focus photograph and a snapshot of clouds and fog banks[45].

Schrödinger's cat is neither dead nor alive. It is in a *superposition* of possibilities until someone takes a peak. Neils Bohr, the author of the Copenhagen Interpretation, thought that that there were two separate domains: the macroscopic world of every day experience and the quantum domain. Events only occurred at the macroscopic level. Consciousness is the link between these domains. Conscious observations force a collapse of the quantum wave function to accord with the observation. Prior to that the wave function embodied all the possible states that might be observed including even a live and dead cat.

If you think this is nonsense you are not alone. There are competing interpretations. The most popular is "shut up and calculate." This says do not worry about interpretations. The mathematics must compute probabilities that conform to experimental observation. That is what is important.

A popular interpretation among physicists is called ‘Many Worlds’. It assumes all the possibilities exist physically in different realities. It takes the wave function as the *primary* reality and experimental observations as secondary. To take the mathematical models one has created as the primary reality is a violation of the spirit of science which makes nature the final arbiter through experiments.

Fundamental physical models are the simplest possible *descriptions* of what we have observed experimentally. We have no way of knowing how accurately they reflect the true structure of physical reality. It is possible to construct radically different models that give nearly identical experimental predictions. Physics was able to progress only by ignoring the religious dogma of the day. Unfortunately each new generation of scientists in every field has a tendency to create new dogmas.

To conclude we have gone beyond classical logic or mathematics requires compelling experimental evidence. Problems that suggest this possibility are for more likely to be a product of the limitations of our existing understanding. Philosophies that magnify those limitations into new realities disguise rather than confront our limitations.

Einstein always felt that the problem was not finding a correct interpretation but in developing a more complete theory. As mentioned in Section 7.1 Einstein came to suspect near the end of his life that a more complete theory must move away from continuous structures. In the next two section we outline a possible approach to developing a more complete theory.

This discussion of a more complete theory is speculative and intuitive. Developing properties of discrete models on a scale that could describe the fundamental particles of physics is far beyond existing technology and available mathematical techniques. Speculation is valuable in suggesting alternative possibilities. This can influence how one looks at existing experimental results and the priorities given to future experiments.

Quantum mechanics was developed by theoreticians and experimenters working together. Results from each side influenced the other. It is inconceivable that a theory as strange and remarkable as quantum mechanics could have emerged as a complete theoretical structure that only needed verification as was the case with special and general relativity.

Developing a more complete theory will almost certainly require a similar collaborative effort of experimentalists and theoreticians feeding each others understanding and intuition. If the more complete theory requires 'digital' space-time than this is almost certainly true. Thus speculation that might help generate experiments that point in a new direction may be essential. Bell's Theorem (see Section 8.4) establishes that a more complete theory of the sort Einstein sought will be experimentally distinguishable from quantum mechanics. It will be not just more complete but correct in instances when quantum mechanics makes incorrect predictions. Recognizing that there is a direction that may lead to an alternative theory can influence how one evaluates the existing experimental tests of Bell's Inequality (see Section 8.6).

7.4 Discretizing the wave equation

A discrete model can approximate a continuous one to any desired degree of accuracy. Developing such approximations is an important field in applied mathematics. These approximations are widely used in quantum mechanics. It is not possible to model a continuous equation on a digital computer. Thus discrete approximations provide the only practical approach to a great many problems.

Discrete wave models can be studied to understand their intrinsic properties and not just how they can be used to model a continuous reality that may not exist. There is little work in this area. No doubt the complexity of such models is one reason. To understand how such models might look we start with the wave equation which might be called the universal equation of physics because it occurs in so many contexts. We can generate a discrete wave model starting with the continuous wave equation.

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f$$

This notation defines a partial differential equation[3]. In spite of its foreboding appearance it says something simple. Think of f as the level of water in at a single point in a lake. This equation describes how the level changes based on conditions in its immediate neighborhood. The equation applies to every point on the lake so we can use it to model the dynamic behavior of a wave.

The term on the left of the equal sign is the rate at which the level is *accelerating*[4] up or down at this point. The term on the right hand sums acceleration across each spatial dimension at the same point[5]. For the surface of a lake there are two spatial dimensions. c is the velocity of the wave. The equation says the rate at which

[3] A differential equation describes how a single variable (such as the level of a lake) changes relative to some other single variable such as time. A partial differential equation involves the rate of change of multiple variable relative to other variables. The wave equation relates change relative to time to change relative to location.

[4] If you are moving 60 miles an hour and travel for 2 hours you will go 120 miles. If you are accelerating at 20 miles per hour per second and go for three seconds from a standing start you will be going 60 miles an hour. Your car almost certainly cannot accelerate that fast but if you have a hot motorcycle it might.

[5] We understand acceleration in time from driving. Acceleration is zero when speed is constant neither increasing or decreasing. Acceleration across distance is similar. A flat

the level is accelerating in time at a give point is proportional to the sum of the rates at which the level is accelerating across each dimension in space at that point.

The wave equation is the universal equation of physics. It works for light, sound, waves on the surface of water and a great deal more. It is the relativistic Schrödinger equation that describes the quantum mechanical evolution of the wave function of a single particle with zero rest mass[6].

There are many ways to discretize the wave equation. One of the simplest is to define a grid or array of points as shown in Figure 7.1. Instead of defining the value of a function everywhere we consider only selected points. The more closely these points are spaced the more accurate an approximation to the continuous case and the more time consuming the computation. To keep things simple we will consider two dimensions in space and one in time. It is straightforward to move to three spatial dimensions.

Indices are used to locate points in the grid. $f_{x,y,t}$ is the location x, y in space at time t. This position has four immediate neighbors in space: $f_{x+1,y,t}$, $f_{x-1,y,t}$, $f_{x,y+1,t}$ and $f_{x,y-1,t}$. See Figure 7.1. Similarly it has two immediate neighbors in time: $f_{x,y,t+1}$ and $f_{x,y,t-1}$.

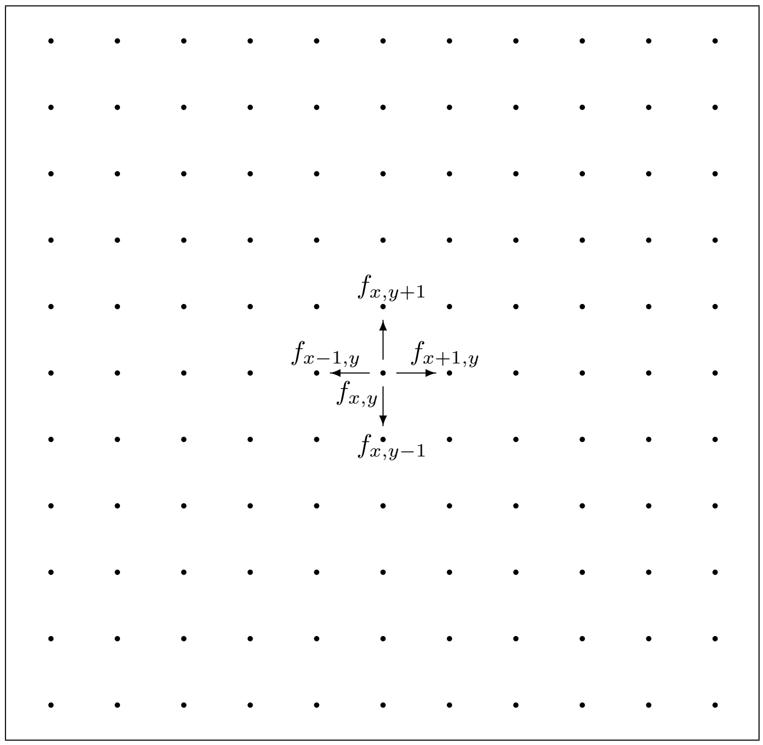
Continuous differential equations are defined by taking the limit[7] of finitely spaced locations as the distance between points goes to zero. The first order difference is computed by subtracting neighboring values along the relevant dimension (time, x position or y position). The first order difference in time is either $f_{x,y,t} - f_{x,y,t-1}$ or $f_{x,y,t+1} - f_{x,y,t}$. The wave equation does not use the first order difference or rate of change. It uses the second order difference or rate of acceleration. To get the second order difference we compute

---

plane or a uniform slope has zero acceleration. It is only when the steepness of the hill is changing that there is acceleration in space. The hill that keeps getting steeper or that bottoms out as you approach level terrain has acceleration.

[6] Any particle has some energy and thus mass. But some particles like photons that make up light travel at the speed of light and are said to have no rest mass. No amount of energy is sufficient to make a particle that has rest mass move at the velocity of light. In contrast a particle with zero rest mass must always move at the speed of light.

[7] Consider a sequence $a_n = \frac{n+1}{2n}$. The limit as n approaches ∞ is $\frac{1}{2}$. No value in the sequence ever equals $\frac{1}{2}$ but each a_n differs from $\frac{1}{2}$ by $\frac{1}{2n}$ which gets arbitrarily close to zero as n goes to ∞.



A finite difference equation uses a grid or array of points. Each point at each time step is affected only by neighboring points as shown by the four arrows from the center point.

Figure 7.1: Discretizing a continuous equation

a difference of differences.[8]

In generating the difference equation from the differential equation we must take into account the time and distance scale of the points on the grid. For this illustration we combine these constants with the velocity of the wave c generating a new constant c_d.

The second order difference is computed by subtracting one first order difference from the other. This is $f_{x,y,t+1}+f_{x,y,t-1}-2f_{x,y,t}$. There are two second order spatial differences for the x and y dimensions. They are computed in a similar way and added together. That is what the notation ∇^2 implies. The result is the following finite difference equation.

$$f_{x,y,t+1} + f_{x,y,t-1} - 2f_{x,y,t} = c_d^2(f_{x+1,y,t} + f_{x-1,y,t} + f_{x,y+1,t} + f_{x-1,y,t} - 4f_{x,y,t})$$

Converting to a form that allows us to compute the next point in time ($f_{x,y,t+1}$) from previous values yields the following equation.

$$f_{x,y,t+1} = c_d^2(f_{x+1,y,t} + f_{x-1,y,t} + f_{x,y+1,t} + f_{x-1,y,t} - 4f_{x,y,t}) - f_{x,y,t-1} + 2f_{x,y,t}$$

To fully digitize the equation we must restrict f to a discrete set of values like the integers. The expression c_d^2 must be less than one if the difference equation is to approximate the continuous wave equation. Thus we must add an element to the equation to eliminate the possibility of non integer values. For simplicity we truncate toward 0. This means 1.8 is truncated to 1 and -1.8 is truncated to -1. We use T to represent this truncation function.

Table 7.1 shows the fully discretized finite difference equation. This figure also shows how an initial state evolves for a few time steps using this equation with c_d^2 set to $1/4$.

---

[8] The second order difference in time is an acceleration. To get an average velocity we divide distance by time. If you go 100 miles in two hours your average velocity is 50 miles per hour. To get an acceleration we divide the change in velocity by time. If you go from 30 miles per hour to 60 miles per hour in 10 seconds the acceleration is 3 miles per hour per second.

Time -1		
0	0	0
0	100	0
0	0	0

Time 0		
0	0	0
0	100	0
0	0	0

Time 1		
0	25	0
25	0	25
0	25	0

Time 2				
0	0	6	0	0
0	12	25	12	0
6	25	-75	25	6
0	12	25	12	0
0	0	6	0	0

Time 3						
0	0	0	1	0	0	0
0	0	4	12	4	0	0
0	4	24	-11	24	4	0
1	12	-11	-50	-11	12	1
0	4	24	-11	24	4	0
0	0	4	12	4	0	0
0	0	0	1	0	0	0

Time 4						
0	0	1	4	1	0	0
0	2	13	6	13	2	0
1	13	9	-34	9	13	1
4	6	-34	14	-34	6	4
1	13	9	-34	9	13	1
0	2	13	6	13	2	0
0	0	1	4	1	0	0

Time 5								
0	0	0	0	1	0	0	0	0
0	0	0	5	5	5	0	0	0
0	0	8	14	-7	14	8	0	0
0	5	14	-25	-14	-25	14	5	0
1	5	-7	-14	30	-14	-7	5	1
0	5	14	-25	-14	-25	14	5	0
0	0	8	14	-7	14	8	0	0
0	0	0	5	5	5	0	0	0
0	0	0	0	1	0	0	0	0

Time 6								
0	0	0	1	2	1	0	0	0
0	0	3	9	2	9	3	0	0
0	3	13	-3	-9	-3	13	3	0
1	9	-3	-34	13	-34	-3	9	1
2	2	-9	13	2	13	-9	2	2
1	9	-3	-34	13	-34	-3	9	1
0	3	13	-3	-9	-3	13	3	0
0	0	3	9	2	9	3	0	0
0	0	0	1	2	1	0	0	0

Time 7								
0	0	1	3	2	3	1	0	0
0	1	8	5	-1	5	8	1	0
1	8	5	-22	0	-22	5	8	1
3	5	-22	-4	9	-4	-22	5	3
2	-1	0	9	-15	9	0	-1	2
3	5	-22	-4	9	-4	-22	5	3
1	8	5	-22	0	-22	5	8	1
0	1	8	5	-1	5	8	1	0
0	0	1	3	2	3	1	0	0

$$f_{x,y,t+1} = 2f_{x,y,t} - f_{x,y,t-1} + T\left(\frac{f_{x+1,y,t} + f_{x-1,y,t} + f_{x,y+1,t} + f_{x-1,y,t} - 4f_{x,y,t}}{4}\right)$$

T is truncation toward 0. For example $T(1.7) = 1$ and $T(-12.9) = -12$.

Table 7.1: Discretized wave equation example

7.5 Unique properties of discrete models

The finite difference equation can only approximate the continuous differential equation for a limited number of time steps. Eventually it will behave quite differently. In the continuous case the amplitude of the wave spreads out at ever greater distances decreasing in amplitude to arbitrarily small levels. In the discrete case there is a limit beyond which this cannot happen.

Discrete systems either diverge or go through a repeating sequence of states. A system that does not diverge has only a finite number of states and therefore must at some point loop through the same sequence. However even small discrete models have an enormous number of states. One hundred integers each with a range of 100 allows for 100^{100} possible states. The time before one must repeat a state in even a very small system can easily exceed the age of the universe.

For a finite difference equation to be a candidate for a physical model it must form stable dynamic structures that go through a repeated sequence of similar states. Such structures could lead to a more complete theory of the fundamental particles of physics. They would exhibit chaotic like behavior.[9] The truncation function

[9] Chaos theory studies the very complex behavior that can be exhibited by continuous nonlinear systems. These are usually far more complex than linear systems. Discretized linear finite difference equations can be made nonlinear by forcing them to assume only

T defined in Table 7.1 is nonlinear. This could induce chaotic like effects. It is plausible that some fully discrete approximations to the wave equation would lead to a variety of dynamically stable structures. These are structures that repeat a similar but not necessarily identical sequence of states. They would be relatively stable in that small perturbations would not significantly affect their average behavior. An initial burst of energy would break up into such structures. These structures could transform into one another under appropriate conditions and with constraints on what transformations were allowed. The chaotic like randomness of their behavior would be fully deterministic but knowing the exact integer value at every point in space would in most circumstances be impossible. It is only these exact values that would support fully deterministic predictions.

The model in Table 7.1 has exact time symmetry. That imposes a strong conservation law that puts limits on possible transformations It implies that all transformations are reversible. Swap the values at two successive time steps and the previous sequence of events will reoccur in reverse order.

This is all speculative but such structures could provide an explanation for wave particle duality and support a physical wave function collapse. There are many difficulties with this possibility. One aspect of quantum mechanics, quantum entanglement (discussed in Section 7.11), contradicts any model of this class. We discuss the experiments to test this in Section 8.6.

The great difficulty with these class of models is their enormous complexity. The basic rules are simple but any attempt to model even the smallest of fundamental particles would require an enormous simulation. Of necessity mathematicians and physicists work with mathematical models they can solve. Nature is under no such constraint. We further discuss the possible behavior of discrete models in Section 8.7.

One objection to this class of models is the crude truncation toward zero. No doubt if such models are the way nature works there

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integer values as we did using the truncation function T. This can make the behavior of the discretized difference equation for more complex than the linear differential equation from which it was derived although it is not chaotic because it is a discrete and not continuous system.

is something more elegant involved. Edward Fredkin has pursed a potentially more elegant approach in using Cellular Automaton as models[54]. The problem with cellular automaton is simulating the wave equation that can grow to very large amplitudes. There are solutions but they may not be elegant either.

7.6 Einstein's approach to physics

Einstein felt that the fundamental structure of the universe must be simple and elegant. Of course the universe does not need to conform to anyone's preconceptions. But Einstein's physical intuition was profound and extraordinarily fertile.

> There is no doubt that quantum mechanics has seized hold of a beautiful element of truth and that it will be a touchstone for a future theoretical basis in that it must

> be deducible as a limiting case from that basis, just as electrostatics is deducible from the Maxwell equations of the electromagnetic field or as thermodynamics[10] is deducible from statistical mechanics.
>
> I do not believe that quantum mechanics will be the starting point in the search for this basis, just as one cannot arrive at the foundations of mechanics from thermodynamics or statistical mechanics[20].

Einstein regarded quantum mechanics as a statistical theory that ignores the detailed mechanisms that generates those statistics just as thermodynamics ignores the details of mechanics. Thermodynamics is a statistical theory that describes the average behavior and global effects of large numbers of particles. Dynamics is the detailed theory that describes the behavior of individual particles. Derivations can only go from the mechanistic theory to the statistical theory and never the other way.

As we mentioned at the beginning of this chapter, near the end of his life Einstein came to suspect that this more complete theory lay in discrete as opposed to continuous structures. In a paper[21] so famous it is referred to by the initials of its authors (EPR for Einstein, Podolsky and Rosen) Einstein and his colleagues started a chain of argument that has led to a possible experimental path to the more complete theory Einstein felt must exist. Those experiments are discussed in Section 8.4.

[10] Thermodynamics is the study of heat. Initially heat was thought of as a liquid that flows. Eventually it was discovered that heat is a measure of the average random motion of molecules. Thermodynamics studies the macroscopic aspects of heat as if it were a fluid. It ignores the motion of individual molecules. Thus it is a statistical theory.

7.7 Special relativity

We have an intuitive notion of space and time that seems natural. But space and time have a structure just as objects that exist in space and time do. That structure is not quite as it appears. For example it seems obvious that the shortest distance between two points is a straight line. However we live not on a plane but on the surface of a sphere and the shortest distance is not necessarily a straight line. It is a great circle route. Special relativity is a theory about the structure of space and time that is a radical departure from our intuitive ideas. Relativistic effects are only noticeable at relative velocities much faster then we encounter in every day activities. The first relativistic effects were observed in trying to measure the earth's motion through space.

The waves that ripple out from a rock thrown in a pool are structurally similar to sound waves. When you speak the air vibrates with a pattern of pressure changes in the form of waves. Light was shown to have many of the properties of waves. Unlike sound light travels through empty space. Whatever contains the changing levels of pressure associated with a wave is called its medium of propagation. With sound this is air and with water waves it is water. Scientists naturally wondered what supports the propagation of light in empty space. They assumed there must exist such a medium and they called it the ether.

The earth circles the sun which in turn circles the galaxy which in turn moves away form neighboring galaxies. The absolute motion of the earth relative to the ether must be very fast. By measuring the speed of light in one direction and then the opposite direction one should be able to determine the absolute motion of the earth through the ether. When the earth is moving in the same direction as the light beam the speed relative to the earth will be slower because light has to travel not only the distance between two points on earth but also the distance the earth moved in the time between the measurements at the two locations. When moving in the opposite direction the speed is higher because the speed of the earth is added to, not subtracted from, the speed of light. Take the difference of the two speeds and divide by two and you have the absolute speed of the earth.

Michelson and Morley devised an ingenious experiment to make these measurements. It was difficult because light travels at approximately 300,000 kilometers a second and the motion of earth through space was expected to be only a tiny faction of that speed. The result was that one could detect no difference. The speed of light was the same in all directions.

Einstein explained this mystery by assuming the experiment results were correct and generalizing them. He assumed the speed of light is the same no matter how fast one is moving relative to a light beam. A system that is moving at uniform speed (not accelerating or decelerating) in a constant direction is called an inertial frame of reference. Einstein assumed that not just the speed of light but all physical measurements would be the same as long as the experiment was carried out in an inertial frame of reference. This implies that our measurements change depending on how fast we are traveling relative to the object being measured. Time itself slows down as we approach the speed of light. If we could travel as fast as light time would stop completely.

At first this seems absurd. We think of distance as being absolute and not relative to how we measure it. Mathematically topology is independent of distance. We can set up a topology or mathematical set of points and then impose on this any distance function we choose. To specify location one must describe the connectedness of the geometry. For example we can use pairs of real numbers to specify the ordering of points in a two dimensional space. The connectedness of the points is determined by the ordering of pairs of real numbers (x, y). We can then assign a distant function. The standard Euclidean distance function between (x_1, y_1) and (x_2, y_2) is as follows.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

There are many other possible distance functions. For example if one lived in a world where diagonal motion was not possible and one could only move parallel to the x or y axis the distance function would be as follows.

$$d = |x_2 - x_1| + |y_2 - y_1|$$

Note $|x|$ denotes the absolute value of x. If x is negative $|x| = -x$ otherwise $|x| = x$.

The relativistic distance function is strange because the measurements two astronauts make of each others space ships depend on the relative speed of the two ships. In relativity there is no absolute speed and no absolute frame of reference against which absolute speed could be defined. This does not mean that relativity is inconsistent with assuming there is an absolute reference frame. On the contrary one can assume any inertial frame of reference is the absolute reference frame. Relativity treats all inertial frames the same.

In special relativity the Euclidean distance function works as long the object being measured is not at motion relative to the measuring apparatus. If it is in motion than the measured distance will be less than the measurement made of the same object at rest by a factor of $\sqrt{1 - v^2/c^2}$ where v is the relative velocity and c is the speed of light. This comes from the Lorentz transformation named for its inventor. Einstein did not create the equations of special relativity. He interpreted them. Different observers moving at different velocities (different frames of reference) will get different measurements of the same object and all of them will be correct. The Lorentz transformation shows how to translate the measurement in one frame of reference to a measurement of the same object in a different frame of reference.

7.8 General relativity

General relativity is about gravity or the attraction that any object has for any other object. We think of gravity as what keeps us on the ground, The earth's gravity is a major factor in our lives. But gravity is universal. Every object attracts every other object. The prior Newtonian theory of gravity was a bit of a mystery. It required action at a distance. Einstein solved the mystery by showing that gravity warps space and time. It does so in a way that is called local. Gravity propagates through space at the speed of light.

Special relativity is based on the simplifying assumption that the laws of physics are the same in any inertial frame of reference. General relativity is based on the simplifying assumption that gravity and acceleration are indistinguishable. If your ship in deep space accelerates at just the right rate you will feel the same force pushing you toward the floor that you feel on earth.

Special relativity suggests that time and space measurements transform so that one cannot detect absolute motion. Philosophically (but not mathematically) special relativity denies the existence of an absolute frame of reference. General relativity suggests that mass warps space and time to appear just as they do to someone under uniform acceleration in deep space.

General relativity implies that space can be warped by a massive object to the point space turns in on itself. An object that produces a gravitational field this strong is called a black hole. External objects can fall into a black hole but nothing can escape. Black holes come in all sizes. The creation of a black hole depends on both the mass of an object and how small it is. The force of Gravity falls off with the square of the distance. Thus in theory any mass no matter how small could create a black hole if it was packed into a small enough space. This is part of the problem in combining relativity and quantum mechanics. At small enough distances the quantum mechanical uncertainty in density can be large enough to allow for the creation of black holes.

7.9 Quantum mechanics

> While the mathematical developments which constitute quantum mechanics have been outstandingly successful in describing all manner of observed properties of matter, it is fair to say that the conceptual basis of the theory is still somewhat obscure. I myself do not properly understand what it is that quantum theory tells us about the nature of the physical world, and by this I mean to imply that I do not think anybody else understands it either, though there are respectable scientists who write with confidence on the subject[39, p 95].

The above quote by Ian Lawrie is, no doubt, a minority opinion among professional physicists. But I suspect Lawrie may be more of a minority in his frankness than in his opinion. What quantum mechanics says is very strange. If you think you understand what it means you are almost certainly wrong. What one can understand is the structure of the mathematical theory and how experimental techniques are used to test the predictions of the model. In this and the next two sections we give an outline of that structure.

Models in Newtonian physics start with an initial state that is assumed or observed. To this state one applies a mathematical model that describes how the state evolves or changes over time. For example $x = x_0 + vt$ says the position x of an object is given by the initial position x_0 plus the velocity (speed and direction) of the object multiplied by time. If a ball is one foot away from you and moving further away at 2 feet per second it will be 3 feet away one second later.

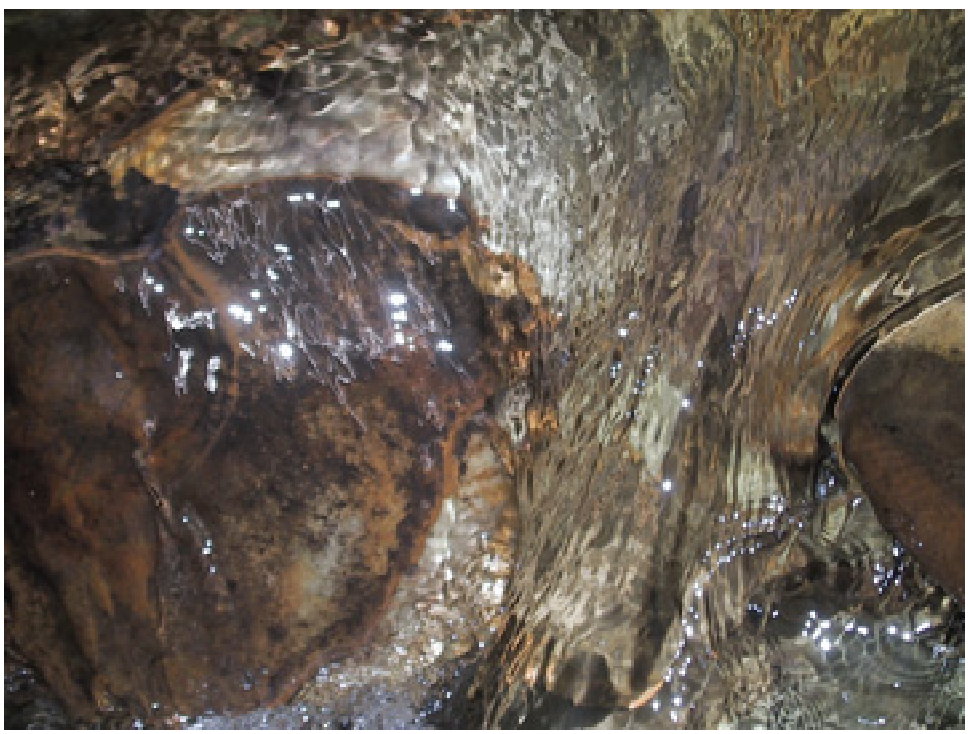
In classical physics the state of a particle is continuously described by an equation. We can directly map variables in the equation to physical quantities that we can measure. x is position and t is time in the model $x = x_0 + vt$. The variables in quantum mechanics do not represent physical state but rather probability densities. What evolves over time is not the physical state but the probability density that the object will be observed in a given state.

Probability density gives the likelihood that a physical observable like position will assume a given value. Thus the probability density function must range over all possible values that the observable might be seen at. Consider a ball rolling down a frictionless inclined plane or ramp. A simple equation describes the ideal motion of the ball. However real balls are never perfectly round and real ramps are never frictionless. In an accurate enough experiment both these effects would be observable and we could no longer use an exact equation to predict the position of the ball. Instead we would need to use a probability density that would describe the most likely position of the ball as well as any position the ball might be found at no matter how unlikely.

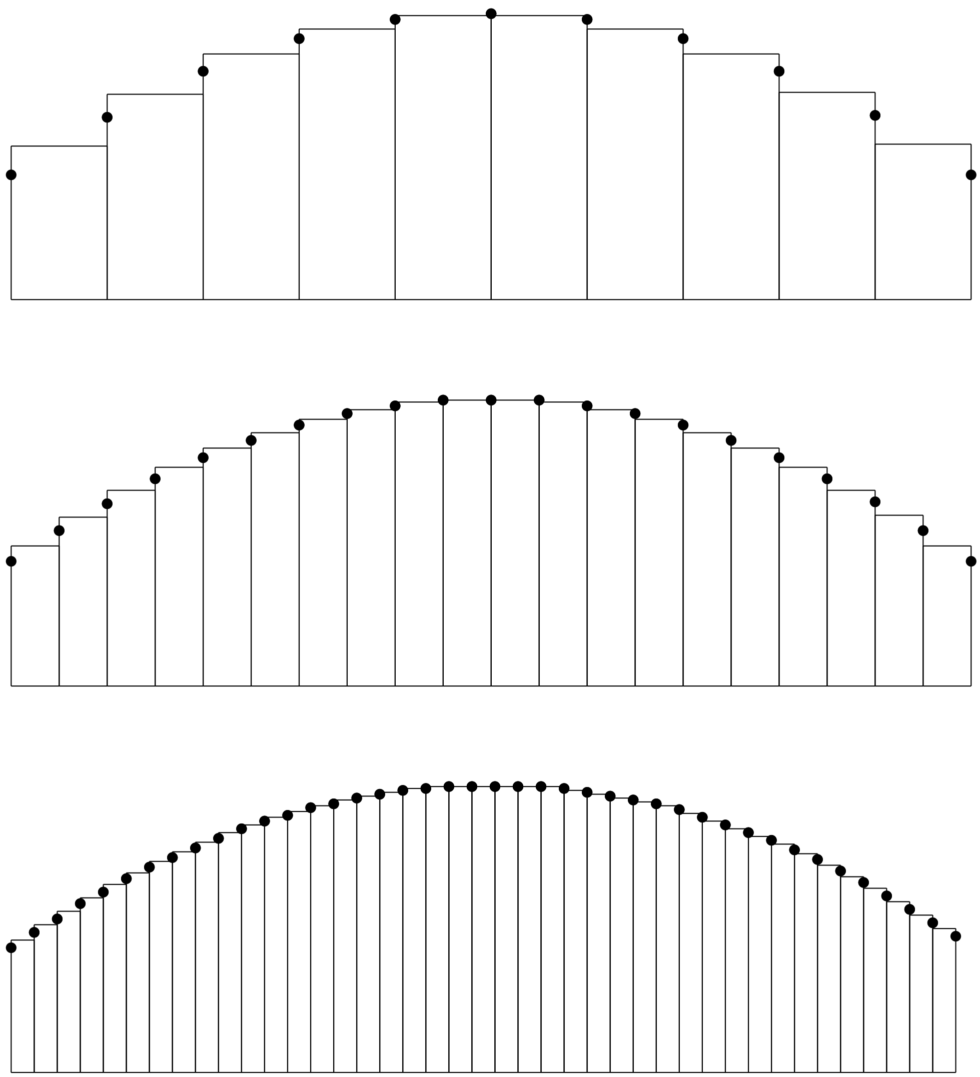
A probability density function $p(x)$ gives the relative probability that the ball will be at location x. If it is twice as likely to be at x_0 as it is to be at x_1 then $p(x_0) = 2p(x_1)$. The probability that the ball will be at any exact location is usually 0 since there are an infinite number of possible locations where the ball might be. Instead of asking for the probability that the ball will be at an exact location we can ask for the probability that it will be in some range for example between x_0 and x_1.

To compute that we evaluate $\int_{x_1}^{x_2} p(x)\,dx$. An integral ($\int$) is the limit of a sequence of additions. One can think of an integral as the computation of the area under a curve. One way to estimate the area is to break the region into rectangles that approximate the area. See Figure 7.2. The more rectangles we use the more accurately we can estimate the area. Some integrals can only be evaluated numerically by doing something similar to that shown in the figure. But for many functions we can compute an exact solution.

7.10 The uncertainty principle

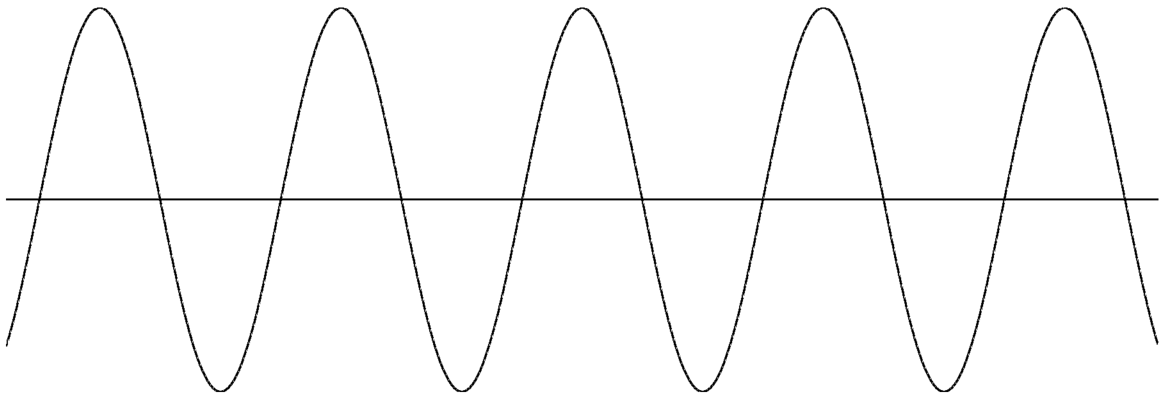


One naturally links the uncertainty principle with the probabilistic nature of quantum mechanics. They are connected but



We approximate the area under a continuous function by adding up the area of a finite number of rectangles. The dots represent the actual value of the function. The height of each rectangle is the average value over the width of the rectangle. As we increase the number of of rectangles we get a more accurate estimate. For many functions we can compute the integral exactly. Others functions can only be integrated numerically by using a technique like the above.

Figure 7.2: Integration as the limit of sums



A pure tone or pure sine wave extends infinitely far in both directions with no change in the size of the peaks or valleys. The frequency in cycles per second is the number of peaks that occur each second.

Figure 7.3: A sine wave or pure tone

they involve different principles. The probabilistic nature of quantum mechanics comes from the lack of a physical state model. Only the evolution of probabilities is modeled. The uncertainty principle comes from the wave structure of these probability densities.

In classical physics one cannot simultaneously measure both the frequency and location of a wave. Figure 7.3 is the graph of a sine wave or pure tone. Objects that vibrate or resonate tend to move in a pattern resembling this figure. A musical instrument that produces a pure tone will be very close to the graph. Frequency in cycles per second is the number of peaks that occur each second. More complex waves can be regarded as the sum of many pure tones of different frequencies. Notice that the peaks and valleys are all the same size. A pure tone or pure frequency extends infinitely far in each direction with no fall off of amplitude. Thus no real sound is a pure tone.

Pure tones have an exact frequency and impulses have an exact position. An impulse is a vertical line at a single point. It is zero except at one point. An explosion will generate a sound approximating an impulse. There are no exact impulses just as there are no pure tones. Real explosions extend over some time. An impulse is the sum of *all* frequencies. Figure 7.4 shows how an impulse can be approximated by adding many pure tones.

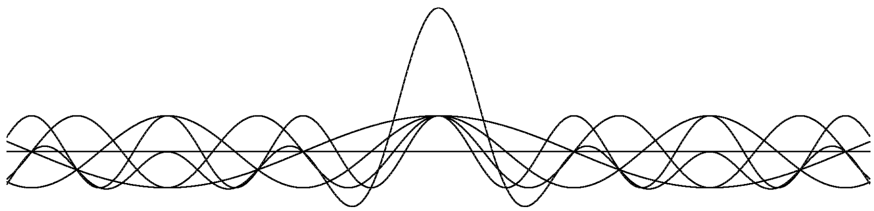
A wave that is narrowly constrained in position has a broad range of frequencies. A wave that is constrained to a narrow range of frequencies extends over a large distance in position as Figure 7.4 illustrates. Mathematicians and scientists do not say the exact position or frequency of a wave is uncertain. Rather they are undefined or meaningless. There is no uncertainty in the structure of the wave or its evolution in time in either classical or quantum mechanics. However in quantum mechanics a wave is used to compute the probability that some event will be observed.

We can control the shape of the wave by the structure of the experiment. If we use a massive particle we can measure position accurately but not velocity because the wave function for a massive particle is focused in a narrow' range of positions but has much more variance in velocity. Mathematically this tradeoff is identical to the tradeoff between narrowly constraining a classical wave's location and narrowly constraining its frequency.

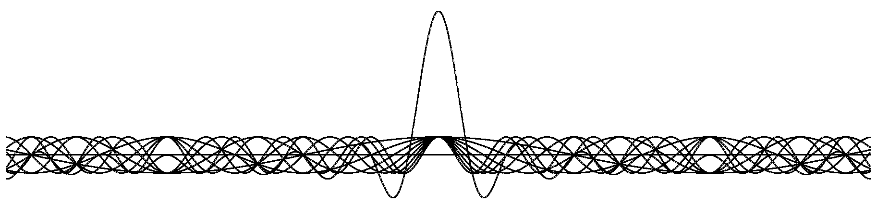
Uncertainty in quantum mechanics is not connected to the probabilistic nature of the wave function. It is inherent in any wave function including those in classical physics. The inability to assign exact position and momentum to a particle may mean that there is no such thing. The inability to make those assignments need not be an obstacle to deterministic predictions. For classical waves, frequency and location cannot be simultaneously assigned, but those models are completely deterministic.

We mentioned in Section 7.5 that a discrete model would not have point like particles but dynamically stable structures that approximate classical waves. Such structures would not have an exact position. However when they interact and transform their structure there would be a focal point of that transformation. Experimental conditions would determine how precise that focal point is just as the precision of the lenses in a camera determine how much detail is resolvable in pictures the camera takes.

The inability to simultaneously constrain a particle's position and momentum is fundamental to the wave structure of quantum



4 sine waves and their sum.



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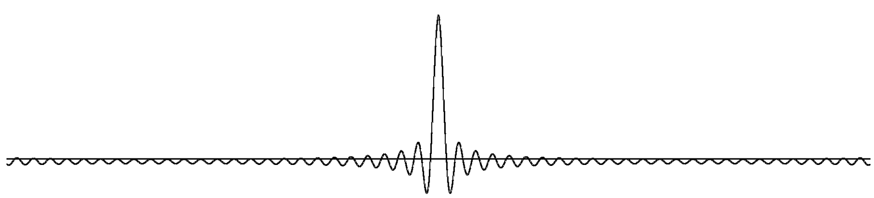
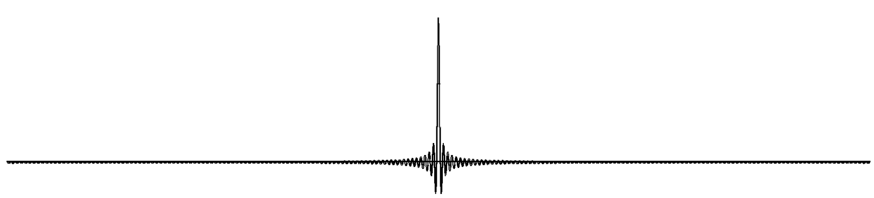


The sum of 32 sine waves.



The sum of 128 sine waves.

Figure 7.4: An impulse has all frequencies

mechanics. The question of predicting the outcome of experiments is an independent one. For a hypothetical model based on stable dynamic structures it might be possible to predict the exact outcome of experiments even though the exact position and momentum of particles is not known because it has no more meaning than the exact location and position of a classical wave. If the evolution of such structures is deterministic then one might, in some special cases, know enough to make exact predictions.

In 1932 the renowned mathematician and physicist von Neumann published a proof that no more complete theory could be consistent with the predictions of quantum mechanics[51][50]. Von Neumann's reputation was so great that the proof stood for thirty years in spite of Bohm's publication in 1952 of a more complete theory that was consistent with quantum mechanics [8]. Bohm thought at the time there were subtle differences in the predictions of the two theories and thus his result did not contradict von Neumann's proof.

In 1966 Bell published a paper revealing a problem with von Neumann's proof[6]. The mathematics was fine but the assumptions von Neumann made about the constraints a more complete theory had to meet were not justified. Bell went on to show that quantum mechanics was not a local theory. Bohm's theory was an explicitly nonlocal theory and Bohm's work was an important influence for Bell. This story continues in Section 8.4.

There are many properties like frequency and location that cannot be simultaneously measured with high accuracy. Such pairs are said to be non commuting. None of this says anything about uncertainty or lack of predictability. That is a separate issue. Bohm showed one way a more complete theory can be constructed. If there is one there are many.

7.11 Quantum entanglement

Quantum mechanics make two seemingly incompatible assumptions. It assumes conservation laws[11] are absolute and it assumes probabilities are irreducible. In classical physics there are mechanisms that explains how the conservation laws are 'enforced'. By claiming probabilistic laws are absolute one precludes the possibility of an enforcing mechanism.

[11] We have all heard for every action there is an equal an opposite reaction. This is an informal statement of the law of conservation of momentum. Momentum is the product of velocity and mass. Assume a 1000 pound object traveling at 10 miles an hour smashes head on to a 100 pound object traveling at 100 miles an hour. The two objects will have equal and opposite momentum. They will both come to a dead stop. This is required by the conservation of momentum. If a large truck smashes head on into a massive concrete building the earth itself (or at least a portion of it connected to the buildings foundation) will move to conserve the momentum of the truck. There are many other conservation laws.

This leads to a serious issue. When a pair of particles are created and travel off in opposite directions they must have exactly equal momentum in opposite directions. Yet observing the momentum of either of these particles is statistical. The result can vary over a range of values. But once an observation is made, we know with similar accuracy the momentum of the other particle which may be far away even on the other side of the universe.

Einstein explained this difficulty in the previously cited paper known by the initials of its authors EPR[21]. Einstein and his colleagues concluded that quantum mechanics must be incomplete. For momentum must have an objective reality independent of each observation if momentum is conserved absolutely. Nature is doing a sort of cosmic bookkeeping to make sure momentum is never created or destroyed and their needs to be a mechanism not part of any existing theory that implements this accounting procedure. The principle on which Einstein's argument is based is uniformly true in classical physics. It was argued that this principle does not apply to quantum mechanics. The debate is closed for most physicists and decided against Einstein. I suspect Einstein will ultimately be proved correct.

Probabilistic observations and absolute conservation laws leads to quantum entanglement. In classical physics state evolution is local. If particles are far apart they cannot affect each other except by transferring some signal at a speed that cannot exceed that of light. Because of quantum entanglement this is not true of quantum mechanics. Observations of one particle can instantaneously put constraints on observations of a second particle with which the first has become entangled *even if the two particles are a billion light years apart*. Every corner of the universe can be instantaneously influenced by every other corner of the universe. The physicist David Bohm embraced this nonlocal model as support for a philosophy that saw the universe as whole[9]. But is it true? And how can it be true and relativity also be true?

An absolute contradiction with special relativity is avoided through quantum randomness. The effects transmitted instantaneously are in effect encrypted with quantum uncertainty. Such effects always involve two events at distant locations A and B. Violations of classical locality that are consistent with the predictions of relativity always happen in a way that it is impossible to know if the effect goes from location A to B or vice versa. In some inertial frames of reference it will be seen to go in one direction and in other frames of reference it will be seen to go in the opposite direction. Neither relativistic view of the situation contradicts the predictions of quantum mechanics because quantum randomness is claimed to be absolute and not reducible to some causal model that would show how the effect operates.

This factor also makes it impossible to send a faster than light signal using quantum entanglement. One can only tell that information has been transferred instantaneously by comparing the results at A and B and that comparison requires that information be transferred no faster than the speed of light.

This is a very strange situation. The inability to separate space into local regions that are causally independent is fundamental to the idea of special relativity yet it is egregiously violated in quantum mechanics. Yet there is no contradiction in the predictions of the two theories. The next chapter explores these and other issues in integrating these two theories that have an uneasy coexistence at the core of contemporary physics.

7.12 The real line and configuration space

At the start of this chapter we observed that the continuous real line that we see is a creation of our brain and nervous system. Everything we see and touch is made up of fundamental particles. Although objects in space-time are discrete, space-time itself remains continuous in special and general relativity. Those theories are so dependent on the classical continuum that Einstein recognized any fully discrete theory would imply relativity was only approximately true and would make false predictions at the scale of space-time discreteness (see Section 7.1).

Space-time is very strange in quantum mechanics. It remains continuous but it has a peculiar connectivity because of quantum entanglement. In classical physics and relativity space is separable. You can fully describe what happens in any localized region over a brief time interval without taking into account distant events. This is not possible in quantum mechanics.

The nonrelativistic version of quantum mechanics exists, not in physical space, but in an abstract higher dimensional structure known as configuration space where there is a single time dimension and a separate set of spatial dimensions for every particle. See Figure 7.5. The connection between configuration space and physical space is through a probability distribution which gives the probability that a given *configuration* of particles will be observed.

In physical space we do not have anything like the classical real line. What exactly we do have is not clear since the actualization of probabilities in configuration space to events in physical space is not part of any existing scientific theory.

A mathematical model from a scientific theory may have little to do with how nature is structured. Obviously it must provide an accurate approximation in its experimental predictions. Classical mechanics is very accurate for a wide range of experiments but quantum mechanics has shown that the structure of physical reality must differ radically, Some physicists argue that it is naive realism to expect a correspondence between nature and our mathematical models. While one must admit that anything is possible, and there are many aspects of existing theory that make it seem difficult to construct such a correspondence, I suspect that those are problems in the existing theories and our understanding of nature. Mathematics can model what nature does to extraordinarily accuracy and this leads me to suspect that nature has a mathematical structure.

Quantum entanglement is at the core of the strangeness in contemporary physics. The evidence that distant events influence each other, in ways that can never be explained by a local mechanism, Two particles p_1 and p_2 in physical space have trajectories defined by equations with parameters for the three spatial dimensions and one dimension in time.

$$p1(x, y, z, t)$$

and

$$p2(x, y, z, t)$$

In contrast quantum mechanics models a joint probability $p_{(1,2)}$ that one can detect *both* particles in a particular *configuration* that has p_1 at location x_1, y_1, z_1 and p_2 at location x_2, y_2, z_2.

$$p_{(1,2))}(x_1, y_1, z_1, x_2, y_2, z_2, t)$$

This equation has 7 parameters to give probabilities for all the possible *combinations* of locations of the two particles. For n particles configuration space requires $3n + 1$ dimensions. There are three for the position of every particle and one for time. In general it is not possible to separate out probability models for p_1 and p_2. In many instances particles become entangled in a way that observations of one particle imposes constraints on observations of the other particle. This entanglement is discussed in Sections 7.11 and 8.4.

Figure 7.5: Configuration space

is dramatic and compelling but not totally conclusive (see Section 8.6). Experiments, as always, will decide the issue but what we make of experiments and how decisive they need to be is a matter of judgment. One of the factors that goes into such judgments is our sense of what alternative possibilities exist. Fully discrete models would be radically different than anything previously investigated. They hold the possibility of the more complete theory Einstein sought. There may be an experimental path that leads to such a theory. All of these issues fall under the problem of integrating relativity and quantum mechanics which is the subject of the next chapter.

Chapter 8

Relativity plus quantum mechanics

... Einstein never had a good word for the relativity version of quantum mechanics known as quantum field the-

> ory. Its successes did not impress him. Once, in 1912, he said of the quantum theory the more successful it is the sillier it looks. When speaking of successful physical theories, he would, in his later years, quote the example of the old gravitational theory. Had Newton not been successful for more than two centuries? And had his theory not turned out to be incomplete.[41, p 24]

There is no theory that combines quantum mechanics and general relativity. In quantum mechanics the greater the accuracy of a measurement of location the more uncertainty there is a in the measurement of energy. The uncertainty principle applies not just to particles but also to empty space. Over very short intervals phantom or virtual particles can appear. The shorter the time, the more massive the particles can be. At very short intervals, virtual particles will be massive enough to form black holes. One cannot extrapolate simultaneously both quantum mechanics and general relativity to minute distances. The theories explode or diverge.

Quantum field theory combines *special* relativity and quantum mechanics in a problematic structure. Practical experiments can test the questionable aspects of quantum field theory using Bell's inequality (see Section 8.4). It is only the predictions of quantum field theory and not the *mechanism* that generates those predictions that are relativistic.

This is possible if probabilities are absolute. For that allows two events to affect each other without being able to determine which is the cause and which is the effect. The direction of the causal arrow is masked by quantum randomness. A mathematical model (like quantum field theory) that creates these predictions has to make a choice and, in making that choice, it violates relativity. Many physicists would not agree with this statement claiming only *realistic* models have this problem. Such arguments depend on one's philosophical view of quantum randomness. The absolute probabilities claimed for quantum mechanics have no mathematical definition and this leads to philosophical debate (see Section 8.2).

General relativity and quantum mechanics have disjoint experimental domains. General relativity is only observable with massive objects. Quantum effects are only observable with minute particles. Thus these incompatible theories can coexist in a temporary truce. Eventually something has to change.

This has created an unfortunate situation in contemporary foundations research. The hottest research area for extending theoretical physics is combining these theories. The experimental domain in which such combinations could be tested is unreachable with existing and foreseeable technology. The situation is not unlike that in mathematics where fundamental research focuses on properties of large cardinals when no infinite sets let alone large cardinals may exist. Reconciling the two fundamental physical theories is a mathematical exercise that may be devoid of physical content.

8.1 Locality and quantum mechanics

Locality is the denial of action at a distance It requires that all the information useful in predicting what will happen at a given location and time is contained in a sphere of influence. For an event that will occur in one second the sphere has a radius of 300,000 kilometers, the distance light travels in one second[1].

[1] In contemporary physics the speed of light is assumed to define locality. In general

Locality is the most powerful simplifying assumption in physics. Without it any event no matter how distant can influence any other event. Prediction would be impossible without locality or some other powerful restriction on what events can affect other events. Otherwise one would need to know the state of the universe to predict anything. Quantum mechanics is a local theory in configuration space but not in physical space.

As mentioned in Section 7.10 Bell refuted von Neumann's proof that no more complete theory could be consistent with quantum mechanics. In proving this Bell was influenced by Bohm's development of a more complete theory that was explicitly nonlocal. This led him to a proof that no local theory with hidden variables could reproduce the statistical predictions of quantum mechanics. Hidden variables were defined by Bell in a general way to include any more complete theory with a mechanism for explaining the conservation laws. He suggested that it should be possible to test some of the nonlocal predictions experimentally[5].

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locality is satisfied if there is any speed that limits the rate at which effects can propagate.

8.2 Realistic theories and randomness

Often Bell's result is presented as showing that quantum mechanics is not a realistic theory rather than showing that it is non-local. The focus is on the reference to hidden variables in Bell's proof. Eberhard developed a version of Bell's argument that did not involve hidden variables[19]. In turn some physicists objected to Eberhard's proof because he assumed "contrafactual definiteness". That is he assumed one could argue about all possible outcomes of an experiment including those that did not happen.

Arguments like those about hidden variables and contrafactual definiteness are philosophical. They have no clear resolution unlike problems that can be formulated mathematically. Such arguments are rare in the hard sciences. They occur here because of the claim in quantum mechanics that probabilities are fundamental or irreducible.

There is no mathematical model for irreducible probabilities.

There is not even a mathematically definition of a random number sequence. There are sequences that are *recursively random*. Loosely speaking this means that no recursive process can do better than chance at guessing the next element in the sequence. The problem with recursively random sequences is that they are more complex than any recursive sequence. If somehow one could generate such a sequence one could use it to solve recursively unsolvable problems.

This suggests that a *truly random* sequence cannot exist. Any sequence that is 'truly' random must be recursively random. Otherwise there is some computer program that can guess with some degree of accuracy the elements in the sequence. Yet *no* recursive random sequence can be *truly random*. This presents a philosophical problem for the claim that quantum mechanics is truly random.

The randomness claimed for quantum mechanics has no foundation in mathematics and it appears to be impossible to construct such a foundation. This does not make it wrong but suggests there are problems in our existing conceptual framework. It also means that physicists when arguing about these issues are debating philosophy with no objective way of deciding the issue. My prejudice is with Einstein. I do not see a need to go beyond conventional logic or mathematics. I only see a problem with developing a better theory.

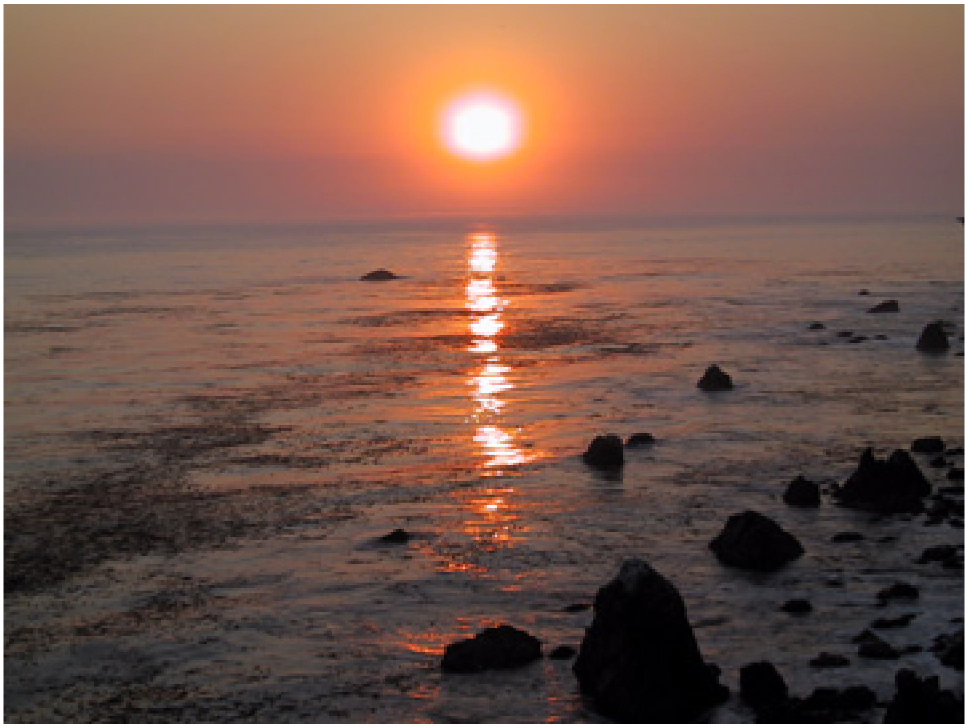
Understanding Bell's result can be difficult even though it is simple. It involves phenomena that can only exist in a theory like quantum mechanics in which probabilities are irreducible. At the macroscopic level of everyday experience the world seems to be causal. What is meant by an experimenter influencing a result is straight forward. In quantum mechanics this is not the case. An observation can be influenced by an experimenter but it usually also has a probabilistic component. Thus we can never tell how much the experimental manipulation contributed to the final observation.

Consider an experiment in which a pair of photons (particles of light) are emitted in a single event such as a particle decay. The conservation of momentum requires that the two photons be emitted in *exactly* opposite directions. Yet one cannot measure the position of a particle perfectly. Measuring the position of one particle puts constraints on the position of the other particle. Of course the same thing is true in classical physics. Information about each particle is implicit in the trajectory of the other particle. The difference is that quantum mechanics denies the existence of a particle trajectory independent of a series of position measurements. Depending on how the experiment is setup, the measurement will fall within some range of possible values that could be large. Measuring one particle's position with high accuracy gives the other particles position to a similar accuracy. Before that first measurement there was far more uncertainty in the second particles position. So if the particle does not have a classical trajectory, does our first measurement actually influence the second measurement? It cannot do so in a direct causal way without violating relativity. But suppose a pair of measurements are made simultaneously on both particles. Could there be a correlation between those two measurements that implies non local influence without either measurement affecting the other in a way that would violate relativity?

Bell proved the answer to that question is yes. An experimental manipulation, like changing the angle of a polarizing filter, can be involved in a measurement. Bell proved that experimental manipulations can influence a distant detection instantaneously if quantum mechanics is correct. This is only possible when there are a pair of experimental manipulations and distant detections. It is never possible to know which experimental manipulation affected which detection but it is possible to measure the influence that one of them had. The probabilistic element in the measurements is sufficient to mask any information about which measurement influenced which distant observation, but the mathematics requires that one of the two measurements did influence the more distant observation. The order of the measurements can be different in different frames of reference[2] but it does not matter because the causal affect goes in a direction that is indeterminate.

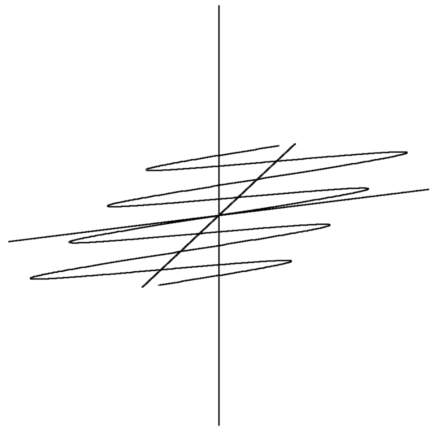
[2] In special relativity two events are said to be space-like separated if their separation in space exceeds the distance light can travel in the time between the two events. The order that such events seem to occur depends on the inertial frame of reference. Thus two events, like the measurements in tests of locality in quantum mechanics, will occur in a different order in different frames of reference.

8.3 Polarized light

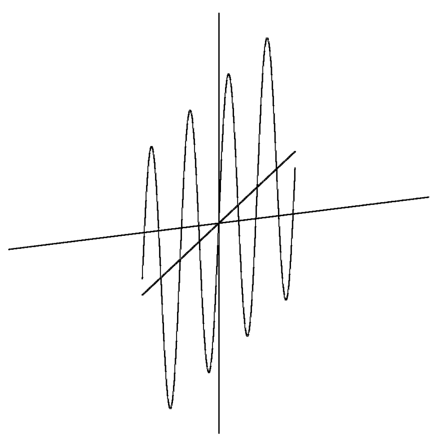


Bell's result can be explained using polarized light. There are two motions associated with a wave. Light travels in one direction and the field strength changes as light passes through a fixed point. The field strength change can occur in any direction perpendicular to the direction of motion. Light is polarized in the direction that the field level changes. See Figure 8.1. Each photon or particle of light has an angle of polarization. We say a source of light is polarized when most of the photons are aligned in a single direction. There are many ways light can become polarized. Light reflected at a shallow angle is polarized to some degree. That is why polarizing sun glasses can reduce glare.

An ideal polarizing filter only allows that component of light to be transmitted that is parallel to the axis of polarization of the filter. If the angle between the axis of polarization of light and the polar-



Wave moving along the Z axis and polarized horizontally or parallel with the X axis.



Wave moving along the Z axis and polarized vertically or parallel with the Y axis.

Figure 8.1: Polarized waves

izing filter is θ then the amplitude of the transmitted light is $\cos(\theta)$[3]. See Figure 8.2. If a a single photon encounters a polarizing filter it must either completely traverse the filter or be completely blocked. It cannot split into smaller particles. However the classical relationship must hold in a statistical sense. The probability that a single photon will traverse the filter must be such that statistically the predictions of quantum mechanics and classical physics will agree.

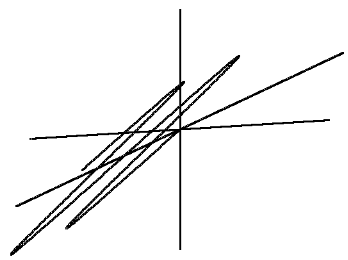
The strangeness of quantum mechanics makes it difficult to describe these experiments coherently. On the one hand the photon does not have a definite polarization until and unless it is detected. Yet it is difficult to avoid talking about the angle between the photon's polarization and the filter. There is no good way to deal with this.

Consider the experiment illustrated in Figure 8.3. There are two polarizers at a $90°$ angle. This blocks all light since the light coming out of the first polarizer behaves as if it is polarized at a $90°$ angle relative to the second polarizer. $\cos(90°) = 0$. Now insert a third polarizer between the two existing polarizers at a $45°$ angle relative to both of them. The amplitude coming out of the second polarizer is proportional to $\cos(45°) = 1/\sqrt(2)$. The amplitude coming out of the third polarizer is $\cos(45°)^2 = 1/2$.

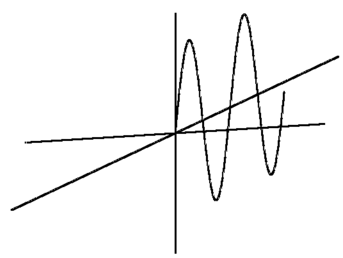
Thus it would seem that the second polarizer changes the angle of polarization of the photons that passed through it Otherwise nothing would make it through the third polarizer. But that assumptions leads to problems that will be explained shortly.

Many physicists believe it is not meaningful to talk about what is happening in physical space between observations. Of course that does not prevent them from doing so. Its almost impossible not to, but one has to be careful about taking such talk too seriously. At best it is metaphor and intuitive guide. Bohm succeeded in giving a consistent theory that talks about what the particle is

[3]The sin and cos functions are common in physics. For example they give the amplitude of a perfect tone as a function of time. Even the change in the length of a day over the course of a year is approximately a sin function with the 0 crossings (where the change in length goes from positive to negative or vice versa) occurring at the summer and winter solstices. The two functions are identical in shape but start with different initial values. $\sin(0) = 0$ and $\cos(0) = 1$. Figure 7.3 is a sine function.

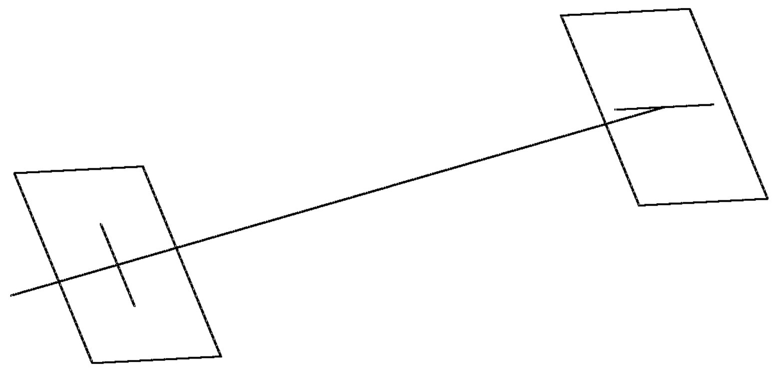


The input wave is traveling along the Z axis. Its polarization angle is $30°$ relative to the polarizing filter which is aligned with the Y axis.

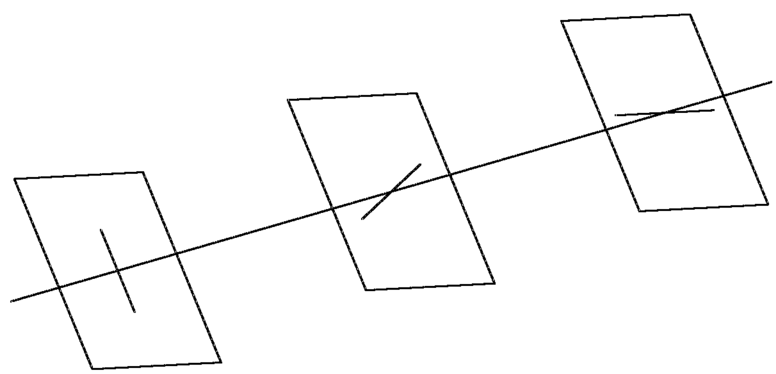


The output wave is polarized along the Y axis aligned with the filter. Its amplitude is $\cos(30°)$ times the input amplitude.

Figure 8.2: Polarizing filter



With polarizers at a 90° angle no light gets through.



Add a third polarizer in the middle at an intermediate 45° angle and half the light gets through.

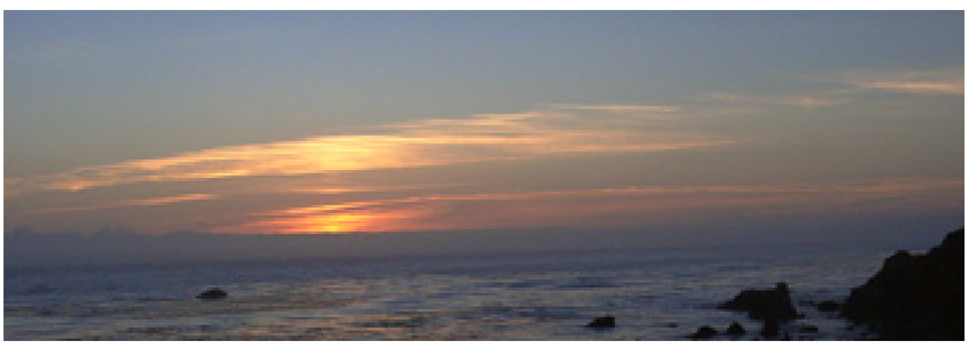
Figure 8.3: *Adding* a polarizing filter allows light to get through

doing between measurements[8]. However any casual talk about what is happening to the particle between measurements, if taken too seriously, is almost certain to lead to wrong results.

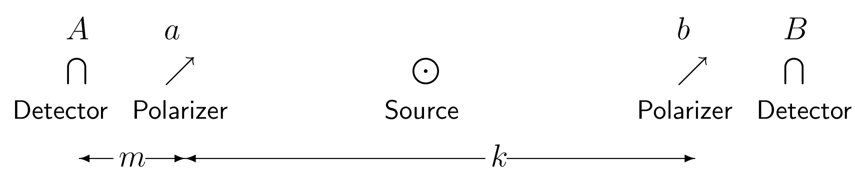
Consider a single quantum event that creates a pair of photons. Conservation laws require a correlation in properties like polarization for the elements of such pairs. The probability that both will pass though a pair of polarizers is $\cos(\theta)$ where θ is the angle between the polarizers. Note this says nothing about the polarization angle of the photons. That does not *exist* until it is observed!

In quantum mechanics it is as if, once one of the photons traverses a polarizer, the other becomes aligned with that polarizer. Before either particle traverses a polarizer neither particle had a polarization angle. Afterwards they have perfectly correlated polarization angles. This is the type of talk that can be so misleading. Yet quantum mechanics predicts that the detection of one of the photons must influence the detection of the other as if something like this happened.

8.4 Bell's theorem simple limited proof



Consider the experiment illustrated in Figure 8.4. Two exactly correlated photons are created simultaneously. They move off in opposite directions for a distance that could be a billion light years in a thought experiment but must be considerably less in any practical experiment. Eventually each photon encounters an experimental setup consisting of a polarizing filter and a detector. Experimenters at each location can vary the angle of the polarizer.



A photon pair created by a single decay event is emitted by the source. The photons travel in opposite directions for a distance of $k/2$ when they encounter variable polarizers set to values a and b. A short distance m later they encounter a detector that records observations A and B. The probability of detecting both photons is correlated with the two polarizer angles. Quantum mechanics predicts a correlation that cannot be explained by a local hidden variables theory. A hidden variables theory like that of Bohm[8] requires that information about one of the polarizer settings affects the more distant detection instantaneously. Because quantum uncertainty masks any indication of which detection is affected by the nonlocal transfer of information, the predictions are consistent with relativity.

Figure 8.4: Experiment to test locality

Quantum mechanic implies that the two photons do not have a definite polarization until one of them is detected but once they are detected they will have identical polarizations. Thus quantum mechanics predicts that the probability of a joint detection is $\cos(\theta)$ where θ is the angle between the polarizers. In a local theory one can compute independent functions $P(a)$ and $P(b)$ that give the probability of detection of each particle as a a function of the angle of the local polarizer. Bell proved it was impossible to separate out the probabilities in this way. He derived an inequality by assuming the two detections were determined by some arbitrary local hidden variables. He showed quantum mechanics predicts the inequality is violated.

Following is a simplified version of Bell's proof that assumes a particular hidden variables model. It is based on an idea of d'Espagnat[17][16]. The more general proof is simple but requires more mathematics. These two derivations are strictly mathematical. They have nothing to do with physics. Physics enters the picture only in deriving predictions that violate the constraints derived in these proofs..

Consider three properties a, b and c that an object might have. The objects and properties could be anything. For example the objects could be words and the three properties could be whether a word contains the letter 'a', 'b' or 'c'. Another example might be pictures containing the colors red, green and blue. Now consider three categories of objects: $(a \wedge \overline{b})$, $(b \wedge \overline{c})$ and $(a \wedge \overline{c})$. Assume we have a collections of objects that are candidates for each category. Denote the number of objects in a category by $N(a \wedge \overline{b})$ etc. The following must hold.

$$N(a \wedge \overline{b}) + N(b \wedge \overline{c}) \geq N(a \wedge \overline{c}) \tag{8.1}$$

The above and later equations are numbered to make them easy to refer to.

It is simple to prove Equation 8.1. Consider any object x that satisfies $(a \wedge \overline{c})$. Now either x satisfies b or it does not. In the first case it also satisfies $(b \wedge \overline{c})$ and if it does not satisfy b it satisfies $(a \wedge \overline{b})$. Every element counted in $N(a \wedge \overline{c})$ must also be counted in either $N(a \wedge \overline{b})$ or $N(b \wedge \overline{c})$. Thus the above relationship holds.

This is one version of many results that are referred to as Bell inequalities. These are constraints that any hidden variables theory must satisfy. The proofs have nothing to do with quantum mechanics. The results depend only on logic and mathematics.

Quantum mechanics predicts equation 8.1 is violated. Polarized photons cannot be used in this simplified proof because it requires determining that a particle does not have some property. Particle spin does work.

Some particles behave as if they were tiny spinning magnets. The spin is quantized. It has a fixed amplitude in either a clockwise or counterclockwise direction. The assumption that the particles have definite spin is a local hidden variables model. It contradicts the quantum mechanical assumption that the state has no well defined value until it is observed. This assumption is inconsistent with the predictions of quantum mechanics.

When two particles are created together in an appropriate event they are said to be in a singlet state. Their spins when observed must be equal and opposite in direction. Depending on the orientation of the particles spin relative to a detector it will be deflected up or down. Its paired twin will have opposite spin and be deflected in the opposite direction. The particles are spin up or spin down.

Assume each particle has a definite spin *before it is detected*. Divide the particles into three categories.

1. Particles detectable with spin up at $0°$ but not at $45°$.
2. Particles detectable with spin up at $45°$ but not at $90°$.
3. Particles detectable with spin up at $0°$ but not at $90°$.

One can determine if a particle is not detectable with spin up at a given angle by looking at the particle it is paired with. If a particle is detectable with spin up at some angle than the particle it is paired with cannot be because they have opposite spin. Count the detections of paired particles x_i and x'_i that fall into the following three categories in which all detections are with spin up.

1. x_i detectable at $0°$ and its pair x'_i at $45°$.
2. x_i detectable at $45°$ and its pair x'_i at $90°$.

3. x_i detectable at $0°$ and its pair x'_i at $90°$.

These three categories fit the constraints of Equation 8.1. Quantum mechanics predicts that the probability that two particles will be detected at spin up with an angle ϕ between the detectors as follows.

$$\frac{1}{2}(\sin(\frac{\phi}{2}))^2$$

Substituting into Equation 8.1 gives the following.

$$\frac{1}{2}(\sin(\frac{45°}{2}))^2 + \frac{1}{2}(\sin(\frac{45°}{2}))^2 \geq \frac{1}{2}(\sin(\frac{90°}{2}))^2$$

Which evaluates as follows.

$$0.0732 + 0.0732 \geq 0.25$$

Clearly the local hidden variables model with the particles having a definite spin orientation before they are detected is wrong.

8.5 Bell's theorem general proof

This is the only section with substantial mathematics. It is not essential. One can understand the idea of Bell's result from the previous section. However it is worth making the effort to follow if you are at all inclined to do so.

Bell's proof that *no* hidden variables model will work involves some simple integral equations. The following repeats the treatment with additional commentary from Appendix 2 of Bell's article "Bertlmann's socks and the nature of reality"[7]. The commentary is intended to make the argument understandable even for those allergic to mathematical notation.

For the general case, two adjustments a and b and two observations A and B as are needed. These are shown in Figure 8.4. Assume there is some unknown set of hidden variables represented by a single parameter λ. The assumption of locality is that the joint probability of simultaneous detections factors into two independent local probabilities. The joint probability of detections A and B with experimental settings a and b is $P(A, B|a.b.\lambda)$. The locality assumptions is given by the following.

$$P(A, B|a.b.\lambda) = P_1(A|a, \lambda)P_2(B|b, \lambda) \tag{8.2}$$

This implies that the probability of a detection at A is only dependent on the local setting of a and a local unknown variable λ. The same is true at the other detector. λ can be removed from the left hand side by integrating or averaging over all possible values of λ. To do this requires the probability density function $f(\lambda)$ that gives the likelihood that λ will have a particular value.

$$P(A, B|a.b) = \int P_1(A|a, \lambda)P_2(B|b, \lambda)f(\lambda)d\lambda$$

This says that the joint probability of observing A and B at settings a and b is the average value of a product of three terms. The first two are the likelihood of making observations A and B locally for a given value of λ. The third $f(\lambda)$ is the likelihood that λ will have this value.

Practical experiments detect or fail to detect a particle. Thus outcomes $(A,\ B)$ cam replace the four possible detection combinations of $(yes,\ yes)$, $(yes,\ no)$, $(no,\ yes)$ and $(no,\ no)$ representing a detection or no detection at the two distant sensors. The following sum of these possible outcomes is particularly useful in developing a version of Bell's inequality that can be tested experimentally.

$$\begin{aligned} E(a, b) \;=\; & P(yes, yes|a, b) - P(yes, no|a, b) \\ & -P(no, yes|a, b) + P(no, no|a, b) \end{aligned} \tag{8.3}$$

So instead of computing the probability of possible outcomes $P(A, B|a, b)$ compute the probability that $E(a, b)$ will have a given value. E is not a probability and will be negative if it is more likely that there are detections in only one of the two sensors rather than in both or neither.

Write E as a function of local variables using Equation 8.2. The idea is to replace the nonlocal dependency in (a, b) with dependency only on local values (only a or only b) and the hidden local variable λ. This results in the following.

$$\begin{gathered} E(a,b) = \int \{P_1(yes|a,\lambda)P_2(yes|b,\lambda) \\ -P_1(yes|a,\lambda)P_2(no|b,\lambda) \\ -P_1(no|a,\lambda)P_2(yes|b,\lambda) + P_1(no|a,\lambda)P_2(no|b,\lambda)\}f(\lambda)d\lambda \end{gathered} \tag{8.4}$$

Factor the part of 8.4 involving P_1 and P_2 as follows.

$$\begin{gathered} E(a,b) = \int \{P_1(yes|a,\lambda) - P_1(no|a,\lambda)\} \\ \times\{P_2(yes|b,\lambda) - P_2(no|b,\lambda)\}f(\lambda)d\lambda \end{gathered} \tag{8.5}$$

Rewrite 8.5 as follows.

$$E(a,b) = \int \overline{A}(a,\lambda)\overline{B}(b,\lambda)f(\lambda)d\lambda \tag{8.6}$$

By making the following substitutions.

$$\overline{A}(a,\lambda) = \{P_1(yes|a,\lambda) - P_1(no|a,\lambda)\}$$

$$\overline{B}(b,\lambda) = \{P_2(yes|b,\lambda) - P_2(no|b,\lambda)\}$$

Use 8.6 to construct a formula for $E(a,b) \pm E(a,b')$. The symbol '$\pm$' is used as a shorthand for two equations. The first uses $+$ and the second $-$. You have to make the substitution every place $\pm$ occurs. Shortly Use $\mp$ which substitutes the signs in the reverse order. For example $a \pm b \pm c \mp d$ represents the two equations $a + b + c - d$ and $a - b - c + d$.

$$E(a,b) \pm E(a,b') = \int \overline{A}(a,\lambda)\{\overline{B}(b,\lambda) \pm \overline{B}(b',\lambda)\}f(\lambda)d\lambda \tag{8.7}$$

Since P_1 and P_2 are probabilities the following holds.

$$0 \leq P_1 \leq 1 \tag{8.8}$$
$$0 \leq P_2 \leq 1$$

Thus $|\overline{A}(a, \lambda)| \leq 1$ and $|\overline{B}(b, \lambda)| \leq 1$. For these are both differences between two numbers between 0 and 1 and thus their absolute value must be less than or equal one. Using this can remove $\overline{A}(a, \lambda)$ from 8.7 by converting the equality to an inequality. It is convenient to do generate two versions of the result.

$$|E(a, b) \pm E(a, b')| \leq \int |\overline{B}(b, \lambda) \pm \overline{B}(b', \lambda)| f(\lambda) d\lambda \tag{8.9}$$

$$|E(a, b) \mp E(a, b')| \leq \int |\overline{B}(b, \lambda) \mp \overline{B}(b', \lambda)| f(\lambda) d\lambda$$

Using 8.8 gives the following.

$$|\overline{B}(b, \lambda) \pm \overline{B}(b', \lambda)| + |\overline{B}(b, \lambda) \mp \overline{B}(b', \lambda)| \leq 2 \tag{8.10}$$

Because $f(\lambda)$ is a probability density, the following must hold.

$$\int f(\lambda) d\lambda = 1 \tag{8.11}$$

Applying 8.11 and 8.10 to 8.9 yields the following.

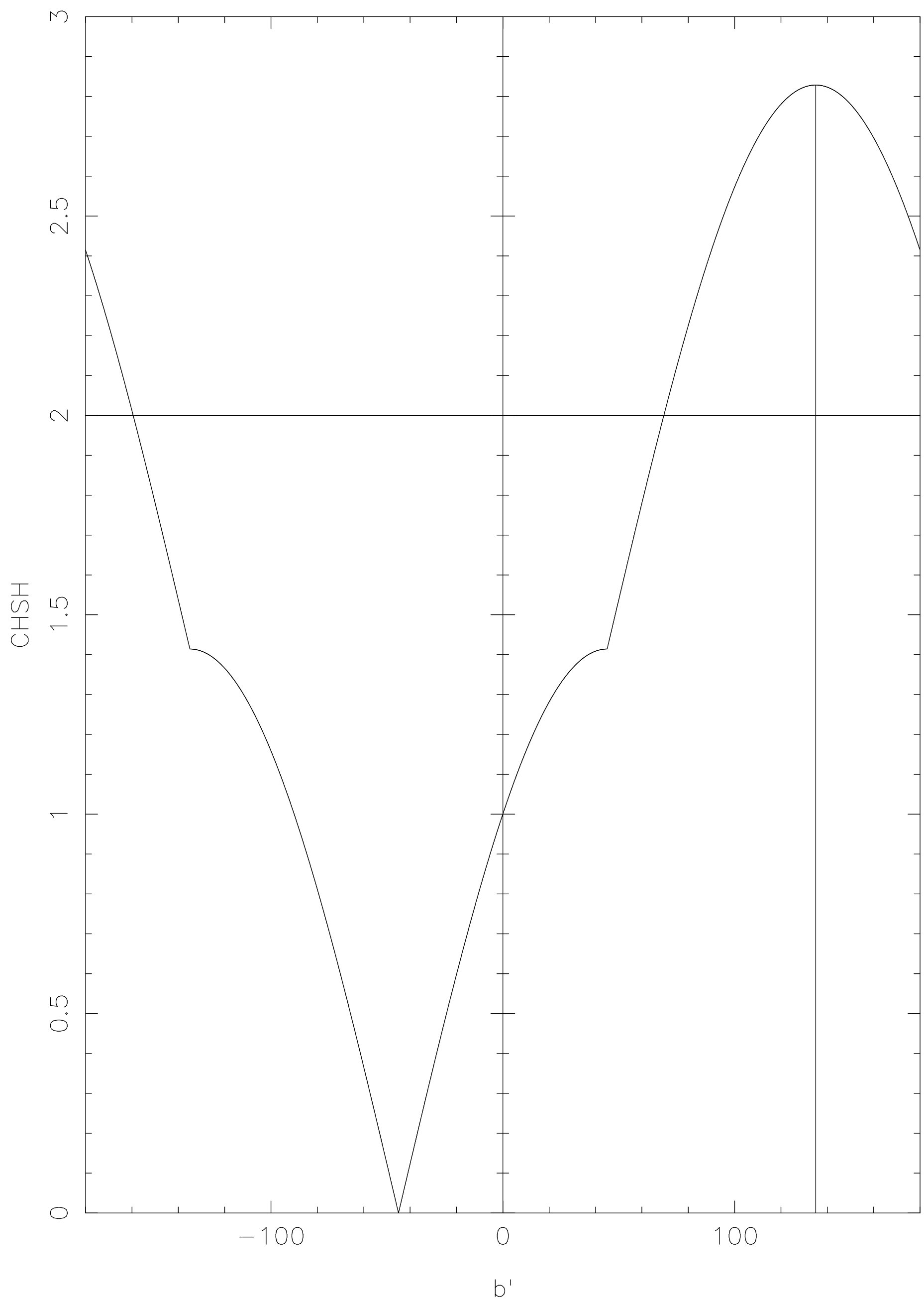
$$|E(a, b) \pm E(a, b')| + |E(a', b) \mp E(a', b')| \leq 2 \tag{8.12}$$

This is called the CHSH inequality from the initials of the authors who first derived it[13] as an inequality that could be tested experimentally.

For the experiment involving magnetic spin in Section 8.4 provides a good example of how this inequality is violated. Quantum mechanics predicts that that $E(a, b) = -\cos(a - b)$ where a and b are the angles of orientation of the two polarizers. Thus from 8.12 we have the following.

$$|\cos(a - b) \mp \cos(a - b')| + |\cos(a' - b) \pm \cos(a' - b')| \leq 2$$

Assume $a = 0$, $b = 90°$ and $a' = 45°$. Figure 8.5 gives a plot of 8.12 as a function of b'. It also plots the classical limit of 2. The peak occurs at $135°$ where the value for CHSH predicted by quantum mechanics is 2.828.



The horizontal straight line at 2 represents the maximum correlation from local hidden variables theories. The curve is the prediction of quantum mechanics.

Figure 8.5: Correlation predictions

8.6 Experimental tests of Bell's inequality

When Bell's result was published, the experimental record was reviewed for evidence against locality. None was found. Thus an effort began to develop tests of Bell's inequality. A series of experiments conducted by Alain Aspect and his colleagues ended with one reported in 1982 in which polarizer angles were changed at a rapid enough rate to prevent light from either polarizer from reaching the more distant detection in time to influence the result[2]. The experimental design was like that shown in Figure 8.4.

The only reservation Aspect and his colleagues expressed for this experiment concerned the lack of randomness in the polarizer settings.

> A more ideal experiment with random and complete switching would be necessary for a fully conclusive argument against the whole class of supplementary-parameter theories obeying Einstein's causality[2].

This is a little surprising. In a previous paper Aspect pointed out another problem.

> Only two loopholes remain open for for advocates of realistic theories without action at a distance. The first one, exploiting the low efficiencies of detectors, could be ruled out by a feasible experiment. The second one, exploiting the static character of all previous experiments, could also be ruled out by a "timing experiment" with variable analyzers now in progress[3].

The experiment "now in progress" was the one widely considered to be conclusive at the time. This was true even though the detector efficiency problem remained until an experiment in 2001[44]. That experiment does not have adequate timing constraints and thus is still inconclusive.

Aspect claimed conclusive results in an experiment prior to both of the above reports.

> Our results in excellent agreement with the quantum mechanical predictions, strongly violate the generalized Bell's inequalities, and rule out the whole class of realistic local theories[1]

Why all the confusion about what constitutes a conclusive experiment? There are no doubt many reasons. Perhaps the biggest problem is the conviction among most physicists that quantum mechanics is correct about this. When one is certain about the expected results of an experiment, one's critical faculty is handicapped. Its hard to fully consider all the possible ways that an experimental claim may be overly broad.

Many physicists have an emotional investment in the outcome of these experiments that compromises their objectivity. Einstein is universally regarded as the greatest physicist of the twentieth century although he barely made it to the middle of the century and all of his major work was done by the 1935 date of the paper known as EPR which led to these experiments. There is a strong conviction that physics has moved way beyond the naive realism of Einstein and these experiments are the objective proof of that. If they ultimately turn out to vindicate Einstein this will be an enormous blow to the ego of many physicists working in the foundations of quantum mechanics,

Aspect's experiment was widely regarded at the time as conclusive, especially in the popular press. The reservation about randomly varying the polarizer angles seemed like nitpicking. Were the photons suppose to figure out the pattern and use it to time their detections?

In 1985 James D. Franson published a paper showing that the timing constraints in this experiment were not adequate to confirm that locality was violated[24]. The difficulty is in establishing the time of detection. For that starts as a microscopic event like the atom decay that determines the fate of Schrödinger's cat. (See Section 7.3.) While few believe the cat's fate remains undecided until one opens the box the exact time at which that fate has become certain is unclear. For timing in a test of Bell's inequality to be conclusive requires that we time the occurrence of a macroscopic event. The trouble is there is no clear definition of what a macroscopic event is. Franson observed the following.

> The time interval over which the probability amplitudes discussed above may simultaneously exist and interact in the experiment by Aspect Dalibard and Roger could conceivably be comparable to the 89-nsec lifetime of the excited atomic state which produces the pair of photons. If the photon emission time remains indeterminate for this length of time than it is plausible that the final outcome of the event may remain indeterminate for a comparable amount of time[24].

Franson introduced the phrase "delayed determinism". This sounds very strange but, as he was at pains to point out in his paper, this is an integral part of quantum mechanics and may well be part of a local realistic theory. There is nothing in existing theory that says when an event is finally determined. Microscopic systems can exist in a superposition of states like the dead and live cat because interference effects are observed from both states simultaneously. Any conclusive result must involve an unambiguously macroscopic measurement of time. On that grounds Aspect's experiment failed.

In spite of Franson's objections and the additional problem of detector efficiencies the belief remained widespread that Aspect's experiment was decisive, "Proposal for a Loophole-free Bell Inequality Experiment[38]" was published in 1993 detailing the problems in existing experiments and how they might be overcome.

A major stride in addressing the timing was described in "Violation of Bell's inequality under strict Einstein locality"[52]. This provided a much tighter (although not absolutely conclusive) constraint on timing by separating the detections by 400 meters. Fiber optics made this practical. It take light only $1.3\mu s$ ($1\mu s$ is one millionth of a second) to travel 400 meters. The time between when the polarizer settings were changed and a detection was registered was less than $100ns$. ($1ns$ is one billionth of a second.) The paper draws the following conclusions.

> While our results confirm the quantum theoretical predictions, we admit that, however unlikely, local realistic or semi-classical interpretations are still possible. Contrary to all other statistical observations we would then have to assume that the sample of pairs registered is not a faithful representative of the whole ensemble emitted. While we share Bell's judgment about the likelihood of that explanation[4], we agree that an ultimate experiment should also have higher detection/collection efficiency, which was 5% in our experiment.
>
> Further improvements, e.g. having a human observers choose the analyzer directions would again necessitate major improvements of technology as was the case in order to finally, after more than 15 years, go significantly beyond the beautiful 1982 experiment of Aspect et al[2]. Expecting that any improved experiment will also agree with quantum theory, a shift of our classical philosophical positions seems necessary. Among the possible implications are nonlocality or complete determinism or the

[4] The observation that the photons in a pair, as used by us, are always found to have different polarization can not as easily be understood as the fact that the socks in a pair, as worn by Bertlmann, are always found to have different color[7].

> abandonment of contrafactual conclusions. Whether or not this will finally answer the eternal question: "Is the moon there, when nobody looks?"[40], is certainly up to the reader's personal judgment[52].

The authors admit that the detector efficiencies make the experiment less than conclusive yet they are completely confident that no future experiment in this area will contradict quantum mechanics. The problem with this attitude is illustrated by their speculation on what this means. They can ask "Is the moon there when nobody looks?" because quantum mechanics says nothing about physical state between observations. The only thing that evolves between observations is a wave function of probability densities in configuration space. I suspect something physical does happen between observations and that alone makes quantum mechanics incomplete. Given the complete ignorance about what is happening between the creation of the photon pairs and their detection it would seem that a higher degree of skepticism about a truly conclusive experiment is called for.

There was a similar experiment about the same time that achieved even greater separation of detectors[47].

A recent experiment finally addressed the detector efficiency problem. The following portion of the abstract describes what was accomplished in this experiment.

> Here we have measured correlations in the classical properties of massive entangled particles ($^9Be^+$ ions): these correlations violate a form of Bell's inequality. Our measured value of the appropriate Bell's 'signal' is $2.25 \pm .03$, whereas a value of 2 is the maximum allowed by local realistic theories of nature. In contrast to previous measurements with massive particles, this violation of Bell's inequality was obtained by use of a complete set of measurements. Moreover, the high detection efficiency of our apparatus eliminates the so-called 'detection' loophole[44].

At the end of the article the authors say they were not able to overcome the timing or "light cone loophole" in this experiment.

Thus there are experiments that individually address these two loopholes but no single conclusive experiment. In addition the timing loophole is somewhat ill defined because of the lack of a a clear distinction between microscopic and macroscopic.

A recent theoretical paper[23] analyzes noise in these experiments and raises new and difficult issues. Following is the abstract and conclusion of this paper.

> We emphasize the difficulties of an experiment that can definitely discriminate between local realistic hidden variables theories and quantum mechanics using the Bell CHSH inequalities and a real measurement apparatus. In particular we analyze some examples in which the noise in real instruments can alter the experimental results, and the nontrivial problem to find a real "fair sample" of particles to test the inequalities.
>
> [...]
>
> Bell's inequality tests necessitate major improvements of technology in order to finally, after more than 15 years, go significantly beyond the 1982 experiment of Aspect et al. [2]. While expecting that any improved experiment will also agree with quantum theory, actually the final answer to the eternal question: "Is the moon there, when nobody looks?", is certainly up to our judgement capability. But sometime also the question "Is the moon there when we look at it by a noisy telescope?" appears very hard to address.

It is not difficult to see why most physicists are confident in their expectations for these experiments. Quantum mechanics is a spectacularly successful theory producing extraordinary predictions many of which have astounding accuracy far surpassing anything possible with classical physics. One can easily understand Bell's skepticism about the detection efficiency loophole.

> . . . it is hard for me to believe that quantum mechanics works so nicely for inefficient practical set-ups and is yet going to fail badly when sufficient refinements are made[7].

How can a theory that has been so spectacularly reliable and successful suddenly falter because of improved detector efficiency? That is one way to look at things and the way most physicists do.

An alternative view focuses on how extraordinary these predictions are and on how convoluted and improbable a theory quantum mechanics is. Locality is the most powerful simplifying assumption in physics. Without it any event in the universe can influence any other and physical theories become problematic if not impossible. How is it that the universe violates locality but only does so in obscure and difficult experiments that retain significant loopholes? One would expect that a universe, containing the complexity required for nonlocality, would be spectacularly nonlocal. One would hardly expect a theory like relativity, that is local at its core, to be one of the two dominant theories in such a universe. Of course the universe does not have to live up to our expectations, but simplicity and elegance have often been a guide to deeper and richer physical theory and these predictions of quantum mechanics are about as far from simplicity and elegance as one can get.

Bell proved that the configuration space model of quantum theory cannot be mapped into physical space except with an explicitly nonlocal model such as Bohm's[8]. The alternatives are action at a distance (that is not relativistic) or a model like configuration space that is not relativistic in its structure, but can be, to a limited degree, in its predictions. In addition quantum mechanics says nothing about what happens in physical space between observations. If it is meaningful to talk about an objective physical state that changes continuously, then quantum mechanics, which lacks such a description, is incomplete.

Quantum mechanics fails to define measurement. Does it take a human conscious observation? Is a macroscopic event enough? If so how big is macroscopic? Is a gauge than can be read by a human enough? How about the change in state of a single bit in a computers memory? If there is an objective definition of measurement than quantum mechanics is an incomplete theory.

One can argue that the philosophy of contemporary physics is to define as meaningless what one does not understand. It is one thing to avoid such problems because one has no idea how to deal with them and quite another to say fundamental philosophical principles need to be changed. These predictions deserve a high level of skepticism no matter how many inconclusive experiments agree with quantum mechanics.

The conviction that quantum mechanics will not be falsified by these experiments stems in part from the difficulty of imagining how an alternative local theory could account for the existing experimental record. In the next section I speculate about the properties of discrete models based on the wave equation. Such a model might account for the existing experimental results. This speculations is far from being a new theory. That is how one must start if the models can only be developed in conjunction with experiments as seems likely.

8.7 Exploring discretized wave equations

The simplest dynamic discrete system involves a single scalar value that changes at each time step or iteration. A simple time symmetric[5] finite difference equation for this is as follows.

$$f_{t+1} = T(nf_t/d) - f_{t-1} \tag{8.13}$$

n and d are integers (numerator and denominator). T is truncation toward 0 defined in Table 7.1. The equation says that the next value f_{t+1} is obtained by multiplying the current value f_t by a factor (n/d) truncating the result and then subtracting the previous value f_{t-1}.

The corresponding differential equation is as follows.

$$\frac{d^2 f}{dt^2} = (n/d - 2)f(t) \tag{8.14}$$

The -2 comes in because the second order difference equation subtracts $(f_t - f_{t-1})$ from $(f_{t+1} - f_t)$ generating a term $-2f_t$.

For $-2 > n/d > 2$ 8.14 has a solution as follows.

$$f(t) = cos(t\sqrt{2 - n/d}) \tag{8.15}$$

t is in radians[6]. Figure 8.6 plots a solution for $n = 19$ and $d = 10$. The solution increments the angle of the $\cos$ $.3162$ radians or $18.11°$ each time step. It takes about 20 time steps to complete one cycle of the $\cos$ wave.

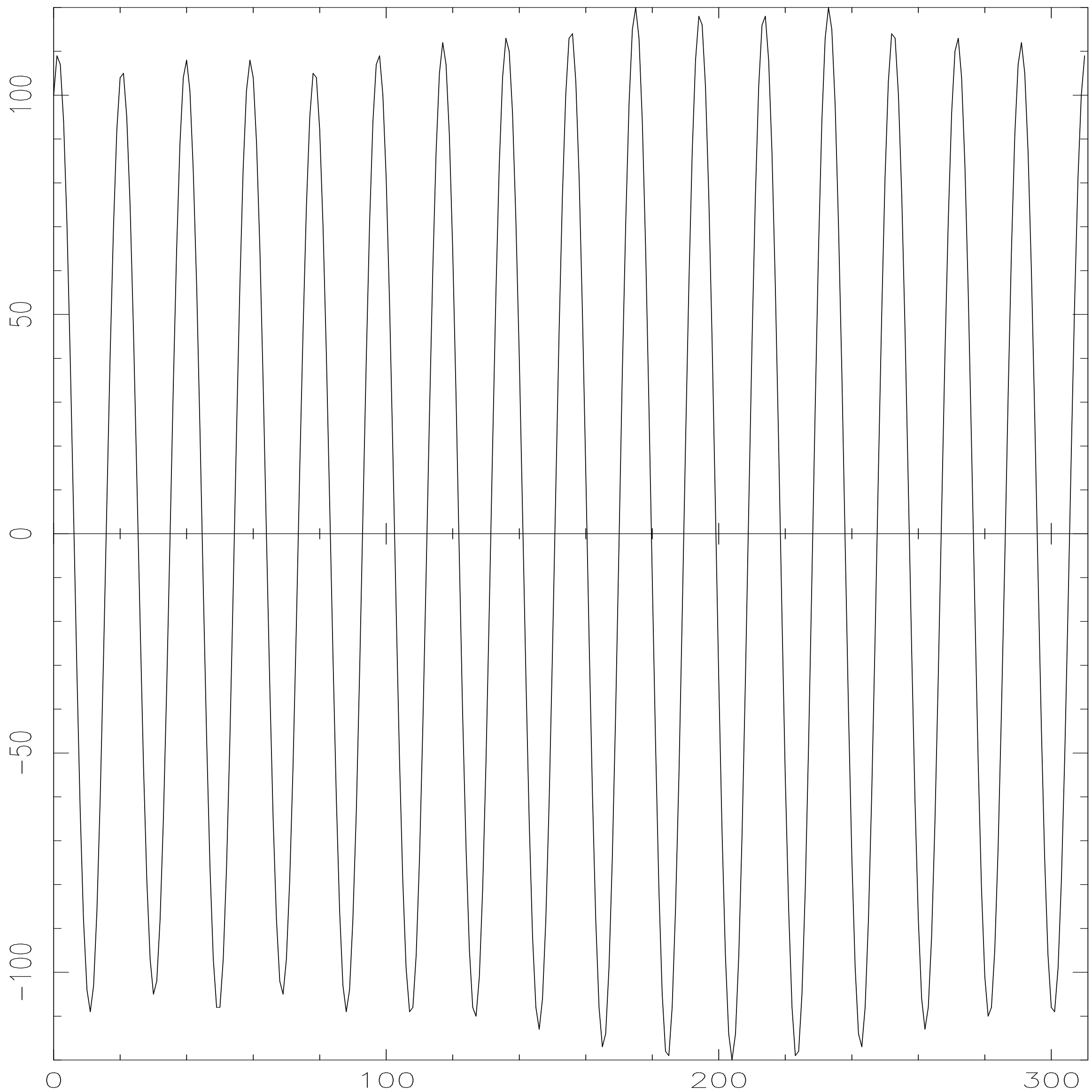
The solutions to the finite difference equation 8.13 are more complex than the solutions to the corresponding differential equation 8.14. The former are completely described by equations like 8.15. The latter have a rich structure that varies with the initial conditions. Table 8.1 gives the length until the sequence starts to repeat of the solution to 8.13 (again with $n = 19$ and $d = 10$) for various initial conditions.

Could the rich structure of the discretized difference equations account for both the weirdness of quantum mechanics and the fundamental constants of physics including those not derivable from an existing theory?

---

[5] An equation is symmetric in time if the solution for f_{t-1} is the same as the solution for f_{t+1}. The fundamental laws of physics are symmetric in time with some exceptions. Time symmetric models are reversible. Reverse the order in time of the initial conditions and the sequence of states goes in the opposite direction in time.

[6] Radian is a measurement of angle like degrees. There are 2π radians in $360°$.



The above is a plot of the solution to $f_{t+1} = T(nf_t/d) - f_{t-1}$ with $f_0 = 100$, $f_1 = 109$, $n = 19$ and $d = 10$. T is truncation toward 0 (see Table 7.1). It completes about 5 cycles for every 100 iterations. It departs significantly from the solution to the differential equation.

Figure 8.6: Simple discretized finite difference equation plot

	f_1												
f_0	100	101	102	103	104	105	106	107	108	109	110	111	112
100	154	269	154	154	269	328	328	328	289	309	174	116	116
101	269	77	250	328	250	289	328	77	309	116	309	174	174
102	154	250	154	289	328	309	250	77	77	58	174	289	77
103	154	328	289	77	328	328	328	328	77	309	116	289	289
104	269	250	328	328	77	309	289	250	309	309	289	174	174
105	328	289	309	328	309	328	77	77	289	58	289	77	309
106	328	328	250	328	289	77	116	77	77	116	116	174	77
107	328	77	77	328	250	77	77	58	289	309	174	289	309
108	289	309	77	77	309	289	77	289	309	309	309	289	174
109	309	116	58	309	309	58	116	309	309	289	174	174	289
110	174	309	174	116	289	289	116	174	309	174	250	116	77
111	116	174	289	289	174	77	174	289	289	174	116	174	174
112	116	174	77	289	174	309	77	309	174	289	77	174	116
113	309	135	289	251	309	116	309	174	309	116	309	251	289
114	58	174	58	309	135	135	174	289	77	289	174	135	135
115	174	174	251	174	174	174	251	116	174	58	174	116	251
116	58	174	309	135	77	251	58	77	58	135	174	135	58
117	251	174	251	251	251	58	135	174	309	135	251	174	251
118	58	251	406	58	135	309	251	251	309	174	58	77	135
119	484	368	174	58	58	174	174	58	309	251	174	251	251
120	368	232	484	406	368	368	58	251	58	174	135	251	58
121	58	232	232	406	406	484	58	174	97	251	174	58	174
122	58	58	484	232	484	58	406	368	406	58	174	58	58
123	406	194	484	232	484	368	58	484	484	368	58	406	58
124	213	155	174	368	232	232	58	97	484	368	58	58	368
125	194	155	174	406	97	213	484	58	368	232	484	406	58
126	349	136	213	174	174	97	232	368	58	97	484	232	232

The above gives the length until repetition of the sequence generated by $f_{t+1} = T(nf_t/d) - f_{t-1}$ with $n = 19$ and $d = 10$. T is truncation toward 0 (see Table 7.1). The table is symmetric about a diagonal because reversing the order of the initial two values does not affect the sequence or its length. It reverses the sequence order because the equation is symmetric in time.

Table 8.1: Cycle lengths for discretized finite difference equation

The remainder of this section is about intuitive possibilities. We need to develop the skills for collective work at the intuitive level. Western culture is good at focusing intellectual talent on a project beyond the capacity of an individual. The same is not true at early intuitive stages. One way to start is intuitive brainstorming in print.

The ultimate goal is to write on a half a sheet of paper a single discretized finite difference equation that explains all of physics and thus all of creation. One suspects this is possible in part because of the universality of the wave equation and in part because of the added complexity and nonlinearity that discretization produces. Such a model would only explain the structure of our conscious experience and not its essence. (See Chapter 3.) But such a model would be the Holy Grail of physics. It is the ultimate explanation Einstein was seeking. So let us brainstorm about this possibility.

The nonlinearity introduced by discretization may produce chaotic like behavior. Chaos theory is the study of continuous nonlinear systems that are so sensitive to initial conditions that an exponential increase in knowledge of initial conditions only allows a linear increase in predictability. For example to predict one second into the future might require an accuracy of 10, but to predict five seconds into the future would require an accuracy of 10^5 or $10,000$. Typically computing resources needed for prediction grows exponentially as well. These systems are not predictable in any practical sense. It is not possible to obtain sufficient knowledge of initial conditions and the computing power rapidly exceeds what would fit in the known universe. While the detailed behavior of these systems is not predictable many global aspects of them may be.

Chaotic systems often have attractors. These are states the system converges toward over time. They are like the point at the bottom of a circular bowl. If you drop a marble it will eventually settle down at the bottom of the bowl. One can often determine the attractors in a chaotic system and use these to predict the system's behavior. There may be multiple attractors and it may be impossible to predict which will win out, but one can be sure the system will wind one in one of these states. This is like a double bottomed bowl. If you drop a marble in at point midway between the two bottoms allowing the marble to roll in any direction you cannot predict where it will wind up but you know it will be in one of the two bottom points.

Discrete systems cannot be chaotic. There is an upper limit to the information it takes to fully characterize a discrete system and that alone disqualifies them. They can approximate chaotic behavior just as they can approximate any continuous system. If the universe is discontinuous then no truly chaotic systems exist. The sequences in Table 8.1 are a little like attractors. If a sequence is perturbed by slightly changing the current value it may start a new sequence. Longer sequences are stronger attractors. It is more likely to fall into or stay in such a sequence after a perturbation.

Going from a finite difference equation at a single point in space to one spatial dimension (or a line) greatly complicates matters. A single spatial dimension has an enormous number of states. There must still be loops of repeated sequences of states because the total number of possibilities is finite. However these loops could easily exceed the age of the universe in units of Planck time[7]. A one dimensional line of only 100 integers between -100 and 100 involves 201^{100} possible combinations.

Discretizing the wave equation makes it nonlinear. That is reflected in the varying amplitude of the peaks in Figure 8.6. In larger three dimensional examples it is expected that this can introduce chaotic like behavior that *appears* to be random. Yet there will be structural conservation laws, if the discretized finite difference equation is symmetric in time. This makes it reversible. In a sense nothing can ever be created or destroyed. The history of the universe is contained in the most recent states. Reverse their order and time will evolve backwards.

I speculate that there will be “stable dynamic structures” in these models that are somewhat like attractors in chaos theories. These are state sequences that repeat themselves approximately and that are relatively immune to external disturbances. However these structures can transform into each other either is a result of

[7] The Planck time is $(G\hbar/c^5)^{(1/2)} \cong 10^{-43} seconds$ where G is the gravitational constant, $\hbar$ is Planck's constant divided by 2π and c is the speed of light.

interactions or spontaneously. They are the model for particles.

Absolute conservation laws and probabilistic laws of observation are characteristics of this class of models. Could that account for the existing experiments? The transformation of particles would be a physical quantum collapse process. But it is a process spread out in time and space. There is *no* point at which the process is definitely complete. The conservation laws can prevent a transformation from being complete or even cause it to reverse after it seems complete. So the objections raised by Franson[24] make it difficult to know when a measurement is complete.

Envision a microscopic world of attractor like stable states. Occasionally particles are perturbed and transform between states. Time reversibility imposes a strong form of conservation that must be honored in the long run but can be deviated from significantly in the short run because of the nonlinear effects needed to discretize the wave equation. Transformations start to happen and reverse far more often than they complete. Multiple transformations can start at different parts of the same particle but at most one of them can complete. In this model strange things can happen.

Even with such a radically different discrete model, it can be hard to imagine how the more recent experimental results can be consistent with classical locality. In this model quantum collapse is a process of converging to a stable state consistent with the conservation laws. This can happen in many and very indirect ways. Nature may seem to conspire to remain consistent with classical locality and quantum mechanics until every possible loophole is plugged.

We now turn to the problem of explaining gravity in this class of models. The hope is to use only a discretized version of the wave equation. There is a different wave equation for a single particle with rest mass.

$$\frac{\partial^2\psi}{\partial t^2} = c^2\nabla^2\psi - \frac{m^2c^2\psi}{\hbar^2} \tag{8.16}$$

This is known as the Klein Gordon equation or the relativistic Schrödinger equation. It is identical[8] to the wave equation in Sec-

[8] The substitution of ψ for f between the two equations has no effect.

tion7.4 except for the term $-\frac{m^2c^2\psi}{\hbar^2}$ where m is the rest mass of the particle and $\hbar$ is Planck's constant or approximately $6.62606891 \times 10^{-34}$ Joule-seconds.[9] The rest mass decreases the rate at which the level of ψ accelerates in time.

How can 8.16 be derived from the same rule of evolution that approximates the classical wave equation? This may be possible if there is a high carrier frequency near the highest frequencies that can exist in the discrete model. The Schrödinger wave equation for particles with rest mass would represent the average behavior of the physical wave. It would be the equation for a wave that modulates the high frequency carrier. The carrier itself is not a part of any existing model and would not have significant electromagnetic interactions with ordinary matter because of its high frequency.

Such a model may be able to account for the Klein Gordon equation for a particle with rest mass. A high frequency carrier wave will amplify any truncation effect. Because of this the differential equation that describes the carrier envelope is not necessarily the same as the differential equation that describes the carrier. If the carrier is not detectable by ordinary means, only effects from the envelope of the carrier will be observable, not the carrier itself. The minimum time step for the envelope may involve integrating over many carrier cycles. If round off error accumulates during this time in a way that is proportional to the modulation wave amplitude, this will lead to an equation in the form of the Klein Gordon equation.

The particle mass squared factor in the Klein Gordon equation can be interpreted as establishing an amplitude scale. The discretized wave equation may describe the full evolution of the carrier and the modulating wave that is a solution of the Klein Gordon equation. However, since no effects (except mass and gravity) of the high frequency carrier are detectable with current technology, only the effects of the modulating wave will be observable. No matter how localized the particle may be it still must have a surrounding field that falls off in amplitude as $1/r^2$. It is this surrounding field that embodies the gravitational field.

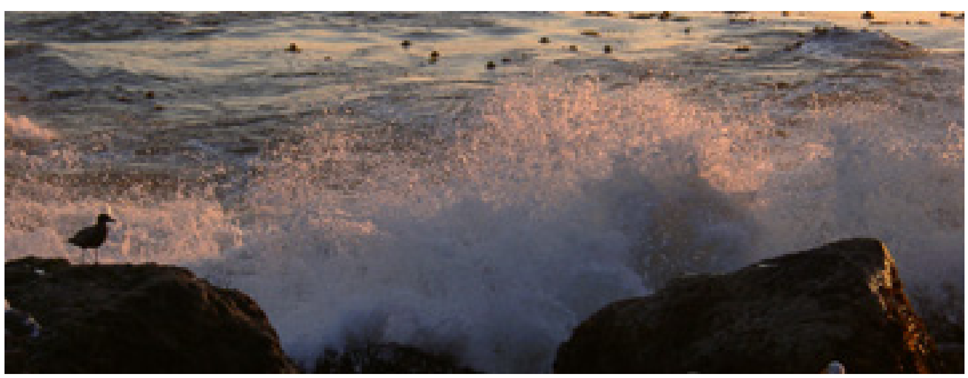
If discretization is accomplished by truncating the field values this creates a generalized attractive force. It slows the rate at which

[9] A Joule is unit of energy. One Joule is 0.2388 calories.

a structure diffuses relative to a solution of the corresponding differential equation by a marginal amount. Since the gravitational field is a high frequency electromagnetic field it will alternately act to attract and repel any bit of matter which is also an electromagnetic field. Round off error makes the attraction effect slightly greater and the repulsion slightly less than it is in solutions of the continuous differential equation.

Because everything is electromagnetic in this model special relativity falls out directly. If gravity is a perturbation effect of the electromagnetic force as described it will appear to alter the space time metric and an approximation to general relativity should also be derivable. It is only the metric and not the space time manifold (lattice of discrete points) that is affected by gravity. Thus there is an absolute frame of reference. True singularities will never occur in this class of models. Instead one will expect new structures will appear at the point where the existing theory predicts mass will collapse to a singularity.

8.8 The structure of the universe



Central to this discussion of physics is the assumption that the structure of the universe is simple, understandable, mathematical and finite. If completed infinite totalities exist our philosophical views are false. The universe may however be unbounded and potentially infinite. Simplicity and in particular the simplicity implied by locality is not essential to the position I take but it is in the spirit that leads to that position. Quantum mechanics as it stands is consistent with a finite universe. A finite amount of information can fully characterize the state of any finite space time region.

Many physicists label assumptions like the above about locality and simplicity as naive realism. They would claim that as long as a theory has an elegant mathematical formulation it is simple. Philosophical and aesthetic debates can never be conclusive. Ultimately experiments will decide the issue and tests of Bell's inequality are one arena where such tests are possible. There are others. Quantum computing takes advantage of the superposition of states to compute not a single result but many possible results simultaneously. There are important practical problems that quantum computing can in theory solve that would require far too much computation for a practical solution with conventional computers. These are not *recursively unsolvable* problems. They just require computing power that grows exponentially with the size of the problem on a standard computer but not on a quantum computer.

There is significant research in this area. How far quantum computing can go depends on how accurately physical reality is described by quantum mechanics. Quantum computing will eventually fail if time and space have a discrete and not continuous structure.

Quantum cryptography comes in several forms some of which depend on quantum entanglement. In theory quantum cryptography can provide levels of security and tamper detection that are not possible with more conventional approaches. The attempt to develop practical devices in this area can run into problems if quantum entanglement results from an underlying mechanistic process that enforces the conservation laws.

The potential for economically important devices in these two areas provides an incentive to test the limits of physical theory that does not exist in tests of Bell's inequality.

Science provides simple explanations for complex systems. That has been its history. Two related aspects of quantum mechanics, irreducible probabilities and quantum entanglement, seem at odds with the historical trend. Quantum entanglement leads to spacetime connections between events far more complex than the spatially separable universe of classical physics. Absolute randomness seems to be impossible to define mathematically since an absolutely random sequence would have to be recursively random but recursively random sequences cannot be truly random. They are higher up on the scale of mathematical complexity than recursive sequences.

To some quantum mechanics seems like a marvelous structure To this author it looks like a house of cards waiting for a gentle breeze to collapse it. There are too many interconnected implausible assumptions. Consistency with special relativity requires irreducible probabilities which implies no intermediate state evolution between observations. There can be no objective definition of observation although physics is dependent on experimental technique which has a well developed practical approach to objectively defining observation.

The claim is made that physicists have come to understand the weird and wonderful way that nature is. No one can know for certain but skepticism is called for. One can always invent new philosophical principles to deal with the inexplicable. That was the standard approach before science.

8.9 A digital physics fantasy

There is no digital theory of physics. All efforts in this direction are in a primitive state. Developing a digital theory that makes macroscopic predictions is likely to be far more difficult than developing quantum mechanics was. For any digital theory that approximates the continuum will be impossible to simulate at a macroscopic scale with existing or foreseeable technology (see Section 7.1). Quantum mechanics was created by experimenters and theoreticians feeding each other. A more complete digital theory may require a trio of experimenters, theoreticians and engineers. The engineers will design the computers made possible by a deeper understanding of physics and thus create the simulation tools to further expand that understanding.

Although there is nothing close to a complete theory it is possible to construct an intuitive fantasy that illustrates how such a theory might account for the existing experimental record. At best the description that follows will turn out to be vaguely right in some respects. No doubt in others it will be precisely wrong. Still this description may be useful in giving physicists a sense of what might be possible. Quantum mechanics is so successful and so strange it has become hard to see how an alternative more complete theory might be possible. The hope is that the following description can convince at least a few that there could be an alternative.

Start with a discretized wave equation as described in Section 7.4. This is the relativistic Schrödinger equation for a single particle with zero rest mass. This is a very simple model that can be easily described on a half sheet of paper. I speculated in Section 8.7 about how this same model might lead to the relativistic Schrödinger equation for a single particle with rest mass.

Discrete systems can generate dynamically stable structures. These structures can transform into each other when perturbed something like a chaotic system moving between attractors. These structures are the particles of quantum mechanics. Reversibility without divergence produces a sort of structural conservation law. Reversibility can exist without absolute conservation at each time step. Thus individual observations can be pseudo random. But reversibility imposes global conservation laws. These laws are enforced in a combinatorially complex way that can only be understood with detailed knowledge of how the discrete states behave. From a macroscopic view the conservation laws are enforced with no visible mechanism to enforce them.

In classical physics higher dimensional models are required for deterministic systems in which we have only limited knowledge and can only model statistically. I think the same thing is true of quantum mechanics. The big difference with quantum mechanics are things like violations of Bell's inequality and quantum computing which are inconsistent with a simple local model in physical space.

Superposition seems to exist physically. Perhaps this can be explained by the chaotic like structural transformation of particles that can start any place. Keep in mind that these must be huge structures relative to the discretization of physical space if they approximate continuous structures to high accuracy. The same particle can start to transform at two or more different places simultaneously. Ultimately only one transformation can complete but for a while there is a physical superposition of states. Quantum collapse is the physical process of these structural transformations. These transformations have focal points in physical space and state space and the location of these focal points are the values observed experimentally. The uncertainty principle constrains how tightly focused these transformations can be in a given experiment. Ultimately spontaneous quantum collapse puts a limit on quantum computing. For problems that are complex enough you will only get a linear speed up. Bell's inequality is not violated at least not relative to the speed of causality in the discrete model. But the conservation laws are enforced by this extraordinarily complex combinatorial process that has reversibility at its core. One cannot understand how the correlations observed in existing experiments occur without a complete understanding of the discrete model.

There are two aspects of a fundamental theory that are generally considered to be independent. These are state evolution and initial conditions. Discrete models offer the possibility that a simple model could fully account for both. In a continuous solution to the wave equation an initial disturbance spreads our over an arbitrarily large volume becoming arbitrarily small in amplitude. A discrete model cannot do this. There is a lower limit to how small the amplitude can become. There are two possibilities as to how a fully discrete model that approximates the wave equation can behave in the long run. It can break up into independent components that separate from each other but do not themselves further disperse. This behavior could explain the quantization of electromagnetic energy. The other thing that may happen is that the wave front fills all of space with a residue of small but nonzero values.

These are not mutually exclusive alternatives. The diffusing wave function can do both. It can break up into components that have structural integrity and do not further diffuse and it can it also fill all of space with a small residue of nonzero values. The latter behavior can only happen in virgin space. A region of space can only become permeated with these residual values once.

This suggests the possibility of a divergent model that creates energy but only on the surface of an expanding sphere. It is possible that a very simply transformation of state model combined with a very simple initial state could account for all aspects of the universe. This would be the ultimate in simple explanations.

It is clearly possible to start with some simple discrete model that supports a Universal Turing Machine and program it so that eventually it will fill all of space with simulations that will grow in arbitrary complexity because for example they simulate every possible program. There is a simplest set of rules and initial conditions that will grow into arbitrary complexity. Perhaps it is precisely this simplest possible model that is the basis for all of physics.

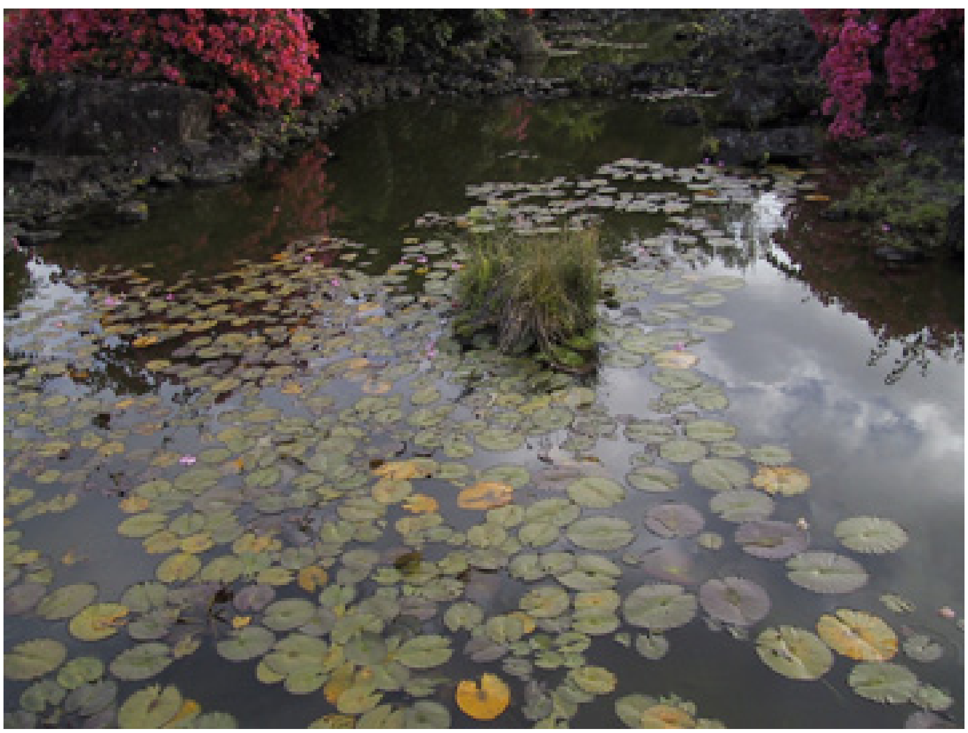
Chapter 9

Applying mathematics to consciousness

Materialism see consciousness as an epiphenomena of matter. Some Eastern philosophies see the material world as an epiphenomena of consciousness. I avoid duality by assuming they are identical.

With that assumption I can apply scientific understanding to the problem of values. Conscious experience has intrinsic value. There is a relationship between the quality of experience and physical structure. This relationship is creative. Fundamentally new structures evolve with dramatically expanded conscious experience. Mathematics is the key to understanding the structure of this creative process and the conditions that make it possible. This chapter explores that relationship.

9.1 Truth is an art



Our ignorance vastly exceeds our knowledge. As we learn the horizons of our ignorance expand far more rapidly then those of our understanding. The Human Genome Project, for example, has completed the mapping of a human genome, but the result is primarily an expanded outline of the depths of our ignorance. The genes encode an enormous number of proteins. The next step is to understand what these proteins are and how they interact with each other and the rest of the body. That is a problem that is ultimately unbounded. For we are interested not simply in what existing structures do, but also in how we can alter them and what the consequences will be. That problem is recursively unsolvable. It will continually expand as we more deeply understand it.

Gödel's result applies to mathematics, but almost any "interesting" physical system, like the operation of our genes, is powerful enough to embed a Universal Turing Machine and thus subject to the limits of Gödel's proof. Of course we need to be interested in the implications into an indefinite and unbounded future to meet the requirements of unlimited time and storage.

The search for truth in any nontrivial field is a divergent not a convergent process. We cannot find *the truth*. We can at best explore all the possibilities and thus insure that no truth is ignored.

How do we deal with an ever expanding ignorance in a universe in which we must make decisions? Truth is an art. It is an experience that must be lived and not a goal to be conquered. Evolution has been working this problem for billions of years. The solutions it has come up with are encoded in our genes.

Inevitably our instincts move us in ways we do not understand. We are beginning to explore some of the cruder and simpler ways that our instincts operate through evolutionary psychology. Long before this analytical discipline existed Carl Jung used the term archetype to denote aspects of the evolutionary structures of our mind[31]. Jung approached this topic with the most profound respect for the depth and power of these instincts.

Those instincts that we have developed and integrated into our ego can make us very uncomfortable. We cannot explain why we feel a certain way or have a certain attitude. There is a powerful human tendency to create fantasies to explain the incomprehensible. Doing this with our instincts is dangerous. The fantasy can displace the deeper reality the instincts are striving for. Avoiding such fantasies is difficult. We like to have good clear reasons for our decisions. To go with what feels right without understanding why can be alarming. Truth is an art that we are learning slowly and painfully.

9.2 Boundary conditions for creativity

Understanding the connection between mathematics and consciousness can help refine this art. Consciousness has a finite logical or mathematical structure. This structure does not exhaust its essence. It hardly touches on it, but it is an essential aspect of conscious experience. Mathematical limits are limits on structural possibilities and these in turn are limitations on consciousness. These limits include boundary conditions for creativity. Within those boundaries and with sufficient physical resources, consciousness can expand without limit.

There is a conceptual leap between the problem of mathematical creativity discussed in previous chapters and practical problems of creativity in our political and economic institutions. In mathematics we are dealing with logically determined systems. Real world systems are far too complex to be fully understood or characterized.

Much of practical creativity involves coping with uncertainty. There is a great practical description of this problem in *Guns, Germs and Steel*[18]. Jared Diamond investigates why certain cultures came to dominate the planet while others remained relatively stagnant. There were a variety of reasons, but two essential ones were diversity and concentration of resources. One needed a dynamic tradeoff between these two for modern civilization to arise. A culture dominated by a single ruling elite, like China, inevitably failed to pursue possibilities essential to future development. Similarly a region, like Africa, with so many small communities could never marshal the resources needed for certain kinds of progress. Europe presented the ideal combination of diversity and concentration of resources.

Competition and cooperation are equally essential elements of creative development. Mathematics teaches us that we must continually increase both diversity and the resources available along any given path if we are not to stagnate. It is possible to investigate this tradeoff in more detail to get tighter constraints on the balance between these two. This problem is of great practical concern in this age of globalization. The fear many have of this process is more than justified. Globalization is inevitable, but establishing the boundary conditions that do not cause creativity to stagnate in a unified planet is an an enormous practical and theoretical problem.

It is all too easy to stray outside of the region where creativity is unlimited. The current bias toward megacorporations is potentially dangerous. It focuses on concentration of resources at the price of diversity. Everyone wants the biggest guns for competing in the global free for all. As we are increasingly dominated by global institutions we must incorporate the boundary conditions for creativity in the structure of those institutions. We must understand and prohibit the conditions that will stifle creativity. We need to better understand the conditions that will cause it to flourish.

9.3 The evolution of consciousness

There are many reasons to not limit creativity. The deepest of these is spiritual. It connects with the ongoing, ever expanding evolution of consciousness. To limit creativity is to limit the richness and depth of possible experience.

Structure is an aspect of essence. Through it we can understand and to some degree control experience. Similar structures correspond to similar experiences. We suspect other human beings have experiences like we do. If two people have similar personalities and histories we assume their experiences are closer than more dissimilar people. Human beings have a richer capacity for experience than beings with less complex and subtle minds.

Human experience is not just richer than that of insects. Human experience is beyond the comprehension of insects. The creative nature of evolution and the hierarchy of mathematical structures suggests that this process of creating values is an ever expanding one. We can evolve into beings as far beyond us as we are beyond insects and *all* of those descendants can do the same no matter how far beyond us they are.

We can understand aspects of the evolution of structure. We cannot even begin to imagine the evolution of experience that manifests that structure. The universe is truly and deeply creative in ways that transcend any attempts to comprehend it. God, as this creative process, is beyond anything any being will ever be able to imagine. For God is becoming through being. She is a potential that can never be fully realized but always is.

Man naturally thinks of himself as the center of the universe and the focal point of creation. That arrogance has repeatedly been proved wrong. We may represent the leading edge of evolution on one small planet in one solar system in one galaxy. We are an incomprehensibly tiny speck in the universe. It is likely that we are far from being at the leading edge of evolution in our galaxy let alone in the known universe. Mathematics teaches us a lesson similar to astronomy. In the grand scheme of what will be we are not even at the beginning.

9.4 Levels of structure and consciousness

In this section we begin to describe how the structures of mathematics connect to conscious experience. Central to this notion and to the structure of our brain is the idea of feedback. There is a general progression of information processing that starts with data from our senses and proceeds to processes that perform ever higher levels of integration. A low level process might detect a sharp change in color at one point in the visual field. A higher level process might integrate these into a line or edge. A higher level process might recognize a door which may be what we experience consciously. We are not aware of the lines that make up the door unless something shifts our attention from the scene as a whole to such details. The process of recognizing a door is not strictly one way. There are feedback loops where higher level processes generate inputs to lower level processes. One speculative theory suggests that the brain, at each level of processing, is continually making predictions of what to expect[25]. Feedback to lower level processes is generated from those predictions. If the expectations are not met then signals to higher level processes are generated. Whatever the detailed structure of the brain, feedback plays a central role. The ordinal numbers in mathematics can be thought of as characterizing the subtlety of feedback in a mathematical system. They are the tool I use to begin to connect mathematical structure with conscious experience.

The ordinal numbers described in Section 5.6 characterize the power of feedback, iteration or self reflection that a system is capable of. There is an enormous richness of possible mathematical structures that can be defined at higher levels of the ordinal hierarchy. Exploring ordinals is not the primary focus of mathematics. Ascending to higher levels of structure is not the primary focus of evolution. However ordinals characterize the power of a mathematical system and the limit of sophistication of a physical structure. This suggests they may do the same for conscious experience embodying such structures.

There is no precise mapping of ordinals to physical systems or biological structures, but biological structures for modeling external and internal state can be assigned ordinals that characterize them. The goal is to develop a feel for the connection between these mathematical structures and biological structures.

Table 9.1 gives some simple examples. The first entry is for a fixed response for fixed input. There is no iteration and thus no sense of the potentially infinite. Thus the limit ordinal for these structures is the first non finite ordinal ω. A limit ordinal encompasses all smaller (in this case finite) ordinals, but is not reachable by the structures that define these smaller ordinals. Next is the amplification of an input. The intensity of the response is determined by the intensity of the input. For example if a creature is running to escape a predator it will try to travel proportionately faster than its attacker. The limit ordinal for this is $\omega \times \omega$. The next two entries involve stringing together amplifiers. The response is proportional to the product of two inputs or more inputs. Finally we consider a variable number of memories with the ability to mutually reinforce each other. If the number of these has no fixed limit then the ordinal that characterizes this is ω^{ω}.

feedback structure	ordinal
fixed response for input	ω
single amplifier	$\omega \times \omega$
double amplifier	$\omega \times \omega \times \omega$
series of n amplifiers	ω^{n+1}
variable number of distinct memories with ability to connect and mutually reinforce	ω^{ω}

Table 9.1: Ordinals that characterize biological structures

It is only simple toy structures like the above that we can easily relate to ordinals. There is a certain artificiality to this process, but it gives a feel for the relationship between feedback mechanisms and mathematical structure. More complex thought processes have ordinal limits, but it is more difficult to determine these.

The most complex structures in the human brain are not for dealing with simple external stimulus. They are for dealing with our fellow creatures. Subtle forms of iteration and self reflection evolve because evolution creates an environment in which they are valuable.

9.5 Evolution feeds on itself

In *The Red Queen* Matt Ridley[43] gives a fascinating example of the evolution creating capabilities to deal with other products of evolution. He describes the evidence that sex evolved to deal with pathogens. The value of quickly spreading a gene that confers immunity justifies the many disadvantages of having to find a mate to reproduce. The extraordinary long term advantages to evolution are irrelevant in the short term. The evidence suggests that in the absence of pathogens asexual reproduction always wins out in the short run eliminating those individuals that need to find a partner to reproduce.

Most of the environment that we interact with to survive is created by evolution. The capacity for language is one example. It has become increasingly clear that we are primed to learn language at a particular age. If a child is not exposed to language at this time it becomes impossible to lean language later. Language is useful only for dealing with people who share a common language.

Evolution seems to boot strap itself to higher levels of creativity. With sufficient diversity it creates environments in which complex structures evolve that have no meaning outside of the environment populated by highly evolved beings. There is no way to extrapolate from any stage of evolution to what may be meaningful and important at much later stages.

It is inevitable that evolutionary structures have evolved to facilitate the creative nature of evolution. Perhaps the most obvious is human culture. By creating beings capable of both cooperation and competition evolution has tapped into an enormous creative force that may completely transform evolution itself. It is this fundamentally creative aspect of evolution that suggest the possible importance of *all* levels in the hierarchy of mathematical truth. Carl Jung had an intuitive sense of the connection between the creative forces of the psyche that he called archetypes and mathematics.

9.6 Number and Archetype

After C. G. Jung had completed his work on synchronicity in "Synchronicity: An Acausal Connecting Principle," he hazarded the conjecture, already briefly suggested in his paper, that it might be possible to take a further step into the realization of the unity of psyche and matter through research into the archetypes of the natural numbers. He even began to note down some of the mathematical characteristics of the first five integers on a slip of paper. But, about two years before his death, he handed the slip over to me with the words: "I am too old to be able to write this now, so I hand it over to you." —

Marie-Louise von Franz, from the preface of *Number and Time*[49].

The mathematical properties of numbers are discovered, absolute and creative. At some point in time each of them may be discovered, but at any point in time only an infinitesimal fragment of them can be known. These seemingly paradoxical properties come from mathematics' concern with the potentially infinite in a universe in which everything that exists is a particular finite experience.

The archetypes have been built from an unfathomable history of experience. The details of those experiences are different, but there are structural similarities that are universal enough to find their way into our genes. The generality that makes these experiences important enough to incorporate in our genetics makes it problematic to apply the experience to specific situations. To a large degree life is a process of refining archetypal material into ideas, intuitions, art and behavior that have value in our life and times. Jung saw medieval alchemy as providing both a metaphor for this process and as an intuitive and intellectual study of the process[28, 27, 30].

Archetypal material related to sex, birth and family is among the most basic and direct. The refinement of this archetypal material to deal with contemporary reality is extremely difficult as the immense problems we are having today in family structure confirm. The difficulty is rooted in the contradictory and competitive nature of the archetypes.

The problem is not just to refine archetype images individually to golden nuggets of practical value. Their deepest values can only be realized through a union of contradictory claims[30]. For example the competition between career and family that creates so many problems is a real one. It does not have a *solution*. The problem is a creative force that can lead us to a deeper development of self or to destruction.

9.7 Archetypes and the infinite

We can gain insight into the archetypes through the finite and infinite in mathematics. There is no finite way to encompass mathematical truth even when that truth is constrained to statements about the future states of a computer following the precise deterministic steps of a program. This is important not only for questions that refer to an indefinite future. Exact mechanistic predictions are rare for complex real world problems. We need to come up with general principles that help us predict the consequences of our actions when we are not able to model precisely what those consequences will be. This can involve the same mathematical principles that allow us to decide that some programs will never halt without being able to observe their behavior for an infinite time.

The unknowable creative aspect of the properties of numbers and the unknowable creative aspect of matter are the same thing.

It is this creativity that has expressed itself in our world as it is today and that continually unfolds in ways that we can never predict or control. The archetypal images of the human psyche are formed from this creative process and point toward it.

The mathematics of creativity as described in Chapter 6 allow us to know with mathematical precision some of the properties and constraints of creativity. It allows us to make connections between some human instincts and general mathematical properties. It opens the Jungian notion of archetype to mathematical analysis. This does not lessen the divine mystical nature of archetype. On the contrary it shows the divine mystical nature of mathematics.

9.8 The necessity of archetypes

How does evolution deal with the mathematical constraints on an organisms ability to control or predict its environment? There is no optimum solution. One should expect many, approaches and many tradeoffs with complex feedback mechanisms between them. It is important for the fit to survive, but is also important that the survivors do not become too narrow in their approach to survival. What may be the best approach in one set of circumstances may be a disaster as things change. This requires instincts for many different approaches and a decision algorithm to select between them depending on circumstances. No selection is ever certain to be correct. Thus decisions are tentative and the path not chosen still clamors for attention.

One can begin to understand at the level of mathematics why the human psyche must be so diverse and seemingly chaotic. All the approaches to patterns in life that have been built up over the eons are there. It is primarily intuition that first recognizes a match and brings something to consciousness. Both intellect and feeling are essential in evaluating the content both in terms of seeing if it applies to the current situation and understanding how to use it. This process is concerned with individual survival, but it also concerned with survival in the broadest sense. Thus some of the images and ideas may be harmful to the individual, but helpful in a broader sense.

The psychic structures that motivate and inspire us cannot be characterized in any simple way any more than mathematical truth can be. They do not have well defined goals because they are concerned with creative evolution. The struggle between the elements of the psyche is necessary. There is no way to decide what is to be done in a given situation. The different possible approaches must all be given a hearing. Then one decides based on an individual approach to life.

9.9 Continual progress can be stagnation

No matter how much our consciousness evolves through the creation of more subtle and complex structures, we will have covered only an infinitesimal fragment of an ever expanding possibilities. As with the evolution of sex, the incentive for evolving general approaches to problems comes from evolutionary competition and cooperation. One only needs powerful self reflecting structure to deal with beings at a level comparable to one's own level. The only process that can evolve such structures without a definite limit is a divergent one following an ever increasing number of paths. Any system that focuses on a finite number of alternatives will, with mathematical certainty, hit a brick wall limiting levels of structure and levels of consciousness. Such a system can make continual progress. However it will have a specific limit that a divergent process can eventually discover. There are finite structures that will, in some sense, encompass the entire infinite sequence of progressive steps that the single path process might follow over an eternity.

9.10 We have just begun

There is a limit ordinal that characterizes the iteration and self reflection we are *ultimately* capable of understanding. We may be able to understand any of the modes of iteration below this ordinal, but we can never understand this ordinal It is only by evolving into higher beings that we can transcend this limitation.

The limit ordinal for humanity is a limit not on the product of an individual, but the product of culture. Our understanding of mathematics is the product of the work of thousands of major players over thousands of years. Their work depended on millions of others who created the wider cultures in which study of mathematics was possible.

It may seem frightening and blasphemous to see humanity not as the culmination of evolution, but as a stepping stone in a never ending process. We are unique in evolutionary history in our ability to influence the planet. We have a achieved a level of self reflection that allows us to build models of the world and ourselves and to test and manipulate those models. We have progressed from marks in the earth to supercomputers. But nothing of our achievements suggests that we are beyond improvement or have reached some ultimate capability.

The decision we have to make is not whether we will continue to evolve, but whether we will be conscious of what we are doing. Only with conscious attention can we take responsibility for what we are doing. Only through understanding the structure and boundary conditions of evolution do we have hope of dealing with the enormous responsibility of controlling evolution. Forces now in motion make our direct control of evolution on multiple levels inevitable. There is no practical way to go back. We need to go forward with the greatest caution and the deepest understanding we can muster. Beyond all this we must develop our values and sense of self so we not only know what is right, but are motivated to do what is right.

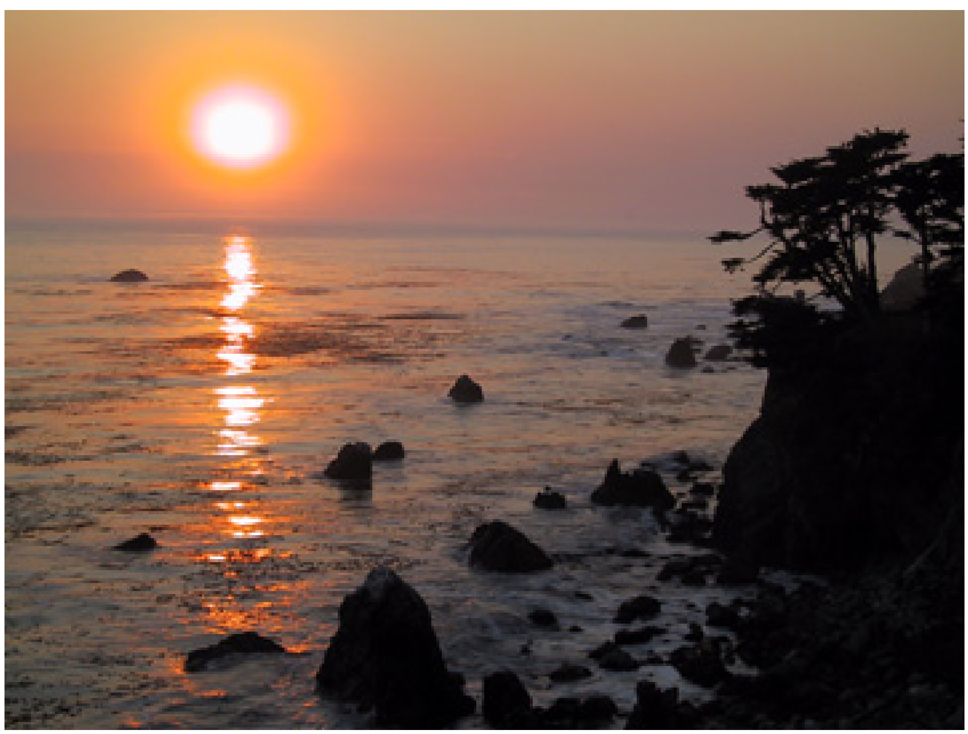
Chapter 10

Values and evil

We are increasingly able to explain the structure, function and evolution of values, but that does not make them less real. A mother's love for her newborn has a biological explanation in the helplessness of the baby and the consequent need to motivate the mother to care for her child. The experience and feelings of the mother are the irreducible primary reality. Our understanding of how those feelings developed functionally can be useful, but it can never explain the primary reality of the experience itself or reduce it to anything other than what is is. Our universe creates values that are real and meaningful because it creates the conscious experience in which all meaning lies.

We do not so much have values as we have a valuing system. This system is all that motivates us to action and enables us to choose. In this chapter we give a brief overview of our valuing system based largely on Jungian theory. We relate this structure to the creative nature of mathematics. We investigate the problem of evil and its role in creative evolution. We discuss the degree to which evolution that has become conscious of itself can eliminate evil without limiting creative evolution.

10.1 Pleasure and pain



In their simplest form values move us toward pleasure and away from pain. Plants have values at this level. It is widely reported that the deepest and most profound pleasure comes through spiritual awakening. The highest forms of pleasure or peak experiences seem to involve a deeply felt connection to the creative process. For me these have come either through insights that seem important or through moments of personal growth and transformation. This makes evolutionary sense. We understand the connection with creativity in the profound feelings that parents have for their children. But our impact on future generations is not limited to passing on our genes. How we live our lives influences others. Culture and technology are creative human products that will determine if we survive as a species and evolve to something higher.

The conscious experience of pain and pleasure is associated with brain structures. Things we are designed to avoid are painful and those we are designed to move toward are pleasurable. The more subtle and complex these systems and the overall structure they are embedded in, the more profound the experience may be. Instincts and archetypal forces that connect us to the creative evolutionary process seem to be capable of generating the deepest and most profound experiences. This is compatible with our sense that subtlety of structure is subtlety of consciousness.

Our valuing system is complicated by our often contradictory motivations. Spiritual experience may involve a sense of transcending these contradictions. Conflict is a central element in our valuing system because it is essential for creativity. We can never escape this reality, but we can be at peace with it. Fundamental to this conflict in ourselves and in the wider culture is the inherent conflict in attitude and function types described by Carl Jung.

10.2 Jung's typology

Jung saw universal 'types' in human personality These types are all present in all of us, but there tends to be one predominant type or normal mode of organizing our experience. The types are both complementary and competitive. One can gain insight into oneself and others by understanding the structure that Jung described, but one must not interpret it too narrowly or literally. The reality that underlies this simple intellectual model is far more complex and problematic than any description of it can suggest. I urge you to read Jung, but I will give a brief summary to provide context for my remarks.

Jung begins his description by noting two approaches to evolutionary success. One can have many offspring with few defenses and a limited chance or survival or one can, at the expense of lower fertility, invest more in the individual, equipping each with more defenses and a better shot at surviving to reproduce themselves. This fundamental tradeoff can appear in many forms. Some individuals limit their activities and carry them on intensively. They are inner directed or introverts. Others are extensive in their activities and of necessity less intense. They are extroverts. We live in a strongly extroverted culture and thus introversion tends to be viewed in inferior terms and seen almost as a defect. Jung, who considered himself an introvert, did not see it that way. Introversion and extroversion are *attitude-types*.

> The attitude-types ... are distinguished by their attitude to the object. The introvert's attitude is an abstracting one; at bottom, he is always intent on withdrawing libido from the object, as though he had to prevent the object from gaining power over him. The extrovert, on the contrary, has a positive relation to the object. He affirms its importance to such an extent that his subjective attitude is constantly related to and oriented by the object[32, ¶557].

In contrast to the attitude-types Jung defined the *function-types*. These refer to the predominant mode of processing information and the orientation of that mode. The rational types process information somewhat like a von Neumann computer. They organize experience in a framework of cause and effect. The irrational types process information somewhat like a neural net. They organize experience in a framework of patterns with more complex and higher dimensional structures than the linear processing of a von Neumann computer.

Thinking uses rational processes, and its own laws or models, to bring elements of both internal and external experience into conceptual connection with one another. *Feeling* uses rational processes to recognize the *value* of an experience or situation. Thinking relates experience to a conceptual framework in which internal consistency and coherence are primary. Feeling relates experience to a framework of what is valuable or important. One strives for a coherence of results rather than a conceptual coherence. Saying contradictory things in different situations may be the best way to get coherency of results. The two functions may be in accord. For example, if there is a physical threat, understanding how to avoid this is what is important. In dealing with other people these functions are often in conflict. Statements that will enhance the feeling situation are often not in accord with ones own conceptual framework. Saying what you believe to be objectively correct can get you in a lot of trouble or alienate people who's opinion is important to your well being.

At a superficial level thinking and feeling types can be mutually attractive and compatible. The feeling type reacts to enhance the feeling situation and thus it seems to the thinking type that they have compatible conceptual frameworks. The feeling type is able to move the situation to what is of value because the thinking type is largely unaware of how these movements are being controlled by the feeling type. This superficial attraction and compatibility can lead to a fundamental impasse if too deep a relationship is attempted. The superficial compatibility comes from the inherent differences that gives each their own sphere of influence. If either tries to move out of their sphere the fundamental difference becomes apparent.

The sensation types are oriented by the patterns they recognize in internal or external experience. The intuitive types are oriented by patterns that indicate where a situation came from or where it is leading to. "In intuition a content presents itself whole and complete, without our being able to explain or discover how this content came into existence[32, ¶770]." The same is true of sensation. When we recognize our friend's face we cannot say what steps we went through to do this. Intuition and sensation are pattern recognition processes. The difference is that sensation focuses on the content of an internal or external experience. Intuition focuses on the precursors of an experience or where an experience might lead. We cannot explain pattern recognition the way we can explain a rational process. That does not mean it is beyond rational or causal explanation. We can describe how a neural net comes to recognize a pattern. We can break this up into causal steps although these are nothing like the causal steps in a rational deduction.

This typology is the basis of the widely used Myers-Briggs personality assessment. The rational and irrational types of Jung bear a stinking resemblance to the comparatively recent left brain and right brain discoveries in biology.

We all have all of these capabilities. We have different strengths and weaknesses. We develop and differentiate them to various degrees. We orient ourselves and our experience in different ways.

To the degree that we one sidedly develop one of these attitudes and functions in our consciousness there will be a compensating effect from our unconscious. The function types form pairs of opposites. Thinking is opposed to feeling and sensation is opposed to intuition. Of course we can think about both the value of an action and its objective meaning. However feeling is not thinking about value. Feeling is *organizing* experience from the standpoint of feeling. A Star Trek episode illustrates this in a way that puts a very negative light on it as one would expect in a thinking dominated culture. Captain Picard is being tortured by an alien that wants to break him. He is shown five lights, but told that there are only four and asked how many lights he sees. Each time he answers five he experiences intense pain. He never gives in, but, when he discusses the incident later, he confesses that at the end he saw only four lights.

Any one of these primary functions can be the dominant approach we use to deal with the world. As we grow with experience we are better able to integrate all of these functions into conscious decision making. To the degree that we have failed to do so the function exists in an undeveloped and unconscious state. It still influences our actions. One way this happens is though projection where we see magnified in others what we are blind to in ourselves.

The functions are necessary and all active in each of us. They are the source of great internal and external conflict in life. They are how nature deals with the complexity of the world and our fellow creatures and the limited knowledge than any of us can have of that world. Conflict is a necessary precondition for creativity.

Conflict is what motivates us to try a new approach.

10.3 Feeling versus thinking

We can illustrate the inherent conflict between thinking and feeling through the character of recent presidents. President Clinton is a thinking type with his objectivity and extraordinary intellect dominating his psyche. Yet he understood that he could seldom accomplish what he wanted with other people by saying what he thought. Thus out of necessity he became a consummate liar. He adopted a legalistic approach to lying so that he could always deny an outright lie. This is also an example of thinking over feeling. His intention and what he achieved was deception. The fact that he accomplished this without saying anything that was literally untrue was irrelevant from the standpoint of feeling. Clinton's method of reconciling his instincts with the demands of a political career can easily be condemned, but one should keep in mind that, however flawed his pattern of behavior, it was enormously successful.

In contrast President Reagan was a feeling type. He did not need to lie because he saw the world through a subjective window that warped truth to support accomplishing what he cared about. This was made explicit in his admission of mistakes in the Iran-Contra scandal. He did not believe there was an arms for hostages deal, but his advisers told him there was. Thus he admitted it without believing it.

President Carter was a scrupulously honest thinking type who was able to accomplish little in the presidency. It is extremely difficult to be objective, honest and successful politically. Consider President Roosevelt's duplicity about involvement in World War II. Perhaps only Lincoln under extraordinary circumstances and with almost unbelievable insight, intuition and wisdom was able to pull it off.

The conflict between feeling and thinking are a good illustration of the problem of decision making. Both viewpoints have validity. Jung said that wisdom was the integration of thinking and feeling. By this he did not mean that the conflict between them could be resolved. He only meant that a wise individual is able to use either function as the situation demanded. Its not a matter of being aware of subjective and objective issues. That is all thinking. It is a matter of shifting functions and standpoints.

10.4 Culture and values

As mentioned in Section 9.6 the full spectrum of our evolutionary past is reflected in what Jung called archetypes[31]. Because of their universal nature, archetypes are a starting point for cultural values. The problem of integrating the disparate and often conflicting forces that motivates us is complex. We need to create a coherent sense of the world and our place in it that is a reference point for decision making. It can be an ideal we strive for like the good life that Christianity promotes or the complete cessation of striving that is the Buddhist ideal. It can be material success that seems to be the religion of many people today. It is something we choose because it fits (however awkwardly) our instincts and place in the world.

This individual system must function in a wider culture. Common to all cultures is the existence of some codified value system. Just as an individual needs a valuing system to make decisions so does a group. Such systems are older than humanity. Value systems are needed for any animals that at times act collectively to defend themselves, obtain food or care for their young.

The construction of such value systems is the greatest practical problem facing humanity today. Wars are fought over conflicting values. What resource we devote to protecting the environment, combating AIDS and educating the impoverished all depend on our values.

As globalization increases a world government is just beginning to impinge on national sovereignty as for example in the World Trade Organization and the World Court. A unified world is essential to prevent war, deal with our global impact on the environment and, in time, eliminate world poverty. But a world government could impose limits on creativity with no competing governments to pursue the path not followed.

We could repeat on a global scale the historical mistake of China. It was far ahead of the rest of the world, but squandered its lead because of dictatorial decisions to cease developing some technologies like ship building [18]. A global government that restricts some paths of development will have no countervailing force to correct the problem.

The key to avoiding this danger is an objective system of values that establishes in a testable verifiable way the boundary conditions that support unlimited creative evolution. Such a system might obtain the same wide spread acceptance as science through its demonstrable practical value.

10.5 Political and economic institutions

The declaration of independence and constitution of the United States codify a successful political structure. They established a limited central authority that enhances rather than circumscribes individual freedom. The central government attempts to minimize forms of competition (like murdering your opponent) that do not enhance creativity while supporting those that do. Through democracy it creates at least the possibility of an effective voice to any group no matter how little power or influence they may have aside from the vote.

These documents reflect an intuitive recognition that freedom of individuals and institutions is crucial to creative development and the pursuit of happiness. We have a creative instinct that is always seeking something new and different. The founding documents of the United States reflect and respect human striving as a universal instinct not limited to a ruling elite.

The political structure of the United States established boundary conditions that have supported enormous creativity. Yet our history is scarred by equally enormous injustices starting with slavery and continuing with fraud and exploitation of the weak by the strong up to the present moment. We can do far better then we have. The increasing disparity between the haves and have nots and especially the increasing impoverishment of those near the bottom is a serious threat to democracy itself.

Globalization has made the poor through out the world competitors for an increasing range of jobs. This is eroding the need for and bargaining power of low skilled workers. At the same time the political influence of money has expanded with the growing importance of mass media and the rising cost of advertising.

Our rapidly increasing productivity should be making life better and easier for everyone. We should have more choices and options. But the opposite seems to be happening and not just to those at the bottom. Increasingly jobs that will support a middle class life style are making greater demands on the employee's time and energy.

These trends are a corrosive process that could destroy the foundations of democracy and with it human creativity. Part of the problem is the erosion of a value system that saw the family as the primary source of meaning. When I was growing up in the 1950's a job was seen primarily as a means to support a family and every *man* who played by the rules was entitled to a descent job at a living wage. Of course that value system was permeated with injustices against women, blacks, gays and everyone that did not fit a narrow Ozzie and Harriet stereotype. Those advocating a return to that value system are moving in the wrong direction.

But there was something important in that value system that we do need to revive. The focus on the family was also a focus on quality of life. The purpose of work and corporations was to provide a higher quality of life for the companies employees, customers and investors. Economic gains through greater productivity is one *but only one* means to a better quality of life. A good income means little if one has no time to enjoy it and to nurture all the dimensions of ones humanity.

Competition is essential to creativity, but diversity is equally essential. A winner take all economy in the long run is less productive because it destroys diversity. Productivity and economic wealth have no intrinsic value. They are means to the end of enriching and expanding conscious experience. The forces that put too much emphasis on winning are equally a threat to diversity and quality of life. We need to develop some cultural counterweights to insure the diversity that is at the core of our enormous economic success will continue. One way to do this would be through the tax system as described in Section 14.2.

One source of perverted values in Western culture is the search for logical absolutes where none exist. In part the winner take all economy stems from this. If productivity is good than more productivity is better even if does serious damage to long term creativity. An extreme example of this is the bumper sticker: "He who dies with the most toys wins." Life is not a game to win or loose, but an experience to enjoy. Measuring ourselves against others in limited ways is important, but it is not what life is about. We are different from each other for good reasons.

An important example of this fallacious logic that comes from requiring absolute distinctions where none exist is the argument that human life begins when sperm meets egg. It is obviously absurd to call the resulting single cell a human being. It is human only in its potential and not in its being. As an embryo and fetus develops it becomes increasingly human with no absolute dividing point. The seriousness with which one must take this developing life depends on its stage of development. That is an uncomfortable reality for many on both sides of the abortion debate. But it is an obvious fact if one is willing to simply look at what is and recognize that human life has no absolute defining set of conditions. Insisting on an absolute boundary leads to absurd philosophical positions by creating boundaries where there is a convenient absolute dividing point even though that boundary has nothing to do with the distinction one is trying to make. It is the modern equivalent of arguing about how many angels fit on the head of pin. One can construct elegant arguments about this, but in the end they are pointless and meaningless.

Morality is an unsolvable problem. No simple absolute rules can be adequate to codify moral behavior. Life creates values. It makes as much sense for there to be an absolute final morality as there does for there to be an an absolute final biology or mathematics. These fields have no bounds and neither does the evolution of values.

10.6 Religious institutions

Religion connects our individual existence to a deeper reality. It touches some of the deepest and most powerful values in the human spirit. Religious institutions have often done more harm than good. I can speak about the Roman Catholic Church that I was raised in. Today I consider it an immoral institution.

As a teenager I was taught that I could be damned to hell for all eternity for masturbating. That remains part of church teaching although the Second Vatican Council shifted the focus of the church toward God's love and away from hell and damnation. Teaching a pubescent teenager that unforgiven masturbation leads to damnation is psychological child abuse. It damages teenagers, like I was, that foolishly take it seriously. That the nuns and priests who were doing the abusing firmly believed they were doing God's will does not make it less abusive or less immoral. Morality is hard. It requires understanding the consequences of ones actions and that is something one can never be certain of. It demands that one studies and understands what science teaches us about human nature. When the evidence contradicts pridefully held ancient dogma the dogma must go.

The current scandal in the Church about priestly child abuse has its roots in ignorant dogma taking priority over objective understanding. The scandal is not that some priests abused children. That is horrible, but it is the sin a few and not the scandal of a church. The scandal is the way the church dealt with the problem. Those who understood how to deal with it did not share the Church's ignorance of human sexuality. Those that shared the Church's ignorance, and thus were acceptable advisers to the church, were of course incompetent.

What the church calls natural law is in violent conflict with human nature. As a result the church is schizophrenic. In the United States the laity largely ignores the Church's teaching on sexuality. The clergy is torn between the need to minister to their flock as they actually behave and a dogma that insists that some of what the flock does, with no sense of guilt or sin, is cause for hell and damnation. Many of the clergy are caught in a struggle between the reality of their own sexuality and the Church's sexual fantasies codified in church teaching.

Outside of the developed world the prideful ignorance of the church leads to immorality on a grand scale. The bible says "By their fruits you shall know them." By that standard the sins of the Catholic Church are responsible for untold human suffering and death. Preaching against condom use in an Africa being destroyed by AIDS is to commit mass murder, if you are an institution whose teachings are taken seriously. It is irrelevant that what the church advocates does not spread AIDS. It is irrelevant that the church has no intention to kill. The only thing that is relevant is the predictable consequence of what the church does. "By their fruits you shall know them." Similarly preaching against birth control in a country that cannot support their existing population is an unconscionable cruel and evil act by those that have the trust and faith of the people.

The immorality of crashing a plane full of innocent people into a building full of innocent people is obvious. Claims that this is justified by God's will deserve utter contempt. The Catholic Church's immoral teachings on human sexuality have and are producing far more evil than the destruction of 3,000 innocent lives. The Church's invocation of God's will as justification deserves the same contempt we give to the religious claims of Bin Laden. This is not to imply that the two cases are similar. Murdering people with a religious rationale is different than preaching what one honestly believes even though it has disastrous consequences. The correct response to the former is police or military action. The correct response to the latter is to try to convince those doing the damage and their supporters of the evil results of their actions.

The church recognizes the immorality in its past when it was torturing in the cruelest possible manner and burning at the stake heretics, witches and anyone else that the church, in its psychotic paranoia, saw as a threat to its authority. When it was murdering people up close and personal there was a limit to the damage it could inflict. The cruelty it inflicts today is far worse.

We are living in a world that can no longer afford the prideful ignorance of such a powerful institution. It is immoral to support the corrupt Catholic Church financially or in any other way. Of course there are many truly good works the church does. Supporting these is not necessarily supporting the church as an institution. Making the distinction is not easy. Morality seldom is.

I single out the Catholic Church because of my personal involvement and because of their size and importance. Many other Christian churches have teachings as abhorrently immoral if not more so than those of the Catholic Church. Orthodox Jews, many Muslims and branches of practically every religious tradition cling to obviously false dogma about human nature.

Science has done vastly more to ease human suffering that all the religions of history combined. I say this recognizing the enormous importance of spirituality to so many lives including my own. Spirituality is as natural as sex and does not require religious institution to experience and develop. More often than not institutions get in the way of authentic spirituality. The liturgy and symbolism of the Catholic Church can aid spiritual experience, but the obstacles of arbitrary and false dogma seriously hinder such experience.

Religious freedom is essential to a democracy, but so is free speech. Those that cling to a religious tradition do not have a monopoly on the language of morality. When religion claims precedence over scientific understanding whether it is about the sun orbiting the earth or about human sexuality it is the moral obligation of those who know better to speak up. It is vitally important that they speak out when those beliefs lead to evil consequences.

'Evil' is the correct word, Science is not value free. It values the truth. To claim moral precedence for absurd beliefs violates the morality of science and scientists have a moral obligation to respond to such ignorance. When the consequences are human suffering than the language of morality is not just appropriate but essential.

10.7 Evil

The concept of evil serves a valuable purpose. It helps us deal with threats. If one village is attacking another, it will be hard to fight effectively, if one sees the man one is about to kill as like oneself. Objectifying an enemy allows us to be warm and loving with our family and and coldly brutal to an attacker. The worst evils perpetrated by human beings are associated with this instinct. The genocides of the 20th century all used it.

The concept of evil is not an absolute. It is a created concept that can all too easily be misapplied. We need to think of and deal with evil like we deal with infections disease. Evil results from processes and instincts that are an essential part of creative evolution. Evil is contextual not innate.

Evil is not a matter of intention but of consequences. We cannot insure the morality of our own actions, but we can increasingly learn to improve the odds. However risk is inevitable in a divergent creative process. To be creative one must follow untried paths and the result can be disaster. Complete elimination of pain and suffering can only come at the cost of creative stagnation and that would be the greatest evil of all.

> The Tao doesn't take sides;
> It gives birth to both good and evil.[48, v 5]

> What is a good man but a bad man's teacher?
> What is a bad man but a good man's job?[48, v 27]

Evil is inevitable but it can be conquered. Historically and to this day the greatest single evil facing humanity is infections disease. We have made enormous progress in conquering that evil in the developed world. There is reason to suspect that within the next few decades our genetic understanding will lead to the nearly complete victory over this most deadly of human scourges. Total victory will requires political reform as much as scientific advancement and the former is less predictable than the latter.

The organisms that cause disease in humans do not have evil intentions. As we discussed in Section 9.5 pathogens seem to have been responsible for the evolution of sexual reproduction without which human life never could have evolved. The competition for resources that is at the heart of so much evil is a necessary element for creativity. Evolution does not care what pain or pleasure leads to survival and creativity, but conscious creatures do.

Consciousness changes the game. We can eliminate much of the pain of the struggle for survival. It is debatable whether we have made true progress. We have eliminated many sources of physical pain and suffering, but we seem to have amplified psychological pain and suffering in the process. Many "primitive" cultures seem to provide happier lives than that of the harried commuter trapped in an isolated suburban home and money obsessed corporate culture .

It takes more than technological or economic advances to limit evil. It takes a deep understanding of human nature and the wisdom to construct institutions that are both economically productive and life enhancing. We do not know how to do this very well.

The enormous success of the West has come from a one sided emphasis on intellectual development. It is the source of our great scientific and engineering achievements. But this has created an often empty culture in which life can seem meaningless and be oppressive. The contradictory forces in the human psyche that are essential for creativity are also the roots of a fulfilling life. We need to achieve more of a balance by developing more fully the other psychic functions. The place to start is intuition and this is the subject of Chapter 11.

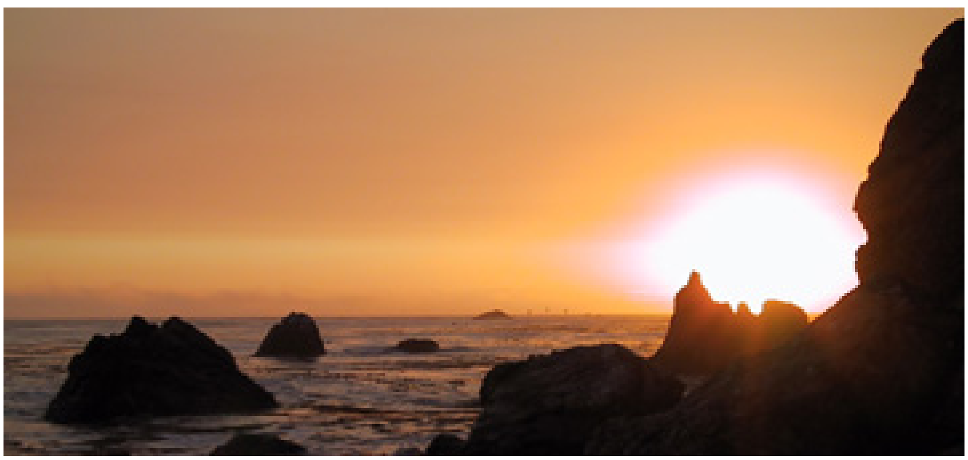
10.8 Conquering evil

We conquer evil by recognizing that what we interpret as evil always has roots in the creative evolutionary process. When disease was seen as the work of the devil little progress was made. As we have come to understand it as an integral and even essential part of the life process we have made enormous progress. Our instinctual sense of good and evil serves important practical purposes in limited circumstances when we can only act effectively by objectifying an opponent. Our ancestors needed that instinct to survive, but we seldom do. It is like our desire for sweet and fatty foods. It served our ancestors well, but can easily kill us. As weapons of mass destruction become more deadly and more widely accessible the danger that this primitive instinct will create mass horror is increasing every day.

We need to look at our economic, political and social institutions as processes that serve practical ends and require continual refinement and optimization. We need to abandon our idealogical and religious prejudices because no simplistic set of ideals is adequate to deal with the practical complexities of life and culture. First and foremost we need to recognize that the end all these institutions serve is that of enhancing and expanding conscious experience both now and into an indefinite future.

Chapter 11

Intuition



Intuition is defined as knowing or sensing without the use of rational processes. This negative definition is a sign of our limited understanding of this faculty. We do not know all the dimensions of human sensation and communication. Intuition defined in this negative way may involve capacities that are not part of existing science. One aspect of this I have experienced involves staring at someone especially with some sexual energy. Many people seem to sense this. People often look directly at the person doing the staring even when there seems no way they could have sensed this.

In the absence of careful experiments it is difficult to know if one is incorrectly attributing significance to random occurrences.

A recent experiment on staring[4] has produced no significant effects. However, if such an effect exists, it may be subtle, selective and easily missed by experiments not properly focused. We need to be open about such possibilities, while being skeptical of the significance of anecdotal or personal experiences. It is equally important that we be careful in not overinterpreting experiments. In science it is most important to be clear about what you do *not* know.

Whatever other nonrational ways of knowing may be a part of intuition, pattern recognition plays a major role. As we better understand the brains ability to find meaningful patterns and other nonrational means of knowing, we will differentiate these capacities and describe them in positive language. This chapter focuses on intuition as pattern recognition. The source of those patterns is not limited to an individual life experience. They include genetic knowledge revealed as archetypal images as described in Section 9.6.

11.1 Intuition as pattern recognition

Pattern recognition like intuition has a vague definition. We know what it means to recognize a face, but we cannot explain how we do it. Intuition is like that. An idea or solution to a problem pops into our mind, but we have no idea where it came from. Intuition is often perceived as an external force. A writer may speak of the characters in her novel as creating their own story.

We have recently developed two technologies for pattern recognition, neural nets and genetic algorithms, that do not use rational or deductive processes. The field of neural nets originated in a desire to better understand the human nervous system and to apply that understanding. There is no precise definition of a neural net, but they generally consist of a large number of simple processors connected to near neighbors. There are input and output connections and simple algorithms determine their relationship. The networks are trained through some process that adjusts the relationship between inputs and outputs to enhance some global result. In a simple example there are weights on each of the inputs. An output level is computed as the sum of the products of a weight for each input and signal level on that input. During a training period weights are increased on inputs with a strong signal when the system is getting closer to the desired response and decreased when it is moving away from it. Such simple devices can be extremely effective at solving problems for which there is no simple analytical solution. Of course they do not usually produce an optimal result.

Genetic algorithms, in a very simplified way, mimic biological evolution. A population of individuals with various traits is created. They are evaluated for fitness against some criteria. Those that rank highest have their properties mixed to create the next generation. A substantial portion of automated investing uses genetic algorithms[35, p. 87].

There is no simple way to explain why a neural net or genetic algorithm produces one response rather than another. One can do a detailed analysis of the state and explain exactly why this history and input produces this response, but that does not explain why one alternative is better than another. Because intuition is a generalized pattern recognition process, you cannot break up the result into a series of steps or analyze the process for mistakes. The way you discipline and develop intuition is different then the way you develop intellect.

Neural nets and genetic algorithms are increasingly important technologies that recognize patterns without a rational deductive process. We know in complete detail what neural nets and genetic algorithms do, but we do not understand how they work in the way we understand a rational process. These processes depend on a limited uniformity in the world yet they are robust in the face of anomalies. Our concern here is not the structure of these processes or the much harder problem of understanding the structure of human intuition. We want to look at the practical questions of how we develop intuition individually and culturally.

With intuition the search for patterns often includes archetypal material. Little in our lives is fundamentally original. Almost every situation we encounter is similar to an immense number of previous situations. These similarities are not limited to the human species. They go back through the history of evolution. For example walking past a dog that feels you are violating its territory raises instincts in the dog that are not so far removed from similar human instincts.

Many traditional approaches to developing intuition, like astrology, the *I Ching* and Tarot connect with archetypes. Evolution molds life to respond to recurring situations. The *I Ching*[53] is a catalog of recurring life patterns. It can strengthen our awareness and sensitivity to the archetypal patterns that intuition recognizes. With a better conscious understanding we know more about what to make of these patterns and we can better focus our intuition.

11.2 Einstein's intuition

Einstein is universally regarded as the greatest physicist of the 20th century. He alone was responsible for relativity and was a major contributor to quantum mechanics. These two theories have dominated twentieth century physics. Yet for most of his adult life Einstein was at odds with the majority of his colleagues about the nature of quantum mechanics. Was this disagreement simply a matter of different opinions or was something deeper involved? Einstein was an intuitive genius. He was of course intelligent, but by no means an intellectual genius. It is no accident that he was working in the patent office when he developed special relativity. He was not considered competitively qualified for a professorial appointment until he had revolutionized our understanding of time and space.

It is worth looking closer at this quarrel. Extroverted thinking, that dominates our culture including science, draws its energy from the external facts or experimental results. Quantum mechanics is extraordinarily successful at explaining those facts. The refinements that his colleagues made to the theory while Einstein was pursuing his futile quest for a more complete theory have made quantum mechanics, and specifically quantum field theory the most accurate theory man has ever developed by a wide margin. Certainly his colleagues had reason to complain when they accomplished so much and Einstein so little. Einstein respected the enormous achievement but felt we must start over.

> There is no doubt that quantum mechanics has seized hold of a beautiful element of truth and that it will be a touchstone for a future theoretical basis in that it must be deducible as a limiting case from that basis, just as electrostatics is deducible from the Maxwell equations of the electromagnetic field or as thermodynamics is deducible from statistical mechanics. I do not believe that quantum mechanics will be the starting point in the search for this basis, just as one cannot arrive at the foundations of mechanics from thermodynamics or statistical mechanics(461)[20].

We must start over because you cannot derive a causal theory from a statistical one. Einstein had an inner vision or intuition about what was and was not a good fundamental theory. A theory that did not match that inner vision was sadly lacking no matter how successful it became. Quantum mechanics did not match this vision and no amount of doctoring it to cover a wider range of effects or achieve greater accuracy could help. Quantum field theory, which combines special relativity and quantum mechanics, was anathema to him.

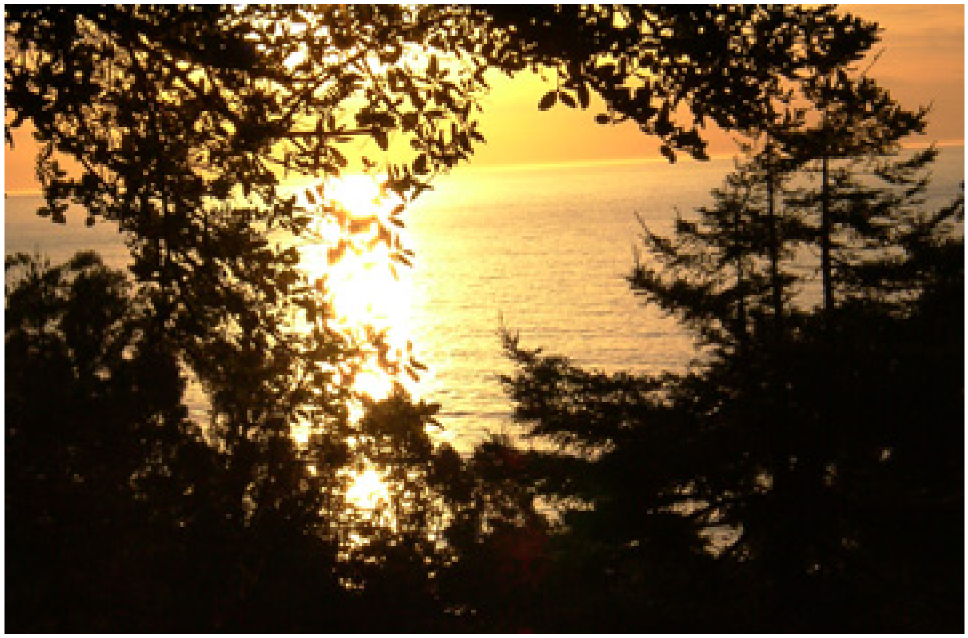
> Einstein never had a good word for the relativity version of quantum mechanics knows as quantum field theory. It successes did not impress him. Once in 1912, he said of the quantum theory the more successful it is, the sillier it looks(24)[41].

His colleagues, impressed by the enormous success of quantum mechanics, did not share his view. They understood how the theory fell short of what had been accepted principles for a physical theory. Their solution was to modify these principles. Thus we have a host of interpretations of quantum mechanics each with its own special metaphysics and new principles for a fundamental theory. For the extrovert the idea must succumb to the data. For the introvert it is the opposite. Neither principle works universally. That is why an opposition is needed.

Why do I insist that the idea will ultimately win out in this contest? It is the accumulation of intuitive problems with the theory. They are what make the theory look sillier the more successful it becomes. The problems are listed in Section 8.8. Beyond this intuition is able to consider possibilities that intellect cannot deal with. Intuition is always ready to start over. Intellect is loathe to do so because without its existing conceptual framework it is lost. It has nothing to orient itself with.

For intellect to proceed in physics it must have or work out the mathematics in some detail. Intuition can play with ideas at a looser level. Intuition can leave the conceptual framework of classical particles that quantum mechanics is trapped in. Without knowing the details it can match patterns and see where connections are possible in a different framework. Of course this process is far more error prone then a more narrow intellectual approach, but for many problems it is the only possible approach.

11.3 The dawning age of intuition



In Jungian theory the process of individuation or becoming an individual is that of developing and differentiating ones psychic capacities to create an integrated whole or self that is adapted to both external reality and the creative life forces one has inherited genetically. Central to this process is the development of the often inherently conflicting forces of the attitude-types and function-types described in Section 10.2.

The greatest difficulty in this process is developing the function that is the opposite of one's dominant function. Thinking and feeling are opposed pairs as are intuition and sensation. The opposite function-type also has the opposite attitude-type. The opposite of extroverted thinking is introverted feeling. The opposed or inferior function, because it is the least developed, is the most firmly entrenched in the raw archetypal material of the unconscious. This can give it a magical and menacing tone. The path to the inferior function is through the other pair of functions. For a thinking type this means intuition or sensation. Because these functions are not in direct opposition to the dominant function they are not so difficult to develop.

Cultures also have a dominant type. They go through a process of cultural integration of more of the genetic psychic inheritance of their people. Cultural evolution is not individuation, but it comes from the individuation of its people. Western Culture is dominated by extroverted thinking. We individually and collectively project the undifferentiated elements of our psyche unto others. "And why beholdest thou the mote that is in thy brother's eye, but considerest not the bean that is in thine own eye?"

Projection serves the purpose of making us aware of personal issues in a way we can deal with. That purpose is only served if we can move beyond the projection to begin to see in ourselves what seems so transparent in others. This process can cause a great deal of harm before it achieves its natural ends. Projections can give energy to the inferior elements of those we project our own inferior elements onto. This is especially true if we are in a position of power or dominance over the target of our projection. Because Western Culture and the United States is so dominant in the world our collective projections are capable of enormous harm.

The second half of the twentieth century provides a depressing example of this. Communism was both a real menace and a magnet for our projections. Our zeal to combat it came not just from the real threat it posed, but also from the numinous chthonic power that we attributed to it through projections. Any crime was justified to combat such a menace. We could easily justify policies advantageous to powerful special interests that cruelly and inhumanely exploited populations of other countries. We masked our own power grabbing cruelty by attributing such base instincts only to our adversary. As a result we supported some of the bloodiest and most evil dictators on earth. When communism collapsed through the weight of its inefficiency and corruption the horrors we helped to create remained to menace us.

The battle against terrorists is the easier part of the struggle we face. The harder part is getting beyond our projections and the horrors those projections feed. Until we do that new horrors will emerge no matter how successful we are at destroying our current real enemies. These horrors have many roots. We do not create them. We feed them. Our power can greatly amplify the corrosive forces that already exist throughout the world.

The starting point for Western cultural evolution must be the development of intuition. It is the path to integrating our inferior feeling. It is impossible to approach that task directly. Feeling and thinking are too antagonistic. One must develop culturally the complimentary functions first and intuition is the most important of these.

11.4 Developing intuition

Many of the most difficult problems have been and are not susceptible to a primarily intellectual approach. Intuition aids us in dealing with these problems as a sort of poor relation to intellect. We recognize the need for inspiration and creativity, but treat these as magical gifts and not as a talent that can be developed or neglected.

The creative arts are further along at developing intuitive talent than academia and the worlds of science and technology. Intuition is at the core of the creative arts, but it is equally central to any creative endeavor. In the creative arts, intuition is often holds center stage. There are disciplined approaches to developing and extending it. Method Acting is one example.

How do we recognize and develop intuitive talent? When I entered the University of Illinois at Urbana as a freshman I took a test for the Math Honors Program. The test consisted of three problems one of which one needed to solve to pass the test. A straightforward solution to the problems required a course in mathematics one level beyond what one had taken in high school. I only got as far as analytic geometry at my small Catholic high school and so a straight forward solution to the problems required a knowledge of calculus. Of course you did not need to reinvent the calculus, but you did need to invent some aspect of the ideas that led to the creation of the calculus. A thorough understanding of the mathematics one had studied was not enough to pass the test. One had to come up with a creative approach to the problem.

This is one approach to testing for intuitive talent. Can one extend what one has learned to solve problems that do not have a straightforward solution? When I taught a course in logical design at UCLA I included a question that most students missed even though it was a 'who is buried in Grant's tomb?' type of question. One can construct logical circuits like those described in Section 5.3 with multiple levels of logic. The outputs of lower levels are inputs to higher levels as shown in Table 5.2. If one has a truth table for such a circuit one can construct the minimal two level solution, i. e. the one that has the fewest logical operators: AND and OR. The problem started with the diagram of a logical circuit. There were three parts to the problem. First construct the truth table for the circuit. Second construct the the minimal two level circuit that implements the truth table. Third note that the circuit you constructed has more logical elements than the original circuit. Explain how it can be the minimal two level solution. The answer was that the original circuit had three levels.

Why did so many students miss this obvious answer? I think it is because they are not taught to make connections. They are taught to apply methods. If you are exploring possible connections than the phrase 'minimal two level circuit' suggests that a three level circuit gives you more options to try and might be better than a two level solution. Intuition is always making connections and seeing possibilities. Invariably most of these connections and possibilities are meaningless, nonsense or false. But that is how intuition must work. If it is constrained by what makes sense logically than it cannot do its job. Intellect and other functions are needed to evaluate the work of intuition, but they must not limit the scope of its functioning. Often that is precisely what we are taught in formal education.

After all intuition can be extremely distracting. Instead of focusing minds on the material at hand it leads off in all directions. We need the discipline of focus, but we equally need the discipline of intuitive rambling. We need to give space for and encouragement to both. Formal education almost universally discourages intuitive wandering. No doubt one of the effects of Ritalin widely prescribed to children for 'Attention Deficit Disorder' is to weaken intuition.

There are three components to developing intuition. First is providing the personal material in terms of learning and life experience that intuition uses. Next is exercising the intuitive muscle by using intuition. This should involve both random rambling and focused problem solving. The problems must always be ones for which one knows no canned intellectual approach. It is a challenge to create such problems. One cannot give a standardized test for intuition because one can always educate for the test. Finally there is the need to develop the archetypal images that intuition relies on. The best of Fairy Tales is one way to do this for small children.

Astrology, the I Ching and Tarot are examples of ancient methods of developing intuition. They focus on archetypal images. They describe the seeds of transformation that exist in a current state. They are are immersed in superstition. No attempt has been made to integrate them with scientific understanding or to create similar new forms that are compatible with contemporary science. This is particularly difficult because archetypal material have a numinous chthonic aspect.

How can we develop intuition, let it lead the way and yet hold it back from leaping into the abyss. For intuition to become more universal it must become more developed and differentiated. We must know when and how to use it and we must know with some, albeit imperfect, reliability when it leads us too far afield from what is practically possible.

The one sided culture I am so critical of has provided one important tool for this. The computer allows us to create artificial universes to play with ideas and refine our intuition. I can learn complex technical material best if I can program it and play with the program. A mass of equations without the opportunity to make them alive in a computer is virtually meaningless to me. It is not that I am unable to understand them, but the mode that I can understand them has to involve an element of playfulness and has to be tolerant of many silly errors which I continuously make. Although a computer is completely intolerant of mistakes, it allows as many tries as you are willing to make to get it right.

Intuition is not as quick as intellect but it is deeper. Intellect can easily grasp things as a series of complex operations. This is impossible for intuition. Intuition must know how the operations relate to each other and to a host of similar operations that are already understood. This takes time and it takes playing with ideas. For complex systems this is impossible without a computer to handle the details. Of course there is no intuitive only or intellectual only learning. All learning involves sequences of steps, playing with ideas and relating new ideas to old ones. The difference is one of emphasis.

The computer combined with communication technology is a powerful aid to intuition in another way. It can create learning and dialog networks of people concerned about a particular issue. The misnamed newsgroups on Internet serve this purpose. Although they do contain some news the vast majority of traffic involves networks of people exchanging ideas and learning from each other material that is far from new. For me this was an effective way to learn the language and some of the technical content of quantum mechanics. It helped me to extend my ideas and put them in a context that others could more easily understand.

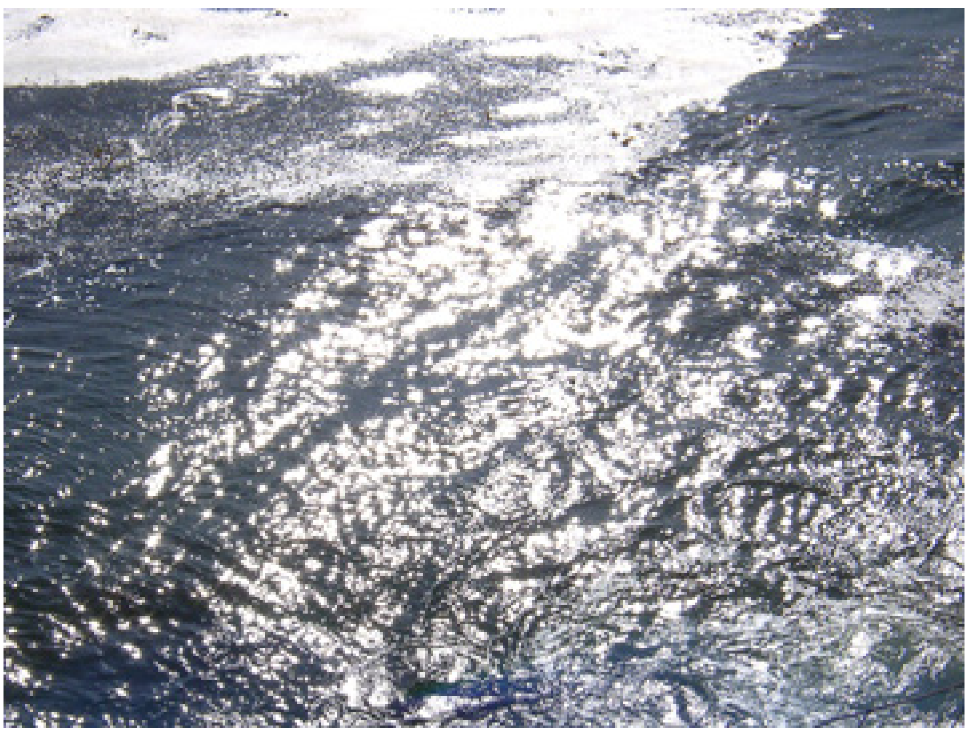
Technology can change the value of human talents. Gauss had an advantage over his colleagues in being a skilled calculator. That was an important asset for a mathematician in his time, but is of little use today. No matter how good a calculator you are you can buy a better one for a few dollars.

Computer technology allows us to automate many of the simpler intellectual skills such as calculation. Inevitably this lowers the value of those skills while opening new possibilities to those with different skills. We are just beginning to understand what can be done and still view this opportunity too narrowly. We want to automate mathematical proofs so we try to create completely automated theorem provers. We want to automate chess so we make a computer program that can beat a grand master. Technology is far from being able to replace the human mind. The enormous calculating power of modern computers is sufficient to defeat the best of human chess players with the brute force methods that such chess programs use. That is not the way to make the best chess player. To do that combine the special skills of the computer with the subtle skills of the human. Let the human use a computer program to aid play just as you let a student use a calculator during a physics exam. The best computer aided chess player will almost certainly not be the same person as the best unaided chess player.

Finding the worlds best computer aided chess player may not be important to cultural development, but effectively using the computer to amplify human mental skills is. This is starting to happen with intuitive graphical user interfaces, programs to do mathematical analysis as well as computation and tools for scientific visualization. However we must recognize how primitive our understanding is. People with powerful intuition that have played a major role in science like Einstein and Jung are usually in Jung's terminology thinking types. Their greatest strength is their powerful intuition, but it is only through the dominance of intellect that they are able to digest the fruits of that intuition to a form that can be appreciated by our intellectually dominated culture. To get beyond this stage is no small task. We have *regressed* in the institutional structures to develop intuition since the middle ages. It is not possible for anyone to say what a world with intuition and intellect in more equal roles would be like other than it will be markedly different and far richer than the world we know.

Chapter 12

The deeper self



Sense of self is a creative product of evolution. The “me now” sense of self is essential to protect our fragile bodies. Evolution has created other senses of self. These include identification with ones immediate family, extended family, community or tribe, nation and religious heritage.

These identities have evolutionary roots. Only instincts that support survival in both the short and long term can evolve. As the sense of self becomes more inclusive the focus shifts to a longer time scale. Finding enough to eat is a matter of immediate survival. Building a stable family involves the time scale of a generation. Identifying with a nation connects one to a history beyond human life span and an indefinite future. Religious and spiritual instincts connect us to the creative evolutionary process.

In exploring these senses of self we will be guided by the psychology of Jung. Jung's work is an intuitive guide and not an intellectual theory. We are traversing an arena where there are no definitive answers. Indeed our collective task is to create the answers that do not exist.

12.1 Attachment and the ego

We start to acquire a sense of self, probably in the womb, as we learn to distinguish me from not me. ‘Me’ is what I feel directly and what can hurt. It is what I can control with an act of will. ‘Not me’ is everything else. This sense of self is a pragmatic invention of evolution called the ego. It is attached to many things. That is its nature. It needs to preserve its body and propagate the genes that created it. The story of the ego is inevitably a tragedy. It ends in death often preceded by disability, pain and suffering.

We have evolved other senses of self. We are part of families and wider communities and we identify with these. There are solid scientific explanation for these wider senses of self. There are formulas for maximizing the propagation of ones genes. These compute how much of our resources we should be willing to give up or risk for others. The formulas depend on how much doing so will contribute to the reproductive fitness of those we help. They also depend on the proportion of our genes that are likely to be shared by that person.

Buddhism sees detachment through enlightenment as the only escape from the suffering of life. Buddhism has a rich and specific tradition and enlightenment in the Buddhist sense can only be approached in that tradition. But one can, without following any specific tradition, move ones emotional center and focus away from the ego and toward the self. This is a natural process that comes with the maturity of accepting life as it is. The result is a detachment from ones individual existence and a wider sense of identity.

Identity with a family and community are powerful human instincts obvious to everyone. The wider identities we all share are not so connected to personal survival and short term reproductive success. They are more subtle. They have evolved with a focus on long term evolutionary success. This inevitably involves a wide diffusion of an individuals genes. They are harder to measure or test experimentally. Whatever one says about this is based on intuition and thus highly speculative. But it is a topic one cannot afford to avoid for this reason. For it is becoming increasingly central to the most fundamental questions of cultural development.

The problems of civilization today are far more a battle of values than a struggle for resources. Technology has transformed this equation as it has so many other things. Of course resources like oil still play a major role, but not the primary one. The major cultural struggles today are about values, religion and spirituality. They stem from our deeper instincts for identity and how those instincts form the attitudes and beliefs of individual and societies.

12.2 The archetypal self of Jung

> As an empirical concept, the self designates the whole range of psychic phenomena in man. It expresses the unity of the personality as a whole. ... it is a *transcendental* concept, for it presupposes the existence of unconscious factors on empirical grounds and thus characterizes an entity that can be described only in part, but for the other part, remains at present unknowable and illimitable[32, ¶789].

Jung suggests in the above quote that we have evolved a wider sense of self that is transcendental for it is at present unknowable and illimitable. Does this make any sense?

As we develop and individuate the psychic functions we pull more of the unconscious forces of life into the domain of consciousness. This expands our sense of self. Are there limits to that expansion? Our individual existence is an integral part of the evolution of consciousness on this planet and in the universe. Most of the chemical elements in our body were created in stars that exploded eons ago. We could not exist without the history of those stars. Any boundaries we draw around the self are arbitrary. We need a limited sense of self for practical reasons. Consciousness only exists in the particular. But it also only exists in the wider context of an evolving universe. From the narrow ego of "me now" to the all encompassing spiritual evolution of consciousness is a vast array of possible selves. They are all valid but limited views of reality.

Jung saw the mandalas created by every culture as a symbolic representation of the self.

> Their [mandalas'] basic motif is the premonition of a center of personality, a kind of central point within the psyche to which everything is related, by which everything is arranged, and which itself is a source of energy. The energy of the central point is manifested in the almost irresistible compulsion and urge to *become what one is*, just as every organism is driven to assume the form that is characteristic of its nature, no matter what the circumstances. This center is not felt or thought of as the ego but, if one may so express it, as the *self*. Although the center is represented by an innermost point, it is surrounded by a periphery containing everything that belongs to the self—the paired opposites that make up the total personality. This totality comprises consciousness first of all, then the personal unconscious, and finally an indefinitely large segment of the collective unconscious whose archetypes are common to all mankind. A certain number of these, however, are permanently or temporarily included within the scope of the personality and,

> through this contact, acquire an individual stamp as the shadow, anima and animus, to mention only the best known figures. The self though on the one hand simple, is on the other hand, an extremely composite thing, a "conglomerate soul," to use the Indian expression.[31, ¶634]

The human psyche in our bodies and brains is the most complex structure in the known universe. We are at the earliest stages of acquiring the tools that will make it possible to gain a rigorous scientific understanding of the psyche. In the absence of the necessary tools the natural instinct is to try to fit the immense complexity of the psyche into an overly narrow intellectual model. Two of Jung's contemporaries, Freud and Adler, constructed such models. Jung was motivated to write *Psychological Types*[32] by the limited truth he saw in both of their perspectives and the wider terrain he had observed in working with patients and through introspection.

The price paid for such a broad view is lack of precision and rigor. Jung's work is largely intuitive, it is at best vaguely correct and no doubt often precisely wrong. Still I believe it the best attempt to date to probe the depths of psychic structure especially its profoundly creative instincts.

In evolving an extraordinarily flexible psyche it was necessary to evolve an equally flexible system of motivation or emotions. Without the flexibility of motivation the flexibility of the psyche itself would never be used. The key to flexibility of motivation is inherently conflicting motivational structures. We have discussed a little of this structure in Sections 10.2 and 10.3. The self as Jung defines it is the core or central element that keeps these contradictory forces operating as an integrated whole.

To what end does this process operate? It was created by evolution and so survival is the architect. But it is survival not just of the next generation but into an indefinite future. The self as Jung describes it is the psychic image of this unlimited potential for future development. As such it focuses on the many dimensions of human functioning that contribute to survival including creativity in all its forms.

> Sensing the self as something irrational, as an indefinable existent, to which the ego is neither opposed nor subject, but merely attached, and about which it revolves very much as the earth revolves round the sun—thus we come to the goal of individuation. I use the word "sensing" to indicate the apperceptive character of the relationship between ego and self. In this relationship nothing is knowable, because we can say nothing about the contents of the self. The ego is the only content of the self that we do know. The individuated ego senses itself as the object of an unknown and supraordinate subject. It seems to me that our psychological inquiry must come to a stop here, for the idea of a self is itself a transcendental postulate which, although justified psychologically, does not allow of scientific proof. This step beyond science is an unconditional requirement of the psychological development I have sought to depict, because without this postulate I could give no adequate formulation of the psychic processes that occur empirically. At the very least, therefore, the self can claim the value of an hypothesis analogous to that of the structure of the atom. And even though we should once again be enmeshed in an image, it is none the less powerfully alive, and its interpretation quite exceeds my powers. I have no doubt at all that is an image, but one in which we are contained.[26, ¶405]

The self is transcendent because it points to an unlimited future and unbounded creative expansion of the evolutionary process. This is something that no being can comprehend. Of course we can have some sense of the future structure of the evolutionary process, but that tells us nothing of its essence. It tells us nothing of what it is like to be a more highly evolved being.

Is it plausible that such a psychic structure would evolve and if so how can we accept Jung's claim that this structure does not "allow of scientific proof". The key to this riddle may lie in the previously mentioned intuition of Jung that number is the archetypal mediator between the physical and the transcendent.

> The role that number plays in mythology and in the un-

> conscious gives food for thought. They are an aspect of the physically real as well of the psychically imaginary. They do not only count and measure, and are not merely quantitative; they also make qualitative statements and are therefore a mysterious something midway between myth and reality, partly discovered and partly invented. Equations, for instance, that were invented as pure mathematical formulae have subsequently proved to be formulations of the quantitative behavior of physical things. Conversely owing to their individual qualities, numbers can be vehicles for psychic processes in the unconscious. The structure of the mandala, for instance, is intrinsically mathematical. We may exclaim with the mathematician Jacobi: "In the Olympian host Number eternally reigns,"
>
> These hints are merely intended to point out to the reader that the opposition between the human world and the higher world is not absolute; the two are only relatively incommensurable, for the bridge between them is not entirely lacking. Between them stands the great mediator. Number, whose reality is valid in both worlds, as an archetype in its very essence.[29, ¶777]

Mathematics allows us to gain some understanding of the evolution of structure over time. It connects with the transcendent. We can know about structural aspects of what will be. But structures of the psyche that have evolved to facilitate human creativity do not have a precise or scientifically comprehensible goal. If they did they would not be creative.

One thing to keep in mind in interpreting Jung's intuitions about Number is that he never understood mathematics.

> My intellectual morality fought against these whimsical inconsistencies, which have forever debarred me from understanding mathematics. Right into old age I have had the incorrigible feeling that if, like my schoolmates. I could have accepted without a struggle the proposition

> that a = b, or that sun = moon, dog = cat, then mathematics might have fooled me endlessly–just *how* much I only began to realize at the age of eighty-four. All my life it remained a puzzle to me why it was that I never managed to get my bearings in mathematics when there was no doubt whatever that I could calculate properly. Least of all did I understand my own *moral* doubts concerning mathematics.[33, p 28]

Perhaps Jung's intuitive sense that structure never captures or even touches on essence underlies his difficulty with mathematics. Mathematical identity is structural identity as made explicit in the Axiom of Extensionality given in Section 5.5.1. But mathematical identity is not existential identity. In the physical world every object has a location. Even if two objects at different locations have identical internal structure their relationship to time and space keep them from being identical. They are two essences and not one.

Perhaps Jung,s intellectual morality would not allow the artificial separation of structure and essence that is at the core of contemporary mathematics and science. That separation is an artificial game that is essential in the hard sciences that have become purely mathematical. He no doubt could have learned to play the game. Perhaps he would have been happy to do so if he understood it in these terms. But he was living at a time, as we still are today, when the discoveries of science about physical structure are all too often taken as the primary or ultimate reality. The problem with that is that science and mathematics deal only with structure. Seeing structure as ultimate reality leads to a dead and meaningless universe. For Jung the universe is overflowing with meaning.

12.3 The divine self

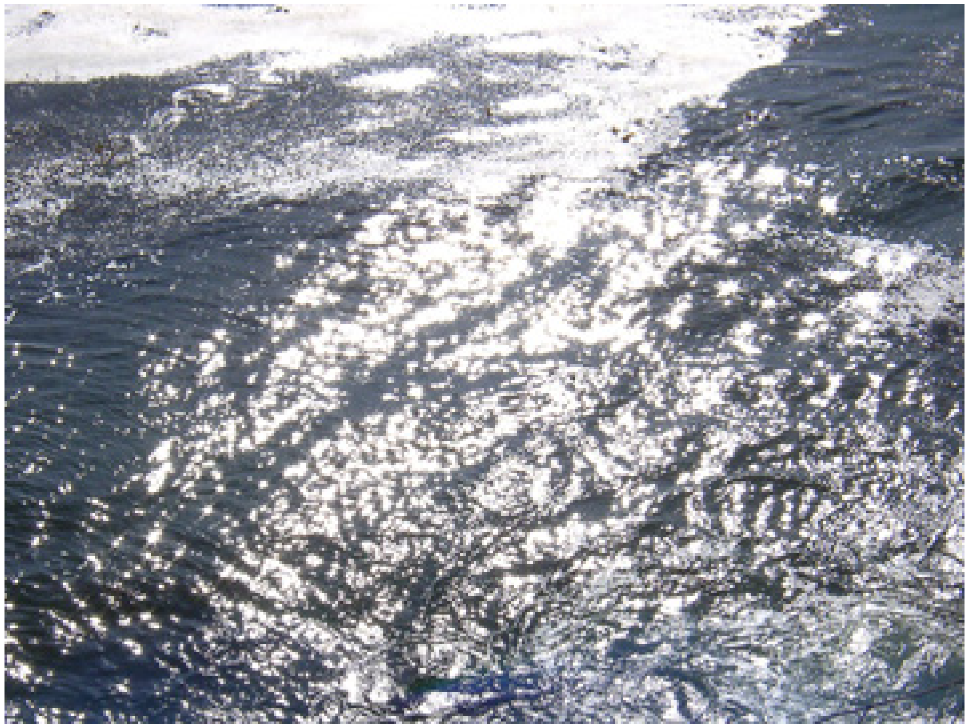
We are for more the divine self than the limited ego. As the divine self, we have been alive throughout the history of man and of life itself and we will live on into an indefinite future. It is no more arbitrary to think this way, than it is to believe a particular collection of cells that existed a decade ago or will exist a decade into the future is the same being I am now. It is memory and similarity of experience that connects us with our future and previous individual existence. But we have more than individual memory. We have cultural memory that has played in an essential role in creating our present selves. And we have archetypal memory that has played a central role in forming our individual being.

In recognizing that we are the divine and unbounded self it is essential that the ego not identify with that self. That leads to a dangerous inflation of the ego. The eternal unbounded self only exists as embodied in particular individual experience. We have an arc of consciousness that is our individual existence. It is an integral inseparable part of a wider existence, but it is equally a particular stream of consciousness with a beginning and end. *After the Ecstasy, the Laundry*[34] is a book that expresses this point elegantly. The practical realities of any individual life fade away through the mists of time, but they are the essential basis for all life and experience.

We all need to live in both dimensions and to recognize that they are different perspectives on a single reality. We must come from where we are, but where we are has the seeds of what can be. The road from what is to what will be is not easy. It demands far more than good intentions. The rich complexity of the human psyche evolved because it solved practical problems of survival. We need all of the resources evolution has bestowed on us, including the inherent conflict that is an essential part of our instincts.

There is no single path to salvation nor single salvation. Evolution is a divergent process. Pain and struggle *at some level* will always be an essential part of the journey. Spiritual growth creates evil. When only singled celled organisms existed on earth, none of the evil we experience today existed. We can mitigate or eliminate evil at one level only by evolving to new levels. This create new levels of meaning and new problems. We cannot prevent the resulting evil without destroying the creative process.

All searches for perfection or the *absolute* lead to the hell of stagnation or worse. But heaven is obtainable, not as the final triumph of good over evil, but as an ever expanding richness of experience. This expansion, by its very nature, creates new possibilities for pain and suffering. One deals with evil not be eliminating it, but by expanding ones own consciousness. As ones center shifts away from the ego and towards the divine self, one becomes more at peace with the circumstances of ones own life and more able to contribute to the ongoing evolution of ever more wondrous conscious experience.



Chapter 13

The eyes of God

Existence and consciousness are mysteries beyond explanation. They are where explanation starts. God is the traditional name for these mysteries and I use it. God is a work in progress. God is the creative evolution of consciousness. This evolution is a physical process, directed not by a higher intelligence, but by simple laws. These laws have molded the amorphous consciousness of inanimate matter into the extraordinary subtlety and richness of human experience. In the process that have laid the groundwork for the most profound change in the structure of evolution. Human understanding has the potential to direct the future evolution of consciousness.

We are the eyes of God with the power to create the world. We are acquiring the power to consciously control evolution. To deal with this enormous responsibility, requires a sense of spirituality firmly rooted in the archetypal wisdom expressed through the worlds religions and equally rooted in objective scientific and mathematical understanding. This chapter relates the ideas of this book to religion and spirituality. It lays the groundwork for a modern mythology that grows out of our mathematical and scientific understanding and connects to the universal themes that unite the mythologies of all cultures.

13.1 Spirit is physical

The spiritual journey is a physical journey. There do not exist distinct physical and spiritual realms. The evolutions of physical structure is *the* spiritual journey. Matter is the transformation of consciousness through time. The essence and substance of the physical exists only in the irreducible reality of immediate experience. Spiritual intuition points to the infinite and eternal, but spirit transcends the finite and the physical only be being realized in the finite and physical. No finite realization fulfills the spiritual potential, but there is no spirituality without the actualization of the spirit in finite conscious experience and thus finite physical structures.

We cannot transcend physical reality or evolve to a higher non-physical level of consciousness. But this does not mean that mythical ideas and religious intuitions that suggest we can are nonsense. The spiritual transcends the physical through physical evolution over time. Our spiritual intuition connects us with this process. Every realization of the potential of existence is necessarily finite and limited, but the process of evolutionary development need have no finite limit.

Equating the physical and the spiritual does not impose limits on the spiritual. It may seem that we are making the laws of physics paramount, but that is a misconception. The laws of physics are the simplest assumptions that explain what we know of the structure of the world. Physical reality is not defined by the existing laws of physics. Its the other way around. The laws of physics come from our imperfect attempts to explain the structure of human experience.

Science deals with anything that is repeatable and measurable. Science does not claim that the existing laws of physics will be the final word on our understanding of fundamental physical structure. Science does not even deny the existence of entities that are not repeatable and measurable. It only pleads helplessness in dealing with them for good reasons. Knowledge is useless as a guide to the future unless it leads to measurable repeatable effects. For example one might have the insight that touching a person in a certain way will cure an intractable disease. If this is a one time event, it has not practical value for guiding future actions.

Seeing the spiritual as physical enriches rather than restricts spirituality. As discussed at the end of Section 12.2, Jung saw number as the archetypal mediator between the physical world and the spiritual world. Seeing spirit as always embodied in the physical allows us to apply mathematical understanding to spiritual questions. The unbounded hierarchy of mathematical structures discussed in Section 9.4 suggests that whatever ecstatic conscious experience anyone ever has it is the merest hint of a shadow of what can be and that will always be the case. The most powerful argument for this possibility is the history of evolution and the extraordinary beings it has created.

13.2 Ethics and Spirituality

The idea that an all powerful father figure created the universe and the rules of ethics seems absurd from even a casual study of history. The immense cruelty in the world would be a reason for utter despair, if one thought it the result of the design of an all powerful being. Yet attempts to create ethical systems without reference to the creative nature of the universe, like Utilitarianism, fail to connect with the spiritual instincts that play an essential role in ethical feelings and thought. The Totality Axiom suggests an ethical approach similar to Utilitarianism aimed at maximizing happiness. But the Totality Axiom and Gödel's result imply that the capacity for experience is continually expanding. We need not only to make the world less cruel and more joyful. We need to support the evolution of beings whose experience of joy is beyond anything we can imagine.

The cruelty of evolution, including the evolution of culture, that dominates the world stage today is an inevitable part of a creative evolutionary process that is *not* directed or designed by a higher intelligence. Consciousness has evolved through a random and cruel struggle for survival. In the process values were created that can lead us into a less cruel more joyful future. The Totality Axiom is a starting point for integrating our evolved values with an objective view of the universe. We can create an ethics that strives not just to minimize suffering and maximize happiness, but also to extend the capacity for joy through the creative evolution of consciousness. Ethics in tune with the full range of human instincts must have a spiritual vision.

The spiritual vision suggested by the philosophy expressed here can be summarized as follows. God is the creative universe. She does not have a further explanation or creator. God is not an ultimate being or final destination. She is the unbounded evolution of consciousness. God is infinite in potential but not in actuality.

This view of spirituality has parallels with Buddhism, which sees our kinship with all sentient beings. Ultimately it sees all such beings as one. All that exists is the evolution of consciousness, which is an indivisible whole. There is no unique soul that defines one's individuality. We are not ultimately separate from the rest of humanity or the rest of the physical world. Our soul is the world soul with its ever evolving consciousness.

Traditionally Buddhism seeks enlightenment as a final or ultimate goal although many contemporary Buddhist thinkers see it more as a process with no end point[11]. There is no ultimate goal in the framework described here. There is only a continual striving for a higher level of consciousness. By treating Buddhist philosophy as metaphor we can speculate about how enlightenment can be reinterpreted.

The cells in the human body do not compete for survival as do single celled animals. They have a steady supply of food and nearly ideal living conditions. The price they pay for this is to loose their freedom to reproduce independently of the needs of the body they are part of. If they renege on that bargain and become cancer cells, they may destroy the environment that gives them life.

One can argue that cells in an evolved animal have reached a form of enlightenment. They have not eliminated the problems of survival, but they have pushed those problems to a new level. As long as the organism they comprise survives they live in a protected environment.

One cannot end all suffering or all attachment, but one can, to a large degree, push them to a higher level. Enlightenment is not an ultimate achievement but a continual progression.

Christian notions of heaven can be connected to this sense of enlightenment. We have or are developing the technology to create something approaching heaven on earth. We can eliminate most forms of suffering and it seems likely that we will learn to greatly extend human life. Much of what one may imagine in heaven may become a practical reality. Even the sense of communion with God, that is central to Christian beliefs, is obtainable. Understanding and *feeling* that we are an integral part of an unbounded creative process is communion with the great mystery and power of existence.

Like the cell we will pay a price in limiting our reproductive potential. A stable ordered world can only support a finite population although probably one far greater than exists today or in the foreseeable future. (This is not to question the enormous problems that over population creates today. It is only to suggest that, in time, we will develop the technology to support a much larger population in a fully sustainable way.) We can use our understanding of mathematical creativity to minimize the limitations that a finite planet imposes on the evolution of consciousness, but we cannot eliminate those restrictions. It is likely that we will start to reproduce as an entire world sending unmanned probes into space with enough technology and biological material to create new civilizations on worlds in which life has not evolved. We will be somewhat like the cells in an organism. We will live in a protected environment, but we will still be creative individuals striving to expand our own consciousness and that of future generations. We will want to seed the galaxy and universe with evolving creative life. We will have reached the heaven of Christianity, but see it as a single stepping stone on an endless unbounded divergent path.

If we get beyond the current crises and figure out out how to live together in peace and cooperation on this tiny planet, than we will push much of the strife that has dominated the history of mankind to a new level. Our galaxy and certainly our universe is likely to have more than one reproducing world. We will be competing with them to seed the universe and we will also be 'mating' with them to accelerate the creative process. Any civilization that reaches that level of development will understand the implications of Gödel's Incompleteness Theorem for the evolution of consciousness and structure. This should instill a respect for all possible paths of development and limit the desire to dominate the universe.

Chapter 14

What will be

We cannot know the future, but we will collectively create it. This has never been truer. Human activity is transforming the world often in dangerous ways. Science and technology continually expand our ability to do so. What will we do with this power?

We can create a second Eden. The enormous environmental problems of today all have solutions. The continual expansion of productivity through technology means that we can work less and live better. We will probably, in the next few decades, develop an understanding of how our genetic code shapes the proteins that in turn create our bodies. Such an understanding should lead to magic bullets tailored not just to a disease, but to the individual with the disease. Much human suffering can be eliminated. Even aging may be forestalled, stopped or even reversed[37]. We can through technology create a second Eden where much of human suffering is eliminated and we have the time and resources to live joyful creative lives.

But that does not seem to be the direction we are going. In the United States we are developing a winner take all climate that lavishes resources on a handful of super stars to such a degree that the entire economy suffers. Instead of increasing productivity allowing all of us to work less and live better, it is seen as a threat to job growth and deflationary. If the benefits of productivity were widely shared with the workers that are made more productive, this would not be a problem. But if wealth is increasing concentrated in the hands of the few, than productivity hurts the economy by siphoning a greater percentage of total wealth into the hands of those who will not spend it on themselves and will not invest it in unneeded new production.

The myth of Eden speaks of a time when our ancestors were living in harmony with the natural world and their internal nature. It was self consciousness and the power of decision that comes with self consciousness that separated us from Eden. Instead of simply reacting, we started thinking about our decisions and in the process created a new reality. The world became split between good and evil. Our harmony with nature and ourselves was shattered. Cultural evolution accelerated this process. Evolution created a mind able to shape culture too rapidly for evolution to adjust. Eden had been lost as the price of rapid progress.

We cannot evolve our instincts to adjust to a rapidly changing culture. We need to recognize that simplistic ideologies that appeal to those instincts are deadly dangerous. There is no easy or simple solution. Love is not *the* answer. Unbridled capitalism is not *the* answer. No fundamentalist religion holds *the* answer. No ideology or doctrine is *the* answer. We have to create solutions for our problems one at a time. As we succeed at this, there will be a multiplication of opportunity with new problems. These will require more creative solutions and that will in turn expand our possibilities and our problems.

The second Eden, if it comes to pass, will be a time when we again live largely in harmony with ourselves and our environment. We we will have mastered the art of using the bounty of nature expanded through technology without lessening that bounty. We will have learned to live together respecting all our varied instincts and we will have social structures that makes it practical to meet most needs of most people. But the second Eden is not the end of the line or the ultimate goal. It is base for future evolutionary steps.

This chapter is about the path to a second Eden and beyond. It starts with a discussion of how to avoid chaos and stagnation using terrorism as an example. It then discusses how tax policy can be used to serve the interest of creativity by helping to strike a balance between diversity and concentration of resources. The chapter then moves from the practical to the speculative discussing globalization and its ultimate endpoint a global organism. The chapter ends with speculation on the perhaps unending evolution of consciousness.

14.1 Terrorism

When the World Trade Center buildings were destroyed with massive loss of human life. President Bush saw the struggle against terrorism as good versus evil. Such a simplistic view is wrong and dangerous. The United States literally created the Taliban and Al-Qaeda. In the language of Jungian psychology they are a physical embodiment of the shadow of Western culture.

In the development of an individual the shadow is a metaphor for all the poorly developed and poorly differentiated elements of the personal psyche. We project these primitive and threatening aspects of ourselves onto our image of *the enemy*. The Taliban and Al-Qaeda are real enemies created by the inferior elements of our cultural development and theirs. We gave them the weapons that allowed them to defeat the Soviet Union and then we abandoned Afghanistan to chaos. We helped to finance the anarchy and war-lords in Afghanistan with our archaic and destructive drug policy.

We need to do what we can to prevent terrorism and punish those responsible, but ultimately the war is not against an external enemy. The war is within our culture and within each individual just as the real Jihad is a struggle within the individual. We can only conquer terrorism by developing and integrating the inferior elements of our culture. This is the most difficult of tasks in that it can only be accomplished through the development of each individual.

Joseph Stiglitz, a Nobel Prize winning economist, wrote a damning indictment on the policy of international financial institution such as the World Bank and World Trade Organization. *Globalization and its Discontents*[46] describes how these institutions under the guise of a capitalist orthodoxy has forced many third world countries into destructive policies that were long ago recognized as such by the vast majority of economists. The United States is the dominant power in these organizations. One of the most troubling examples is the way third world countries are forced to open their markets to manufactured goods largely destroying indigenous industries while simultaneously their agricultural products are forced to be noncompetitive because of the huge agricultural subsidies in first world countries. The policy can lead only to impoverishment.

An even more devastating indictment of United States Policy is contained in John Perkins' *Confessions of an Economic Hit Man*[42]. Perkins was employed as an economist. His job was to analyze the impact of large projects on developing countries. It was made clear to him that he was to vastly over estimate the growth that these projects could create. The purpose was to economically enslave the target country by saddling them with debts they could never repay. The resulting economic dependency would create a new form of empire whereby the United States could rule the world. The result has been economic devastation across the developing world.

The sickness in this over reaching desire to control the world is a symptom of the one sided focus on intellectual development just as orthodox capitalism is. Economic freedom is an essential element of liberty, but economic freedom is a complicated thing. No simple formula such as providing free markets leads to economic freedom. Only regulated markets can be truly free. Otherwise fraud and legal thievery will dominate. The same is true of the market for labor. It is not freedom to allow a worker to be employed in a job that is likely to kill or injure him. It is not freedom to allow a plant to pollute the air and water. Excessive or needless regulation can destroy freedom as effectively as unfettered markets. Excessive regulation throughout much of the developing world makes it impossible for the poor to access the capital they have accumulated in their homes as Hernando de Soto explains in *The Mystery of Capital*[15].

International financial institutions must recognize the enormous complexity of the problem of development. Instead of tying to fit reality into a narrow intellectual model, they must accept their limited understanding and ability to fix things. The must stop trying to control the world and start trying to empower it. They must stop putting the top priority on securing loans made by powerful first world institutions. First priority should go to the welfare of the people in the countries they are claiming to help. This does not mean a policy that leads to dependence, but it must be a policy that does not further impoverish the people by extracting repayment of debt incurred by dictators that stole and squandered the original loans. This is doubly true because these loans were based on deliberately fraudulent economic forecasts.

Policies that enrich special interests in the West while further impoverishing the desperately poor are murder by other means. They are as morally reprehensible as terrorism because they equally destroy lives of innocents. Admitting that we are doing this is a morally difficult act. Any politician who suggests it will be crucified. But it is the truth and until we can, as a culture, see it as such, we will keep doing it and keep creating the hate that can lead to terrorism.

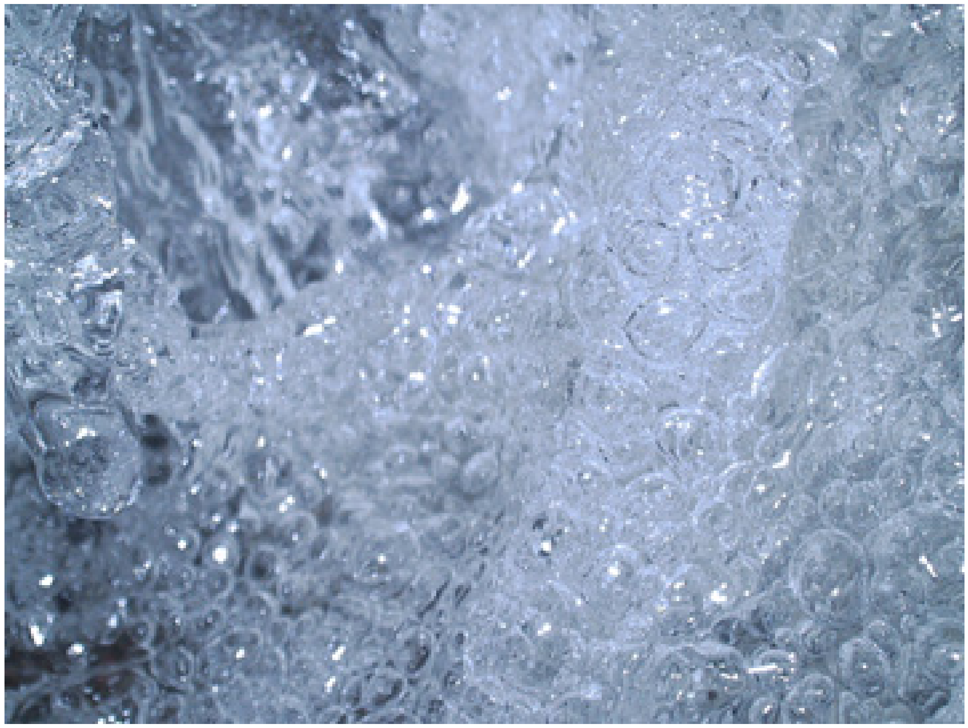
It is not an impossible task. The starting point is to recognize that the problem exists. We have purchased the power of our technology at the price of developing a very one sided culture where logic and intellect reign supreme on the surface. Evolution developed those functions because of their practical value, but simultaneously it evolved complementary and countervailing forces. These are equally active, but we are not so aware of them. These underground forces are often the dominant factor in our decisions.

Because we are unaware of their functioning we repeatedly misinterpret and project these forces.

In Section 10.3 we touched on Jung's concept of the dominant function of our age, thinking, and its shadow feeling. Our cultural inferior feeling is the source of many of our problems. It leads to the arrogant selfishness that has largely characterized our international policy since the war on communism became our dominant international concern. The fall of communism has not changed things much. Of course we have on occasion taken action for humanitarian reasons, but these instances have been all too infrequent given our enormous power and the problems the world faces.

The cultural path to development and differentiation of feeling starts with the development of intuition. Though intuition we can deal with situations that are too complex or poorly understood for intellect. A neural net can converge to an effective solution without any concept of how or why the solution works. Intuition can make wild inspired guesses for almost any problem. Most such guesses will be wrong or ineffective, but with experiments it can converge on truly effective solutions for seemingly unsolvable problems. Intuitions is the starting point for recognizing the full complexity of the problems we face and the inadequacy of intellectual solutions to these problems.

14.2 Diversity and taxation



Intuition requires diversity. Evolution has made us extremely diverse, but the winner take all culture of the West makes it increasingly difficult to tap into the creativity inherent in human diversity. We have to reverse this. Perhaps the single most important thing that could be done would be a asset tax. It should be a progressive tax with a very large deductible (in the millions of dollars). It could replace the corporate income tax, capital gains tax and taxes on dividends. The deductible for individuals should be set high enough so a family can become financially independent, with enough wealth to meet expenses for an indefinite future with reasonable assumptions about market appreciation and cost of living.

For Corporations the deductible should be large enough to encourage the formation of new companies. At the high end the rate should discourage megacorporations and huge mergers. The deductible limits and rate brackets should grow with inflation and also with per capita income, but at a lower rate than growth of that income. The idea is from the founding fathers of the U. S. We want to encourage people to obtain economic freedom and discourage excessive concentrations of power.

Such a tax would be more just than almost all existing taxes. Much of the federal government exists to protect wealth and thus it is only fair that wealth be taxed to pay for that protection. When one has accumulated more wealth than one will need in a lifetime than wealth becomes a tool for doing good by wise investing. It is socially valuable for those that are most efficient at such investing to accumulate wealth most rapidly and those that are not good at this to loose the wealth they cannot make productive. This is best accomplished by taxing wealth itself and not the income obtained from wealth.

When corporations become too powerful they are able to leverage their power in ways that give them an unfair advantage over smaller competitors. The antitrust laws have a limited ability to prevent this. Because of this phenomena very large corporations impose a bigness tax on all of us by limiting the ability of competitors to develop innovative and cost effective alternatives. An outrageous example of this is the “Microsoft tax” on desk top operating system and office productivity tools. A bigness tax on corporations will cause corporations to voluntarily limit their size through spinoffs. It will recapture some of the excess profits of excessively large and powerful corporations that refuse to break up into smaller independent entities.

14.3 A Global Organism

The perennial conflict between diversity and concentration of resources is entering a new phase. The creation of global economic and political institutions to resolve disputes between nations peacefully is an essential, but also very dangerous development. History suggests that such institutions will inevitably serve the interests of the powerful in opposition to the general interest. The situation is complicated by the corruption of the vast majority of governments in the developing world.

In addition to the problem of dominance by the powerful globalization presents us with the danger of a race to to the bottom in areas such as wages for the lightly skilled, environmental protection, working conditions, right to unionize and legal protection against corporate abuse and corruption. Unfettered free enterprise will seek the greatest profit without regard to the human cost.

For all of these problems it is not that difficult to provide practical solutions. We need to price in all the costs of doing business including things like environmental degradation and we need to establish a universal legally enforceable code of human rights. The difficulty is implementing the solutions in a world dominated by special interests. Whether or when we will succeed is an open question. The appeal of demagogues and simple minded solutions is all too compelling to far too many people.

I am an optimist and for the remainder of this chapter I assume we will overcome the immense difficulties of preserving diversity and creativity in globally governed world. Where will this take us? Where do we want to go? A second Eden is the starting point. The work needed to provide food, shelter and good health will continue to decline to the point where such work is largely automated. The world's population will be independently wealthy with a minimal need to work for a living. The creation of beauty, art and knowledge will be engrossing tasks. The exploration and expansion of consciousness will be the primary source of meaning and value as it becomes increasingly clear this is what spirituality is about.

There are multiple ways in which consciousness can be expanded. Traditional methods of spiritual practice are one. Far more people will have the time for such pursuits. Mind altering drugs are another. In time our puritanical attitude to such drugs will fade and our understanding of neural physiology will allow us to explore this arena more safely than is possible now. Directly interfacing physical devices with the nervous system is just beginning to be experimented with to mitigate handicaps such as blindness or deafness. As this research expands it may some day be used to expand the consciousness of those with normal sensation.

In the not too distant future the combinatorial complexity of relatively inexpensive computers will exceed the combinatorial complexity of the human brain. That does not mean that such computers will be intelligent, but it does open the door to that possibility. It is impossible to predict when machines will reach something approaching human intelligence because we still have such a limited understanding of the functioning of the brain. That it will happen in an environment of peace and creativity is almost inevitable. However that does not necessarily mean that machines will come to dominate. For simultaneously with this development will be expanded interfaces between computers and the human neural network so that human intelligence and consciousness will be expanded.

Perhaps the strangest possibility is the emergence of a global intelligent conscious organism. Inevitably the global network of connected computers and human beings will vastly exceed the complexity of any single human brain. This opens the possibility for the emergence of a world intelligence and consciousness that is as far removed from human consciousness as our consciousness is from that of the cells we are composed of. This may sound frightening and threatening and potentially it is. Any single path approach to development, no matter what resources it has, is doomed to a Gödelesque stagnation. Seeing our individual existence as being like cells in a global organism raises the specter of a monstrously cruel totalitarian state that views its people as a resource to be exploited.

The dangers are real and there is no sure way to avoid them. But understanding evolution in the widest possible context is one essential element in minimizing the risk. We need to recognize that evolution is an ongoing process and cultural and technology are both a product of evolution and essential elements in its expansion. They are the key to accelerating evolution. They are major milestones in evolutionary history like the creation of DNA the development of multicellular organisms or sexual reproduction. Seeing technology as opposed to nature or other than nature is to miss the forest for the trees. Technology is a natural product and an essential stepping stone in evolutionary creativity.

Equally essential to avoiding the dangers inherent in a unified planet is the cultivation of a wider sense of self discussed in Chapter 12. We need to *feel* that we are the evolutionary process. A global consciousness is our future and destiny. One must feel that their history is not just that of a individual life, but of life on this planet. One must feel that ones individual future is the future of life. When this happens one's perspective and values shift. One becomes more saintly or enlightened or whatever it is called in your spiritual tradition. One must *feel* it in the way one identifies with oneself as a child and with the person one will be tomorrow.

So what will we do when we become a conscious planet? That is a bit like asking what does God do. The question is way beyond us, but one can speculate about how evolution will proceed beyond this planet. Barring a revolution in the laws of physics that leads to faster than light speeds we will travel to other worlds only through robotically controlled vehicles. But these can contain much of the knowledge that human culture has created up to the time they are launched and enough biological and mechanical devices to create evolving live on barren planets. We will, as a global organism, seed the galaxy with life. Are there any limits to the evolution of consciousness?

14.4 The merest hint of a shadow

Is the universe infinite? That is a question that can never be answered conclusively. I suspect it is potentially infinite. It could, for example, be expanding and creating new mass at its perimeter as discussed in Section 8.9. Other possibilities require that the universe be limited to some particular finite value. Why would it be that particular value?

Whether or not the universe is infinite it is big enough! Evolution can continue for billions of years and develop technologies and consciousness that is beyond our imagination. Any notion of God as an ultimate final and complete entity seems tiny and small to me in comparison to a continually evolving God-consciousness that expand with perhaps no limit.

What will be? Whatever ecstatic wondrous experience anyone ever has it is the merest hint of a shadow of what will be and that will always be the case. That is what I like to think and as strange as it may sound that is the simplest possible assumption consistent with what we know to be true.

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